Isaacson's thesis and Wilkie's theorem

MSc Thesis

written by

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under the supervision of **Dr. Luca Incurvati**[1] and **Dr. Sean Walsh**[2], and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:
June 29th, 2017

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Abstract

In this thesis, I explore Isaacson's thesis and Wilkie's theorem, providing philosophical and formal results on how they relate to each other. At a first approximation, Isaacson's thesis claims that *Peano arithmetic is sound and complete with respect to genuinely arithmetical statements*. Using internalist notions familiar from recent work on internal categoricity theorems, I provide a formal definition of *genuinely arithmetical* statements. As for Wilkie's theorem, it roughly says that, from an external perspective, Peano arithmetic is minimal, in that it is entailed by all categorical axiomatisations of the natural numbers satisfying a certain syntactic restriction. After expositing Wilkie's theorem and the relation of its proof to other known techniques, I discuss its relation to Isaacson's thesis, in particular whether Peano arithmetic is a maximal theory obtained from the categorical characterisation.

Acknowledgement

Only when I had a complete draft of the thesis, I noticed that there was not a single woman cited in my references. This could have been caused by my own lack of initiative in the writing process, but there are many other reasons why this is the case. Without the previous generations of women and marginalised individuals in academia, it would have been even more challenging for everyone now. I, therefore, would really like to dedicate this to all the individuals who have fought against the stereotype of being in a particular society. It is not just about being a woman, but also about other identities these individuals have. During my master's I have learnt so much about how important my own personal identities are. I would like to especially thank my friends for sharing their wisdom during this journey.

I also thank the Evert Willem Beth Foundations for the financial support during the two years of my master's degree. Without the support, I would not have been able to begin my master's, nevermind finish it. There are so many students who are so passionate about their research, but they cannot pursue this due to the financial difficulties. The support the Beth Foundations has provided has meant a great deal for me.

My parents have always been supporting me from a distance. Without them I would not be in this world, and I would not be who I am today.

Luca, I thank you for being an amazing supervisor at a distance, across the different time zones. I was not fully aware how interesting I would find this topic when you suggested it to me. But I was very happy and excited trying to think about all these questions and also being able to philosophically and mathematically process some of the answers I have found.

Sean, without you I would have not been able to finish this master's. You have been so supportive academically and personally. I am truly grateful for every single thing you have done for me, and all the support you have provided for me. One day, when I have a chance to supervise students, I aspire to be a supervisor like you have been to me.

Benedikt, Dick, Benno and Peter, thank you so much for being on the thesis committee. I value your feedback and time, and I appreciate the energy you all have invested on evaluating my work. I especially thank Benedikt for travelling to Amsterdam to chair my defence, and also Dick, for the hand-written feedback.

I also send special thanks to Fedor Pakhomov for multiple email exchanges discussing about the notion of n-fulfillability. Thank you, Fedor, for the detailed comments on the result in chapter 6 of this thesis.

Many people have been supportive during the process of writing this thesis. The department of Logic and Philosophy of Science at University of California, Irvine has provided me with an office space and an opportunity to interact with the community – thank you for welcoming me to the community and I am looking forward to becoming an official LPS-er. I also thank my MoL friends for the support. Without their valuable support and understanding, I would not have been able to become a master of logic – 'there is life after MoL'.

Prior to starting this master's, my knowledge of logic has been all from the teaching of Walter Dean, at the University of Warwick. Thank you for encouraging a young student to pursue an interest in logic. Similarly, without Will Stafford, I would not have learnt so much logic prior to starting at Amsterdam. Thank you for reading some chapters of my thesis, and sparing your time to listen to my naïve philosophical ideas since our time together at Warwick. I'm looking forward to asking you further naïve questions in California! Also thanks to Martina Kluvancova for the comments on the thesis.

Most importantly, I thank myself for not giving up! It has been a very difficult few years, with each day seeming like a roller-coaster ride. I have become more skilled at handling various situations through the difficult experiences, and have certainly become wiser to share them with others.

Contents

Chapter 1

Introduction

In this thesis, I will give a survey of Daniel Isaacson's philosophical positions and give a precise formulation of Isaacson's thesis. A naïve understanding of Isaacson's thesis challenges the well-known Gödel's incompleteness theorems. The aim is to find a precise formulation of Isaacson's thesis that is motivated from Isaacson's philosophy of mathematics. In doing so, I will give some formal accounts of Isaacson's claims and discuss what this tells us about which statements are *genuinely arithmetical*.

There are various formulations of Isaacson's thesis, which can be found in (Smith [2008]) and (Incurvati [2008]). One formulation, close to Incurvati's, is that *Peano arithmetic is sound and complete with respect to genuinely arithmetical statements*. However, it is unclear what the notion of *genuinely arithmetical* means, and thus what Isaacson's thesis claims. The term *genuinely arithmetical* was used by Isaacson in (Isaacson [1987]). Another formulation, by Smith, focuses more on the notion of *higher-order concepts*, stating Isaacson's thesis to be the claim that *in order to prove statements that are independent of Peano arithmetic, we must access the hidden higher-order concepts*. Isaacson claims that the hidden *higher-order concepts* are the notions that go beyond finite arithmetic, but this formulation gives us no better understanding of Isaacson's thesis.

I will focus on Isaacson's philosophical motivations behind his thesis, and I will argue that Incurvati's formulation of Isaacson's thesis is superior to Smith's formulation. If we were to understand the notion of *higher-order concepts* to be going beyond finite arithmetic (or beyond Peano arithmetic), Smith's formulation of Isaacson's thesis appears to be a trivial statement. Further diving into understanding the notion of *higher-order concepts*, we will see that Smith's formulation of Isaacson's thesis is circular.

Isaacson's structuralism is an important part of understanding Isaacson's philosophy of mathematics. He claims that we understand mathematical structures to be mathematical concepts, and those concepts are parts of our process of thinking. From this idea, the notion of categoricity captures the unique concept we have of (e.g.) arithmetic or set theory. This is a crucial part of our interpretation of Isaacson's thesis, where it is precisely the categorical characterisation of arithmetic that allows us to capture what is *genuinely arithmetical*.

From the natural language formulation of Isaacson's thesis, we attempt to formalise the notion of a *genuinely arithmetical* statement, and give a formalisation of Isaacson's thesis. This provides us with some mathematical results on Isaacson's thesis and challenges whether Isaacson's thesis is true of arithmetic. I will give six different definitions of *genuinely arithmetical* statements, and we will see that some of these definitions turn out to be inconsistent, while some others are extensionally equivalent to each other. The formal definition of *genuinely arithmetical* statements challenges the current formulation of Isaacson's thesis and demands a modified statement, which I will call *neo-Isaacson's thesis*.

Neo-Isaacson's thesis is obtained from formalising Isaacson's thesis from our interpretation of *genuinely arithmetical* statements which comes from an internalist perspective. Given Isaacson's structuralism and his concept of the *reality of mathematics*, internalism gives a natural formalisation of Isaacson's thesis. In fact, the philosophical insights we obtain from neo-Isaacson's thesis gives us a more precise understanding of Isaacson's perspective. In the original Isaacson's thesis, he focuses on first-order logic as the deductive system for arithmetic, while second-order logic is the system that captures the structures. However, internalist ideas use second-order logic to capture both the structures and deductive ideas.

There is another formalisation of Isaacson's thesis, which I will refer to as *Wilkie's theorem* (Wilkie [1987]). If we understand Isaacson's thesis to claim that *Peano arithmetic is sound and complete with respect to genuinely arithmetical statements*, Wilkie's theorem captures the soundness part of the statement. Briefly, Wilkie's theorem gives us that Peano arithmetic is the minimal theory we can obtain from categorical axiomatisations of the natural numbers. This is equivalent to the soundness part of the thesis, since what is provable from Peano arithmetic will be what is captured from the categorical characterisations. However, there is no known existing proof that tells us whether there is a maximal theory that is captured by the categorical characterisations. I will give a proof that there is no maximal such theory extending from Wilkie's result.

1.1 Outline of the thesis

The outline of the thesis is as follows. We divide the thesis into two parts – part I focuses on Isaacson's philosophical motivations and gives a precise formulation of Isaacson's thesis; and part II will focus on Wilkie's theorem. Part I consists of chapters 2, 3 and 4, and part II consists of chapters 5 and 6.

In chapter 2, I will state Isaacson's structuralism. Starting in section 2.1 by giving a historical motivation behind structuralism as a position in philosophy of mathematics, I will introduce some of the contemporary positions on structuralism. Sections 2.2, 2.3 and 2.4 will be on *eliminative structuralism*, *contextual structuralism* and *ante rem structuralism*, respectively. We will see that Isaacson's *concept-realist structuralism* differs from all of these positions and stands on its own as an *epistemic* position on structuralism. We conclude chapter 2, by discussing the importance of the notion of *categoricity* for Isaacson's structuralism.

After exploring Isaacson's structuralism, we will give a precise formulation of Isaacson's thesis in chapter 3. In section 3.1, we will give Incurvati [2008] and Smith [2008]'s formulations of Isaacson's thesis. I will argue that Incurvati's formulation is superior and more reflective of Isaacson's structuralism. In order to achieve this, I will go over Isaacson's examples of *higher-order concepts*. In (Isaacson [1987]), he argues against the idea that Gödel sentences, Goodstein's theorem and Paris-Harrington sentences are counterexamples to his thesis. These are statements that can be formulated in the language of arithmetic but are independent of Peano arithmetic. Isaacson argues that the proofs in obtaining the truths of such statements must rely on hidden *higher-order concepts*.

We also touch on Isaacson's distinction between the notion of *structures* and *semantics*. In fact, the internalist perspective can capture the structures without committing to some model-theoretic or set-theoretic notions about structures. But more importantly, understanding Isaacson's notion of *genuinely arithmetical* statements requires an understanding of how we can *directly perceive* the statements from the structures. The notion of *directly perceiving* is philosophically informal but important in understanding Isaacson's thesis. It appears natural to follow internalism as a formal approach to understanding the notion of *direct perceivability*.

In chapter 4, we give some formalisations of the notion of *genuinely arithmetical* statements that are in line with Isaacson's structuralism and the internalist ideas. In fact, there is an internal variation of Dedekind's categoricity theorem of arithmetic, that captures that there is a unique internal structure of arithmetic. This coincides with Isaacson's demand for categorical characterisation, and we can obtain the notion of *genuinely arithmetical* statements by using internal notions of *categoricity* and *equivalence*. We will see that the most natural formalisation demands a change in Isaacson's thesis. We conclude this chapter by giving *neo-Isaacson's thesis* that claims that *second-order Peano arithmetic is sound and complete with respect to genuinely arithmetical statements*, and we also conclude the first part of the thesis.

In part II, chapter 5, I will give a full proof of Wilkie's theorem. I will introduce some new definitions and tools that are necessary in providing the proof of the theorem. In particular, Wilkie [1987] refers to the notion of *fulfillability*. Using this notion, Wilkie shows that Peano arithmetic is the minimal theory that is captured from the categorical second-order characterisation of $\mathbb{N}$. In fact, the notion was developed by Saul Kripke to provide a semantic proof of Gödel's incompleteness theorems. This notion is also relevant for Isaacson since he claims that Gödel sentences are not *genuinely arithmetical* due to the way their proof is established. But if we are able to find another way to obtain the truth of Gödel sentences, Isaacson's existing argument becomes less plausible. We will touch on this issue alongside the discussion of the notion of *fulfillability*.

In chapter 6, I will give a proof that there is no maximal theory of arithmetic that can be captured from second-order categorical axiomatisations. This result challenges the completeness statement of Isaacson's thesis, since even if we can provide a formal result satisfying the soundness thesis, we cannot do so for the completeness thesis. But the proof I will give in this chapter relies on capturing the statements that are already independent from Peano arithmetic. This feature of the proof suggests that the formal result does not establish that Isaacson's completeness thesis is incorrect. The main focus of this chapter will be to answer the following question:

Question 1. *Are there restricted categorical axiomatisations whose first-order counterparts are mutually independent? Wilkie's Theorem in effect says that the first-order counterpart of second-order Peano arithmetic is minimal amongst such axiomatisations. Can it be shown that that there is no maximal such one?*

We will conclude the thesis, in chapter 7, with a summary of what we obtained, and also with some open questions related to Isaacson's thesis and the results. One potential direction is to extend the results in the thesis to ZFC[1]. If this were achieved, it might give us a new way to understand what is part of *mathematical knowledge* and *understanding*.

I have assumed some reasonable background knowledge in first-order Peano arithmetic throughout the thesis. This includes the materials covered in (Kaye [1991, ch.1 – 9]). If relevant, I will give the definitions and theorems of important notions as we move along.

In general, the thesis will begin with more philosophical chapters, gradually becoming more technical and formal. 'Mathematical maturity' will be assumed in following the materials from chapter 4 onwards. The most mathematical chapter of the thesis is chapter 5, where we introduce new technical tools in order to prove Wilkie's theorem.

[1] ZFC refers to the Zermelo-Fraenkel axioms of set theory. For the list of axioms, see (Kunen [1980]).

Part I

Formalising Isaacson's thesis

Chapter 2

Situating Isaacson's structuralism within contemporary structuralism

Isaacson's argument for his thesis is motivated by a form of structuralism. Although at some point, structuralism was regarded as an ontological position, contemporary structuralists hold varying views on the matter of ontology of mathematics. This chapter will introduce what structuralism is and the historical motivation in the position. Furthermore, I will discuss different types of structuralism due to Parsons, Shapiro, and Isaacson.

In the current chapter, we talk about the varying views in structuralism. In section 2.1, I will present a historical motivation for structuralism starting from Frege and also Benacerraf. From those, *eliminative structuralism* is born. This view will be discussed in section 2.2. As introduced earlier, there are various positions on structuralism including the views from Parsons, Shapiro and Isaacson. Each of their views will be discussed in detail in sections 2.3 for Parsons, 2.4 for Shapiro and 2.5 for Isaacson.

What holds the different views in structuralism together are the notions (introduced by Shapiro) of *coherence* and *categoricity*. Coherence can be understood as the notion of existence of a structure, while categoricity is the statement about uniqueness. For Isaacson, Shapiro and Parsons, categoricity appears to be an important discussion in their argument. But without coherence, or the existence of such structures, the categoricity claim is vacuous. We say that a theory is *categorical* just in case for any two structures of the theory, there is an isomorphism between them. But this claim universally quantifies over structures, thus without the existence of a structure, categoricity will trivially hold. Sometimes the existence and uniqueness claims can be expressed as categoricity – there is a unique model (up-to-isomorphism) – but for the sake of emphasising coherence and categoricity to be separate notions, I will not use the definition of categoricity that claims existence, in an attempt to separate the existence claim from the uniqueness claim.

Although we can formally define categoricity and understand it to pick out a unique structure, coherence is a more difficult notion to understand. Simply put as an existence claim, one could interpret that a system or a theory is coherent to mean that it is consistent in the deductive system of first-order logic, which is sound and complete. But the problem arises here that a model existence claim is dependent on first-order logic, which due to the completeness theorem or the Löwenheim-Skolem theorem give us many non-isomorphic structures. This shows that the categoricity claim for theories of infinite structures can never be attained along with the notion of coherence, defined in terms of consistency of the theory in first-order logic.

In section 2.6, I will go into details about *coherence* and *categoricity*. I will describe how Isaacson, Shapiro and Parsons differ on these notions. In fact, we will see later that Isaacson's structuralism is a crucial part of understanding Isaacson's thesis. His structuralism is centred around his concept of the *reality of mathematics*, which will be introduced in detail in section 2.7. But in the next section I will introduce the historical motivation for structuralism.

2.1 The historical roots of contemporary structuralism: Frege and Benacceraf

In *The Foundations of Arithmetic*, Frege [1974] argues that the natural numbers are objects that belong to concepts. Simply put, consider the concept *Jupiter's moon*. Then the number four belongs to the concept 'Jupiter's moons', hence 'the number of Jupiter's moons' is identical to the number four. Frege argues that numbers cannot be concepts, but objects, and this is reflected in the use of number words as predicates in natural language. The most intuitively obvious way to formulate an answer to how many lions are at the zoo in our natural language is 'there are seventeen lions in the zoo'. This sentence contains the number word 'seventeen' as a predicate that applies to 'lions in the zoo'. However, we can simply rephrase the sentence as 'the number of lions in the zoo is seventeen' so the number word is expressed as an object, rather than a predicate. Viewing the numbers as mathematical objects gives arise to an infamous problem denoted the *Caesar problem*:

> [...] but we can never – to take a crude example – decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a number or is not. (Frege [1974, §57])

For we know that Julius Caesar designates an object, Frege was concerned about whether we can meaningfully ask the question 'is Julius Caesar identical to the number 4?'. Intuitively, we do not accept Julius Caesar (a person) to be identified with any natural number, and intuitively, we might be willing to even say that such an identification question has an indeterminate truth-value. This is a problem in Frege's ontology, where we could question 'for any objects a and b, is it true that $a = b$?' Frege desired identity to be a notion so general, it can be a relation applied to any two objects. But a person clearly is not a number, and why should we be concerned whether a person is identical to a number?

This is where structuralism was presented as a possible solution to resolving such a problem. Benacerraf [1965] argues that Frege's Caesar problem arises from having a single universe of objects, where identity questions can be posed on any two designators:

> It made sense for Frege to ask of any two names (or descriptions) whether they named the same object or different ones. Hence [...] one could not tell from his definitions whether Julius Caesar was a number. (Benacerraf [1965, p. 64])

Since the problem is based on Frege's ontological position about objects – that they all belong in one world of objects – Benacerraf suggests that this view is too broad, and hence demands a context in which two designators in the same context can be asked of whether they are identical. He claims that a context is given by a predicate, and one can only ask the identity question between those objects that satisfy the predicate. Since it is indeterminate whether Caesar is a number, the question whether the number-four is identical to the non-number-Caesar must have an indeterminate truth-value:

> I am arguing that questions of the identity of a particular "entity" do not make sense. [...] (Benacerraf [1965, p. 65])

The question of identity between two designators such as 'Caesar' and 'four' is not just a matter of the fact that one is a non-mathematical object, while the other is an abstract mathematical object. What Benacerraf attempted to solve is the problem of *multiple realisation* of mathematical objects. With the demands for set-theoretic interpretations of non-set-theoretic mathematical objects, the natural numbers have gained *at least* two different extensional definitions. Within set theory, we call the natural numbers *finite ordinals*, where an ordinal is a set with a well-order ¡[1].

One realisation of the natural numbers as finite ordinals in set theory is known as the *Zermelo ordinals* – $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \cdots$ – and the other, is the *von Neumann ordinals* – $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \cdots$. Note that these systems have different definitions of the successor function. In the Zermelo ordinals, for any number n, the successor of n is defined by $n + 1 := \{n\}$, so each number (except for *zero*, or *one* in the case of Benacerraf) only contains the immediate predecessor. On the other hand, for the von Neumann ordinals, $n+1 := n \cup \{n\}$, thus containing every predecessor as an element. Note that the two definitions of the number *three* are not extensionally identical. For Zermelo is $\{\{\{\emptyset\}\}\}$, and for von Neumann, it is $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, we can see that the number $0 := \emptyset$ satisfies $0 \in 3$ for von Neumann, but it is not satisfied for the Zermelo *three*.[2]

There are more issues with these two possible definitions of the natural numbers than there was for Julius Caesar. Firstly, the relation $\in$ is not an arithmetical relation, thus it is questionable whether $0 \in 3$ is a question that is meaningful in arithmetic. One attempt to resolve this problem is to ask the question 'why should a property of a natural number be only formulated in the language of arithmetic?'. For example, 0 is a number that is

[1] $<$ is a well-order on a set X just in case for any $x, y \in X$, one of the following holds – $x < y$, $y < x$ or $x = y$; $<$ is a transitive relation; and for any non-empty subset Y of X, there is a $<$-least element Y.

[2] In many set-theoretic textbooks we simply define an ordinal to be the von Neumann ordinal. Since the transitivity of $\in$-relation does not hold for Zermelo ordinals, in practice von Neumann ordinals are preferred.

more talked about by people than the number 927376393923, but 'more talked about by people than' cannot be formulated in the language of arithmetic.[3] The problem with this attempt is that this is not an essential property of the number 0 – whether or not 0 is talked about more than some other number, it does not change that this number *is* 0.

Secondly, if one commits to ontological realism, then one will ostensibly have to commit to a unique definition of the natural numbers. But if one does not commit to ontological realism, given the two extensionally different definitions of the natural numbers, and with distinct definitions of the arithmetical operations and relations on these definitions, these two definitions might satisfy the same properties that we desire from the natural numbers.[4] The multiple realisation problem certainly makes Benacerraf's so-called eliminative structuralism more appealing than Frege's ontology that takes the numbers to be objects.

Benacerraf concludes his paper by asserting that arithmetic is the *science of structures*, rather than the *science of particular number objects* (Benacerraf [1965, p. 70]). Therefore, Frege's analysis of arithmetic is inadequate and the multiple realisation problem cannot be avoided. Thus diverging away from Frege's ontology, and moving towards Benacerraf's structuralism resolves the issues at hand. What is more pleasing is the fact that regardless of whether one still commits to Frege's ontology, we can view the Zermelo ordinals and von Neumann ordinals as structures of the natural numbers that are equivalent. This equivalence basically shows that regardless of what the numbers *are*, the way they relate to other numbers in the structure is invariant even if these objects are replaced with other objects. Thus Benacerraf's structuralism abstracts away from the question of 'what are numbers' to the questions 'how do numbers relate to each other', 'is the number n identical to a given sum', etc. and modifies the ontological perspective and commits to *eliminative structuralism*.

The following few sections will introduce various views on structuralism. Starting with *eliminative structuralism*, motivated by Benacerraf, we will look into Parsons's *conceptual structuralism*, Shaprio's *ante rem structuralism* and Isaacson's *concept-realist structuralism*.

2.2 Eliminative structuralism

In this section, I will expand on Benacerraf's *eliminative structuralism*. As shown in the previous section, Benacerraf was motivated to resolve Frege's Caesar problem, by leaving behind *object-based* ontology, and focusing on structural notions. I will discuss some problems in *eliminative structuralism*, regarding the *multiple realisation*. It appears that the multiple realisation of the numbers as a solution for Frege's Caesar problem, rather ignores it than attempts to solve it.

It is possible to characterise the fundamental part of structuralism as the relativity of ontology. Regardless of what ontological position one takes, different structuralists' will

[3] I have not attempted to formulate it myself, but I would be surprised if there was a formulation.

[4] One could argue against this by raising the questions relating language and reality. For example, there could be some indeterminancy of reference in our mathematical language. I will not touch on this issue here.

share some equivalent views on the ontology of the structures. As mentioned before, the two set-theoretic extensions of the natural numbers are still structurally equivalent and that is *enough* for the practice of mathematics. A position in structuralism that is only concerned with relative ontology is *eliminative structuralism*. This term was introduced by Parsons [1990], but he rejected the position. Benacerraf's position in structuralism can be considered to be part of eliminative programme, as he abstracts away from the nature of the objects but is only concerned with the internal relations the objects have within the given structure.

Parsons [1990, p. 307] describes eliminative programme as 'a way of eliminating reference to mathematical objects'. This way, the multiple realisation problem is no longer in question because one can simply reject the question of identity to be 'meaningless' as Benacerraf has done. However, there are several problems with this position. Eliminative structuralism ignores the ontological nature of the objects in discussion, rather than solving the multiple realisation problem, by committing to the view that there are no objects to discuss.

Shapiro [2000, p.86] describes that for the eliminativists, the '*nature* of objects' does not matter, as long as it has 'a lot of objects there'. If we take some Φ to be a sentence in the language of arithmetic, eliminativists can take a natural number structure S such that $\Phi(S)$ is obtained by 'interpreting the nonlogical terminology'.

This means that we can take any system (say the von Neumann ordinals) representing the natural numbers, and abstracting way by some operation Φ from the set-theoretic ontology, and focus only on how the objects relate to each other. But the question is whether set theory is the adequate ontology for the eliminativists. Parsons's claims that the extensional nature of set theory shows that it is not the right kind of background ontology for the eliminative programme.

Note that even if we can ignore the ontological nature of the objects within a structure, we still need the background ontology to be available. But if we don't require the background ontology to be structural, then it fails to give a full structural account of what structuralism is about. This way we end up committing to the mathematical structures to exist in the set-theoretic hierarchy, which is already within the structure of sets. So in the end, this position would inevitably conclude with that all mathematical structures are sets.

One way to escape this problem is to talk about a *possible* ontology, rather than committing to a specific ontology. But this position requires us to better understand the notion of *possibility* is, logical or metaphysical. And if we decide that it is a logical notion, which logic can we take to accurately hold the position? These kinds of questions give us reasons to look at other forms of structuralism.. With many issues unanswered for the *eliminative structuralism*, we move onto other views of structuralism that are faced with different problems.

2.3 Parsons's contextual structuralism

If *eliminative structuralism* had problems from the ontological perspectives, some views in structuralism do not have to do with ontology. One non-ontological position within structuralism can be attributed to Parsons (Parsons [1990]). We will call Parsons's position *contextual structuralism* – it is a meta-linguistic thesis claiming that structures are given by predicates that are interpreted based on our interaction with mathematics. Parsons's introduces his structuralism after arguing against eliminative structuralism – his position is dubbed by Button and Walsh [2016] as *contextual structuralism*. In this section, I will outline Parsons's structuralism, highlighting the difference between Parsons and Benacerraf.

