Cardinal spaces and topological representations of bimodal logics

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Abstract

We look at bimodal logics interpreted by cartesian products of topological spaces and discuss the validity of certain bimodal formulae in products of so-called cardinal spaces. This solves an open problem of van Benthem *et al.*

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Introduction

Topological interpretations of modal logics have been introduced by McKinsey and Tarski [2] long before the advent of Kripke semantics. The authors of [1] have introduced an interpretation of bimodal logics on cartesian products of topological spaces: you have a modal language with two modalities, $\Box_1$ and $\Box_2$, and interpret them as interior operators on horizontal and vertical sections of the cartesian product of two topological spaces. It is clear that both the $\Box_1$ fragment and the $\Box_2$ fragment satisfy the axioms of $\mathsf{S4}$.

In [1, § 2], there is a list of results on validity of mixed formulas, in particular the mixed formulas

- $\Diamond_1\Diamond_2\, p \to \Diamond_2\Diamond_1\, p$ (**left commutativity**, $\mathrm{com}_\leftarrow$),

- $\Diamond_2\Diamond_1 p \to \Diamond_1\Diamond_2 p$ (**right commutativity**, $\mathrm{com}_{\rightarrow}$), and
- $\Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$ (**Church-Rosser**, chr).

Theorem 1 (van Benthem, Bezhanishvili, ten Cate, Sarenac) *For first-countable spaces* $\mathbf{X}$ *and* $\mathbf{Y}$*, the following equivalences hold:*

- $\mathbf{X}$ *is Alexandroff* $\iff$ $\mathbf{X}, \mathbf{Y} \models \mathrm{com}_{\leftarrow}$*,*
- $\mathbf{Y}$ *is Alexandroff* $\iff$ $\mathbf{X}, \mathbf{Y} \models \mathrm{com}_{\rightarrow}$*, and*
- *at least one of* $\mathbf{X}$ *and* $\mathbf{Y}$ *is Alexandroff* $\iff$ $\mathbf{X}, \mathbf{Y} \models \mathrm{chr}$.

Moreover, in each of the equivalences, the forward direction ("$\Rightarrow$") holds for all topological spaces.[1]

Question 2 (van Benthem, Bezhanishvili, ten Cate, Sarenac) *Do the backwards directions ("$\Leftarrow$") of the equivalences in Theorem 1 hold for arbitrary topological spaces* $\mathbf{X}$ *and* $\mathbf{Y}$*?*

In this note, we answer Question 2 negatively.

Definitions

Let $\mathbf{X} = \langle X, \tau_X \rangle$ and $\mathbf{Y} = \langle Y, \tau_Y \rangle$ be topological spaces. If $A \subseteq X \times Y$, $x^* \in X$ and $y^* \in Y$, we can look at **vertical sections** $A^{x^*} := \{y \in Y\,;\, \langle x^*, y \rangle \in A\}$ and **horizontal sections** $A_{y^*} := \{x \in X\,;\, \langle x, y^* \rangle \in A\}$. Vertical and horizontal sections are subsets of Y and X, respectively, and hence we can look at their closures and interiors in the spaces $\mathbf{Y}$ and $\mathbf{X}$. We define the **horizontal** (**vertical**) **closure** (**interior**) of A as follows:[2]

$$
\begin{aligned}
\langle x, y \rangle \in \mathrm{hcl}(A) &: \iff x \in \mathrm{cl}_{\tau_X}(A_y), \\
\langle x, y \rangle \in \mathrm{hint}(A) &: \iff x \in \mathrm{int}_{\tau_X}(A_y), \\
\langle x, y \rangle \in \mathrm{vcl}(A) &: \iff y \in \mathrm{cl}_{\tau_Y}(A^x), \\
\langle x, y \rangle \in \mathrm{vint}(A) &: \iff y \in \mathrm{int}_{\tau_Y}(A^x).
\end{aligned}
$$

Now look at the modal language with two modalities $\Box_1$ and $\Box_2$. The cartesian product interpretation of $\Box_1$ and $\Box_2$ is given by the following recursion[3]: suppose we have already defined the meaning of $\mathbf{X}, \mathbf{Y}, x, y \models \varphi$ for all $x \in X$

