Modal Deduction in Second-Order Logic and Set Theory

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Abstract

We investigate modal deduction through translation into standard logic and set theory. Derivability in the minimal modal logic is captured precisely by translation into a weak, computationally attractive set theory Ω. This approach is shown equivalent to working with standard first-order translations of modal formulas in a theory of general frames. Next, deduction in a more powerful second-order logic of general frames is shown equivalent with set-theoretic derivability in an 'admissible variant' of Ω. Our methods are mainly model-theoretic and set-theoretic, and they admit extension to richer languages than that of basic modal logic.

1 Introduction

We are interested in analyzing general deduction for modal formulae [4]. The standard systems used for this purpose are the so-called "minimal modal logic" K or, if one wants to work over general frames (as we do), the system K_s obtained from K adding a substitution rule. We do not consider special purpose calculi for stronger modal logics over K_s, such as $S4$ or $S5$, although we expect that our results and method can be specialized to frame classes obeying their additional specific constraints. Such logics are treated as particular sets of assumptions in our general deduction mechanisms.

Our aim is to analyze general modal deduction via translations into standard logics. What already exists for this purpose in the logical tradition are "standard translations" into first-order and monadic second-order logic (for their extensive theory, cf. [4]). In recent years, the computer-science community has also become interested in this approach, (see, for example, [14]), with results that are mainly restricted to cases where specific modal logics have "nice" translations (first-order, or well-behaved second-order). A new theme in the latter approach has been variations in translation formats, e.g., special operator-based 'path languages' have been used to profit optimally from resolution methods [12, 13].

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This paper takes its point of departure in [8], where a general *set-theoretic* translation, whose main idea is to treat the universal modality as a Power-set operator, was proposed for modal logic. The engine driving soundness and completeness of this translation was a weak set theory Ω, having pleasant computational properties [7]. The method was shown adequate only for frame complete modal logics, using an argument heavily relying on non-well-founded set theory [1].

Against this background, the present paper makes the following new contributions. Standard modal deduction via the standard translation is supported by a simple two-sorted first-order theory μ of general frames. Comparing μ with Ω, we establish an effective equivalence between the latter and a suitable extension of μ, that we call μ^+. Such an extension is obtained adding to μ an axiom reflecting the "uniformity" of our set-theoretic approach which uses just $\in$ for both set-membership and the accessibility relation. We then prove that μ^+ is non-conservative over μ. On the ground of the equivalence between Ω and μ^+, this allows us to conclude that the translation method given in [8] is not adequate for logics which are not frame complete. By varying our translation, however, we manage to obtain a full equivalence between general modal deduction in K_s and Ω-deduction. This is the desired general result, which is based on standard set-theoretic methods (including a suitable adaptation of Fraenkel-Mostowski permutations of the universe [10]).

Next, we turn to extensions on a broader perspective. One can consider various principled strengthenings of minimal modal deduction; from a second-order point of view, a good candidate is L_2 [3]. L_2 is based on a second-order language containing, besides equality and a binary constant R, unary predicate variables. The logic is provided with a suitable form of substitution that allows to replace universal quantified second-order variables with first-order formulae in the language of R and equality. L_2 turns out to be rather expressive; in particular, it allows one to derive the first-order correspondents of Sahlqvist formulae. We show how to match L_2-derivability of modal formulae with derivability in a suitably modified version Ω_c of Ω, which adds (some of) the well-known operations of admissible set theory.

In conclusion we briefly consider what happens when we vary the source language for these translations and deductive engines, from the basic modal language to stronger (temporal or first-order) formalisms.

The paper is organized as follows. Section 2 contains basic results about the standard translation method and the set-theoretic one. Section 3 compares the two methods providing the basic equivalence result. In Section 4 the full equivalence between general modal deduction in K_s and Ω is proved. In Section 5 such an equivalence is generalized to the case of L_2. Finally, Section 6 presents some variations of the proposed framework suitable to deal with languages other than the basic modal one.

2 Translation methods for modal logic

Let us briefly recall some notation and basic results about modal logic (see [4]) for details). *Propositional modal formulae* are constructed using propositional letters $P_1, P_2, \ldots$, boolean operators $\vee, \neg$ and the $\Box$ operator. The minimal modal logic K_s consists of a set of propositional axioms complete for classical logic, the modal axiom $\Box(P \rightarrow Q) \rightarrow (\Box(P) \rightarrow \Box(Q))$, and the rules of substitution, modus ponens and necessitation (infer $\Box(\phi)$ from ϕ). Deducibility of ψ from ϕ in K_s is defined as usual and denoted by $\phi \vdash_{K_s} \psi$.

A *frame* is a pair $F = (W, R)$, where W is a set (of *worlds*) and R is a binary relation on W (the *accessibility relation*). A *valuation* $\models$ in the frame (W, R) is a subset of $W \times \{P_1, \ldots\}$. For all $w \in W$, $w \models \phi$ is defined by induction on the structural complexity of the modal formula ϕ, as usual. A formula ϕ is *valid* in a frame (W, R) if and only if, for all $w \in W$ and for all valuation $\models$, $w \models \phi$ holds. A formula ψ is a *frame logical consequence* of a formula ϕ ($\phi \models_f \psi$) if and only if, for all frames F, if ϕ is valid in F, then ψ is valid in F. A formula ϕ is said to be *complete* if and only if, for all ψ, $\phi \vdash_{K_s} \psi \Leftrightarrow \phi \models_f \psi$. Examples of incomplete formulae can be found in the literature (see for example [4]).

A semantic characterization of K_s is obtained via general frames. A *general frame* is a pair $(F, \mathcal{W})$, where F is a frame and $\mathcal{W}$ is a set of subsets of W, which is closed under the boolean operations of union,

complementation w.r.t. W, and $\Box(X) = \{w \in W : \forall v\,(wRv \rightarrow v \in X)\}$. Validity in general frames is defined as for frames except for the fact that we only consider valuations taking values on $\mathcal{W}$. Logical consequence in general frames ($\phi \models_{gf} \psi$) is defined accordingly. For all modal formulae ϕ, ψ we have

$$\phi \vdash_{K_s} \psi \Leftrightarrow \phi \models_{gf} \psi.$$

2.1 The standard translation method

The standard translation of modal formulae into L_2-formulae is the following [4, 5]:

- $ST(P_i) = \mathtt{P_i}(x)$,
- $ST(\phi \vee \psi) = ST(\phi) \vee ST(\phi)$,
- $ST(\neg\phi) = \neg ST(\phi)$,
- $ST(\Box\phi) = \forall y\,(xRy \rightarrow ST(\phi)(y|x))$,

where $y|x$ denotes uniform substitution of the variable x with the variable y.

The *closed standard translation* $\overline{ST}(\phi)$ of a formula ϕ is defined as the L_2-sentence $\forall \mathtt{P_1} \ldots \forall \mathtt{P_n} \forall x ST(\phi)$, and it is is easy to see that

$$\phi \models_f \psi \text{ if and only if } \overline{ST}(\phi) \models \overline{ST}(\psi),$$

where the $\models$ on the right-hand side denotes second-order consequence.

In [3], axiomatic theories for deduction in the L_2-language are introduced. In particular, a system of weak second-order logic is defined by means of a suitable form of substitution. Let an L_0-formula be an L_2-formula with a free variable x but without occurrences of second-order quantifiers. Denote by $\alpha(\gamma|\mathtt{P})$ the L_2-formula obtained from the L_2-formula α by replacing subformulas of the form $\mathtt{P}(u)$ by an L_0-formula $\gamma(u|x)$ (modulo some technicalities about free variables [3]). L_0-substitution is expressed by the axiom

$$\forall \mathtt{P}\alpha \rightarrow \alpha(\gamma|P).$$

Weak second-order logic contains a set of axioms complete for first-order predicate logic, plus L_0-substitution.

It is possible to obtain a semantic counterpart of deducibility in weak second-order logic by means of closure under L_0-definitions in general frames. A general frames $(F, \mathcal{W})$ is *closed under L_0-definitions* if, for all L_0-formula γ with free world-variables $x, x_1, \ldots, x_n$, free set-variables $X_1, \ldots, X_m$, and for all $w, w_1, \ldots, w_n \in W$, $A_1, \ldots, A_m \in \mathcal{W}$, the set $\{w \in W : F \models \gamma(w, w_1, \ldots, w_n, A_1, \ldots, A_m)\}$ belongs to $\mathcal{W}$. It follows that $\alpha \vdash_{L_2} \beta$ if and only if, for all general frames $(F, \mathcal{W})$ closed under L_0-definitions, if $(F, \mathcal{W}) \models \alpha[f]$, then $(F, \mathcal{W}) \models \beta[f]$ (where f is an assignment of worlds in W to individual variables and set of worlds in $\mathcal{W}$ to unary predicate variables). Weak second-order logic is a nonconservative extension of K_s, namely, $\phi \vdash_{K_s} \psi \Rightarrow \overline{ST}(\phi) \vdash_{L_2} \overline{ST}(\psi)$, but there exist ϕ and ψ such that $\overline{ST}(\phi) \vdash_{L_2} \overline{ST}(\psi)$ but $\phi \nvdash_{K_s} \psi$ [3].

2.2 The set-theoretic translation method

The basic idea behind the set-theoretic translation is to represent any Kripke frame as a set, with the accessibility relation modeled using the membership relation $\in$. The theory in which the translation is carried out is a very weak[1] *finitely* (first-order) axiomatizable set theory called Ω [7, 8]. The axioms, in the language with relational symbols $\in$ and $\subseteq$, and functional symbols $\cup, \setminus$, and Pow, are the following:

$x \in y \cup z \leftrightarrow x \in y \vee x \in z$;

[1] Compare this theory with more classical finite axiomatizations of set theory, such as NBG [11].

$x \in y \setminus z \leftrightarrow x \in y \wedge x \notin z$;

$x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y)$;

$x \in Pow(y) \leftrightarrow x \subseteq y$.

Notice that neither the extensionality axiom nor the axiom of foundation are in Ω.

Given a modal formula $\phi(P_1, ..., P_n)$, we define its translation as the set-theoretic *term* $\phi^*(x, x_1, ..., x_n)$, with variables $x, x_1, ..., x_n$, built using $\cup, \setminus$, and Pow. Intuitively the term $\phi^*(x, x_1, ..., x_n)$ represents the set of those worlds (in the frame x) in which the formula ϕ holds. The inductive definition of $\phi^*(x, x_1, ..., x_n)$ is the following:

- $P_i^* = x_i$;
- $(\phi \vee \psi)^* = \phi^* \cup \psi^*$;
- $(\neg\phi)^* = x \setminus \phi^*$;
- $(\Box\, \phi)^* = Pow(\phi^*)$.

For all modal formulae ϕ, ψ, the following results showing the adequacy of the translation hold:

completeness:

$$\phi \vdash_{K_s} \psi \;\Rightarrow\; \Omega \vdash \forall x(Trans(x) \wedge \forall x_1 \ldots x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n)) \rightarrow \forall x_1 \ldots x_n(x \subseteq \psi^*(x, x_1, \ldots, x_n))),$$

soundness:

$$\Omega \vdash \forall x(Trans(x) \wedge \forall x_1 \ldots x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n)) \rightarrow \forall x_1 \ldots x_n(x \subseteq \psi^*(x, x_1, \ldots x_n))) \;\Rightarrow\; \phi \models_f \psi,$$

where $Trans(x)$ stands for $\forall y\ (y \in x \rightarrow y \subseteq x)$.

It is worth noting that, for frame-complete theories, the above translation captures exactly the notion of K_s-derivability.

3 A comparison of the two methods

In this section, we compare the two translation methods. To this end, consider a two-sorted first-order language, with *worlds* (denoted by small letters) and *sets* (denoted by capital letters), with binary predicates R (on worlds) and $\in$ (between worlds and sets), as well as operations $-, \cup$ and $\Box$ on the set sort. This language is adequate for the description of the general frame semantics. The minimal logic describing it is obtained by considering the following *theory* μ:

- First-Order Principles for both sorts;

- $\forall w(w \in -\mathtt{P} \leftrightarrow \neg w \in \mathtt{P})$;

- $\forall w(w \in \mathtt{P} \cup \mathtt{Q} \leftrightarrow w \in \mathtt{P} \vee w \in \mathtt{Q})$;

- $\forall w(w \in \Box\mathtt{P} \leftrightarrow \forall v(wRv \rightarrow v \in \mathtt{P}))$

- $\forall \mathtt{P} \forall \mathtt{Q}(\forall w(w \in \mathtt{P} \leftrightarrow w \in \mathtt{Q}) \rightarrow \mathtt{P} = \mathtt{Q})$ (extensionality).

Since Ω models both set membership and the accessibility relation in terms of $\in$-relationship, let μ^+ be the theory obtained by adding to μ the axiom

- $\forall w \exists \mathtt{P} \forall v \ (v \in \mathtt{P} \leftrightarrow wRv)$

that links R and $\in$.

In the following, we will show that the standard translation method w.r.t. μ^+ and the set-theoretic one w.r.t. Ω are equivalent. More precisely, we will prove that, for all modal formulae ϕ, ψ,

$$\mu^+ \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi)$$

if and only if

$$\Omega \vdash \forall x \ (Trans(x) \wedge \forall \overline{y} \ (x \subseteq \phi^*(x, \overline{y})) \rightarrow \forall \overline{y}(x \subseteq \psi^*(x, \overline{y}))).$$

We first prove that derivability in μ^+ implies derivability in Ω. Given an Ω-model based on a set U and a transitive set x in that model, we show how to construct a model for μ^+.

DEFINITION 3.1 *Let $(U, \in^U, \ldots)$ be a model for Ω, and let x be a transitive element of U. The corresponding model of μ^+ is a a general frame $(W, R, \mathcal{W})$ such that:*

- *the world sort W is given by all elements (w.r.t. the model U) of x ;*
- *the set sort $\mathcal{W}$ is given by equivalence classes in U with respect to the equivalence relation of "having the same elements of the world sort"[2]. More precisely, we state that $[y] = [z]$ if and only if, $w \in^U y \Leftrightarrow w \in^U z$, for every $w \in^U x$. An element of the set sort is an equivalence class $[y]$, for $y \in U$;*
- *the operations over $\mathcal{W}$ are defined as follows: $[y] \cup [z] = [y \cup^U z]$, $-[y] = [x \setminus^U y]$, and $\Box[y] = [Pow^U(y)]$.*
- *for worlds y, z we define yRz iff $z \in^U y$, while if y is a world and $[z]$ is a set, we define $y \in [z]$ iff $y \in^U z$.*

It is easy to show that the resulting model verifies all principles of μ^+.

THEOREM 3.2 *For each pair of formulae ϕ, ψ,*

$$\mu^+ \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi) \Rightarrow \Omega \vdash \forall x \ (Trans(x) \wedge \forall \overline{y} \ (x \subseteq \phi^*(x, \overline{y})) \rightarrow \forall \overline{y}(x \subseteq \psi^*(x, \overline{y}))).$$

Proof. First of all, we prove that, for any modal formula $\phi(P_1, .., P_n)$, each $y \in x$, and $y_1, ..., y_n \in U$,

$y \in \phi^*(x, y_1, ..., y_n)$ holds in the Ω-model iff $ST(\phi)(y, [y_1], ..., [y_n])$ holds in the two-sorted one.