Parsons is motivated by what he calls *quasi-concrete* objects, which are abstract objects that can be represented in the concrete (see (Shapiro [2000, pp. 100-105])). For example, geometric figures can be represented in the physical space, and the natural numbers can also be represented by strokes on a surface. What is important in this position is the context in which these quasi-concrete objects that are being discussed, thus Parsons describes his structuralism to be *meta-linguistic*:

> I have resisted the interpretation of structuralism that would make it an interpretation of mathematical statements as *about* structures, thus giving rise to the question what manner of objects theses are. [...] The most fundamental notion of structure of this purpose is metalinguistic: the 'domain' is given by a predicate, and the relations and functions by further predicates and functors. (Parsons [1990, pp. 335-336])

In order for us to understand the domain of our structures, we need to have a context which we interpret the predicate in. The identity question between Zermelo ordinals and von Neumann ordinals, within the context of set theory, is a very much meaningful question that has a determinate truth-value. But within the context of arithmetic, this is not the case. Hence in different contexts, the same structure (e.g. the structure of finite numbers) can differ whether an identity question has a determinate truth-value or not:

> When, however, the structuralist thesis says that talk about mathematical objects presupposes a background structure, what this structure is is context-dependent. [...] Attribution of a 'background structure' to a mathematical discourse is not only context-dependent; it is also a matter of interpretation. (Parsons [1990, pp. 333-335])

In this position our interpretation of the predicate that determines the structure in discussion is very important. It is possible that we interpret the natural numbers to be some set-theoretic objects. In this interpretation the question of 'which set-theoretic objects' is highly relevant. But when we discuss arithmetic, we are concerned only with the arithmetical operations, and not about which set-theoretic objects the numbers are identical to.

When quasi-concrete objects are in discussion, it is clear that there must be some external relation between the abstract objects and the concrete representation. The ontological picture of structuralism does not indicate how these different kinds of structures are related to each other. Eliminative structuralists are only concerned about the internal structure of a given structure, and not with how these structures are related to other representations. Parsons's contextualism tried to capture this problem. And hence the context in which we are discussing the predicates is important because we can obtain some knowledge about the objects through their concrete representations.

We could question whether there is another way to rescue structuralism from the ontological problems Benacerraf's structuralism suffered from. Unlike Parsons, Shapiro engages directly with the ontological questions about structures. Hence we will move on to Shapiro's structuralism in the following section.

2.4 Shapiro's ante rem structuralism

In this section, I will introduce another form of structuralism that tackles the ontological questions directly. Shapiro's *ante rem structuralism* is an ontological claim about the structures, but differs very much from Benacerraf's structuralism.[5] For Shapiro, mathematical objects and mathematical structures 'exist'. He commits to the claim that the places in a structure are the mathematical objects, and a structure is 'an abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system' (Shapiro [2000, p. 74]). For this view of structuralism, Shapiro claims what he requires is 'the *theory* of structures' (Shapiro [2000, p. 92]).

Shapiro also gives an account of what objects and structures are. He holds the view that objects in structures are *positions in structures* (Shapiro [2000, p. 92]). The notion of 'positions' here can be understood to be similar to *types* in model theory. [6] In model theory, a *type* for a constant a in a model M is defined to be the set of formulas in the language of the model, that is satisfied by the constant a in M (see (Marker [2006, p. 115]) for the definition). In some sense, we can consider this to be a collection of descriptions that can be assigned to an object in the domain.

When two things in the domain satisfy the same description but we do not have the constants representing the objects in the domain, it is only their *types* that capture these objects. In this case, the model is not capable of distinguishing between the two objects since the only information that is available to the model is the types of these objects, which are identical. In this sense, in the context of natural number structures, we cannot distinguish between each of the numbers of Zermelo and von Neumann ordianals, as e.g. the object 3 is merely the position in each of the structures. We are not equipped with the set-theoretic language which is needed to give the description of Zermelo and von

[5] In some sense, Shapiro's structuralism is an ontological thesis that is more similar to Frege than Benacerraf. Shapiro believes in mathematical objects and believes that the natural number structure N is exemplified by 0 and the successor function $s : N \to N$.

[6] This should not be confused with the *types* in *type-token* distinction.

Neumann ordinals as positions in our structures. For more discussions on *objects-as-types*, see (Burgess [1999]) and (Keränen [2001]).

Although Shapiro holds a very different position to Parsons in his structuralism, he claims that even the distinct positions in structuralism are all equivalent:

> Indeed, it follows from the thesis of structuralism that, in a sense, all [...] options are equivalent. (Shapiro [2000, p. 90])

He further supports this position by giving an axiomatic description of what a structure is[7]. If every structuralist can agree that these axioms do capture what structures are, and what matters for a structure is to satisfy these axioms, and it is so that the given collection of axioms is 'categorical', then there is one unifying understanding of structuralism. This idea can seem very pleasing for structuralists because it gives them a way to unify the distinct views of structuralism. If they differ on their accounts of structures, this might be from some non-structural description of the structures. By giving some axiomatic descriptions and capturing structures in this way, structuralism can stay faithful to their commitment to structures without committing to some non-structural background ontology. Thus Shapiro's axiomatisation could be a way Benacerraf, Parsons and Shapiro can be brought together. But we will see in the next section that this is not the case for Isaacson's structuralism.

2.5 Isaacson's concept-realist structuralism

Isaacson's structuralism differs from the previous accounts of structuralism in many ways. Isaacson argues that structures cannot be sets because sets are already in the structure of sets, namely the set-theoretic hierarchy. For Isaacson, the *reality of mathematics* is what is important in describing his structuralism, which I will call *concept-realist structuralism.* The *reality of mathematics* is not concerned about the ontology of mathematics but rather the truths of mathematics. In this sense, Isaacson's structuralism stands on its own among the other structuralists' positions.

In this section, we discuss Isaacson's structuralism based on his concept of the *reality of mathematics.* His structuralism is a fundamental position in understanding his thesis. He is motivated by his *conceptual reality*, which bridges variations of Platonism and intuitionism. And one could interpret Isaacson's structuralism to be motivated from views in philosophy of mind:

> The compelling and immediate reason for rejecting the idea that mathematics is about particular objects is that for any mathematical theory the domain of objects which that theory is taken to be about can always be replaced by a

[7] The axioms given by Shapiro are named 'infinity', 'subtraction', 'addition', 'powerstructure', 'replacement', 'coherence' and 'reflection'. See (Shapiro [1991, p.90–97]) for more details on the axioms, but in general, the axioms are similar to the usual set-theoretic axioms of second-order ZFC with 'coherence' referring to the existence of a structure from categorical characterisation and 'reflection' representing the idea of the reflection principles in set theory.

> domain consisting of different objects[....] What structure must be preserved by isomorphism depends, of course, on the notions being used in that mathematics, but in all cases mathematics is inherently to do with structure. [...] The particular individuals do not make a difference to the mathematics. What we are concerned with is structure. (Isaacson [1994, pp. 122-123])

Isaacson describes himself to be a *concept realist*, where mathematicians come into contact with the structures by their *concepts*. According to Isaacson, concepts here refer to the 'process of thinking', which should be understood epistemologically. And thus the questions regarding the ontology of mathematics is less important because the ontological nature can always be replaced, but the structures that describe how objects relate to each other cannot be. In this sense, isomorphisms between structures are still important for Isaacson, because isomorphic structures not only preserve the internal relations between the objects in the structure, but also the truths of these objects.

The importance of concepts in Isaacson's structuralism can be seen to be similar to Parsons's contextualism, as they both appear to rely on the notions of *concepts* and *intuitions*. However they differ in a few ways. For Parsons, the context the structures are described in depended a lot on the interpretation. One of the ways we can come into contact with the mathematical structures was by the representations of the quasi-concretes as concrete objects. Hence the interpretations of what the predicates that capture the structures are are an important feature of this position.

Contrastingly, Isaacson's concept realism is concerned with the *concepts* which are, according to Isaacson, 'a process of thinking'. We do not interact with mathematical structures through the quasi-concretes but rather via the concepts, that is, via our thoughts:

> Because a structure is given by concepts, it favors an account of mathematical reality in terms of the reality of mathematical concepts. [...] The locus of our contact with concepts is the process of thinking about, or with, them. Concepts are the sort of thing with which the mind engages. [...] I shall call such a philosophy of mathematics, based upon the objective reality of mathematical concepts, "concept [realism]" (Isaacson [1994, p. 125])[8]

The view that we engage with mathematics via our process of thinking can be seen to be similar to Brouwer's *intuitionism*. Brouwer's intuitionism claims that mathematics is the construction of our mind, and thus there is no real independent realm of mathematics. Hence, mathematics can exist only as much as mathematicians are constructing them mind dependently. However, Isaacson's position differs from Brouwer on the importance of mathematical truths. For Brouwer, the truth-values of mathematical statements are not important notions – he was not concerned about truths, but rather the process a mathematician's mind takes to capture whether a statement is correct. Hence Brouwer did not accept the law of excluded middle on the grounds that this involves some omniscience of the mathematician. On the other hand, Isaacson believes and commits to bivalent

[8] He initially dubbed his position to be 'concept Platonism', but later he changed the position to 'concept realism' as Platonism can be confused with an ontological position.

truth-values, and that is how we can grasp the unique *intended* structure via the concepts in the 'reality'. See (Brouwer [1912]) for more on Brouwer's intuitionism.

Isaacson's structuralism differs from other views on structuralism that we have seen so far. Recall Shapiro's suggestion that axiomatising the notion of structure is a way to unify different views on structuralism. But Isaacson strongly argues against Shapiro that this is what the axiomatisation achieves. Isaacson claims that 'Shapiro's [axioms do] *not determine* any notion', but he is '*using* the notion as understood' (Isaacson [2011b, p. 28]).

One could argue that we can just take the notion of structures to be a primitive notion. But this will fall back into the problems of eliminative structuralism which Parsons had argued against. Furthermore, Shapiro had suggested that objects in structures are similar to *types* in models, and in his set-theoretic hierarchy, a structure can be thought of as a model in the hierarchy because our background theory is set theory for our structures. But Isaacson claims that 'the models proved to exist are not, in general, the mathematical structures that structuralism is about' (Isaacson [2011b, p. 28]).[9] There are many models of arithmetic that are non-isomorphic to the standard model. Isaacson argues that committing to models to give the structures would be problematic because the models do not capture the unique structure of arithmetic that we are looking for. Shapiro might resort to second-order logic, if set theory appears to be problematic. But it still remains for Shapiro to argue what the real background ontology for the structures must be as he remains to be an ontological realist.

In summary, Isaacson's structuralism can be considered to be an epistemic position, that differs from Parsons and Shapiro. Isaacson is concerned about the concepts we understand, and these concepts are in the *reality of mathematics* which are *processes of thinking*. Although what Isaacson means by a *process of thinking* is unclear, we understand so far that Isaacson's *reality of mathematics* gives the *intended particular-structure*.

Although there are different views on structuralism that are incompatible with each other, they agree on the importance of categoricity within their positions. One well-known contrarian to this position is Resnik (Resnik [1981]), where he claims the we could have structures of a theory in first-, second-order logics or in modal logics, that give us more access to different kinds of structures for a theory. We will not discuss this position in depth, but we concentrate, in the following section, on Shapiro and Isaacson's positions on the importance of *coherence* and *categoricity*. By having a system that is categorical, we can pick out a unique structure that captures the system. But without some form of existence, categorical systems are vacuous, since it claims that any two structures are equivalent. But if there are no structures at all, the statement will hold trivially. The existence of structures can be captured by the notion of *coherence*.

[9]Note that Isaacson distinguishes 'models' from 'mathematical structures' here. This view of Isaacson can be seen in his earlier writing such as in (Isaacson [1987]). We will come back to this issue later in the thesis, when we formalise the notion of *genuinely arithmetical* statements, in an attempt to give a precise formulation of Isaacson's thesis.

2.6 Coherence and categoricity

In this section, I will outline how we can formally understand the notions of *coherence* and *categoricity*. Focusing on Shapiro and Isaacson mainly, it will be highlighted that they not only differ on the process of grasping these structures, but also on the importance of *coherence* of the captured structures.

In the case of geometry, Isaacson argues that we have models of non-Euclidean geometry which enlighten us that there is no one unique structure of geometry. We can grasp the distinct concepts of Euclidean and non-Euclidean geometry, to both be about geometry (and space). In this sense, we cannot give a categorical characterisation for *geometry*. One could question whether this goes against Isaacson's notion of *reality of mathematics* where truth-values are preseved about certain concepts. In fact, this is where Isaacson distinguishes between the *particular* and *general* structures:

> The view [...] taken that the discovery of non-Euclidean geometry showed that the axioms of Euclidean geometry are not to be thought of as "evident truths" because they are not, as it turns out truths at all. [... The] fifth postulate holds in Euclidean geometry and not in non-Euclidean geometries. Since no sentence and its negation can both be true, the fifth postulate is, on this other view point, neither true nor false. There is no true geometry. It is, of course, correct that there is no *one* true geometry. (Isaacson [2011b, p. 25])

Particular structures are unique structures that are captured, but the general structures only pick out a generalisation of certain structures.[10] Of course, general structures can be captured by a particular structure in category theory, by taking the category of groups, for example. Axiomatic approaches have been used to study general structures, as a given collection of axioms can give many structures that satisfy the axioms. But the *reality of mathematics* is about particular structures of mathematics (Isaacson [2011b, p. 25]), thus having a unique *intended* structure of arithmetic or set theory is important.

However, this distinction between *general* and *particular* structures is not always clear. What Isaacson wishes to claim is that there are different ways of studying structures. When we practice group theory, we are interested in the *general* structure of groups. But when we study arithmetic, we focus on the *particular* structure which we have captured from the categorical characterisations.

We can best understand Isaacson's *reality of mathematics* to be some conceptual thing that contains the structures that are captured by categorical characterisations. And, only by giving the categorical characterisation, we can come to understand that these *particular* structures *exist*.

Shapiro claims that capturing the existence of a structure is to have a coherent axiomatisation of it. Since his position relies on model-theoretic structures, he considers (but rejects) that it could be seen that a consistent axiomatisation will imply existence. However, consistency cannot be used to make an existence claim, for there are infinitely

[10] The *particular* and *general* structures are equivalent to Shapiro's *token-structures* and *type-structures*, respectively.

many first-order models of Peano arithmetic (PA), so it would be difficult to pick out the relevant structure. Furthermore, using consistency to mean coherence can be problematic when we have models of PA that claim that 'PA is not consistent'. If coherence gives a coherent model of the theory, it is hard to see how a model, whose existence comes from PA's consistency, claiming that 'PA is not consistent' can be a coherent model. If we attempt to take the consistency statements to claim the existence of structures, then we must accept that we should be able to intuitively accept that a model of PA can think of itself to be inconsistent. Thus Shapiro concludes that coherence is not a formal notion, but should be understood to be a primitive and informal notion (Shapiro [2000, p. 135]), and we should obtain coherence via axiomatisation.

Isaacson disagrees with Shapiro on the notion of coherence. He claims that we do not obtain coherence from axiomatisation but by 'reflecting on the development of mathematical practice by which particular mathematical structures come to be understood, the natural numbers'. He further claims that [there] is nothing we can do to establish that particular mathematical structures exist apart from articulating a coherent conception of such a particular structure' (Isaacson [2011b, p. 29]).

For both Shapiro and Isaacson, coherence is not a mere existence claim about the structures, but the existence of the *intended* structure. One important position for both of them is that there is a categorical characterisation for the particular (or *token* for Shapiro) structure. This can be seen as the axiomatisation for Shapiro in second-order logic or set theory, and the proofs of the categoricity of the axioms in higher-order language for Isaacson. In fact, Isaacson notes that 'each particular structure developed in the practice of mathematics' has a 'categorical characterization' which can be captured by 'full second-order logic' (Isaacson [2011b, p. 30]). He further emphasises that first-order logic is inadequate for capturing categoricity, but its completeness makes it more desirable as a deductive system it is only by higher-order language that the structures come to 'existence'. But it is important not to confuse Isaacson's use of 'existence' as an ontological position.

When it comes to the existence claims about mathematical objects, Isaacson resists committing to independently 'existing' objects. He claims that:

> I accept and must insist upon the existence of various abstract entities (I am not a nominalist). [...] So why not also numbers and sets as abstract entities? The point is that an individual natural number [...] exists only through its determination by the categorical theory of the structure of the natural numbers. It has no independent existence. (Isaacson [2011b, p. 37])

As mentioned earlier Shapiro believes that mathematical structures *exist* in some ontological sense. And similar to Isaacson's claim, Shapiro [2000, p. 72] claims that numbers don't have 'any independence from the structure in which they are positions', and the essence of the numbers are defined by its relation to other numbers.

Isaacson continues on to say that axiomatisation comes *after* the structures which exist in the conceptual way:

> The basis of mathematics is conceptual and epistemological, not ontological, and understanding particular mathematical structures is prior to axiomatic

> characterization. When such a resulting axiomatization is categorical, a particular mathematical structure is established. Particular mathematical structures are not mathematical objects. They are characterizations. (Isaacson [2011b, p. 38])

Thus the 'existence' claim is only established by giving the characterisation, but the characterisation can only be obtained after the understanding of the concept. The importance of the categorical characterisation is that it can be seen as the evidence of the *intended* structure in the *reality of mathematics*. There is an *intended* structure of arithmetic in the *reality of mathematics*, and we can give a categorical characterisation of arithmetic in the language of second-order logic. The proof of categoricity of the characterisation shows that this does capture our *intended* structure. In establishing the coherent characterisation, we also require the 'informal rigor' that allows us to see that the charactersation is 'coherent' (Isaacson [2011b, p. 41]).

Furthermore, Isaacson argues that we 'must reflect on our conceptual understanding of a given structure' when it comes to a successful characterisation. It is clear that first-order logic does not capture the coherent structure that we desire from our conceptual understanding. First-order logic fails at capturing the *intended* infinite structure due to results such as the compactness theorem and the Löwenheim-Skolem theorem. The Löwenheim-Skolem theorem states that for any theory with countable cardinality, there are models of cardinality κ for every uncountable cardinal κ. But second-order logic also has problems. While the deductive system of first-order logic is sound and complete, this is not the case for second-order logic.

Full second-order logic is where the second-order quantification is applied to all subsets of the first-order domain. This can be seen as the intended meaning of second-order logic against the Henkin semantics that takes some arbitrary interpretation of the quantifications. There are some discussions to whether full semantics really is the *intended* semantics, but we will not discuss this here. Although full second-order logic is not complete, i.e. there are second-order validities in full-semantics that cannot be proven in the second-order deductive system, it captures the *intended* coherent structures from Isaacson's *reality of mathematics*. What this feature of second-order logic shows is that second-order logic cannot capture a strong deduction system that we might desire:

> The very success of first-order logic for deduction unfits it for characterization, and the very success of full second-order logic for characterization unfits it for deduction. The completeness theorem for first-order logic implies the compactness theorem and proofs of the completeness theorem establish the Löwenheim-Skolem Theorem. These two theorems show that first-order languages are not suited to characterizing structures. [... The] success of characterizing the structure of the natural numbers by means of full second-order quantification establishes that there can be no complete system of deduction with respect to second-order validity. (Isaacson [2011b, p. 47])

For deductive purposes, it is not hard to see that second-order logic is not the best fit. In fact, the existence of second-order categorical characterisations of arithmetic or set theory shows that these systems must be semantically decidable. More specifically, second-order ZFC is quasi-categorical[11], thus the continuum hypothesis is semantically decidable. But the continuum hypothesis cannot be decided deductively in second-order ZFC.

It is unclear how important the deductive system is for Isaacson's structuralism. What the first-order logic shows is that it is not the axioms that can decide the undecided problems of arithmetic or set theory. For Isaacson claims that the undecidability of certain statements cannot be solved by adding new axioms but rather shows that some questions are not important in certain theories. Perhaps Isaacson would argue that this is because these questions are not 'real' arithmetical or set-theoretic sentences.

The adequacy of second-order logic to capture categoricity is what attracts all three structuralists to discuss second-order logic in depth. For Parsons's contextualism, we can consider our structure to be captured by a second-order predicate, and the description of the structure is given by some second-order axioms. The semantic interpretation of these axioms will be left to the intuitions, but if the axioms are categorical, we can successfully discuss the equivalent structures independently.

By contrast, Shapiro's ante rem structuralism motivates categoricity by ontological realism. When there is a categorical theory that captures infinite models, then we can use the axioms of the theory to be exemplifying the *intended* infinite structure. As mentioned earlier, this can be done successfully in second-order logic, but not in first-order logic due to Löwenheim-Skolem. The categorical theory also characterises the 'same structure' no matter what the axiomatisation might be, hence categoricity is very useful in exemplifying the same structure.

Isaacson's motivation for categoricity to take an important role in his structuralism is due to his belief in bivalent truth-values, that are preserved in the concepts. Being able to provide a categorical characterisation is showing that there is a unique conceptual structure in the *reality of mathematics*. The concept of these structures comes prior to the axiomatisation in the reality, but categorical characterisations show that the *intended* structure exists in the reality.

In the case of arithmetic, the use of second-order *axiom of induction* has been very successful. The axiom captures the *intended* structure of the natural numbers. One can formulate the axiom as the following:

$$\forall X((X0 \ \wedge \ \forall x(Xx \rightarrow Xsx)) \rightarrow \forall x \ Xx).$$

In Parsons's motivation of categoricity, this is captured by the predicate X (in the above formulation). Hence, X satisfies the property that there is some initial entity 0 that is the smallest, and if it is successively closed, then it captures the whole X. Hence when we discuss the natural numbers, we know that we are referring to the same structure that satisfies the above property. For Shapiro, there is a unique natural number structure that we want to characterise, and the above axiom (with some other axioms describing the successor) satisfies that by Dedekind's categoricity theorem (see (Shapiro [1991])). If

[11]Second-order ZFC is *quasi-categorical* because for any two structures of second-order ZFC, either they are isomorphic or one is an initial segment of the other.

the axiom had failed to be categorical, then we just fail to capture the unique structure that really exists. In fact, this unique structure is extensionally identical to to Dedekind's *simply infinite system*, which is captured as the minimal structure containing an initial element, say 0, such that it is closed by the successor operation. But the fact we can categorise it gives arise to its existence in Isaacson's epistemological *reality of mathematics*. The *intended* structure of the natural numbers exists because we can give a categorical charactersation. And we are able to give a categorical characterisation because we have our conceptual understanding of the structure.

The importance of categoricity is exemplified in all three structuralists' positions. Shapiro claims that coherence is more important than categoricity because coherence gives the existence of the structures, but Isaacson obtains the *reality of mathematics* from categoricity. In this sense, Shapiro might tie his notion of *coherence* to Isaacson's notion of the *reality of mathematics*.

For Parsons, categoricity can be a way of capturing the unique structure in the meta-linguistic way. The predicate that we describe our domain of the structure to be, can be interpreted depending on the intuition we have of the structure. But this does not restrict us to capture the one structure that we'd like to refer to.

For the rest of the thesis, we only focus on Isaacson's structuralism and the *reality of mathematics*. Hence, we understand categoricity to be the crucial notion, and through categoricity, we can obtain coherence.

2.7 Summary

To remind the readers of Isaacson's structuralism and the *conceptual reality of mathematics*, I will give a brief summary of what we have discussed so far in this chapter. Isaacson's structuralism stands out from the other forms of structuralism we touched on as it is independent of some background system.

Parsons's structuralism was labelled *contextual structuralism*, a view that claims mathematical structures give a partial picture of the mathematical ontology. Note that contextual structuralists are not concerned whether these structures that they talk about exist. But they hold the view that contextual structuralism is a *meta-linguistic* thesis, where the *interpretation* on the predicate, dependent on our *intuition*, defines the structure. This is to say that the understanding of structures described by the predicate comes from our own interpretation.

In contrast to contextual structuralism, Shapiro's structuralism is an ontological thesis about what structures are. We called Shapiro's position *ante rem structuralism*. His commitment to the existence of mathematical structures and objects is a fundamental part of the position. In an attempt to fully describe his structuralism, Shapiro gives an axiomatisation of what a structure is. With the axiomatisation, Shapiro argues that all distinct positions regarding 'what structuralism is' are equivalent. Isaacson however argues that this is not the case.

Isaacson's structuralism, which I have called *concept-realist structuralism* is neither ontological nor meta-linguistic. He differs on the fundamentals of what the *reality of* *mathematics* is to Parsons and Shapiro – it is based on mathematical *truths*, rather than mathematical *objects*. Thus, unlike Shapiro, Isaacson is not concerned about mathematical ontology, but only with the truth-values. And unlike Parsons, the *intuition* about the predicates is not as important as the *concepts* of mathematics, which are a part of our *process of thinking*. Isaacson's position could rest on his position within philosophy of mind about what concepts are, which is not discussed by Parsons nor Shapiro.

There is also Feferman's *conceptual structuralism* which varies from Isaacson's *concept-realist structuralism*. We did not discuss this position in the chapter, but it is important not to be confused between the two views of structuralism as Feferman [2009] explicitly asserts that his position is different from Isaacson's. Feferman holds the view that mathematical thoughts exist 'as mental conceptions', and the conceptions are gained through 'everyday experience'. This differs from Isaacson's structuralism since Isaacson is motivated by the truth-values of mathematical statements, rather than the method of gaining these conceptions. Furthemore, Feferman's structuralism cannot be considered independently of empiricism, while Isaacson is not explicit on this issue.

In order to understand the mathematical structures Isaacson is referring to, we merely have to look into the concepts of the mathematics we have. What is problematic with this position is we are unsure of how we gain the 'concepts'. All we have is that the 'concepts' are a 'processes of thinking'.

The *conceptual reality* is where all the mathematical structures are. Among those the particular structures can be captured by giving some categorical charactersation. From the proof of categoricity of the characterisation, we give evidence for capturing the *intended structure* and its existence in the *conceptual reality*. It can be seen that Isaacson has chosen second-order logic to capture the structures, and first-order logic for deductive ideas. If *concepts* are *process of thinking*, perhaps that is why Isaacson put some importance on deduction – deduction can be seen to reflect some notion of *process of thinking*.

Isaacson remarked that capturing a categorical characterisation has to be done in higher-order logic, such as full second-order logic. But full second-order logic fails to be complete, thus we required first-order logic for deduction. On the other hand, first-order logic is unfit for capturing structures, so we require both systems in order to capture the structure and also the deductive ideas. What Isaacson teaches us is that we do not have to rely on faith in ontology in order to attain a view of structuralism. The way we obtained the '*intended* structure' of arithmetic, set theory and other particular structures, is via second-order characterisation. From that, we can obtain the first-order theory of arithmetic, that is an extension of the sound and complete deductive system.

