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ON THE STRUCTURE OF KRIPKE MODELS OF HEYTING ARITHMETIC

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ON THE STRUCTURE OF KRIPKE MODELS[1] OF HEYTING ARITHMETIC

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ABSTRACT

Since in Heyting Arithmetic (HA) all atomic formulas are decidable, a Kripke model for HA may be regarded classically as a collection of classical structures for the language of arithmetic, partially ordered by the submodel relation. The obvious question is then: are these classical structures models of Peano Arithmetic (PA)? And dually: if a collection of models of PA, partially ordered by the submodel relation, is regarded as a Kripke model, is it a model of HA? Some partial answers to these questions were obtained in [6], [3], [1] and [2]. Here we present some results in the same direction, announced in [7]. In particular, it is proved that the classical structures at the nodes of a Kripke model of HA must be models of $I\Delta_1$ (PA^- with induction for provably Δ_1 formulas) and that the relation between these classical structures must be that of a Δ_1-elementary submodel.

§0. Introduction

It is easy to see that in a Kripke model of a theory with decidable atomic formulas, old elements can not acquire new atomic properties in later worlds. From a classical point of view, a Kripke model of such a theory may be regarded as a partially ordered collection of classical structures for the same language, where the partial order is introduced by the submodel relation (as opposed to a homomorphic embedding in the most general case). The intuitionistic satisfaction relation in such a model, usually called forcing, may be compared to Robinson's model theoretic forcing,

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except that two new logical connectives are also considered: $\rightarrow$ and $\forall$, different from their classical counterparts. However, intuitionistic formulas may be regarded as having a meaning also in the classical structures, by the obvious identification of $\varphi \rightarrow \psi$ with $\neg\varphi \vee \psi$ and of $\forall x \varphi(x)$ with $\neg\exists x \neg\varphi(x)$.

Kripke models for Heyting arithmetic (HA) were explicitly considered for the first time by Smorynski in [8], where he defined a powerful collection operation, which he used to prove a number of metatheoretic results about HA. Collection enables one to construct new Kripke models for HA starting with some given models and even to construct completely new Kripke models starting with models of PA. However, the new models thus obtained can only have ordering with finitely many levels. Further, the universes at all but the terminal nodes (endpoints) must necessarily consist of the standard natural numbers, unless we also introduce the notion of "Kripke model of HA definable in a nonstandard model of PA", which complicates matters considerably.

It looks as if a better understanding of Kripke models of HA requires a better understanding of the classical structures at their nodes. However, forcing at a node coincides with the truth in the corresponding classical structure for a very restricted class of formulas only (it is shown here to be Σ_1). Each formula whose decidability is not forced at some node may give rise to a whole Rieger-Nishimura lattice of intuitionistically non-equivalent formulas. Therefore the problem of what must hold in all classical structures at the nodes of a Kripke model of HA does not appear to be easy.

The natural assumption is that the classical structures are models of PA. Indeed, it was shown in [3] that a Kripke model of HA on a finite frame (with finitely many nodes) must have a model of PA at each node. However, in a recent paper [2] Buss shows that a Kripke model consisting of classical models for PA, need not even be a model of Π_1 induction. Obviously, the classical structures at the nodes, besides being models of PA or a significant fragment thereof (certainly all prenex theorems of HA have to be satisfied), must also be interrelated in some ways. It is shown here that if a node s is smaller than the node t (in the partial ordering of a Kripke model of HA), the classical structure associated with s, must be a Δ_1-elementary submodel of the structure associated with t.

The approach taken here is purely classical and we make an effort to use standard model-theoretic terminology wherever possible. That the results obtained in such a manner may still have intuitionistic relevance (via arithmetization and Π_2-conservativeness of PA over HA) has been argued in [8] and [9].

§1. Notation

For simplicity, we may define a Kripke model for HA to be a structure

$$\mathfrak{M} = \langle\langle T, 0, \leq\rangle;\ \mathfrak{A}_t : t \in T\rangle$$

where $\langle T, 0, \leq \rangle$ is a tree with the least element 0 (cf.[5] and [8]) and for each $t \in T$, $\mathfrak{A}_t$ is a classical structure for the language of arithmetic such that for any $s, t \in T$, $s \leq t$ implies $\mathfrak{A}_s \subseteq \mathfrak{A}_t$ ($\mathfrak{A}_s$ is a submodel of $\mathfrak{A}_t$). Elements of T are called nodes.

