A right semimodel structure on semisimplicial sets

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In this thesis we investigate to what extent the Kan-Quillen model structure on simplicial sets can be transferred along the left adjoint of the free-forgetful adjunction with semisimplicial sets. We establish the novel result that, while the full model structure cannot be transferred, the underlying right semimodel structure can. Using earlier results we show that the adjunction becomes an equivalence between right semimodel structures, and that the fibrant objects and fibrations between fibrant objects are characterised by having the right lifting property against the semisimplicial horn inclusions. We show by a counterexample that this is not the case for all fibrations, for not every morphism that maps to a simplicial anodyne extension is a semisimplicial anodyne extension. Finally, we show that the *strong* anodyne extensions of simplicial and semisimplicial sets do coincide.

Contents

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Introduction

In [EZ50], S. Eilenberg and J.A. Zilber first introduced *simplicial sets*, then called *complete semisimplicial complexes*. Meant as an addition to the already familiar simplicial complexes, they are abstractions of topological spaces that serve as technical tools for the calculation of certain topological properties.

Most of the applications of simplicial sets lie within the field of homotopy theory, which is concerned with the study of continuous deformations of continuous maps. In the language of category theory (also co-invented by Eilenberg), [Qui67] provides an axiomatic framework for studying homotopy theory, distinguishing three fundamental classes of morphisms: *cofibrations*, *fibrations* and *weak equivalences*. Categories that satisfy the given axioms are called *(Quillen) model categories*. Among the three key examples given by Quillen are the category of topological spaces and the category of simplicial sets. In fact, he showed that, as model categories, these two are equivalent under the relevant notion of equivalence.

In recent years this abstract homotopy theory has attracted the attention of logicians. Interpreting types as spaces, notably identity types as path spaces, a connection can be made between the formal language of intensional type theory, and homotopical structures, like Quillen model structures. The project of investigating this connection is referred to as *homotopy type theory* and is a very active field of research at the moment. Unsurprisingly, the first category found to be a suitable model for the specific type theory studied in this regard, is the category of simplicial sets (see [KL12]).

A downside of the model of homotopy type theory in simplicial sets (and also of the Quillen model structure on simplicial sets) is that its standard existence proof is non-constructive. This fact is proven in [BC15] and has the consequence that there is no universal method for constructing this model *internally* to some other model of type theory, for type theory is generally constructive.

The key fact preventing this existence proof to be constructive, is that simplicial sets come with certain maps called *degeneracy maps*. An element of a simplicial set is called *degenerate* whenever it lies in the image of a degeneracy map. However, constructively it is not always decidable whether a given element is degenerate, a fact that is needed in the proof.

This brings us to the subject of this thesis. The weaker notion of *semisimplicial sets* (in [EZ50]: *semisimplicial complexes*) refers to objects like simplicial sets, but without degeneracy maps. Investigating the homotopical properties of these weaker objects might eventually lead to the construction of a model of homotopy type theory that can be performed within type theory itself. We will investigate the possibility of imposing a Quillen model structure on semisimplicial sets using classical methods, leaving the further requirements for a model of homotopy type theory and the generalisation to a constructive metatheory for future work. It should be noted there is another important difficulty with internalising homotopy type theory. Namely, even if semisimplicial sets could provide a model of type theory constructively, it is still far from trivial to construct semisimplicial sets within homotopy type theory. Work on this topic has been done by for instance [ACK16].

There are many ways in which a category can be equipped with a model structure and therefore we must make one further restriction to the scope of our question. There is a free-forgetful adjunction between simplicial and semisimplicial sets of which the left adjoint preserves the abstracted topological space. We will concentrate on a model structure that is *left-induced* by this adjunction, meaning that certain Quillen model-theoretic properties will be preserved and reflected by the 'free' functor from semisimplicial into simplicial sets.

Our main result will be that, within the bounds of left-induced structures, there is no model structure on semisimplicial sets, but there is something slightly weaker, a *right semimodel structure*.