We conclude this chapter with a summary of Isaacson's structuralism and the *conceptual reality of mathematics*. In the next chapter, I will state Isaacson's thesis and outline how his thesis is motivated by his structuralism. Isaacson's has never explicitly stated a formulation of his thesis, so I devote parts of next chapter in accurately formalising Isaacson's thesis.

Chapter 3

Isaacson's Thesis, higher-order concepts and direct perceivability

In chapter 2, we introduced Isaacson's structuralism and his notion of *conceptual reality of mathematics*. Isaacson's structuralism is an epistemological position about mathematical structures. The structures exist as *concepts* in the *reality of mathematics*, and we can capture these structures by providing a categorical characterisation of the structure.

Isaacson's structuralism is explained in (Isaacson [2011b]), but the discussion on Isaacson's thesis comes prior in (Isaacson [1987, 1992, 1994]). Isaacson has never formulated his thesis precisely, which gave arise to multiple distinct formulation of Isaacson's thesis. In this chapter, I will introduced two distinct formulations of Isaacson's thesis. Of course, both of these formulations have been motivated from (Isaacson [1987, 1992, 1994]), but I will argue that one formulation is more fundamental and superior to the other.

3.1 Isaacson's thesis

In this section, we outline two different formulations of Isaacson's thesis. The idea behind Isaacson's thesis has been developed through (Isaacson [1987]), (Isaacson [1992]) and (Isaacson [1994]). The formulations of Isaacson's thesis I will discuss here can be found in (Smith [2008]) and (Incurvati [2008]). Incurvati states Isaacson's thesis as:

> PA is sound and complete with respect to arithmetical truths' (Incurvati [2008, p. 3]);

while Smith formulates Isaacson's thesis as:

> If we are to give a proof for any true sentence of $\mathcal{L}_a$ which is independent of PA, then we will need to appeal to ideas that go beyond those that are required in understanding PA (Smith [2008, p. 1]),

where $\mathcal{L}_a$ denotes the language of first-order arithmetic. In some sense, we can consider these statements to be equivalent to each other – we can take 'understanding PA' to mean 'arithmetical', and obviously if PA is 'sound and complete' then anything that is 'independent of PA will require appealing to ideas going 'beyond those that are required in understanding PA'. But with these formulations we still need to know what 'arithmetical truths' (Incurvati [2008]) and 'ideas [going beyond] understanding PA' (Smith [2008]) really mean.

I will introduce Isaacson's examples of *higher-order concepts* and show how they can be interpreted to be 'non-genuinely arithmetical' – where I use *genuinely* to emphasise that it is not just a statement that can be expressed in the language of arithmetic, but satisfies Isaacson's notion of *arithmetical truths* in the sense of Incurvati's formulation. I will argue in section 3.1.2 that relying on *higher-order concepts* in formulating Isaacson's thesis seems circular. However, using the notion of *genuinely arithmetical* can be a more informative way to understand Isaacson. In this sense, Incurvati's formulation of Isaacson's thesis is more fundamental than Smith's formulation. We start with a discussion on what *genuinely arithmetical* (or in Incurvati's terms, *arithmetical truths*) means.

Incurvati [2008, p. 262] gives a formulation of what 'arithmetical truths' are as the following:

> Arithmetical Truth: A proposition is arithmetically true if and only if (i) it is expressible in L_{PA} and (ii) it can be perceived as true directly on the basis of our grasp of the structure of the natural numbers or directly from arithmetical truths.

Here, L_{PA} refers to the first-order language of arithmetic $\{s, +, \cdot, <, 0\}$. What is still unexplained from the above formulation is what it means for a statement/proposition to be *directly perceivable*. In Smith's formulation, what appears to be essential in understanding Isaacson's thesis is the *higher-order concepts*, which is still left unexplained. By the end of this chapter, I hope the readers will have some understanding of what it means for a statement to be *directly perceivable*. And from this understanding, we will be able to express what *genuinely arithmetical* means according to Isaacson. For now, we focus our attention to the notion of *higher-order concepts*. I will introduce the examples from Isaacson [1987], so we can have a better understanding of Smith's formulation of Isaacson's thesis.

3.1.1 Higher-order concepts

Isaacson [1987] is where we first see the ideas of 'higher-order concepts' and Isaacson's thesis. In the paper, Isaacson argues that Peano arithmetic 'occupies an intrinsic, conceptually well-defined region of arithmetical truth' (Isaacson [1987, p. 147]). This resonates well with where Smith's formulation of Isaacson's thesis could have come from. Isaacson claims that we can access 'hidden-higher-order concepts' in PA via the method of coding. Isaacson describes that the phenomenon of coding contains 'essentially hidden higher-order' concepts (Isaacson [1987, p. 148]), while Peano arithmetic is 'complete for finite mathematics' (Isaacson [1987, p. 148]). He continues on to claim that 'those mathematical truths expressible in the language of arithmetic but not provable in PA contain "hidden-higher-order concepts", where what is hidden is revealed by the recognition of the phenomenon of coding' (Isaacson [1987, pp. 154-155]). But it is not only coding that reveals these *higher-order concepts*.

It is natural to ask what Isaacson means precisely by 'hidden higher-order concepts'. There are well-known statements expressed in the language of arithmetic that are independent of PA, but we can still decide their truth-values by going beyond the methods within the system PA. These methods are what Isaacson's refers to as *higher-order* – it is important to not confuse between Isaacson's 'higher-order concepts' with 'higher-order logic'. The examples Isaacson discusses are (1) Gödel sentences; (2) Goodstein's theorem; (3) the Paris-Harrington sentence; and (4) finitisation of Kruskal's theorem. In the next sections, I will discuss Isaacson's notions of 'hidden higher-order concepts' in the above examples, namely Gödel sentences and Goodstein's theorem. I will also discuss the Paris-Harrington sentence briefly to illustrate how provable equivalence to non-arithmetical statements deems the statement non-arithmetical, according to Isaacson.

(1) Gödel sentences: coding

Assuming that the readers are familiar with the formal statement and proofs of Gödel's incompleteness theorems, I will informally remind the readers how Gödel sentences are obtained from the system PA before discussing Isaacson's 'hidden higher-order concepts'. For more technical details on Gödel's incompleteness theorems, one should see (Kaye [1991]). Gödel (and later with Rosser's modification) famously proved that any consistent formal system containing some basic arithmetic, that can be recursively axiomatised, cannot prove its own consistency. The result is obtained by defining a predicate that represents 'provability' in the language of arithmetic for the given system, and using the method of coding to represent a sentence composed in finitely many symbols as a single natural number. This method allows the system to refer to its own theorems, and thus formulate a sentence that represents 'This sentence is not provable' via *diagonalisation*. We call the sentences of this form, or those that are equivalent in PA to these sentences, *Gödel sentences*.

With some further derivations in PA, we can see that Gödel sentences of PA are equivalent to the consistency statement of PA, namely $\neg Prv_{PA}(\ulcorner \bot \urcorner)$[1]. But since we have that the consistency statement of PA is equivalent to the sentence 'this sentence is not provable', PA cannot prove its own consistency.

Isaacson claims that '[T]he phenomenon of coding reveals fixed links between two situations or facts, one in the structure of arithmetic, the other in the realm of syntax of a formal system' (Isaacson [1987, p. 159]). What is hiding behind the coding in Gödel sentences is our interpretation of what these statements mean. We take the sentences to refer to itself and say something about itself, but '[i]t "says" nothing about itself. What it asserts is that a certain universal relation holds of all natural numbers (given that the Gödel sentence is of Π^0_1-form)' (Isaacson [1987, p. 158]). Isaacson goes on to say that it is

[1]I will use the notation $\ulcorner \cdot \urcorner$ as the function that represents the code of a sequence of symbols from the language of arithmetic. Thus $\neg Prv_{PA}(\ulcorner \bot \urcorner)$ is a sentence formulated in the language of arithmetic, that represents '$\bot$ is not provable', which is equivalent to claiming that PA consistent.

our interpretation in our natural language that allows us to see what the Gödel sentences mean, even if PA cannot.

The coding method has revealed what the Gödel sentences mean as it connects the syntax and structure, and allows us to give an interpretation to some statement in the language of arithmetic. In this sense, the connection from syntax to structure is the 'hidden-higher-order concept' that we have used to find the truth-value of Gödel sentences.

For Incurvati's formulation of Isaacson's thesis, we want to argue that the consistency statement of PA is not *directly perceivable* from the concept of the structure of the natural numbers. In fact, we need to argue that the axioms of PA are *directly perceivable* but the consistency statement of PA is not. One could say that each (finite) instance of the axioms of PA is *directly perceivable* as they are finite statements about finite arithmetic. But to claim the consistency of PA, we must be able to refer to all the infinitely many statements at the same time. Even if we can commit to the belief that each axiom of PA is *genuinely arithmetical*, because every one of them is *directly perceivable*, claiming that 'the collection of infinitely many statements about arithmetic is consistent' appears to be going beyond the concept of the finite structure. Not only that, but it also seems challenging how we can understand all of these infinitely many statements at once, in order to claim that these are consistent together.[23] Thus, Gödel sentences are not examples of *genuinely arithmetical* sentences – (a) we need coding (a *higher-order concepta*) to determine its truths, and (b) we cannot *directly perceive* them from the concept of finite structures.

(2) Goodstein's theorem: ϵ_0

There are other statements that are independent from PA, that do not reply on coding to express itself. A well-known example is Goodstein's theorem, which states roughly that *a Goodstein sequence always converges to zero.* See (Kaye [1991, p. 220]) and (Goodstein [1944]) for the details on Goodstein's theorem.

We define for each $n \in \omega$ a *Goodstein sequence* for n, as $G(n)(1) = n$, and for $G(n)(2)$, we write the number n in hereditary base-2 notation, replace all the 2's to 3's, then subtract 1. For each $m \in \omega_{>1}$, $G(n)(m)$ is the number such that $G(n)(m-1)$ written in hereditary base-m notation, replace all the instances of m's with $(m+1)$'s, and then subtract 1. For example if we start with $n = 2$, we will have $G(2)(1) = 2$, $G(2)(2) = (2+1)^1 - 1 = 2, G(2)(3) = (2 \cdot 3^0) - 1 = 2 \cdot 4^0 - 1 = 1, G(2)(4) = 4^0 - 1 = 0$. When the sequence reaches 0, the sequence terminates. For $n = 2$ and 3, it was easy to see that the sequence reaches 0 in some finitely many steps, but for 4 this is unclear.

The sequence $G(4)$ proceeds as 4, 26, 41, 60, 83, 109, $\cdots$. Since we have changed the base number to its successor and subtracted 1, it seems hard to believe that the

[2] Of course, one could argue that due to compactness theorem of first-order logic, we know that the consistency of every finite theory gives us the consistency of the whole theory. However, this is a theorem about *first-order logic*, not about arithmetic. The compactness theorem clearly is not a statement that we can perceive directly from the concept of the structure of finite arithmetic.

[3] In chapter 5, I will discuss briefly the notion of *fulfilability* and a semantic proof of Gödel's incompleteness (See (Kochen and Kripke [1982], Putnam [2000], Quinsey [1980])). This will suggest that coding nor Gödel's fixed-point theorem are essential for Gödel's incompleteness theorems. We can argue from this that Gödel's incompleteness can be understood from a semantic perspective.

Goodstein sequence of 4 would reach zero, and the first few numbers suggest that it is an increasing sequence. And whether the Goodstein sequence for an arbitrary n terminates is undecidable in PA:

> Indeed, Goodstein's interest in studying these sequences of natural numbers was a way of giving arithmetical expressions to ordinal inductions of order types less than ϵ_0. Hence by the adequacy of ϵ_0-ordinal induction for proving the consistency of PA, combined with Gödel's second incompleteness theorem, Goodstein's theorem must be unprovable in PA. (Isaacson [1987, p.160])

The proof of the theorem, however, uses ϵ_0-induction, where $\epsilon_0 := sup\langle \omega, \omega^\omega, \omega^{\omega^\omega}, \cdots \rangle$. Since ϵ_0-induction is not available within PA, Isaacson argues that ϵ_0-induction is a higher-order notion that is hidden within PA:

> The fact that it codes ϵ-induction tells us that there is no way to perceive the truth of Goodstein's theorem which does not also establish the correctness of ϵ_0-induction. The question raised by Goodstein's thereom in this context comes then to the following: is it possible that we should manage to establish this result using only purely arithmetical notions, that is without the use of any 'higher-order' notions, such as 'arbitrary subset', or 'well-ordering', or 'sound axiomatization of arithmetical truth'? Could we have some basis just within our understanding of arithmetic on the natural numbers for taking Goodstein's theorem as an axiom of true arithmetic? What is at least clear is that the way in which we do know that Goodstein's thereom is true is not such a basis. (Isaacson [1987, pp. 160–161])

Isaacson continues on to claim that PA is about *finite* structures, thus one 'can then think of this demonstration of independence as modeling the fact that the process of generating Goodstein sequences goes essentially beyond finite arithmetic' (Isaacson [1987, p. 161]).

At first it is not clear why ϵ_0-induction is infinitary merely due to the independence result of Goodstein's theorem. We actually have transfinite induction of order-type $\alpha < \epsilon_0$, which might be argued to be infinitary since α can be an ordinal greater than ω. But PA allows for α-induction for any $\alpha < \epsilon_0$, so it seems like there are theorems of PA that does demand some infinitary notions.

This is the difference between Smith's and Incurvati's formulations of Isaacson's thesis. Smith's formulation is only concerned about *what is independent of* PA and the requirement of *higher-order concepts* for obtaining their truths. But when Isaacson claims in (Isaacson [1987, p. 60]), 'by the adequacy of ϵ_0-ordinal induction for proving the consistency of PA, [...] Goodstein's theorem must be unprovable in PA', it suggests Isaacson is referring to the converse statement. That is, Goodstein's theorem is independent of PA, *because* we have to use ϵ_0. In this sense, Smith's formulation is only half of Incurvati's formulation, corresponding to the *soundness* of PA. So Isaacson's thesis can be split into:

> **Isaacson's soundness thesis.** An arithmetical statement provable in PA is *genuinely arithmetical*;

and

> **Isaacson's completeness thesis.** A *genuinely arithmetical* statement is provable in PA.

Thus, it becomes natural to ponder whether ϵ_0 is *genuinely arithmetical*. If it is *genuinely arithmetical*, this will suggest that ϵ_0-induction is *genuinely arithmetical*, and therefore Goodstein's theorem will be *genuinely arithmetical*. To answer whether ϵ_0 is *genuinely arithmetical*, we must ask whether ϵ_0 can be *directly perceived* from the concept of finite structures. Note that, we have assumed that the notion of *genuinely arithmetical* is closed under deduction. In fact, Isaacson claims that a statement is *(genuinely) arithmetical* if it is *directly perceived* from the *conceptual reality*, or it is derivable from other *genuinely arithmetical* statements:

> [... A] truth expressed in the language of arithmetic is arithmetical just in case its truth is directly perceivable so expressed, <u>or</u> on the basis of other truths in the language of arithmetic which are themselves arithmetical. (Isaacson [1987, p. 162])

Now, an argument against the claim that ϵ_0 is not *directly perceivable* could be that we can arithmetically represent ϵ_0 in PA, since it is a computable ordinal. However, not all properties of ϵ_0 can be shown within PA; for example, 'ϵ_0 can be well-ordered' is not a statement that PA can prove about the arithmetical representation of ϵ_0.

With the above objection, we can note that the notion of *direct perceivability* has a *de dicto* reading rather than *de re* reading. We understand the *de dicto* reading to be about the *statement*, while *de re* reading to be about the *object*. The claim is that ϵ_0 is not *directly perceivable* because not all of its properties can be proven to hold in the language of arithmetic. In this sense, the essence of ϵ_0 is not really captured by its arithmetical representation. What we can perceive from the *arithmetical representation* of the ordinal ϵ_0 is that *there is some object that satisfies certain properties (which are some of the properties of ϵ_0).*

Therefore, ϵ_0 is not *directly perceivable*, thus it is a *higher-order concept* hidden in PA. This gives us that ϵ_0-induction is also not *directly perceivable*, and the proof of Goodstein's theorem cannot be *directly perceived*. This allows us to conclude that Goodstein's theorem is not *genuinely arithmetical*.

(3) Paris-Harrington sentence: reflection principles

The Paris-Harrington sentence is another example of true (in the standard model) but undecidable within PA. It is a variation of finite Ramsey's theorem, which was discussed to be the 'arithmetical' example of an independent statement from PA. However, this notion of 'arithmetical' is different from Isaacson's notion of 'arithmetical', which he argues is about 'direct perceivability'. Isaacson's further comments that Paris-Harrington sentence is 'provably equivalent in PA to Σ^0_1-reflection for Peano Arithmetic', which he argues to be an 'implicit (hidden) higher-order content' (Isaacson [1987, p. 163]). Thus it follows from this that Paris-Harrington is also non-arithmetical because of the equivalence to Σ_1^0-reflection. For more details on the Paris-Harrington sentence, see (Kaye [1991, ch. 14.3]) and (Paris and Harrington [1977]).

Σ_1^0-reflection, Isaacson argues, is non-arithmetical because it is 'an expression of the soundness of PA as an axiomatization of arithmetical truths'. This way to interpreting Paris-Harrington shows that the statement is actually about the formal system PA, as Σ_1^0-reflection is, and thus it reveals some 'higher-order' notions.

In fact, Σ_1^0-reflection states that *every* Σ_1^0 *statement, that is true in the standard model* $\mathbb{N}$*, is provable in* PA. In this sense, Σ_1^0-reflection talks about *truths* and *provability*. The statement must hold together referring to all the *true* Σ_1^0-statements of arithmetic. If we were to commit to a form of *deflationism* about truths – *truth* is insubstantial; we do not add any more meaning by using the *truth-predicate* or the notion of *truth* – then it seems to be a statement about mere provability about Σ_1^0-statements. But there are many discussions on the *nature of truth*, which we will not go into in this thesis. See (Künne [2003]) for some further discussions on the *nature of truth*.

Due to the questions on the *nature of truth*, whether Σ_1^0-reflection is *directly perceivable* is left unanswered. Furthermore, what is different about the Paris-Harrington statement from the other two is Isaacson's emphasis on the 'implicit-ness' of the equivalence. We will discuss shortly that Isaacson's focus on the hidden *higher-order concepts* is about the way we obtain certain truths. In that sense, it is not fully clear why the equivalence would show that Paris-Harrington is immediately higher order. But we can understand Isaacson's argument to be saying that Σ_1^0-reflection is a property that claims a connection between the structure and the syntax, which makes it a 'higher-order' notion. Thus the equivalence between Paris-Harrington sentences and reflection principles show that these sentences cannot be *genuinely arithmetical*.

We have looked through Isaacson's examples in demonstrating the *hidden higher-order concepts*. These were precisely the methods we use in order to obtain the truths that are undecidable in PA. Smith's formulation focuses on independent statement of PA and how we obtain these truths. But it seems that Isaacson is mostly concerned about what is really *arithmetical*, rather than the independent statements themselves. And to understand what is *genuinely arithmetical*, we must understand how to *directly perceive* them. The statements that are not *directly perceivable* are those that require *hidden higher-order concepts* to perceive them. In the next section, I will go into details about the connection between *higher-order concepts*, *genuinely arithmetical* statements and *direct perceivability*.

3.1.2 Isaacson's thesis, higher-order concepts and direct perceivability

From the examples of *higher-order concepts*, we can understand the *higher-order concepts* to be something that allows us to establish a connection between the structure and the syntax. Coding revealed the connection between syntax and the consistency of all the statements which are *directly perceivable* from the structure. Our structure here is only the standard structure of the natural numbers $\mathbb{N}$. In this sense, we cannot perceive ϵ_0, because of the infinitary ideas hidden in ϵ_0. Thus if a statement *requires* some *higher-order concepts* in order to show that it holds true in arithmetic, it cannot be *genuinely arithmetical*. The *higher-order concepts* reveal things that we cannot *perceive directly* from the concept of the finite structure, but only by revealing some explicit connection between the structure and syntax. In this sense, it appears that *direct perceivability* is the more essential and crucial notion for understanding Isaacson's thesis. *Higher-order concepts* allow us to *perceive* the truths that we cannot access from *direct perceivability* from the structure, but rather via recognising the connection between the structure and the syntax.

Isaacson's thesis

We can now understand Isaacson's thesis to be the claim that

Isaacson's thesis *PA is sound and complete with respect to genuinely arithmetical statements.*

A statement in the language of arithmetic can be understood to be *genuinely arithmetical arithmetical* iff it is *directly perceivable* from the concept of the structure of the (finite) natural numbers. What we have observed is that provable equivalence to statements about the formal system PA shows that some statements expressed in the language of arithmetic are not *genuinely arithmetical* in Isaacson's sense. Furthermore, what is not *directly perceivable* requires *hidden higher-order concepts* that allow us to see the statement's determinate truth-value.

We also want to emphasise that what is important for Isaacson is *how* we perceive these truths, rather than *what* the statements are about:

> I am concerned with the way[4] in which arithmetical truths can be established. The point about the examples of truths unprovable in PA considered in this paper is that they are, in each case, shown to be true by an argument in terms of truths concerning some higher-order notion, and in each case also a converse holds, so that the only way in which the arithmetical statement can be established is by an argument which establishes the higher-order truth. (Isaacson [1987, pp. 164–165])

For Isaacson, the hidden higher-order concepts allow us to perceive these undecidable statements that are not arithmetical, but can be expressed in the language of arithmetic. And our ability to perceive these non-arithmetical statements in PA allows us to express undecidable statements of PA, but in the end, PA is sound and complete with respect to *genuinely arithmetical* statements, which can only be perceived without the *hidden higher-order concepts*:

> It is in this sense that truths in the language of arithmetic which lie beyond what is provable in Peano Arithmetic must be perceived in terms of hidden higher-order concepts. (Isaacson [1987, p. 167])

[4]The underline is added by the writer.

Relying on the *higher-order concepts* to capture what is *genuinely arithmetical* seems a bit circular. Isaacson claims that if we require the *higher-order concepts* in order to establish the truth of an arithmetical statement, then it is not *genuinely arithmetical*. But what does this really mean? *Higher-order concepts* go beyond finite arithmetic, and finite arithmetic is *genuinely arithmetical*. And Isaacson's thesis claims that what is *genuinely arithmetical* is exactly the theorems of PA, so in that sense, using the *higher-order concepts*, that go beyond arithmetic, is obviously not *genuinely arithmetical.*

It seems that from this formulation, we do not have a new piece of information on what Isaacson wants to claim. Of course, we require notions going beyond Peano arithmetical itself in order decide statements that are undecidable within PA – this is trivial.

Hence, we can conclude that *higher-order concepts* are not fundamental enough to give us a better understanding of Isaacson's thesis. The formulation of Isaacson's thesis via *higher-order concepts* can be circular, and does not give us a better understanding of the claim Isaacson desires to make. But rather, we should focus on Isaacson's structuralism and *direct perceivability* to comprehend without (or at least with minimal) circularity what he wants to share with us.

The advantage of using the notion of *direct perceivability* also connects directly with Isaacson's structuralism. In chapter 2, we discussed the importance of the notion of categoricity in picking out the intended structure. In fact, we captured the intended structure, via second-order categorisation and by showing that Dedekind's categoricity theorem holds. From this, we can argue that the first-order induction scheme can be *perceived directly* from the concept of the intended structure, and that's how we can have infinitely many *directly perceivable* statements from the structure.

Before moving on, I would like to point out some similarity between Isaacson's thesis and the Church-Turing thesis. The Church-Turing thesis claims that a function $f : \mathbb{N} \to \mathbb{N}$ is computable iff it can be implemented by a Turing machine. Note that the informal notion of *computability* is being identified with the formal/mathematical notion of *Turing machine*. In a similar way, Isaacson's thesis can be seen to equate the informal notion of *genuinely arithmetical* statements with the formal notions of *theorems of first-order Peano arithmetic*. Both of the theses give some conceptual analysis, providing some formal necessary and sufficient conditions for the informal concepts, that allow us to treat these notions mathematically.

What I aim to do is to provide a more precise understanding behind Isaacson's thesis by providing a mathematical definition of what is *genuinely arithmetical* without relying on, but being a relatively natural replacement for, the informal notion of *direct perceivability*. Thus the mathematical definitions I provide are giving some conceptual analysis of the notion of *genuinely arithmetical* statements, motivated by Isaacson's philosophy. From this, we can understand why it must/must not be a first-order theorems of PA, or why it must/must not be first-order deductive system, that captures the *genuinely arithmetical* statements. The following section will give some further details on my move from Isaacson's philosophy into *internalism*, in providing a natural formalisation of *genuinely arithmetical* statements.

3.2 Structures, semantics and internalism

In this section, I will briefly discuss what internalism is and why it is a natural position to be adopted in formalising Isaacson's notion of *genuinely arithmetical* statements. More detailed arguments will be presented in the next chapter, where I will give a mathematical definition for *genuinely arithmetical* statements. In doing so, I will focus on Isaacson's careful use of the word 'structure' over 'semantics' or 'models', avoiding to commit to some set-theoretic or model-theoretic notion of structures.

We have discussed how what is more important for Isaacson's thesis is *direct perceivability* rather than the *hidden higher-order concepts*. Recall the claim that

> [T]he phenomenon of coding reveals fixed links between two situations or facts, one in the structure of arithmetic, the other in the realm of syntax of a formal system. (Isaacson [1987, p.159])

Note that Isaacson has chosen the word 'structure' of arithmetic, rather than 'semantics' or 'models' of arithmetic. For Isaacson, coding (and other *higher-order concepts*) is a tool that links the structures in his *reality of mathematics* with the syntactic language that we adopt in order to characterise the structures. I'll discuss some important aspects of the distinction between 'structure' and 'semantics' in formalising the notion of *genuinely arithmetical*. But before this, there is some more support for taking Incurvati's formulation of Isaacson's thesis, over Smith's formulation.