[1] Corollary 6.11, Proposition 6.14, Proposition 6.1, and Proposition 6.9 of [1].
[2] Note that these operations are closure and interior in the topology $\mathsf{discrete}_X \otimes \tau_Y$ and $\tau_X \otimes \mathsf{discrete}_Y$, respectively.
[3] For details, see [1].

and $y \in Y$, then we let

$$\mathbf{X}, \mathbf{Y}, x, y \models \Box_1 \varphi \iff \langle x, y\rangle \in \mathrm{hint}(\{\langle v, w\rangle\,;\, \mathbf{X}, \mathbf{Y}, v, w \models \varphi\}), \text{ and}$$

$$\mathbf{X}, \mathbf{Y}, x, y \models \Box_2 \varphi \iff \langle x, y\rangle \in \mathrm{vint}(\{\langle v, w\rangle\,;\, \mathbf{X}, \mathbf{Y}, v, w \models \varphi\}).$$

The derived modalities $\Diamond_1 = \neg\Box_1\neg$ and $\Diamond_2 = \neg\Box_2\neg$ then correspond to horizontal and vertical closure. As usual, we write

$$\mathbf{X}, \mathbf{Y} \models \varphi$$

if the formula φ holds at all points. Consequently, for topological spaces $\mathbf{X}$ and $\mathbf{Y}$, the mentioned bimodal formulae transform into the following topological statements

$$\begin{aligned}
\mathrm{com}_{\rightarrow} &\rightsquigarrow \forall A \subseteq X \times Y\,(\,\mathrm{hcl}(\mathrm{vcl}(A)) \subseteq \mathrm{vcl}(\mathrm{hcl}(A))\,),\\
\mathrm{com}_{\leftarrow} &\rightsquigarrow \forall A \subseteq X \times Y\,(\,\mathrm{vcl}(\mathrm{hcl}(A)) \subseteq \mathrm{hcl}(\mathrm{vcl}(A))\,), \text{ and}\\
\mathrm{chr} &\rightsquigarrow \forall A \subseteq X \times Y\,(\,\mathrm{hcl}(\mathrm{vint}(A)) \subseteq \mathrm{vint}(\mathrm{hcl}(A)),
\end{aligned}$$

respectively. Note that chr is symmetric:

$$\begin{aligned}
\mathbf{X}, \mathbf{Y} \models \mathrm{chr} &\iff \mathbf{X}, \mathbf{Y} \models \Diamond_1\Box_2\, p \to \Box_2\Diamond_1\, p\\
&\iff \mathbf{Y}, \mathbf{X} \models \Diamond_2\Box_1\, p \to \Box_1\Diamond_2\, p\\
&\iff \mathbf{Y}, \mathbf{X} \models \neg\Box_2\Diamond_1\, \neg p \to \neg\Diamond_1\Box_2\, \neg p\\
&\iff \mathbf{Y}, \mathbf{X} \models \Diamond_1\Box_2\, \neg p \to \Box_2\Diamond_1\, \neg p\\
&\iff \mathbf{Y}, \mathbf{X} \models \mathrm{chr}.
\end{aligned}$$

We call a topological space **Alexandroff** if arbitrary intersections of open sets are open. Examples are the discrete or the indiscrete topologies.

As mentioned in the introduction, we are looking for non-Alexandroff spaces that validate the bimodal formulae $\mathrm{com}_{\rightarrow}$, $\mathrm{com}_{\leftarrow}$, and/or chr.

For this, we define the **cardinal space** of cardinality κ, denoted by $\mathbf{card}_\kappa$ as follows: The underlying set of the space is $\kappa \cup \{\infty\}$ where $\infty \notin \kappa$. The open neighbourhood base for each $\alpha \in \kappa$ is $\{\{\alpha\}\}$, the open neighbourhood base for ∞ is

$$\{\{\xi\,;\, \alpha < \xi < \kappa\} \cup \{\infty\}\,;\, \alpha \in \kappa\}.$$

The topology of $\mathbf{card}_\kappa$ is the discrete topology on κ and a point at infinity that is infinitely far away (can be reached only by sequences cofinal in κ). An alternative way of viewing these spaces is as the ordinal topology on the ordinal $\kappa + 1$ with all limit points below κ removed.

Note that for infinite cardinals κ, the cardinal space $\mathbf{card}_\kappa$ is not Alexandroff, and for uncountable cardinals κ, it is not first-countable.