The proof is by induction on the structure of ϕ.

- if $\phi = P_i$, then $U \models y \in \phi^*(x, y_i)$ iff $U \models y \in y_i$ iff $y \in [y_i]$ holds in the two-sorted model iff $ST(\phi)(y, [y_i])$ holds;
- if $\phi = \neg\psi$, then $U \models y \in \phi^*(x, y_1, ..., y_n)$ iff $U \models y \in x \setminus \psi^*(x, y_1, ..., y_n)$ iff $U \models y \in x \wedge y \notin \psi^*(x, y_1, ..., y_n)$ iff $ST(\psi)(y, [y_1], ..., [y_n])$ does not hold in the two-sorted model iff $ST(\phi)(y, [y_1], ..., [y_n])$ holds;
- the case of $\cup$ is left to the reader;

[2] We cannot restrict ourselves to x, because x may be not closed under the set operations. For instance, the transitive set $\{\emptyset, \{\emptyset\}\}$ is not closed under the *Pow* operation.

- if $\phi = \Box\psi$ then $U \models y \in \phi^*(x, y_1, ..., y_n)$ iff $U \models y \in Pow(\psi^*(x, y_1, ..., y_n))$ iff $U \models \forall w\ (w \in y \rightarrow w \in \psi^*(x, y_1, ..., y_n))$ iff $\forall w\ (yRw \rightarrow ST(\psi)(w, [y_1], ..., [y_n]))$ holds in the two-sorted model iff $ST(\phi)(y, [y_1], ..., [y_n])$ holds.

This implies that, for any modal formula $\phi(P_1, ..., P_n)$,

$$U \models \forall y_1 ... \forall y_n\ (x \subseteq \phi^*(x, y_1, ..., y_n)) \text{ iff } \overline{ST}(\phi) \text{ holds in the two-sorted model.}$$

If x is transitive and $U \models \forall \overline{y}\ (x \subseteq \phi^*(x, \overline{y}))$, then, from what preceeds, it follows that $\overline{ST}(\phi)$ holds in the two-sorted model. Thus, by hypothesis, we obtain $\overline{ST}(\psi)$ and again $U \models \forall \overline{y}(x \subseteq \psi^*(x, \overline{y}))$. $\dashv$

Let us prove that derivability in Ω implies derivability in μ^+. Let $(W, R, \mathcal{W})$ be a general frame satisfying μ^+. This means that $\mathcal{W}$ includes all subsets of W defined by the additional axiom characterizing μ^+, plus the subsets that can be obtained from them by union, (relative) complementation, and the $\Box$ operation. Moreover, without loss of generality, we suppose that the elements of W are sets of the same rank α. An Ω-model can be obtained from $(W, R, \mathcal{W})$ in the following way:

DEFINITION 3.3 *The Ω-model $(U, \in^U, \ldots)$ generated by $(W, R, \mathcal{W})$ is defined as follows.*
Let

$$\begin{aligned} U_0 &= W, \\ &\vdots \\ U_n &= Pow(U_{n-1}) \cup U_{n-1}, \end{aligned}$$

and let $U = \bigcup_{n=0}^{\omega} U_n$.
Define a function $F : U \rightarrow U$ as:

$$F(x) = \begin{cases} \{v \in W : xRv\} & \text{if } x \in W \\ x \setminus W & \text{if } x \notin W \wedge x \cap W \notin \mathcal{W} \\ x & \text{if } x \notin W \wedge x \cap W \in \mathcal{W} \end{cases}$$

where $\{v \in W : xRv\} \in \mathcal{W}$ for any $x \in W$.
A structure for the language $\{\in\}$ is obtained by taking U as domain, and defining

$$x \in^U y \text{ if and only } x \in F(y).$$

The definition of $\in^U$ is very similar to that of the so-called Fraenkel-Mostowski's permutation of the universe [10].
The following properties are easily verified:

- if $y \in W$, then $x \in^U y$ iff yRx;
- if $y \in \mathcal{W}$, then $x \in^U y$ iff $x \in y$.

LEMMA 3.4 *The interpretation $(U, \in^U, \ldots)$ generated by a general frame $(W, R, \mathcal{W})$ satisfying μ^+ is an Ω-model.*

Proof. To show that $(U, \in^U)$ is a model of Ω, it is enough to prove that

$$\begin{aligned}(U,\in^U) &\models \forall x\forall y\exists z\forall u\ (u \in z \leftrightarrow u \in x \vee u \in y);\\ (U,\in^U) &\models \forall x\forall y\exists z\forall u\ (u \in z \leftrightarrow u \in x \wedge u \notin y);\\ (U,\in^U) &\models \forall x\exists z\forall y\ (y \in z \leftrightarrow \forall s\ (s \in y \rightarrow s \in x)).\end{aligned}$$

Notice that the range of F is the set

$$\{y \in U \setminus W : y \cap W \in \mathcal{W}\}.$$

For any $x, y \in U$, the set $F(x) \cup F(y)$ is in the range of F. Indeed $(F(x) \cup F(y)) \cap W = (F(x) \cap W) \cup (F(y) \cap W)$, and both $(F(x) \cap W)$ and $(F(y) \cap W)$ belong to $\mathcal{W}$, which is closed under union. Therefore, take a z such that $F(z) = F(x) \cup F(y)$, and observe that

$$t \in^U z \leftrightarrow t \in F(z) \leftrightarrow t \in F(x) \vee t \in F(y) \leftrightarrow t \in^U x \vee t \in^U y.$$

Similarly, we show that the set $F(x) \setminus F(y)$ is in the range of F. In fact, $(F(x) \setminus F(y)) \cap W = (F(x) \cap W) \setminus (F(y) \cap W)$, and both $(F(x) \cap W)$ and $(F(y) \cap W)$ belong to $\mathcal{W}$, which is closed under difference. Any z such that $F(z) = F(x) \setminus F(y)$ will verify the axiom for $\setminus$.

Consider now the set $\{y \in U : F(y) \subseteq F(x)\}$. This set is in the range of F because

$$(\{y \in U : F(y) \subseteq F(x)\}) \cap W = \{y \in W : F(y) \subseteq F(x)\} =$$

$$\{y \in W : \forall z(yRz \rightarrow z \in F(x))\} = \Box(F(x) \cap W).$$

Since $F(x) \cap W$ is in $\mathcal{W}$ and $\mathcal{W}$ is closed under $\Box$, the set $\Box(F(x) \cap W)$ is in $\mathcal{W}$. Any z such that $F(z) = \{y \in U : F(y) \subseteq F(x)\}$ will verify the axiom for $\Box$, because

$$F(y) \subseteq F(x) \Leftrightarrow \forall s(s \in^U y \rightarrow s \in^U x).$$

$\dashv$

THEOREM 3.5 *For each pair of formulae ϕ, ψ,*

$$\Omega \vdash \forall x\ (Trans(x) \wedge \forall \overline{y}\ (x \subseteq \phi^*(x, \overline{y})) \rightarrow \forall \overline{y}(x \subseteq \psi^*(x, \overline{y}))) \Rightarrow \mu^+ \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi).$$

Proof. For any modal formula $\phi(P_1, ..., P_n)$, we prove that, for each $y \in W$ and $y_1, ..., y_n$ in U,

$$y \in \phi^*(W, y_1, .., y_n) \text{ holds in } (U, \in^U) \text{ iff } ST(\phi)(y, F(y_1) \cap W, .., F(y_n) \cap W) \text{ holds in } (W, R, \mathcal{W})$$

(recall that $F(y_i) \cap W \in \mathcal{W}$).

The proof is by induction on the structure of ϕ:

- if $\phi = P_i$, then $y \in \phi^*(W, y_i)$ in the model $(U, \in^U)$ iff $y \in^U y_i$ iff $y \in F(y_i)$ iff $y \in F(y_i) \cap W$ iff $ST(\phi)(y, F(y_i) \cap W)$;
- if $\phi = \neg\psi$, then $y \in \phi^*(W, y_1, .., y_n)$ iff $y \in W \setminus \psi^*(W, y_1, .., y_n)$ iff $y \in W \wedge y \notin \psi^*(W, y_1, .., y_n)$ iff not $ST(\psi)(y, F(y_1) \cap W, .., F(y_n) \cap W)$ iff $-ST(\psi)(y, F(y_1) \cap W, .., F(y_n) \cap W)$ iff $ST(\phi)(y, F(y_1) \cap W, .., F(y_n) \cap W)$;
- the case of $\cup$ is left to the reader;

- if $\phi = \Box\psi$, then $y \in \phi^*(W, y_1, .., y_n)$ in $(U, \in^U)$ iff $y \in Pow(\psi^*(W, y_1, .., y_n))$ in $(U, \in^U)$ iff $\forall v(v \in y \rightarrow v \in \psi^*(W, y_1, .., y_n))$ in $(U, \in^U)$ iff for all $v \in W$, if yRv then $v \in \psi^*(W, y_1, .., y_n))$ in $(U, \in^U)$ iff for all $v \in W$, if yRv then $ST(\psi)(v, F(y_1) \cap W, .., F(y_n) \cap W)$ iff $ST(\phi)(y, F(y_1) \cap W, .., F(y_n) \cap W)$.

This implies that, for any modal formula $\phi(P_1, ..., P_n)$,

$$\forall y_1 ... \forall y_n \ (W \subseteq \phi^*(W, y_1, ..., y_n)) \text{ holds in the } \Omega\text{-model}$$

$$\text{if and only if}$$

$$\overline{ST}(\phi) \text{ holds in the original one.}$$

Let us prove that W is transitive, that is,

$$\forall x(x \in W \rightarrow \forall y(y \in x \rightarrow y \in W)).$$

From $x \in W$ (in the Ω-model), it follows that x is an element of the world sort (in the model of μ^+), and from $y \in x$ (in the Ω-model), it follows that xRy in the model of μ^+; therefore, $y \in W$ (in the Ω-model).

Thus, if $\overline{ST}(\phi)$ holds (in the original model), then $\forall \overline{y} \ (W \subseteq \phi^*(W, \overline{y}))$ holds (in the Ω-model) and W is transitive. By hypothesis, we obtain $\forall \overline{y}(W \subseteq \psi^*(W, \overline{y}))$ and then $\overline{ST}(\psi)$. $\dashv$

It is worth noting that it is possible to remove the extensionality axiom from μ^+ without affecting the validity of the established result. To this end, it suffices to slightly modify Definition (3.1).

4 Capturing derivability in K_s

In Section 3, we have proved that Ω and μ^+ are equiparable, provided that the translation for Ω is the one defined in Section 2.2. If μ^+ was conservative over μ, then Ω would exactly correspond to K_s. It is indeed easy to see that:

$$\mu \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi) \Leftrightarrow \phi \models_{gf} \psi,$$

and then

$$\phi \vdash_{K_s} \psi \Leftrightarrow \mu \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi).$$

In Section 4.1, we will show that μ^+ is a nonconservative extension of μ. This gives rise to a natural question: can we change the translation to make Ω and μ equiparable? In such a case, we would be able to capture derivability in K_s not only for complete formulae, as in [8], but in general. The answer is positive. The generalization of the proposed set-theoretic translation method to any (possibly incomplete) modal formula is given in Section 4.2.

4.1 Is μ^+ conservative over μ?

PROPOSITION 4.1 *The theory $\mu^+ = \mu + \forall x \exists \mathtt{P} \forall y \ (y \in \mathtt{P} \leftrightarrow xRy)$ is non-conservative (w.r.t. modal formulae) over μ.*

Proof. The proof is a variant of the proof given in [3], where it is proved that the theory μ^s equal to μ plus the axiom $\forall x \exists \mathtt{P} \forall y \ (y \in \mathtt{P} \leftrightarrow x = y)$ is nonconservative over μ. The modal incomplete logic and the formula showing its incompleteness are indeed the same as in [3].

We prove that, if $\psi = \Box \diamond \top \rightarrow \Box(\Box(\Box P \rightarrow P) \rightarrow P)$ and $\phi = \Box \diamond \top \rightarrow \Box \bot$, then

$$\mu \not\vdash \overline{ST}(\psi) \rightarrow \overline{ST}(\phi) \quad \text{(or, equivalently, } \psi \not\models_{gf} \phi),$$

but

$$\mu^+ \vdash \overline{ST}(\psi) \rightarrow \overline{ST}(\phi).$$

The proof that $\psi \not\models_g \phi$ is given in [3]. Let us prove now that $\mu^+ \vdash \overline{ST}(\psi) \rightarrow \overline{ST}(\phi)$. The proof is model-theoretic: we suppose that a general frame $(W, R, \mathcal{W})$ satisfying μ^+ is given in which ϕ is not valid, and then prove that such a general frame invalidates ψ too.

If ϕ is not valid, then there exists a world x in the general frame such that $\Box \diamond \top$ is true at x and $\exists y\ xRy$. There are two possible cases:

- all y such that xRy are reflexive;
- there exists y such that xRy and y is not reflexive.

Let us consider the first case. We know that there exists $X = \{y : xRy\}$, together with its relative complement $-X$. Let us take a valuation $\models$ such that $\{w \in W : w \models \neg P)\} = X$. Consider a world y such that xRy. For all z such that yRz, either $z \in -X$, or xRz and z is reflexive; in both cases $z \models \Box P \rightarrow P$. Thus, $y \models \Box(\Box P \rightarrow P)$, but $y \not\models P$.

Consider now the second case. Let $\models$ be such that $\{w \in W : w \models P\} = \{z : yRz\}$. Since y is not reflexive, again $y \not\models P$; moreover, for all z such that yRz, $z \models P$, and then $y \models \Box(\Box P \rightarrow P)$.

In both cases, there exists at least one y such that xRy and $y \not\models \Box(\Box P \rightarrow P) \rightarrow P$. Thus,

$$x \not\models \Box \diamond T \rightarrow \Box(\Box(\Box P \rightarrow P) \rightarrow P).$$

$\dashv$

We can actually prove something stronger, namely, that μ^+ plus $\overline{ST}(\psi)$ is complete with respect to the class of frames for ψ.

PROPOSITION 4.2 *For any formula θ,*

$$\psi \models_f \theta \textit{ implies } \mu^+ \vdash \overline{ST}(\psi) \rightarrow \overline{ST}(\theta)$$

Proof. Consider the formula ϕ of Proposition (4.1). We already proved that

$$\mu^+ \vdash \overline{ST}(\psi) \rightarrow \overline{ST}(\phi),$$

and it is straightforward to show that $\phi \models_f \psi$. Therefore, in order to prove that $\psi \models_f \theta$ implies $\mu^+ \vdash \overline{ST}(\psi) \rightarrow \overline{ST}(\theta)$, it suffices to show that $\mu \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\theta)$. Since μ^+ is an extension of μ, the thesis indeed follows from $\mu^+ \vdash \overline{ST}(\psi) \rightarrow \overline{ST}(\phi)$.

