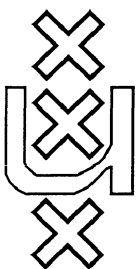


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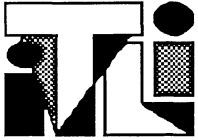
**ROSSER ORDERINGS
AND
FREE VARIABLES**

Dick de Jongh
Franco Montagna

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Instituut voor Taal, Logica en Informatie
Institute for Language, Logic and Information

Faculteit der Wiskunde en Informatica
(Department of Mathematics and Computer Science)
Roetersstraat 15
1018WB Amsterdam

Faculteit der Wijsbegeerte
(Department of Philosophy)
Nieuwe Doelenstraat 15
1012CP Amsterdam

ROSSER ORDERINGS

AND

FREE VARIABLES

Dick de Jongh
Department of Mathematics and Computer Science, University of Amsterdam

Franco Montagna
Dipartimento di Matematica, Università di Siena

1. Introduction. In Guaspari-Solovay (1979) the arithmetical completeness theorem of Solovay (1976) for the modal system L (PRL in Smoryński 1985) was extended to arithmetical completeness for a system R which includes witness comparison symbols \prec and \preceq and which thereby to a certain extent takes Rosser sentences into account. The system R is obtained by first adding Σ -completeness as well as ordering axioms for \prec and \preceq to L to obtain a system R^- , and subsequently adjoining the rule $\Box E$ (i.e. $\Box A/A$) to R^- .

Syntactically as well as semantically the system R compares unfavorably to the system R^- . The rule $\Box E$ makes that the system lacks the subformula property (possibly the best way to think about it is that $\vdash_R A$ iff $\vdash_{R^-} \Box^n A$ for some n , cf. de Jongh 1987), and modal completeness of R can only be proved with respect to a rather awkward subset of the natural class of Kripke-models for R^- : $\vdash_R A$ iff A is valid on all A-sound Kripke-models for R^- (an A-sound Kripke-model is one the root of which forces $\Box B \rightarrow B$ for each subformula $\Box B$ of A).

In the first part of this paper we will show that, if one extends arithmetical interpretations by allowing free variables, it is R^- that gets to be the system that is arithmetically complete. It is a well-known fact that for such interpretations the system R is not sound; the rule $\Box E$ is not valid if the restriction to closed formulae is dropped. A counterexample is, for example, the fact that $\vdash_{PA} \Box \neg \text{Prf}(x, \ulcorner \perp \urcorner)$, whereas not $\vdash_{PA} \neg \text{Prf}(x, \ulcorner \perp \urcorner)$ (we write $\Box A$ for $\exists y \text{Prf}(y, \ulcorner A \urcorner)$).

At first sight one might think it impossible for R^- to be arithmetically complete, since $\vdash_{PA} \Box T \prec \Box \perp$ and $\vdash_R \Box T \prec \Box \perp$, but $\not\vdash_{R^-} \Box T \prec \Box \perp$ (Guaspari-Solovay 1979). But that difficulty can be overcome by taking a formulation of R^- without such closed formulae, or alternatively, as will be our choice, by interpreting \perp in PA as $1=0 \wedge x=x$ and T as $0=0 \wedge x=x$. To get the general idea, note, for example that, where $\not\vdash_{R^-} \Box(p \vee \neg p) \prec \Box \perp$, indeed, for each formula $A(x)$ of PA which actually contains x as a free variable, $\not\vdash_{PA} \Box(A(x) \vee \neg A(x)) \prec \Box \perp$ (we use the convention of automatically interpreting variables like x as dotted: \dot{x} , unless the opposite is

stated explicitly). The latter is the case, since PA can recognize that proofs of $A(n) \vee \neg A(n)$ do get arbitrary large simply by the size of n , so that $\Box(A(n) \vee \neg A(n)) \prec \Box \perp$ for each n would in PA imply that $\neg \Box \perp$, which is not provable in PA. In Section 2 this arithmetical completeness result will be proved. The consequent non-validity of some rules for PA will be discussed in Section 3. In Section 4 it is shown that some principles concerning witness comparisons and free variables cannot be decided, even when one restricts oneself to "usual" proof predicates.

