Institute for Language, Logic and Information

SEMANTIC PARALLELs
IN NATURAL LANGUAGE AND COMPUTATION

Johan van Benthem

ITLI Prepublication Series

for Logic and Philosophy of Language  LP-88-06

University of Amsterdam
Table of Contents

Semantic Parallels

1 Convergence

2 Minimality

2.1 Non-Standard Inference

2.2 The Logic of Circumscription
   2.2.1 Minimal Models
   2.2.2 Conditional Logic
   2.2.3 Dynamics of Minimality
   2.2.4 First-Order Reduction

2.3 Data Types and Theories
   2.3.1 Logic Programs
   2.3.2 Abstract Data Types ... 
   2.3.3 ... and Scientific Theories

3 Dynamics

3.1 Dynamics of Interpretation
   3.1.1 Operational Semantics for Programs
   3.1.2 Operational Semantics for Assertions

3.2 Dynamics of Information Flow
   3.2.1 Reducing Possibilities
   3.2.2 Building Up Information
   3.2.3 Relational Calculus and Categorical Grammar
   3.2.2 Propositional Dynamic Logic

4 Discussion

5 References
of the discipline as studying the variety of modes of inference which human beings have available for various intellectual tasks. With Bolzano, this led to the logical study of several notions of valid consequence, differing in their formal properties, without a pious commitment to any particular calculus. And that is certainly also the proper spirit to approach the recent developments to be discussed here.

The paper also makes a number of technical contributions to the topics discussed. The first group of these concerns minimal models. We provide a mathematical analysis of the minimization operator on classes of models (Section 2.2.3), while also investigating several special systems in which minimal models play a central role. Many such systems are found in the areas of Logic Programming and Abstract Data Types (Section 2.3). Here, we develop an analogy with earlier work in the Philosophy of Science on so-called Ramsey eliminability of theoretical terms in scientific theories (Section 2.3.2), showing how the latter can incorporate minimal models (Section 2.3.3). The central example, however, is the notion of circumscription, recently developed in Artificial Intelligence: which may be viewed as a semantic generalization of minimal Herbrand modelling for Horn clause formalisms. A technical connection is found between the general inferential properties of circumscription and more traditional conditional logic (Section 2.2.2). We also consider possible reductions of circumscriptive inference to standard first-order logic, establishing a high complexity for the question just when this is possible (Section 2.2.4).

Then, there is also a number of results on dynamical semantics. We provide several reductions of proposed dynamic systems to, again, standard ('static') first-order logic (Sections 3.1.2, 3.2.1). Also, various proposals are made for connecting up what may be called 'abstract information structures' with already existing types of models, notably in relational algebra (Section 3.2.3) and propositional dynamic logic (Section 3.2.4). Especially, the latter system seems to provide a promising tool for investigating dynamic modes of handling propositions.
2. MINIMALITY

2.1 Non-Standard Inference

Logically valid inference in the standard sense allows us to draw those conclusions from a set of premises which are true in all models for those premises. This is an important, and safe procedure. But, conclusions drawn in this way may be few in number, when we have to act upon a small amount of explicit information. Accordingly, we may observe the presence of additional mechanisms in cognitive activity, allowing us some further 'contextual' inferences. One pervasive mechanism of this kind is the assumption of a certain completeness: the premises as stated give us 'the whole truth' about the matter [and of course, to continue the juridical analogy, hopefully also nothing but the truth']. In the Philosophy of Language, this phenomenon was observed by Paul Grice, whose "Maxim of Quantity" expresses that what we say should state our complete information (as far as relevant to the matter at hand). Here is a concrete example of how this convention works. If my partner in a conversation tells me that Basra lies either in Iran or in Iraq, I can take it that she does not know anything more definite about its location than that. So, I can conclude that she does not know that it lies in Iraq - even though this does not logically follow. (She might be just pretending ignorance, of course: but, this is not assumed in normal conversation.) A less epistemically oriented example would be this. If I am told that event A happened because of event B, I will tend to assume that, as far as the speaker is concerned, B is the only cause of A. Again, there is no logical necessity here: but rather the assumption that I have been supplied with the complete relevant information.

These examples are still rather vague - and indeed, one recurrent complaint about pragmatic principles like Grice's Maxims has been that they can be twisted to suit too many a purpose. Nevertheless, there are already two interesting features of non-standard inference to be observed here. First, we increase the number of available inferences, by making some general assumptions about the 'status' of our premises. Next, as a price to be paid for this, we have to be vigilant, since the additional conclusions are defeasible, and may have to be given up after all, in the light of further evidence. (I may learn subsequently of another possible reason for event A.) The latter phenomenon is often called non-monotonicity, in contrast to the 'monotonicity', or cumulativity, of inferences in standard logic.

One way of making these procedures more precise consists in the following modification of the standard semantic picture. A piece of text (story, proof or
program) is often concerned with, not all its models in the usual logical sense, but rather with one intended, or at least a severely restricted class of intended models. One reason for this seems to lie in the operation of a general principle allowing us to exploit absence as well as presence of explicit information:

the world is the smallest,

or at least a minimal situation verifying our data.

And then, of course, more conclusions will be valid (generally speaking) in just the minimal models of our data than in all of its conceivable models. This perspective turns out to admit of precise mathematical treatment, and indeed, it runs through a number of technical proposals for implementing reasoning in a computational setting. We shall look at a few examples, while pointing at a more general background in the philosophy of language, and also the philosophy of science.

The above idea is prominent in Logic Programming. Notably, PROLOG programs are supposed to describe a so-called 'minimal Herbrand model' for their assertions (see Lloyd 1985). In fact, such models already exhibit two aspects to minimality. First, they contain no individuals / objects except for those which are explicitly named in the language of the program. But also, they contain no facts about these objects except for those explicitly enforced by the program ('Negation as Failure'). Both aspects are of independent interest. Individual minimality gives a minimal delineation of our objects; which can be cashed in, so to speak, as a principle of 'induction':

If a property holds of all objects nameable in our language,
then it is universally true.

In fact, this is close to ordinary mathematical senses of induction. The defeasibility here shows in that addition of further operations forming objects may actually weaken the method of induction (one has to prove more 'inductive steps'). This is also true in ordinary mathematics. Next, predicate minimality is actually closer to the original examples given above. In PROLOG terms, its inferential value lies in reading a program as standing for its own 'completion', where a condition P is not only implied by all possible antecedents available for it, but also itself implies their disjunction. (There are no other reasons for P than those explicitly stated.)

Again, it is worth-while to think of similar phenomena in natural language. For instance, many implications function like the above conditions. When the doctor says that you will recover if you do as he says, he does not just utter a promise, but also a threat ('only if'): 'if you don't ..., you won't ...'.
[Similar conventions may be observed, incidentally, in the juridical argument called 'a contrario', which looks like the formal fallacy of deriving \( \neg A \rightarrow \neg B \) from \( A \rightarrow B \): but which in fact resembles the preceding pattern.] But individual minimality occurs too. For instance, answers to questions are often taken 'exhaustively' (see Groenendijk & Stokhof 1985). In the dialogue

"Who are crying?"
"John and Mary"

the answer will normally be interpreted as a complete list, even though the direct logical information is only that "John cries and Mary cries". And this phenomenon persists to more complex answers, such as "a girl", which suggests that exactly one person is crying, and that a girl. In other words, we are invited to form a minimal model in which the answer is true.

Another area of Computer Science where explicit formal models of minimality have emerged is in Artificial Intelligence. Here again, action by intelligent agents, whether alive or mechanical, presupposes inference on the basis of incomplete information. In particular, many practical rules which guide us in action involve a 'ceteris paribus' clause: other things being equal, doing this or that will produce such and such effect (cf. Shoham 1988). Making certain that no 'other things' have changed will in general require an infinite amount of information about the state of the world: something that we cannot obtain. Instead, we tend to assume the absence of disturbing circumstances, as long as they do not appear explicitly. Put differently, until forced to the contrary, we will assume that the case at hand is not an exceptional, but rather a 'normal' one. This strategy, though obviously defeasible, is clearly sound: the odds being in its favour (by the definition of normal versus exceptional cases).

Ceteris paribus rules are quite common in juridical procedures, in ethics (cf. Aqvist 1984), but also in the philosophy of science, in connection with counterfactual statements or dispositional terms ('soluble', 'inflammable'): see Sosa 1975. So, the strategies we are considering are not just little short-cuts for self-deception, but rather necessities, even of scientific life.

Some familiar examples from Artificial Intelligence motivated one basic semantic way of extending the earlier notion of minimality, which is due to McCarthy 1980. All birds can fly, barring some exceptional cases like penguins. Tweety is a bird: now, can Tweety fly? Logically speaking, nothing can be concluded here. But practically, we will want to conclude that Tweety can fly. One solution here is to assume the following formalization:
∀x((Bx ∧ ¬ax) → Fx)

Bt

That is, 'not abnormal' birds can fly. Now, reading this as describing models with a minimal extension for the abnormality predicate, Tweety will not be abnormal, and hence she can fly. In general, however, there may be more abnormality predicates involved, corresponding to different defeasible rules. Then, there may be a genuine ambiguity, as in another well-known example:

- Quakers are pacifists : ∀x(Qx ∧ ¬ab₁x) → Px
- Republicans are non-pacifists : ∀x(Rx ∧ ¬ab₂x) → ¬Px
- Nixon is a Quaker : Qn
- Nixon is a Republican : Rn

Now, will Nixon be a pacifist, or not? There are two incomparable minimal models here: one with 'ab₁' empty and 'ab₂' consisting of Nixon only, the other with the situation reversed. And this may reflect a genuine intuitive uncertainty here. On the other hand, there may also be cases where we have clear priorities as to which abnormality predicate should be minimalized first, so that we should have to break the tie by prioritizing rules.

There is a general technical notion of minimal model for these cases, which makes sense for arbitrary predicate-logical formulas (not just the Horn clauses of the PROLOG formalism), and which allows for the above multiplicity of minimal models. This is the so-called circumscription of McCarthy 1980, which will be considered in greater detail below. We can view it as a generalization of the minimal model semantics for logic programs, in which we now consider all those models for a set of premises whose proper submodels (in some suitable sense) no longer verify those premises.

Again, it is of interest to observe that there were earlier, similar attempts in the Philosophy of Science. Philosophers had already studied such weaker notions of consequence as confirmation of a hypothesis by certain evidence. For instance, a universal generalization ∀xPx is, if not implied, at least confirmed by observing instances of it: Pa, Pb, ... . Interestingly, Hempel 1965 suggests the following explanation for this. When observing the evidence, we tend to form a minimal model in which it holds: and that is one in which the universal generalization is valid. (Again, of course, there is non-monotonicity: a well-confirmed regularity can be refuted after all, by new evidence.) This particular analogy is well in line with the earlier reference to Bolzano, whose logical main work was in fact called 'Wissenschaftslehre'. It was especially in scientific thinking that Bolzano found diverse forms of rational argument, with varying logical properties. Confirmation
itself is just one example of this. Another one would be *explanation* of some fact from certain scientific assumptions. Here, for instance, we would want another kind of minimality, namely the use of only some minimal set of relevant premises deriving the conclusion. In this case too, there will be non-monotonicity: as adding irrelevant premises will disturb the explanatory character of an inference. This observation emphasizes the many possible sources of non-monotonic behaviour: which is a *symptom*, rather than the essence of non-standard inference.

More generally, in this perspective, the choice of a particular mode of reasoning, monotonic or non-monotonic, will be dictated by the specific kind of application intended. What is suitable in one context, need not be suitable in another. (For instance, in faculty politics, many people would be in much better health if they did not jump non-monotonically to ill-informed conclusions all the time.) Thus, human rationality becomes 'parametrized': we have to select some sensible mode of reasoning, prior to performing rationally within that mode. This attitude also fits well with two main aspects to current theories in Artificial Intelligence (of which the above-mentioned circumscription is an example). They are sometimes meant to *describe* intellectual activity, but sometimes also to *design* intelligent systems. And at least the latter purpose requires conscious selection of intellectual tools.

In the following parts of this Section, we shall now consider some technical implementations of minimal modelling in more detail, pointing at various interrelations, open questions, etcetera.

### 2.2 The Logic of Circumscription

#### 2.2.1 Minimal Models

*Individual-minimal* models for a set $\Sigma$ of predicate-logical (or indeed, any kind of) formulas may be defined as those $M$ such that

1) $M \models \Sigma$

2) for no $M' \subsetneq M$, $M' \models \Sigma$.

Likewise, *predicate-minimal* models are those models $M$ for $\Sigma$ whose predicate extensions cannot be decreased without losing the truth of $\Sigma$. (The latter notion is somewhat like 'Pareto Optimality' in economics.) More generally, we can also consider models only some of whose predicates are minimized in this fashion.
One test of the adequacy of this notion is an application to the earlier topic of exhaustive answers to questions. It is natural to assume that, in such a case, the answer is read as describing a Q-minimal model, where Q is the predicate that the query is about. Thus, the C-minimal models of CjACm are precisely those in which only John and Mary cry. And likewise, the C-minimal models of \( \exists x (Gx \land Cx) \) are those in which exactly one girl forms the whole extension of 'crying'. (This simple analysis obviates the need for the complex 'minimalized generalized quantifier' analysis found in Groenendijk and Stokhof 1985. The point is, so to speak, that the complexity does not reside in the answer, but in the query.) Incidentally, the claim here is not that answers are invariably taken in this minimal sense. For instance, on the present account, the two answers 'no girl' and 'no boy' to the previous question would both state that no one is crying. And that is certainly unreasonable: these answers mean no more than they say in standard terms, being \( \neg \exists x (Gx \land Cx) \), \( \neg \exists x (Bx \land Cx) \), respectively. More generally, in applications of circumscription, whether more linguistic or more computational, there is always a decision to be made, not prescribed by the formal theory, as to which predicates are to be minimalized.