If $\mu \leq \nu$ are ordinals, and $\gamma \in \nu$, we can form the Cantor Normal Form of γ to the base μ:

$$\gamma = \mu^{\alpha_n} \cdot \gamma_n + \mu^{\alpha_{n-1}} \cdot \gamma_{n-1} + \ldots + \mu^{\alpha_1} \cdot \gamma_1 + \gamma_0.$$

We write $\mathrm{S}_\mu(\gamma) := \gamma_0$ and call it the **scalar term** of γ to the base μ.

Lemma 3 *If $\mu \leq \nu$ are cardinals, $\gamma < \mu$ and $\beta < \nu$, then there is some $\beta < \eta < \nu$ such that $\mathrm{S}_\mu(\eta) \geq \gamma$.*

Proof. Let $\xi := \mathrm{S}_\mu(\beta)$. If $\xi \geq \gamma$, then $\eta := \beta + 1$ does the job. Otherwise, there is a unique $0 < \sigma < \mu$ such that $\gamma = \xi + \sigma$. Let $\eta := \beta + \sigma$. q.e.d.

Theorem 4 *Let κ and λ be cardinals. Then the following are equivalent:*

(i) $\mathbf{card}_\kappa, \mathbf{card}_\lambda \models \Diamond_1 \Diamond_2 p \to \Diamond_2 \Diamond_1 p$, *and*
(ii) $\lambda < \mathrm{cf}\, \kappa$.

Proof. "*(i)⇒(ii)*". Suppose $\mathrm{cf}\, \kappa \leq \lambda$. We'll construct a subset of $\mathbf{card}_\kappa \times \mathbf{card}_\lambda$ that constitutes a counterexample to $\mathrm{com}_\to$. Let $A = \{\alpha_\gamma\, ;\, \gamma < \mathrm{cf}\, \kappa\} \subseteq \kappa$ be an increasing enumeration of a cofinal subset of κ. We define a subset of $\kappa \times \lambda$ as follows:

$$\langle \alpha_\gamma, \beta \rangle \in X :\Longleftrightarrow \gamma \leq \mathrm{S}_{\mathrm{cf}\, \kappa}(\beta).$$

Note that $\mathrm{S}_{\mathrm{cf}\, \kappa}(\beta) < \mathrm{cf}\, \kappa$, so if you fix an element $\beta \in \lambda$ and look at the horizontal section $X_\beta = \{\alpha\, ;\, \langle \alpha, \beta \rangle \in X\}$, then each of these sets has cardinality less than $\mathrm{cf}\, \kappa$. In particular, none of these can be cofinal in κ $(\star)$.

Moreover, if you fix $\alpha_\gamma \in A$ and look at the vertical section

$$X^{\alpha_\gamma} = \{\beta\, ;\, \langle \alpha_\gamma, \beta \rangle \in X\},$$

then this set is cofinal in λ $(\star\star)$ by the following argument: Take an arbitrary $\beta < \lambda$. By Lemma 3 applied to $\mathrm{cf}\, \kappa \leq \lambda$, we find $\beta < \eta < \lambda$ such that $\mathrm{S}_{\mathrm{cf}\, \kappa}(\eta) \geq \gamma$. But that means that $\langle \alpha_\gamma, \eta \rangle \in X$, so $\beta < \eta \in X^{\alpha_\gamma}$.

By $(\star)$, the horizontal closure of X is X itself: none of the elements of the form $\langle \infty, \beta \rangle$ are reached by horizontal sections of X. By $(\star\star)$, the vertical closure of X is $X \cup A \times \{\infty\}$. Of course, since A is cofinal in κ, the horizontal closure of $A \times \{\infty\}$ includes the point $\langle \infty, \infty \rangle$.