We assume that the reader is familiar with the basic notions of category theory, including functors, natural transformations, (co)limits and adjoints, as well as some standard constructions, such as the (co)monad of an adjunction and the colimit-diagonal adjunction. Beyond this only some very elementary knowledge of algebraic topology is assumed, in particular the notion of contractible spaces and some of their properties.

Structure and original contributions

- In Chapter 1 we introduce semisimplicial sets and show how they relate to their simplicial counterparts. Most of the results in this chapter appear already in [RS71], but are here presented from a more modern categorical point of view. A particular observation that, to our knowledge, is an addition to the existing literature is that the free-forgetful adjunction is both monadic and comonadic.

- Chapter 2 introduces the necessary abstract homotopy theory. We prove some standard results on model categories and extend them to right semimodel categories. The standard reference for semimodel categories is the thesis [Spi]. While there are some discussions of this kind of structure in the literature, it has been studied considerably less than standard model structures. Simple, but to our knowledge novel, are the sufficient conditions for two interacting weak factorisation systems to induce a right semimodel structure, and the characterisation of the weak equivalences of such structures in terms of the other constituent morphisms.

  The rest of the Chapter covers standard Quillen model-theoretic techniques that will be needed in Chapter 3, including the *small object argument*, some results on *locally finitely presentable categories* and the *Leibniz adjunction*. Particularly interesting is a relatively recent conceptual proof from [RR15] that in suitable model categories weak equivalences are closed under *directed colimits*.

- Chapter 3 begins with a proof that there is no model structure on semisimplicial sets of the kind we are looking for. We then adapt several construction methods for model structure on simplicial sets (mainly from [JT08]) to the semisimplicial case, recording the ones that work in the general Theorem 3.3.1. This new theorem gives a range of structures on *ssSet* that are almost left-induced right semimodel structures. Combined with a powerful result from [MR14] concerning left-induced weak factorisation systems, our theorem can be used to establish the desired right semimodel structure, which is the main result of this thesis.

We then use a result from [KS17] to show that this structure is equivalent to the right semimodel structure on semisimplicial sets and a result from [Sat18] to prove a recognition lemma for a certain class of its morphisms. We demonstrate with an example from [Mos15] that, in the category of semisimplicial sets, the class of *simplicial anodyne extensions* is strictly larger than the class of *semisimplicial anodyne extensions*. It follows that the semisimplicial structure considered in [Sat18] (which can also be obtained using Theorem 3.3.1) is not a right semimodel structure. Finally, we prove the novel result that the smaller classes of *strong semisimplicial anodyne extensions* and *strong simplicial anodyne extensions*, in contrast, do coincide.

During the writing of this thesis two preprints on the same topic, [Hen18] and [Sat18], appeared on the arXiv. The only part of this thesis for which either of the two preprints is consulted, is the part in Chapter 3 in which a result from [Sat18] is explicitly used to derive a property of the right semimodel structure. Though the rest of this thesis is written independently from them, the results from the two preprints combined subsume almost every novel contribution in Chapter 1 and Chapter 3, with as notable exception the conclusion that the right semimodel structure exists.

Notations and conventions

In this thesis we use the following notation. In any category:

- 0 is the initial object and $*_A : 0 \longrightarrow A$ is the unique morphism from 0 to A;
- 1 is the terminal object and $!_B : B \longrightarrow 1$ is the unique morphism from B to 1.

In the context of model structures, we shall often use the following decorated arrows.

	Morphism	Fibration	Cofibration
Morphism	$\longrightarrow$	$\longrightarrow\!\!\!\!\!\twoheadrightarrow$	$\rightarrowtail\!\!\longrightarrow$
Trivial		$\overset{\text{triv}}{\twoheadrightarrow}$	$\overset{\text{triv}}{\rightarrowtail}$
Weak equivalence	$\overset{\sim}{\longrightarrow}$	$\overset{\sim}{\twoheadrightarrow}$	$\overset{\sim}{\rightarrowtail}$
Trivial and weak equivalence		$\underset{\sim}{\overset{\text{triv}}{\twoheadrightarrow}}$	$\underset{\sim}{\overset{\text{triv}}{\rightarrowtail}}$

Chapter 1

Semisimplicial sets

In this chapter we introduce the main objects of study of this thesis, semisimplicial sets. These objects once arose from the study of topological spaces by considering them as the result of gluing together several n-dimensional generalisations of triangles, called *n-simplices*. Though not all topological spaces admit such a *triangulation*, the study of those that do is simplified by the nice combinatorial properties that simplices possess.