Let $(W, R, \mathcal{W})$ be a general frame satisfying ϕ. Since ϕ does not involve any propositional variable, the underline frame (W, R) validates ϕ too. Hence, it validates ψ and, from $\psi \models_f \theta$, it follows that it validates θ. This implies that the general frame $(W, R, \mathcal{W})$ validates θ too. $\dashv$

4.2 A modified translation for K_s

In this section, we show how to change the translation to capture derivability in K_s. Let $Trans(x)$ be the formula $\forall y\ (y \in x \rightarrow y \subseteq x)$, and define $Cl(y, x)$ as the conjunction of the following formulas:

- $\forall z\ (z \in y \rightarrow Pow(z) \cap x \in y)$;
- $\forall z\ (z \in y \rightarrow x \setminus z \in y)$;

- $\forall z \forall u\ (z \in y \wedge u \in y \rightarrow z \cup u \in y)$.

We also modify the translation of the $\Box$:

$$(\Box\phi)^* = Pow(\phi^*) \cap x.$$

We prove that, for all modal formulae ϕ, ψ, the following holds:

$$\phi \vdash_{K_s} \psi$$

if and only if

$$\Omega \vdash \forall x \forall y\ (Trans(x) \wedge Cl(y,x) \wedge \forall \overline{z} \in y\ (x \subseteq \phi^*(x,\overline{z})) \rightarrow \forall \overline{z} \in y\ (x \subseteq \psi^*(x,\overline{z}))),$$

where if $\alpha(P_1, \ldots P_n)$ is a modal formula, $\forall \overline{z} \in y\ (x \subseteq \alpha^*(x,\overline{z}))$ stands for $\forall z_1 \ldots \forall z_n\ (z_1 \in y \wedge \ldots \wedge z_n \in y \rightarrow x \subseteq \alpha^*(x, z_1, \ldots, z_n))$.

The proof of completeness is similar to the proof of completeness for frame-complete logics given in [8].

THEOREM 4.3 (Completeness of the translation method) *For each pair of formulae ϕ, ψ,*

$$\phi \vdash_{K_s} \psi \Rightarrow \Omega \vdash \forall x \forall y\ (Trans(x) \wedge Cl(y,x) \wedge \forall \overline{z} \in y\ (x \subseteq \phi^*(x,\overline{z})) \rightarrow \forall \overline{z} \in y\ (x \subseteq \psi^*(x,\overline{z}))).$$

Proof. The proof is by induction on the derivation $\phi \vdash_{K_s} \psi$. We skip the verification of closure under modus ponens, tautologies, necessitation rule, and K_s-axiom. We only prove the case of closure under substitution rule.

Let $\psi(P_1, ..., P_n)$ such that

$$\Omega \vdash \forall x \forall y\ (Trans(x) \wedge Cl(y,x) \wedge \forall \overline{z} \in y\ (x \subseteq \phi^*(x,\overline{z})) \rightarrow \forall \overline{z} \in y\ (x \subseteq \psi^*(x,\overline{z}))).$$

and let $\tau_1(P_1, .., P_m), ..., \tau_n(P_1, .., P_m)$ be modal formulae. We want to prove that

$$\Omega \vdash \forall x \forall y\ (Trans(x) \wedge Cl(y,x) \wedge \forall \overline{z} \in y\ (x \subseteq \phi^*(x,\overline{z})) \rightarrow \forall \overline{z} \in y\ (x \subseteq \psi(\tau_1, \ldots, \tau_n)^*(x,\overline{z}))).$$

This is easily shown because:

(i) For any modal formula $\sigma(P_1, ..., P_m)$ it holds that

$$\Omega \vdash Cl(y,x) \rightarrow \forall x_1, ..., \forall x_m\ (x_1 \in y \wedge ... \wedge x_m \in y \rightarrow \sigma^*(x_1, ..., x_m) \in y);$$

(ii) $\psi(\tau_1, ..., \tau_n)^*$ is the same (syntactically) as $\psi^*(\tau_1^*|x_1, ..., \tau_n^*|x_n)$ (both are easily proved by induction).

By simultaneous substitution, it holds that

$$\Omega \vdash \forall x_1, ..., \forall x_n\ (x_1 \in y \wedge ... \wedge x_n \in y \rightarrow x \subseteq \psi^*(x, x_1, .., x_n)) \rightarrow$$

$$\forall x_1, ..., \forall x_m\ (\tau_1^* \in y \wedge ... \wedge \tau_n^* \in y \rightarrow x \subseteq \psi^*(x, \tau_1^*, .., \tau_n^*)),$$

By applying *(i)* to terms $\tau_1^*, .., \tau_n^*$, it follows that

$$\Omega \vdash Cl(y,x) \wedge \forall x_1, ..., \forall x_n\ (x_1 \in y \wedge ... \wedge x_n \in y \rightarrow x \subseteq \psi^*(x, x_1, ...x_n)) \rightarrow$$

$$\forall x_1, ..., \forall x_m\ (x_1 \in y \wedge ... \wedge x_m \in y \rightarrow x \subseteq \psi^*(x, \tau_1^*, .., \tau_n^*)),$$

and from *(ii)* the desired result follows. $\dashv$

Let us prove now the soundness of the translation method. First of all, we show how an Ω model can be obtained from $(W, R, \mathcal{W})$. Without loss of generality, we suppose that the elements of W are sets of the same rank α.

DEFINITION 4.4 *The Ω-model $(U, \in^U, \ldots)$ generated by $(W, R, \mathcal{W})$ is defined as follows.*
Let

$$\begin{aligned} U_0 &= W, \\ &\vdots \\ U_n &= Pow(U_{n-1}) \cup U_{n-1}, \end{aligned}$$

and let $U = \bigcup_{n=0}^{\omega} U_n$.
Define a function $F : U \to U$ as

$$F(x) = \begin{cases} \{v \in W : xRv\} & \text{if } x \in W \\ x & \text{otherwise} \end{cases}$$

A structure for the language $\{\in\}$ is obtained by taking U as domain, and defining

$$x \in^U y \text{ if and only } x \in F(y).$$

First notice that, if $y \in W$, then we have:

$$x \in^U y \text{ iff } x \in W \wedge yRx.$$

Moreover, it is immediate to see that the range of F is the set $U \setminus W$. This set is closed under union and difference, and F is the identity on $U \setminus W$. Therefore, we can define

$$x \cup^U y = F(x) \cup F(y) \qquad x \setminus^U y = F(x) \setminus F(y)$$

and the axioms for $\cup$ and $\setminus$ will be satisfied. Finally, the set $\{y \in U : F(y) \subseteq F(x)\}$ is in $U \setminus W$ too, and we can safely define

$$Pow^U(x) = \{y \in U : F(y) \subseteq F(x)\}.$$

THEOREM 4.5 (Soundness of the translation method) *For each pair of formulae ϕ, ψ,*

$$\Omega \vdash \forall x \forall y \ (Trans(x) \wedge Cl(y, x) \wedge \forall \overline{z} \in y \ (x \subseteq \phi^*(x, \overline{z})) \to \forall \overline{z} \in y \ (x \subseteq \psi^*(x, \overline{z}))) \Rightarrow \phi \vdash_{K_s} \psi$$

Proof. We prove that $\phi \vdash_{K_s} \psi$ by showing that, for any general frame $(W, R, \mathcal{W})$, if ϕ is valid in $(W, R, \mathcal{W})$, then ψ is valid too.

As before, $(U, \in^U) \models Trans(W)$. Moreover, it is straightforward to show that $Closure(\mathcal{W}, W)$ holds in $(U, \in^U)$: closure under union and complementation w.r.t. W follows from the corresponding closure properties of $\mathcal{W}$, while, for all $x \in \mathcal{W}$,

$$Pow^U(x) \cap W = \{y \in W : F(y) \subseteq F(x)\} = \{y \in W : \{v \in W : yRv\} \subseteq x\} = \Box(x) \in \mathcal{W}.$$

It is now easy to show that, for all $y_1, ..., y_n$ in $\mathcal{W}$ and for any modal formula $\phi(P_1, \ldots, P_n)$,

$$W \subseteq \phi^*(W, y_1, ..., y_n)) \text{ holds in } (U, \in^U, \ldots)$$

if and only if

$$(W, R) \models \phi(P_1, .., P_n), \text{ where } \{w \in W : w \models P_i\} = y_i, \text{ for all } i = 1, ..., n.$$

By using this correspondence, one easily shows that, if ϕ is valid in $(W, R, \mathcal{W})$, then ψ is valid too. $\dashv$

5 On the strengthening of K_s

In this section we consider a finite set of operations and a translation suitable to capture L_2-derivability in a set-theoretic context. We will work with a set theory whose axioms are closely related to the so-called Gödel operations for defining the constructible universe. Among all possible approaches, we choose (essentially) the one proposed by Barwise in [2]. We introduce some natural functions defining suitable cartesian products ($\times, \times_=, \times_\in$) and their projections (Dom, Rng), plus some operations allowing us to manipulate argument positions in ordered sequences (C_1, C_2). As far as the translation is concerned, we will need a slight modification of the previously introduced ones in order to properly distribute the closure requirements providing the set-theoretic counterpart of L_0-definability (see Theorem (5.4) below).

Let us consider the theory Ω_c defined as follows. The language of Ω_c has $=, \in$, and $\subseteq$ as predicate symbols, $\{\}$, Dom, and Rng as unary functional symbols, and $\cup, \setminus, \times, \times_\in, \times_=, C_1$, and C_2 as binary functional symbols. The axioms for Ω_c are the identity axioms and the axioms, already in Ω, describing $\subseteq, \cup$, and $\setminus$ in terms of $\in$, plus the axioms defining $\times, \times_\in, \times_=, Dom, Rng, C_1$, and C_2. There are two main differences between our set operators and more classical ones for the constructible universe. First, we use the *binary* union instead of the *unary* union. Moreover, even though there is no axiom defining the singleton operator (see the list of axioms below), a unary functional symbol $\{\}$, behaving as the usual singleton operator on part of the domain, is in the language of Ω_c.

Besides the axioms for equality, the axioms for Ω_c are the following:

$t \in x \cup y \leftrightarrow t \in x \vee t \in y$;
$t \in x \setminus y \leftrightarrow t \in x \wedge t \notin y$;
$x \subseteq y \leftrightarrow \forall t\ (t \in x \rightarrow t \in y)$;
$t \in x \times y \leftrightarrow \exists a \in x \exists b \in y\ (t = \langle a, b\rangle \wedge Pair(a, b))$;
$t \in x \times_\in y \leftrightarrow \exists a \in x \exists b \in y\ (t = \langle a, b\rangle \wedge Pair(a, b) \wedge a \in b)$;
$t \in x \times_= y \leftrightarrow \exists a \in x \exists b \in y\ (t = \langle a, b\rangle \wedge Pair(a, b) \wedge a = b))$;
$t \in Dom(x) \leftrightarrow \exists s\ (\langle t, s\rangle \in x \wedge Pair(t, s))$;
$t \in Rng(x) \leftrightarrow \exists s\ (\langle s, t\rangle \in x \wedge Pair(s, t))$;
$t \in C_1(x, y) \leftrightarrow \exists a \exists b \exists c\ (\langle a, b\rangle \in x \wedge Pair(a, b) \wedge c \in y \wedge Pair(b, c) \wedge t = \langle a, \langle b, c\rangle\rangle \wedge Pair(a, \langle b, c\rangle))$;
$t \in C_2(x, y) \leftrightarrow \exists a \exists b \exists c\ (\langle a, c\rangle \in x \wedge Pair(a, c) \wedge b \in y \wedge Pair(b, c) \wedge t = \langle a, \langle b, c\rangle\rangle \wedge Pair(a, \langle b, c\rangle))$.

The meaning of $Pair(x, y)$ is that the elements x, y can be used to build an ordered pair in the Kuratowski's style. More formally,

- $\langle x, y\rangle = \{\{x\}\} \cup \{\{x\} \cup \{y\}\}$, and
- $Pair(x, y) = Sing(x) \wedge Sing(y) \wedge Sing(\{x\}) \wedge Sing(\{x\} \cup \{y\})$,

where $Sing(x)$ is a shorthand for $\forall t\ (t \in \{x\} \leftrightarrow t = x)$.

If $Pair(a, b)$ holds, then $\langle a, b\rangle$ behaves as the usual encoding of ordered pairs. In particular, notice that $Pair(a, b) \wedge Pair(c, d) \Rightarrow (\langle a, b\rangle = \langle c, d\rangle \leftrightarrow a = c \wedge b = d)$.

Let us now define a translation of modal formulae into Ω_c-terms that captures derivability in L_2 (in Section 6, we will show how this result can be generalized to the full L_2-language). We will prove that

$$L_2 \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi)$$

if and only if

$$\Omega_c \vdash \forall x(Cl(x) \wedge \forall \overline{z}\ (x \subseteq \phi^*(x, \overline{z})) \rightarrow \forall \overline{z}\ (x \subseteq \psi^*(x, \overline{z}))),$$

where $Cl(x)$ stands for $\exists y(y \in x)$ (x is not empty), and

$$\forall y \forall z\ (y \in x \wedge z \in x \rightarrow Pair(y,z)) \wedge \forall y \forall u \forall v\ (Pair(u,v) \wedge y \in x \rightarrow Pair(y, \langle u,v \rangle) \wedge Pair(\langle u,v \rangle, y)).$$

The translation function $(\cdot)^*$ is the same as before except for the $\Box$ operator, and this is obviously the case, since we do not have Pow among the symbols in the language of Ω_c. Moreover, notice that $Trans(x)$ disappeared from the antecedent of the translated sentence. It can be easily checked that, in the case of Ω, it would have made no difference to work with $(\Box\, \phi)^*$ defined either as $Pow(\phi^*)$ or as $Pow(\phi^*) \cap x$. Actually, we chose the first alternative only to maintain the translated terms simpler. It is also easy to see that $Pow(\phi^*) \cap x = \{y \in x : y \cap x \subseteq \phi^*\}$, whenever x is transitive. As a matter of fact, we will see that the set $\{y \in x : y \cap x \subseteq \phi^*\}$ can always be used to translate $\Box\phi$ (even in the case in which x is not transitive). Moreover, the reader can easily check that

$$\{y \in x : y \cap x \subseteq \phi^*\} = x \setminus Rng((x \setminus \phi^*) \times_{\in} x),$$

and hence we put:

$$(\Box\, \phi)^* = x \setminus Rng((x \setminus \phi^*) \times_{\in} x).$$

It is worth noting that the introduction of an axiom for the singleton in the theory and the addition of *Cl(x)* as a new conjunct in the antecedent of the translation (as we did) are not equivalent. In the first case, there is the singleton of any set (in any model of the theory); in the second case, the existence of the singletons is only guaranteed for the elements of any set x for which *Cl(x)* holds. As a consequence, we cannot exclude that the set of subsets of a given x (such that *Cl(x)* holds) in a model of Ω_c could be (strictly) smaller than the set of subsets of the corresponding x in a model of Ω_c plus the singleton axiom. Assuming the singleton axiom makes indeed models richer, and this may turn out to force the presence of subsets of W which are not L_0-definable. In Lemma (5.13) we prove that the subsets of a suitable x (such that *Cl(x)* holds) in a suitable model of Ω_c are exactly the L_0-definable sets.