2. Arithmetical completeness of R^- .

2.1 Definition. An *open arithmetical interpretation* of the language of R is a mapping $*$ from R -sentences into PA-formulae such that:

$\top^* \equiv 0 = 0 \wedge x = x$, $\perp^* \equiv 1 = 0 \wedge x = x$, $*$ commutes with \wedge , \prec , \preceq , and $(\Box A)^* \equiv \Box^f A^*$, where $\Box^f B \equiv \exists v (\ulcorner B \urcorner \in f(v))$ is such that $\vdash_{PA} \Box^f B \leftrightarrow \Box B$ for each B .

2.2 Theorem. $\vdash_R A$ iff, for each open interpretation $*$, $\vdash_{PA} A^*$.

Proof. \Rightarrow : Standard.

\Leftarrow : We will mainly follow the Guaspari-Solovay argument, but there are some differences.

Let $\not\vdash_R A$ and let $M \equiv \langle \{1, 2, \dots, n\}, R, \Vdash \rangle$ be a Kripke-model of R^- for a finite adequate set Γ containing A , such that $M \not\vdash A$. Add 0 as a new root to M , but in this case without defining any forcing on it. (Trying to define forcing with respect to the node 0 would lead to insuperable problems with Σ -persistency; indeed, we will not need forcing on it, but we shall need the node in the definition of the Solovay-function h .)

Define Solovay's function h as usual (writing $\ell = i$ for $\exists u \forall v > u. h(v) = i$):

$$h(0) = 0$$

$$h(n+1) = \begin{cases} i & \text{if } h(n) R i \text{ and } \text{Prf}(n, \ulcorner \ell \neq i \urcorner) \\ h(n) & \text{else} \end{cases}$$

Also define, simultaneously, f and an interpretation $*$ (using the recursion theorem):

- (1) $p_j^* \equiv \bigvee_{i \Vdash p} (1=i) \wedge j=j \wedge x=x$,
- (2) $\top^* \equiv 0=0 \wedge x=x$, $\perp^* \equiv 1=0 \wedge x=x$,
- (3) $*$ commutes with $\vee, \wedge, \neg, \rightarrow, \prec, \preceq$,
- (4) $(\Box B)^* \equiv \Box^f B^*$,

where f is defined in stages as explained below. The idea is that as long as the function h equals 0 everything proceeds normally. During that period certain formulae which are translations of \Box -formulae in the adequate set Γ , but with numerals substituted for x , are being proved. If h moves, then at that point there is a largest n for which such a formula with the numeral n has been proved. Afterwards we continue treating the relevant formulae with numerals $m \leq n$ according to the occurrence of their usual proofs, but we start handling the formulae with numerals $m > n$ more or less as in Guaspari-Solovay. The idea is to show

$\not\vdash_{PA} \forall x (x > t \rightarrow A^*(x))$ with t describing n , and hence $\not\vdash_{PA} \forall x A^*(x)$.

Let us set $p = \mu z. h(z) \neq 0$ and n equal to the maximal number such that for some subformula $\Box B$ of A and some $p' \leq p$, $\text{Prf}(p', \ulcorner B^*(n) \urcorner)$.

STAGE m for $m < p$: If, for no C , $\text{Prf}(m, \ulcorner C \urcorner)$, let $f(m) = \emptyset$; if $\text{Prf}(m, \ulcorner C \urcorner)$, let $f(m) = \{C\}$.