Conceptually, and, as can be seen in [EZ50], also historically, semisimplicial sets form the middle link of a three-part sequence of developments:

$$\text{Simplicial complexes} \;\longrightarrow\; \text{Semisimplicial sets} \;\longrightarrow\; \text{Simplicial sets.}$$

For the purpose of completeness, we begin this chapter in Section 1.1 by introducing the objects that form the leftmost link.

In Section 1.2, we generalize to semisimplicial sets. We will see that semisimplicial sets can not only be studied geometrically, as topological spaces, or combinatorially, as a collection of simplices, but also categorically, for the category of semisimplicial sets conveniently arises as a presheaf category. This last point of view will be the dominant one for the rest of this thesis.

In Section 1.2.2 we use categorical methods to show how to construct the topological space corresponding to some semisimplicial set. The categorical product of semisimplicial sets unfortunately does not respect this *geometric realisation*, which is why, in Section 1.2.3, we introduce an operation, called the *geometric product*, that does. We show that under this product operation semisimplicial sets form a symmetric closed monoidal category.

Finally, in Section 1.3, we generalize further to simplicial sets and show how they relate to their 'semi'-counterparts. Simplicial sets have some additional desirable properties, which has made them more popular tools for several applications, and therefore also more well-studied. In Chapter 3 we will use known results stemming from this line of research to derive some (Quillen) model-theoretic results for semisimplicial sets.

1.1 The combinatorial study of geometry

The purpose of this section is to provide the reader with some background knowledge that will make it easier to intuitively follow the reasoning throughout the rest of this thesis. Everything in this section is standard, but we roughly follow [Fri12].

We begin by defining the building blocks of any triangulation of a topological space, the n-dimensional simplices. First let us consider them as concrete spaces living in the category **Top** of topological spaces and continuous functions between them.

Definition 1.1.1. The *standard geometric n-simplex* Δ^n is the object of **Top** given by

$$\Delta^n = \{(t_0, \dots, t_n)\} \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\},$$

endowed with de subspace topology. ■

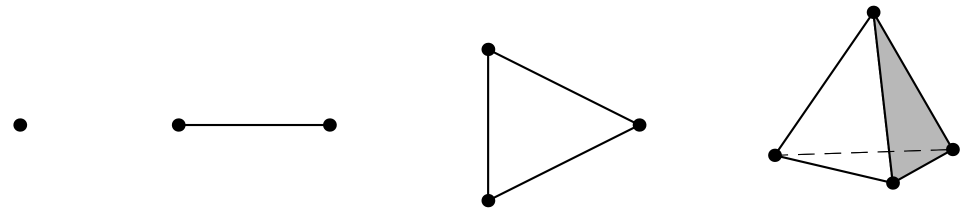


Figure 1.1: The standard geometric 0-, 1-, 2- and 3-simplices.

In Figure 1.1 the standard simplices in the first four dimensions are depicted. Note that simplices are solid objects and not merely boundaries. Consequently, every n-simplex is homeomorphic to the n-ball.

A standard geometric simplex is the convex hull of the points corresponding to the n-dimensional unit vectors. In general, we define:

Definition 1.1.2. A *geometric n-simplex C* is a topological space that can be written as

$$C = \{(t_0 u_0, \dots, t_n u_n)\} \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\}$$

with $u_0, \dots, u_0$ affinely independent. ■

Again, another way to describe a geometric simplex C is as the convex set spanned by the points $u_0, \dots, u_n$. We call the simplex that is the convex spanned by a subset $u_{i_0}, \dots u_{i_k}$ of these points a *face* of C. In particular, the empty set is a face of every simplex.