Minimal models form an interesting object of mathematical study by themselves. In fact, their behaviour is full of surprises, once infinite structures are taken into account. For instance, the integers are a predicate-minimal model for their own first-order theory, being that of unbounded discrete linear orders. (Any removal of a pair in the order relation would destroy linearity.) But, they are not an individual-minimal model for it: the proper substructure of the even integers would serve just as well. In fact, this ordering theory has no individual-minimal model at all. On the other hand, what are classically small changes in the presentation of a theory may have telling effects now. If one adds two Skolem functions, witnessing immediate successors and predecessors, then, viewed as a structure for the expanded similarity type, the integers will become a minimal model for their ordering theory (expanded with the two relevant definitions). This reflects a general phenomenon. With function symbols around, submodels will have to be closed under the corresponding operations: which increases the chances of a model's being minimal. We shall not pursue this mathematical direction here, which would call for comparison with other notions of minimal model in the literature (such as 'prime models': cf. Chang and Keisler 1973).
As for semantic consequence, there are several options now, summed up in
the following definition:

\[ \Sigma \models_{\text{ind}} \phi \quad \text{if } \phi \text{ is true in all individual-minimal models of } \Sigma \]
\[ \Sigma \models_{\text{pred}} \phi \quad \text{if } \phi \text{ is true in all predicate-minimal models of } \Sigma \]
\[ \Sigma \models \ast \phi \quad \text{if } \phi \text{ is true in all structures which are both individual- and} \]
\text{predicate-minimal models of } \Sigma .

Of the major properties of classical consequence, these new notions preserve some,
while losing others. We shall illustrate this for the case of \( \models \ast \).

- \[ \Sigma \models \ast \phi \quad \Rightarrow \quad \Sigma \models \ast \phi \lor \psi \]

I.e., one may 'weaken consequents'. But, one may not 'strengthen antecedents:

- \[ \Sigma \models \ast \phi \quad \not\Rightarrow \quad \Sigma, \alpha \models \ast \phi . \]

This is the earlier-mentioned non-monotonicity: the minimal models for \( \Sigma \) plus \( \alpha \)
may have shifted from those of \( \Sigma \) by itself.

Likewise we lose the classically valid 'transmission of truth':

- \[ \Sigma \models \ast \phi, \quad \phi \models \ast \psi \not\Rightarrow \quad \Sigma \models \ast \psi \]

A counter-example is: \( \Sigma = \exists x P x \land \exists x \neg P x, \quad \phi = \exists x P x, \quad \psi = \forall x P x . \)

On the other hand, we keep such classically admissible operations on premises as
Permutation, or Contraction of identicals.

Actually, could there be any gain in general properties for \( \models \) too, as
compared to classical consequence \( \models \) ? As we shall see later on, the answer is
negative.

Before continuing with the general logic of circumscription, it may be useful
to emphasize a new aspect of reasoning in this situation. Even if a certain pattern of
inference is invalid in general, it may still be admissible for certain types of
statement. For instance, monotonicity will in fact be valid for those additional
premises which are preserved under the model relations employed in the definitition
of minimality. Thus,

for universal formulas \( \alpha , \)

\[ \Sigma \models_{\text{ind}} \phi \quad \Rightarrow \quad \Sigma, \alpha \models_{\text{ind}} \phi . \]

For , if \( M \) is an individual-minimal model for \( \Sigma + \alpha , \) and \( M' \) is a submodel of \( M \)
verifying \( \Sigma , \) then it also verifies \( \alpha \) (by preservation of universal statements under
submodels), and so \( M' = M . \) This proves that \( M \) is also an individual-minimal model
for \( \Sigma \) itself, and hence that \( M \models \phi , \) by the assumption. \( \Box \)
By a similar argument, $\vdash_{\text{med}}$ is monotone for added negative formulas, constructed from negations of atoms (as well as identities) using $\land, \lor, \forall, \exists$.
Thus, minimal reasoning may require closer attention to syntactic 'fine-structure' of premises.

2.2.2 Conditional Logic

As a notion of predicate-logical consequence, circumscription is highly complex. For instance, as the only minimal model of the usual Peano Axioms (even minus Induction) consists of the standard natural numbers, minimal consequence from these premises coincides with actual arithmetical truth: hence, non-axiomatizability (and worse) follows by Tarski's Theorem.

On the other hand, there is also a general logic of circumscription, not referring to specific predicate-logical forms, but only to simple (at most) Boolean Structure. Examples of these inferences were such earlier patterns as Transitivity or Weakening Consequents. This raises the issue of what might be called the propositional logic of circumscription. The latter system is not trivial: especially, since the break-down of classical principles can often be compensated for by more refined substitutes. For instance, the following form of Transitivity is in fact valid:

$$\Sigma \vdash \phi, \quad \Sigma, \phi \vdash \psi \quad \Rightarrow \quad \Sigma \vdash \psi.$$

And, thinking of progressive stages in an argument, this makes sense.

In fact, a further study of valid patterns for notions of minimal consequence brings to light a striking analogy with systems of reasoning developed earlier in Conditional Logic (cf. Lewis 1973), which started out as a study of so-called counterfactual assertions 'if A had been the case, then B would have been the case'. [Again, this (non-monotone) notion has strong connections with the philosophy of science: cf. Harper et al., eds., 1981.] For instance, the basic axioms of Conditional Logic are as follows (Burgess 1981, Veltman 1986):

\begin{align*}
i & \quad \phi \Rightarrow \phi \\
ii & \quad \phi \Rightarrow \psi, \quad \psi \Rightarrow \chi \quad \text{imply} \quad \phi \Rightarrow \psi \land \chi \\
iii & \quad \phi \Rightarrow \psi, \quad \chi \Rightarrow \psi \quad \text{imply} \quad \phi \lor \chi \Rightarrow \psi \\
iv & \quad \phi \Rightarrow \psi \quad \text{implies} \quad \phi \Rightarrow \psi \land \chi \\
v & \quad \phi \Rightarrow \psi, \quad \phi \Rightarrow \chi \quad \text{imply} \quad \phi \land \psi \Rightarrow \chi.
\end{align*}

Of these five, all except the last are valid with respect to all three notions of minimal consequence. For the validity of the last, one has to assume a universe of models in
which the submodel relation is \textit{well-founded} for any set of premises. For instance, the universe of all \textit{finite} models would do. (Compare Shoham 1988 on the plausibility of this requirement.) Further principles which are valid turn out to be derivable from these.

\textbf{Example:} Refined Transitivity derived:
\[
\begin{align*}
\varphi \land \neg \psi & \Rightarrow \varphi \land \neg \psi \quad \text{(i)} \\
\varphi \land \neg \psi & \Rightarrow \neg \psi \quad \text{(iv)} \quad \varphi \land \psi & \Rightarrow \chi \quad \text{(assumption)} \\
\varphi \land \neg \psi & \Rightarrow \neg \psi \lor \chi \quad \text{(iv)} \quad \varphi \land \psi & \Rightarrow \neg \psi \lor \chi \quad \text{(iv)} \\
(\varphi \land \neg \psi) \lor (\varphi \land \psi) & \Rightarrow \neg \psi \lor \chi \quad \text{(iii)} \\
\text{i.e.,} & \quad \varphi \Rightarrow \neg \psi \lor \chi \\
\text{also,} & \quad \varphi \Rightarrow \psi \quad \text{(assumption)} \\
\text{so} & \quad \varphi \Rightarrow \psi \land (\neg \psi \lor \chi) \quad \text{(ii)} \\
\text{whence} & \quad \varphi \Rightarrow \chi \quad \text{(iv)}.
\end{align*}
\]

Note the use of Boolean equivalences throughout.

\textbf{Digression:} Incidentally, here is one other computational way of viewing the minimal conditional logic. The inferences displayed above present a mixture of the usual Boolean rules for \textit{classical} implication $\rightarrow$ and a \textit{non-classical} implication $\Rightarrow$.

For instance, axiom iv may be regarded as
\[
\varphi \Rightarrow \psi, \quad \psi \rightarrow \chi \quad \text{imply} \quad \varphi \Rightarrow \chi;
\]
and other rules concern suprema and infima in the $\rightarrow$ ordering combined with $\Rightarrow$.

This interaction between simple rules for classical entailments and non-monotone \textit{default} rules is reminiscent of the situation in so-called \textit{semantic networks}. In the latter area, one of the key issues is the creation of sound and computationally tractable inferential algorithms. Perhaps, Conditional Logic can also serve as a useful model for the latter activity.

On the other hand, principles not derivable in the minimal conditional logic turn out to be typically invalid for circumscription too. In a more general perspective, this correspondence is not surprising. There has been a tendency in the literature on circumscription to minimize over other ordering relations between models than just the above notions involving submodels. (Compare the above reference to prioritized circumscription, cf. Lifschitz 1985, or also Shoham 1988). And in fact, only some general properties of such orderings seem relevant to evaluating validity of propositional circumscriptive patterns. But then, we are
precisely back with the original possible worlds semantics for conditional statements, due to David Lewis and Robert Stalnaker:

Models are structures \((W,R,w)\), where \(w\) is the actual world, from whose perspective we are viewing the other worlds in \(W\), and \(R\) orders the latter as to greater or lesser similarity to \(w\). (Technically, \(R\) is a strict partial order.) Then, at least on finite models (the infinite case is somewhat more subtle), conditionals are evaluated as follows:

\[ \varphi \Rightarrow \psi \text{ is true at } w \text{ if all } R\text{-closest} \text{ worlds verifying the antecedent } \varphi \text{ also verify the consequent } \psi. \]

This explains the general analogy between Circumscription and Conditional Logic. What remains is a number of more specialized questions. For instance, can minimality as defined above serve as an adequate concrete model for Lewis Semantics; e.g., in the sense of the following conjecture:

A conditional inference \( \varphi \Rightarrow \psi \) is derivable in the minimal conditional logic if and only if it is valid as a circumscriptive consequence \( \varphi \models^c \psi \)

in a language whose proposition letters stand for formulas of monadic predicate logic ?

What would be needed here is a representation of abstract Lewis models in terms of finite models with the above submodel relation, under a suitable translation of proposition letters into monadic sentences.

Another question in this perspective would concern special logics for special relations to be minimized over. From Conditional Logic, we know that additional conditions on strict partial orders may produce additional validities. For instance, making the relation \( R \) linear will validate so-called 'conditional excluded middle':

\[ \varphi \Rightarrow \psi \lor \varphi \Rightarrow \neg \psi. \]

So, variants of circumscription may also induce weaker or stronger general logics.

Toward a proof of the above conjecture, at least one suggestive result may be cited [for convenience, attention will be restricted to individual-minimality]:

**Proposition:** The following two assertions are equivalent,

for any propositional inference from \( A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n \) to \( C \Rightarrow D \):

1. \( C \Rightarrow D \) is derivable from \( A_i \Rightarrow B_i \) \((1 \leq i \leq n)\) in the minimal conditional logic,
2. for every substitution \( \sigma \) replacing proposition letters by sentences of a monadic predicate logic with identity, and every sentence \( \varphi \) of such a language, if \( \varphi, \sigma(A_i) \models_{\text{ind}} \sigma(B_i) \) for all \( i \) \((1 \leq i \leq n)\), then \( \varphi, \sigma(C) \models_{\text{ind}} \sigma(D). \)
**Proof:** From (1) to (2). This is a straightforward induction on the length of derivations in the minimal conditional logic. The additional premise \( \varphi \) does not affect the earlier soundness argument. (Note here that, for a monadic predicate logic with identity, attention may be restricted to finite models, without any changes in semantic satisfiability.)

From (2) to (1). Suppose that \( C \Rightarrow D \) is not derivable from the \( A_i \Rightarrow B_i \). Then there exists some finite possible worlds model \( W \) containing a world \( w \) from whose perspective all formulas \( A_i \Rightarrow B_i \) are true, whereas \( C \Rightarrow D \) is false. Passing on to the relevant binary order \( \prec \) on worlds ('greater similarity to \( w \)'), we can view this counter-model as some finite ordered set of propositional valuations (sitting on worlds), with conditional statements evaluated as in the Stalnaker-Lewis semantics. Now, take distinct unary predicates \( P_v \) for each world \( v \), and set

\[
\alpha_v(x) := P_v x \land \bigwedge_{u \in W, u \not= v} \neg P_u x.
\]

Let there be exactly \( u_1, \ldots, u_n \leq v \). Set

\[
\beta_v := \exists x_1 \ldots \exists x_n (\alpha_{u_1}(x_1) \land \ldots \land \alpha_{u_n}(x_n) \land \forall y (y = x_1 \lor \ldots \lor y = x_n)).
\]

Moreover, let

\[
\varphi := \bigwedge_{v \in W} \beta_v.
\]

Finally, let \( \sigma \) be the substitution which assigns to each proposition letter \( p \) the formula

\[
\bigwedge_{v \in \{p\}} \beta_v.
\]

**Claim:** Let \( M \) be a structure for monadic predicate logic which verifies \( \beta_v \) for some world \( v \). Then, for each purely propositional formula \( \alpha \),

\( M \models \sigma(\alpha) \) if and only if \( \alpha \) is true in \( v \).