STAGE $p + 2m$:

Let k_{2m} be the smallest number for which f has not been defined (so, $k_0 = p$). If $\text{Prf}(p+m, \ulcorner C \urcorner)$ for no C let $k_{2m+1} = k_{2m}$ and go to the next stage. Else, if $\text{Prf}(p+m, \ulcorner C \urcorner)$, let $f(k_{2m})$ be $\{C\}$, let $k_{2m+1} = k_{2m} + 1$ and go to the next stage, unless the following happens: $C \equiv B^*(n')$ for some subformula $\Box B$ of A and some $n' > n$. For B with respect to such a "large" n' do nothing and go to the next stage.

STAGE $p + 2m + 1$: Write k for $n + m$. Define $Y = \{B \mid h(m) \Vdash \Box B\}$. Do nothing, unless $Y \neq \emptyset$. If the latter is the case, let E_0, \dots, E_q be the equivalence classes in Y with respect to the relation $\equiv_{h(m)}$ (i.e. $h(m) \Vdash \Box B \preceq \Box C \wedge \Box C \preceq \Box B$) of \Box -subformulae of A forced at $h(m)$, enumerated according to their ordering by \prec . (Comment: that the \Box -formulae forced already "before" $h(m)$ are in Y contrary to the

way Guaspari and Solovay do it, makes no difference, since, by Σ -persistency, the right order is preserved.) If $E_i = \{\Box B_1, \dots, \Box B_r\}$, then writing $E_i^*(m)$ for

$\{B_1^*(S_n), \dots, B_1^*(k), B_2^*(S_n), \dots, B_2^*(k), \dots, B_r^*(S_n), \dots, B_r^*(k)\}$,

we define

$f(k_{2m}) = E_0^*(m), \dots, f(k_{2m+q}) = E_q^*(m)$

and set

$k_{2m+2} = k_{2m+1} + q + 2$.

It is immediately clear that

(a) $\vdash_{PA} \ell = 0 \rightarrow \forall v (\Box(v) \leftrightarrow \Box^f(v))$,

(b) $\vdash_{PA} \ell \neq 0 \wedge y \leq t \rightarrow (\Box(B^*(y)) \leftrightarrow \Box^f(B^*(y)))$ (with t a term describing n as above).

To obtain the theorem it is as in Guaspari-Solovay sufficient to prove the following claims:

CLAIM 1: For each subformula B of A and $i \neq 0$:

$\vdash_{PA} \ell = i \wedge i \Vdash B \rightarrow \forall u > t. B^*(u)$

$\vdash_{PA} \ell = i \wedge i \nVdash B \rightarrow \forall u > t. \neg B^*(u)$

CLAIM 2: For each subformula $\Box B$ of A and $i \neq 0$:

$\vdash_{PA} \ell = i \wedge i \Vdash \Box B \rightarrow \forall u > t. \Box B^*(u)$

$\vdash_{PA} \ell = i \wedge i \nVdash \Box B \rightarrow \forall u > t. \neg \Box B^*(u)$

CLAIM 3: $\vdash_{PA} \forall v (\Box(v) \leftrightarrow \Box^f(v))$

CLAIM 4: $PA + \ell = i$ is consistent.

The last claim can be proven as in Solovay (1976). We will prove claims 1–3.

Proof of CLAIM 1: By induction on the complexity of B . Reason in $PA + \ell = i$. Only the \Box -, \prec - and \preceq -case are of interest. We treat \Box and \preceq (\prec is similar to \preceq):

If $i \Vdash \Box C$, then according to the definition of the odd stages f outputs $C^*(S_n), C^*(SS_n), C^*(SSS_n), \dots$. If $i \nVdash \Box C$, then $C^*(j)$ for $j > n$ is not output by f (not at the even stages by the manner of the construction and not at the odd ones, because, for no m , $h(m) \Vdash \Box C$, since $h(m) = i$ from a certain point onwards).

If $i \Vdash \Box C \preceq \Box D$, then, for $j > n$, $C^*(j)$ is output at some odd stage, and, if $D^*(j)$ is output by f , then this occurs at an odd stage as well

and the ordering agrees with the forcing in i . If $i \not\Vdash \Box C \leq \Box D$, then, if $i \not\Vdash \Box C$, then $C^*(j)$ is never output by f , and if $i \Vdash \Box C$, then also $i \Vdash \Box D$ and $C^*(j)$ is output after $D^*(j)$.