But then

$$\mathrm{vcl}(\mathrm{hcl}(X)) = X \cup A \times \{\infty\}, \text{ yet}$$
$$\mathrm{hcl}(\mathrm{vcl}(X)) = X \cup A \times \{\infty\} \cup \{\langle \infty, \infty \rangle\}.$$

But this means that

$$\mathbf{card}_\kappa, \mathbf{card}_\lambda \not\models \Diamond_1\Diamond_2\, p \to \Diamond_2\Diamond_1\, p.$$

"*(ii)⇒(i)*". Assume that $\lambda < \operatorname{cf}\kappa$. We have to show that $\mathbf{card}_\kappa, \mathbf{card}_\lambda \models \Diamond_1\Diamond_2\, p \to \Diamond_2\Diamond_1\, p$, so we have to show for every subset X of the product that $\operatorname{hcl}(\operatorname{vcl}(X)) \subseteq \operatorname{vcl}(\operatorname{hcl}(X))$. Note that the only point for which the order of horizontal and vertical closures matters is the point $\langle\infty,\infty\rangle$, so the we are done if we can show that

$$\langle\infty,\infty\rangle \in \operatorname{hcl}(\operatorname{vcl}(X)) \text{ implies } \langle\infty,\infty\rangle \in \operatorname{vcl}(\operatorname{hcl}(X)).$$

Without loss of generality, $X \subseteq \kappa \times \lambda$.

If $\langle\infty,\infty\rangle \in \operatorname{hcl}(\operatorname{vcl}(X))$, there is a cofinal set $C \subseteq \kappa$ of cardinality $\operatorname{cf}\kappa$ such that for all $\gamma \in C$, we have

$$\langle\gamma,\infty\rangle \in \operatorname{vcl}(X).$$

This in turn means that for each such γ, the vertical section $X^\gamma = \{\beta\,;\, \langle\gamma,\beta\rangle \in X\}$ must be cofinal in λ. In other words, if you fix $\eta \in \lambda$, then

$$X^*_{>\eta} := \{\langle\alpha,\beta\rangle \in X\,;\, \beta > \eta\} \cap (\kappa \times C)$$

must have cardinality at least $\operatorname{cf}\kappa$.

For each $\beta \in \lambda$, let

$$P_\beta := \{\langle\alpha,\beta\rangle \in X\,;\, \alpha \in C\}.$$

The family $\{P_\beta\,;\, \eta < \beta < \lambda\}$ is a partition of $X^*_{>\eta}$ into at most λ many pieces. Consequently, by the pigeon hole principle, there must be a $\beta^* > \eta$ such that P_{β^*} has $\operatorname{cf}\kappa$ many elements. But since $P_{\beta^*} \subseteq X_{\beta^*}$ and C was cofinal in κ, this means that $\langle\infty,\beta^*\rangle \in \operatorname{hcl}(X)$.

Since η was arbitrary, we just showed that the set of such β^* is cofinal in λ, and thus $\langle\infty,\infty\rangle \in \operatorname{vcl}(\operatorname{hcl}(X))$. This was the claim. q.e.d.

Corollary 5 *For $\aleph_0 \le \lambda < \operatorname{cf}\kappa$, $\mathbf{card}_\kappa$ and $\mathbf{card}_\lambda$ are non-Alexandroff spaces such that $\mathrm{com}_\leftarrow$ holds in $\mathbf{card}_\kappa \times \mathbf{card}_\lambda$. In particular, this is true in $\mathbf{card}_{\aleph_1} \times \mathbf{card}_{\aleph_0}$. Also, $\mathrm{com}_\rightarrow$ holds in $\mathbf{card}_{\aleph_0} \times \mathbf{card}_{\aleph_1}$.*

Theorem 6 *Let κ and λ be cardinals. Then the following are equivalent:*

(i) $\mathbf{card}_\kappa, \mathbf{card}_\lambda \models \Diamond_1\Box_2\, p \to \Box_2\Diamond_1\, p$, *and*
(ii) $\operatorname{cf}\lambda \ne \operatorname{cf}\kappa$.

Proof. "*(i)⇒(ii)*". Suppose that $\vartheta := \mathrm{cf}\,\kappa = \mathrm{cf}\,\lambda$. Let $A = \{\alpha_\gamma\,;\,\gamma < \vartheta\} \subseteq \kappa$ and $B = \{\beta_\gamma\,;\,\gamma < \vartheta\} \subseteq \lambda$ be increasing enumerations of cofinal subsets. Define

$$X := \{\langle\alpha_\gamma, \beta\rangle\,;\,\beta \geq \beta_\gamma, \gamma < \vartheta\} \cup \{\langle\alpha_\gamma, \infty\rangle\,;\,\gamma < \vartheta\}.$$

Then for each $\gamma < \vartheta$, $\{\infty\} \cup \{\beta\,;\,\beta_\gamma \leq \beta < \lambda\} \subseteq X^{\alpha_\gamma}$ which is an open neighbourhood of ∞ in $\mathbf{card}_\lambda$. Consequently, $\langle\alpha_\gamma, \infty\rangle \in \mathrm{vint}(X)$. Since A was cofinal in κ, this means that $\langle\infty, \infty\rangle \in \mathrm{hcl}(\mathrm{vint}(X))$.