Note, and this can be seen as the first step in our sequence of abstractions, that the only relevant information determining a simplex is the set of its vertices. This means that we may refer to the simplex spanned by the points $v_0, \dots, v_n$ simply as the set $\{v_0, \dots, v_n\}$.

Example 1.1.3. If $x = \{v_0, \dots, v_n\}$ is an n-simplex, then its $(n-1)$-dimensional faces are precisely the $(n-1)$-simplices that are determined by all but one of its vertices. We write

$$\{v_0, \dots, \widehat{v_k}, \dots, v_n\}$$

for the simplex determined by all but the k-th vertex and call it the *k-th face* of x. ▲

The following standard topological definition encapsulates those spaces that can be obtained by taking a set of simplices and gluing them together along their faces.

Definition 1.1.4. A *geometric simplicial complex* X is a set of geometric simplices such that

1. Every face of a simplex of X is in X;
2. For any two simplices $x, y \in X$, their intersection is a face of both x and y. ■

Example 1.1.5. Every simplex $\{v_0, \dots, v_n\}$ gives rise to a simplicial complex, consisting of itself and its faces. ▲

We now have everything we need to abstract away from the realm of topological spaces. Note that the data determining a simplicial complex X may be organised by, for each $n \in \mathbb{N}$, letting X_n be the set of all n-simplices of X. Then, by the fact that X is a simplicial complex, each face of an n-simplex $\{v_0, \dots, v_n\} \in X_n$ occurs in some X_m for $m \leq n$. For instance, we have $v_i \in X_0$ for every $i \in [0, n]$.

Definition 1.1.6. An *abstract simplicial complex* is a graded set $(X_n)_{n \in \mathbb{N}}$ such that every X_k consists of subsets $x \subseteq X_0$ with $|x| = k + 1$, and such that for every non-empty $x' \subseteq x$ with $|x'| = j + 1$, we have $x' \in X_j$. ■

It is easy to see that every geometric simplicial complex naturally corresponds to an abstract simplicial complex. There is then a canonical way to recover a geometric simplicial complex (up to homeomorphism) from the corresponding abstract simplicial complex. Note also: the one (-1)-simplex, the empty set, is only implicitly part of every abstract simplicial complex.

In case the set X_0 is totally ordered, every k-simplex of an abstract simplicial complex X_k can be uniquely written as a tuple $[v_{i_0}, \dots, v_{i_k}]$ with $v_{i_0} < \dots < v_{i_k}$. We call this an *ordered simplicial complex.*

Example 1.1.7. Any partially ordered set P can be made into an ordered simplicial complex $C(P)$ set by setting

$$C(P)_k = \{[\alpha(0), \dots, \alpha(k)] \mid \alpha : [k] \longrightarrow P \text{ is injective and order-preserving}\}.$$

The simplicial complex $C(P)$ is called the *nerve* of P.

In particular, every finite ordinal $[n]$ gives rise to an ordered simplicial complex of which the k-simplices are the tuples

$$[\alpha(0), \dots, \alpha(k)]$$

given by some injective order-preserving map $\alpha : [k] \longrightarrow [n]$. This ordered simplicial complex, which we call the *standard abstract n-simplex*, consists of precisely one n-simplex, together with all of its faces. ▲

Definition 1.1.8 (Face maps). Let X be an ordered simplicial complex. Then for every n and k there is a map $d^k : X_{n+1} \longrightarrow X_n$ given by

$$d^k([v_{i_0}, \dots, v_{i_n}]) \longmapsto [v_{i_0}, \dots, \widehat{v_{i_k}}, \dots, v_{i_n}].$$