**Proof:** For proposition letters \( \alpha \), 'if' follows by the definition of the substitution \( \sigma \).

Conversely, suppose that \( \alpha = p \) is false in \( v \). So, \( \sigma(p) \) is a disjunction of formulas \( \beta_u \) with \( u \not= v \). It suffices to observe that such \( \beta_u \) exclude \( \beta_v \), by their definition.

The remainder of the argument is a routine induction on Boolean operators. 

The desired conclusion then follows from the next assertion:

**Claim:**

(i) \( \varphi, \sigma(A_i) \models_{\text{ind}} \sigma(B_i) \), for all \( i \) \( (1 \leq i \leq n) \),

(ii) not \( \varphi, \sigma(C) \models_{\text{ind}} \sigma(D) \).

**Proof:** (i). Let \( M \) be a minimal model for \( \varphi \land \sigma(A_i) \). As \( \varphi \) is true, \( M \) verifies some \( \beta_v \). So, by the first claim, the truth of \( \sigma(A_i) \) implies that \( A_i \) holds in the world \( v \).
Moreover, \( v \) is a \(<\)-minimal world where \( A_i \) holds. For, if \( A_i \) were true at some \( u<\nu \), then an obvious proper submodel of \( M \) could be extracted which verifies \( \beta_u \) and (hence) \( \sigma(A_i) \), contradicting the minimality of \( M \) itself. So, by the truth of \( A_i \Rightarrow B_i \) in the original possible worlds model \( W \), \( B_i \) holds in \( v \), whence \( \sigma(B_i) \) holds in \( M \).

(ii). Since \( C \Rightarrow D \) fails in the possible worlds model \( W \), there must be some \(<\)-minimal world \( v \) where \( C \) holds, while \( D \) does not. Now, take the obvious minimal model \( M \) for \( \beta_v \). As before, \( M \) verifies \( \sigma(C) \), while falsifying \( \sigma(D) \). It suffices to show then that \( M \) is a minimal model for \( \phi \land \sigma(C) \). So, suppose that some proper submodel \( M' \) also verified \( \phi \land \sigma(C) \): say, \( \beta_u \) holds, as well as \( \sigma(C) \). Then, by the definition of the formulas \( \beta_u \) must be a proper \(<\)-predecessor of \( v \), where \( C \) is true. But, this would contradict the \( C \)-minimality of \( v \) in \( W \).  

Remark: The above characterization essentially describes a more general form of circumscription, where the universe of models being compared can be restricted by some prior condition \( \phi \).

The proof presented here derives from an argument by Frank Veltman, produced in response to an earlier version of this paper.

Finally, we return to the earlier question why circumscription should not also validate new inference patterns, not found in classical logic. Here, by 'inference patterns' we mean schemata of the form

\[ \phi_1 \Rightarrow \psi_1 , \ldots , \phi_n \Rightarrow \psi_n \text{ imply } \phi \Rightarrow \psi. \]

Here, the \( \phi \)'s and \( \psi \)'s are Boolean forms - as was the case in all earlier examples, and '\( \Rightarrow \)' stands for the particular notion of consequence being considered. (Thus, in a sense, we are concerned with the Horn clause logic of valid consequence.)

Now, if such a schema is not classically valid, then the following single consequence cannot be valid either:

\[ (\phi_1 \Rightarrow \psi_1) \land \ldots \land (\phi_n \Rightarrow \psi_n) \models \phi \Rightarrow \psi. \]

Otherwise, the schema would be derivable. So, there exists some valuation \( V \) making all \( \phi_i \Rightarrow \psi_i \) true (1 \( \leq i \leq n \)), but \( \phi \Rightarrow \psi \) false. Now, consider the following substitution \( \sigma \) of formulas for variables in the above schema:

\[ \sigma(p) = T , \text{ if } V(p)=1 \]
\[ \bot , \text{ if } V(p)=0 \]

It is easy to check that, then, all \( \phi_i \Rightarrow \psi_i \) (1 \( \leq i \leq n \)) are valid consequences, whereas \( \phi \Rightarrow \psi \) is not. But, for Boolean combinations of \( T \) and \( \bot \), circumscriptive
consequence and classical consequence coincide. Hence, the above schema is not valid for circumscriptive consequence either.

We have proved the following

**Proposition:** All valid Horn clause principles of circumscriptive consequence are also valid for classical logic.

**Remark:** In this connection, note that at the level of meta-properties of logical systems, non-monotonicity can indeed occur, and has even been well-known for a long time. A notable example is Intuitionistic Logic, which satisfies the so-called Disjunction Property:

\[
\text{if } \vdash \varphi \lor \psi, \text{ then } \vdash \varphi \lor \psi.
\]

The stronger system of Classical Logic, however, has lost this property:

\[
\vdash C \varphi \lor \neg \varphi, \text{ but neither } \vdash C \varphi \text{ nor } \vdash C \neg \varphi.
\]

### 2.2.3 Dynamics of Minimality

We now turn to another aspect of circumscription. As was remarked before, what we are dealing with in general are various modes of taking propositions, as they come in. For instance, answers to questions could be taken 'at face value', but also 'exhaustively'. And likewise, ordinary statements can be taken routinely, so to speak, or 'minimally'. (In fact, the latter mode itself turned out to have at least three varieties.) Here are a few comments on this situation, preparatory to the discussion of 'dynamics' in Section 3 below.

A class of models may be regarded as a state of information, localizing 'the real world' within its range. The ordinary way of taking propositions \( \varphi \) amounts to the following transformation of such states:

\[
X \mapsto X \cap \text{MOD}(\varphi).
\]

In general, thus, each successive new premise decreases our ignorance, or equivalently, increases our knowledge. Various logical operators then acquire new overtones in this setting. One example is 'conjunction as composition':

\[
X \cap \text{MOD}(\varphi \land \psi) = X \cap (\text{MOD}(\varphi) \cap (\text{MOD}(\psi))
\]

\[
= (X \cap \text{MOD}(\varphi)) \cap \text{MOD}(\psi).
\]

Here, we are only interested in the general properties of what may be called a classical transformation \( \tau \), given by

\[
\lambda X . X \cap F, \text{for some fixed class of models } F.
\]

The following two properties are easily seen to be necessary:
1 $\tau(X) \subseteq X$ \hspace{1cm} (Introversion)

2 $\tau(\cup \{X_i \mid i \in I\}) = \cup \{\tau(X_i) \mid i \in I\}$ \hspace{1cm} (Continuity)

They are also sufficient:

Any introvert continuous operator on sets can be represented by an intersection with some fixed ("information") set.

The argument is this. Define $F$ as $\{x \mid \tau(\{x\}) = \{x\}\}$.

Then $\tau(X) = \tau(\cup \{\{x\} \mid x \in X\}) = \cup \{\tau(\{x\}) \mid x \in X\} = X \cap F$. 

Of course, such operators, when defined by formulas of some specific language, may have many additional structural properties. But, the present analysis provides a convenient comparison with our second non-classical mode of transformation. What happens with circumscription may be described as follows. The universe of models now carries some comparative order $R$. Starting from a certain information state $X$, one again restricts attention first to $X \cap \text{MOD}(\varphi)$, as before, but next only to all $R$-minimal items in here. Thus, another basic transformation on knowledge states is the following minimization:

$\mu(X) = \{x \in X \mid \neg \exists y \in X \ R y x\}.$

Can we also determine the basic properties of this new process?

Here are again some obvious features:

1 $\mu(X) \subseteq X$ \hspace{1cm} (Introversion)

Of the earlier Continuity, however, only one half remains:

3 $\mu(\cup \{X_i \mid i \in I\}) \subseteq \cup \{\mu(X_i) \mid i \in I\}$

(The side which is lost here is actually equivalent to monotonicity.)

To compensate, we do have a new property, namely:

4 $\cap \{\mu(X_i) \mid i \in I\} \subseteq \mu(\cup \{X_i \mid i \in I\})$.

Again, these three conditions are sufficient:

Any operation on sets satisfying 1,3,4 may be represented as a minimization operator for some suitable model relation $R$. To see this, define the following relation among models:

$R_{xy}$ iff $y \notin \mu(\{x,y\})$.

Then we have, for all $X$, $\mu(X) = \{x \in X \mid \neg \exists y \in X \ R y x\}$.

From left to right. Let $x \in \mu(X)$, $y \in X$. Then $x \in \mu(\{x,y\} \cup (X-\{x,y\})) \subseteq \mu(\{x,y\}) \cup \mu(X-\{x,y\})$ (by 3). By 1 then, $x \in \mu(\{x,y\})$, and hence $\neg R y x$ (by definition).

From right to left. Let $x \in X$ with $\neg \exists y \in X \ R y x$. So for all $y \in X$, $x \in \mu(\{x,y\})$, i.e., $x \in \cap \{\mu(\{x,y\}) \mid y \in X\}$. By 4 then, $x \in \mu(\cup \{\{x,y\} \mid y \in X\})$, i.e., $x \in \mu(X)$. 

\hfill \square
Minimization has several other interesting properties, derivable from the above. One convenient way to proceed here is by organizing the necessary calculations in a modal logic, with the following axioms:

1. \( \mu p \rightarrow p \)
2. \( \mu(p \lor q) \rightarrow \mu p \lor \mu q \)
3. \( \mu p \land \mu q \rightarrow \mu(p \land q) \).

Formal deduction in this modal logic brings out various useful theorems.

**Example:** i. \( p \land \mu q \rightarrow \mu(p \land q) : \)

\[
\mu q \rightarrow \mu((p \land q) \lor (\neg p \land q)) \rightarrow (by \ 3) \ \mu(p \land q) \lor \mu(\neg p \land q), \so \\
(p \land \mu q) \rightarrow (p \land \mu(p \land q)) \lor (p \land \mu(\neg p \land q)) \; \text{but} \\
\mu(\neg p \land q) \rightarrow (by \ 1) \ \neg p \land q \rightarrow \neg p, \so \\
p \land \mu q \rightarrow p \land \mu(p \land q) \rightarrow \mu(p \land q). \\
\]

ii. \( \mu p \rightarrow \mu \mu p : \)

\( \mu p \land \mu p \rightarrow \mu(\mu p \land p) \) (by part i), and

\( \mu p \land p \leftrightarrow \mu p \) (by 1), so

\( \mu p \rightarrow \mu \mu p. \)

We can derive an obvious possible worlds semantics for this modal operator \( \mu \). But perhaps the easiest observation is just this:

\( \mu \) is completely described by the following definition

inside the minimal modal logic K:

\( \mu p := p \land \neg p \).

We conclude by mentioning a few facts about the interplay of a classical transformation \( \tau \) with minimization \( \mu \). Most iterations collapse, via the following equations:

\[
\begin{align*}
\tau\tau(X) &= \tau(X) \\
\mu\mu(X) &= \mu(X) \\
\tau\mu(X) &= \tau(X) \land \mu(X) \\
\mu\tau\tau(X) &= \mu\tau(X) \\
\mu\tau(X) &= \tau\mu(X)
\end{align*}
\]

Proofs can again be given in the above modal logic.

**Example:** The last identity can be transcribed as the equivalence

\( \mu(\mu p \land q) \leftrightarrow \mu p \land q. \)
Here, → follows by axiom 1, and ← from the preceding example:
μp∧q → μμp∧q → μ(μp∧q).

Thus, the only non-equivalent modes of transforming knowledge states generated by the present perspective are precisely as expected:
τ, μ and μτ.

2.2.4 First-Order Reduction

Circumscriptive inference was already quite complex for predicate-logical sentences, as we have seen. The reason for this lies in its second-order nature. Using the minimal models of a set of premises Σ amounts to using all models in the standard sense for an associated second-order formula. For individual circumscription, that formula is
\[ \forall X \ (\text{ClyXy} \land \Sigma^X) \leftrightarrow \forall yXy \]
where \( \Sigma^X \) is the relativization of \( \Sigma \) to the subdomain \( X \). [In the presence of function symbols in the language, some more care will actually be needed here.]
For predicate circumscription, the corresponding formula is this [written up, for convenience, for some \( \Sigma \) involving only one unary predicate letter \( P \)]:
\[ \forall X \ (\forall y(Xy \rightarrow Py) \rightarrow (\Sigma(X) \leftrightarrow \forall y(Py \rightarrow Xy))) \]
where \( \Sigma(X) \) is obtained by substituting \( X \) for \( P \) in \( \Sigma \).
Both of these second-order formulas are relatively simple, so-called monadic \( \Pi^1_1 \)-sentences - but still, their standard notion of consequence is already quite complex.

Now, Vladimir Lifschitz has pointed out that this complexity need not worry us, if we can show that all, or most natural applications employ premises falling within some narrower syntactic class of formulas, for which these circumscription formulas are in fact equivalent to first-order ones. For then, we can employ standard reasoning systems to simulate circumscription after all. And, he does manage to isolate large such classes in Lifschitz 1985a, 1985b, with respect to predicate circumscription. One useful general result is, e.g., that circumscription with respect to \( P \) on \( \Sigma \) will define a first-order class of minimal models, in case \( P \) occurs only positively in \( \Sigma \). This covers such cases as the following:
\[ \forall x(Qx \rightarrow Px), \text{ circumscribed to: } \forall x(Qx \leftrightarrow Px) \]
\[ \exists x(\neg Qx \land Px), \text{ circumscribed to: } \exists x(\neg Qx \land \forall y(Py \leftrightarrow y=x)) \]

Similar questions arise for individual-minimality.

Put in general logical terms:
• What is the complexity of 'first-order definability' for circumscriptive monadic $\Pi^1_1$-sentences?

• What are large syntactic classes of premises $\Sigma$ which guarantee the existence of a(n effectively obtainable) first-order equivalent?