Proof of CLAIM 2: If $i \Vdash \Box B$, then $\forall j (iRj \Rightarrow j \Vdash B)$, so, by claim 1, $\forall j (iRj \Rightarrow \vdash_{PA} \ell = j \rightarrow \forall u > t. B^*(u))$. Hence

$$\vdash_{PA} \bigvee_{iRj} \ell = j \rightarrow \forall u > t. B^*(u), \quad \vdash_{PA} \Box \bigvee_{iRj} \ell = j \rightarrow \Box \forall u > t. B^*(u) \text{ and}$$

$$\vdash_{PA} \Box \bigvee_{iRj} \ell = j \rightarrow \forall u > t. \Box B^*(u). \text{ By Solovay (1976),}$$

$$\vdash_{PA} \ell = i \rightarrow \Box \bigvee_{iRj} \ell = j \wedge \bigwedge_{iRj} \neg \Box \ell \neq j, \text{ so}$$

$$\vdash_{PA} \ell = i \rightarrow \forall u > t. \Box B^*(u).$$

If $i \not\Vdash \Box B$, there is a j with iRj such that $j \not\Vdash B$. Hence:

$$PA \vdash \ell = j \rightarrow \forall u > t. \neg B^*(u),$$

$$PA \vdash \forall u > t (B^*(u) \rightarrow \ell \neq j),$$

$$PA \vdash \forall u > t (\Box B^*(u) \rightarrow \Box \ell \neq j); \text{ but we have } PA \vdash \ell = i \rightarrow \neg \Box \ell \neq j;$$

$$\text{hence } PA \vdash \ell = i \rightarrow \forall u > t \neg \Box B^*(u).$$

Proof of CLAIM 3: Argue in PA : if v does not have the form $B^*(q)$ for some subformula $\Box B$ of A , the claim is obvious. If $\ell = 0$ the claim follows from (a). If $\ell \neq 0$ and $v \equiv B^*(q)$ for some subformula $\Box B$ of A with $q \leq t$, the claim follows from (b). So, assume $\ell = i \neq 0$, $v \equiv B^*(q)$ for some subformula $\Box B$ of A , $q > t$. We have:

$$i \Vdash \Box B \leftrightarrow \Box B^*(q) \text{ by claim 2, and } i \Vdash \Box B \leftrightarrow \Box^f B^*(q) \text{ follows from claim 1 applied to } \Box B. \quad \square$$

3. The non-validity of some rules.

An application of the arithmetic completeness theorem proved in Section 2 is the following Corollary.

3.1 Corollary. For each $n \geq 0$ there is a formula A_n such that $\vdash_{PA} \Box^{n+1} A_n$, but $\not\vdash_{PA} \Box^n A_n$.

Proof. As shown in de Jongh (1987), $\vdash_{R^-} \Box^{n+1} (\Box T < \Box^{n+1} \perp)$, but $\not\vdash_{R^-} \Box^n (\Box T < \Box^{n+1} \perp)$. By arithmetic completeness this implies that, for some standard proof predicate \Box^* ,

$$\vdash_{PA} \Box^{*n+1} (\Box^*(T \wedge x=x) < \Box^{*n+1} (\perp \wedge x=x)), \text{ but}$$

$\not\vdash_{PA} \Box^{*n}(\Box^*(T \wedge x=x) \prec \Box^{*n}(\perp \wedge x=x))$. Since \Box^* is standard, it immediately follows that $\vdash_{PA} \Box^{n+1}(\Box^*(T \wedge x=x) \prec \Box^{*n+1}(\perp \wedge x=x))$ and $\not\vdash_{PA} \Box^n(\Box^*(T \wedge x=x) \prec \Box^{*n+1}(\perp \wedge x=x))$. \square