The forcing ($\Vdash$) relation between a node t and a sentence φ in the extended language containing names for all elements of A_t (the universe of $\mathfrak{A}_t$), is defined in the usual way: forcing for atomic sentences coincides with truth ($\models$) in $\mathfrak{A}_t$ (more precisely: $(\mathfrak{A}_t, A_t)$), inductive clauses for $\vee, \wedge$ and $\exists$ are just like in standard truth definitions and:

$$t \Vdash \varphi_1 \rightarrow \varphi_2 \quad \text{iff} \quad \text{for every } t' \geq t \ (t' \nVdash \varphi_1 \text{ or } t' \Vdash \varphi_2),$$
$$t \Vdash \forall x \varphi(x) \quad \text{iff} \quad \text{for every } t' \geq t \text{ and every } a \in A_{t'} \ (t' \Vdash \varphi(a))$$

(We use the same notation for $a \in A_t$ and its name in $\varphi(a)$.)

By Heyting arithmetic we understand the intuitionistic first-order logic with the usual axioms for PA^- (care should be taken to put the obvious bound on the only existential quantifier) and the induction schema. Thus PA is obtained by adding to HA the Principle of Excluded Middle (or some other appropriate schema). HA may be formulated with symbols for all primitive recursive functions (cf. [10] or [11]) or with predicate symbols only (cf. [8]), but for the results presented here this would not make any difference.

If the free variables of a formula φ are not explicitly stated, we shall use $\forall \bar{x} \varphi$ to denote its universal closure (i.e. we assume that all the free variables of φ are contained in the finite sequence $\bar{x}$). Thus we denote the decidability of φ by HA$\vdash \forall \bar{x}(\varphi \vee \neg\varphi)$. In such contexts we shall use $\bar{a} \in A_t$ to denote an appropriate finite sequence of elements of the universe A_t, and $\varphi(\bar{a})$ to denote the formula in which the appropriate names for the elements of A_t are substituted.

All the other notation is as in [10] or [11], where also all the results that are invoked may be found.

§2. Δ_0 - Formulas

It was proved in [6] that in a Kripke model of a theory with decidable atomic formulas, the following holds for every node t:

Lemma 1. (i) *If $\varphi(\bar{x})$ is a quantifier-free formula and $\bar{a} \in A_t$ then*

$$t \Vdash \varphi(\bar{a}) \vee \neg\varphi(\bar{a})$$

(ii) *If $\varphi(\bar{x})$ is an existential formula (i.e. $\varphi(\bar{x}) = \exists y_1 \ldots \exists y_n \psi(\bar{x}, y_1, \ldots, y_n)$, where ψ is quantifier-free) and $\bar{a} \in A_t$:*

$$t \Vdash \varphi(\bar{a}) \quad \textit{iff} \quad \mathfrak{A}_t \models \varphi(\bar{a})$$

(iii) *If* $\varphi(\bar{x})$ *is in prenex normal form and* $\bar{a} \in A_t$ *then:*

$$t \Vdash \varphi(\bar{a}) \quad \textit{implies} \quad \mathfrak{A}_t \models \varphi(\bar{a}).$$

As an immediate consequence we have the following:

Corollary 1. *If* $\mathfrak{M} = \langle\langle T, 0, \leq\rangle; \mathfrak{A}_t : t \in T\rangle$ *is a Kripke model of HA, then for every* $t \in T$*:*

$$\mathfrak{A}_t \models PA^-.$$

However, in Intuitionistic predicate calculus formulas do not necessarily have equivalent prenex normal forms, so (iii) can not provide much information on the induction schema. Using an old Kleene's result that decidable formulas in HA are closed under bounded quantification, we prove:

Theorem 1. *If* $\mathfrak{M} = \langle\langle T, 0, \leq\rangle; \mathfrak{A}_t : t \in T\rangle$ *is a Kripke model for HA and* $\varphi(\bar{x})$ *is a* Δ_0 *formula (i.e., with all the quantifiers bounded), the following holds for any* $s, t \in T$ *and* $\bar{a} \in A_t$*:*

(i) $t \Vdash \varphi(\bar{a}) \vee \neg\varphi(\bar{a})$

(ii) $t \Vdash \varphi(\bar{a})$ *iff* $\mathfrak{A}_t \models \varphi(\bar{a})$

(iii) $t \leq s$ *implies* ($\mathfrak{A}_t \models \varphi(\bar{a})$ *iff* $\mathfrak{A}_s \models \varphi(\bar{a})$) *(i.e.* $\mathfrak{A}_t \prec_{\Delta_0} \mathfrak{A}_s$*)*

Proof (i) Starting with Lemma 1(i), we prove that the set of decidable formulas is closed under propositional connectives and bounded quantifiers.