As it happens, there is a connection here with earlier work in Modal Logic. Modal axioms on possible worlds structures may be viewed in general as monadic $\Pi^1_1$-sentences, and their possible first-orderness has been studied extensively (see van Benthem 1984, 1985a). For instance, we know that the general question if an arbitrary monadic $\Pi^1_1$-sentence has a first-order equivalent is of hyper-arithmetic complexity. Even so, there are indeed large classes of first-order cases with an effective syntactic description. And, this analogy can be exploited to prove such results as the following.

**Proposition:** Individual circumscription is first-order for all first-order formulas $\Sigma$ which are of the syntactic shape

$$\exists x_1 \ldots \exists x_m \forall y_1 \ldots \forall y_n <\text{quantifier-free matrix}>.$$ 

**Proof:** By the obvious preservation properties of such a formula, its minimal models can be enumerated as a finite set of finite models whose size does not exceed $m$. $\blacklozenge$

Simple though it is, this proposition has a useful consequence for a large class of cases where circumscription is actually used:

**Corollary:** Individual circumscription is first-order for all $\Sigma$ formulated in monadic predicate logic with identity.

**Proof:** All formulas in this language have a normal form of the above syntactic shape. $\blacklozenge$

First-orderness is no longer guaranteed with other quantifier prefixes, such as $\forall \exists$. Then, first-order cases have to be located by more sensitive analysis.

**Remark:** There is an interesting boundary case of first-order definability for circumscription. For certain first-order formulas $\Sigma$, ordinary models and minimal models already coincide. For instance, $\forall x (Qx \leftrightarrow Px)$ has only $P$-predicate-minimal models: and so do all explicit definitions for minimized predicates in terms of the
remaining vocabulary. But, there are further instances of the phenomenon too, witness the following $\Sigma$:

' $<$ is a discrete unbounded strict linear order with immediate successors and predecessors, and $\forall x \forall y ((x < y \land \neg \exists z (x < z \land z < y)) \rightarrow (Qx \leftrightarrow Py))$ '.

Models of $\Sigma$ are linear orders of copies of the integers, with $P$ interpreted as either the odd or the even integers in each copy. These models are all $P$-minimal; but even so, $P$ is not definable in terms of just $<$ on the basis of $\Sigma$. [E.g., the odd numbers are not explicitly definable in the pure ordering theory of the integers.] Is there a good characterization of those first-order formulas $\Sigma(P)$ which automatically enforce $P$-predicate-minimality?

Although the general complexity of detecting first-orderness in modal logic may be high, there is a natural specialization. In many cases, the first-order equivalent of a monadic $\Pi_1^1$-sentence $\forall X \varphi(X)$ comes in the form of a conjunction of first-order substitution instances $\varphi(\psi)$. Now, the class of second-order formulas having such a first-order equivalent is recursively enumerable. For, it suffices to enumerate all possible finite conjunctions of first-order instances, checking in each case if

$$\varphi(\psi_1) \land \ldots \land \varphi(\psi_n) \models \forall X \varphi(X).$$

(Note that the other direction is automatic.) As the predicate variable $X$ does not occur on the left-hand side of the turn-stile here, the latter consequence is equivalent to the (recursively enumerable) standard first-order consequence

$$\varphi(\psi_1) \land \ldots \land \varphi(\psi_n) \models \varphi(X).$$

Quite probably, however, this property is not decidable.

Again, these considerations may be transferred to the case of circumscription.

This time, we shall consider the case with minimal predicates.

**Proposition:** Having a first-order predicate circumscription is not a decidable, in fact not even an arithmetical property of first-order formulas.

**Proof:** We give an effective reduction of arithmetical truth to first-orderness of predicate circumscriptions. The proposition then follows by Tarski's Theorem.

Consider a finite relational formulation $PA^-$ for Peano Arithmetic minus the Induction Axiom, including the first-order theory of $<$. Let $\varphi$ be an arbitrary
arithmetical sentence in this format. Moreover, let $N$ be a new unary predicate letter not occurring in $PA^-$.

**Claim:** The next two assertions are equivalent:

1. $\text{IN} \models \varphi$ (i.e., $\varphi$ is arithmetically true)
2. the following sentence $\Phi$ has a first-order predicate circumscription:

   
   \[ \langle PA^- \land \neg \varphi \land 'N \text{ contains } 0, \text{ and is } S\text{-closed and cofinal in } ' \rangle \]

**Proof:** $1 \Rightarrow 2$. Let $\text{N} \models \varphi$. Suppose that $D$ is any model for $\Phi$. Because $D \models PA^-$, it consists of some initial copy of $N$ followed by a tail of copies of the integers $Z$. (As $D \models \neg \varphi$, this tail must be non-empty.) Now, the extension of $N$ in $D$ must intersect at least one copy of $Z$ (by cofinality). Then, removing one $N$-point together with all its predecessors in that copy of $Z$ from the extension of $N$ will leave $\Phi$ true. It follows that $\Phi$ has no predicate-minimal models at all; whence its predicate circumscription is defined by the first-order sentence FALSE.

$2 \Rightarrow 1$. Let $\Phi$ have a first-order predicate circumscription, say $\alpha$.

Suppose that $\text{N} \not\models \varphi$. We derive a contradiction. The model $D$ consisting of $N$ itself, with $N$ interpreted as the set of all natural numbers, is a model for $\Phi$. Moreover, it is even a predicate-minimal model: any decrease in some predicate extension would disturb either the first or the last conjunct of $\Phi$. So, $D \models \alpha$. Now take any proper elementary extension $D^+$ of $D$. Still, $D^+ \models \alpha$, but $D^+$ can no longer be a predicate-minimal model for $\Phi$ (which is the desired contradiction), for the same reason as above. \(\blacklozenge\)

**Remark:** The idea of this argument is that $\Phi$ can only have predicate-minimal models on $N$ (if it has them at all). Note that the proof works equally well for general predicate-minimality and for minimality only with respect to the special predicate $N$.

**Corollary:** There exist formulas having first-order predicate circumscriptions without the latter being obtainable through first-order substitution instances.

**Proof:** The first class of formulas contains the second, obviously. Moreover, it is not arithmetical, whereas the second is (being recursively enumerable). Hence, the inclusion must be proper. \(\blacklozenge\)
2.3 Data Types and Theories

2.3.1 Logic Programs

Next, we shall briefly consider another tradition of studying minimal models in Computer Science. First, there is an extensive literature on semantic properties of logic programming (see Lloyd 1985, Apt 1987). Usually, this area carries a restriction to a fragment of first-order logic, namely universal Horn clauses. This restriction leads to some better logical (and computational) behaviour, such as the existence of unique minimal (Herbrand) models. The special status of the Horn Clause formalism has been investigated from various angles. For instance, by an early result of Malcev's, as a fragment of first-order predicate logic, Horn clauses are characterized by two model-theoretic preservation properties: they are preserved under the formation of submodels and direct products of models. There is also a theorem in Mahr & Makowsky 1983, however, which characterizes the Horn clause framework as the maximal one in which every specification is guaranteed to have an initial model. We shall return to the latter notion below.

One of the attractions in having unique minimal models for specifications, from an intuitive point of view, is that it corresponds with a natural tendency which many people have. When reading a piece of text, they assume that some unique world is being built up - and it always takes some time to persuade students in a logic class to shift to the standard technical perspective of huge classes of different models for sentences. (See also van Benthem and van Eyck 1982 on such issues.)

Semantic consequence $\vdash^+$ for logic programs $\Sigma$ may be defined either in the standard fashion, or via the corresponding minimal Herbrand Model $\mu_\Sigma$:

$$\Sigma \vdash^+ \varphi \text{ if } \mu_\Sigma \models \varphi.$$  

For atomic formulas $\varphi$, this choice is irrelevant, as $\Sigma \models \varphi$ will be equivalent to $\mu_\Sigma \models \varphi$. This well-known observation implies correctness and completeness for answers to the usual types of question $\varphi$ computed in logic programming. But, for general conclusions $\varphi$, truth in $\mu_\Sigma$ will be a stronger notion than standard consequence from $\Sigma$ - and characteristically, it is the latter notion being referred to, when we prove certain correctness statements about the behaviour of PROLOG programs. Additional principles, over and above classical consequence, which are available for reasoning in the latter setting reflect the earlier two aspects of minimality: individual minimality sanctions certain forms of induction, predicate
minimality rather the use of the so-called completion of the relevant program (cf. Lloyd 1985).

The connection with our earlier notion of circumscriptive consequence is found in the following simple observation:

\[ \Sigma \vdash \varphi \text{ only if } \Sigma \vdash *\varphi \text{ only if } \mu_\Sigma \vdash \varphi. \]

Again, this is because minimal Herbrand models combine individual and predicate minimality. None of these implications can be reversed, however, for arbitrary formulas \( \varphi \). For instance, \( \Sigma = \{ R0S(0) \} \) will imply \(-R0S(0)0\) in its minimal Herbrand model, whereas it does have a trivial one-point minimal model refuting \(-R0S(0)0\). This difference reflects the decision, when working with Herbrand models, not to identify any more terms than is absolutely necessary given the premises. [Similar proposals have been made for circumscription too ('uniqueness of names') - but we will not go into the latter variant here.]

Another point of comparison between the two approaches to semantic consequence concerns the earlier completion \( \text{compl}(\Sigma) \) of a logic program \( \Sigma \). It is easy to show that, for all Horn programs \( \Sigma, \Sigma \vdash *\text{compl}(\Sigma) \). But, in a way, circumscription provides a more stable semantic account here, untroubled by some of the more curious syntactic details of forming completions. This may be seen in the following comparative list:

\[
\begin{align*}
\Sigma: \{ p \rightarrow q \} & \quad \text{compl}(\Sigma): \{ q \leftarrow p \} & \text{q-minimal models: } q \leftarrow p, \\
\Sigma: \{ p \rightarrow q, q \rightarrow q \} & \quad \text{compl}(\Sigma): \{ q \leftarrow (p \lor q) \} \quad \text{(i.e., } p \rightarrow q!) & \text{q-minimal models: } q \leftarrow p.
\end{align*}
\]

**Digression:** This example may be viewed as illustrating the superiority of a semantic notion like circumscription over a syntactic one like completion: in line with current public opinion. Nevertheless, from a different angle, there is also a good deal of interest to the more syntactic reaction of PROLOG to adding 'harmless' rules like \( q \rightarrow q \). As is well-known, such rules can trick the inference engine into entering a vicious loop, unsettling earlier proofs - and that is analogous to a phenomenon to be observed in ordinary argumentation. Someone who commits the informal fallacy of **Begging the Question**, by offering \( q \) itself as a reason for \( q \), vitiates whatever else she might have advanced already in favour of \( q \). Various logical commentators have found it hard to explain why this should be so, since the new statement is logically true. But again, the explanation lies in the procedural aspects of argumentation and discourse.

We conclude with another observation about circumscription suggested by logic programming. It follows from an earlier remark that, at least for Horn clause
premises and *atomic* conclusions, classical consequence and circumscriptional consequence *coincide*. One interesting question then becomes if this collapse can be *generalized* to larger fragments of predicate logic. For instance, standard consequence and circumscriptive consequence also coincide for all universal \( \Sigma \), with positive quantifier-free conclusions \( \phi \), at least, in languages without function symbols. [For, if \( \Sigma \not\models \phi \), then there exists a *finite* model for \( \Sigma \) where \( \phi \) fails - and we can obtain a minimal model for \( \Sigma \) with that same failure by stepwise decreasing domain and interpretation. It is crucial here that \( \phi \) be positive quantifier-free, so that the truth value of its negation cannot change in this process.]

In fact, the universal fragment of predicate logic has many things to recommend it (see also below). It would therefore be of interest to determine the precise complexity of the notion \( \Sigma \models * \phi \) as restricted to universal statements.

### 2.3.2 Abstract Data Types ...

The theory of logic programs is hard to distinguish from that of so-called *abstract data types*. Here again, the emphasis is on special models for specifications \( \Sigma \), namely those which are *initial* in \( \text{MOD}(\Sigma) \). That is, such a model must have a *unique homomorphic embedding* into every other model for \( \Sigma \). Part of the motivation for this notion again amounts to various forms of minimality (be it now with different slogans; such as 'no junk' or 'no confusion': cf. Goguen and Meseguer 1983). But, there is also the interesting idea that 'minimality' should consist in some homomorphic relation to other models of the specification. This might be worth pursuing for the case of circumscription too. In particular, the approach in the theory of abstract data types characterizes its distinguished objects only *up to isomorphism*. But, from that viewpoint, it would be natural to weaken the notion of, say, individual minimality as follows:

- \( \text{D} \) is a model for \( \Sigma \) and it has no proper *non-isomorphic* submodels verifying \( \Sigma \).

In the latter case, e.g., the integers as they stand would be a minimal model for their own ordering theory - which is certainly reasonable.

Another interesting aspect to the theory of abstract data types is the greater attention paid to further *syntactic fine-structure* of premises. For instance, in specifications, it often makes sense to distinguish between 'visible variables', standing for features of the system which are accessible to, or at least observable by, the user, and *hidden variables*, helping to structure the specification, without becoming public. Such a division at once introduces a new range of interesting
logical questions. For instance, given the full specification $\Sigma(L_V, L_H)$, what would be a good independent description of its 'observable content'? The literature contains both syntactical proposals here, such as

$$\Sigma \models L_V = \{ \varphi \in L_V \mid S \vdash \varphi \}$$

and semantical ones, such as

$$\text{MOD}(\Sigma) \models L_V = \{ (D \models L_V) \mid D \models \Sigma \}$$

The latter class consists of all $L_V$-reducts of models for $\Sigma$. (Or equivalently, it consists of all structures for the observable language which can be expanded to models of the full specification $\Sigma$.) And accordingly, questions concerning minimal models become more complex too. Notably, how are minimal models for the full $\Sigma$ related to minimal models for its observable part?