We ask whether there is a way to mathematically formalise the notion of *genuinely arithmetical* so it is relevant to Isaacson's *direct perceivability*, and also highlights on the distinction between 'structures' and 'semantics'. I propose *internalism* as one position that we could adopt in formalising Isaacson's notion of *genuinely arithmetical*. By internalism, we can understand it to be grounded in the following manifesto (Button and Walsh [Forthcoming, ch. 10]):

> **The internalist manifesto.** We should investigate mathematical structure, informally construed, using second-order deductions, rather than semantic ascent. Indeed, the metamathematics of second-order theories should be dealt with inside the object language of the theories themselves.

The internal categoricity result shows that within a second-order deductive system, we can show Dedekind's *simply finite system* to be categorical. With the internal categoricity result, it seems natural for Isaacson to adopt this view. The distinction between 'structure' and 'semantics' can be made internally, where 'semantics' often is associated with model-theoretic ideas, but internally we merely talk about the *structures* without referring to some set-theoretic/model-theoretic structures.

We have so far that (1) a statement in the language of arithmetic is *directly perceivable* if it is entailed by categorical axiomatisation of the natural numbers; and (2) the notion of 'entailment' is best understood in an 'internal' way, that is, where one focuses on viewing second-order logic as a deductive system, because we can access the concept of the structure of finite numbers directly from our syntax. If what is a *genuinely arithmetical* statement must be *directly perceived*, then we would look for the structures directly from our deductive system.

Within second-order logic, we can talk about structures of second-order theories, including second-order Peano arithmetic. But by taking a structure within the second-order logical language, we do not have to rely on the model-theoretic notion of structure, but only on a second-order object that satisfies the characterisation/axiomatisation of the structure we are looking for.

For more details about internalism, see (Button and Walsh [Forthcoming, ch. 10]). We do not need to go into details about Button and Walsh's consideration on adopting the internalist stance for Isaacson, and neither do we need to go in further to develop a philosphical position for it. In fact, my claim is entirely independent of Button and Walsh's claims about internalism. I merely want to suggest that this could be one direction for formalising Isaacson's thesis, so we could understand Isaacson's claim better.

I conclude this chapter with a fruitful outcome, suggesting that Isaacson's thesis is better understood via the notion of a *genuinely arithmetical* statement and *direct perceivability*. Understanding these notions relies on understanding Isaacson's structuralism. And furthermore, we could try and formalise the notion of *genuinely arithmetical* statements using the internalist manifesto.

In the next chapter, we give a mathematical definition of *genuinely arithmetical* statements using the internalist manifesto. Since we want that a *genuinely arithmetical* statement to be *directly perceivable* from the concept of the finite structures of arithmetic, the internalist approach allows us to access these structures from the object-language perspective, without relying on set-theoretic or model-theoretic ideas about structures.

Chapter 4

Formalising Isaacson's notion of *genuinely arithmetical*

In this chapter, we will provide a formalisation of the notion of *genuinely arithmetical* and discuss whether this more formal notion might be able to support Isaacson's thesis. After defining what it means for a formula to be *genuinely arithmetical*, I will show that any statement provable in PA satisfies this new notion of *genuinely arithmetical*. Second-order logic differs from first-order logic in syntax and semantics. For the syntax, we extend the syntax of first-order logic with second-order variables, denoted by capital Latin alphabet, $X, Y, Z, \cdots$, while the first-order variables are denoted by the lower-case Latin alphabet, $x, y, z, \cdots$. We also extend the deduction rules of first-order logic by adding the rules for the second-order quantifiers.

Most importantly, second-order logic contains the *(full) comprehension schema* that allows us to express for any second-order formula Φ, some second-order symbol R that is extensionally equivalent to Φ:

Full comprehension schema

$$\forall w\ \forall x\ \forall Y\ \forall Z\ \exists R\ \forall u\ \forall v\ [Ruv \leftrightarrow \Phi(u, v, w, x, Y, Z)].$$

In the case of arithmetic, this allows us to assert that for any formula Φ, there is a second-order variable R that is satisfied exactly by u, v that satisfies Φ with some parameters w, x, Y, Z. We could also restrict the comprehension schema to certain types of formulas for Φ, for example, Φ that only contains second-order variables that are free and any first-order arithmetical formulas – we call this *arithmetical comprehension schema*. The second-order theory that extends first-order PA with the arithmetical comprehension schema is denoted ACA_0, where 0 refers to the fact that there is no restriction on the *first*-order formulas. To obtain second-order PA from first-order PA, we can extend first-order PA with the full comprehension schema, which is as described above. We can also define the theory ATR_0 by extending ACA_0 with *arithmetical transfinite recursion*: let $\theta(n, X)$ be an arithmetical formula with free variables n and X (with parameters); fixing the parameters, we can define some Θ such that

$$\Theta(X) = \{n \in \omega \mid \theta(n, X)\};$$

let A with a countable well-order $<_A$ on A, we have the set $Y \subseteq \omega$ such that Y is the result of transfinitely iterating the operator Θ on A; and the arithmetical transfinite recursion statements that for any $\theta(n, X)$ there is such Y as described. See Simpson [2009, pp. 39–40] for more details on ATR_0. In fact, we have that

$$ACA_0 \subsetneq ATR_0 \subsetneq PA^2,$$

and we will use this fact in some later arguments in the thesis.

We could also extend second-order logic with the axiom of choice, which can be stated in the language of second-order logic. In the case of arithmetic, it is not so important whether we adopt the axiom of choice as a second-order axiom, since it is equivalent to the induction axioms. However, there are stronger variations of the axiom of choice for arithmetic, and also if we were discussing other second-order theories going beyond arithmetic, this could be highly relevant. Furthermore, since second-order logic is not a complete deductive system, it is important what axioms we choose for our second-order logic. But for this thesis, I will not discuss this issue here.

When we refer to a theory with a superscript, T^1 and T^2, the numerals denote the order of the deductive system. Hence $T^1 \vdash \varphi$ means that *φ is deducible in first-order logic from T*. We also denote *first-order Peano arithmetic* and *second-order Peano arithmetic* as PA^1 and PA^2 respectively. Occasionally we will drop the superscript for *first-order PA*. Similarly we use M^2 to mean the *second-order semantics for the structure M*. For second-order semantics, there are two notions that we will use. *Full semantics* means that second-order quantification ranges over all subsets of the first-order domain. If we take some arbitrary second-order quantification, we call this *Henkin semantics*.

Before proceeding we define the second-order induction axiom as the following:

$$Ind(X) := (X0 \wedge \forall x(Nx \rightarrow (Xx \rightarrow Xsx)) \rightarrow \forall x\ (Nx \rightarrow Xx).$$

This corresponds to the way Dedekind defined the *simply infinite system* that was categorical. The first-order version of the above axiom is obtained by replacing the second-order variable X with formulas in the first-order language of arithmetic. We will denote the *first-order language of arithmetic* $\{<, s, +, \cdot, 0\}$ as L_1 and the *second-order language of arithmetic* $\{N, s, 0\}$ as L_2.

Thus we have the axioms of second-order Peano arithmetic PA^2 to be defined by the following axioms:

$$
\begin{aligned}
&P0: \ \forall x(Nx \rightarrow Nsx)\\
&P1: \ \forall x(Nx \rightarrow \neg sx = 0)\\
&P2: \ \forall x \forall y(Nx \wedge Ny) \rightarrow (sx = sy \rightarrow x = y)\\
&P_{ind}: \ \forall X\ Ind(X).
\end{aligned}
$$

By replacing P_{ind} with its first-order schematic variation, we obtain the axioms of first-order Peano arithmetic PA^1, with $+$ and $\cdot$ defined to be some recursive binary functions operating on the terms. Before we proceed, we will give some examples of Henkin structures of PA^2. These examples will make it clear how a Henkin structure, that is not a full structure, is different from a full structure.

Example 2. *Let* L *be the constructible universe and let* V *be the von Neumann universe of* ZFC. *Suppose (externally) that* $\mathsf{L} \neq \mathsf{V}$. *This means that 'there are sets in* V*, that are not constructible in the language of* ZFC *set theory.' Consider the structure* $(\omega, \mathcal{P}(\omega) \cap \mathsf{L})$*, so we have* ω *as the first-order domain, and* $\mathcal{P}(\omega) \cap \mathsf{L}$ *as the second-order domain. In the second-order domain,* $\mathcal{P}(\omega) \cap \mathsf{L}$ *highlights that we only consider the constructible subsets of* ω. *This is a Henkin structure of* PA^2 *that is not a full structure, since we do not have all subsets of* ω *in our second-order domain.*

Note that the above structure will continue to satisfy the Comprehension Schema, due to set-theoretic reflection in L. Briefly, this means that we have

$$(\omega, \mathcal{P}(\omega)) \vDash \exists X \ \forall n(n \in X \leftrightarrow \varphi(n)).$$

By the set-theoretic reflection theorem, we have some $\alpha > \omega$ such that for all $n \in \mathsf{L}_\alpha \cap \mathbb{N}$,

$$\mathsf{L} \vDash n \in \omega \wedge \varphi(n) \Leftrightarrow \mathsf{L}_\alpha \vDash n \in \omega \wedge \varphi(n).$$

Let $Y = \{n \in \mathsf{L}_\alpha \mid \mathsf{L}_\alpha \vDash n \in \omega \wedge \varphi(n)\}$. Since $Y \subseteq \mathsf{L}_\alpha$, it follows that $Y \in \mathsf{L}_{\alpha+1} \subseteq \mathsf{L}$. It follows then that

$$(\omega, \mathcal{P}(\omega)) \vDash \forall n \ n \in Y \leftrightarrow \varphi(n).$$

This shows that the Comprehension Schema is still satisfied in $(\omega, \mathcal{P}(\omega))$. For more details on the reflection theorem, see (Kunen [1980, p. 137, theorem 7.5]).

In the above example, we defined a Henkin structure by constructing a structure such that the second-order domain does not contain all subsets of the first-order domain. The next example will show that we could also obtain a Henkin structure from a full structure.

Example 3. *Consider the full structure,* $(\omega, \mathcal{P}(\omega))$. *Apply downward Löwenheim-Skolem on the two-sorted structure, so we obtain a second-order structure with size of the second-order domain less than* $2^{\aleph_0}$. *This will give us a Henkin structure, that is not a full structure since we cannot quantifier over all subsets of* ω*, as the second-order domain will have the number of elements strictly less than the size of the set of all subsets of* ω*,* $2^{\aleph_0}$.

Notice that what differentiates the full structure of PA^2 from other Henkin structures is the second-order domain. Full structures only allow second-order quantification over all subsets of the first-order domain. But Henkin structures allow some arbitrary second-order quantification. This means that a Henkin structure, that is not a full structure, will have the second-order domain to be different from $\mathcal{P}(\omega)$, supposing that ω is our first-order domain. Following on from the notation mentioned earlier for *first-order* and *second-order*, we will use $M^2 = (M, \mathcal{P}(M))$ to denote a full structure. Now, we can start to formulate the notion of *genuinely arithmetical*. The next section will remind the readers of the internalist approach which we adopt in defining the notion of *genuinely arithmetical*.

4.1 Internalist approach

The new formulation of the notion of *genuinely arithmetical* relies on internalist ideas about theories and structures. This is to say that the usual notions of equivalence can be defined *internally* within (e.g.) second-order logic. Recall the *internalist manifesto* as stated in (Button and Walsh [Forthcoming, ch. 10]):

> **The internalist manifesto.** We should investigate mathematical structure, informally construed, using second-order deductions, rather than semantic ascent. Indeed, the metamathematics of second-order theories should be dealt with inside the object language of the theories themselves.

This satisfies Isaacson's structuralist perspective that mathematical structures are not model-theoretic structures, but rather they exist in the *reality of mathematics*. Isaacson was extremely careful to not use the term 'semantics' in place of 'structures'. The internalist approach might happen to be what Isaacson was demanding in his structuralism without explicitly dedicating himself to it.

If coding was a higher-order concept that linked the syntax (object language) and structures, it appears that we can simply replace coding with internalist ideas. What is different about the internalist ideas is that we reveal the connection implicitly, rather than explicitly. Button and Walsh [Forthcoming, p. 236] claim that internalist ideas are about 'saying something in a second-order object language', thus capturing the structures in the object language, rather than using meta-theoretic semantics. This appears to capture the notion of *direct perceivability* faithfully, because we are directly accessing the structures from our perspective.

Väänänen and Wang [2015, pp. 123–124] give a definition of *internal categoricity* which we will use throughout this chapter. They state that internal categoricity is useful because it shows that 'non-standard models and categoricity can exist in harmony' (Väänänen and Wang [2015, p. 122]). Often the distinction between full semantics and Henkin semantics is emphasised when discussing categoricity and structures, but using the internal notions, we are free from the two distinct semantics, and simply focus on the structures within our object language. The following is the definition of an *internally categorical* theory. We will see that there are multiple $\vDash$'s in stating the internal categoricity of T^2. I will use the $\vDash$ on the left-end of the statement to denote the *external* entailment, while other $\vDash$ that follows after denotes the *internal* entailment. Also $ISO(R, X, Y)$ relation can be seen to represent that 'R is a bijection between X and Y that preserves the structure X to Y'.

Definition 4. *Let $L = \{N, s, 0, N', s', 0'\}$ consist of two copies of the language of L_2. Let M^2 be an arbitrary Henkin structure of second-order logic, and let T^2 be a theory. We say that T^2 is* internally categorical *if we have that whenever,*

$$M^2 \vDash \big((N, s, 0) \vDash T^2 \wedge (N', s', 0') \vDash T^2\big),$$

stating that if M^2 contains two copies of a structure satisfying the axioms of T^2 in the language L_2, then it follows that

$$M^2 \vDash \exists R\ ISO(R, (N, s, 0), (N', s', 0')).$$

The above definition claims that a theory T^2 is *internally categorical*, whenever we have that in any Henkin structure, the structure sees that any two structures of T^2 are isomorphic to each other. Furthermore, this definition holds for our system PA^2, thus we have that PA^2 is internally categorical. (See (Väänänen and Wang [2015, p.124]) and (Button and Walsh [Forthcoming, ch. 10]) for a proof of internal categoricity.)

Theorem 5 (Internal categoricity theorem)**.** *In the second-order deductive system with comprehension axiom, we have*

$$\vdash \big((N, s, 0) \vDash PA^2 \wedge (N', s', 0') \vDash PA^2 \rightarrow \exists R\ ISO(R, (N, s, 0), (N', s', 0')\big).$$

There is another internal notion that we can adopt for Isaacon's notion of *genuinely arithmetical*. This notion, which we will call *internal equivalence* claims that a theory can have exactly the same structures as PA^2.

Definition 6 (Internally equivalent)**.** *For a theory T^2, we say that T^2 is* internally equivalent *to PA^2 when*

$$\vdash \forall N\ \forall s\ \forall 0\ (N, s, 0) \vDash T^2 \leftrightarrow (N, s, 0) \vDash PA^2.$$

What is mathematically motivating for the internal notions is that from internal categoricity we can always obtain the external results. Thus, even without the philosophical motivation of internalism, we can be mathematically motivated to pursue internalist ideas in order to answer the questions externalists and model-theorists desire. We will use the following facts/lemmas throughout the chapter, in proving some of the theorems regarding the internalist versions of the notion of *genuinely arithmetical*.

Lemma 7. *If T^2 is internally categorical, then T^2 is externally categorical*

Lemma 8 (Externalisation lemma)**.** *If T^2 is internally equivalent to T'^2, then T^2 is externally equivalent to T'^2.*

Now, we have defined the internal notions that we will adopt in order to give some formal definitions of *genuinely arithmetical*. We will give six distinct definitions of *genuinely arithmetical* in this section, and prove how some of them are extensionally equivalent to the first-order theorems of PA^2. This is a fascinating result since the internalist idea was to link syntax and structures, and in a sense, the equivalence result shows it to be correct. The categoricity and equivalence give us the deductive equivalence. We will spend the following two sections defining the notion of *genuinely arithmetical*.

4.2 Genuinely arithmetical

In this section, we will give two definitions of *genuinely arithmetical*. In fact, these turn out to be equivalent. For the first definition, we focus on the second-order deductive system, which might appear to be problematic for Isaacson – Isaacson claimed that first-order logic is about deduction, and second-order logic is about structures. Thus we also look at the definition with the first-order deductive system. Here is the first definition of *genuinely arithmetical*. We will also give a more thorough intuition on how some external assumptions don't apply to internal notions as we go through some results for the collection of *genuinely arithmetical* statements.

Before the first formal definition of *genuinely arithmetical* statements, we define the notation $T = \langle \omega_T, \Omega_T(X) \rangle$ to denote the theory T such that ω_T is some set of finite first-order arithmetical sentences, and $\Omega_T(X)$ is some second-order formula. Using this notation, we can understand T^2 to denote the second-order version of the theory T, where we take the axioms of T^2 to be $\forall X \Omega_T(X) \wedge \omega_T$. Similarly, the axioms of T^1 are $\omega_T \wedge Scheme(\Omega_T)$, where $Scheme(\Omega_T)$ denotes the collection of first-order formulas such that the second-order free-variable X has been replaced by instances of L_1-formulas. Here are some examples of T that we are already familiar with.

Example 9. *Let $T = \langle Q, Ind(X) \rangle$, where Q denotes Robinson arithmetic with the following axioms:*

$$
\begin{aligned}
&s(x) \neq 0 \\
&s(x) = s(y) \rightarrow x = y \\
&x \neq 0 \rightarrow \exists y \; x = s(y) \\
&x + 0 = x \\
&x + s(y) = s(x + y) \\
&x \cdot 0 = 0 \\
&x \cdot s(y) = (x \cdot y) + x \\
&x \leq y = \exists z \; (z + x = y).
\end{aligned}
$$

The above eight axioms together state that $+$ and $\cdot$ are recursive binary functions and $\leq$ is a recursive relation (See (Lindström [2003, p.7])). However, there are some properties of $+$, $\cdot$ and $\leq$ that cannot be proven in Q. For these, we require the induction scheme of first-order arithmetic. In our T, it contains $Ind(X)$ that denotes the second-order induction axiom. Thus T^1 is equivalent to PA^1 that contains all the arithmetical instances of induction. Similarly T^2 is equivalent to PA^2.

In fact, we will use the notion of T again in the next chapter when proving Wilkie's theorem. The advantage of this notation is we can quickly talk about the first-order theory and the second-order theory from some $T = \langle \omega_T, \Omega_T(X) \rangle$, and this becomes important in defining the notion of *genuinely arithmetical* statements.

Definition 10. *Let ϕ be an L_1-formula. ϕ is* genuinely arithmetical *iff there is some $T = \langle \omega_T, \Omega_T \rangle$, where ω_T is an L_1-formula and Ω_T is an L_2-formula, such that*

$T^2 \vdash \phi$;
T^2 *is internally categorical; and*
T^2 *is internally equivalent to* PA^2.

And we define the collection of genuinely arithmetical *sentences as*

$$GEN := \{\phi \in L_1 \mid \phi \text{ is genuinely arithmetical } \}.$$

Before proceeding, we will check that this definition is consistent and satisfies the fundamental properties we desire. After all, if this definition does not contain theorems of PA^1, we have already failed at successfully finding what Isaacson would desire for the soundness part of his thesis.

Lemma 11. *Let $GEN = \{\phi \in L_1 \mid \phi$ is genuinely arithmetical $\}$, and let TA denote the collection of sentences that are true in the standard structure $\mathbb{N}$.*[1] *Then the following hold:*

$$GEN \subseteq TA; \tag{4.1}$$

$$GEN \text{ is consistent; and} \tag{4.2}$$

$$PA \subseteq GEN. \tag{4.3}$$

Proof. • $GEN \subseteq TA$.

Suppose for a contradiction that $GEN \not\subseteq TA$. This means that there is a formula $\phi \in GEN$ such that $\phi \notin TA$. Since TA is maximally consistent and complete, we must have $\neg\phi \in TA$. Note that it also follows that $\mathbb{N} \vDash \neg\phi$, because $\mathbb{N}$ is a model of TA.

Since $\phi \in GEN$, there must be some $T = \langle \omega_T, \Omega_T(X) \rangle$ such that T witnesses that ϕ is *genuinely arithmetical*. This includes that T^2 is internally equivalent to PA^2, so by the externalisation lemma (lemma 8), T^2 is externally equivalent to PA^2. This means that for any structure M of PA, M is also a structure of T^2. But by external equivalence, $\mathbb{N}^2 \vDash T^2$, so $\mathbb{N}^2 \vDash \phi$, since $T^2 \vdash \phi$.

This means that $\mathbb{N} \vDash \phi \wedge \neg\phi$, which is a contradiction.

- GEN is consistent.

Since $GEN \subseteq TA$, and TA is maximally consistent, it follows that GEN must be consistent.

- $PA \subseteq GEN$.

Let $\phi \in PA^1$, and let $T = \langle \omega_{PA}, Ind(X) \rangle$, where ω_{PA} consists of the eight axioms of Robinson's Q. This trivially means that

$T^2 \vdash \phi$;
T^2 is internally equivalent to PA; and
T^2 is internally categorical.

Hence, ϕ must be genuinely arithmetical because PA witnesses ϕ to be *genuinely arithmetical*. Thus $\phi \in GEN$, so we have shown that $PA \subseteq GEN$. □

[1] This gives us that TA is a complete theory of arithmetic.

We have so far obtained that $PA \subseteq GEN \subseteq TA$, however we have not shown yet whether $PA = GEN$, or $GEN = TA$. In the case $PA = GEN$, this new formulation of *genuinely arithmetical* will show that Isaacson's thesis will hold true, because everything that is *genuinely arithmetical* is provable in PA, and everything PA proves is *genuinely arithmetical* (we already have the latter proposition since $PA \subseteq GEN$, so PA is sound with respect to *genuinely arithmetical* statements). But if $GEN = TA$, then Isaacson's thesis fails because there are still independent statements of PA that are *genuinely arithmetical*. In order to decide whether $GEN = PA$, we first define the following:

Definition 12. $FO(PA^2) = \{\varphi \in L_1 \mid PA^2 \vdash \varphi\}$. *Hence* $FO(PA^2)$ *denotes the collection of first-order* L_1 *formulas that are provable in* PA^2.

Since this collection is defined by second-order deducibility, it must be recursively enumerable. And since PA^2 is stronger than PA^1, $PA^1 \subsetneq FO(PA^2)$, because $Con(PA) \notin PA$, but $Con(PA) \in FO(PA^2)$. For further details on $Con(PA) \in FO(PA^2)$, please see (Simpson [2009, p. 313 Corollary VII.14]), but simply by the fact that PA^2 can formalise the semantics for first-order structures and in particular PA^2 can show that the structure of the natural numbers $\mathbb{N}$ is a model of PA^1 shows that $Con(PA) \in FO(PA^2)$.

Now, if we can show that $FO(PA^2) \subseteq GEN$, this is enough to show that $PA \subsetneq GEN$. However, GEN is defined using internal notions of categoricity and equivalence, which is much stronger than the equivalent external notions. Although if we just consider the external notions, it appears easy to show that $GEN \subseteq FO(PA^2)$ or that $GEN \nsubseteq FO(PA^2)$, there are small details that we cannot apply to the internal notions.

Lemma 13. $GEN \subseteq FO(PA^2)$.

I will first provide an argument for the above lemma which is actually fallacious. But this can help us to better understand the differences between the internal and external perspective in an informative way:

Let $\varphi \in GEN$ with some witness T. This means that

$$\vdash \forall M^2, M^2 \vDash PA^2 \leftrightarrow M^2 \vDash T^2$$

by internal equivalence to PA. Since $T^2 \vdash \varphi$, this means that

$$\vdash \forall M^2, M^2 \vDash T^2 \rightarrow M^2 \vDash \varphi.$$

By the externalisation lemma (lemma 8), for any structure $M^2 = (M, \mathcal{P}(M)) \vDash T^2$, we have that $M^2 \vDash \varphi$. And due to internal equivalence, this arbitrary (full) structure $M^2 \vDash PA^2$. Thus it follows that $PA^2 \vdash \varphi$.

The problem with the above proof is at the last step – we deduced that $PA^2 \vdash \varphi$ since any arbitrary full structure $M^2 = (M, \mathcal{P}(M)) \vDash PA^2$ is such that $M^2 \vDash \varphi$. But second-order logic is not complete with respect to full semantics, and we cannot obtain external Henkin categoricity from internal categoricity results. However, we can modify the above 'proof' slightly to obtain the result that we are looking for. We start the proof with the exactly the same steps, but we will use Henkin semantics externally instead of full semantics.

Proof. Let $\varphi \in GEN$ with some witness T. This means that

$$\vdash \forall M^2, M^2 \vDash PA^2 \leftrightarrow M^2 \vDash T^2$$

by internal equivalence to PA. Since $T^2 \vdash \varphi$, this means that

$$\vdash \forall M^2, M^2 \vDash T^2 \rightarrow M^2 \vDash \varphi.$$

.

Let $N^2 \vDash PA^2$ be an arbitrary Henkin structure of PA^2. This gives us that

$$N^2 \vDash \forall M^2, M^2 \vDash PA^2 \leftrightarrow M^2 \vDash T^2.$$

Hence namely, it gives us that

$$N^2 \vDash N^2 \vDash PA^2 \leftrightarrow N^2 \vDash T^2.$$

Since $T^2 \vdash \varphi$, we have that

$$N^2 \vDash N^2 \vDash \varphi.$$

Here, it is important to see that the left most N^2 is the external Henkin structure of PA^2, while the N^2 following $\vDash$ is the internal *full* structure of PA^2 interpreted by the *external* N^2. In Henkin semantics, we have soundness and completeness, and since N^2 was an arbitrary Henkin structure of PA^2, it follows that $PA^2 \vdash \varphi$. □

By using arbitrary Henkin semantics, we were able to prove that $GEN \subseteq FO(PA^2)$. In fact, the internal structures are actually 'full structures' according to the external Henkin structure. Hence, depending on the external Henkin structure, the second-order internal quantifiers range over all subsets of the first-order domain. Thus, we must be extra careful when going between internal and external notions, as well as full and Henkin semantics. If we show that $FO(PA^2) \subseteq GEN$, then we have successfully shown that $FO(PA^2) = GEN$.