Albert Visser suggested in reaction to this proof that a direct example is supplied by: (1) $\vdash_{PA} \Box^{n+1}(\ulcorner \neg \text{Prf}(x, \ulcorner \Box^n \perp \urcorner) \urcorner)$, but (2) $\not\vdash_{PA} \Box^n(\ulcorner \neg \text{Prf}(x, \ulcorner \Box^n \perp \urcorner) \urcorner)$. To see that this does indeed check out, reason in PA:

(1) By cases: if $\neg \text{Prf}(x, \ulcorner \Box^n \perp \urcorner)$, then $\Box^{n+1}(\ulcorner \neg \text{Prf}(x, \ulcorner \Box^n \perp \urcorner) \urcorner)$ by Δ_0 -completeness. If $\text{Prf}(x, \ulcorner \Box^n \perp \urcorner)$, then $\Box^{n+1} \perp$, so again $\Box^{n+1}(\ulcorner \neg \text{Prf}(x, \ulcorner \Box^n \perp \urcorner) \urcorner)$.

(2) Assume to the contrary that $\vdash_{PA} \Box^n(\ulcorner \neg \text{Prf}(x, \ulcorner \Box^n \perp \urcorner) \urcorner)$. Assume moreover, $\Box^{n+1} \perp$. Then, for some y , $\text{Prf}(y, \ulcorner \Box^n \perp \urcorner)$, hence $\Box^n(\ulcorner \text{Prf}(y, \ulcorner \Box^n \perp \urcorner) \urcorner)$ which together with the initial assumption immediately gives $\Box^n \perp$. We have proved $\Box^{n+1} \perp \rightarrow \Box^n \perp$, which by Löb's theorem is an impossibility.

One might hope to generalize the non-validity of the rule $\Box^{n+1}A/\Box^nA$ to more examples. What comes to mind are those formulae $\Box A(p) \prec \Box B(p)$, with $A(p)$ and $B(p)$ L-formulae which have only provable fixed points in R (see de Jongh-Montagna 1988), since $\Box^{n+1}p \prec \Box^n p$ is a particular example of such a formula. This idea indeed does turn out to lead to a generalization.

3.2. Theorem. If $A(p)$ and $B(p)$ are L-formulae with only p free and $\Box A(p) \prec \Box B(p)$ has only provable fixed points in R, then the rule $\Box A(p)/\Box B(p)$ is not valid under arithmetic interpretations.

Proof. Suppose $\Box A(p) \prec \Box B(p)$ has only provable fixed points in R. Then, by theorem 3.4 of de Jongh-Montagna (1988), $\vdash_R A(T)$, $\vdash_R \Box A(\perp) \leftrightarrow \Box^m \perp$, $\vdash_R \Box B(\perp) \leftrightarrow \Box^n \perp$, $m > n$. It is sufficient to show that $\vdash_R \Box A(\Box T \prec \Box^{n+1} \perp)$, $\not\vdash_R \Box B(\Box T \prec \Box^{n+1} \perp)$. On any L-model with a root of depth $= n+1$ (end points having depth 1), $\Box T \prec \Box^{n+1} \perp$ may be made false everywhere. This means then that $\Box B(\Box T \prec \Box^{n+1} \perp)$ is everywhere on that model equivalent to $\Box B(\perp)$ and hence to $\Box^n \perp$ and hence false in the root. So,

$\not\vdash_{R^-} \Box B(\Box T \prec \Box^{n+1} \perp)$ has been shown. Next, take an arbitrary Kripke-model M for R^- . Now, either the depth of the root $\geq n+2$, in which case $\Box T \prec \Box^{n+1} \perp$ is verified everywhere in the model, so $\Box A(\Box T \prec \Box^{n+1} \perp)$ is equivalent to $\Box A(T)$ everywhere in the model, and hence, since $\vdash_{R^-} A(T)$, is validated by the model; or else the root has depth $\leq n+1$ in which case $\Box T \prec \Box^{n+1} \perp$ is, either verified everywhere in the model and the same reasoning obtains again, or falsified everywhere in the model in which case $\Box A(\Box T \prec \Box^{n+1} \perp)$ is everywhere in the model equivalent to $\Box A(\perp)$ and hence to $\Box^m \perp$ and therefore, since $m \geq n+1$, is validated by the model. \square