### 2.3.3 ... and Scientific Theories

Before going into these questions, this is the proper place to point out yet another wider analogy. Within an entirely different tradition, similar questions have come up in the *Philosophy of Science*. There too, it is customary to analyze *scientific theories* as being formulated in a two-tier language $L_e + L_t$. There is an *empirical*, or observational vocabulary corresponding to what we can observe or measure, on top of which one finds a *theoretical vocabulary* corresponding to postulated theoretical entities. For instance, in mechanics, observables would include predicates of position, velocity, etcetera, whereas, e.g., forces would be theoretical constructs. In general, the statements of a theory will then fall into three kinds:

- purely within $L_e$: empirical facts and regularities,
- purely within $L_t$: theoretical axioms,
- mixed $L_e$, $L_t$: 'bridge principles'.

For empirical theories, this view goes back to Frank Ramsey and Rudolf Carnap: but, it may even be observed for mathematical theories with David Hilbert (where $L_e$ would be the 'finitistically' computable part).

Many logical questions have been studied in this perspective, often concerning the status of the theoretical terms. (See Przelecki 1969, van Benthem 1982). The *practical* value of introducing such terms is not in question, but one would like to know how far this procedure is necessary *in principle*. The latter question brings us back to the issue of adequate independent definitions of the *empirical content* of a theory, which was already mentioned above. Both the above possibilities have been considered in the literature. In particular, $\text{MOD}(\Sigma) \models L_V$ has a
vivid interpretation now. When applying, say, mechanics to some empirical system which we have measured, we have to stipulate suitable mass and force functions so as to satisfy Newton's Laws: in order to start mechanical calculations, which can then be used again to explain, or predict, further empirical facts.

Here is one question which has received a good deal of attention. When are the theoretical terms eliminable from a theory $\Sigma$, in the sense that $\text{MOD}(\Sigma) \restriction L_e$ has a first-order definition purely inside $L_e$? The answer is that this happens if and only if the operators $\text{MOD}$ and restriction commute:

$$\text{MOD}(\Sigma) \restriction L_e = \text{MOD}(\Sigma \restriction L_e).$$

Such commutation does not always occur (in fact, it is an undecidable matter when it happens). What we always have is one inclusion:

$$\text{MOD}(\Sigma) \restriction L_e \subseteq \text{MOD}(\Sigma \restriction L_e).$$

The converse, however, will only be guaranteed to hold for certain special syntactic classes of theories $\Sigma$. In particular, we always have it for universal theories.

The latter restriction even has some further motivation on the above analysis of theories. As soon as we allow quantifier combinations such as $\forall \exists$ inside the 'observational' language, we are really smuggling in theoretical terms through logical complexity. This may be seen explicitly using Skolem forms:

$$\forall x \exists y Rxy \iff \exists f \forall x Rxf(x).$$

So, it is quite natural to consider only Skolemized theories, keeping the axioms universal, while making our ontological commitments explicit in the function symbols.

This move will make theories more algebraic than has been suggested in our discussion of circumscription. Nevertheless, that would be quite in line with research into logic programs and abstract data types, which is usually confined to almost algebraic formalisms.

Now let us return to the issue of minimal models. This has not been very prominent in the philosophical literature (with the earlier-mentioned exception of Hempel 1965). In fact, there might be an interesting new research line into elimination of theoretical terms on minimal models. Here, we shall only consider some special cases - while also slightly changing course.

Let $\Sigma$ be a universal theory in a language $L_e + L_t$ with identity, where each language contains at least one individual constant. What kind of connection can we expect between minimal models for $\Sigma$ itself and those for its restriction $\Sigma \restriction L_e$ (only with respect to universal formulas)? We define a new, and perhaps more appropriate, notion of model-theoretic restriction:
For any $L_\epsilon + L_\tau$-structure $D$, $D \models L_\epsilon$ is the $L_\epsilon$-structure whose domain consists of all closed $L_\epsilon$-term interpretations in $D$, with the inherited interpretation for $L_\epsilon$-functions and predicates.

Then, we can define an operation $X \models L_\epsilon$ on model classes $X$, with an obvious meaning. Finally, let the operator $\mu$ select (say) individual-minimal models.

**Proposition:** For $\Sigma$ and $\Sigma / L_\epsilon$ as above,

$$(\mu \text{MOD}(\Sigma)) \models L_\epsilon = \mu \text{MOD}(\Sigma / L_\epsilon).$$

**Proof:** '\subseteq'. Let $D$ be a minimal model for $\Sigma$. A fortiori, $D \models \Sigma / L_\epsilon$. Now, the operation $\models L_\epsilon$ preserves truth of universal $L_\epsilon$-formulas. So, $\Delta \models L_\epsilon \models \Sigma / L_\epsilon$.

Moreover, individual-minimality of this model is automatic, because the universe has only definable objects.

'\supseteq'. Let $D$ be a minimal model for $\Sigma / L_\epsilon$. Consider the following set of formulas:

$$(\Delta) \quad \Sigma \cup \langle \text{all true } L_\epsilon \text{-atoms or true negated } L_\epsilon \text{-atoms in } D \rangle.$$

Every finite subset of $\Delta$ has a model. [Otherwise, $\Sigma$ would classically imply a formula $\neg \Lambda \delta_i$; where the $\delta_i$ are (negated) $L_\epsilon$-atoms occurring in $\Delta$. But then, this formula would belong to $\Sigma / L_\epsilon$ and be true in $D$, whereas by definition, $\Lambda \delta_i$ is true in $D$.] By compactness, then, $\Delta$ itself has a model, say $M$. Now, consider the submodel $M^*$ of $M$ whose domain is generated by all closed term interpretations from $L_\epsilon + L_\tau$. By the universal form of $\Sigma$, $M^* \models \Sigma$. Moreover, $M^*$ is a minimal model for $\Sigma$, given its term construction. Finally, $D$ is isomorphic to $M^* \models L_\epsilon$, since $M^*$ satisfies a faithful description of its atomic structure. But then, the required extension of $D$ may be constructed by isomorphic copying from $M^*$.

**Remark:** This type of analysis can be pushed further. For instance, for arbitrary theories $T_1 \subseteq T$, the following statements can be proved equivalent:

- $(\mu \text{MOD}(T)) \models L_1 = \mu \text{MOD}(T_1)$
- $T$ is $m$-conservative over $T_1$,
  i.e., for all quantifier-free $L_1$-sentences $\phi$, $T \models \phi$ only if $T_1 \models \text{ind } \phi$.

Finally, we note one further similarity between the theory of abstract data types and the philosophy of science. Philosophers have emphasized the existence of a huge network of scientific theories, connected by various relations, such as being an 'extension' of another theory, or being 'interpretable' in it. Again, such relations have also been proposed and studied for abstract data types. In fact, a very
interesting *calculus* of operations on and relations between data types has been
developed in Bergstra, Heering and Klint 1986. We conclude with one example,
which is relevant to the useful property of *modularity*.

The *sum* $\Sigma_1 + \Sigma_2$ of two abstract data types is just the union of their axioms,
in their combined language. Again the question arises as to the connection between
the minimal models for the various components and for the whole. As before, one
direction is easy to establish (again, for the purposes of illustration, we consider the
case of individual-minimality):

- if $D$ is a minimal model for $\Sigma$,
  then $D \parallel L_i$ is a minimal model for $\Sigma_i$ ($i = 1,2$).

For a converse, we would need an *amalgamation* property:

- If $D_1$ is a minimal model for $\Sigma_1$, and $D_2$ one for $\Sigma_2$,
  then there exists some minimal model $D$ for $\Sigma$ such that $D \parallel L_i = D_i$ ($i = 1,2$)

This will hold only if $\Sigma$ satisfies a strong splitting condition:

- if $\Sigma \models \phi_1 \lor \phi_2$ for quantifier-free $L_i$-sentences $\phi_i$ ($i = 1,2$),
  then $\Sigma_1 \models_{ind} \phi_1$ or $\Sigma_2 \models_{ind} \phi_2$.

In such a case, $\Sigma_1$ and $\Sigma_2$ do not really 'interact' in $\Sigma$.
As soon as they do, however, the above may be violated.

**Example:** $\Sigma_1 = \{ Pa \lor Qa \}$, $\Sigma_2 = \{ \neg Qa \lor Ra \}$, $\Sigma = \Sigma_1 \cup \Sigma_2 \models Pa \lor Ra$,
but neither $\Sigma_1 \models_{ind} Pa$ nor $\Sigma_2 \models_{ind} Ra$.

The interest in the fine-structure of axioms $\Sigma$, both as to different kinds of
vocabulary involved and as to various syntactic modules, represents a notable
tendency in current research. Against the background of the earlier general semantic
results, actual performance of logic programs or data specifications will still be
crucially affected by their syntactic design. It is important to bring to light useful
structures here. (See also Apt and Pugin 1987 on 'stratified' PROLOG programs.)
This is not just a technical concern within computer science. Also in general logic,
there is a great need for a better theory of the *structure* of premises, if we are to
arrive at a deeper understanding of at present intangible phenomena like good
*organization* and structuring of arguments.
3. DYNAMICS

The study of minimal models and non-monotonicity almost forces one to acknowledge the more dynamic aspects of knowledge acquisition and revision. But even so, it is only one strand among various motives pointing in the latter direction. In this Section, we shall also consider dynamics at the level of building up interpretations of single statements, and related forms of information flow.

3.1 Dynamics of Interpretation

'Processing mechanisms' play a pervasive role in programming languages, obviously - but also in natural language. For instance, it seems natural to understand various possibilities and impossibilities in anaphoric linkage from an algorithmic perspective. And, such considerations are also appropriate in the analysis of conditionals (cf. Stalnaker 1972) or bare plurals when viewed as expressing default rules (cf. Pelletier and Schubert 1985). Here, we shall concentrate on the example of anaphora (see Section 3.1.2).

There are various new semantic theories invented especially for their 'dynamic flavour'; but, in fact, ordinary predicate logic itself provides an excellent initial model for studying such phenomena. To see this, we take our starting point in a well-known approach to the semantics of programs, due to Floyd and Hoare.

3.1.1 Operational Semantics for Programs

The basic format of interpretation for predicate logic is Tarski's relational schema

\[ M, I \models \varphi [a] \]

('\( \varphi \) is true in \( M \) under \( I, a \)');

where \( M \) is a model structure, \( I \) an interpretation of vocabulary into suitable items of \( M \), \( \varphi \) some formula, and \( a \) an assignment to the variables occurring freely in \( \varphi \).

Here, the assignment is a modest 'auxiliary interpretation function', needed in order to get a recursion going on the structure of \( \varphi \). Later on, however, this Cinderella met her prince. In Computer Science, assignments may be viewed as memory states of a computer (functions from identifiers / addresses to data values) - and as such, they play the leading role in computationally oriented introductions to Logic (such as Gries 1981).
One fundamental generalization of the Tarski schema to the semantics of programming languages has taken place in so-called operational semantics. In addition to the set of descriptive formulas, one also defines a set of programs inductively, and then interprets these according to the following schema (in the context of some model M, I):

$$[[\pi]]$$ is the set of successful state transitions associated with the program $$\pi$$.

Here are some typical steps encountered for simple program constructions:

1. $$[[x:=t]] = \{ (a,a^x_a) \mid$$ a any assignment $$\}$$

2. $$[[\pi_1;\pi_2]] = \{ (a,b) \mid$$ for some c, $$(a,c) \in [[\pi_1]]$$ and $$(c,b) \in [[\pi_2]]$$ $$\}$$

3. $$[[\text{IF } \epsilon \text{ THEN } \pi_1, \text{ ELSE } \pi_2]] = \{ (a,b) \mid M,I \models \epsilon[a] \text{ and } (a,b) \in [[\pi_1]],$$ or $$M,I \not\models \epsilon[a] \text{ and } (a,b) \in [[\pi_2]] \}$$

Predicate-logical formulas function as static assertions in this context. They may be tests as in (3), or statements of specification for programs, or statements about program behaviour, such as the well-known correctness assertions

$$\{ \phi \} \pi \{ \psi \} :$$

for all assignments a such that $$M,I \models \phi[a],$$

and all assignments b such that $$(a,b) \in [[\pi]],$$

it holds that $$M,I \models \psi[b].$$

('pre-condition $$\phi$$ for $$\pi$$ implies post-condition $$\psi$$').

Although the semantic format is still very much in the usual spirit, even this application yields its own questions. For instance, the logical meta-theory of this framework derives its interest to a large extent from the fact that its predicate-logical counterpart does not transfer smoothly. Notably, in searching for completeness theorems in this area, it turns out that even the correct formulation of what should be regarded as 'completeness' is a tricky issue (cf. Cook 1978, Bergstra and Tucker 1982).