Lemma 14. *$FO(PA^2) \subseteq GEN$.*

Proof. Let $\varphi \in FO(PA^2)$. We want to show that $\varphi \in GEN$. Let $T := \langle \omega_T, Ind(X) \rangle$, where ω_T is the eight axioms of Q. Then it follows that

- T^2 is internally equivalent to PA^2;
- T^2 is internally categorical; and
- $T^2 \vdash \varphi$.

Thus, T is the witness of φ as a *genuinely arithmetical* sentence, so $FO(PA^2) \subseteq GEN$. □

From the above lemmas, we can now deduce that

Theorem 15. $FO(PA^2) = GEN$.

. ☐

We have so far that

$$PA \subsetneq GEN = FO(PA^2) \subsetneq TA.$$

This result shows that our formal definition of *genuinely arithmetical* is too strong to show that Isaacson's thesis holds. What we would have desired is for $GEN = PA^1$, thus $GEN \subseteq PA^1$ corresponds to the completeness thesis, and $PA^1 \subseteq GEN$ to the soundness thesis.

It is natural to ask whether we could weaken our current definition of GEN in order to find a definition of *genuinely arithmetical* so that we have a collection of L_1-formulas that is extensionally equivalent to the theorems of PA^1.

Before moving on, we should remark the following theorem stating that 'the consistency statement of PA^2 is not *genuinely arithmetical*'.[2] What might be interesting here is that in external full semantics, we have the theorem that for any second-order structure $M^2 = (M, \mathcal{P}(M))$ such that

$$M^2 \vDash PA^2 \Leftrightarrow M^2 \vDash PA^2 + Con(PA^2).$$

The idea behind this is due to the fact PA^2 is categorical. Trivially, from a structure of $PA^2 + Con(PA^2)$, we have a structure of PA^2. For the converse direction, there is only one structure of PA^2 up to isomorphism, and this structure $\mathbb{N}$ satisfies that PA^2 is consistent. Therefore, we can say that there is an external equivalence between PA^2 and $PA^2 + Con(PA^2)$. It is even more interesting because, by Gödel's incompleteness theorems, we do not have that $PA^2 \vdash Con(PA^2)$. But extending PA^2 with its own consistency statement, it does not restrict us to fewer structures of PA^2, when we are working externally in the full semantics.

However, this equivalence does not hold at an internal level. Once again, this shows us that internal notions should be very carefully handled. Our intuitions behind the external notions should not be naïvely applied to the internal notions. The internal notions are based on the internal structure, and thus a careful attention on the internal structure is necessary.

Corollary 16. $Con(PA^2) \notin GEN$.

Proof. Note that $GEN = FO(PA^2)$, and $PA^2 \nvdash Con(PA^2)$ due to Gödel's incompleteness theorems. Hence $Con(PA^2) \notin GEN$. ☐

This means that the equivalence between structures of PA^2, and $PA^2 + Con(PA^2)$ fails internally. If it does not fail, then we can simply choose a witness T such that $T^2 \vdash PA^2$, and we will trivially have internal equivalence, and therefore internal categoricity.

[2] Let us take some suitable provability predicate for PA^2 in L_1 so we have $Con(PA^2)$ to denote the L_1-formula, '$\perp$ is not provable in PA^2'.

Furthermore, noting that the consistency statement for the formal system second-order Peano Arithmetic is not *genuinely arithmetical* appears to be a positive result for Isaacson. It appears natural to ask whether we can modify our definition to be more about the first-order deductive system, rather than the second-order deductive system. Afterall, Isaacson claimed that second-order logic is about the mathematical structures, and first-order logic is about deduction, so we could modify the clause $T^2 \vdash \varphi$ into $T^1 \vdash \varphi$ to be *genuinely arithmetical*.

Definition 17. *An L_1-formula φ is* genuinely arithmetical^1 *if there is a witness T such that*

- *T^2 is internally categorical;*
- *T^2 is internally equivalent to PA^2; and*
- *$T^1 \vdash \varphi$.*

We will denote that class of L_1-formulas that are *genuinely arithmetical*^{1} as GEN^1. Our hope is that by changing to the first-order deductive system, we should be able to obtain what Isaacson wants, and give a definition that excludes the consistency statement of PA as *genuinely arithmetical*^{1}. However, unfortunately this was not the case.

The only part where the fact we have a second-order deductive system was important was when we wanted to state that

$$\vdash \forall M^2, M^2 \vDash T^2 \to M^2 \vDash \varphi.$$

But the above follows also from assuming that $T^1 \vdash \varphi$ since a theorem of T^1 will also be a theorem of T^2. Hence this actually gives us

Theorem 18. *$GEN = GEN^1$.*

We should remark at this point that internal categoricity of a witness T always follows from the internal equivalence of T^2 to PA^2. By the internal categoricity theorem, we have that any T^2 that is internally equivalent to PA^2 must be internally categorical. Thus, it seems necessary to modify the definition further in order to give a more concise definition, that does not carry some unnecessary weight. Thus in the next section, we modify the definitions further. This will show us some unexpected result leading to an inconsistent definition, as well as some equivalence to the previous definitions.

4.3 More *genuinely arithmetical*

I will give some further four new definitions of *genuinely arithmetical*, where there are some changes to the part of the definition about internal equivalence. We will consider both second-order and first-order deductive systems as we have done previously.

Definition 19 (Genuinely arithmetical$_1$)**.** *An L_1-formula φ is* genuinely arithmetical$_1$ *if there is a witness T such that*

- T^2 *is internally categorical;*
- $\vdash \forall M^2, M^2 \vDash T^2 \to M^2 \vDash PA^2$*; and*
- $T^2 \vdash \varphi$.

This definition is a reduction of the first definition given, where internal equivalence to PA^2 has been replaced with a weaker statement, that any internal structure of T^2 is an internal structure of PA^2, but the converse does not necessarily hold. Surprisingly, this simple modification gives us that this definition is inconsistent.

Lemma 20. *GEN_1 is inconsistent.*

Proof. We will show that GEN_1 is inconsistent by showing that both $Con(PA^2)$ and $\neg Con(PA^2)$ are both in GEN_1.

Let $T = \langle \omega_T, \Omega_T(X) \rangle$ such that $\omega_T = Q$, and $\Omega_T(X) := Ind(X) \wedge Con(PA^2)$. Then since $T^2 \supset PA^2$ as it contains $Ind(X)$, it follows that T^2 is internally categorical. Trivially we have that

$$\vdash \forall M^2, M^2 \vDash T^2 \to M^2 \vDash PA^2, \text{ and } T^2 \vdash Con(PA^2).$$

Thus, $Con(PA^2)$ is *genuinely arithmetical*$_1$.

Now, we apply the same idea for $\neg Con(PA^2)$. Let $T' = \langle \omega_{T'}, \Omega_{T'}(X) \rangle$ such that $\omega_{T'} = Q$ and $\Omega_{T'}(X) := Ind(X) \wedge \neg Con(PA^2)$.

Since $T'^2 \supset PA^2$, we have that T'^2 is internally categorical. Once again, trivially we have that

$$\vdash \forall M^2, M^2 \vDash T'^2 \to M^2 \vDash PA^2, \text{ and } T'^2 \vdash \neg Con(PA^2).$$

Thus it follows that $Con(PA^2) \in GEN_1$ and $\neg Con(PA^2) \in GEN_1$. Hence, GEN_1 is inconsistent. □

It is somewhat surprising that modifying one small notion can change the consistency of the definition. And it raises the question, what does it mean for an internal structure of T to be an internal structure of PA^2?. We will discuss this question briefly after the following definition, if weakening the deductive system would rescue the definition from inconsistency. From now on, we will follow the superscript notation on GEN, such as GEN^1 or GEN^2, to denote the deductive system we will use for the witness T. Thus in the following definition, we change the above definition of *genuinely arithmetical*$_1$ to *genuinely arithmetical*$_1^1$ since we focus on $T^1 \vdash \varphi$, rather than $T^2 \vdash \varphi$.

Definition 21. *An L_1-formula φ is* genuinely arithmetical$_1^1$ *if there is a witness T such that*

- T^2 *is internally categorical;*
- $\vdash \forall M^2, M^2 \vDash T^2 \to M^2 \vDash PA^2$*; and*

- $T^1 \vdash \varphi$.

Since our initial definition of GEN and its first-order variant GEN^1 were extensionally identical, it would be interesting to see whether the same will hold for GEN_1 and GEN_1^1. Unsurprisingly, once again, from $T^1 \vdash \varphi$, we can deduce that $T^2 \vdash \varphi$. Hence this modification cannot save the definition of *genuinely arithmetical*$_1$ from inconsistency.

Our modification from internal equivalence to PA^2 to a weaker form that only indicates an internal containment in one direction failed in an unexpected way.

The part that led the previous definition to inconsistency was precisely the direction of internal containment. What we had previously was that any internal structure of T^2 would be a structure of PA^2. This automatically allows us to extend any T with statements φ that are deductively independent of PA^2 to witness that φ is *genuinely arithmetical*^{1}. The next question to ask is *what if we change the direction of the containment?*

The next definitions do not have such property. By changing the direction of the containment, it appears that we have consistently found a weaker definition of *genuinely arithmetical* than the one we proposed at the beginning of this chapter.

Definition 22 (*genuinely arithmetical*$_2$)**.** *An L_1-formula φ is* genuinely arithmetical$_2$ *if there is a witness T such that*

- T^2 *is internally categorical;*
- $\vdash \forall M^2, M^2 \vDash PA^2 \rightarrow M^2 \vDash T^2$*; and*
- $T^2 \vdash \varphi$.

As remarked earlier, here we take any internal structure of PA^2 to be an internal structure of T^2. When we defined GEN_1 previously, we took the converse of the above statement. That definition was inconsistent, as we could find a T that witnessed a statement undecidable by deduction in PA^2. We will see that this new definition is actually extensionally equivalent to our original definition GEN. Thus restricting the notion of equivalence with PA^2 does not seem to alter what statements can be picked out to be *genuinely arithmetical*. But we will actually see that there are philosophical differences in the two definitions. We will discuss this briefly after demonstrating some mathematical results regarding the notion of *genuinely arithmetical*$_2$.

Lemma 23. *$GEN_2 \subseteq FO(PA^2)$.*

Proof. Let $\varphi \in GEN_2$. We want to show that $\varphi \in FO(PA^2)$. Since φ is *genuinely arithmetical*, there must be a witness $T = \langle \omega_T, \Omega_T(X) \rangle$, such that

$$T^2 \vdash \varphi;$$
T^2 is internally categorical; and
$$\vdash \forall M^2, M^2 \vDash PA^2 \rightarrow M^2 \vDash T^2.$$

Since $T^2 \vdash \varphi$, we must have

$$\vdash \forall M^2, M^2 \vDash T^2 \rightarrow M^2 \vDash \varphi.$$

This gives us that

$$\vdash \forall M^2, M^2 \vDash PA^2 \to M^2 \vDash \varphi,$$

since any internal structure of PA^2 is a structure of T^2.

Now let N^2 be an arbitrary Henkin structure such that $N^2 \vDash PA^2$. Then we must have, by Henkin soundness, that

$$N^2 \vDash \forall M^2, M^2 \vDash PA^2 \to M^2 \vDash \varphi.$$

Namely, we could interpret the arbitrary internal structure of PA^2 to be N^2:

$$N^2 \vDash N^2 \vDash PA^2 \to N^2 \vDash \varphi.$$

Since N^2 is an arbitrary Henkin structure of PA^2, it follows that

$$N^2 \vDash \varphi.$$

And by Henkin completeness, this gives us that

$$PA^2 \vdash \varphi.$$

This gives us that $\varphi \in FO(PA^2)$, as we wanted. □

Recall that $GEN = FO(PA^2)$, where GEN was the first definition of *genuinely arithmetical* we gave. Since

$$FO(PA^2) = GEN \subseteq GEN_2 \subseteq FO(PA^2),$$

we have the following theorem.

Theorem 24. *$GEN = GEN^1 = GEN_2 = FO(PA^2)$.*

What might appear to be surprising is that even by weakening the initial definition of *genuinely arithmetical* to *genuinely arithmetical*$_2$, we still obtain exactly the same set of sentences. The inconsistency of *genuinely arithmetical*$_1$ appears to suggest that one direction of the internal equivalence to PA^2 is too strong to capture *genuinely arithmetical truths*. But why is it the case? The methods we've used in the proof to show the inconsistency relied on the fact if we take the witness T to be such that it already contains PA^2, then we can take a statement false in $\mathbb{N}$ but undecidable deductively in PA^2, and construct a new witness T that includes the statement.

For the converse direction of the equivalence, we take that any internal structure of PA^2 is an internal structure of T^2. But from the internal categoricity theorem, we have that PA^2 satisfies internal categoricity. In this sense, it is not surprising that the only internal structure of PA^2 (up to internal isomorphism), ends up being the only internal structure of T^2.

Once again, weakening to first order deduction, we have the following definition:

Definition 25. *An L_1-formula φ is* genuinely arithmetical$^1_2$ *if there is a witness T such that*

- *T^2 is internally categorical;*
- *$\vdash \forall M^2, M^2 \vDash PA^2 \rightarrow M^2 \vDash T^2$; and*
- *$T^1 \vdash \varphi$.*

However, just as before, we have from $T^1 \vdash \varphi$ that $T^2 \vdash \varphi$, so we can still have the same results from this definition.

There are still some more questions that we have not yet answered about the definition of GEN_2.

Question 26. *Is there a $T = \langle \omega_T, \Omega_T(X) \rangle$ such that*

- *T^2 is internally categorical;*
- *$\vdash \forall M^2, M^2 \vDash PA^2 \rightarrow M^2 \vDash T^2$; and*
- *$\nvdash \forall N^2, N^2 \vDash T^2 \rightarrow N^2 \vDash PA^2$?*

Note that the third condition requires that we find a Henkin structure M^2 such that

$$M^2 \vDash \exists N^2, N^2 \vDash T^2 + \neg PA^2.$$

And from the second condition, T^2 must be such that $PA^2 \vdash T^2$, since any structure of PA^2 will be a structure of T^2. But from the first point, we must have internal categoricity, so any PA^2 internal structure, that is also T^2 internal structure is unique up to internal isomorphism, and thus from the Henkin structure M^2, the existing internal structure N^2 must be the unique one. However, N^2 cannot be an internal structure of PA^2 in our external structure M^2. Thus this would mean that there is no internal structure of PA^2 relative to the external Henkin structure M^2.

With this result, it is questionable whether such a definition of *genuinely arithmetical*$_2$ is something that we would like to pursue. What this gives us is that there must be a structure M^2 of T^2, that witnesses a formula φ to be *genuinely arithmetical*$_2$, will not have an internal structure of PA^2. In fact, what this means for Isaacson is that *there is no internal structure of* PA^2, which would make Isaacson's thesis redundant. We use the internal categorical characterisations to capture the intended structure of arithmetic and to show that such particular structure exists. But if we can show that there is no particular structure, then the question we have about what is *genuinely arithmetical* cannot be answered.

The fact that there is no internal structure of PA^2 relative to M^2 places great constraints on M^2, since it is pretty easy to build models of PA^2. Dedekind's original construction of a structure of PA^2 was established using the concept of *Dedekind-infinite sets.* We say that a set X is *Dedekind-infinite*, if *there is a proper subset Y such that there is a bijection between X and Y*. The outline of the proof that one can build a model of PA^2 from a Dedekind-infinite set goes as follows: Let $(A, s, 0)$ be a Dedekind-infinite structure, so A is a Dedekind-infinite set. We define that a set X is *inductive* if $0 \in X$ and for any $x \in X$ we have that $sx \in X$. Take $N := \bigcap \mathcal{A}$, where $\mathcal{A}$ is the *set of inductive subsets of* A. In the second-order setting, this intersection can be replaced by a universal quantifier over inductive sets. It is not then hard to show that $(N, s, 0)$ is a structure of PA^2.

We have outlined how from a Dedekind-infinite set, we can show that there is a structure of PA^2. So by the fact that there is no internal structure of PA^2 within the structure M^2 of the T defined above shows that there are no Dedekind-infinite sets in M^2. It is not obvious that there is an easy route to theorem 24, which goes through an equivalence between the second conditions in each of the theorems.

It seems that the most appropriate definition of capturing the notion of *genuinely arithmetical* via internalism is the very first definition we introduced. Although internal equivalence to PA^2 implies internal categoricity, the weakened variations have philosophical problems. GEN_1 is defined by taking that all internal structures of T^2 are internal structures of PA^2, but this definition turned out to be inconsistent.

4.4 Neo-Isaacson's thesis and higher-order concepts

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Recall our formulation of Isaacson's thesis to be

Isaacson's thesis PA is sound and complete with respect to genuinely/essentially arithmetical statements.

In the previous chapter, we discussed what is *genuinely arithmetical* is what is *directly perceivable* from the concept of the structure of arithmetic. The internalist manifesto is a position that allowed us to express the structures of arithmetic in the object language, which allowed us to *directly* access the structure in our language.

One might question whether we've assumed Isaacson's thesis in defining our notion of *genuinely arithmetical*, as we have questioned before about the circularity of *higher-order concepts*. The way to understand the claim that PA *is sound with respect to genuinely arithmetical statements* is to claim that any provably true statement of PA is a *genuinely arithmetical* statement. In assuming for the definitions of GEN and GEN_2 that any internal structure of PA^2 is an internal structure of T^2, it might appear that we have assumed Isaacson's thesis. However, this is not the case since the internal structure of PA^2 does not inform us whether the statements, that are true in the internal structure, are deducible in PA^2. We have merely expressed what our internal structures should show us about what we could understand about deduction from them.

Similarly for the converse direction, one can mistake it to be the completeness thesis of Isaacson's thesis – anything that is *genuinely arithmetical* is derivably true in PA. Once again, this is not the case, because we only refer to the internal structures of the witness T^2 to be structures of PA^2. Isaacson's thesis was about what is deductively true about arithmetic can be identified with what is *genuinely/essentially* arithmetical. In defining what is *genuinely arithmetical*, we adopted structural ideas in the hope that this will reveal the connection with the syntax of arithmetic without relying on an explicit statement saying so.

What our formalisation of *genuinely arithmetical* has shown is that the statements $\varphi \in L_1$ that are *genuinely arithmetical* are equivalent to those that are deducible in PA^2. Hence we re-formulate Isaacson's thesis in the following way:

neo-Isaacson's thesis*: Second-order Peano arithmetic is sound and complete with respect to* genuinely arithmetical *statements.*

The original statement of Isaacson's thesis desired to claim that *genuinely arithmetical* statements are exactly those that are theorems of first-order Peano arithmetic. But what we claim now to be *neo-Isaacson's thesis* is that *genuinely arithmetical* statements are exactly those that are first-order arithmetical statements that are theorems in PA^2. The fact our consistent notions of *genuinely arithmetical* are extensionally equivalent to $FO(PA^2)$ shows that Isaacson was correct to see the connection between syntax and structures. Where Isaacson may have gone wrong is to take *first-order logic* as the correct theory for deduction, as opposed to *second-order logic*.

By formalising the notion of *genuinely arithmetical*, we do not have to appeal to the notion of perception, except perhaps as a metaphor for 'deduced from a categorical axiomatisation'. This precise formulation of *genuinely arithmetical* statements and neo-Isaacson's thesis allows us to have a more solid understanding without committing us to any notion of *perceivability*. Our notion of *genuinely arithmetical* is a deductive consequence of categorical axiomatisation, and we do not need to philosophically commit to Isaacson's views.

Another deep advantage of the formalisation of Isaacson's thesis to the neo-Isaacson's thesis is that we have an explanation for why the induction axioms are *genuinely arithmetical*. Induction can be seen to be trivially *genuinely arithmetical* since it captures the way the natural number structure looks like. From starting with the minimal element, any property that is satisfied by every successor will be a property satisfied by all the natural numbers. In neo-Isaacson's thesis, we simply obtained this fact by providing a categorical characterisation of the arithmetical structure, without demanding some exposition to why this is *directly perceivable*.

We can recall Isaacson stating that due to the failure of completeness for full semantics, we should not consider second-order logic to demonstrate the deductive features of a theory. However, it appears that Isaacson [2011b, p. 47] was too quick with this. He wanted to emphasise that second-order logic was about structures, but first-order logic was about deduction: 'The very success of first-order logic for deduction unfits it for characterization, and the very success of full second-order logic for characterization unfits it for deduction'. For him, what is important about the system we chose for structures was *categoricity*, which led him to choose second-order logic, but we were able to replace this with *internalist semantics*. For deduction, he desired a *complete* deductive system. What we have established so far is that we can establish *completeness* for internal structures in second-order deductive system.

Hence we can conclude this chapter, showing that Isaacson was *half-right* and *half-wrong*. His choice of systems for structures and syntax was where his thesis went wrong. But I have argued that from Isaacson's structuralism and the importance of categoricity, we are persuaded to take the internalist approach. Furthermore, looking into the internalist approach regarding structures have enlightened us to see that, with the formalisation of *genuinely arithmetical*, we can argue that some independent statements $\varphi \in L_1$ in PA^1 are theorems of PA^2, and therefore *genuinely arithmetical*.

In chapter 3, we discussed that Gödel sentences, Goodstein's theorem and Paris-Harring sentences were not *directly perceivable*, especially noting that they require some *hidden higher-order concepts* in obtaining their truths. It remains to question whether these statements are counter-examples to our *neo-Isaacson's thesis*. We will go through the examples in chapter 3, arguing that they are not counter-examples to neo-Isaacson's thesis.

Gödel sentences: coding

Our previous argument that the consistency statement of PA was not *directly perceivable* was about the challenges in claiming consistency of infinitely many statements at once. In contrast with our new definition of *genuinely arithmetical* statements, we can see that the consistency statement of PA is *genuinely arithmetical*. It follows from the fact $Con(PA^1)$ is a provable consequence of PA^2.

This suggests that Isaacson was wrong to think that $Con(PA^1)$ is not *genuinely arithmetical* because it is a statement about the formal system, or it is our natural language interpretation that allows us to see that Gödel sentences are equivalent to the consistency statement (see (Isaacson [1987]) for the details). However, Isaacson was not entirely wrong on this issue. There are still some consistency statements that are not *genuinely arithmetical*. In particular, $Con(PA^2)$ is not *genuinely arithmetical*, because we can apply Gödel's incompleteness theorems to PA^2, thus PA^2 cannot prove $Con(PA^2)$ (even if it is expressible in L_1). The advantage of using a formal definition of *genuinely arithmetical* statements is that it gives us a formal argument showing why certain statements are *genuinely arithmetical* and others are not. The consistency of PA^1 is seen to be *genuinely arithmetical* under our new definition that relies on the internalist manifesto, which is arguably motivated by Isaacson's philosophical positions, while the consistency of PA^2 is not *genuinely arithmetical*.

Goodstein's theorem: ϵ_0

We claimed earlier that Goodstein's theorem cannot be *directly perceived* because ϵ_0 cannot be *de dicto perceived* from the concept of finite structure. But in fact, we can understand infinite limits from finitistic ideas, by starting from some finite n and applying the successor operation repeatedly. When we don't reach an end, it seems quite natural to claim some limit object of the successors. Given that exponentiation can be understood on finite numbers to be such that $x^y = (x^{y-1}) \cdot x$, we can also define ${}^y x = x^{x^{\cdot^{\cdot^{\cdot^x}}}}$ where x is exponentiated $y-1$-times. Furthermore, we consider ${}^x x$, and the limit as the successor operation has been applied to x. This clearly shows that we can understand ϵ_0 from the concept of finite structure, and thus it is *directly perceivable*.

In fact, we can define transfinite induction in PA^2 by saying that 'X is well-ordered if $\forall x_1 \in X\ \exists x_2 \in X\ \forall x_3 \in X\ (x_3 \leq x_2)$'. We can denote this as $WO(X)$. So from this notation, we formulate transfinite induction as: for all first-order arithmetically definable formula $\psi(x)$,

$$WO(X) \to \left(\forall x \in X\ \forall y \in X\ \Big((x < y \to \psi(x)) \to \psi(y)\Big) \to \forall x \in X\ \psi(x)\right).$$

Hence this defines that ϵ_0 is well-ordered, if $WO(\epsilon_0)$ is provable.

It actually follows from theorems 11 and 31 in (Hirst [2005]) that PA^2 proves that ϵ_0 is well-ordered. Theorem 11 states that for any sequence of well-orders, ATR_0 proves there is a well-order on the supremum of the sequence, and theorem 31 states that ACA_0 proves countable well-orders are closed under exponentiation. Since ϵ_0 is the supremum of the sequence $\langle \omega, \omega^\omega, \omega^{\omega^\omega}, \cdots \rangle$, it follows from the theorems that the well-orderedness of ϵ_0 is provable in PA^2, and this shows that ϵ_0-induction is provable in PA^2. This means that we can prove Goodstein's theorem in PA^2, hence this is not a counter-example to neo-Isaacson's thesis.

Our argument that the Goodstein's theorem is a *genuinely arithmetical* statement comes from the fact that the definition of GEN is such that we capture the statements in GEN from the categorical characterisation. From this, we have deduced that GEN is equivalent to $FO(PA^2)$, and ϵ_0-induction is deducible in PA^2 (and we can express it in L_1). Thus we can claim that Goodstein's theorem is *genuinely arithmetical* since it is obtained from ϵ_0-induction, which is also *genuinely arithmetical*.

Paris-Harrington sentences: reflection principles

The earlier argument against the *direct perceivability* of Paris-Harrington sentences was about its equivalence to the reflection principles of PA^1. Reflection principles were not considered *directly perceivable* because they were explicit statements about the connection between the structure and syntax, in that sense were 'implicit' *higher-order concepts*. In fact, PA^2 proves the infinite Ramsey theorem, so it follows that Paris-Harrington sentences are also provable in PA^2. In this sense, Paris-Harrington sentences are also *genuinely arithmetical*.