We start proving the completeness of the translation. First of all, we construct a general frame corresponding to a suitable element of a model of Ω_c.

DEFINITION 5.1 *Let $(U, \in^U, \ldots)$ be a model for Ω_c, and let x_0 be an element of U such that $U \models Cl(x_0)$. The general frame* generated *by x_0 in U is the triplet $(W, R, \mathcal{W})$ where:*

- $W = \{w \in U : w \in^U x_0\}$,
- *for all $w, w' \in W$, $wRw' \Leftrightarrow w' \in^U w$,*
- $\mathcal{W} = \{\{w \in U : w \in^U y\} : y \in U \wedge y \subseteq^U x_0\}$.

In order to show that a generated general frame is a general frame closed under L_0-definitions, we prove that the set of all n-tuples of worlds, defined by a first-order formula in $(W, R, \mathcal{W})$, can be obtained by means of the set-theoretic operations of Ω_c.

In the following, given $y \in U$ such that $y \subseteq^U x_0$, we will identify it with the element $\{w \in U : w \in^U y\}$ of $\mathcal{W}$ in the generated general frame; in particular, we will identify x_0 with W.

LEMMA 5.2 *For every L_0-formula $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$, whose free variables are among $x_1, \ldots, x_n$ (world variables) and $Y_1, \ldots, Y_k$ (set variables), there exists a term $F_\alpha(X_1, \ldots, X_n, Y_1, \ldots, Y_k)$ in the language of Ω_c such that, for all $V_1, \ldots, V_n, W_1, \ldots, W_k$ in $\mathcal{W}$, and $u \in U$,*

$$U \models u \in F_\alpha(V_1, \ldots, V_n, W_1, \ldots, W_k)$$

if and only if
there exist $w_1 \in V_1, \ldots, w_n \in V_n$ such that

a) $U \models u = \langle w_n, w_{n-1} \ldots, w_1 \rangle$,
b) $(W, R) \models \alpha(w_1, \ldots, w_n, W_1, \ldots, W_k)$.

The proof of Lemma (5.2) (as well as all the missing proofs of the following lemmas) is given in the Appendix.

The next lemma shows that any generated general frame is actually a general frame closed under L_0-definitions.

LEMMA 5.3 *Let $(U, \in^U, \ldots)$ be a model for Ω_c, and let x_0 be an element of U such that $U \models Cl(x_0)$. The general frame $(W, R, \mathcal{W})$ generated by x_0 in U is a general frame closed under L_0-definitions.*

Proof. Let $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ be an L_0-formula, $a_2, \ldots, a_n \in W$, and $W_1, \ldots, W_k \in \mathcal{W}$. We need to show that the set $\{a_1 \in W : (W, R) \models \alpha(a_1, a_2 \ldots, a_n, W_1, \ldots, W_k)\}$ belongs to $\mathcal{W}$. Since $Cl(x_0)$ holds in U, the sets $\{a_2\}, \ldots, \{a_n\}$ belong to $\mathcal{W}$ and, applying Lemma (5.2) with $V_1 = W, V_2 = \{a_2\}, \ldots, V_n = \{a_n\}$, we have that, for every $u \in U$,

$$U \models u \in F_\alpha(W, \{a_2\}, \ldots, \{a_n\}, W_1, \ldots, W_k)$$

if and only if
there exists $a_1 \in W$ such that

a) $U \models u = \langle a_n, a_{n-1} \ldots, a_1 \rangle$,
b) $(W, R) \models \alpha(a_1, \ldots, a_n, W_1, \ldots, W_k)$.

It follows that, for every $a_1 \in U$,

$$U \models a_1 \in Rng^{n-1}(F_\alpha(W, \{a_2\}, \ldots, \{a_n\}, W_1, \ldots, W_k)$$

if and only if

$$a_1 \in W \text{ and } (W, R) \models \alpha(a_1, a_2 \ldots, a_n, W_1, \ldots, W_k).$$

Since $Rng^{n-1}(F_\alpha(W, \{a_2\}, \ldots, \{a_n\}, W_1, \ldots, W_k)) \subseteq^U W$, the set

$$\{a_1 \in W : (W, R) \models \alpha(a_1, a_2 \ldots, a_n, W_1, \ldots, W_k)\}$$

is equal to the set $\{a_1 \in U : a_1 \in^U Rng^{n-1}(F_\alpha(W, \{a_2\}, \ldots, \{a_n\}, W_1, \ldots, W_k)\}$, and thus it belongs to $\mathcal{W}$. Hence $(W, R, \mathcal{W})$ is closed under L_0-definability. In particular, notice that the closure under union, relative complementation, and $\Box$ follows, since, if $W_1, W_2 \in \mathcal{W}$, then $W_1 \cup W_2$, $W \setminus W_1$, and $\Box(W_1)$ are characterized by L_0-definitions from parameters in $\{W_1, W_2\}$. $\dashv$

On the basis of the previous results, we can now prove the completeness of the translation.

THEOREM 5.4 (Completeness of the translation method) *For any pair of formulae ϕ, ψ,*

$$L_2 \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi) \Rightarrow \Omega_c \vdash \forall x(Cl(x) \wedge \forall \overline{z}\ (x \subseteq \phi^*(x, \overline{z})) \rightarrow \forall \overline{z}\ (x \subseteq \psi^*(x, \overline{z}))).$$

Proof. Let $(U, \in^U, \ldots)$ be a model for Ω_c, and let x_0 be an element of U such that $U \models Cl(x_0)$. We already proved that the general frame $(W, R, \mathcal{W})$, generated by x_0, is closed under L_0-definitions. To prove the thesis, it suffices to show that a modal formula ϕ is valid in $(W, R, \mathcal{W})$ (equivalently $(W, R, \mathcal{W}) \models \overline{ST}(\phi)$) if and only if $U \models \forall \overline{z}\ (x_0 \subseteq \phi^*(x_0, \overline{z}))$.

We can proceed, basically, as in Section 3. It is not difficult to prove, indeed, that, for all $w \in W$ and $y_1, \ldots, y_n \in U$,

$$U \models w \in \phi^*(W, y_1, \ldots, y_n) \Leftrightarrow (W, R, \mathcal{W}) \models ST(\phi)(w, y_1 \cap W, \ldots, y_n \cap W),$$

from which the thesis easily follows.

We just check the property for the $\Box$ operator, leaving the remaining cases to the reader:

$$U \models w \in (\Box\,\phi)^*(W, y_1, \ldots, y_n) \Leftrightarrow U \models w \in W \setminus Rng((W \setminus \phi^*) \times_\in W) \Leftrightarrow$$

$$U \models w \notin Rng((W \setminus \phi^*) \times_\in W) \text{ (since } U \models w \in W) \Leftrightarrow U \models \forall s\ (Pair(s,w) \rightarrow \langle s,w\rangle \notin (W \setminus \phi^*) \times_\in W) \Leftrightarrow$$

$$U \models \forall s\ (Pair(s,w) \wedge s \in W \wedge s \in w \rightarrow s \in \phi^*) \Leftrightarrow U \models \forall s\ (s \in W \wedge s \in w \rightarrow s \in \phi^*)$$

$$\text{(because } U \models Cl(W) \text{ and, if } s, w \in W \text{ then } U \models Pair(s,w)) \Leftrightarrow$$

$$\forall s \in W\ (wRs \rightarrow (W,R,\mathcal{W}) \models ST(\phi)(s, y_1 \cap W, \ldots, y_n \cap W)) \Leftrightarrow (W,R,\mathcal{W}) \models ST(\Box\,\phi)(w, y_1 \cap W, \ldots, y_n \cap W).$$

$\dashv$

Let us prove now the correctness of the translation (Theorem (5.14) below). The structure of the proof is similar to the one of Theorem (5.4): we will first define a suitable notion of Ω_c-model *generated* by a general frame $(W,R,\mathcal{W})$ closed under L_0-definability; then, we will prove the following two claims:

- the generated Ω_c-model is actually a model of Ω_c;
- a modal formula ϕ is valid in the general frame $(W,R,\mathcal{W})$ if and only if $\forall \overline{z}(W \subseteq \phi^*(W,\overline{z}))$ holds in the generated Ω_c-model.

In the following, $\in, \subseteq, \{\}, Dom, Rng, \cup, \setminus, \times, \times_=, C_1, C_2$ denote standard (meta-theoretic) relations and operations. For example, $C_1(x,y)$ is the set having as elements all sets t such that $t = \langle a, \langle b, c\rangle\rangle$ with $\langle a,b\rangle \in x$ and $c \in y$, where $\langle -,-\rangle$ denotes the standard pair. Moreover, if R is a binary relation, then $x \times_{\in,R} y$ denotes the set $\{\langle a,b\rangle : a \in x \wedge b \in y \wedge a \in b\} \cup \{\langle a,b\rangle : a \in x \wedge b \in y \wedge bRa\}$.

From now on we will denote by $(W,R,\mathcal{W})$ a general frame closed under L_0-definitions. Without loss of generality, we can assume that the elements of W are sets having the same rank, and no element of W is a pair or a singleton. From such a general frame, we obtain an interpretation $(U_c, \in^{U_c}, \ldots)$ of Ω_c as follows. The domain of the interpretation is built starting from all ordered sequences of W, adding the distinguished subsets of $\mathcal{W}$, and closing under the Ω_c set operations. The membership relation and the interpretation of the function symbols are defined in the same way as in Definition (4.4).

DEFINITION 5.5 *The Ω_c-model $(U_c, \in^{U_c}, \ldots)$ generated by $(W,R,\mathcal{W})$ is defined as follows.*

Domain of interpretation U_c:

$W^s = \overline{W_0^s} = \{\{w\} : w \in W\}$;

$\overline{W_{(n+1)}^s} = \{\{t_1\}, t_1 \cup t_2 : t_1, t_2 \in \bigcup_{i=0}^n \overline{W_i^s}\}$;

$\overline{W^s} = \bigcup_{i=0}^\omega \overline{W_i^s}$;

$\mathcal{W}^c = \overline{\mathcal{W}_0^c} = \mathcal{W} \cup \overline{W^s}$;

$\overline{\mathcal{W}_{(n+1)}^c} = \{Dom(t_1), Rng(t_1), t_1 \circ t_2 : \circ \in \{\cup, \setminus, \times, \times_{\in,R}, \times_=, C_1, C_2\}, t_1, t_2 \in \bigcup_{i=0}^n \overline{\mathcal{W}_i^c}\}$;

$\overline{\mathcal{W}^c} = \bigcup_{i=0}^\omega \overline{\mathcal{W}_i^c}$;

$U_c = W \cup \overline{\mathcal{W}^c}$.

Membership relation $\in^{U_c}$*:*

$$x \in^{U_c} y \text{ iff } x \in F(y) \text{ for } x, y \in U_c,$$

where

$$F(y) = \begin{cases} \{w \in W : yRw\} = R(y) & \text{if } y \in W, \\ y & \text{otherwise.} \end{cases}$$

Interpretation of function symbols:

- $\{z\}^{U_c} = \begin{cases} \{z\} & \text{if } z \in W \cup \overline{W^s}, \\ \emptyset & \text{otherwise} \end{cases}$;
- $z \circ^{U_c} r = F(z) \circ F(r)$, *for* $\circ$ *in* $\{\cup, \setminus, \times, \times_=, C_1, C_2\}$;
- $z \times_{\in}^{U_c} r = F(z) \times_{\in,R} F(r)$;
- $Dom^{U_c}(z) = Dom(F(z))$, *and* $Rng^{U_c}(z) = Rng(F(z))$.

where the operations on the right-hand side of the equalities are the standard ones.

Notice that we used standard operations to build the domain of U_c, but the interpreted functions of the model are non-standard. Moreover, since $W \in \mathcal{W}$, we have that W is an element of the generated model U_c.

OBSERVATION 5.6 *If* $a, b \in W \cup \overline{W^s}$*, then we have* $U_c \models Pair(a, b)$ *and* $\langle a, b\rangle = \langle a, b\rangle^{U_c}$*. If* $a \in W \cup \overline{W^s}$ *(resp.* $b \in W \cup \overline{W^s}$*), but* $b \notin W \cup \overline{W^s}$ *(resp.* $a \notin W \cup \overline{W^s}$*), then* $U_c \not\models Pair(a, b)$ *and* $\langle a, b\rangle^{U_c} = \langle a, a\rangle^{U_c} = \langle a, a\rangle$ *(resp.* $\langle a, b\rangle^{U_c} = \langle b, b\rangle^{U_c} = \langle b, b\rangle$*). If* $a, b \notin W \cup \overline{W^s}$*, then* $U_c \not\models Pair(a, b)$ *and* $\langle a, b\rangle^{U_c} = \emptyset$.

In order to show that the Ω_c-model generated by $(W, R, \mathcal{W})$ is indeed an Ω_c-model, we first need to prove the following Lemma.

LEMMA 5.7 *The following properties hold:*

a) $t \in \overline{\mathcal{W}^c} \Rightarrow t \subseteq W \cup \overline{W^s}$;

b) $t \in U_c \Rightarrow t \subseteq^{U_c} W \cup \overline{W^s}$.

Proof. a) The proof is by induction on the structure of t. If $t \in \mathcal{W}^c$, then either $t \in \mathcal{W}$ or $t \in \overline{W^s}$. If $t \in \mathcal{W}$, then $t \subseteq W$ (by definition of $\mathcal{W}$). If $t \in \overline{W^s}$, then it is easy to prove that $t \subseteq W \cup \overline{W^s}$ (by induction).

Consider now the inductive step. The cases of $t = t_1 \cup t_2$ and $t = t_1 \setminus t_2$ are straightforward, and thus omitted.

If $t = t_1 \times t_2$ and $u \in t$, then there exist $a \in t_1$ and $b \in t_2$ such that $u = \langle a, b\rangle$. By induction, we have $a, b \in W \cup \overline{W^s}$ and thus $u = \langle a, b\rangle \in W \cup \overline{W^s}$, because this set is closed for ordered pairs. The case of $t_1 \times_{\in,R} t_2, t_1 \times_= t_2$ are immediately solved by noticing that both are subsets of $t_1 \times t_2$.

If $t = Dom(t_1)$ and $u \in t$, then there exists s such that $\langle u, s\rangle \in t_1$ and thus $\langle u, s\rangle \in W \cup \overline{W^s}$. Since no element of W is a pair, $\langle u, s\rangle \in \overline{W^s}$, and thus $\langle u, s\rangle \subseteq W \cup \overline{W^s}$. Moreover, since $\{u\} \in \langle u, s\rangle$, we have $\{u\} \in W \cup \overline{W^s}$. Finally, since $\{u\} \notin W$, we have $\{u\} \in \overline{W^s}$, and therefore $\{u\} \subseteq W \cup \overline{W^s}$. This allows us to conclude that $u \in W \cup \overline{W^s}$. The case of $t = Rng(t_1)$ is similar, and thus omitted.