■

Lemma 1.1.9. *Of every ordered simplicial complex X, the face maps satisfy the relation*

$$d^i d^j = d^{j-1} d^i \quad \text{if } i < j.$$

Proof. For any k-simplex $[v_{i_0}, \dots, v_{i_k}]$, we have

$$\begin{aligned} d^i d^j [v_{i_0}, \dots, v_{i_k}] &= d^i [v_{i_0}, \dots, \widehat{v_{i_j}}, \dots, v_{i_n}] \\ &= [v_{i_0}, \dots, \hat{v}_{i_i} \dots, \widehat{v_{i_j}}, \dots, v_{i_n}] \\ &= d^{j-1} [v_{i_0}, \dots, \hat{v}_{i_i}, \dots, v_{i_n}] = d^{j-1} d^i [v_{i_0}, \dots, v_{i_n}], \end{aligned}$$

as required. $\square$

1.2 Semisimplicial sets

Semisimplicial sets are simply a generalisation of the ordered simplicial complexes defined in Section 1.1.

Definition 1.2.1. A *semisimplicial set* X is an $\mathbb{N}$-graded set $X = (X_n)_{n \in \mathbb{N}}$, together with for every $n \geq 0$ and $0 \leq i \leq n$ a map $d^i : X_{n+1} \longrightarrow X_n$ such that

$$d^i d^j = d^{j-1} d^i \text{ if } i < j.$$

The d^i are again called *face maps* and can be intuitively thought of as sending an n-dimensional simplex to its i-th $(n-1)$-dimensional face.

It follows from Lemma 1.1.9 that the ordered simplicial complexes are semisimplicial sets. It is, however, not the case that all semisimplicial sets are also simplicial complexes, for we no longer require the faces of a given simplex to be distinct. Consider, for instance, the semisimplicial set A, given by:

$$A_0 = \{a\} \qquad d^0 I = d^1 I = a$$
$$A_1 = \{I\}$$

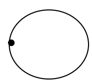


Figure 1.2: The semisimplicial set A.

This semisimplicial set may be pictured as a line of which the endpoints are glued together and is not a simplicial complex. Another property of simplicial complexes that fails for semisimplicial sets is that in the former a given simplex is uniquely determined by its vertices. In contrast, two distinct simplices of a semisimplicial set may have precisely the same faces.

There is a natural definition of morphisms between semisimplicial sets:

Definition 1.2.2. Let X and Y be semisimplicial sets. A *semisimpicial morphism* $f : X \longrightarrow Y$ is a graded morphism that commutes with the face maps, *i.e.* such that

$$d^k f_{n+1}(x) = f_n(d^k x),$$

for all $n \geq 0$ and $0 \leq k \leq n+1$.

1.2.1 The presheaf category ssSet

We are now ready to take a more categorical approach. The maps $d^k : C[n]_n \longrightarrow C[n]_{n-1}$, from the top-dimensional simplex of $C[n]$ to its $n-1$ dimensional faces, are dual to the injective order-preserving maps $\epsilon^k : [n-1] \longrightarrow [n]$ skipping the k-th element. This gives rise to a directed graph

$$\begin{tikzcd} {[0]} \arrow[r, shift left, "\epsilon^0"] \arrow[r, shift right, "\epsilon^1"'] & {[1]} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \end{tikzcd} \tag{1.1}$$

Note that the ϵ^k verify a relation dual to the d^k, namely

$$\epsilon^j \epsilon^i = \epsilon^i \epsilon^{j-1} \ \text{ if } i < j. \tag{1.2}$$

Now let Δ_i be the category of nonempty finite ordinals $[n] = \{0, 1, \dots, n\}$ and injective order-preserving functions between them.

Lemma 1.2.3. *The category Δ_i is freely generated by the graph (1.1) subject to the relation (1.2).*

Proof. We claim that every morphism $\epsilon : [n] \longrightarrow [m]$ in Δ_i admits a unique decomposition

$$\epsilon = \epsilon^{i_k} \epsilon^{i_{k-1}} \cdots \epsilon^{i_1} \tag{1.3}$$

such that $k = m - n$ and $m \geq i_k > \dots > i_1 \geq 0$. Indeed, a straightforward inductive argument shows that each i_k must be the k-th highest number skipped by ϵ, which suffices.