Cases where $\varphi = \exists x < y\psi$ or $\varphi = \forall x < y\psi$ for decidable ψ are theorems *150. and *151. in [4].

If $\varphi = \varphi_1 \vee \varphi_2$ where φ_1, φ_2 are decidable formulas and $t \nVdash \varphi_1 \vee \varphi_2$, then $t \nVdash \varphi_1$ and $t \nVdash \varphi_2$ and by the decidability of φ_1 and φ_2, $t \Vdash \neg\varphi_1 \wedge \neg\varphi_2$, which is equivalent to $t \Vdash \neg(\varphi \vee \varphi_2)$.

If $\varphi = \varphi_1 \wedge \varphi_2$ and $t \nVdash \varphi_1 \wedge \varphi_2$, then $t \nVdash \varphi_1$ or $t \nVdash \varphi_1$. In the first case $t \Vdash \neg\varphi_1$ and in the second $t \Vdash \neg\varphi_2$ so $t \Vdash \neg\varphi_1 \vee \neg\varphi_2$ which implies $t \Vdash \neg(\varphi_1 \wedge \varphi_2)$.

If $\varphi = \varphi_1 \rightarrow \varphi_2$ and $t \nVdash \varphi_1 \rightarrow \varphi_2$, it follows that for some $s \geq t$, $s \Vdash \varphi_1$ and $s \nVdash \varphi_2$. By decidability of φ_1 and φ_2 it follows that $t \Vdash \varphi_1$ and $t \Vdash \neg\varphi_2$, so for every $t' \geq t$, $t' \Vdash \varphi_1$ and $t' \nVdash \varphi_2$, i.e., $t \Vdash \neg(\varphi_1 \rightarrow \varphi_2)$.

(ii) Induction on the complexity of φ. By Lemma 1(ii) the Theorem holds if φ is quantifier-free.

Let $\varphi = \exists x < a\psi(x)$ and suppose $t \Vdash \exists x < a\psi(x)$. By definition this means that for some $b \in A_t$, $t \Vdash b < a$ and $t \Vdash \psi(b)$. This is equivalent to $\mathfrak{A}_t \models b < a$, and, by induction hypothesis, $\mathfrak{A}_t \models \psi(b)$, which means $\mathfrak{A}_t \models \exists x < a\psi(x)$.

Let $\varphi = \forall x < a\psi(x)$ and suppose $t \Vdash \forall x < a\psi(x)$. This implies that for every $b \in A_t$, $t \Vdash b < a$ implies $t \Vdash \psi(b)$. So, if $t \Vdash b < a$ then, by induction hypothesis,

$\mathfrak{A}_t \Vdash \psi(b)$. If $t \nVdash b < a$, we have $\mathfrak{A}_t \Vdash \neg b < a$. In any case $\mathfrak{A}_t \Vdash b < a \to \psi(b)$, for every $b \in A_t$, so $\mathfrak{A}_t \models \forall x < a\psi(x)$. Suppose now $\mathfrak{A}_t \models \forall x < a\psi(x)$ and consider the formula $\exists x < a\neg\psi(x)$. If $t \Vdash \exists x < a\neg\psi(x)$ then for some $b \in A_t$, $t \Vdash b < a \wedge \neg\psi(b)$. By induction hypothesis then $\mathfrak{A}_t \not\models \psi(b)$, so $\mathfrak{A}_t \models b < a \wedge \neg\psi(b)$ contradicting $\mathfrak{A}_t \models \forall x < a\psi(x)$. Therefore, by (i), $t \Vdash \neg\exists x < a\neg\psi(x)$. By intuitionistic logic, this is equivalent to $t \Vdash \forall x < a\neg\neg\psi(x)$. As ψ is decidable (by (i)) it follows that $t \Vdash \forall x < a\psi(x)$.