Note that Theorem 3.2 is not a full generalization of Corollary 3.1, since the case $n=0$, $\Box A/A$, is not subsumed under it. A way out may be here to consider formulae like $\Box p \prec p$ having only provable fixed points (consider a fixed point $Cx \leftrightarrow \exists y < x (\text{Prf}(y, \ulcorner \exists x Cx \urcorner))$ and look at $\exists x Cx$), but we have not pursued the subject.

4. Usual proof predicates, Rosser formulae.

A question that comes to mind in connection with this is, whether, by restricting oneself to more "usual" proof predicates, there are not many more principles than embodied in R^- that can be decided. This may well be the case for some principles, but we will show now that it is not so for one of the obvious candidate principles: $\Box(T \wedge x=x) \prec \Box(\perp \wedge x=x)$. Of course, the whole concept of "usual" proof predicate as introduced by Guaspari-Solovay is somewhat vague. We will accept any proof predicate as *usual* if it is based on a proof system which might be not unreasonably used in practice (logical, not mathematical) to generate the theorems of PA.

4.1 Proposition. (a) There are usual proof predicates such that

$$\not\vdash_{PA} \Box(T \wedge x=x) \prec \Box(\perp \wedge x=x).$$

(b) There are usual proof predicates such that

$$\vdash_{PA} \Box(T \wedge x=x) \prec \Box(\perp \wedge x=x).$$

There are some requirements on the coding too. We will, for example, have to take a coding of formulae as sequences of symbols and proofs as sequences (or trees) of formulae and basing the sequence coding on a monotonic pairing function j , i.e.

$$a \leq j(a, b), b \leq j(a, b), a \leq a' \wedge b \leq b' \Rightarrow j(a, b) \leq j(a', b').$$

The point is that we have to be sure that, if n is considerably larger than the gödel number $\ulcorner \Pi \urcorner$ of the given proof Π of \perp , then all proofs of $T \wedge n = n$ will have to have a proof with a higher gödel number than that of $\langle \Pi, \perp \wedge n = n \rangle$. This will certainly be the case if $\langle \Pi, \perp \wedge n = n \rangle$ is coded by $j(m, j(\ulcorner \Pi \urcorner, j(\ulcorner \wedge \urcorner, j(\ulcorner \perp \urcorner, \ulcorner n = n \urcorner))))$, where m codes the application of the *ex falso* rule, and if the following conditions are fulfilled: (1) m is smaller than the gödel numbers of the other rules, (2) \perp has a smaller gödel number than T .

It is obvious that this whole proof is a purely combinatorial matter and can easily be executed in PA.

(b) With the above in mind the reader will see that, if we take an axiomatization of PA with, say, all true Δ_0 -sentences included, which makes $T \wedge n = n$ an axiom for each n , give T a smaller gödel number than \perp , and have the same coding machinery as above, then PA will be able to prove that $T \wedge n = n$ has shorter proofs than any $\perp \wedge n = n$ might have. \square

A similar reasoning as in the above proof may be put to use in the consideration of the following problem. Consider a formula Rx to be a Rosser formula, if

$$\vdash_{PA} Rx \leftrightarrow \Box \neg Rx \prec \Box Rx$$

The question is, whether there exist usual proof predicates for which PA cannot prove all Rosser formulae to be equivalent. We will answer this question in the affirmative. One may consider this to be a very weak positive answer to the difficult problem Guaspari and Solovay stated: do there exist non-equivalent Rosser-sentences with respect to "the" usual proof predicate.