Now let Δ_i' be the freely generated category in question. Because the relation (1.2) is verified in Δ_i, the unique canonical mapping $\Delta_i' \longrightarrow \Delta_i$ that is the identity both on objects and on morphisms, is a functor. It is clearly bijective on objects and, by the above, surjective on morphisms. We will show that it is also injective on morphisms, after which we are done. To this end, suppose that two arrows u, v of Δ_i' get sent to the same morphism of Δ_i. Repeatedly using the relation (1.2), we can rewrite (not necessarily uniquely) both u and v into the form (1.3). But then, by the uniqueness of the decomposition in Δ_i, these decompositions must be the same in Δ_i' and thus $u = v$. □

Let **ssSet** denote the category of semisimplicial sets and semisimplicial morphisms between them.

Lemma 1.2.4. *The presheaf category $[\Delta_i^{op}, \mathbf{Set}]$ is isomorphic to* **ssSet**.

Proof. Consider the assignment $[\Delta_i^{op}, \mathbf{Set}] \longrightarrow \mathbf{ssSet}$, sending a presheaf P to the simplicial set X^P with $X^P_n = P([n])$ and face maps $P(\epsilon^k) : X^P_{n+1} \longrightarrow X^P_n$. For $i < j$, we have

$$P(\epsilon^i)P(\epsilon^j) = P(\epsilon^j \epsilon^i) = P(\epsilon^i \epsilon^{j-1}) = P(\epsilon^{j-1})P(\epsilon^i),$$

as required. This assignment extends to arrows in the obvious way and is clearly functorial.

The described functor is easily seen to be injective on objects. To see that it is also surjective, we define, for a given X in **ssSet**, the corresponding presheaf P^X as follows. Of course we set $P^X([n]) = X_n$ and $P^X(\epsilon^k)(x) = d^k x$. By Lemma 1.2.3, this uniquely extends to a definition of P_X on all arrows in Δ_i. Finally, it defines a functor because, by the dual to Lemma 1.2.3, each sequence of $d^{k'}s$ admits a unique dual decomposition. It is easily verified that $X^{(P^X)} = X$.

The last thing to show is that the described functor is bijective on morphisms. This is once more a direct consequence of Lemma 1.2.3. □

In this thesis we shall take both points of view on the category **ssSet**, depending on which is more convenient in the given context. As seen in Example 1.1.7, every standard abstract n-simplex lives in the category of semisimplicial sets as the ordered simplicial complex $C[n]$. We shall denote it by $\Delta_i[n]$. Another way to describe $\Delta_i[n]$ is as the representable functor $\Delta_i[n] = \mathrm{Hom}_{\Delta_i}(-, [n])$. In fact, the representable presheaves of **ssSet** are precisely the standard simplices. Because of this, we will write $\Delta_i : \Delta_i \longrightarrow \mathbf{ssSet}$ for the Yoneda embedding into the category of semisimplicial sets.

By the Yoneda lemma, the set X_n of n-simplices of a semisimplicial set X corresponds to the set of morphisms $x : \Delta_i[n] \longrightarrow X$. If, for a map $\alpha : [m] \longrightarrow [n]$ in Δ_i such that $X(\alpha)(x) = y$, we view the simplex $y : \Delta_i[m] \longrightarrow X$ as a morphism, then, by naturality, we have $y = x\Delta_i(\alpha)$. We frequently omit the Δ_i and write $y = x\alpha$. For $\epsilon^i : [n] \longrightarrow [n+1]$ the function that skips the i-th element, we will often write ϵ^i where we actually mean its image under Δ_i, *i.e.* the morphism $\Delta_i\epsilon^i : \Delta_i[n] \longrightarrow \Delta_i[n+1]$.

The following fact is easy to take for granted, but does not hold in every functor category. It does hold in every presheaf category.