We will touch on the reflection principles of PA^1 again in chapter 5. In fact, some tools we will use in proving Wilkie's theorem have some interesting connections to Isaacson's examples of statements that were not *genuinely arithmetical*.

We conclude this chapter with our suggestion of neo-Isaacson's thesis, and how this is supported from the internalist manifesto. Isaacson was motivated by his structuralism and his concept of the *reality of mathematics*. From that he argued that giving a categorical characterisation of the particular structures in the *reality* shows that these structures exist and are unique. Therefore, we adopted the internalist approach that takes some direct access to the structures we conceptually understand.

If second-order logic was a good tool to capture categorical structures, so was the internalist's adoption of second-order logic for internally categorical structures. For Isaacson, it was also important to have a deductive system that was complete, but also one that can reflect that the *concepts* are some *process of thinking*.

Regardless of taking that a statement φ was provably true by some witness T in first/second-order deductive system, the completeness property with arithmetical statements held faithfully with what we wanted.

We conclude Part I of the thesis by suggesting a new approach to Isaacson's philosophical ideas and thesis. The next part will consist of a discussion on Wilkie's theorem. Isaacson often references Wilkie when discussing his thesis. In fact, we could interpret Wilkie's thereom to be support parts of Isaacson thesis. Wilkie's theorem shows that PA^1 is the minimal theory that can be captured from appropriate second-order characterisation. However, it remains open whether there is a maximal such theory.

In Part II, I will give mathematical answers for whether PA^1 is the maximal such theory as above. I will also argue how this relates to Isaacson's philosophy and his thesis.

Part II

Wilkie's theorem

Chapter 5

Wilkie's Theorem

In the first part of thesis, we introduced Isaacson's structuralism, and how we can understand Isaacson's thesis from his structuralism. We adopted internalism to formalise Isaacson's notion of *genuinely arithmetical*, thus showing that there is a formal way to understand Isaacson's thesis. Through the formalisation, we understood that following Isaacson's arguments for his position, we could also adopt what I called *neo-Isaacson's thesis*. Neo-Isaacson's thesis states that PA^2 *is sound and complete with respect to genuinely arithmetical statements*. This is different from the original Isaacson's thesis, which referred to the first-order theory PA^1 to capture the *genuinely arithmetical* statements.

In this part of this thesis, we go back to the original Isaacson's thesis, and what Wilkie's theorem means for the original thesis. In fact, the technical tools we will adopt in this chapter are highly relevant to what we introduced in the previous part of the thesis. We discussed how, in formulating Isaacson's thesis, the notion of *genuinely arithmetical* was more fundamental over *higher-order concepts*. In fact, some of the formal tools we take in proving Wilkie's theorem shows that there are different methods in *how* we obtain the independent statements of PA^1. In the discussion of *higher-order concepts*, Isaacson argued that what is important is *how* we understand the truths of the independence results, rather than *what* they say. But the fact that there is a different way to understand, say, Gödel's incompleteness theorem emphasises that the focus on *higher-order concepts* is less significant for Isaacson's thesis.

In the next two chapters, we will focus on Wilkie's theorem and what the result means for Isaacson's thesis. What the theorem suggests is that PA^1 is the minimal theory such that it can be captured from some categorical axiomatisation. Recall that the argument Isaacson provides for PA to be the correct theory of true arithmetic relies on the idea that the second-order induction axiom is categorical for the standard model (over some finite axioms of arithmetic), and the first order counterpart of the axiom gives us the induction schema, and therefore PA. In this chapter, I will present Wilkie's theorem and the *proof* from (Wilkie [1987]). I will continue to use L_1 and L_2 to denote the languages of first-order and second-order arithmetic, respectively, and capital Greek letters Φ, $\Psi, \cdots$ for an L_2 formula and lower-case Greek letters φ, $\psi, \chi, \cdots$ for an L_1 formula.

In the following section of the current chapter, I will introduce the definitions required in stating Wilkie's theorem. After formulating Wilkie's theorem, I will further introduce the definitions and technical tools required for proving Wilkie's theorem. The chapter will conclude with remarks regarding what Wilkie's theorem suggests for Isaacson's thesis. From the theorem, we have that PA^1 is the minimal theory that is captured from categorical axiomatisations. However, it remains an open question whether there is a maximal theory that is captured from categorical axiomatisation. In chapter 6, we will show that there is no such maximal theory, and we will also discuss what these formal results suggest for Isaacson's thesis.

5.1 Wilkie's theorem

In this section, we will introduce the definitions required in formulating Wilkie's theorem. We will conclude this section by stating Wilkie's theorem, and provide the proof in the following section. The proof presented here follows Wilkie's own proof very closely, but adds a little more discussion along the way, verifies more of the details, and provides a few more examples of important notions. One reason for doing this is that Wilkie's theorem is not widely known in the secondary literature. Another reason for doing this is that we appeal to Wilkie's theorem in the next chapter.

We first define what *restricted formulas* are.

Definition 27. *An L_2-formula $\Phi(X)$ is restricted if it is of the form*

$$Q_1 x_1 \in X \; Q_2 x_2 \in X \; \cdots \; Q_k x_k \in X \; \varphi(x_1, \cdots, x_k),$$

where Q_i's are $\exists$ or $\forall$ and $\phi(x_1, \cdots, x_k)$ is an L_1-formula.

From the restricted second-order formulas, we can have infinitely many first-order formulas. We replace the second-order variable X with any arithmetically defined first-order formula φ, which will give us infinitely many first-order formulas.

Definition 28. *Let $\Phi(X)$ be an L_2 formula. The L_1-scheme associated with Φ is the set of L_1-sentences of the form $\forall x_1 \cdots \forall x_n \Phi(\psi)$ such that ψ is an $n+1$-ary L_1-formula with no variables in common with Φ and $\Phi(\psi)$ denotes the L_1-formula, where every instance of "$\forall x \; x \in X$" is replaced with "$\forall x \; \psi(x_1, \cdots, x_n, x) \rightarrow$", and "$\exists x \; x \in X$" is replaced with "$\exists x \; \psi(x_1, \cdots, x_n, x) \wedge$". We denote the scheme associated with Φ as Scheme(Φ):*

$$\begin{aligned}&Scheme(\Phi) := \\ &\{\; \forall x_1 \cdots \forall x_n \Phi(\psi) \in L_1 \mid \psi := \psi(x_1, \cdots, x_{n+1}) \textit{ has no common variables with } \Phi \\ &\;\textit{and } \Phi(\psi) := \Phi(X)[x \in X/\psi(x_1, \cdots, x_n, x)].\}\end{aligned}$$

Both of these definitions are required in formulating Wilkie's theorem. What we want to show by Wilkie's theorem is that for any restricted categorical formula, its first-order scheme contains the first-order induction scheme. In order to formulate this precisely, we need to look at what a *categorical formula* is.

Categoricity was previously defined for theories – a theory T is *categorical* if for any two models of T, there is an isomorphism between them. We will use a minor variation of the definition for restricted-formulas, so the formula will capture the unique (up to isomorphism) model that we are looking for.

Definition 29. *An L_2-sentence $\forall X \Phi(X)$ is categorical over a finite L_1-theory T_0 for the standard model $\mathbb{N}$ iff for any L_1-structure M,*

$$M \models T_0 + \ \forall X \Phi(X)$$
$$\textit{iff}$$
$$M \cong \mathbb{N}.$$

An obvious example of such a sentence would be the induction axiom in L_2 – we already have that the induction axiom in L_2 over the finite L_1 theory of Robinson's Q, so there can only be one model of PA^2 in full semantics. (Shapiro [2000, p. 81])

Example 30. *We can define the induction axiom in L_2 as the following:*

$$IND(X) := (0 \in X \ \wedge \ \forall x \ (x \in X \ \rightarrow \ sx \in X)) \ \rightarrow \ \forall x \ x \in X.$$

The L_1-scheme associated with $IND(X)$ is the usual induction scheme of PA. By letting T_0 to be some finite subset of PA axioms defining successor, we have that $\forall X \ IND(X)$ is categorical over T_0 for the standard model.

Note, however, that $IND(X)$ is not a restricted formula, with first-order quantifiers leading the statements. But we will show shortly that $IND(X)$ is equivalent to some restricted formula. We will state Wilkie's theorem, which uses the notion of restricted formulas. If we can successfully show that $IND(X)$ is equivalent to some restricted formula, we can apply Wilkie's theorem to it.

Theorem 31 (Wilkie's theorem (1987))**.** *Let $\Phi(X)$ be a restricted L_2-formula and $\forall X \Phi(X)$ is categorical for $\mathbb{N}$ for some finite T. Then there is a finite set T_1 of L_1-sentences with $\mathbb{N} \vDash T_1$ such that $T_1 + Scheme(\Phi) \vdash PA^1$.*

The above theorem tells us that Isaacon's argument for PA^1's significance as a *genuinely* complete theory might be based on an unstable ground. If PA is *genuinely* complete, there should not be another restricted categorisation that is stronger than PA. But Wilkie's theorem only gives us that PA is the weakest such theory and leaves open whether it is also the maximal such theory. In the current chapter, I will present the proof of Wilkie's theorem. And in the following chapter, I will present a proof that suggests that PA is not the maximal *genuinely complete* theory for arithmetic.

Given the above theorem, a natural question to ask at this point is whether $IND(X)$ is logically equivalent to a restricted formula. This can be easily achieved by rewriting induction as the least number principle on ω:

$$WO(X) := \ \forall x_1 \in X \ \exists x_2 \in X \ \forall x_3 \in X \ x_2 \leq x_3.$$

It is easy to see that the above formula is a restricted formula such that $\forall X \ WO(X)$ is equivalent to $\forall X \ IND(X)$ over some finite L_1-theory T that contains the axioms for the successor operation. Thus we can apply Wilkie's theorem (theorem 31) on $WO(X)$ over the theory T, to show that we can obtain PA^1 from $WO(X)$. We will gradually cover all the mathematical tools required in establishing Wilkie's theorem.

5.2 Proof of Wilkie's theorem

The proof of Wilkie's theorem requires an approximation of an infinite structure by using an infinite chain of finite structures. The definition of *approximating structures* are only for models without function and constant symbols. This ensures that the structures generated are finite, since no new terms can be generated by a function symbol.

The idea of approximating structure was developed by Saul Kripke, as a method of providing a semantic proof for Gödel's incompleteness theorem, and other independent results from PA. Wilkie remarks that the notion comes from Kripke's *fulfilment* without a reference. But as it turns out, Kripke had not published his notion first (See the acknowledgement in (Putnam [2000])). Kripke had presented his notion of *fulfilment* in a lecture to the Department of Computer Science at Peking University, June 1984 (Putnam [2000, p.58]), and he presents an outline of his ideas in (Kochen and Kripke [1982, section VII(d), pp. 229-230]).

The presentation of the proof of Wilkie's theorem is as it is presented in (Wilkie [1987]). But more details about the notion of *fulfilment* can be found in (Putnam [2000]) and (Quinsey [1980]). Putnam's presentation of the notion of *fulfilment* uses game-semantic ideas. Quinsey's PhD thesis contains many theorems regarding the notion of *fulfilment* (or *fulfillability* because 'the notion of fulfilment' is a longer word than 'the fulfillability' (Quinsey [1980, p. 1])). I will use *fulfilment* and *fulfillability* interchangeably.

In this section, I will present the proof of Wilkie's theorem following the presentation of *fulfilment* in (Wilkie [1987]). After the proof, I will briefly discuss some philosophically motivating features of *fulfilment*, as suggested by Putnam [2000] and Quinsey [1980].

Definition 32 ($\mathcal{I}$-structure)**.** *Let L be a finite relational language, and $\mathcal{I} = \langle I, < \rangle$ be a totally ordered set. An $\mathcal{I}$-structure is a sequence of L-structures $\vec{\mathcal{A}} = \langle \mathcal{A}_i \mid i \in I \rangle$ such that if $i < j \in I$, then $\mathcal{A}_i \subset \mathcal{A}_j$.*

In the above notation, we take $\mathcal{A}_i$ to denote the structure and we will use A_i for the domain of $\mathcal{A}_i$. Thus $\mathcal{A}_i \subset \mathcal{A}_j$ means that $\mathcal{A}_i$ *is a proper substructure of* $\mathcal{A}_j$.

Definition 33 (Language $L(A)$)**.** *Let $\vec{\mathcal{A}}$ be an $\mathcal{I}$-structure in some finite relational language L, and $\mathcal{A} := \bigcup \vec{\mathcal{A}}$. Define $L(A)$ the language L with the constant symbols $\bar{a}$ for each $a \in A$.*

By defining $L(A)$, we can now define a $\Vdash$-semantics (*fulfillability* semantics) on $\vec{\mathcal{A}}$ by induction on $L(A)$-sentences φ. But first we put φ is a prenex-normal-form, so φ consists of a finite sequence of quantifiers followed by a quantifier free formula. Note that every $L(A)$-formula is logically equivalent to an $L(A)$-formula in prenex-normal-form. For convenience, we can reduce all $L(A)$ formulas to prenex-normal-form and define the following notation:

Definition 34. *U is the class of $L(A)$ formulas φ which are in prenex-normal form, and such that all negations are applied to atomics.*

Hence, every formula is equivalent to one in U simply by writing in prenex-normal form and then 'pushing' all the negations through to the atomics. I will give a few examples to show how this can be done.

Example 35. *Let $\varphi := \neg(\psi \wedge \chi)$, where ψ and χ are atomic sentences. Here φ is in prenex-normal form, but it is not U. This is due to the fact that φ has $\neg$ as the leading connective, which is applied to the non-atomic sentence $\psi \wedge \chi$.*

However, we can have another formula that is equivalent to φ above, that is in U.

Example 36. *Let $\varphi' := \neg\psi \vee \neg\chi$, which the same ψ and χ as in the previous example. By de Morgan's law, we can see that $\vdash \varphi \leftrightarrow \varphi'$. Furthermore $\varphi' \in U$, since φ' is in prenex-normal form, and negations are only applied to the atomic ψ and χ.*

We also define the following mapping:

Definition 37. $^* : FORM_{L(A)} \rightarrow U$ *such that* $\varphi^* = \psi$ *such that*

$$\vdash \neg\varphi \ \leftrightarrow \ \psi \text{ and}$$
$$\psi \text{ is in prenex-normal-form.}$$

We can understand the function * to be the function that maps any formula φ to another formula ψ that is equivalent to $\neg\varphi$, and ψ is in prenex-normal-form. This is convenient because we do not have to consider infinitely many formulas which are logically equivalent to each other when want to prove theorems about $L(A)$-formulas. Here are some examples of formulas we can obtain from applying *.

Example 38. *Let φ be an arbitrary atomic formula. Consider $\neg^n\varphi$, where $n \in \omega$. Thus $\neg^n\varphi$ denotes a formula with an arbitrary number of $\neg$ symbols followed by φ.*

Note that for any odd/even numbers $k, l \in \omega$,

$$\vdash \neg^k\varphi \leftrightarrow \neg^l\varphi.$$

Suppose $\psi \in U$ such that $\varphi^* = \psi$, and

$$\vdash \neg\varphi \leftrightarrow \psi.$$

It is clear from above that $\neg\neg\varphi^* = \psi$, since $\neg\neg\varphi$ is logically equivalent to φ, and $\psi \in U$ is also logically equivalent to φ. This can be applied to any $\neg^n\varphi$ for an even $n \in \omega$.

Similarly let $(\neg\varphi)^* = \chi$, such that

$$\vdash \neg\varphi \leftrightarrow \chi.$$

From this, we get that $(\neg^n\varphi)^* = \chi$ such that $n \in \omega$ is an odd number.

The above example demonstrates how the $*$ mapping can be seen as some equivalence relation such that $\varphi^* = \psi$ when $\psi \in U$ and $\vdash \neg\varphi \leftrightarrow \psi$. From now on, we will only consider formulas that satisfy the above condition on U when we talk about $L(A)$-formulas. This way, we can restrict our attention to formulas in certain syntactic forms. We will now define $\Vdash$-semantics (*fulfillability*-semantics) for $\vec{\mathcal{A}}$ on prenex-normal-formulas.

Definition 39. *Let φ be an $L(A)$-formula in prenex-normal-form. $\vec{\mathcal{A}} \Vdash \varphi$ ($\vec{\mathcal{A}}$ fulfils φ) is defined inductively as the following:*

$$
\begin{array}{lll}
\vec{\mathcal{A}} \Vdash \varphi & \textit{iff } \mathcal{A} \vDash \varphi \textit{ for } \varphi \textit{ quantifier free} \\
\vec{\mathcal{A}} \Vdash \forall x \varphi(x) & \textit{iff } \textit{ for all } a \in A,\ \vec{\mathcal{A}} \Vdash \varphi(\bar{a}) \\
\vec{\mathcal{A}} \Vdash \exists x \varphi(x) & \textit{iff } \textit{ if } i \in I \textit{ is such that all } \bar{a} \textit{ occurring in } \varphi,\ a \in A_i, \\
& \quad \textit{then for every } j > i, \textit{ there is } b_j \in A_j \textit{ such that } \vec{\mathcal{A}} \Vdash \varphi(\bar{b}_j).
\end{array}
$$

Note that the $\Vdash$ is a relation between the sequence of structures $\vec{\mathcal{A}}$ and $L(A)$-formulas, while the $\vDash$ is a relation between the limit structure $\mathcal{A}$ and $L(A)$-formulas. Since $\mathcal{A}$ is an infinite structure if I is infinite, we can find an example where $\mathcal{A} \vDash \varphi$ but $\vec{\mathcal{A}} \nVdash \varphi$. From now on, we will call φ an $L(A_i)$-formula, if all constants $\bar{a}$ occurring in φ are such that $a \in A_i$. The following is an example of a formula φ such that $\mathcal{A} \vDash \varphi$ but $\vec{\mathcal{A}} \nVdash \varphi$.

Example 40. *Let $L = \{R\}$ be a finite set of language that consists of one atomic binary relation $R(x, y) := x + 2 < y$, and let $\varphi := \forall x\ \exists y\ R(x, y)$. Let $\mathcal{A}_i = [0, 1, \cdots, i + 1]$, and let $\mathcal{I} := \langle \omega, < \rangle$, where this $<$ defined in $\mathcal{I}$ is the usual well-order on $<$, so $\leq := < \cup =$ is a total order on ω. Then we have $\vec{\mathcal{A}} = \langle \mathcal{A}_i, < \rangle$ an $\mathcal{I}$-structure of L-structures, and $\mathcal{A} \vDash \varphi$, because for any $a \in A = [0, \inf)$, we can take $b = a + 3 \in A$.*

Fix a number, say 5, then we have $\mathcal{A} \vDash \exists y\ R(5, y)$. We want to show that $\vec{\mathcal{A}} \nVdash \exists y\ R(5, y)$, so suppose for a contradiction that $\vec{\mathcal{A}} \Vdash \exists y\ R(5, y)$. Note that $A_4 = [0, 5]$, so A_4 contains all the constants occurring in $\exists y\ R(5, y)$, namely 5. Since $\vec{\mathcal{A}} \Vdash \exists y\ R(5, y)$, we must have some $a \in A_5 = [0, 6]$ such that $\vec{\mathcal{A}} \Vdash \bar{a} > 5 + 2$. This implies that $a > 7$, but there is no element greater than 7 in A_5. Thus we have contradiction that we were looking for, and $\Vdash$ does not coincide with $\vDash$.

For the converse, we have the following lemma:

Lemma 41. *If $\mathcal{I}$ has no greatest element and $\vec{\mathcal{A}} \Vdash \varphi$, then $\mathcal{A} \vDash \varphi$.*

Proof. Suppose that $\mathcal{I}$ has no greatest and $\vec{\mathcal{A}} \Vdash \varphi$. We will show by induction on φ that $\mathcal{A} \vDash \varphi$.

Base case: φ is a quantifier free sentence in $L(A)$. If $\vec{\mathcal{A}} \Vdash \varphi$, then by definition we have $\mathcal{A} \vDash \varphi$.

Inductive cases:

Case 1: Suppose $\vec{\mathcal{A}} \Vdash \forall x \varphi(x)$. By definition, this means that for all $a \in A$, $\vec{\mathcal{A}} \Vdash \varphi(\bar{a})$. By induction hypothesis, it follows that for all $a \in A$, $\mathcal{A} \vDash \varphi(\bar{a})$. Then it follows that $\mathcal{A} \vDash \forall x \varphi(x)$.

Case 2: Suppose $\vec{\mathcal{A}} \Vdash \exists x \varphi(x)$. By definition, for any $i \in I$, such that $\varphi(x)$ is an $L(A_i)$-formula, then for each $j > i$ there exists $a_j \in A_j$ such that $\vec{\mathcal{A}} \Vdash \varphi(a_j)$. Now by induction hypothesis, and from the assumption that $\mathcal{I}$ has no greatest element, we have for each of these A_j, there exists $a_j \in A_j \subseteq A$ such that $\mathcal{A} \vDash \varphi(\bar{a}_j)$. It follows then that $\mathcal{A} \vDash \exists x \varphi(x)$. □

It does not seem so obvious where the assumption that $\mathcal{I}$ has no greatest element is used in the above proof, so consider the case where $\mathcal{I}$ has a greatest element, call it n, so we have that $\mathcal{A}_n$ is a maximal structure. In this case, we can take a formula $\psi(\vec{a}, x)$ such that $\vec{a} \in A_n \backslash A_{n-1}$, so ψ contains constants only occurring in A_n.

Since there are no $j > n$, it follows that $\vec{\mathcal{A}} \Vdash \exists x \psi(\vec{a}, x)$ trivially holds. Consider where $\psi(\vec{y}, x) := x \neq x \ \wedge \ \vec{y} = \vec{y}$. Suppose that $\mathcal{A} \vDash \exists x \psi(\vec{a}, x)$. So there is some $a \in A$ such that $\mathcal{A} \vDash a \neq a$. But this is a contradiction since A is non-empty and no element is not equal to itself.

Now the following lemma plays a crucial role in proving Wilkie's theorem. The lemma roughly states that given a $\mathcal{I}$-structure and a subsequence $\mathcal{I}' \subseteq \mathcal{I}$, we have some $\mathcal{I}'$-structure that *fulfils* any formula φ that is *fulfilable* by the $\mathcal{I}$-structure. This is really useful since we can always find a shorter subsequence to generate another infinite model, that is an elementary substructure of the model of $\mathcal{I}$-structure that we started with.

Lemma 42 (Fulfilling substructure). *Let $\mathcal{I}' \subseteq \mathcal{I}$, $\vec{\mathcal{A}}' = \langle \mathcal{A}_i \mid i \in I' \rangle$ and $A' = \bigcup \vec{\mathcal{A}}'$. If $\varphi \in L(A')$ and $\vec{\mathcal{A}} \Vdash \varphi$, then $\vec{\mathcal{A}}' \Vdash \varphi$.*

Note that $A' \subset A$ since for any $a \in A'$, there is an $i \in I'$ such that $a \in A_i \subset A$. Also for any $R \in L$, $R^{\mathcal{A}'} = R^{\mathcal{A}} \cap (A' \times A')$, so $\mathcal{A}' \subset \mathcal{A}$.

Proof. We also prove the above lemma by induction on the complexity of φ. Base case: Suppose that $\vec{\mathcal{A}} \Vdash \varphi$ where φ is an atomic $L(A')$-formula. By definition, we have that $\mathcal{A} \vDash \varphi$. But since $\mathcal{A}' \subset \mathcal{A}$, it follows that $\mathcal{A}' \vDash \varphi$. Then again by definition, we have $\vec{\mathcal{A}}' \Vdash \varphi$.

Inductive cases:

Case 1: Suppose $\vec{\mathcal{A}} \Vdash \forall x \varphi(x)$. By definition, we have that for any $a \in A, \vec{\mathcal{A}} \Vdash \varphi(\bar{a})$. Since $A' \subseteq A$, we have for any $a \in A', \vec{\mathcal{A}}' \Vdash \varphi(\bar{a})$. Thus this gives us $\vec{\mathcal{A}}' \Vdash \forall x \varphi(x)$.

Case 2: Suppose $\vec{\mathcal{A}} \Vdash \exists x \varphi(x)$. By definition, for every $i \in I$, if $\varphi(x)$ is an $L(A_i)$-formula, then for any $j > i$, we have some $a_j \in A_j$ such that $\vec{\mathcal{A}} \Vdash \varphi(\bar{a_j})$. Let $J := \{j \in I \mid j > i \wedge \varphi \text{ is an } L(A_i)-\text{formula for any } i \in I\}$, and consider $J' := J \cap I'$. Then clearly we have that for all $j' \in J'$, there is $a'_j \in A'_j$ such that $\vec{\mathcal{A}}' \Vdash \varphi(\bar{a}'_j)$. Recall that φ is an $L(A')$-formula, so if i is so that φ is an $L(A_i)$-formula, then $i \in I'$. And since $J' \subseteq I'$, it follows that $\vec{\mathcal{A}}' \Vdash \exists x \varphi(x)$. ☐

Before moving onto another important lemma for the proof of Wilkie's theorem, we define a few more things on U.

Definition 43. *Let U be the class of $L(A)$-formulas defined previously. $S \subseteq U$ is a* closed set *iff S is closed under taking sub-formulas, and if $\phi \in S$ then $\phi^* \in S$.*

It is easy to see that if $S \subseteq U$ is a finite subset, then there is a closed set $\overline{S} \subseteq U$ such that $S \subseteq \overline{S}$. We can simply obtain $\overline{S}$ by taking every formula $\varphi \in S$ and extend S with all the sub-formulas of φ and φ^*. Here is an example of a closed set S:

Example 44. *Let $S = \{x = x, y = s(x)\}$, then $\overline{S}$ is a closed set such that*

$$\overline{S} = \{x = x, \neg(x = x), y = s(x), \neg(y = s(x))\}.$$

If S contains a formula that is not atomic, then $\overline{S}$ must contain all sub-formulas of the non-atomic formula φ and their $*$-negations φ^*.