Let $\varphi = \varphi_1 \to \varphi_2$. If $t \Vdash \varphi_1 \to \varphi_2$ then $t \Vdash \varphi_2$ or $t \nVdash \varphi_1$ and by induction hypothesis we have $\mathfrak{A}_t \models \varphi_2$ or $\mathfrak{A}_t \nVdash \varphi_1$, so $\mathfrak{A}_t \models \varphi_1 \to \varphi_2$. Suppose now that $\mathfrak{A}_t \models \varphi_1 \to \varphi_2$. If $\mathfrak{A}_t \models \varphi_2$, by induction hypothesis we have $t \Vdash \varphi_2$. If $\mathfrak{A}_t \models \neg\varphi_1$, by induction hypothesis we have $t \nVdash \varphi_1$ and by (i), $t \Vdash \neg\varphi_1$. In either case we get $t \Vdash \varphi_1 \to \varphi_2$. The cases where the principal connective of φ is $\vee$ or $\wedge$ are trivial.

(iii) The fact that $\mathfrak{A}_t \subseteq \mathfrak{A}_s$ for $t \leq s$, implies that (iii) holds for quantifier-free formulas (with parameters from A_t). The proof proceeds by induction on the number of (bounded) quantifiers in φ.

Let $\varphi = \exists x < a\psi(x)$ and suppose $\mathfrak{A}_s \models \exists x < a\psi(x)$ $(t \leq s,\ a \in A_t)$. By (ii), it follows that $s \Vdash \exists x < a\psi(x)$. Then, as $t \leq s$, $t \nVdash \neg\exists x < a\psi(x)$, so, by (i), $t \Vdash \exists x < a\psi(x)$. Applying (ii) again, we get $\mathfrak{A}_t \models \exists x < a\psi(x)$. The converse is trivial, by classical model theory and induction hypothesis on $\psi(x)$, $\mathfrak{A}_t$ being a submodel of $\mathfrak{A}_s$.

Let $\varphi = \forall x < a\psi(x)$ and suppose $\mathfrak{A}_t \models \forall x < a\psi(x)$. If $\mathfrak{A}_s \not\models \forall x < a\psi(x)$ then $\mathfrak{A}_s \models \exists x < a\neg\psi(x)$ and we may apply the above argument, obtaining the contradiction. Thus $\mathfrak{A}_s \models \forall x < a\psi(x)$. The converse is trivial, as above.

§3. Δ_1 - Formulas

For arbitrary formulas $\psi(y)$ and $\chi(y)$ in the language of arithmetic let us define the sentences:

$$\Delta(\psi,\chi) \stackrel{\text{def}}{=} \forall\bar{x}(\exists y\psi(y) \leftrightarrow \forall y\chi(y)),$$
$$\mathrm{P}\Delta(\psi,\chi) \stackrel{\text{def}}{=} \forall\bar{x}\forall y\forall z\exists u\exists v((\psi(y) \to \chi(z)) \wedge (\chi(u) \to \psi(v))).$$

Lemma 2. *If ψ and χ are Δ_0 formulas then:*

(i) $\mathrm{HA} \vdash \Delta(\psi,\chi)$ *iff* $\mathrm{HA} \vdash P\Delta(\psi,\chi)$

(ii) $\mathrm{HA} \vdash \Delta(\psi,\chi)$ *iff* $\mathrm{PA} \vdash \Delta(\psi,\chi)$.

Proof It is easy to check that $\mathrm{P}\Delta(\psi,\chi) \to \Delta(\psi,\chi)$ is a theorem of intuitionistic logic. Also, HA being a subtheory of PA, HA$\vdash \Delta(\psi,\chi)$ implies PA$\vdash \Delta(\psi,\chi)$. From PA$\vdash \Delta(\psi,\chi)$ it follows, by classical logic, that PA$\vdash \mathrm{P}\Delta(\psi,\chi)$. Since $\mathrm{P}\Delta(\psi,\chi)$ is a

Π_2 sentence, we may use the fact that PA is conservative over HA with respect to Π_2 sentences (cf[10],3.8.6) to obtain HA$\vdash$ P$\Delta(\psi,\chi)$.