Lemma 1.2.5. *A semisimplicial morphism is monic if and only if it is monic in every dimension.*

Proof. The implication from right-to-left is immediate. The converse is easy, using the notation we have just fixed. Let $f : X \longrightarrow Y$ be a monic semisimplicial morphisms and $x, x' \in X_n$ such that $f_n(x) = f_n(x')$. Then, by the Yoneda lemma, $fx = fx'$ and thus $x = x'$. □

We end this section with two standard results that hold for any presheaf category. While these results are very elementary category theory, we believe that their important role in the theory of semisimplicial sets makes it beneficial to write out the proofs for this specific case.

We begin by defining what is generally called the *category of elements*.

Definition 1.2.6. For X a semisimplicial set, its *category of simplices* (denoted $\Delta_i X$) is the comma category $(\Delta_i \downarrow X)$. ■

Example 1.2.7. The category of simplices $\Delta_i A$ of the semisimplicial set described above has two object, I and a, and may be pictured as follows:

$$\begin{tikzcd} a \arrow[loop left, "id_{[0]}"] \arrow[r, shift left, "\epsilon^0"] \arrow[r, shift right, "\epsilon^1"'] & I \arrow[loop right, "id_{[1]}"] \end{tikzcd}$$ ▲

Proposition 1.2.8. *For any semisimplicial set X,*

$$\varinjlim_{\substack{x : \Delta_i[n] \to X \\ \text{in } \Delta_i X}} \Delta_i[n] \cong X.$$

Proof. It immediately follows from the Yoneda lemma that X forms a cocone to the colimit. Now suppose that we are given another cocone

$$(Y, \{\Delta_i[n] \xrightarrow{\varphi_x} Y\}_{\substack{x : \Delta_i[n] \to X \\ \text{in } \Delta_i X}}).$$

Then a mediating morphism $f : X \longrightarrow Y$ must be such that for each $x : \Delta_i[n] \longrightarrow X$, it holds that $fx = \varphi_x$, which, by naturality, is equivalent to $f_n(x) = \varphi_x$. It suffices to show that this f commutes with the face maps, which is demonstrated by the following chain of equalities:

$$d^k f_{n+1}(x) = d^k \varphi_x = \varphi_x \epsilon^k = \varphi_{x\epsilon^k} = \varphi_{d^k x} = f_n(d^k x),$$

in which the third equality holds due to the fact that Y forms a cocone and the others by the Yoneda lemma. □

Proposition 1.2.9. *Let $\mathscr{E}$ be a cocomplete category and $f : \Delta_i \longrightarrow \mathscr{E}$ a functor. Then there is a unique left adjoint $f_! : \mathbf{ssSet} \longrightarrow \mathscr{E}$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathbf{ssSet} \arrow[r, "f_!"] & \mathscr{E} \\ \Delta_i \arrow[u, "\Delta_i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proof. Since $f_!$ must make the above diagram commute and be cocontinuous, we must set:

$$f_!(X) := \varinjlim_{\substack{x : \Delta_i[n] \to X \\ \text{in } \Delta_i X}} f([n]),$$

where an arrow $X \longrightarrow Y$ is sent to the unique arrow out of the limit. We denote the aimed-for right adjoint by f^*. Because we want $\mathrm{Hom}_{\mathbf{ssSet}}(\Delta_i[n], f^*(E)) \cong \mathrm{Hom}_{\mathscr{E}}(f([n]), E)$, we must set:

$$f^*(E)_n := \mathrm{Hom}_{\mathscr{E}}(f([n]), E).$$

On arrows f^* is defined by postcomposition. It follows that we have the following chain of natural isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{E}}(f_!(X), E) &\cong \mathrm{Hom}_{\mathscr{E}}(\varinjlim_{\Delta_i[n] \to X} f[n], E) && \text{(Definition of } f_!\text{)} \\ &\cong \varprojlim_{\Delta_i[n] \to X} \mathrm{Hom}_{\mathscr{E}}(f[n], E) && \text{(Hom}(-,-) \text{ preserves limits)} \\ &\cong \varprojlim_{\Delta_i[n] \to X} f^*(E)_n && \text{(Definition of } f^*\text{)} \\ &\cong \varprojlim_{\Delta_i[n] \to X} \mathrm{Hom}_{\mathbf{ssSet}}(\Delta_i[n], f^*(E)) && \text{(Yoneda Lemma)} \\ &\cong \mathrm{Hom}_{\mathbf{ssSet}}(\varinjlim_{\Delta_i[n] \to X} \Delta_i[n], f^*(E)) && \text{(Hom}(-,-) \text{ preserves limits)} \\ &\cong \mathrm{Hom}_{\mathbf{ssSet}}(X, f^*(E)), && \text{(Proposition 1.2.8)} \end{aligned}$$