Yet, for each $\beta \in \lambda$ there is an upper bound for X_β: if $\beta_\gamma \leq \beta < \beta_{\gamma+1}$, then

$$X_\beta \subseteq \{\alpha \in \kappa\,;\,0 \leq \alpha < \alpha_{\gamma+1}\}.$$

That means that $\mathrm{hcl}(X)$ doesn't contain any element of the form $\langle\infty, \beta\rangle$, and so $\langle\infty, \infty\rangle \notin \mathrm{vint}(\mathrm{hcl}(X))$.

"*(ii)⇒(i)*". The symmetry of chr makes sure that we only have to check the case $\mathrm{cf}\,\kappa < \mathrm{cf}\,\lambda$.

To start, let us notice that for subsets X of $\mathbf{card}_\kappa \times \mathbf{card}_\lambda$, we always have that

$$\mathrm{hcl}(\mathrm{vint}(X))\backslash\{\langle\infty, \infty\rangle\} \subseteq \mathrm{vint}(\mathrm{hcl}(X)),$$

since the elements of $\kappa \times \lambda$ are not affected by any of the interior and closure operations. Thus, we only have to show

$$\langle\infty, \infty\rangle \in \mathrm{hcl}(\mathrm{vint}(X)) \text{ implies } \langle\infty, \infty\rangle \in \mathrm{vint}(\mathrm{hcl}(X)).$$

Fix X such that $\langle\infty, \infty\rangle \in \mathrm{hcl}(\mathrm{vint}(X))$. This means that there is some cofinal set $A = \{\alpha_\gamma\,;\,\gamma < \mathrm{cf}\,\kappa\} \subseteq \kappa$ such that $A \times \{\infty\} \subseteq \mathrm{vint}(X)$, so for each γ, there is some $\beta_\gamma < \lambda$ such that

$$\{\infty\} \cup \{\beta\,;\,\beta_\gamma \leq \beta < \lambda\} \subseteq X^{\alpha_\gamma}.$$

The set $\{\beta_\gamma\,;\,\gamma < \mathrm{cf}\,\kappa\}$ has cardinality $\mathrm{cf}\,\kappa < \mathrm{cf}\,\lambda$, so $\beta^* := \sup\{\beta_\gamma\,;\,\gamma < \mathrm{cf}\,\kappa\} < \lambda$. But then for every $\beta > \beta^*$, we have that $A \subseteq X_\beta$, and so $\langle\infty, \beta\rangle \in \mathrm{hcl}(X)$. This means that $\{\infty\} \cup \{\beta\,;\,\beta^* < \beta\}$ is an $\mathbf{card}_\lambda$-open neighbourhood contained in $(\mathrm{hcl}(X))^\infty$, so $\langle\infty, \infty\rangle \in \mathrm{vint}(\mathrm{hcl}(X))$. q.e.d.

Corollary 7 *For $\aleph_0 \leq \mathrm{cf}\,\kappa < \mathrm{cf}\,\lambda$, $\mathbf{card}_\kappa$ and $\mathbf{card}_\lambda$ are non-Alexandroff spaces such that* chr *holds in $\mathbf{card}_\kappa \times \mathbf{card}_\lambda$. In particular, this is true in $\mathbf{card}_{\aleph_0} \times \mathbf{card}_{\aleph_1}$.*

Corollaries 5 and 7 together answer Question 2 negatively:

$$\mathbf{card}_{\aleph_0}, \mathbf{card}_{\aleph_1} \models \mathrm{chr}\,\&\,\mathrm{com}_\rightarrow\,\&\,\neg\mathrm{com}_\leftarrow, \text{ and}$$

$$\mathbf{card}_{\aleph_1}, \mathbf{card}_{\aleph_0} \models \mathrm{chr}\,\&\,\mathrm{com}_\leftarrow\,\&\,\neg\mathrm{com}_\rightarrow.$$

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