We may now define Δ_1 to be the set of all Σ_1 formulas $\exists y\psi(\bar{x},y)$ such that for some $\chi \in \Delta_0$

$$\mathrm{PA} \vdash \Delta(\psi,\chi)$$

(or equivalently HA$\vdash$ $\Delta(\psi,\chi)$).

Using the preceding results (which includes the theorem that PA is conservative over HA with respect to Π_2 sentences) we can give a proof of what is sometimes called the Kleene-Post rule, as a formal analogon of the theorem of recursion theory.

Theorem 2. *Let $\varphi(x)$ be a formula in the language of HA such that for some Δ_0 formulas ψ and χ:*
HA $\vdash \forall x(\varphi(x) \leftrightarrow \exists y\psi(x,y))$ *and*
HA$\vdash \forall x(\varphi(x) \leftrightarrow \forall y\chi(x,y))$
Then HA $\vdash \forall x(\varphi(x) \vee \neg\varphi(x))$.

Proof From the assumptions of the Theorem we may immediately derive:

HA$\vdash \forall x(\exists y\psi(x,y) \leftrightarrow \forall y\chi(x,y))$, i.e. HA$\vdash$ $\Delta(\psi,\chi)$.

Assume now that t is a node of an arbitrary Kripke model $\mathfrak{M}$ of HA, and assume for some $a \in A_t$, $t \not\Vdash \neg\varphi(a)$. This means that for some $s \geq t$ in $\mathfrak{M}$, $s \Vdash \varphi(a)$. By the first assumption of the Theorem, since $s \Vdash$HA, it follows that $s \Vdash \exists y\psi(a,y)$, i.e., for some $b \in A_s$, $s \Vdash \psi(a,b)$. By Theorem 1.(ii) this implies $\mathfrak{A}_s \models \psi(a,b)$ and $\mathfrak{A}_s \models \exists y\psi(a,y)$.

For any node t in $\mathfrak{M}$, by Lemma 2.(i), we have $t \Vdash$ P$\Delta(\psi,\chi)$. As P$\Delta(\psi,\chi)$ is a prenex formula, by Lemma 1.(iii), we get $\mathfrak{A}_t \models P\Delta(\psi,\chi)$ and by classical logic $\mathfrak{A}_t \Vdash \Delta(\psi,\chi)$.

Thus we may derive $\mathfrak{A}_s \models \forall y\chi(a,y)$. By classical model theory, using Theorem 1.(iii), it follows that $\mathfrak{A}_t \models \forall y\chi(a,y)$ and so $\mathfrak{A}_t \models \exists y\psi(a,y)$ and $t \Vdash \exists y\psi(a,y)$ as above. Using the first assumption of the Theorem we get $t \Vdash \varphi(a)$.

Remark. It is obvious from the proof that ψ and χ may be taken to be Σ_1 and Π_1 formulas, respectively. Also, each of the exibited quantifiers may be replaced by a string of quantifiers of the same type.

Corollary 2. *If ψ is a Σ_1 formula and χ is a Π_1 formula and PA*$\vdash \forall\bar{x}(\psi \leftrightarrow \chi)$ *then $HA \vdash \forall\bar{x}(\psi \vee \neg\psi)$ and $HA \vdash \forall\bar{x}(\chi \vee \neg\chi)$.*

Proof Using Lemma 2. we obtain HA$\vdash$ $\Delta(\psi,\chi)$ and may then apply the proof of the preceding Theorem.

Remark This argument does not extend any further, to arbitrary formula φ which is, provably in PA, equivalent to a Δ_1 formula, since such φ may contain, for example, subformulas of the type $\forall\bar{x}(\xi \vee \neg\xi)$ for ξ of arbitrary complexity.

The converse to Theorem 2. actually also holds.