Programming languages, at least the traditional imperative ones, exert dynamic control over a series of actions to be performed by the audience, usually the computer. But, this feature is also very conspicuous in natural languages, where speakers direct their listener's representation of information by means of various textual devices. This process occurs explicitly in reasoning, when we set up an argumentative structure using imperatives: 'suppose', 'let', 'take', ... . But it is also present at the sentence level, witness the illustration given below. Thus, the slogan of Language as Action has become a central one in current semantics, and many proposals for 'dynamic formats' of interpretation have appeared (cf. Kamp 1981, Heim 1982, Seuren 1985).
One phenomenon where dynamic aspects emerge is that of anaphoric connection. In actual discourse, pronouns are used to refer back to earlier expressions, in such a way that the listener can pick up links intended by the speaker. There are limits to this process, however: not anything goes - and contemporary dynamic theories are often motivated by the desire to explain these constraints by reference to some processing mechanism. Here are three examples of anaphoric facts whose explanation seems to go beyond predicate logic as ordinarily conceived:

'A speaker arrived. He was late.' (1)

How can the 'existential quantifier' pick up a pronoun in another sentence?
Moreover, there is a clear-cut ordering involved, witness the impossibility of an intended link as in the following sentence:

*He was late. A speaker arrived.' (2)

And finally, within a single sentence, consider the pattern:

'If a speaker arrives, he is late.' (3)

This is traditionally transcribed as follows:

\[
\forall x ((Sx \land Ax) \rightarrow Lx),
\]

interchanging the conditional with the existential quantifier. (It is as if the, invalid, prenex law \((\exists x \phi(x) \rightarrow \psi(x)) \leftrightarrow \forall x (\phi(x) \rightarrow \psi(x))\) had been applied.) Why cannot this sentence be interpreted correctly as it stands: its obvious form being

\[
(\exists x (Sx \land Ax) \rightarrow Lx)
\]

These problems are solved in the cited work by Kamp and Heim through the intermediary of a level of 'discourse representation', in between syntactic form and eventual truth conditions. But, as was pointed out convincingly in Barwise 1987, they can be handled equally well without such a move, by adopting a more dynamic format of interpreting syntactic forms, inspired by the earlier operational semantics.

3.1.2 Operational Semantics for Assertions

A very clean presentation of this idea arises when the language interpreted is predicate logic itself, as is shown in Groenendijk & Stokhof 1987. Thus, the earlier 'static' assertion language for programs itself receives a dynamic interpretation:

\[
[[Px]] = \{ (a,a) \mid M, I \models Px [a] \}:
\]
i.e., atomic formulas are just tests without special side-effects.

\[
[[\phi \land \psi]] = [[\phi]] \circ [[\psi]]:
\]
i.e., conjunction is treated like composition (⊙or ;).
\[[\exists x\varphi]\] = \{(a,b) \mid (c,b) \in [[\varphi]] \text{ for some } c \sim \chi a\}:

i.e, existential statements are made true by finding some witness to their matrix.

These stipulations explain the first two anaphoric facts mentioned above. \(\exists x A x \land L x\) will have the same interpretation as \(\exists x (A x \land L x)\), but not as \(L x \land \exists x A x\). In general, the effect of the above clauses is to widen scopes toward the right.

An explanation of the third phenomenon observed requires a treatment of conditionals. We start with a preliminary notion, however:

\[[\neg \varphi]\] = \{(a,a) \mid \text{for no } b \text{, } (a,b) \in [[\varphi]]\}.

Thus, a negation is a strong denial, without side-effects. Then, a clause for implications \(\varphi \rightarrow \psi\) may be derived via their traditional equivalence with \(\neg (\varphi \land \neg \psi)\),
or by direct introspection:

\[[\varphi \rightarrow \psi]\] = \{(a,a) \mid \text{for all } b \text{ with } (a,b) \in [[\varphi]], \text{there exists } c \text{ with } (b,c) \in [[\psi]]\}.

Finally, we introduce universal quantification, again without side-effects:

\[[\forall x \varphi]\] = \{(a,a) \mid \text{for all } b \text{ with } a \sim x b, (a,b) \in [[\varphi]]\}.

Then, it may be checked that the following two formulas indeed have the same associated transition relation:

\(\exists x A x \rightarrow L x\) and \(\forall x (A x \rightarrow L x)\).

Thus, the anaphoric sentence 3 is vindicated as it stands.

On the other hand, no binding across the conditional occurs with the related syntactic form

\(\neg \exists x A x \rightarrow L x\).

For, the negation in the antecedent 'envelops' the scope of the existential quantifier. But this is as it should be. After all, there is no such link in natural language either, witness the incorrect sequence

*'If no speaker arrives, he is late.'

It should be admitted at once that this account still has many empirical weaknesses. For instance, it describes anaphoric facts in natural language only indirectly, being dependent on a translation into predicate logic. Moreover, those anaphoric facts are much more diverse and subtle than may be apparent from the four examples given here. But, the account does show how standard predicate logic, far from being an obstacle to recognizing the role of dynamical interpretation, can actually be a good testing ground for theorizing about it.

In fact, there is still a close connection between ordinary predicate logic and its interpreted variant. In a sense, the latter amounts to reading 'ordinary' formulas with a different scope convention, extending scopes of existential quantifiers as far
as possible toward the right, until one hits the boundary of an operator like negation, which 'seals off' its subformula. (For a more general discussion of different scoping conventions, as expressing various strategies of interpretation for one single language, cf. van Benthem, 1986b).

More formally, there is a reduction to ordinary predicate logic which works as follows. For each formula $\varphi$, with free variables $x_1, \ldots, x_n$, the following induction defines a predicate $\text{TRANS} (\varphi, x_1, \ldots, x_n, y_1, \ldots, y_n)$ [intuitively, $(x_1, \ldots, x_n; y_1, \ldots, y_n)$ is a successful $\varphi$-transition between partial assignments']:

\[
\begin{align*}
\text{TRANS} (P_{x_i}; x_i, y_i) &= y_i = x_i \land P_{x_i} \\
\text{TRANS} (\varphi \land \psi; \bar{x}, \bar{y}) &= \exists \bar{z} \ (\text{TRANS} (\varphi; \bar{x}, \bar{z}) \land \text{TRANS} (\psi; \bar{z}, \bar{y})) \\
&\quad \text{(here the } \bar{z} \text{ are new free variables)} \\
\text{TRANS} (\exists x \varphi; \bar{x}, \bar{y}) &= \exists z_j \ \text{TRANS} (\varphi; \bar{x}(z_j | x_j), \bar{y}) \\
\text{TRANS} (\neg \varphi; \bar{x}, \bar{y}) &= \bar{y} = \bar{x} \land \forall \bar{z} \neg \text{TRANS} (\varphi; \bar{x}, \bar{z}).
\end{align*}
\]

By an obvious induction, $\text{TRANS} (\varphi; \bar{x}, \bar{y})$ defines the successful transitions for $\varphi$. Therefore, any central semantic notion of dynamic predicate logic can be reduced to static assertions in this way.

Nevertheless, the new formalism as it stands does seem to correspond more closely to practical uses of predicate logic, or semi-formalisms employed in mathematical prose, where we do tend to use scoping conventions closer to the one mentioned above.

Another attraction of the dynamic framework is that it suggests taking a fresh look at defining connectives, and logical operators generally. As has often been observed, logical operators do not have meaning only: they also exert control. (For instance, Jennings 1986 points at the use of "and" and "or" as sequencing, or more generally punctuation devices.) What we can do in the new setting, for instance, is to define both 'dynamic' order-dependent versions of connectives, and more classical 'parallel' ones, studying their interplay. One instructive example is provided by conjunction. The natural 'classical' stipulation, in terms of intersection of successful transition sets, will now express a different option from the preceding one: namely, a requirement of parallel execution. And similarly, a classical negation in terms of complement will suddenly acquire a new operational significance.

Finally, the new system also suggest several ways of defining valid consequence as arising from successive processing of premises. One natural candidate is the following:
\( \varphi_1, \ldots, \varphi_n \vdash \psi \) if, for all models \( M, I \) and assignments \( a_1, \ldots, a_{n+1} \) such that \((a_1, a_2) \in \[[\varphi_1]\], \ldots\), \((a_n, a_{n+1}) \in \[[\varphi_n]\]\), there exists an assignment \( b \) with \((a_{n+1}, b) \in \[[\psi]\]\).

This notion of consequence, like the ones presented in Section 2, lacks several of the main structural properties of the standard one (be it for different reasons).

It is not monotone for instance:

\[ \exists x Ax \vdash Ax; \quad \text{but} \quad \exists x Ax \land \exists x \neg Ax \not\vdash Ax. \]

But this time also, it even lacks such simple 'domestic' properties as insensitivity to permutation, or contraction of identical premises:

\[ \exists x Ax \land Lx \vdash \exists x (Ax \land Lx); \quad \text{but} \quad Lx \land \exists x Ax \not\vdash \exists x (Ax \land Lx), \]

\[ \exists x Ax \land \exists x \neg Ax \land \exists x Ax \vdash Ax; \quad \text{but} \quad \exists x Ax \land \exists x \neg Ax \not\vdash Ax. \]

Still, as in Section 2, certain special cases will remain valid - including the following leftward form of monotonicity:

\[ \varphi \vdash \psi \quad \text{implies} \quad \chi, \varphi \vdash \psi. \]

Once again, the divergences from classical logic here arise from reasons different from those in Section 2. We have defined 'logical consequence' rather close to one particular interpretation algorithm, and are now feeling its effects. Whether this has been a wise policy, will be discussed further in Section 4 below.

### 3.2 Dynamics of Information Flow

Changing assignments means no more than changing our links with a certain model. We have been studying the 'dynamics of adjustment', so to speak. But already in Section 2, we also encountered the dynamics of changing information. The latter perspective is currently receiving a good deal of attention too. For instance, in the philosophy of science, there has been work on the dynamics of changing theories, which has also issued in more general epistemic studies of various operations on knowledge states: in particular, addition and retraction of information (see Gärdenfors 1988.)

There are at least two ways of thinking about the dynamics of changing information states. One is the classical perspective: common to both standard logic and such less standard frameworks as e.g. possible worlds semantics. Here, incoming information is treated as reduction of the space of a priori possibilities - as was done already in Section 2, when discussing 'classical' versus 'minimal' transformations on information states. The other perspective takes some more primitive notion of epistemic state, in particular, one in which partial information need not be represented by a cloud of all possible total (world) extensions. And
then, 'propositions' can act as more abstract operators on such states. We shall consider both approaches.

3.2.1 Reducing Possibilities

First, the method of 'eliminating uncertainty' is easy to implement, witness current folklore. Here is one particular example, due to Veltman 1987 (but compare also Heim 1982, and others). Consider a modal *propositional language* with the ordinary Boolean connectives as well as modality \( \Diamond \) ("might"). Let \( U \) be a set of 'possibilities' (say, ordinary valuations), for which it makes sense to call an atom \( p \) true or false. Then we can define, for each formula \( \varphi \), a corresponding transformation \( [[\varphi]] \) on subsets of \( U \):

\[
[[p]](X) = \{ x \in X \mid x \text{ verifies } p \} \\
[[\varphi \land \psi]](X) = [[\varphi]](X) \cap [[\psi]](X) \\
[[\varphi \lor \psi]](X) = [[\varphi]](X) \cup [[\psi]](X) \\
[[\neg \varphi]](X) = X - [[\varphi]](X)
\]

and

\[
[[\Diamond \varphi]](X) = \begin{cases} 
X & \text{if } [[\varphi]](X) \neq \emptyset \\
\emptyset & \text{otherwise}
\end{cases}
\]

If desired, one might add a sequential conjunction as before:

\[
[[\varphi ; \psi]](X) = [[\psi]]([[\varphi]](X))
\]

For purely propositional formulas \( \varphi \), it is easy to see that \( [[\varphi]](X) \) merely amounts to intersecting \( X \) with the truth range of \( \varphi \) in \( U \) computed in the standard fashion. But, already with the modal operator, some interesting phenomena occur, once *consequence* is introduced on the analogy of the previous subsection:

\[ \varphi_1, \ldots, \varphi_n \vdash \psi \text{ if,} \]

for all \( X \), \( [[\varphi_1; \ldots ;\varphi_n]](X) \subseteq [[\psi]](X) \).

Note, e.g., that \( \Diamond \neg p ; p \) will be a consistent sequence, implying \( p \), whereas its permutation \( p ; \Diamond \neg p \) is *inconsistent*. As Veltman argues, this reflects the facts of life for our ordinary use of the epistemic modality "might". And there are various other applications of this formal system too.

Also as before, the new dynamic consequence can be embedded in 'static' standard logic. For instance, in a sense, the above little system is a part of *monadic* predicate logic. To see this, assign unary predicate letters \( P \) (uniquely) to each proposition letter \( p \), while also taking one distinguished unary predicate letter \( X \). By
induction, we translate each formula $\varphi$ into a syntactic operation $\Delta(\varphi)$ on monadic formulas $\alpha = \alpha(x)$ having a free variable $x$:

$$
\begin{align*}
\Delta(p)(\alpha) &= \alpha \land Px \\
\Delta(\varphi \land \psi)(\alpha) &= \Delta(\varphi)(\alpha) \land \Delta(\psi)(\alpha) \\
\Delta(\varphi \lor \psi)(\alpha) &= \Delta(\varphi)(\alpha) \lor \Delta(\psi)(\alpha) \\
\Delta(\neg \varphi)(\alpha) &= \alpha \land \neg \Delta(\varphi)(\alpha) \\
\Delta(\exists x \Delta(\varphi)(\alpha)) &= \exists y \ [y/x] \Delta(\varphi)(\alpha) \land \alpha \\
\Delta(\varphi;\psi)(\alpha) &= \Delta(\psi)(\Delta(\varphi)(\alpha))
\end{align*}
$$

Evidently, this is a direct transcription of the above 'truth definition' into a simple language quantifying over states in $U$. Thus, we have the following reduction:

$$
\varphi_1, \ldots, \varphi_n \vdash \psi \quad \text{if and only if} \\
\forall x(\Delta(\varphi_1;\ldots;\varphi_n)(Xx) \rightarrow \Delta(\psi)(Xx)) \text{ is valid in monadic predicate logic.}
$$

As a bonus, decidability, the finite model property and other desirable features are immediate for the new dynamic logic.