If $t = C_1(t_1, t_2)$ and $u \in t$ then there are a, b, c such that $\langle a, b\rangle \in t_1$, $c \in t_2$ and $u = \langle a, \langle b, c\rangle\rangle$. As before, it follows $a, b, c \in W \cup \overline{W^s}$, and therefore $u = \langle a, \langle b, c\rangle\rangle \in \overline{W^s}$. The case of $t = C_2(t_1, t_2)$ is similar.

b) Since all elements of W have the same rank, a) implies that $W \cap \overline{\mathcal{W}^c} = \emptyset$. If $t \in \overline{\mathcal{W}^c}$, then $F(t) = t$ and the thesis follows from a). Otherwise, $t \in W$ and $t \subseteq^{U_c} W$.
$\dashv$

The important feature of U_c expressed by b) in Lemma (5.7) is the fact that the extension of any $z \in U_c$ (w.r.t. U_c) is contained in $W \cup \overline{W^s}$.

LEMMA 5.8 *The interpretation $(U_c, \in^{U_c}, \ldots)$ generated by a general frame closed under L_0-definitions is an Ω_c-model.*

Proof. It is easy to see that $(U_c, \in^{U_c}, \ldots)$ satisfies the axioms for union and difference.
Consider the case of $\times$:

$$x \in^{U_c} y \times^{U_c} z \Leftrightarrow x \in F(y \times^{U_c} z) \Leftrightarrow x \in F(F(y) \times F(z)) \Leftrightarrow$$

$$x \in F(y) \times F(z) (\text{since } F(y) \times F(z) \text{ is never in } W) \Leftrightarrow$$

$$\exists a \exists b (a \in F(y) \wedge b \in F(z) \wedge x = \langle a, b \rangle) \Leftrightarrow \exists a \exists b (a \in^{U_c} y \wedge b \in^{U_c} z \wedge x = \langle a, b \rangle^{U_c} \wedge Pair^{U_c}(a, b)).$$

All equivalences are rather straightforward except for (the right direction of) the last one. First, since $y \in U_c$, then $F(y) \in \overline{\mathcal{W}^c}$. From Lemma (5.7 a)) and the definition of $\in^{U_c}$, it follows that $a \in F(y) \rightarrow a \in U_c \wedge a \in^{U_c} y$ - the same for b, z. Moreover, if $a \in^{U_c} y$, $b \in^{U_c} z$, then Lemma (5.7) implies that $a, b \in W \cup \overline{W^s}$, and, by Observation (5.6), we can conclude that $U_c \models Pair(a, b)$, and the standard pair $\langle a, b \rangle$ coincide with $\langle a, b \rangle^{U_c}$.
The proof for the other axioms is given in the Appendix. $\dashv$

In order to prove the second Claim we show in Lemma (5.13) that every set in U_c, constructed by means of the Ω_c-operations, has its members defined by an L_0-formula involving parameters from W and sets from $\mathcal{W}$.

We first need a number of auxiliary definitions and results.

DEFINITION 5.9 *A c-term $t(x_1, .., x_n)$ is either a variable x_i or a term $t'(\{x_1\}, \ldots, \{x_n\})$, where the term $t'(y_1, \ldots, y_n)$ is a term in $\{\}$ and $\cup$.*

Notice that $u \in \overline{W^s}$ if and only if there exists a c-term $t(x_1, \ldots x_n)$ which is not a variable and $w_1, \ldots w_n \in W$ such that $U_c \models u = t(w_1, \ldots w_n)$.

LEMMA 5.10 *Let $t(x_1, .., x_n)$ be a c-term. If it is not a variable, then there exist k c-terms $t_1(x_1, .., x_n), \ldots, t_k(x_1, \ldots, x_n)$ such that, for all $w_1, \ldots, w_n$ in W,*

$$U_c \models t(w_1, \ldots, w_n) = \{t_1(w_1, \ldots, w_n)\} \cup \ldots \cup \{t_k(w_1, \ldots, w_n)\}.$$

Notice that if t_i is a c-term, then $\{t_i(w_1, \ldots, w_n)\}^{U_c} = \{t_i(w_1, \ldots, w_n)\}$.

LEMMA 5.11 *If $t(x_1, .., x_n)$ and $s(y_1, .., y_l)$ are c-terms, then there are two L_0-formulae $\gamma(x_1, .., x_n, y_1, .., y_l)$ and $\delta(x_1, .., x_n, y_1, .., y_l)$, which are boolean combinations of atomic formulae of the forms $z_i = z_j$ and $z_i R z_j$, such that, for all $w_1, \ldots, w_n, v_1, \ldots, v_l$ in W,*

a) $U_c \models t(w_1, .., w_n) = s(v_1, .., v_l)$ if and only if $(W, R) \models \gamma(w_1, .., w_n, v_1, .., v_l)$,

b) $U_c \models t(w_1, .., w_n) \in s(v_1, .., v_l)$ if and only if $(W, R) \models \delta(w_1, .., w_n, v_1, .., v_l)$.

LEMMA 5.12 *For each c-term $t(x_1,\ldots,x_m)$, there exist d c-terms $t^1(x_1,\ldots,x_m),\ldots,t^d(x_1,\ldots,x_m)$, and d^2 L_0-formulae $\theta^t_{e,f}(x_1,...,x_m)$, with $e,f\in\{1,\ldots d\}$, such that, for all $w_1,\ldots,w_m\in W$ and $a,b\in W\cup\overline{W^s}$,*

$$U_c \models t(w_1,\ldots,w_m)=\langle a,b\rangle$$

if and only if
there exist $e,f\in\{1,\ldots d\}$ such that

a) $U_c\models a=t^e(w_1,\ldots,w_m)\wedge b=t^f(w_1,\ldots,w_m)$;
b) $(W,R)\models\theta^t_{e,f}(w_1,\ldots,w_m)$.

We are now ready to prove the main lemma.

LEMMA 5.13 *For any term $\tau(X_1,\ldots,X_n)$ built using the functions $\cup,\setminus,\times,\times_\in,\times_=,Dom,Rng,C_1,C_2$, whose free variables are among $X_1,\ldots,X_n$, and for any $A_1,\ldots,A_n$ in $\mathcal{W}^c$, there are:*

- *k c-terms $t_1(x_1,\ldots,x_m),\ldots,t_k(x_1,\ldots,x_m)$,*
- *k L_0-formulae $\alpha^\tau_1(x_1,\ldots,x_m,Y_1,\ldots,Y_r),\ldots,\alpha^\tau_k(x_1,\ldots,x_m,Y_1,\ldots,Y_r)$,*
- *r sets $W_1,\ldots,W_r$ in $\mathcal{W}$,*

such that, for all $u\in U_c$,

$$U_c\models u\in\tau(A_1,\ldots,A_n)$$

if and only if
there exist $i\in\{1,\ldots,k\}$ and $w_1,\ldots,w_m\in W$ such that

a) $U_c\models u=t_i(w_1,\ldots,w_m)$,
b) $(W,R)\models\alpha^\tau_i(w_1,\ldots,w_m,W_1,\ldots,W_r)$.

Proof. By induction on the structural complexity of the term $\tau(X_1,\ldots,X_n)$.

Base case. $\tau(X_1,\ldots,X_n)=X_1$ and $A_1\in\mathcal{W}^c=\mathcal{W}\cup\overline{W^s}$.

If $A_1\in\mathcal{W}$, then $t_1(x_1)=x_1,\alpha^\tau_1(x_1,Y_1)=x_1\in Y_1$ and $W_1=A_1$ satisfy the required properties.

If $A_1\in\overline{W^s}$, we proceed by induction on the structure of the c-term defining A_1. If $A_1=\{v\}$ with $v\in W$, then the c-term, the L_0-formula, and the set $W_1\in\mathcal{W}$ can be defined as in the preceding case ($\{v\}\in\mathcal{W}$). If $A_1=s_1(v_1,\ldots,v_p)\cup s_2(v_1,\ldots,v_q)$, it is sufficient to take as sets of c-terms, L_0-formulae, and elements in $\mathcal{W}$ the unions of the corresponding sets for $s_1(v_1,\ldots,v_p)$ and $s_2(v'_1,\ldots,v'_q)$. If $A_1=\{s(v_1,\ldots,v_p)\}$, we take $t_1(x_1,\ldots,x_p)=s(x_1,\ldots,x_p)$, $\alpha^\tau_1(x_1,\ldots,x_p,Y_1,\ldots,Y_p)=x_1\in Y_1\wedge\ldots\wedge x_p\in Y_p$, and $W_1=\{v_1\},\ldots,W_p=\{v_p\}$.

Inductive step. The proof for $\cup$ is straightforward, and thus omitted. Let us consider the case of $\tau(X_1,\ldots,X_n)=\mu(X_1,\ldots,X_n)\setminus\nu(X_1,\ldots,X_n)$.

Fix $A_1,\ldots,A_n\in\mathcal{W}^c$.

Consider the term μ. By induction, there are

- k c-terms $m_1(x_1,\ldots,x_m),\ldots,m_k(x_1,\ldots,x_m)$,
- k L_0-formulae $\alpha^\mu_1(x_1,\ldots,x_m,Y_1,\ldots,Y_r),\ldots,\alpha^\mu_k(x_1,\ldots,x_m,Y_1,\ldots,Y_r)$,
- r sets $W_1,\ldots,W_r$ in $\mathcal{W}$,

such that, for all $u \in U_c$,

$$U_c \models u \in \mu(A_1, \ldots, A_n)$$

if and only if

there exist $i \in \{1, \ldots, k\}$ and $w_1, \ldots, w_m \in W$ such that

a) $U_c \models u = m_i(w_1, \ldots, w_m)$,
b) $(W, R) \models \alpha_i^{\mu}(w_1, \ldots, w_m, W_1, \ldots, W_r)$.

Consider the term ν. By induction there are

- h c-terms $n_1(x_1, \ldots, x_l), \ldots, n_h(x_1, \ldots, x_l)$,
- h L_0-formulae $\alpha_1^{\nu}(x_1, \ldots, x_l, Y_1, \ldots, Y_s), \ldots, \alpha_h^{\nu}(x_1, \ldots, x_l, Y_1, \ldots, Y_s)$,
- s sets $V_1, \ldots, V_s$ in $\mathcal{W}$,

such that, for all $u \in U_c$,

$$U_c \models u \notin \nu(A_1, \ldots, A_n)$$

if and only if

for all $j \in \{1, \ldots, h\}$ and for all $v_1, \ldots, v_l \in W$

a) $U_c \models u \neq n_i(v_1, \ldots, v_l)$, or
b) $(W, R) \models \neg\alpha_i^{\nu}(v_1, \ldots, v_l, V_1, \ldots, V_s)$.

It follows that

$$U_c \models u \in \tau(A_1, \ldots, A_n)$$

if and only if

$$U_c \models u \in \mu(A_1, \ldots, A_n) \text{ and } U_c \models u \notin \nu(A_1, \ldots, A_n)$$

if and only if

(there exist $i \in \{1, \ldots, k\}$ and $w_1, \ldots, w_m \in W$ such that

$$U_c \models u = m_i(w_1, \ldots, w_m) \text{ and } (W, R) \models \alpha_i^{\mu}(w_1, \ldots, w_m, W_1, \ldots, W_r))$$

and

(for all $j \in \{1, \ldots, h\}$ and $v_1, \ldots, v_l \in W$,

$$U_c \models u \neq n_j(v_1, \ldots, v_l) \text{ or } (W, R) \models \neg\alpha_j^{\nu}(v_1, \ldots, v_l, V_1, \ldots, V_s))$$

if and only if

there exist $i \in \{1, \ldots, k\}$ and $w_1, \ldots, w_m \in W$ such that

$$(U_c \models u = m_i(w_1, \ldots, w_m) \text{ and } (W, R) \models \alpha_i^{\mu}(w_1, \ldots, w_m, W_1, \ldots, W_r) \text{ and}$$

$$\bigwedge_{j=1}^{h} \forall v_1, \ldots, v_l \in W (U_c \models m_i(w_1, \ldots, w_m) \neq n_j(v_1, \ldots, v_l), \text{ or } (W, R) \models \neg\alpha_j^{\nu}(v_1, \ldots, v_l, V_1, \ldots, V_s))),$$

where the scope of the existential quantifier on i (originally limited to the first conjunct) has become the whole formula, and the u appearing in the second conjunct has been substituted by m_i,

if and only if

there exist $i \in \{1, \ldots, k\}$ and $w_1, \ldots, w_m \in W$ such that

a) $U_c \models u = m_i(w_1, \ldots, w_m)$, and
b) $(W, R) \models \alpha_i^{\mu}(w_1, \ldots, w_m, W_1, \ldots, W_r) \wedge$
$\bigwedge_{j=1}^{h} \forall v_1, \ldots, v_l(\neg\gamma_{i,j}(w_1, \ldots, w_m, v_1, \ldots, v_l) \vee \neg\alpha_j^{\nu}(v_1, \ldots, v_l, V_1, \ldots, V_l))$,

where $\gamma_{i,j}$ is obtained from m_i and n_j as in Lemma (5.12).

The proof for the other cases is given in the Appendix. $\dashv$

On the basis of the previous results, we can now prove the correctness of the translation.

THEOREM 5.14 (Soundness of the translation method) *For each pair of formulae* ϕ, ψ,

$$\Omega_c \vdash \forall x(Cl(x) \wedge \overline{z}\ (x \subseteq \phi^*(x, \overline{z})) \rightarrow \overline{z}\ (x \subseteq \psi^*(x, \overline{z}))) \Rightarrow L_2 \vdash \overline{ST}(\phi) \rightarrow \overline{ST}(\psi).$$

Proof. Consider a general frame $(W, R, \mathcal{W})$ closed under L_0-definitions. The proof is accomplished in two steps:

Step 1: we show that the generated Ω_c-model $(U_c, \in^{U_c}, \ldots)$ satisfies $U_c \models Cl(W)$;

Step 2: we prove that a modal formula θ is valid in $(W, R, \mathcal{W})$ if and only if $U_c \models \forall\overline{z}\ (W \subseteq \theta^*(W, \overline{z}))$.

First of all, if $U_c \models Pair(a, b)$, then $a \in \{a\}^{U_c}, b \in \{b\}^{U_c}$ and, from Lemma (5.7), it follows that $a, b \in \overline{W^s} \cup W$; thus, for all $w \in W$, $U_c \models Pair(w, \langle a, b\rangle)$ and $U_c \models Pair(\langle a, b\rangle, w)$. From this, it follows that $U_c \models Cl(W)$.