Theorem 3. *Formulas decidable in* HA *are in* $\Delta_1(\mathrm{HA})$, *i.e., if* $\mathrm{HA} \vdash \forall\bar{x}(\varphi(\bar{x}) \vee \neg\varphi(\bar{x}))$ *then there exist a* Σ_1 *formula* ψ *and a* Π_1 *formula* χ *such that:*

$$\mathrm{HA} \vdash \forall\bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})) \qquad \text{and} \qquad \mathrm{HA} \vdash \forall\bar{x}(\varphi(\bar{x}) \leftrightarrow \chi(\bar{x})).$$

Proof (provided by de Jongh). Using the standard procedure for eliminating the disjunction in HA we get:

$$\mathrm{HA} \vdash \forall\bar{x}\exists y((y = 0 \rightarrow \varphi(\bar{x})) \wedge (\neg y = 0 \rightarrow \neg\varphi(\bar{x}))).$$

Since $y = 0$ is decidable, we have $\mathrm{HA}\vdash \forall\bar{x}\exists y(y = 0 \leftrightarrow \varphi(\bar{x}))$.

Using the fact that HA is closed under Church's rule (cf [10], 3.1.18. or 4.4.6.) and assuming $\mathrm{HA}\vdash T(x,y,z) \wedge T(x,y,z') \rightarrow z = z'$ (cf.[10], 1.3.10.), with some manipulation, we get that for some $e \in N$,

$$\mathrm{HA} \vdash \forall\bar{x}(\varphi(\bar{x}) \leftrightarrow \exists w(T(e,\bar{x},w) \wedge Uw = 0)).$$

The other part proceeds similarily, starting with $\mathrm{HA}\vdash \forall\bar{x}(\neg\varphi(\bar{x}) \vee \varphi(\bar{x}))$, and obtaining $\mathrm{HA}\vdash \forall\bar{x}(\varphi(\bar{x}) \leftrightarrow \forall w\neg(T(d,\bar{x},w) \wedge Uw = 0))$ for some $d \in N$.

From Theorems 2. and 3. and the proof of Theorem 1.(i) we may immediately derive that the set of formulas $\Delta_1(\mathrm{HA})$ is closed under propositional connectives and bounded quantification.

Lemma 3. *If t and s are nodes of a Kripke model of HA and $t \leq s$ then:*

$$\mathfrak{A}_t \prec_{\Delta_1} \mathfrak{A}_s.$$

Proof Let $\psi, \chi \in \Delta_0$ be such that $\mathrm{HA}\vdash \Delta(\psi,\chi)$ and assume, for some $a \in A_t$, $\mathfrak{A}_s \models \exists y\psi(a,y)$. Then $s \Vdash \exists\psi(a,y)$, so $t \nVdash \neg\exists y\psi(a,y)$. By Theorem 2, it follows that $t \Vdash \exists y\psi(a,y)$ and so $\mathfrak{A}_t \models \exists y\psi(a,y)$.

We may restate now the results of this section as a strengthened version of Theorem 1.

Theorem 4. *If* $\mathfrak{M} = \langle\langle T, 0, \leq\rangle; \mathfrak{A}_t : t \in T\rangle$ *is a Kripke model of HA and* $\varphi(\bar{x}) \in \Delta_1$ *and* $\psi(\bar{x}) \in \Sigma_1$, *the following holds for any* $t, s \in T$ *and any* $\bar{a} \in A_t$*:*

(i) $t \Vdash \varphi(\bar{a}) \vee \neg\varphi(\bar{a})$

(ii) $t \Vdash \psi(\bar{a})$ *iff* $\mathfrak{A}_t \models \psi(\bar{a})$

(iii) $t \leq s$ *implies* $(\mathfrak{A}_t \models \varphi(\bar{a})$ *iff* $\mathfrak{A}_s \models \varphi(\bar{a}))$ (i.e. $\mathfrak{A}_t \prec_{\Delta_1} \mathfrak{A}_s$).

The following two lemmas show that these results are in a sense the best possible, considering that HA+$\neg M_{\mathrm{PR}}$ is consistent (cf.[10], 3.8.3).

Lemma 4. *If in a Kripke model* $\mathfrak{M} = \langle\langle T, 0, \leq\rangle; \mathfrak{A}_t : t \in T\rangle \models \mathrm{HA}$,

$$s \leq t \quad \text{implies} \quad \mathfrak{A}_s \prec_{\Sigma_1} \mathfrak{A}_t \quad \text{for any } s, t \in T,$$

then $\mathfrak{M} \models M_{\mathrm{PR}}$.