### 3.2.2 Building Up Information

Next, implementation of the second approach can actually go in many directions, since we can structure 'knowledge states' in many different ways. (The first approach may in fact be defended as being an elegant way of avoiding such decisions: cf. Stalnaker 1986.) In particular, we have to specify the 'grain size', as it were. Are knowledge states like sets of sentences, with all their syntactic peculiarities? Or, should we smoothen these somewhat by thinking of deductively closed theories in some logic? Could we work on the analogy of Beth tableaus, or should we steer away from their particular notational structure, as is done in Hintikka model sets? Slight differences in presentation may now become logically significant.

We shall not go into any particular proposal here. Rather, we want to point out a certain danger, of merely revamping existing systems. Suppose that we choose the approach via deductively closed theories (as is suggested by Gärdenfors' treatment). Let us work in ordinary classical propositional logic. (Note that this decision itself determines what will be 'deductively closed' theories.) The main idea then becomes just this:

the action of any formula $\varphi$ on any 'state' $T$ consists in
forming the deductive closure of $T \cup \{ \varphi \}$.

Now, these theories come in an obvious relation of inclusion:

$$
T_1 \subseteq T_2 \quad \text{if} \quad T_1 \subseteq T_2.
$$
Moreover, they form a *distributive lattice*, with respect to the available operations of supremum \( \bigvee \) and infimum \( \bigwedge \) on theories. Then, one can set up a recursive definition of operators \( \llbracket \varphi \rrbracket \) by merely providing a convenient decomposition:
\[
\llbracket \varphi \land \psi \rrbracket (T) = \llbracket \psi \rrbracket \llbracket \varphi \rrbracket (T) \bigvee \llbracket \varphi \rrbracket (T), \\
\llbracket \varphi \lor \psi \rrbracket (T) = \llbracket \varphi \rrbracket (T) \bigwedge \llbracket \psi \rrbracket (T).
\]

And other connectives could be treated by postulating additional structure on the set of knowledge states (e.g., for implication, one would have to make it into a Heyting algebra). But obviously, nothing much would be gained in this way. So, there is a danger (to be avoided) of trivially achieving a 'dynamic' presentation.

Another approach might be to start from a very abstract notion of *dynamic information structure*, studying specializations as they arise. For instance, the general type might be this:

\((S, \sqsubseteq, \{\tau_p \mid p \in P\})\),

where \(S\) is a set of 'information states', ordered by increasing strength (\(\sqsubseteq\)), on which a certain family of transformations acts, indexed by propositions. In other words, we assume a perspective from *Group Theory*.

This perspective makes it easy to ask systematic questions. One is how much structure should be imposed on the transformations. Evidently, they should form a *semi-group* under composition. But, should they also be a *group*? i.e., should there be an *inverse* to every proposition (its 'retraction')? [Gärdenfors 1988 has rather looked at this structure of transformations from the point of view of *Category Theory*, demanding the presence of *equalizers*, modelling equivalence between propositions.] But also, this question interacts with the possible structure to be imposed on information states: should they form a lattice, a Heyting Algebra or even a Boolean Algebra? Then, it would be natural to have corresponding closure conditions on transformations too. E.g., there should be some operation on them such that
\[
(\tau_p \sqcap \tau_q)(x) = \tau_p(x) \sqcap \tau_q(x), \text{ for all } x \in S.
\]

Once a class of transformations with certain closure properties has been chosen, we can bring up various additional questions. For instance, what are natural subclasses of transformations satisfying additional mathematical requirements? One natural example are *idempotent* operators, satisfying the condition
\[
\tau_p \circ \tau_p = \tau_p.
\]

This is certainly a very plausible logical requirement too. Another reasonable condition would be *monotonicity*, in the sense of respecting growth of information:
\( x \sqsubseteq y \) only if \( \tau_p(x) \sqsubseteq \tau_p(y) \), for all \( x, y \in S \).

This certainly holds for 'classical' propositions; but also, e.g., for all modal propositions in Veltman 1987 (where \( \sqsubseteq \) is identical with \( \supseteq \)).

Here is one elementary observation in this vein.

**Proposition:** If \( S \) is a Boolean Algebra, and \( \{ \tau_p \mid p \in P \} \) a set of idempotent operators which are also distributive, i.e.,
\[
\tau_p(x \sqcap y) = \tau_p(x) \sqcap \tau_p(y) ,
\]
for all \( x, y \in S \),
then the whole information structure can be represented by a set structure of the 'eliminative' kind described earlier.

**Proof:** (Compare also the analysis of set transformations given in Section 2.)

For each state \( s \in S \), set
\[
\overline{s} = \{ x \mid x \sqsubseteq s \} .
\]

For each proposition \( p \in P \), set
\[
\overline{p} = \{ x \mid \tau_p(x) = x \} .
\]

(This is the common idea of identifying propositions with their fixed points.)

Then it is easy to prove that
\[
\begin{align*}
(1) & \quad x \sqsubseteq y \iff \overline{x} \supseteq \overline{y} \\
(2) & \quad \overline{\tau_p(s)} = \overline{s} \cap \overline{p} .
\end{align*}
\]

But, one could also try to apply more sophisticated mathematical results here, on the representation of algebras or lattices with certain types of endomorphisms. (Compare Jónsson and Tarski 1951/2.)

Finally, one basic attraction of the present framework is that it also suggests new types of question, beyond the classical case. For instance, where transformations are around, invariants cannot be far away (cf. van Benthem 1985b). A relation \( R \) between states is invariant with respect to our class of propositions if, say,
\[
(x, y) \in R \iff (\tau_p(x), \tau_p(y)) \in R , \quad \text{for all } p \in P .
\]

Is there an interesting invariant structure on \( S \)?

### 3.2.3 Relational Calculus and Categorial Grammar

Nevertheless, there are also other possibilities for abstract information structures. Perhaps, propositions are not really functions on information states, but rather relations, which can also take no value, or more than one. For instance,
certain propositions might embody choices, leaving several options. In that case, one natural framework would be to view propositions as forming an algebra in the sense of the Relational Calculus, with basic operations such as composition, intersection, union, etcetera. (Recall the earlier discussion of the dynamics of changing assignments.)

One basic question in this perspective too, is what would be the natural operations on propositions. This amounts to asking for some principled account of operations on binary relations. (For an algebraic study of these matters, see Jónsson 1984.) One way of doing this employs the framework of Type Theory. For instance, there is a natural notion of logicality for this type of operator, in terms of invariance for permutations of states (cf. van Bentham 1987b). Moreover, again, we can introduce additional useful conditions, such as continuity, in the sense of respecting arbitrary unions of families of relations. All possibilities can then be classified (cf. van Benthem 1987a), and they form a neat basic set, including the above-mentioned examples.

**Proposition:** The logical continuous operators on binary relations are exactly those defined by a schema of the form

\[ \lambda R. \lambda xy. \exists u. Ruv \land 'some Boolean condition on identities involving \; x,y,u,v'. \]

Examples are converses:

\[ \lambda R. \lambda xy. \exists u. Ruv \land x = v \land y = u, \]

or diagonal:

\[ \lambda R. \lambda xy. \exists u. Ruv \land x = y = u = v. \]

The result is easily specialized to operations from binary to unary relations; with a typical example such as projection:

\[ \lambda R. \lambda x. \exists u. Ruv \land x = u. \]

It can also be generalized to n-ary operations on binary relations, bringing in (typically) disjunction or conjunction:

\[ \lambda RS. \lambda xy. \exists uv. Ruv \land Szw \land x = u = z \land y = v = w, \]

or composition:

\[ \lambda RS. \lambda xy. \exists uv. Ruv \land Szw \land x = u \land v = z \land w = y. \]

This kind of systematic classification is important if we are to bring some order into the plethora of possibilities for 'dynamic' logical operators.
Another interesting question is how such a relational structure would translate into a corresponding system of inference. Here, we can follow a suggestion arising from Orlowska 1987, and exploit an analogy with Categorial Grammar. The logic will be very much like a so-called 'Lambek Calculus', as developed in that area (cf. van Benthem 1986a, 1987a). The main idea can be explained as follows:

- basic propositions \( p \) are interpreted as binary relations \( R_p \)
- complex propositions \( p, q \) are interpreted through the composition \( R_p \circ R_q \)
- disjunctions may be interpreted by unions

But, what to do about implications? Here, Orlowska notes that two quite plausible 'implicational' operators may be used, introduced recently by Tony Hoare:

\[
\begin{align*}
R \setminus S & = \cup \{X \mid R_0 X \subseteq S\} \\
S / R & = \cup \{X \mid X_0 R \subseteq S\}
\end{align*}
\]

which describe 'weakest pre- and post-specifications'. (This continues the well-known work on pre- and post-conditions in the operational semantics of programming languages. But see also Jónsson 1984 for a purely algebraic introduction of \( \setminus \) and \( / \).) Accordingly, the dynamic perspective suggests introducing two implications, one searching for its argument on the left-hand side, and one searching on the right. [A similar idea has been suggested independently by Gordon Plotkin (private communication).] But, this is precisely standard practice in Categorial Grammar, which has developed systems of proof for directed types

\[ a \setminus b \quad \text{and} \quad b / a. \]

For instance, the basic Lambek Calculus is the following Gentzen-type system:

**Axiom:** \[ a \Rightarrow a \]

**Rules:**

\[
\begin{align*}
& X \Rightarrow a \quad Y, b, Z \Rightarrow c \\
& X \Rightarrow a \quad Y, b, Z \Rightarrow c \\
\hline
& Y, X, a \setminus b, Z \Rightarrow c \\
& Y, b / a, X, Z \Rightarrow c
\end{align*}
\]

\[
\begin{align*}
& X, a \Rightarrow b \\
& a, X \Rightarrow b \\
\hline
& X \Rightarrow b / a \\
& X \Rightarrow a \setminus b
\end{align*}
\]
\( X \Rightarrow a \quad Y \Rightarrow b \quad X, a, b, Y \Rightarrow c \)

\[ \begin{align*}
\text{-------------------------} \\
X, Y \Rightarrow a \cdot b \\
\end{align*} \]

\[ \begin{align*}
\text{-------------------------} \\
X, a \cdot b, Y \Rightarrow c \\
\end{align*} \]

Derivable sequents in this system L include \( a \cdot b \Rightarrow b \), but typically exclude \( a \cdot b, a \Rightarrow b \). [Here, we have deviated somewhat from the standard version, in allowing empty sequences on the left. As a consequence, we can derive, e.g., \( (e/e) \cdot t \Rightarrow t \).]

**Proposition:** The Lambek Calculus L is sound for the above relational calculus.

That is, if the sequent \( a_1, \ldots, a_n \Rightarrow b \) is L-derivable, then,

for any assignment of binary relations \( R_x \) to primitive types \( x \),

with \( R_a \) for complex a computed as above,

\[ R_{a_1} \circ \ldots \circ R_{a_n} \subseteq R_b. \]

It would be very interesting to have a **converse** too (for the operations \( ., \cdot, \\backslash \) and \( / \), that is.) Thus, we would establish a link between basic logics of categories in natural language and plausible systems of 'dynamic logic'.

### 3.2.4 Propositional Dynamic Logic

Indeed, there is also a profitable connection to be found with 'Dynamic Logic' in the usual sense of that phrase (see Harel 1984). In that research program, one adopts an enriched modal logic having both propositions and programs, in which the latter denote transition relations between states, whereas the former stand for functions from states to truth values (their more traditional role). One interesting feature is that there are operators mapping one into the other: modalities take programs to operate on propositions, whereas a test operator takes propositions to programs. This turns out to be a third convenient abstract framework to explore for present purposes.

Note that here, a possibility comes to the fore which has hitherto been neglected. We have been treating propositions themselves as being operations on information states. But in fact, we may want a separation of concerns: into *propositions* expressing a certain informational content, and various *modes of transforming* states (which can use certain propositional contents, to be sure). For
instance, the above test operator is one such mode, which checks if a state has a
certain property, but then leaves it as it is. Another operator might be addition,
which, given a propositional content, transforms any state into a minimal extension
(as measured along some prior inclusion relation among states) having that content.
Thus, we are now interested in Propositional Dynamic Logic for its potential as a
dynamic logic of propositions, rather than programs.

Again, the general situation here can be analyzed in type theory.
Propositional dynamic logic has primitive types t (for truth values) and s (for
states). Propositions have the functional type
(s,t) ('from states to truth values'),
while programs have
(s,(s,t)) ('from pairs of states to truth values').
The above 'switching modes' will then be operators in the type
((s,t), (s,(s,t))).
As before, it makes sense to ask for logical items here, being those which are
invariant for permutations of states. [Note the formal similarity with the earlier
relational calculus case of the converse type ((e,(e,t)), (e,t)).] And in fact, the test
operator is logical, while also satisfying the earlier special requirement of continuity. Also as before, we can classify all possibilities of the latter kind in the
schema
\[ \lambda P. \lambda xy. \exists u. Pu \land 'Boolean condition on identities involving x,y,u'. \]

As a first attempt, one might consider that fragment of dynamic logic which
only has basic programs of the form ?φ (where ? is the test operator) and then the
usual program operations
; (sequencing), ∪ (choice) and * (iteration).
But, this will reduce to ordinary propositional logic, because of such equivalences
as the following:
\[ <π₁ ∪ π₂>φ \iff <π₁>φ ∨ <π₂>φ \]
\[ <π₁; π₂>φ \iff <π₁><π₂>φ \]
\[ <?α>φ \iff α ∧ φ \]
\[ (?φ)* = I \quad \text{(the identity map)} \]

So, it becomes imperative to add some further structure, as was suggested
above. That is, the set Dₘ of states will now carry a binary inclusion relation. Then,
we add two further modes of handling propositions. First, the mode of addition may be defined as follows:

+φ is the relation \{ (s,s') \mid s \geq s' and φ is true at s' and,

either φ is true at s and s'=s, or φ is false at s and φ is false

at all states in between s and s' \}.