4.2 Proposition. There exists a usual proof predicate such that

(a) for all Rosser formulas Rx with respect to that proof predicate $PA \vdash \exists x \forall y \geq x \neg Ry$,

and hence,

(b) for all Rosser formulae Rx and Sx with respect to that proof predicate, $PA \vdash \exists x \forall y \geq x (Ry \leftrightarrow Sy)$,

and

(c) for all such Rosser formulae Rx and Sx , $PA \not\vdash \forall x \forall y (Rx \leftrightarrow Sy)$.

Proof. (a) We again take a usual proof predicate as in (a) of the above proof. First note that in $PA + \neg \Box \perp$, $\forall y \neg Ry$ is provable, and hence also $\exists x \forall y \geq x \neg Ry$. Secondly, note that, arguing in $PA + \Box \perp$, for n large enough, the shortest proofs of $R(n)$ and $\neg R(n)$ are going to consist of the shortest proof of \perp combined with the formula in question. This means that with the gödel numbering discussed above $\Box R(n) < \Box \neg R(n)$ holds for such large n , i.e. $\neg R(n)$. So, in any case, $\exists x \forall y \geq x \neg Ry$.

(b) follows immediately from (a), and (c) follows from (b), since for each Rosser formula $R(x)$, $PA \not\vdash \neg R(n)$ for any $n \in \mathbb{N}$, by the usual Rosser argument. \square

The result is somewhat remarkable in that standard proof predicates \Box^f can be found for which all Rosser-formulas are provably equivalent by extending the Guaspari-Solovay (1979) method (de Jongh A).

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Proof. (a) The idea is the following. If \perp is provable in PA, then, as we will explain, at least for some proof systems and codings thereof by natural numbers the shortest proofs of very large (i.e. non-standard) sentences A which aren't axioms will be $\langle \perp, A \rangle$. (We restrict our attention to natural deduction systems; for, say, Hilbert type systems, even though there is no essential difference, this is of course somewhat more complicated.) If we take $\perp \wedge n=n$ for the sentence A , then we see that from a certain n onwards $\langle \perp, \perp \wedge n=n \rangle$ will in such a case be the shortest proof of $\perp \wedge n=n$. Since in PA it cannot be excluded that \perp is provable, it is now sufficient to make the proof system and coding such that a proof of $T \wedge n=n$ cannot be shorter than the given proof of $\perp \wedge n=n$. For this idea to work it has to be so that the only radical increase in length of formulae in a proof step occurs only in applications of the *ex falso* rule. Usually, however, also application of the $E\forall$ -rule $\forall x A(x)/A(t)$ may give a radical increase in formula length. The problem is then that $T \wedge n=n$ may get short proofs after all, e.g. by concluding it from $\forall x(T \wedge x=x)$. This blocks the argument, since $\forall x(T \wedge x=x)$ might have a shorter proof than \perp (in fact, it certainly will!). The $E\forall$ -rule is the only rule besides the *ex falso* rule with this property however, so if we exclude it the argument will go through. We just have to be sure that $T \wedge n=n$ is not an axiom and that any formulae from which it is concluded will have almost the same length as $T \wedge n=n$ itself.

The above considerations lead to the following solution: choose a formulation of PA with no function symbols (even for the successor function), and take a standard natural deduction system for it. This is slightly awkward for the interpretation of the substitution of numerals, $\Box(T \wedge x=x)$ e.g. gets rather unwieldy, as e.g. $T \wedge 2=2$ stands for $T \wedge \exists x_1 \exists x_2 \exists x_3 \exists x_4 (S0x_1 \wedge Sx_1 x_2 \wedge S0x_3 \wedge Sx_3 x_4 \wedge x_2=x_4)$, but there is no fundamental problem. The effect is that $T \wedge n=n$ cannot have short proofs via $\forall x(T \wedge x=x)$. An alternative, somewhat unnatural, way is to have a natural deduction system with regard to the propositional connectives, together with axioms and not rules for the quantifiers, in particular e.g., $\forall x A(x) \rightarrow A(t)$.

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