Proof We show that in this case Markov's principle holds in $\mathfrak{M}$ for all Σ_1 formulas. Suppose for some $t \in T$, $\bar{a} \in A_t$ and $\varphi \in \Sigma_1$ that $t \Vdash \neg\neg\exists x\varphi(x, \bar{a})$. This means that for every $t' \geq t$ there exists $t'' \geq t'$ such that $t'' \Vdash \exists x\varphi(x, \bar{a})$. Since $\exists x\varphi(x, \bar{a})$ is in Σ_1 it follows that $\mathfrak{A}_{t''} \models \exists x\varphi(x, \bar{a})$. As $t \leq t''$, by the assumption of the lemma, we have $\mathfrak{A}_t \models \exists x\varphi(x, \bar{a})$ and $t \Vdash \exists x\varphi(x, \bar{a})$. Therefore

$$0 \Vdash \forall\bar{y}(\neg\neg\exists x\varphi(x, \bar{y}) \rightarrow \exists x\varphi(x, \bar{y})).$$

Lemma 5. *If in a Kripke model* $\mathfrak{M} = \langle\langle T, 0, \leq\rangle; \mathfrak{A}_t : t \in T\rangle \models \mathrm{HA}$, *we have for every* $t \in T$, *every* Π_1 *formula* φ *and every* $\bar{a} \in A_t$:

$$t \Vdash \varphi(\bar{a}) \qquad \text{iff} \qquad \mathfrak{A}_t \models \varphi(\bar{a})$$

then $\mathfrak{M} \models M_{\mathrm{PR}}$.

Proof We shall show that $\mathfrak{M}$ satisfies the condition of Lemma 4. Let $s \leq t \in T$, $\psi \in \Delta_0$ and suppose $\mathfrak{A}_s \models \forall x\psi(x, \bar{a})$. By the assumption of this Lemma we have $s \Vdash \forall x\psi(x, \bar{a})$ and so $t \Vdash \forall x\psi(x, \bar{a})$ and again $\mathfrak{A}_t \models \forall x\psi(x, \bar{a})$. Thus $\mathfrak{A}_s \prec_{\Sigma_1} \mathfrak{A}_t$.

We end with a theorem which one would not expect to be optimal in the sense in which Theorem 4. is.

Theorem 5. *If t is a node in a Kripke model of HA then* $\mathfrak{A}_t \models \mathrm{I}\Delta_1$

Proof $\mathfrak{A}_t \models \mathrm{PA}^-$, by Corollary 1., so let $\exists y\psi(x, y)$ be a Δ_1 formula and assume $\mathfrak{A}_t \models \exists y\psi(0, y) \wedge \forall x(\exists y\psi(x, y) \rightarrow \exists y\psi(x + 1, y))$. If $\mathfrak{A}_t \not\models \forall x\exists y\psi(x, y)$ then for some $a \in A_t$ we have $\mathfrak{A}_t \not\models \exists y\psi(a, y)$. Since $\exists y\psi(a, y)$ is a Σ_1 formula, we derive $t \not\Vdash \exists y\psi(a, y)$ and since it is also Δ_1, we must have $t \Vdash \neg\exists y\psi(a, y)$, by Theorem 2. This means $t \Vdash \exists x\neg\exists y\psi(x, y)$. But in HA the least number principle holds for decidable formulas (cf.[4],*149a), so we may derive:

$$t \Vdash \exists x(\neg\exists y\psi(x, y) \wedge \forall z < x\neg\neg\exists y\psi(z, y)).$$

Therefore, for some $c \in A_t$ we have $t \Vdash \neg\exists y\psi(c, y)$ and $t \Vdash \forall z < c\exists y\psi(z, y)$, since $\exists y\psi(z, y)$ is decidable. From the first we derive $\mathfrak{A}_t \models \forall y\neg\psi(c, y)$ and from the second $\mathfrak{A}_t \models \forall z < c\exists y\psi(z, y)$. Obviously $c \neq 0$ since we assumed $\mathfrak{A}_t \models \exists y\psi(0, y)$. Then $t \Vdash \exists y(y + 1 = c)$, i.e., $c \dot{-} 1 \in A_t$ and $\mathfrak{A}_t \models \exists y\psi(c \dot{-} 1, y)$. However, by the assumption, this would imply $\mathfrak{A}_t \models \exists y\psi(c, y)$ which is a contradiction. Thus $\mathfrak{A}_t \models \forall x\exists y\psi(x, y)$.

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