Example 45. *Let $S = \{\varphi := \forall y\ \exists x\ (s(x) = y \vee y = 0)\}$. Then the closed set $\overline{S} \subseteq U$ will be as follows:*

$$\begin{aligned}\overline{S} =&\{\forall y\ \exists x\ (s(x) = y \vee y = 0),\\ &\exists x\ (s(x) = y \vee y = 0),\\ &(s(x) = y \vee y = 0),\ s(x) = y,\ y = 0, \neg(s(x) = y),\\ &\neg(y = 0),\ \neg(s(x) = y) \wedge \neg(y = 0),\\ &\forall x\ (\neg(s(x) = y) \wedge \neg(y = 0)),\\ &\exists y\ \forall x\ (\neg(s(x) = y) \wedge \neg(y = 0))\}.\end{aligned}$$

Note that since closed sets are defined to be such that $\overline{S} \subseteq U$, all formulas in $\overline{S}$ must be equivalent to a sub-formula of φ or φ^* and satisfy to be in U.

Since our definition of $\Vdash$ applies to a closed subset, we can define the following notion:

Definition 46. *Let $\varphi(\vec{x}) \in U$ and $\vec{\mathcal{A}}$ be an $\mathcal{I}$-structure. $\vec{\mathcal{A}}$* determines *$\varphi(\vec{x})$ iff when $\vec{a} \subseteq A$, then either (1) $\vec{\mathcal{A}} \Vdash \varphi(\vec{a})$ or (2) $\vec{\mathcal{A}} \Vdash \varphi(\vec{a})^*$. And for a subset $S \subseteq U$, $\vec{\mathcal{A}}$* determines *S iff $\vec{\mathcal{A}}$ determines every formula in S.*

This notion is applied when we need to specify that $\vec{\mathcal{A}}$ can determine the truth-value of $\varphi(\vec{x})$. Since unlike $\vDash$, we do not have that $\Vdash \neg\varphi$ iff $\nVdash \varphi$. Now we will prove a series of lemmas that are needed for proving Wilkie's theorem.

Lemma 47. *Let $\mathcal{I} = \langle \omega, < \rangle$ and $\vec{\mathcal{A}}$ be an $\mathcal{I}$-structure such that A_i is finite for all i. Let $S \subseteq U$ be a closed subset. Then there is an infinite increasing subsequence $\langle i_j \mid j \in \omega \rangle \subseteq \mathcal{I}$ such that the $\mathcal{I}$-structure $\vec{\mathcal{A}}' := \langle \mathcal{A}_{i_j} \mid j \in J \rangle$ determines S.*

Proof. We will prove the above lemma by induction on the size of S, looking at the maximal quantifier depth formula in S. For the base case, if S only contains quantifier free formulas, then clearly every formula in S will be determined.

For the inductive step, consider $\varphi \in S$, a formula with the maximal quantifier depth in S. Without loss of generality, we can suppose that φ and φ^* are the only formulas in S with the maximal quantifier depth, and so $S' := S \backslash \{\varphi, \varphi^*\}$ is a closed subset of U. By assumption, we have the $\mathcal{I}$-structure $\vec{\mathcal{A}}$ such that, for each $i \in \omega$, A_i is finite. And by the induction hypothesis on the size of S, without the loss of generality, we have that $\vec{\mathcal{A}} := \langle A_i \mid i \in \omega \rangle$ determines S'. If we obtain some $\vec{\mathcal{A}}' := \langle A_{i_j} \mid j \in \omega \rangle$ such that $\vec{\mathcal{A}}'$ determines φ, then clearly $\vec{\mathcal{A}}'$ determines S, so we are done.

In order to do this, we define the following: for any $k \in \omega$, consider the sequence $\langle i_0, i_1, \cdots, i_k, 1 + (i_k), 2 + (i_k), \cdots, n + (i_k), \cdots \rangle$. Note that $\langle i_j \mid j \in \omega \rangle$, the sequence we desire to prove the lemma, is a subsequence of the sequence being considered, since they are identical until i_k and then $i_{(k+1)}$ must be one of $n + (i_k)$ – that is, the n-th successor for i_k. If we recursively define the sequence $\langle i_0, i_1, \cdots, i_k, 1 + (i_k), 2 + (i_k) \cdots \rangle$ for each

k, then we can recursively define $\langle i_j \mid j \in \omega \rangle$. In fact the proof generates the sequence $\langle i_j \mid j \in \omega \rangle$ by applying induction on k.

To find the sequence $\langle i_j \mid j \in \omega \rangle$, we define

$$\vec{\mathcal{A}}^{(k)} = \langle \mathcal{A}_{(i_0)}, \mathcal{A}_{(i_1)}, \mathcal{A}_{(i_2)}, \cdots, \mathcal{A}_{(i_k)}, \mathcal{A}_{1+(i_k)}, \mathcal{A}_{2+(i_k)}, \cdots, \mathcal{A}_{n+(i_k)}, \cdots \rangle.$$

Notice that $\vec{\mathcal{A}}^{(0)} = \vec{\mathcal{A}}$, and $\vec{\mathcal{A}}^{(k)}$ is generated by a subsequence of $\mathcal{I}$, so $\vec{\mathcal{A}}^{(k)}$ determines S' by the fulfilling substructure lemma (lemma 42).

We will show by induction on k that $\vec{\mathcal{A}}^{(k)}$ determines $\varphi(\vec{b})$ for any $\vec{b} \subseteq \mathcal{A}_{i_{(k-1)}}$ (with $\mathcal{A}_{i_{(-1)}} = \emptyset$). The induction proof on k gives us which $i_{(k+1)}$ we should have, given i_k, and obtain the sequence $\langle i_j \mid j \in \omega \rangle$ that gives us the structure $\vec{\mathcal{A}}'$ for the induction step on the size of S.

Without loss of generality, let $\varphi(\vec{b}) := \exists y \psi(\vec{b}, y)$. To make notations easier, we will use $\nu(\vec{b})$ to denote the witness when there is some $\nu(\vec{b}) \in A$ such that $\vec{\mathcal{A}}^{(k)} \Vdash \psi(\vec{b}, \nu(\vec{b}))$. We also let i_{k+1} be the least $i > i_k$ such that for all $\vec{b} \subseteq A_{(i_k)}$, if $\nu(\vec{b})$ is defined then $\nu(\vec{b}) \in A_i$. This reassures that whenever there is a witness $\nu(\vec{b})$, it is in $A_{i_{(k+1)}}$.

Base case: For $k = 0$, trivially $\vec{\mathcal{A}}^{(0)} = \vec{\mathcal{A}}$ determines φ.

Inductive case: By the induction hypothesis on k, we have that $\vec{\mathcal{A}}^{(k)}$ determines $\varphi(\vec{b})$, and any formula of quantifier depth less than of $\varphi(\vec{b})$. We want to show that $\vec{\mathcal{A}}^{(k+1)}$ determines $\varphi(\vec{b})$.

Let $\vec{b} \subseteq \mathcal{A}_{(i_k)}$. If $\vec{b} \subseteq A_{i_{(k-1)}}$, then induction hypothesis gives us that $\vec{\mathcal{A}}^{(k)}$ determines $\varphi(\vec{b})$. It follows by fulfilling substructure lemma (lemma 42) that $\vec{\mathcal{A}}^{(k+1)}$ determines $\varphi(\vec{b})$. So suppose that $\vec{b} \subseteq A_{(i_k)} \backslash A_{i_{(k-1)}}$. Then we will have two possible cases: case (1) – $\nu(\vec{b})$ is defined; and case (2) – $\nu(\vec{b})$ is not defined.

Case (1): Suppose $\nu(\vec{b})$ is defined. Then there must be some $d = \nu(\vec{b}) \in A_{i_{(k+1)}}$ such that $\vec{\mathcal{A}}^{(k)} \Vdash \psi(\vec{b}, d)$. Then it follows by the fulfilling substructure lemma (lemma 42) that $\vec{\mathcal{A}}^{(k+1)} \Vdash \psi(\vec{b}, d)$.

Case (2): Suppose $\nu(\vec{b})$ is not defined. This means that for any $a \in A$, we don't have $\vec{\mathcal{A}}^{(k)} \Vdash \psi(\vec{b}, a)$. But since $\psi \in S'$, it must be determined by $\vec{\mathcal{A}}^{(k)}$, so it follows that for all $a \in A$, $\vec{\mathcal{A}}^{(k)} \Vdash \psi(\vec{b}, a)^*$. By definition, this means that $\vec{\mathcal{A}}^{(k)} \Vdash \forall y (\psi(\vec{b}, y)^*)$, which is equivalent to saying that $\vec{\mathcal{A}}^{(k)} \Vdash \varphi(\vec{b})^*$. Thus it follows by the fulfilling substructure lemma (lemma 42), $\vec{\mathcal{A}}^{(k+1)} \Vdash \varphi(\vec{b})^*$.

This shows that for any $k \in \omega$, and for any $\vec{b} \subseteq A_{i_{(k-1)}}$, $\vec{\mathcal{A}}^{(k)}$ determines $\varphi(\vec{b})$. And by the fulfilling substructure lemma (lemma 42), this shows that $\vec{\mathcal{A}}'$ determines $\varphi(\vec{b})$. Thus it follows that $\vec{\mathcal{A}}'$ determines S, and we are done. □

Lemma 48. *Let $\Phi(X)$ be a restricted formula such that*

$$\Phi(X) := Q_1 x_1, \cdots, Q_k x_k \varphi(x_1, \cdots, x_k),$$

where φ is an L_1-formula. Suppose that $\forall X \Phi(X)$ is categorical for $\mathbb{N}$ over some finite L_1-theory T_1. Let L be a language containing only one k-ary relation symbol R, and let σ

be an L-sentence such that

$$\sigma := Q_1 x_1, \cdots, Q_k x_k R(x_1, \cdots, x_k).$$

For all $n \in \omega$, there is a $\langle n+1, < \rangle$ structure $\vec{\mathcal{A}} = \langle \mathcal{A}_i \mid i \leq n \rangle$ such that for all $i \leq n$,

- *A_i is finite;*
- *$A_i \subseteq \omega$ and for all $n_1, \cdots, n_k \in A_i$,*

$$\vec{\mathcal{A}_i} \vDash R(n_1, \cdots, n_k) \text{ iff } \mathbb{N} \vDash \varphi(n_1, \cdots, n_k); \text{ and}$$

- *$\vec{\mathcal{A}} \Vdash \sigma^*$.*

Proof. Let $\mathcal{M}$ (with domain M) be a countable non-standard elementary extension of $\mathbb{N}$. Since $\mathcal{M} \ncong \mathbb{N}$, there must be some $S \subseteq M$ such that $\mathcal{M} \Vdash \neg\Phi(S)$. This is due to the fact that $\forall X \Phi(X)$ is categorical for $\mathbb{N}$. Note that S is a non-empty and infinite subset, because if it is empty or finite, then it follows that $\Phi(S)$ can be expressed in L_1. Since $\mathcal{M}$ is an elementary extension of $\mathbb{N}$, this means that $\Phi(S)$ is also false in $\mathbb{N}$ which is a contradiction. Thus S must be countably infinite, as M is countable. With S being countably infinite, we can have a sequence of finite sets $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$ such that $S = \bigcup\{B_i \mid i \in \omega\}$.

Let $V := \{\langle a_1, \cdots, a_k \rangle \in M^k \mid \mathcal{M} \vDash \varphi(a_1, \cdots, a_k)\}$, so in other words, V is the collection of elements which satisfy φ in M. Let $\mathcal{B}_i$ be the L-structure such that $\mathcal{B}_i := \langle B_i\, ;\, V \cap {B_i}^k \rangle$ so $\mathcal{B}_i$ denotes a finite sequence of finite structures such that R is interpreted as $V \cap {B_i}^k$ in $\mathcal{B}_i$.

By lemma 47, we can pass to a subsequence, and hence assume without loss of generality that we have that $\langle \mathcal{B}_i \mid i \in \omega \rangle$ determines σ, as we can take $\{\sigma, \sigma^*\}$ to be the closed set. So there are two possible cases: case (1) – $\vec{\mathcal{B}} \Vdash \sigma$; and case (2) – $\vec{\mathcal{B}} \Vdash \sigma^*$.

Case (1): If $\vec{\mathcal{B}} \Vdash \sigma$ then it follows that $\bigcup \vec{\mathcal{B}} \vDash \sigma$ by lemma 41. But note that $\bigcup \vec{\mathcal{B}} = \langle S; V \cap S^k \rangle$, so we have that $S \vDash \sigma$. Recall that $\sigma := Q_1 x_1, \cdots, Q_k x_k R(x_1, \cdots, x_k)$. So if $S \vDash \sigma$ then $Q_1 x_1 \in S, \cdots, Q_k x_k \in S$, we have $S \vDash R(x_1, \cdots, x_k)$.

Recall that $R^S = V \cap S^k$, so since $S \vDash R(x_1, \cdots, x_k)$, we have that $\mathcal{M} \vDash \varphi(x_1, \cdots, x_k)$ because $x_1, \cdots, x_k \in V$ so they realised φ in M. This means that $\mathcal{M} \vDash Q_1 x_1 \in S, \cdots, Q_k x_k \in S \varphi(x_1, \cdots, x_k)$, which is equivalent to $\mathcal{M} \vDash \Phi(S)$. This contradicts our assumption on S that $\mathcal{M} \vDash \neg\Phi(S)$.

Thus we must have case (2) to hold: Let $\vec{\mathcal{B}} \Vdash \sigma^*$. By the fulfilling substructure lemma (lemma 42), we have that for any $n \in \omega$, $\langle \mathcal{B}_i \mid i \leq n \rangle \Vdash \sigma^*$. Now we define the following formula:

$$G(x) := \exists y \; y \text{ is a code of a } \langle x+1, < \rangle\text{-structure such that } y \Vdash \sigma^* \text{ and}$$
$$\forall a_1, \cdots, a_k \in \bigcup y, \; \Big(\big(y \Vdash R(a_1, \cdots, a_k) \big) \leftrightarrow \varphi(a_1, \cdots, a_k) \Big).$$

Note that this formula is definable in L_1, since y can be a finite number coding a finite sequence of finite structures, and $\Vdash$ is defined recursively. Thus it is easy to see that for any $n \in \omega$, $\mathcal{M} \vDash G(n)$, because y can code $\langle \mathcal{B}_i \mid i \leq n+1 \rangle$ and we already have that $\langle \mathcal{B}_i \mid i \leq n+1 \rangle \Vdash \sigma^*$, and further more $\bigcup \mathcal{B}_i \subseteq S$ and $R^S \subseteq V$, thus for any $a_1, \cdots, a_k \in \mathcal{B}_i$,

$$\big(y \Vdash R(a_1, \cdots, a_k)\big) \leftrightarrow \varphi(a_1, \cdots, a_k).$$

It follows from that $\mathcal{M}$ is an elementary extension of $\mathbb{N}$ that for any $n \in \omega$, $\mathbb{N} \vDash G(n)$, which is equivalent to saying that $\mathbb{N} \vDash \forall x G(x)$.

We can now conclude the proof because we have shown that for any $n \in \omega$, there is a $\langle n+1, < \rangle$-structure $\vec{\mathcal{B}} = \langle \mathcal{B}_i \mid i \leq n+1 \rangle$ such that

- each B_i is finite;
- $B_i \subseteq \omega$ and by the fact $\forall n G(n)$ holds in $\mathbb{N}$, we have that for any $n_1, \cdots, n_k \in \omega$,

$$\mathcal{B}_i \vDash R(n_1, \cdots, n_k) \text{ iff } \mathbb{N} \vDash \varphi(n_1, \cdots, n_k); \text{ and}$$

- $\vec{\mathcal{B}} \Vdash \sigma^*$.

□

We can sketch out the proof of the above lemma again before moving on to the proof of Wilkie's theorem (theorem 31). From the restricted formula $\Phi(X)$ such that $\forall X \Phi(X)$ is categorical for $\mathbb{N}$, we wanted to construct an ω-long sequence of finite structures only containing the natural numbers such that the elements satisfying R in each finite structure is identical to the elements satisfying φ in $\mathbb{N}$, but the ω-structure $\mathcal{A} \Vdash \sigma^*$. We constructed the sequence of finite structures by taking a non-standard model $\mathcal{M}$ and some $S \subseteq M$ such that $M \vDash \neg\Phi(S)$. By defining the collection of elements V satisfying φ in M, we constructed the sequence of finite structures such that the elements satisfied by the relation R are only those from V. This established the interpretation of R in each finite structure to be equivalent to φ in $\mathcal{M}$. But by the way we constructed the sequence of structures from S, we could not have σ to be satisfied in S as this is equivalent to saying that $\mathcal{M} \vDash \Phi(S)$.

Now we are ready to prove Wilkie's theorem. Recall that the theorem states the following:

Theorem 49. *Let $\Phi(X)$ be a restricted L_2-formula and $\forall X \Phi(X)$ is categorical for* $\mathbb{N}$ *for some finite T. Then there is a finite set T_1 of L_1-sentences with $\mathbb{N} \vDash T_1$ such that $T_1 + Scheme(\Phi) \vdash PA^1$.*

Proof. Let $T_1 := I\Sigma_1 + \forall x G(x)$, where $G(x)$ is the formula defined in the previous lemma relating φ, R and σ. Since $I\Sigma_1$ is finitely axiomatisable, T_1 must be finitely axiomatisable[1]. As we proved in the previous lemma, we have that $\mathbb{N} \vDash \forall x G(x)$, we also have $\mathbb{N} \vDash T_1$.

[1]See (Hájek and Pudlák [1998, pp. 77–81]) for the proof of finite axiomatisability of $I\Sigma_1$.

We want to show that $T_1 + Scheme(\Phi) \vdash PA$. We will show this by taking an arbitrary model $\mathcal{M}$ of T_1 and showing that $\mathcal{M} \vDash PA$. So consider $\mathcal{M} \vDash T_1 + Scheme(\Phi)$. For a contradiction, suppose that $\mathcal{M}$ does not satisfy the induction scheme. Note that if $\mathcal{M}$ satisfies induction, and $\mathcal{M} \vDash I\Sigma_1$, this shows that $\mathcal{M} \vDash PA$.

Let ψ be a formula in L_1 (possibly with parameters from M) such that

$$\mathcal{M} \vDash \psi(0) \wedge \forall x(\psi(x) \to \psi(x+1)) \wedge \exists x \neg\psi(x),$$

i.e. ψ is the formula that falsifies an instance of induction scheme.

Let $J = \{b \in M \mid \mathcal{M} \vDash \forall x \leq b\ \psi(x)\}$. Then J must be a definable initial segment of $\mathcal{M}$. J cannot have a greatest element, since if there is a greatest $b \in J$, then $\mathcal{M} \vDash \forall x \leq b\ \psi(x)$. It follows then that $\mathcal{M} \vDash \psi(b)$, and this gives us that $\mathcal{M} \vDash \psi(b+1)$.

Since induction fails for ψ, it follows that there must be some $a \in M$ such that $a \notin J$. Note that $a > b$ for all $b \in J$ since J is a definable cut on $\mathcal{M}$. From $\mathcal{M} \vDash T_1$, it follows that $\mathcal{M} \vDash \forall x G(x)$. So we must have $\mathcal{M} \vDash G(a)$. Interpreting G inside $\mathcal{M}$, this means that there is a sequence of L-structures (that is coded in $\mathcal{M}$) $\vec{\mathcal{A}} = \langle \mathcal{A}_i \mid \forall i \leq a \rangle$ such that

$$\mathcal{M} \vDash \text{“} \forall i \leq j \leq a, \mathcal{A}_i \subseteq \mathcal{A}_j \text{ and } \vec{\mathcal{A}} \Vdash \sigma^* \qquad (\dagger)$$

$$\text{and } \forall a_1, \cdots, a_k \in \text{domain}(\mathcal{A}_i), \Big(\big(\mathcal{A}_i \vDash R(a_1, \cdots, a_k)\big) \leftrightarrow \varphi(a_1, \cdots, a_k) \Big) \text{”}. \qquad (*)$$

By inspection if φ is Σ^0_n or Π^0_n, then $G(x)$ is provably equivalent to a Σ^0_{n+1} in some sufficiently large fragments of PA (see (Hájek and Pudlák [1998, pp. 13–20])).

Let $\mathcal{I}_a := \langle \{b \in M \mid \mathcal{M} \vDash b \leq a\}, <^{\mathcal{M}} \rangle$. $\mathcal{I}_a$ is a totally ordered set defined out in the meta-theory by taking the constant $a \in M$ in our meta-theory. Thus we can have $\vec{\mathcal{A}}$ to be an $\mathcal{I}_a$-structure in our meta-theory, since the small amount of coding in $(\dagger)$ is absolute.

Since $J \subseteq \mathcal{I}_a$, we have that $\langle \mathcal{A}_i \mid i \in J \rangle \Vdash \sigma^*$ by lemma 41. It follows then by the fulfilling substructure lemma (lemma 42), and by some inspection of definition 39, that $\bigcup \langle \mathcal{A}_i \mid i \in J \rangle \vDash \sigma^*$.

Now define a formula $\chi(x) := \exists y \forall z \leq y\ \psi(z) \wedge x \in \text{domain}(\mathcal{A}_y)$. Note that this actually defines the domain of $\mathcal{A} := \bigcup \langle \mathcal{A}_i \mid i \in J \rangle$, since the index y of $\mathcal{A}_y$ is defined as the elements of J and x is in domain$(\mathcal{A}_j)$.

Note that by the definition of how R is interpreted in $\mathcal{A}_i$, it follows that $\mathcal{A}$ continues to satisfy $(*)$. Hence we can see that

$$R^{\mathcal{A}} := A^k \cap \{(a_1, \cdots, a_k) \in M^k \mid \mathcal{M} \vDash \varphi(a_1, \cdots, a_k)\}.$$

Note that we have $\mathcal{A} \vDash \sigma^*$. By the definition of σ, this means

$$\mathcal{A} \vDash \neg(Q_1 x_1, \cdots, Q_k x_k\ R(x_1, \cdots, x_k)).$$

The following are then equivalent to above, where we denote Q' to denote $\exists$ if Q is $\forall$, and vice versa,

$$\begin{array}{l} Q_1' x_1 \in A, \cdots, Q_k' x_k \in A, \ \mathcal{A} \vDash \neg R(x_1, \cdots, x_k) \\ Q_1' x_1 \in A, \cdots, Q_k' x_k \in A, \ \mathcal{M} \vDash \neg\varphi(x_1, \cdots, x_k) \text{by (*)} \\ \mathcal{M} \vDash Q_1' x_1 \ \chi(x_1), \cdots, Q_k' x_k \ \chi(x_k) \ \neg\varphi(x_1, \cdots, x_k) \text{ since } A \text{ is defined by } \chi(x) \\ \mathcal{M} \vDash \neg Q_1 x_1 \ \chi(x_1), \cdots, Q_k x_k \ \chi(x_k) \ \varphi(x_1, \cdots, x_k). \end{array}$$

Recall that $\mathcal{M} \vDash Scheme(\Phi)$, this means that for any L_1-formula $\xi(x)$,

$$\mathcal{M} \vDash Q_1 x_1 \ \xi(x_1), \cdots, Q_k x_k \ \xi(x_k) \ \varphi(x_1, \cdots, x_k).$$

But χ is an L_1-formula that violates this, thus we have a contradiction. Hence we can conclude that $\mathcal{M} \vDash PA$ since $\mathcal{M}$ satisfies the induction scheme and $\mathcal{M} \vDash I\Sigma_1$. □

The proof of Wilkie's theorem establishes that PA is the minimal theory that we obtain from the restricted second-order axiomatisation of arithmetic. If PA was not the minimal theory of Wilkie's theorem, then there would be a proper sub-theory (call it A) of PA that we obtain by taking the scheme of our second-order characterisation. In this case, there would be at least one statement, say φ that is independent of A, but decidable in PA. It is not so obvious how this connects with Isaacson's thesis immediately. But an obvious question to ask is whether PA is a maximal such theory, and if not, is there a maximal such theory?

5.3 Remarks on *fulfilment*, *coding* and *reflection principles*

Before moving on to answering the above question, I will make some further remarks on the notion of *fulfilment*. As pointed out earlier in the chapter, the notion was introduced by Saul Kripke, to provide an alternative proof of Gödel's incompleteness theorem. This notion stands out among the other proofs for the following reasons: (1) it is a semantic proof of Gödel's incompleteness, which is understood to be a syntactic result; and (2) the proof of Gödel's incompleteness via *fulfilment* does not use self-referential ideas, i.e. it does not rely on the fixed-point theorem/diagonal lemma to show independence using some provability predicate (Quinsey [1980, p. 1] and Putnam [2000, p. 55]).

In fact, the notion of *fulfilment* is similar to Σ_1-soundness of PA. To make this statement more precise we need to define the notion of *n-fulfilment*. For this, we use the definitions from (Putnam [2000]). Here we will use the notation s for the (code of the) sequence, and $(s)_n$ indicating the $(n-1)$-th digit in the sequence. We also abbreviate the *length of the sequence s* as $len(s)$.

Definition 50. *A sequence* $s = \langle (s)_0, (s)_1, (s)_2, \cdots \rangle$ *is* good *if*

$$(s)_0 > len(s);$$
$$\forall i > 0 \ (s)_{i+1} > ((s)_i)^2.$$

Now using the notion of a *good* sequence, we define *n-fulfillability.*

Definition 51. *$\varphi \in L_1$ is n-*fulfilable *if there is a* good *sequence $\mathcal{I}$ of length n such that there is a structure $\vec{\mathcal{A}}$ with the index I, where $\vec{\mathcal{A}}$* fulfils *φ (recall definition 39).*

In fact, what Putnam [2000] shows in his paper is that for any n, the conjunction of the first n axioms of PA^1 is n-fulfilable. He further remarks that n-fulfilment is a Σ_1-statement and that it shows that 'Peano Arithmetic [...] is *true*' (Putnam [2000, p. 56]). Of course, by Tarski's theorem truth cannot be defined in PA, but using the notion of *fulfillability* we can argue that some notions of truth of PA^1 can be obtained regarding any finite subset of PA.