For step 2, it is enough to show that, for all $u \in U_c$, $F(u) \cap W \in \mathcal{W}$. Indeed, if such a condition holds, then for all $u_1, \ldots, u_n \in U_c$,

$$U_c \models w \in \theta^*(W, u_1, \ldots, u_n) \Leftrightarrow (W, R, \mathcal{W}) \models ST(\theta)(w, F(u_1) \cap W, \ldots, F(u_n) \cap W),$$

(the only nontrivial case is $\theta = \Box\gamma$; its proof is similar to the last part of the proof of Theorem (5.4)).

From this, it easily follows that

$$U_c \models \forall\overline{z}\ (W \subseteq \theta^*(W, \overline{z})) \Leftrightarrow (W, R, \mathcal{W}) \models \overline{ST}(\theta) \Leftrightarrow \text{ the formula } \theta \text{ is valid in } (W, R, \mathcal{W}).$$

Hence, let us prove that, for all $u \in U_c$, $F(u) \cap W \in \mathcal{W}$. Consider first the case in which $y \in W$; in this case $F(y) \cap W = R(y) \in \mathcal{W}$, because $\mathcal{W}$ is closed under L_0-definitions. If $y \in \overline{\mathcal{W}^c}$, then $F(y) = y$ and it is easy to show by induction on the structure of the elements of $\overline{\mathcal{W}^c}$ that there exist $A_1, \ldots, A_n$ in W^c such that $U_c \models y = \tau(A_1, \ldots, A_n)$, where τ is a term as in Lemma (5.13). By such a lemma, it follows that there exist

- c-terms $t_1(x_1, \ldots, x_m), \ldots, t_k(x_1, \ldots, x_m)$,
- L_0-formulae $\alpha_1^{\tau}(x_1, \ldots, x_m, Y_1, \ldots, Y_r), \ldots, \alpha_k^{\tau}(x_1, \ldots, x_m, Y_1, \ldots, Y_r)$,
- $W_1, \ldots, W_r$ in $\mathcal{W}$,

such that for all $u \in U_c$,

$$U_c \models u \in \tau(A_1, \ldots, A_n)$$

if and only if

there exist $i \in \{1, \ldots, k\}$ and $w_1, \ldots, w_m \in W$ such that

a) $U_c \models u = t_i(w_1, \ldots, w_m)$,
b) $(W, R) \models \alpha_i^{\tau}(w_1, \ldots, w_m, W_1, \ldots, W_r)$.

Notice that if the c-term t_i is not a variable, then, for any $w_1, \ldots, w_m$ in W, the element $t_i(w_1, \ldots, w_m)$ does not belong to W. Suppose w.l.o.g. that $t_1 = x_{l_1}, \ldots, t_s = x_{l_s}$ $(l_1, \ldots, l_s \subseteq \{1, \ldots m\})$ are all the c-terms between $t_1, .., t_k$ which are variables. Then, $F(y) \cap W$ is equal to

$$\{w \in W : (W, R) \models \bigvee_{j=1}^{s} \exists w_1 \ldots . w_m (w = w_{l_j} \wedge \alpha_j^{\tau}(w_1, \ldots, w_m, W_1, \ldots, W_r))\}.$$

Since $\mathcal{W}$ is closed under L_0-definitions, the latter set belongs to $\mathcal{W}$. $\dashv$

As we said in the introduction, a peculiar point of the technique employed in this section is the fact that the axiomatic requirements on the singleton operator are expressed *locally* on the set x by the formula $Cl(x)$ in the antecedent of the translation. At this point a rather natural question follows: could we prove the correctness of the–more elegant–translation not involving $Cl(x)$ with respect to $\Omega_c + \{\}$ (the extension of Ω_c obtained adding an axiom for the singleton operator)? Even so, we believe that the present division of set-theoretic labour between axioms and translation is a novel of interest by itself.

6 Generalization to other languages

In this section, we briefly consider the problem of applying (suitable variations of) the proposed translation method to other languages than the basic modal one, e.g. temporal, full monadic L_2.

6.1 Tense logic

Let us consider first the case of minimal tense logic TL [6]. TL extends propositional logic with two one-place operators: F (future-tense) and P (past-tense). A sound and complete axiomatic system for TL consists of the usual axioms and rules of propositional logic plus the axioms:

$G(p \to q) \to (Gp \to Gq)$ $\qquad$ $H(p \to q) \to (Hp \to Hq)$
$p \to GPp$ $\qquad$ $p \to HFp$

where G and H stand for $\neg F \neg$ and $\neg P \neg$, respectively, and the rules of (temporal) necessitation:

from ϕ to infer $G\phi$ and $H\phi$

As shown in [16], it is possible to reduce frame validity in tense logic to that in modal logic. Even though somewhat complicated, the composition of such a reduction and our translation method for modal logic would allow us to encode minimal tense logic into Ω as it stands.

As an alternative, a specific translation method for TL can be devised that slightly extends Ω, but keeps the translation simpler. In view of the one-way character of $\in$, the main issue to consider is the treatment of the *backward looking* operator P. To this end, we extend the language of Ω with a binary operator $Pow^{-y}(x)$, and add the following axiom (defining it) to Ω:

$$x \in Pow^{-y}(z) \text{ iff } \; x \in y \wedge (x, y) \subseteq z$$

where $(x, y) \subseteq z$ is a shorthand for the formula $\forall t \; (x \in t \wedge t \in y \to t \in z)$. Let us call the resulting theory Ω_{TL}. Notice that the additional axiom is only a particular instance of the comprehension schema.

The translation of propositional letters and boolean connectives is the usual one. The translation of G is the same as the translation of $\Box$ in basic modal logic. The translation of H can be defined in terms of the $Pow^{-y}(x)$ operator as follows:

$$(H\phi)^* = Pow^{-W}(\phi^*).$$

As an example, consider the translation of the formula Hp_i, which is the term $Pow^{-W}(x_i)$. By definition, $x \in Pow^{-W}(x_i)$ if and only if $x \in W \wedge \forall t(x \in t \wedge t \in W \rightarrow t \in x_i)$, and this is the intendend meaning of Hp_i when tRx corresponds to $x \in t$.

More generally, we have that

$$w \models H\phi \Leftrightarrow \forall t \in W \ (tRw \rightarrow t \models \phi)$$

corresponds to

$$w \in Pow^{-W}(\phi^*) \Leftrightarrow w \in W \wedge (w, W) \subseteq \phi^* \Leftrightarrow w \in W \wedge \forall t \ (w \in t \wedge t \in W \rightarrow t \in \phi^*).$$

It is worth noting that, unlike the case of the operator G, we do not need to require any "(inverse) transitivity" for the operator H, because if $W \subseteq \phi^*$, then we always have $W \subseteq Pow^{-W}(\phi^*)$.

PROPOSITION 6.1 *For any TL-formula $\phi(P_1, \ldots, P_n)$, the following holds:*

$$\vdash_{TL} \phi \ \Leftrightarrow \ \Omega_{TL} \vdash \forall x(Trans(x) \rightarrow \forall x_1 \ldots x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))).$$

As a further alternative, one can look for the proper subset of the operations introduced in section 5 corresponding to the new operator $Pow^{-y}(x)$. This would allow us to single out the set theory for minimal tense logic somewhere between Ω and Ω_c.

More generally, we expect that a similar generalization of the proposed translation method could be devised to obtain a set-theoretic characterization of other extensions of basic modal logic such as the Gabbay's extension of K with the irreflexivity rule [9] and the difference logic [15].

6.2 Full monadic L_2

Let us consider now the case of the full monadic L_2-language. The translation proposed in Section 5 encoded L_2-derivability of formulae of the form $\overline{ST}(\phi) \rightarrow \overline{ST}(\psi)$ into derivability in Ω_c. It is not difficult to see, however, that the proofs of Theorem (5.4) and Theorem (5.14) have a wider scope.

Let $\phi(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ be a generic L_2-formula. Consider the translation τ, introducing a new free variable x, which is inductively defined as follows:

- $\tau(x_i R x_j) = x_j \in x_i$;
- $\tau(x_i = x_j) = \ x_i = x_j$;
- $\tau(x_i \in Y_j) = x_i \in y_j$;
- $\tau(\alpha \vee \beta) = \tau(\alpha) \vee \tau(\beta)$;
- $\tau(\neg\alpha) = \neg\tau(\alpha)$;
- $\tau(\forall x_i \alpha) = \forall x_i \ (x_i \in x \rightarrow \tau(\alpha))$;
- $\tau(\forall Y_i \alpha) = \forall y_i \tau(\alpha)$.

It is possible to prove the following proposition:

PROPOSITION 6.2 *For every pair of L_2-sentences ϕ, ψ, the following holds:*

$$L_2 \vdash \phi \rightarrow \psi \Leftrightarrow \Omega_c \vdash \forall x \ (cl(x) \wedge \tau(\phi)(x) \rightarrow \tau(\psi)(x)).$$

The soundness and completeness of the translation can be proved as in Section 5. Consider, for example, the case of completeness: given an Ω_c-model U and an element x_0 of U satisfying $cl(x_0)$, we first build the general frame of Definition (5.1); then, we prove that, for each L_2-formula $\phi(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$, $w_1, \ldots, w_n \in W$, and $y_1, \ldots y_k \in U$,

$$U \models \tau(\phi)(W, w_1, \ldots, w_n, y_1, \ldots, y_k) \text{ iff } (W, R, \mathcal{W}) \models \phi(w_1, \ldots, w_n, y_1 \cap W, \ldots, y_k \cap W).$$

The same correspondence holds for the proof of soundness.

Moreover, it is possible to show that τ is a *generalization* of the translation given in Section 5: if ψ is the formula $\overline{ST}(\phi)$, where $\phi(P_1, \ldots, P_n)$ is a modal formula, then it holds that: $\Omega_c \vdash \forall x\ (cl(x) \rightarrow (\tau(\phi)(x) \leftrightarrow \forall \overline{z}\ (x \subseteq \phi^*(x, \overline{z})))$.

Conclusions and further directions

The main purpose of this paper has been to build bridges between modal deduction and set-theoretic reasoning. Let us collect some points concerning this enterprise already mentioned in our text. Our program may be extended in several ways, by systematic variation in several 'parameters' of this linkage. For instance, given a fixed translation, one can run through known modal logics extending the minimal one, and see what set theories these involve - and the same may be done in the opposite direction. In the same spirit, one could analyse the set-theoretic strength of currently popular 'additional modal rules of inference'. But also, one can extend the language of the basic modal formalism, all the way up to the full classical logic of frames, as we have seen, obtaining richer reductions. Finally, there is also a constructive aspect. Our semantical arguments can probably be reworked so as to yield effective purely combinatorial transformations between formal derivations in modal logic and our set theory. This will allow a more finely structured comparison of reasoning in both perspectives.

References

[1] P. Aczel, *Non-well-founded sets*, CSLI, Lecture Notes No. 14, 1988.

[2] J. Barwise, *Admissible sets and structure*; Springer-Verlag. 1975.

[3] J. van Benthem, Syntactic aspects of modal incompleteness theorems; *Theoria*, 45, 1979, pp. 67-81.

[4] J. van Benthem, *Modal Logic and Classical Logic*; Bibliopolis, Napoli and Atlantic Heights (N.J.), 1985.

[5] J. van Benthem and K. Doets, Higher-Order Logic, in *Handbook of Philosophical Logic*, Vol. I, D. Gabbay and F. Guenthner (eds.); D. Reidel Pub. Comp., Dordrecht-Holland, 1983, pp. 275-329.

[6] J.P. Burgess, Basic Tense Logic, in *Handbook of Philosophical Logic*, Vol. II, D. Gabbay and F. Guenthner (eds.); D. Reidel Pub. Comp., Dordrecht-Holland, 1984, pp. 89-133.

[7] G. D'Agostino, A. Montanari, and A. Policriti, *Translating Modal Formulae as Set-Theoretic Terms*; Research Report 10/94, Dipartimento di Matematica e Informatica, Università di Udine, May 1994 (also in Logic Colloquium'94).

[8] G. D'Agostino, A. Montanari, and A. Policriti, *A set-theoretic translation method for polymodal logics*; ILLC Prepublication Series, Mathematical Logic and Foundations, Amsterdam, October 1994 (a short version will appear in the Proc. of STACS '95). To appear in the *Journal of Automated Reasoning*.

[9] D. M. Gabbay, An irreflexivity lemma with applications to axiomatizations of conditions on tense frames, in *Aspects of Philosophical Logic*, U. Mönnich (ed.); D. Reidel Pub. Comp., Dordrecht-Holland, 1981, pp. 67–89.

[10] J.L. Krivine, *Introduction to axiomatic Set Theory*; D. Reidel Pub. Comp., Dordrecht-Holland, 1971.

[11] T. Jech, *Set Theory*; Pure and Applied Mathematics Series, Academic Press, 1978.

[12] A. Nonnengart, First-order modal logic theorem proving and functional simulation; in Proc. of 13^{th} International Joint Conference on Artificial Intelligence, IJCAI-93, Chambery, France, 1993, pp. 80-85.

[13] H. J. Ohlbach, Semantic-Based Translation Methods for Modal Logics; *Journal of Logic and Computation*, 1 (5), 1991.

[14] H. J. Ohlbach, Translation Methods for Non-Classical Logics: An Overview; *Bull. of the IGLP*, 1 (1), 1993, pp. 69-89.

[15] M. de Rijke, The modal logic of inequality. *Journal of Symbolic Logic*, 57, 1992, pp. 566–584.

[16] S.K. Thomason, Reduction of Tense Logic to Modal Logic II; *Theoria*, 41, 1975, pp.154–169.

Appendix

LEMMA 5.2 *For every L_0-formula $\alpha(x_1,\ldots,x_n,Y_1,\ldots,Y_k)$, whose free variables are among $x_1,\ldots,x_n$ (world variables) and $Y_1,\ldots,Y_k$ (set variables), there exists a term $F_\alpha(X_1,\ldots,X_n,Y_1,\ldots,Y_k)$ in the language of Ω_c such that, for all $V_1,\ldots,V_n,W_1,\ldots,W_k$ in $\mathcal{W}$, and $u \in U$,*

$$U \models u \in F_\alpha(V_1,\ldots,V_n,W_1,\ldots,W_k)$$

if and only if
there exist $w_1 \in V_1,\ldots,w_n \in V_n$ such that

a) $U \models u = \langle w_n, w_{n-1}\ldots,w_1\rangle$,
b) $(W,R) \models \alpha(w_1,\ldots,w_n,W_1,\ldots,W_k)$.

Proof. The proof is by induction on the structural complexity of α, and it is very similar to the proof of Lemma 6.1, pg. 64, in [2].

By possibly renaming bounded variables we can assume that the formula α verifies the following property: if $\forall x_i$ or $\exists x_i$ occurs in α, then i is the largest index among the indices of the free variables in the scope of the quantifier.

Notice that a formula with free variables among $x_1,\ldots,x_i,Y_1,\ldots,Y_j$ can be considered either as a formula $\alpha(x_1,\ldots,x_i,Y_1,\ldots,Y_j)$ or as a formula $\alpha(x_1,\ldots,x_n,Y_1,\ldots,Y_k)$, for any $n \geq i$, $k \geq j$. Point 1 below shows that we can consider every possible denotation for the same formula without loosing the property expressed in the Lemma.