There is an analogy here with the "until" operator of Tense Logic.
This then suggests having also a dual mode of subtraction (compare "since"): 

-φ is the relation \{ (s,s') \mid s \leq s and φ is false at s' and,

either φ is false at s and s'=s, or φ is true at s and φ is true

at all states in between s' and s \}.

Of course, the test operator ? retains its original definition:

?φ is the relation \{ (s,s) \mid φ is true at s \}.

In addition to these three modes +, -, ?, it will also be useful to have a general modality referring to possible extensions along in the usual way:

\[ \Box \phi \text{ is true at } x \text{ if } \phi \text{ is true at all } y \text{ with } x \leq y \]

\[ \Diamond \phi \text{ is true at } x \text{ if } \phi \text{ is true at some } y \text{ with } x \leq y \]

[Given our dual set-up, it would also make sense to introduce similar 'downward' operators.]

This logic will validate some basic principles for proposition-based transformations.

For instance, all three are idempotent:

\[ ?\phi ; ?\phi = ?\phi \]
\[ +\phi ; +\phi = +\phi \]
\[ -\phi ; -\phi = -\phi \]

But, there are also other validities, such as

\[ \phi \rightarrow (<+\phi> \psi \leftrightarrow \psi) \]
\[ -\phi \rightarrow (<-\phi> \psi \leftrightarrow \psi) \]

or the related

\[ ?\phi ; +\phi = ?\phi \]
\[ ?-\phi ; -\phi = ?-\phi \]

Some non-validities may be worth observing too:

\[ +\phi ; -\phi \neq I \]
\[ -\phi ; +\phi \neq I \]

where I is the identity transformation. The reason is shown in the following two pictures of possible information structures:
Mutatis mutandis, this formalism can also be used to analyse various axioms concerning theory extension and theory revision found in Makinson 1987 or Gardenfors 1988, two studies whose abstract perspective is quite congenial to the one presented here.

The general logic of the dynamic propositional language with only programs formed using \( ?\varphi, +\varphi, -\varphi \) and \( ;, \cup, * \) is probably still effectively axiomatizable. [Without the iteration *, this will even be automatic, because there is a standard translation then of the semantic truth conditions into a first-order language: compare van Benthem 1984.]

Other questions concerning this logic are of a more general model-theoretic nature. In particular, as before, one can also study special types of transformation here. For instance, tests \( ?\varphi \) always denote a (partial) identity function on states, which does not really 'move forward'. But, so do certain additive cases, such as \( +\varphi \),

where \( \diamond \) is the above general modality. If a program \( +\varphi \) always denotes a partial identity function, then this must mean (under a reasonable assumption of well-foundedness for \( \bowtie \)) that we never encounter \( s \bowtie s' \), \( \varphi \) true in \( s' \) but false in \( s \). In other words, \( \varphi \) is downward-persistent, in an obvious sense. But, such formulas \( \varphi \) will be equivalent to \( \diamond \varphi \). So, the above example was indeed characteristic. Thus, we get an interplay between semantical properties of transformations and well-known properties of their underlying propositions.

A further specialization would arise if we were to restrict the available propositions beforehand to cases that are upward- or downward persistent (or perhaps, merely convex).

Another important line of specialization arises, if we impose further restrictions on the structure of states \( (S, \bowtie) \). For instance, it might be reasonable (as earlier on) to regard this as a, possibly distributive, lattice - a lattice which might even be closed under arbitrary joins and meets. (The earlier case of deductively closed theories is one example.) In the latter case, it becomes possible to make propositions uniformly into functions, instead of relations. Namely, one stipulates that
\( +\varphi \) maps any state \( s \) to the meet of the set
\[ \{ s' \mid s \leq s' \} \cup \{ s' \mid \varphi \text{ is true at } s' \} \] (if that meet exists).
And \(-\varphi\) is defined similarly by a join of suitable \( \leq \)-predecessors.
Evidently, the logic of this scheme will be richer than before.
In dynamic logic terms, programs \(+\varphi\) and \(-\varphi\) have now become deterministic.

**Digression: A Connection with Minimality**

Studying the logic of our dynamic propositional operators involves the earlier issue of *minimal modelling* (see Part II). For, \(+\varphi\) may be regarded as a minimized case of a more liberal operator
\[ \text{PLUS } \varphi = \{ (s, s') \mid s \leq s' \text{ and } \varphi \text{ is true at } s' \} . \]
Such an operator would reduce to a general modality as follows:
\[ \langle \text{PLUS } \varphi \rangle \psi \leftrightarrow \Box (\varphi \land \psi) . \]
But, due to the minimality in the truth clause for \(+\varphi\), no similar reduction appears feasible for the latter notion. There is no natural general modal equivalent for the statement \(+\varphi\rangle \psi\). The only plausible reduction would be the special case
\[ \langle +\varphi \rangle T \leftrightarrow \Box \varphi ; \]
which will hold provided that we demand *well-foundedness* for the extension relation \( \leq \).

What further principles are reasonable for the modality \( \langle +\varphi \rangle \)?
Of course, there are the standard modal axioms, such as *Distribution*:
\[ \langle +\varphi \rangle (\alpha \lor \beta) \leftrightarrow (\langle +\varphi \rangle \alpha \lor \langle +\varphi \rangle \beta) . \]
For further axioms, again, we can turn toward earlier systems of minimal logic in Part II: notably, those presented in Sections 2.2.2 and 2.2.3. For instance, conditionals \( \varphi \Rightarrow \psi \) may now be compared with modal formulas
\[ \langle +\varphi \rangle \psi . \]
Of the principles of the minimal conditional logic, then, rightward Monotonicity and Conjunction follow directly from the Distribution axiom. The remaining ones, however, require various additions, such as
**Reflexivity:** \(+\varphi\varphi \).
**Disjunction of Antecedents:** 
\[ ([+\varphi] \lor [+\chi] \psi) \rightarrow [+\varphi \lor \chi] \psi . \]
Deriving the final principle \( \varphi \Rightarrow \psi, \varphi \Rightarrow \chi \rightarrow (\varphi \land \psi) \Rightarrow \chi \) will require a suitable strengthening of the above Wellfoundedness principle.
Thus, properly viewed, the present dynamic logic encompasses the minimality-based logics of our earlier investigation.
Finally, there are still aspects of dynamic interpretation which go beyond the present framework. For instance, commitment to an implication \( \varphi \rightarrow \psi \) in ordinary discourse is often explained as obeying a standing instruction (compare the rules of a Lorenzen dialogue game):

'as soon as one becomes committed, in further discourse or argument, to the antecedent \( \varphi \), the obligation to add \( \psi \) is incurred'.

In the above terms, this would rather say something about the internal structure of a transition process. A successful transition \((s,s')\) gives the two extremal points of a finite sequence of intermediate steps:

\[
s = s_1 \rightarrow s_2 \rightarrow \ldots \rightarrow s_n = s'
\]

which may obey, e.g., the constraint that, whenever \( s_i \) validated \( \varphi \), the next step \( s_i \rightarrow s_{i+1} \) was an addition of \( \psi \). [An alternative would be to program standing instructions explicitly, via Kleene iterations.] This move towards representing a richer internal structure of processes can also be observed in ordinary Dynamic Logic, namely, when one passes on to some kind of Process Algebra (cf. Bergstra and Klop 1984).

Thus, this final perspective too offers us a versatile model for studying the dynamics of information flow.
4. DISCUSSION

The preceding Sections contain enough material to warrant the conclusion that dynamic information structure eminently deserves further logical study. Indeed, the literature already shows some quite promising explorations. Nevertheless, there are also some difficulties to be observed.

One is a kind of general paradox of complexity. Although most systems in the area seem inspired by a desire for modelling simple efficient reasoning (of which our own 'domestic reasoning' is taken to be a paradigmatic example), the actual proposals produced seem to go in the wrong direction. This was striking with Circumscription, which produces a notion of consequence whose complexity is vastly higher than that of classical logic. (Compare the conjecture in van Benthem 1985c, that Tarski's classical notion of consequence is distinguished, among all its rivals, as being the only one to produce an effectively axiomatizable set of validities). But also, the dynamic systems of Section 3 produce logics whose inferential behaviour, upon closer inspection, is not at all as perspicuous as that of standard predicate logic.

Of course, there may be ways-out here. Perhaps, in actual practice, we shall always avoid the complex cases (compare the Lifschitz analysis of 'elementary' circumscription) - and there may be even a system to that practice which can be brought to light.

But, in some ways, these observations also raise the issue of whether we are going about things in the right fashion. For instance, Franz Guenthner has suggested that we should distinguish between (at least) two kinds of inference. One is 'on-line', close to the original linguistic structure of premises, and may be more 'dynamic'. The other operates on more abstract informational content, even after we have forgotten the specific original linguistic formulations. The latter may be closer to classical standard logic.

The latter point is reinforced by the difficulties encountered in analyzing operations on 'information states', where one finds that the properties of particular representations chosen tend to get in the way of logical understanding. Perhaps, what is needed is a clear separation of concerns. [The earlier division into propositional contents and modes of transforming information states was already one attempt in this direction.] In fact, many authors have warned that a genuine deep theory of any phenomenon should stay at a reasonable level of abstraction from its particular implementations. (Compare recursion-theoretic theories of
computation vis-à-vis actual algorithmics, or the plea for a procedurally neutral approach to semantics in Fenstad et al. 1987).

This issue of where to locate what also reminds us of another broad alternative to the semantic approach taken throughout in this paper. Many issues treated here really play at the syntactic level of text and discourse. For instance, the earlier example of 'retracting' a proposition is difficult to grasp semantically: how to recover a unique source for a given value of a transformation? But, this is a self-inflicted perplexity. At the text level, we know what was the previous proposition, and there is no difficulty at all in retracting it. And similar considerations apply to such 'modifying speech acts' as 'conditionalizing', which require undoing the effect of some previous transformation:

"P. If Q, that is ..."

Similarly, the earlier modes of taking propositions are a simple fact of life at this level. We know in a certain discourse whether we are going to take a certain answer exhaustively (see Section 2) or in the normal sense. For instance, when reading a certain Danish children's book to my sons, called "Auntie Thea", I arrived at a passage where questions are posed and answered. Here is one example:

"What was Auntie Thea wearing?"

"A red hat."

On the theory of Groenendijk and Stokhof 1985, who always predict the exhaustive reading, this would hardly be a children's book ... . But of course, my sons realized the 'language game' we were in, and asked what else she was wearing.

So, we may have to look for a theory of text structure and discourse, as indicated by control expressions, such as "so", "but", "suppose", "let", etcetera. And the proper model for that may be not so much Semantics as Proof Theory, with its account of the structure of proofs: themselves already a nice and rich example of textual phenomena. For instance, there is already a fair amount of anaphoric relations, and dynamic dependency structures in Natural Deduction arguments. (For more technical applications of proof theory, to circumscription, cf. Jaeger 1986.)

But even more generally, we also need a pragmatic theory at the level of logical language games, explaining the ease with which we adopt or switch certain modes of reasoning behaviour. (Compare the discussion at the end of Section 2.1.) Logic should not just be about the 'forms' which are the products of reasoning, but also about the 'rules' which guide that activity.

We still have a long way to go.
5. REFERENCES


14. J. van Benthem, 1987a, Categorial Grammar and Type Theory, report 87-07, Institute for Language, Logic and Information, University of Amsterdam. (To appear in Linguistics and Philosophy.)


    Information*, 
    and the Pragmatics of Answers*, 
    dissertation, Filosofisch Instituut, Universiteit van Amsterdam. 
    (To appear with Oxford University Press.)
    Institute for Language, Logic and Information, University of Amsterdam.
32. D. Harel, 1984, 'Dynamic Logic', 
    in D. Gabbay and F. Guenthner, eds., 
    *Handbook of Philosophical Logic, vol. II.*, 
    Reidel, Dordrecht.
    dissertation, Department of Linguistics, 
    Massachusetts Institute of Technology.
    The Free Press, Glencoe (Ill.).
36. G. Jaeger, 1986, Some Contributions to the Logical Analysis 
    of Circumscription, 
    report 86-03, Mathematik, Eidgenössische Technische Hochschule, Zürich.
37. R. Jennings, 1986, 'Logic as Punctuation', 
38. B. Jonsson, 1984, 'The Theory of Binary Relations', 
    Department of Mathematics, Vanderbilt University, Nashville (Tenn.).
    *American Journal of Mathematics* 73, 891-939.
40. B. Jonsson and A. Tarski, 1952, 'Boolean Algebras with Operators. II', 
    *American Journal of Mathematics* 74, 127-162.
44. V. Lifschitz, 1985a, 'Computing Circumscription', *Proceedings IJCAI-85:*1, 121-127.
57. R. Stalnaker, 1986, 'Worlds and Situations',
*Journal of Philosophical Logic* 15:1, 109-123.

dissertation, Filosofisch Instituut, Universiteit van Amsterdam.
(To appear with Cambridge University Press.)

59. F. Veltman, 1987, 'Update Semantics',
Institute for Language, Logic and Information, University of Amsterdam.