Note that Σ_1-soundness of PA is a scheme asserting that for any φ a Σ_1-sentence, 'if φ is provable in PA then φ'. It will also come useful to note the fact that *n-fulfiability* can be defined as a Σ_1-sentence, since it is an existential statement followed by the notion of *fulfillability*, which was recursively defined. With these notions precisely defined, we can state the following theorem.

Theorem 52 (Putnam [2000], Quinsey [1980], Pakhomov)**.** *Let $PA = \{\varphi_i \in L_1 \mid i \in \omega\}$. In PA, we have that*

$$\Sigma_1\textit{-soundness of } PA \Leftrightarrow \textit{for every } n, \bigwedge_{i<n} \varphi_i \textit{ is } n\textit{−fulfilable.}$$

In fact, Putnam [2000, p. 56] makes a remark about *n-fulfillability* by claiming that

> In fact, "n-fulfillable" is a Σ_1 property, so the [*n-fulfillability* of the first n axioms of PA for all n] is only a Π_2-sentence. What it says, however, is that Peano Arithmetic has a weak kind of correctness.

For this result, I would like to send special thanks to Dr. Fedor Pakhomov at Steklov Mathematical Institute, Moscow. In several exchanges of emails, Dr. Pakhomov very generously discussed some ideas about *fulfillability.* Some more details of the notion of *fulfilment* can be found in (Quinsey [1980]).

The theorem shows us how closely related the notion of *fulfillability* and reflection principles are. In fact, Σ_1-soundness is the same axiom as Σ_1-reflection principle. If the reader is interested in this result, I strongly recommend reading (Quinsey [1980]).

As Putnam [2000, p. 54] emphasises the proof via *fulfilment* is not a 'different *version*' of Gödel's proof, but rather a 'different *proof*' in itself. This semantic proof to Gödel's result could be troubling for Isaacson for a few reasons. Isaacson's notion of *higher-order concept* chooses *coding* as an example. But Kripke was able to establish results about arithmetical independence from a semantics perspective, not relying on syntactic ideas.

If coding was a significantly important part in understanding Isaacson's thesis, then this is a significant problem for the thesis. Arguing that Gödel sentences are *non-genuinely arithmetical* on the basis that it requires coding in its proof, on *how* it was obtained, does not hold anymore. We are able to access Gödel sentences from a purely semantic perspective, so coding, in the end, does not show us what might be a *hidden higher-order concept.* If we could grasp the arithmetical structure, and this allows us to grasp the notion of fulfilment, where is the argument suggesting that Gödel sentences are not *genuinely arithmetical*?

This supports the position that it is not the 'higher-order concepts' that are an important part of the statement of Isaacson's thesis, but rather the *direct perceivability*. And our previously suggested definition of *genuinely arithmetical* fits safely with this line of thinking, that Gödel sentences (of PA^1) are still *genuinely arithmetical* because we can perceive them from purely semantic ideas, such as *fulfilment*, without some explicit statement claiming so.

In the next chapter, I will discuss whether there is a maximal theory we can obtain from the conditions of Wilkie's theorem. The formal result that I will provide shows that there is *no* maximal such theory. However, there are some problems with the way we allow the construction of extensions of PA. This problem appears to be unavoidable, and it seems to demand another approach to answer Isaacson's thesis from Wilkie's theorem.

Chapter 6

Isaacson's thesis and Wilkie's theorem

Isaacson [1987, p. 152] makes a remark on Wilkie's theorem in his early papers and how the result can be connected to his thesis. He suggests that the connection can show whether our conceptual understanding of the second-order characterisation can show a form of equivalence to first-order PA.[1]

> I understand that Alex Wilkie has obtained a result which can be interpreted as showing essentially that PA is the weakest first-order system arising from any categorical Π^1_1-characterization of the natural numbers. The question regarding the [intrinsic properties of] Peano Arithmetic might then be explored by assessing the conceptual content of the various categorical Π^1_1-characterizations of the natural numbers, looking to see whether those which yield Peano Arithmetic are recognizable as conceptually equivalent to Dedekind's analysis of the notion of natural number, and whether those which yield stronger first-order systems require some conceptual element which goes beyond our grasp of the natural numbers.

We have seen from the previous chapter that any restricted L_2-formula $\Phi(X)$ that is categorical for $\mathbb{N}$ (over some finite L_1-theory T) is such that $Scheme(\Phi) + T \vdash PA$ (Theorem 31). This suggests that any second-order axiom that is categorical for $\mathbb{N}$ will have its first-order scheme to be at least deductively strong as PA over the finite theory T, hence PA is the minimal such theory. Then the natural question to ask next is whether there can be two distinct restricted L_2-formulas Φ_1 and Φ_2, which their first-order schemes are *deductively orthogonal* – that is, there exist L_1-formulas $\psi_1 \in Scheme(\Phi_1)$ and $\psi_2 \in Scheme(\Phi_2)$ such that

$$Scheme(\Phi_1) \nvdash \psi_2 \text{ and } Scheme(\Phi_2) \nvdash \psi_1.$$

[1] Isaacson cites another paper of Wilkie's on page 152. But from his comments, it appears that Isaacson may have mis-cited Wilkie [1987].

Since both schemes are designed so Wilkie's theorem can be applied to them, if PA proves one of the schemes, it shows that the scheme is provably equivalent to PA. But this equivalence shows that the two schemes cannot be *deductively orthogonal* because one scheme will prove the other. But given the restricted L_2-formula that states the well-ordering on X, we can always extend it to obtain two restricted L_2-formulas that are orthogonal to each other.

Proposition 53. *Let ψ be an L_1-formula without x, y and z occurring as free-variables, and let $WO(X) \wedge \psi := \forall z \in X\ \exists x \in X\ \forall y \in X\ (x \leq y) \wedge \psi$ and $\Phi_\psi(X) := \forall z \in X\ \exists x \in X\ \forall y \in X\ (x \leq y \wedge \psi)$ be L_2-formulas. The $Scheme(WO \wedge \psi)$ is equivalent to $Scheme(\Phi_\psi)$.*

Proof. We will prove this by showing that in second-order logic that

$$\vdash WO(X) \wedge \psi \leftrightarrow \Phi_\psi(X).$$

Note that we can re-write $WO(X) \wedge \psi$ as

$$X \neq \emptyset \rightarrow (\exists x \in X\ \forall y \in X\ x \leq y) \wedge \psi,$$

and similarly, this is equivalent to

$$X \neq \emptyset \rightarrow (\exists x \in X\ \forall y \in X\ x \leq y \wedge \psi),$$

since ψ does not have any free variables occuring in $WO(X)$. Note that if $X = \emptyset$, then we trivially have $WO(X) \wedge \psi \rightarrow \Phi_\psi(X)$. Conversely, if X is empty, then regardless of $\Phi_\psi(X)$, we must have $WO(X) \wedge \psi$. Thus, we only consider the case where X is non-empty. We work in second-order logic in the following proof.

($\Rightarrow$) Suppose that $\exists x \in X\ \forall y \in X\ x \leq y \wedge \psi$. So x be a minimal element and we also have ψ is satisfied. Then trivially we can universally quantify with an arbitrary variable $\forall z$, thus we have $\Phi_\psi(x)$.

($\Leftarrow$) Suppose $\exists x \in X\ \forall y \in X\ (x \leq y \wedge \psi)$. This trivially gives us $WO(X) \wedge \psi$. □

If we can show that there are L_1-sentences ψ_1 and ψ_2 which are both true in the standard model and mutually independent over PA, we can extend $WO(X)$ by ψ_i to obtain two restricted second-order formulas Φ_1 and Φ_2 which are categorical for $\mathbb{N}$ (over some finite T) such that their schemes will be stronger than PA, as they contain L_1-sentences that are independent from PA.

Recall that Isaacson's thesis claims that PA is essentially complete. His argument relied on the fact that PA has a second-order categorical characterisation, and thus the first-order scheme of induction is what captures the correct first-order theory of arithmetic that is sound and complete. However, by finding L_1-sentences which are mutually undecidable over PA, we have given two distinct categorical characterisations of the intended structure of arithmetic that is stronger than PA. If we obtain such characterisations, this goes against Isaacson's claim that PA is essentially complete with respect to *genuinely arithmetical* truths, because there are independent L_1-sentences which can be argued to be essential truths of arithmetic because we have obtained them from some second-order categorical characterisation. We will touch on this issue later in this chapter. In fact, it is an open question whether there could be a natural L_1-sentences which are independent of PA but could be considered *genuinely arithmetical*.

For the remaining part of the chapter, we will show that there are such sentences that we are looking for, and discuss the significance of the results regarding the status of Isaacson's thesis. We introduce a new notation for L_1 formulas $\psi(x)$.

Notation 54. $\psi(x)^0 := \psi(x)$, *and* $\psi(x)^1 := \neg\psi(x)$.

The advantage of using such a notation is that there are infinitely many sentences which are equivalent to ψ or $\neg\psi$ but by using ψ^0 and ψ^1, we only need to pay attention to the two sentences. From the chapter on Wilkie's theorem, we also used ψ^* to mean $\neg\psi$ in prenex normal form. Thus, we can continue to equate ψ^* with ψ^1. Furthermore for a given formula $\psi(x)$, we can consider a sequence s such that $\psi(n)^{s_n}$ is consistent. For example, take $\psi(x) :=$ "x is even". Then $s := 010101\cdots$ will give us the sequence of sentences such that $\psi(n)^{s_n}$ holds in a sufficiently strong arithmetical theory.

The following results are from Lindström [2003, §2].

Definition 55. *Let T be an L_1-theory containing the basic arithmetic Q. We call a formula $\psi(x)$* independent over T *iff for any function $f : \omega \to \{0, 1\}$,*

$$T + \{\psi(k)^{f(k)} \mid k \in \omega\}$$

is consistent.

If we have a formula that satisfies the above definition for PA, we can say that this formula is undecidable in PA for every natural number. This is because we have defined our arbitrary function f on ω and we could take f to be some constant function 0 or 1. Thus we can use this to find a formula that we are looking for.

Proposition 56. *There is a formula $\psi(\vec{x})$ such that $\psi(\vec{x})$ is independent over PA.* (Lindström [2003, p. 33, Theorem 9])

We take our theory to be PA, and we can find infinitely many sentences such that for any $f : \omega \to \{0, 1\}$, for $n \in \omega$, $\psi(n)^{f(n)}$ is independent of PA. We take a function f, and consider $\psi(0)^{f(0)}$ and $\psi(1)^{f(1)}$, which are undecidable sentences of PA. If at least one was decidable (without loss of generality, say $\psi(0)^{f(0)}$, then could merely choose a function g such that $f(0) \neq g(0)$. Then the theorem should give us that $\psi(0)^{g(0)}$ is independent of PA, which will give us a contradiction.

Thus we have two L_1-sentences which are undecidable in PA. But we must establish that they are true in $\mathbb{N}$ and are mutually independent over PA. For being true sentences in $\mathbb{N}$, we can choose our function f to be such that $\psi(0)^{f(0)}$ and $\psi(1)^{f(1)}$ are true in $\mathbb{N}$. And we must show that they are mutually independent over PA. The following proposition is obtained by an elementary proof, but it is significant because it will equip us with *deductively orthogonal* schemes that we are looking for.

Proposition 57. *Let $\psi(n)$ be the formula that is independent over PA. Then for any function $f : \omega \to \{0,1\}$, $\psi(0)^{f(0)}$ and $\psi(1)^{f(1)}$ are mutually independent over PA.*

Proof. Let $f : \omega \to \{0,1\}$ be a function, then we have that $PA + \{\psi(n)^{f(n)} \mid n \in \omega\}$ is consistent. Assume for a contradiction that $\psi(0)^{f(0)}$ and $\psi(1)^{f(1)}$ are not mutually independent over PA. Without loss of generality, suppose that $PA + \psi(1)^{f(1)} \vdash \psi(0)^{f(0)}$.

Let $f' : \omega \to \{0,1\}$ be a function such that $f'(0) \neq f(0)$ and $f'(n) = f(n)$ for any $n > 0$. Then by the previous proposition, we must have $PA + \{\psi(n)^{f'(n)} \mid n \in \omega\}$ to be consistent. Thus let $\mathcal{M} \vDash PA + \{\psi(n)^{f'(n)} \mid n \in \omega\}$.

We assumed that $PA + \psi(1)^{f(1)} \vdash \psi(0)^{f(0)}$, which is equivalent to saying that $PA + \psi(1)^{f'(1)} \vdash \neg\psi(0)^{f'(0)}$. And since $\mathcal{M} \vDash PA + \psi(1)^{f'(1)}$, we must have that $\mathcal{M} \vDash \neg\psi(0)^{f'(0)}$. But we defined our $\mathcal{M} \vDash PA + \{\psi(n)^{f'(n)} \mid n \in \omega\}$, so we must have $\mathcal{M} \vDash \psi(0)^{f'(0)}$. It follows by definition that $\mathcal{M} \vDash \psi(0)^{f'(0)} \wedge \neg\psi(0)^{f'(0)}$, but this is a contradiction. Hence $\psi(0)^{f(0)}$ and $\psi(1)^{f(1)}$ must be mutually independent over PA. □

Note that the above two propositions will apply to any recursively axiomatisable arithmetically sound theory extending PA, thus we can apply them to $PA + T$, where T is an arbitrary finite theory.

Theorem 58. *There are restricted L_2 sentences $\Phi_1(X)$ and $\Phi_2(X)$ such that they are both categorical for $\mathbb{N}$ (over some finite T_1 and T_2 respectively), but $\Phi_1 \neq \Phi_2$. This then gives us that $Scheme(\Phi_1) + T_1 \nvdash Scheme(\Phi_2)$, and $Scheme(\Phi_2) + T_2 \nvdash Scheme(\Phi_1)$.*

Proof. Let $\psi(x)$ be a formula that is independent from PA. Let $f : \omega \to \{0,1\}$ be a function such that $\psi(0)^{f(0)}$ and $\psi(1)^{f(1)}$ holds true in the standard model $\mathbb{N}$. By theorem 57, we have that $\psi(0)^{f(0)}$ and $\psi(1)^{f(1)}$ are mutually independent over PA.

Call the sentences $\psi(0)^{f(0)}$ and $\psi(1)^{f(1)}$, ψ_0 and ψ_1 respectively, and consider the L_2-formulas $\Phi_{\psi_0}(X)$ and $\Phi_{\psi_1}(X)$. Recall that these are restricted formulas that are categorical for $\mathbb{N}$ over the finite theories T_1 and T_2 respectively, because they are equivalent to $WO(X) \wedge \psi_i$, for $i = 0, 1$ by proposition 53. Hence we have two distinct L_2-formulas that are restricted and categorical for $\mathbb{N}$.

Now, we can apply Wilkie's theorem on both L_2-formulas to attain that the first order schemes of the formulas can prove PA. However, the formulas contain ψ_0 and ψ_1 respectively which are mutually independent over PA. Thus neither we have L_2-formulas $\Phi_1 := \Phi_{\psi_0}$ and $\Phi_2 := \Phi_{\psi_1}$ such that $Scheme(\Phi_1)$, $Scheme(\Phi_2) \vdash PA$, but $Scheme(\Phi_i) + T_i \nvdash Scheme(\Phi_j)$ for $i \leq j$. □

We have obtained there to be two categorical axiomatisations which their first-order counterparts derive PA but neither one can derive the other. Isaacson appears partial to the claim that arithmetical truths are to be identified with the first-order counterparts of second-order categorical axiomatisations. If Isaacson's position is that the categorical axioms can be good ersatz for the intended model, and the first-order consequences are bring out the essentially true arithmetical sentences, then it becomes an important interest to find more second-order categorical axiomatisations that are orthogonal in their first-order consequences.

Recall the following question which we proposed to answer in the introduction:

Question 59. *1 Are there restricted categorical axiomatisations whose first-order counterparts (perhaps with the additional single sentences) are mutually independent (in that neither proves the other)? Wilkie's Theorem in effect says that the first-order counterpart of PA^2 is minimal amongst such axiomatisations. Can it be shown that that there is no maximal such one?*

It appears that we have successfully answered the above question because we have provided two restricted categorical formulas, whose first-order schemes are *deductively orthogonal*, and the proof of the following corollary shows that we can resolve the maximality question:

Corollary 60. *There is no maximal restricted categorisation that satisfies Wilkie's theorem.*

Proof. Suppose for a contradiction that there is a maximal one, and call it $\Phi(X)$. By assumption, $\Phi(X)$ is a restricted categorical L_2-formula. This means that $Scheme(\Phi) \supseteq PA$ by Wilkie's theorem, so let us call the theory $Scheme(\Phi)$, T, which is a first order theory. We can apply proposition 56 to T to obtain an independent formula $\psi(x)$ over T. Then by proposition 57, we have two independent sentences from T – call them ψ_1 and ψ_2. Following the idea of the proof for 58, we have obtained theories that are strictly stronger than T, so the second-order statements are stronger than Φ. Hence, we have a contradiction. □

The above corollary shows that there cannot be a maximal categorical axiomatisation, since we can always generate new independent sentences from any categorical axiomatisations and obtain a stronger categorical axiomatisation. However there is something 'unnatural' about the restricted axiomatisations that we obtained. Firstly, it is unclear whether Isaacson's would consider these axiomatisations to be grasping the intended structure, since they contain L_1-sentences that may not be essentially arithmetical sentences. Secondly, we do not know what these L_1-sentences assert and in the proof used in Lindström [2003], the use of coding can be challenged by Isaacson as accessing the *higher-order concepts*.

In order to really solve the question, we need to look out for more *natural* expressions that do not go beyond arithmetic. We will re-formulate the question by adding the *naturality* condition on the restricted axiomatisation. I call an expression of second-order language *natural* if they only express constraints on the order relation $\leq$ in L_1. The obvious example of a restricted natural expression is $WO(X)$, as it only contains $\leq$ from L_1 and does not use other constants. Thus we can re-state question 1 as the following:

Question 61. *Are there* natural *restricted categorical axiomatisations whose first-order counterparts are mutually independent?*

In conclude this chapter with the open question. If we can answer the question positively, then one could argue that Wilkie's theorem fails to be an accurate formalisation of Isaacson's thesis. What we have achieved so far is that given the existence of orthogonal sentences for PA^1, there cannot be a maximal theory that is captured from the categorical characterisation. The final chapter will summarise the results we have developed in this thesis, with some further open questions.

Chapter 7

Conclusions and directions for further work

We began the thesis with the historical motivation for structuralism in philosophy of mathematics and where Isaacson's structuralism is positioned with respect to contemporary structuralism. In chapter 2, we saw that Isaacson's structuralism is motivated by his concept of the *reality of mathematics*. We understand mathematical structures to 'exist' in the *reality of mathematics*, when we can successfully give a categorical characterisation of the structure. In this sense, Dedekind's *simply infinite system* is a successful account capturing the structure of the finite numbers. From this, Isaacson argues that we can obtain Peano arithmetic to be the accurate first-order theory of arithmetic, appropriate for deductive reasoning about the natural numbers.

In order to fully understand Isaacson's ideas and what is known as Isaacson's thesis, we needed to explicate Isaacson's notions of *directly perceivability* and *genuinely arithmetical* statements. Isaacson argued that statements that are independent of first-order Peano arithmetic are not *directly perceivable* from the concept of finite numbers, and therefore they were not *genuinely arithmetical*. The vague analysis of the notion of *direct perceivability* left the notion of *genuinely arithmetical* statements to be equally vague.

In chapter 3, we expanded on the notion of *directly perceivability* with the motivation to give a more precise formulation of Isaacson's thesis. Motivated by Isaacson's view that we can grasp the concept of the finite structure by giving a categorical characterisation, and the implicit connection between syntax and structures is what allows us to *directly perceive* these structures, it seemed natural to move towards the ideas in internalism in order to give a formal definition of *genuinely arithmetical* statements.

From the formal definition of *genuinely arithmetical* statements, we were able to give a formal statement of Isaacson's thesis, which I have called *neo-Isaacson's thesis*. The shift towards formalising Isaacson's notions made the arguments regarding which statements are *genuinely arithmetical* more precise. For example, it was unclear why every instance of induction axiom was consider *genuinely arithmetical* from Isaacson's perspective, but our formalisation of *genuinely arithmetical* statements picked out exactly the first-order theorems of second-order arithmetic.

Another formalisation of Isaacson's thesis is Wilkie's theorem (Wilkie [1987]). Roughly it states that first-order Peano arithmetic is the minimal theory that can be captured by any restricted categorical characterisation of the natural numbers. In chapter 5, I have presented the proof of Wilkie's theorem, and concluded with a brief discussion on the notion of *fulfilability*.

The technical tools used in Wilkie's theorem are related to Kripke's notion of *fulfilability*, which is used to give a semantic proof of Gödel's incompleteness theorems. Moreover, the notion of *fulfilability* turns out to be equivalent to the reflection principles in PA^1. What is important about this notion is that it actually shows that Isaacson's argument about *how* we obtain the *genuinely arithmetical* statements, rather than *what* need to be more grounded. Isaacson's argument against the consistency statement of PA^1 being *genuinely arithmetical* relied on the use of coding that gives us an access to the connection between the syntax and structures. But with the semantic proof of Gödel's incompleteness, it shows us that there is no unique method of understanding these *genuinely arithmetical* statements. In this sense, the formal definition of *genuinely arithmetical* statements, and therefore neo-Isaacson's thesis are beneficial in understanding Isaacson's philosophical positions and the remarks that he is trying to make. The formalisation gives us a better understanding of the statement of Isaacson's thesis.

Even though Wilkie's theorem suggests that PA^1 is the minimal theory obtained from the categorical characterisation of the natural numbers, it had remained open whether there could be a maximal theory. In fact, what we have seen in chapter 6 is that there cannot be a maximal theory that is captured by the restricted categorical characterisation. In order to prove this, I have used statements that are orthogonal and independent from PA^1 such that we have at least four distinct models of PA^1 that claims different truth-values for the statements. I have shown that we could take any independent statement φ from PA^1 that is true in $\mathbb{N}$ and the theory extending PA^1 with φ could be captured from Wilkie's theorem.

It seems evident that formalisation of Isaacson's thesis gives us a better insight to understanding Isaacson's claims and therefore which statements should be considered *genuinely arithmetical* or even *genuinely mathematical*. The results we have found by adopting internalist ideas suggest that PA^1 is not a complete theory for *genuinely arithmetical* statements. But also implies that first-order logic might not be a suitable theory for deduction.

The struggle with using Wilkie's theorem as a formalisation of Isaacson's thesis is that even if it captures PA^1 to be a minimal suitable theory for arithmetic from the categorical characterisation, it does not capture it to be the maximal theory. In fact, this suggests that any theorem of PA^1 is *genuinely arithmetical*, which also holds by neo-Isaacson's thesis. It appears that what Isaacson's thesis was motivated to do all along was to specify what is *genuinely arithmetical*, and perhaps extend this notion to *genuinely mathematical* statements.

If we have successfully captured what is *genuinely arithmetical*, and perhaps we could apply the same ideas to set theory. Since ZFC is quasi-categorical, we might be able to capture what is *genuinely set-theoretic* by using the internalist ideas. Thus the natural open question would be:

Question 62. *Can we extend neo-Isaacson's thesis and the notion of* genuinely arithmetical *statements to set theory, more specifically ZFC?*

An answer to the question could help us decide whether the continuum hypothesis should be considered a *genuinely set-theoretic* statement. The quasi-categoricity of ZFC suggests that for Isaacson, there is no new axiom for set theory that extends ZFC that is *genuinely set-theoretic*. However, it is unclear how much weight the quasi-categoricity of ZFC^2 carries compared to the categoricity of PA^2.

In fact, this idea has been discussed in (Incurvati [2008]), which he dubs the thesis claiming first-order theorems of ZFC to be *genuinely mathematical – Horsten's thesis*. In the end, Incurvati argues that Horsten's thesis is not a ZFC variation of Isaacson's thesis. But whether the ideas developed in this thesis for neo-Isaacson's thesis can be applied to ZFC would be an interesting development towards understanding mathematical epistemology better.

Although I have argued that Wilkie's theorem is not an adequate formalisation of Isaacson's thesis, it is not obvious how Wilkie's theorem would look like in the internalist setting. In theory, the statement of Wilkie's theorem can be formalised in second-order logic. And since *fulfilability* can be defined in L_1, it should be reasonable to assume that Wilkie's theorem can be proved in PA^2. Thus I leave another open question regarding the tools used in neo-Isaacson's thesis and Wilkie's theorem.

Question 63. *Can we prove Wilkie's theorem in the internalist setting?*

Furthermore, if Wilkie's theorem can be proven to be true in second-order logic, this would claim that Wilkie's theorem is a *genuinely arithmetical* statement. The philosophical consequence of this claim is not obvious, and this could be the direction forward for future research on Isaacson's thesis and Wilkie's theorem.

I conclude this thesis, with the two open questions. The first open question regarding neo-Isaacson's thesis and ZFC could tell us more about what is really a *mathematical knowledge*. This is philosophically motivating for understanding the concept of knowledge as well as how we claim knowledge in the practice of mathematics. Hopefully this is another way in understanding our mathematical cognition and the notion of mathematical proof.

The latter question is interesting for understanding Isaacson's thesis better, and perhaps the notion of *fulfilment*. In fact, the notion of *fulfilment* has shown that there is a connection between many of the independent statements. We can obtain Gödel sentences of PA^1 to be true by the semantic proof via *fulfilment*, and the notion of *n-fulfilment* is equivalent to Σ_1-soundness of PA^1. Furthermore Σ_1-soundness is equivalent to Σ_1-reflection principle.

Isaacson [2011a] has considered different notions of consistency for arithmetic, where 1-consistency is equivalent to Σ_1-soundness for PA^1. It would be fascinating to investigate the connections further and how *fulfilment* can reveal some new rich results about Peano arithmetic.

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