1) For any m,r,n,k, if $\beta(x_1,\ldots,x_m,Y_1,\ldots,Y_r) = \alpha(x_1,\ldots,x_n,Y_1,\ldots,Y_k)$ and the thesis is true for $\alpha(x_1,\ldots,x_n,Y_1,\ldots,Y_k)$, then it is true for $\beta(x_1,\ldots,x_m,Y_1,\ldots,Y_r)$.

The statement 1) follows from 1a),1b), and 1c) below.

1a) For any $r \geq 1$, if $\beta(x_1, \ldots, x_n, Y_1, \ldots, Y_k) = \alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_r)$ and the thesis is true for the formula $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_r)$, then it is true for $\beta(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$, with

$$F_\beta(X_1, \ldots, X_n, Y_1, \ldots, Y_k) = F_\alpha(X_1, \ldots, X_n, Y_1, \ldots, Y_r)$$

(the verification is left to the reader).

1b) If $\beta(x_1, \ldots, x_n, x_{n+1}, Y_1, \ldots, Y_k) = \alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ and the thesis is true for the formula $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$, then it is true for $\beta(x_1, \ldots, x_n, x_{n+1}, Y_1, \ldots, Y_k)$, with

$$F_\beta(X_1, \ldots, X_{n+1}, Y_1, \ldots, Y_k) = X_{n+1} \times F_\alpha(X_1, \ldots, X_n, Y_1, \ldots, Y_k).$$

Indeed, for all $V_1, \ldots, V_{n+1}, W_1, \ldots, W_k$ in $\mathcal{W}$, and $u \in U$,

$$U \models u \in F_\beta(V_1, \ldots, V_{n+1}, W_1, \ldots, W_k)$$

if and only if

$$U \models u \in V_{n+1} \times F_\alpha(V_1, \ldots, V_n, W_1, \ldots, W_k)$$

if and only if

$$U \models \exists w_{n+1} \exists b \ (w_{n+1} \in V_{n+1} \wedge b \in F_\alpha(V_1, \ldots, V_n, W_1, \ldots, W_k) \wedge Pair(w_{n+1}, b) \wedge u = \langle w_{n+1}, b \rangle)$$

if and only if
there exist $w_1 \in V_1, \ldots, w_{n+1} \in V_{n+1}$ such that

a) $U \models u = \langle w_{n+1}, w_n, \ldots, w_1 \rangle$,
b) $(W, R) \models \beta(w_1, \ldots, w_{n+1}, W_1, \ldots, W_k)$

(since, for all $w_1, \ldots, w_{n+1} \in W$, $Pair(w_{n+1}, \langle w_n, w_{n-1}, \ldots, w_1 \rangle)$ holds in U).

1c) If $\beta(x_1, \ldots, x_{n-1}, Y_1, \ldots, Y_k) = \alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ and the thesis is true for the right-hand side formula $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$, then it is true for $\beta(x_1, \ldots, x_{n-1}, Y_1, \ldots, Y_k)$, with

$$F_\beta(X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_k) = Rng(F_\alpha(X_1, \ldots, W, Y_1, \ldots, Y_k))$$

(the proof is left to the reader, and relies on the fact that W is not empty).

Two more preliminary steps are needed before starting the inductive proof. In each step we just determine the term F_β, leaving the verification of the thesis to the reader. The notation $\alpha(\ldots, x_i | x_j, \ldots, \ldots)$ is used to denote uniform substitution of x_j with x_i in the formula α.

2) If $\beta(x_1, \ldots, x_{n+1}, Y_1, \ldots, Y_k) = \alpha(x_1, \ldots, x_{n+1} | x_n, Y_1, \ldots, Y_k)$ and the thesis is true for α, then it is true for β. If $n = 1$, it is enough to take $F_\beta(X_1, X_2, Y_1, \ldots, Y_k) = F_\alpha(X_2, Y_1, \ldots, Y_k) \times X_1$. If $n > 1$, let $F_\beta(X_1, \ldots, X_{n+1}, Y_1, \ldots, Y_k) = C_2(F_\alpha(X_1, \ldots, X_{n-1}, X_{n+1}, Y_1, \ldots, Y_k), X_n)$.

3) If $\beta(x_1, \ldots, x_n, Y_1, \ldots, Y_k) = \alpha(x_{n-1} | x_1, x_n | x_2, Y_1, \ldots, Y_k)$, with $n \geq 2$, and the thesis is true for α, then it is true for β, with $F_\beta(X_1, \ldots, X_n, Y_1, \ldots, Y_k) = C_1(F_\alpha(X_{n-1}, X_n, Y_1, \ldots, Y_k), X_{n-2} \times \cdots \times X_1)$.

Let us prove now the thesis by induction on the structural complexity of $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$. We start with atomic formulae.

4a) If $\alpha(x_1, x_2)$ is the formula $x_1 = x_2$, then let $F_\alpha = X_2 \times_= X_1$. If we need to consider $x_1 = x_2$ as a formula in the variables $x_1, \ldots, x_n, Y_1, \ldots, Y_k$, then the thesis follows from 1).

4b) If $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ is the formula $x_{n-1} = x_n$, then the thesis follows from 4a) and 3).

4c) If $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ is the formula $x_m = x_r$, with $m \leq r \leq n$, there are two possibilities: if $m = r$, then $F_\alpha = X_n \times (\ldots \times X_1)$; if $m < r$, then the thesis follows from 4a) and 2) by induction on r.

5a) If $\alpha(x_1, x_2)$ is the formula $x_1 R x_2$, then $F_\alpha = X_2 \times_\in X_1$. If we need to consider $x_1 R x_2$ as a formula in the variables $x_1, \ldots, x_n, Y_1, \ldots, Y_k$, then the thesis follows from 1).

5b) If $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ is the formula $x_{n-1} R x_n$, the thesis follows from 5a) and 3).

5c) If $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ is the formula $x_i R x_j$, consider the formulae $\gamma_1(x_1, \ldots, x_{n+2}, Y_1, \ldots, Y_k) = x_i = x_{n+1}$, $\gamma_2(x_1, \ldots, x_{n+2}, Y_1, \ldots, Y_k) = x_j = x_{n+2}$, and $\gamma_3(x_1, \ldots, x_{n+2}, Y_1, \ldots, Y_k) = x_{n+1} R x_{n+2}$. From 4c) and 5b), it follows that the thesis is true for γ_i, $i = 1, 2, 3$. Then, it is easy to verify that the thesis is true for α, with F_α equal to $Rng(Rng(F_{\gamma_1}(X_1, \ldots, X_n, X_i, X_j, Y_1, \ldots, Y_k) \cap F_{\gamma_2}(X_1, \ldots, X_n, X_i, X_j, Y_1, \ldots, Y_k) \cap F_{\gamma_3}(X_1, \ldots, X_n, X_i, X_j, Y_1, \ldots, Y_k)))$.

6) If $\alpha(x_1, Y_1)$ is the formula $x_1 \in Y_1$, then $F_\alpha(X_1, Y_1) = Y_1$. It can be easily generalized to the case of $\alpha(x_1, \ldots, x_n, Y_1, \ldots, Y_k) = x_i \in Y_j$ as in 5c).

In the inductive step, the case of boolean combinations of formulae is dealt with $\cup$ and $\setminus$ (w.r.t. $X_n \times \ldots \times X_1$, where n is the number of world variables displayed in the formula), and left to the reader. Consider the case in which $\beta(x_1, \ldots, x_n, Y_1, \ldots, Y_k) = \exists x_{n+1} \alpha(x_1, \ldots, x_{n+1}, Y_1, \ldots, Y_k)$. Let $F_\beta(X_1, \ldots, X_n, Y_1, \ldots, Y_k) = Rng(F_\alpha(X_1, \ldots, X_n, W, Y_1, \ldots, Y_k))$. Then, for all $V_1, \ldots, V_n, W_1, \ldots, W_k$ in $\mathcal{W}$, and $u \in U$,

$$U \models u \in F_\beta(V_1, \ldots, V_n, W_1, \ldots, W_k)$$

if and only if
there exist $w_1 \in V_1, \ldots, w_n \in V_n, s \in W$ such that

a) $U \models u = \langle w_n, w_{n-1}, \ldots, w_1 \rangle$,
b) $(W, R) \models \alpha(w_1, \ldots, w_n, s, W_1, \ldots, W_k)$.

if and only if
there exist $w_1 \in V_1, \ldots, w_n \in V_n$ such that

a) $U \models u = \langle w_n, w_{n-1}, \ldots, w_1 \rangle$,
b) $(W, R) \models \exists x_{n+1} \alpha(w_1, \ldots, w_n, x_{n+1}, W_1, \ldots, W_k)$.

$\dashv$

LEMMA 5.8 *The interpretation $(U_c, \in^{U_c}, \ldots)$ generated by a general frame $(W, R, \mathcal{W})$ closed under L_0-definition is an Ω_c model.*

Proof. The verification of the axioms regarding $\cup$ and $\setminus$ is left to the reader, while the case of $\times$ can be found in the text. Consider now the case of $\times_\in$:

$$x \in^{U_c} y \times_\in^{U_c} z \Leftrightarrow x \in F(y \times_\in^{U_c} z) \Leftrightarrow x \in F(F(y) \times_{\in,R} F(z)) \Leftrightarrow x \in F(y) \times_{\in,R} F(z) \Leftrightarrow$$

$$\exists a \exists b (a \in F(y) \wedge b \in F(z) \wedge x = \langle a, b \rangle \wedge (a \in b \vee b R a)) \Leftrightarrow$$

$$\exists a \exists b (a \in^{U_c} y \wedge b \in^{U_c} z \wedge x = \langle a, b \rangle^{U_c} \wedge Pair^{U_c}(a, b) \wedge a \in^{U_c} b)$$

(since, for all $a, b \in U_c$, $a \in^{U_c} b \leftrightarrow (a \in b \vee bRa)$). The case of $\times_=$ is left to the reader.

Consider the case of Dom:

$$x \in^{U_c} Dom^{U_c}(y) \Leftrightarrow x \in F(Dom^{U_c}(y)) \Leftrightarrow x \in F(Dom(F(y)) \Leftrightarrow x \in Dom(F(y)) \Leftrightarrow$$

$$\exists a(\langle x, a\rangle \in F(y)) \Leftrightarrow \exists a \in U_c(\langle x, a\rangle^{U_c} \in^{U_c} y \wedge Pair^{U_c}(x, a)).$$

All equivalences are rather straightforward, execpt for (the right direction of) the last equivalence. If $\langle x, a\rangle \in F(y)$, then $a \in Rg(F(y))$ and $Rg(F(y)) \in \overline{\mathcal{W}^c}$. Hence, from Lemma (5.7), it follows that $a \in W \cup \overline{W^s}$. Similarly, $x \in W \cup \overline{W^s}$ and, by Observation (5.6), $U_c \models Pair(x, a)$, and the standard pair $\langle x, a\rangle$ coincides with $\langle x, a\rangle^{U_c}$.

The cases of C_1, C_2 are left to the reader. $\dashv$

LEMMA 5.10 *Let $t(x_1, .., x_n)$ be a c-term. If it is not a variable, then there exist k c-terms $t_1(x_1, .., x_n), \ldots, t_k(x_1, \ldots, x_n)$ such that, for all $w_1, \ldots, w_n$ in W,*

$$U_c \models t(w_1, \ldots, w_n) = \{t_1(w_1, \ldots, w_n)\} \cup \ldots \cup \{t_k(w_1, \ldots, w_n)\}.$$

Proof. If $t(x_1, .., x_n)$ is not a variable, then $t(x_1, .., x_n) = t'(\{x_1\}, \ldots, \{x_n\})$ (by definition of c-term). The proof of the lemma is by induction on the structure of the term $t'(y_1, \ldots y_n)$.

If $t' = y_i$, then $t(x_i) = \{x_i\}$, and the result follows with $t_1(x_i) = x_i$.

If $t'(y_1, \ldots y_n) = t'_1(y_1, \ldots y_n) \cup t'_2(y_1, \ldots y_n)$, then the union of the two sets of terms obtained by induction for $t_1(x_1, \ldots, x_n) = t'_1(\{x_1\}, \ldots, \{x_n\})$ and $t_2(x_1, \ldots, x_n) = t'_2(\{x_1\}, \ldots, \{x_n\})$ can be taken as the set of terms for $t(x_1, .., x_n)$.

If $t'(y_1, \ldots y_n) = \{t'_1(y_1, \ldots y_n)\}$, then the result follows assuming $t_1(x_1, .., x_n) = t'_1(\{x_1\}, \ldots, \{x_n\})$. $\dashv$

LEMMA 5.11 *If $t(x_1, .., x_n)$ and $s(y_1, .., y_l)$ are c-terms, then there are two L_0-formulae $\gamma(x_1, .., x_n, y_1, .., y_l)$ and $\delta(x_1, .., x_n, y_1, .., y_l)$, which are boolean combinations of atomic formulae of the forms $z_i = z_j$ and $z_i R z_j$, such that, for all $w_1, \ldots, w_n, v_1, \ldots, v_l$ in W,*

a) $U_c \models t(w_1, .., w_n) = s(v_1, .., v_l)$ if and only if $(W, R) \models \gamma(w_1, .., w_n, v_1, .., v_l)$,

b) $U_c \models t(w_1, .., w_n) \in s(v_1, .., v_l)$ if and only if $(W, R) \models \delta(w_1, .., w_n, v_1, .., v_l)$.

Proof. Consider first the case of $=$. The proof is by induction on the maximum of the heights of t and s.

If both terms are variables, say $t = x_1$ and $s = y_1$, take $\gamma(x_1, y_1)$ equal to $x_1 = y_1$.

If one is a variable, but the other is not, then, for every $w_1, \ldots, w_n, v_1, \ldots, v_l \in W$, it is impossible that $U_c \models t(w_1, .., w_n) = s(v_1, .., v_l)$; thus, take $\gamma(x_1, y_1, \ldots, y_l)$ equal to $x_1 \neq x_1$.

If t and s are not variables, then Lemma (5.10) allows us to obtain k c-terms $t_1(x_1, .., x_n), \ldots, t_k(x_1, \ldots, x_n)$ and h c-terms $s_1(y_1, .., y_l), \ldots, s_h(y_1, \ldots, y_l)$ such that, for all $w_1, \ldots, w_n, v_1, \ldots, v_l$ in W,

$$U_c \models t(w_1, \ldots, w_n) = \{t_1(w_1, \ldots, w_n)\} \cup \ldots \cup \{t_k(w_1, \ldots, w_n)\},$$

and

$$U_c \models s(v_1, \ldots, v_l) = \{s_1(v_1, \ldots, v_l)\} \cup \ldots \cup \{s_h(v_1, \ldots, v_l)\}.$$

Therefore,

$$U_c \models t(w_1, \ldots, w_n) = s(v_1, \ldots, v_l)$$

if and only if

$$U_c \models \{t_1(w_1, \ldots, w_n)\} \cup \ldots \cup \{t_k(w_1, \ldots, w_n)\} = \{s_1(v_1, \ldots, v_l)\} \cup \ldots \cup \{s_h(v_1, \ldots, v_l)\}.$$

For this kind of sets, extensionality holds in U_c. Thus, we may replace the above equality by a boolean combination of equalities between $t_i(w_1,\ldots,w_n)$ and $s_j(v_1,\ldots,v_l)$, with $1 \leq i \leq k$ and $1 \leq j \leq h$; the thesis follows from the inductive hypothesis.

Consider now the case of $\in$.

If both terms are variables, say $t = x_1$ and $s = y_1$, take $\delta(x_1, y_1)$ equal to $y_1 R x_1$.

If t is not a variable, but s $(= y_1)$ is a variable, then, for all $w_1,\ldots,w_n,v_1 \in W$, it is impossible that $U_c \models t(w_1,..,w_n) \in v_1$; thus, take $\delta(x_1,\ldots,x_n,y_1)$ equal to $x_1 \neq x_1$.

If s is not a variable, then, by Lemma (5.10), it follows that, for all $w_1,\ldots,w_n,v_1,\ldots,v_l \in W$, $U_c \models t(w_1,\ldots,w_n) \in s(v_1,\ldots,v_l)$ if and only if $U_c \models t(w_1,\ldots,w_n) \in \{s_1(v_1,\ldots,v_l)\} \cup \ldots \cup \{s_h(v_1,\ldots,v_l)\}$ if and only if $U_c \models t(w_1,\ldots,w_n) = s_1(v_1,\ldots,v_l) \vee \ldots \vee t(w_1,\ldots,w_n) = s_h(v_1,\ldots,v_l)$. The thesis follows from a). $\dashv$

LEMMA 5.12 *For each c-term $t(x_1,\ldots,x_m)$, there exist d c-terms $t^1(x_1,\ldots,x_m),\ldots,t^d(x_1,\ldots,x_m)$, and d^2 L_0-formulae $\theta^t_{e,f}(x_1,...,x_m)$, with $e,f \in \{1,\ldots d\}$, such that, for all $w_1,\ldots,w_m \in W$ and $a,b \in W \cup \overline{W^s}$,*

$$U_c \models t(w_1,\ldots,w_m) = \langle a,b\rangle$$

if and only if
there exist $e,f \in \{1,\ldots d\}$ such that

a) $U_c \models a = t^e(w_1,\ldots,w_m) \wedge b = t^f(w_1,\ldots,w_m)$,
b) $(W,R) \models \theta^t_{e,f}(w_1,\ldots,w_m)$.

Proof. Consider the c-term $t(x_1,\ldots,x_m)$. If $t(x_1,\ldots,x_m)$ is a variable, say x_1, then, for all $w_1 \in W$, $t(w_1)(= w_1)$ is not an ordered pair in the model U_c (cfr. Observation (5.6)). Therefore, take d equal to 1, any c-term as t^1, and $w_1 \neq w_1$ as $\theta^t_{e,f}$. The same argument works if $t(x_1,\ldots,x_m)$ is not a variable, but one of the c-term obtained by applying Lemma (5.10) to it is a variable.

Otherwise, apply Lemma (5.10) twice. At the first step, we obtain $t_1(x_1,\ldots,x_m),\ldots,t_k(x_1,\ldots,x_m)$ such that, for all $w_1,\ldots w_m \in W$,

$$U_c \models t(w_1,\ldots,w_m) = \{t_1(w_1,\ldots,w_m),\ldots,t_k(w_1,\ldots,w_m)\};$$

at the second step, for $i = 1,\ldots,k$, we determine n_i terms $t_{i,1}(x_1,\ldots,x_m),\ldots,t_{i,n_i}(x_1,\ldots,x_m)$ such that

$$U_c \models t_i(w_1,\ldots,w_m) = \{t_{i,1}(w_1,\ldots,w_m),\ldots,t_{i,n_i}(w_1,\ldots,w_m)\}.$$

By replacing each $t_i(w_1,\ldots,w_m)$ with the corresponding set of terms, we obtain

$$U_c \models t(w_1,\ldots,w_m) = \{\{t_{1,1}(w_1,\ldots,w_m),\ldots,t_{1,n_1}\}(w_1,\ldots,w_m),\ldots,$$

$$\{t_{k,1}(w_1,\ldots,w_m),\ldots,t_{k,n_k}(w_1,\ldots,w_m)\}\}.$$

Let $d = n_1 + \ldots + n_k$ and $t^1 = t_{1,1},\ldots,t^d = t_{k,n_k}$. For all $w_1,\ldots,w_m \in W$, the element $t(w_1,\ldots,w_m)$ is a pair, with first component $t^e = t_{i_0,j_0}$ and second component $t^f = t_{i_1,j_1}$, if and only if in U_c it holds that

$$(\bigvee_{i=1}^{k}\bigwedge_{j=1}^{n_i} t_{i,j} = t_{i_0,j_0}) \wedge (\bigwedge_{i=1}^{k}\bigwedge_{j=1}^{n_i}(t_{i,j} = t_{i_0,j_0} \vee t_{i,j} = t_{i_1,j_1})) \wedge (\bigwedge_{i=1}^{k}\bigvee_{j=1}^{n_i} t_{i,j} = t_{i_0,j_0}),$$

where the first conjunct says that there exists an i such that t^i is the singleton $\{t_{i_0,j_0}\}$, the second conjunct says that, for all i, t^i is a subset of $\{t_{i_0,j_0}, t_{i_1,j_1}\}$, and the third one says that, for all i, t^i contains t_{i_0,j_0}. By Lemma (5.11), this expression corresponds to an L_0-formula. $\dashv$

LEMMA 5.13 *For any term $\tau(X_1,\ldots,X_n)$ built using the functions $\cup,\setminus,\times,\times_\in,\times_=,Dom,Rng,C_1,C_2,$, whose free variables are among $X_1,\ldots,X_n$, and for any $A_1,\ldots,A_n$ in $\mathcal{W}^c$, there are:*

- *k c-terms $t_1(x_1,\ldots,x_m),\ldots,t_k(x_1,\ldots,x_m)$,*
- *k L_0-formulae $\alpha_1^\tau(x_1,\ldots,x_m,Y_1,\ldots,Y_r),\ldots,\alpha_k^\tau(x_1,\ldots,x_m,Y_1,\ldots,Y_r)$,*
- *r sets $W_1,\ldots,W_r$ in $\mathcal{W}$,*

such that, for all $u\in U_c$,

$$U_c \models u\in\tau(A_1,\ldots,A_n)$$

if and only if

there exist $i\in\{1,\ldots,k\}$ and $w_1,\ldots,w_m\in W$ such that

a) $U_c\models u=t_i(w_1,\ldots,w_m)$,
b) $(W,R)\models\alpha_i^\tau(w_1,\ldots,w_m,W_1,\ldots,W_r)$.

Proof. The base case and the inductive step for $\cup$ and $\setminus$ have been already proved.

To complete the proof, we must consider both terms of the form $\mu(X_1,\ldots,X_n)\circ\nu(X_1,\ldots,X_n)$, with $\circ\in\{\times,\times_\in,\times_=,C_1,C_2\}$, and terms of the forms $Dom(\mu(X_1,\ldots,X_n))$ and $Rng(\mu(X_1,\ldots,X_n))$.

For $i\in\{1,\ldots,k\}$ (resp. $j\in\{1,\ldots,h\}$), let $m_i(x_1,\ldots x_m)$, $\alpha_i^\mu(x_1,\ldots x_m,Y_1,\ldots,Y_r)$, and $W_1,\ldots,W_r$, (resp. $n_j(y_1,\ldots,y_l)$, $\alpha_j^\nu(y_1,\ldots,y_l,Z_1,\ldots,Z_s)$, and $V_1,\ldots V_s$) be obtained by induction for $\mu(A_1,\ldots,A_n)$ (resp. $\nu(A_1,\ldots,A_n)$).

Consider the case of $\times$: $\tau(X_1,\ldots,X_n)=\mu(X_1,\ldots,X_n)\times\nu(X_1,\ldots,X_n)$.

$$U_c\models u\in\tau(A_1,\ldots,A_n)$$

if and only if

$$U_c\models\exists a\exists b\,(a\in\mu(A_1,\ldots,A_n)\wedge b\in\nu(A_1,\ldots,A_n)\wedge u=\langle a,b\rangle\wedge Pair(a,b))$$

if and only if

there exist $i\in\{1,\ldots,k\},j\in\{1,\ldots,h\}$ and $w_1,\ldots,w_m,v_1,\ldots,v_l\in W$ such that

a) $U_c\models u=\langle m_i(w_1,\ldots,w_m),n_j(v_1,\ldots,v_l)\rangle$
b) $(W,R)\models\alpha_i^\mu(w_1,\ldots w_k,W_1,\ldots,W_r)\wedge\alpha_j^\nu(v_1,\ldots,v_l,V_1,\ldots,V_s))),$

(because, from $m_i(w_1,\ldots,w_m)\in^{U_c}\mu(A_1,\ldots,A_n)$ and $n_j(v_1,\ldots,v_l)\in^{U_c}\nu(A_1,\ldots,A_n)$, it follows that $U_c\models Pair(m_i(w_1,\ldots,w_m),n_j(v_1,\ldots,v_l)))$.

The cases $\tau(X_1,\ldots,X_n)=\mu(X_1,\ldots,X_n)\times_\in\nu(X_1,\ldots,X_n)$ and $\tau(X_1,\ldots,X_n)=\mu(X_1,\ldots,X_n)\times_=\nu(X_1,\ldots,X_n)$ are easily proved by using Lemma 5.11.

Consider the case of Dom: $\tau(X_1,\ldots,X_n)=Dom(\mu(X_1,\ldots,X_n))$.

For all $i\in\{1,\ldots,k\}$, the application of Lemma (5.12) to the c-term m_i (for μ) allows us to find d_i c-terms $m_i^1(x_1,\ldots,x_m),\ldots,m_i^{d_i}(x_1,\ldots,x_m)$ and d_i^2 L_0-formulae $\theta_{e_i,f_i}^{m_i}(x_1,...,x_m)$, with $e_i,f_i\in\{1,\ldots d_i\}$, such that, for all $u,b\in W\cup\overline{W^s}$,

$$U_c\models m_i(w_1,\ldots,w_m)=\langle u,b\rangle$$

if and only if

there exist $e_i,f_i\in\{1,\ldots d_i\}$

a) $U_c\models u=m_i^{e_i}(w_1,\ldots,w_m)\wedge b=m_i^{f_i}(w_1,\ldots,w_m)$;
a) $(W,R)\models\theta_{e_i,f_i}^{m_i}(w_1,\ldots,w_m)$.

It follows that the c-terms, the L_0-formulae, and the sets in $\mathcal{W}$ for $\tau(A_1, \ldots, A_n)$ can be chosen as follows:

- the c-term are $m_1^1, \ldots, m_1^{d_1}, \ldots, m_k^1, \ldots, m_k^{d_k}$;
- the L_0-formula relative to the c-term $m_i^{e_i}$ is $\alpha_i^\mu \wedge \bigvee_{f_i=1}^{d_i} \theta_{e_i,f_i}^{m_i}(w_1, \ldots w_m)$;
- the sets in $\mathcal{W}$ are $W_1, \ldots W_m$ (the same we took for μ).

Indeed, for all $u \in U_c$,

$$U_c \models u \in Dom(\mu(A_1, \ldots, A_n))$$

if and only if

$$U_c \models \exists b(\langle u, b\rangle \in \mu(A_1, \ldots, A_n) \wedge Pair(u, b))$$

if and only if

$u \in W \cup \overline{W^s}$ and $\exists b \in W \cup \overline{W^s}$ such that

$$U_c \models \langle u, b\rangle \in \mu(A_1, \ldots, A_n)$$

(since $U_c \models Pair(u, b)$ if and only if $u, b \in W \cup \overline{W^s}$)

if and only if

$u \in W \cup \overline{W^s}$, and there exist $b \in W \cup \overline{W^s}$, $i \in \{1, \ldots, k\}$ and $w_1, \ldots, w_m \in W$ such that

a) $U_c \models \langle u, b\rangle = m_i(w_1, \ldots, w_m)$;
b) $(W, R) \models \alpha_i^\mu(w_1, \ldots w_m, W_1, \ldots, W_r)$.

if and only if

there exist $i \in \{1, \ldots, k\}$, $e_i, f_i \in \{1, \ldots, d_i\}$ and $w_1, \ldots, w_m \in W$ such that

a) $U_c \models u = m_i^{e_i}(w_1, \ldots, w_m)$;
b) $(W, R) \models \alpha_i^\mu(w_1, \ldots w_m, W_1, \ldots, W_r) \wedge \theta_{e_i,f_i}^{m_i}(w_1, \ldots w_m)$

if and only if

there exist $i \in \{1, \ldots, k\}$, $e_i \in \{1, \ldots, d_i\}$ and $w_1, \ldots, w_m \in W$ such that

a) $U_c \models u = m_i^{e_i}(w_1, \ldots, w_m)$;
b) $(W, R) \models \alpha_i^\mu(w_1, \ldots w_m, W_1, \ldots, W_r) \wedge \bigvee_{f_i=1}^{d_i} \theta_{e_i,f_i}^{m_i}(w_1, \ldots w_m)$.

The case of Rng is similar, and therefore it is left to the reader.
Consider now the case of C_1: $\tau(A_1, \ldots, A_n) = C_1(\mu(A_1, \ldots, A_n), \nu(A_1, \ldots, A_n))$.
For all $u \in U_c$, we have

$$U_c \models u \in C_1(\mu(A_1, \ldots, A_n), \nu(A_1, \ldots, A_n))$$

if and only if

there exist a, b, c in $W \cup \overline{W^s}$ such that

$$U_c \models u = \langle a, \langle b, c\rangle\rangle \wedge \langle a, b\rangle \in \mu(A_1, \ldots, A_n) \wedge c \in \nu(A_1, \ldots, A_n)$$

(we can omit the conjuncts $Pair(a, b), Pair(b, c), \ldots$ for the same reason as in the case of Dom)

if and only if

there exist $i \in \{1, \ldots, k\}$, $e_i, f_i \in \{1, \ldots, d_i\}$, $j \in \{1, \ldots, h\}$ and $w_1, \ldots, w_m, v_1, \ldots, v_l \in W$ such that

a)$U_c \models u = \langle m_i^{e_i}(w_1, \ldots w_m), \langle m_i^{f_i}(w_1, \ldots w_m), n_j(v_1, \ldots, v_l)\rangle\rangle$

b) $(W, R) \models \alpha_i^\mu(w_1, \ldots w_m, W_1, \ldots, W_r) \wedge \alpha_j^\nu(v_1, \ldots, v_l, V_1, \ldots, V_r) \wedge \theta_{e_i,f_i}^{m_i}(w_1, \ldots, w_m)$,

(by applying Lemma (5.12) again to each c-term m_i (for μ))

and the thesis easily follows.

The case of C_2 is similar, and therefore it is left to the reader. $\dashv$