as desired. □

The morphism $f_!$ is called the *left Kan extension* of f (along Δ_i).

An important consequence of the previous proposition is that if one wants to define a functor **ssSet** $\longrightarrow \mathscr{E}$ from semisimplicial sets to some cocomplete category, it suffices to define it on just the standard simplices. The following section gives an example of such a construction.

1.2.2 Geometric realisation

Just like the abstract simplicial complexes, a semisimplicial set can be seen as a recipe to build a corresponding topological space by gluing together its constituent simplices. To make that precise, we use the functor r from Δ_i to **Top** that sends the finite ordinal $[n]$ to the standard geometric n-simplex. On arrows, r is defined by

$$r(\alpha : [n] \longrightarrow [m])(t_0, \dots, t_n) = (s_0, \dots, s_m),$$

where $s_i = \sum_{j \in \alpha^{-1}(i)} t_j$.

Proposition 1.2.10. *This describes a covariant functor $r : \Delta_i \longrightarrow$ **Top**.*

Proof. First note that for $\alpha : [n] \longrightarrow [m]$, the function $r(\alpha)$ indeed maps into Δ^m. After all, for $\vec{t} \in \Delta^n$ and $\vec{s} = r(\alpha)(\vec{t})$, we have

$$\sum_{i=0}^{m} s_i = \sum_{i=0}^{m} \sum_{j \in \alpha^{-1}(i)} t_j = \sum_{j=0}^{m} t_j = 1,$$

which means that $\vec{s} \in \Delta^m$. To show continuity, let $\epsilon > 0$ and $\vec{x} \in \Delta^n$. We abbreviate

$$\mathrm{t} = (t_0 - x_0)^2 + \dots + (t_m - x_m)^2,$$

and

$$\mathrm{s} = \left(\sum_{j \in \alpha^{-1}(0)} t_j - \sum_{j \in \alpha^{-1}(0)} x_j \right)^2 + \dots + \left(\sum_{j \in \alpha^{-1}(m)} t_j - \sum_{j \in \alpha^{-1}(m)} x_j \right)^2 .$$

Then for every $i \in [0, n]$, we have $\mathrm{t} \geq (t_i - x_i)^2$ and thus $\sqrt{\mathrm{t}} \geq t_i - x_i$. But then

$$\mathrm{s} < (m+1)((n+1)\sqrt{\mathrm{t}})^2 = (m+1)(n+1)^2\mathrm{t}.$$

So let $\delta = \frac{\epsilon}{(m+1)(n+1)}$ and suppose $\sqrt{\mathrm{t}} = d(\vec{t}, \vec{x}) < \delta$. We find

$$\begin{aligned} d(r(\alpha)(\vec{t}), r(\alpha)(\vec{x})) &= \sqrt{\mathrm{s}} \\ &< \sqrt{(m+1)(n+1)^2\mathrm{t}} \\ &< (m+1)(n+1)\sqrt{\mathrm{t}} < (m+1)(n+1)\delta = \epsilon. \end{aligned}$$

Thus indeed, $r(\alpha)$ is a morphism in **Top**. Moreover, r clearly preserves identity morphisms. To see that it also preserves composition, let $\alpha : [n] \longrightarrow [m]$ and $\beta : [m] \longrightarrow [k]$ be morphisms in Δ_i. We write

$$\begin{aligned} (s_0, \dots, s_m) &= r(\alpha)(t_0, \dots, t_n) \text{ and} \\ (u_0, \dots, u_k) &= r(\beta)(s_0, \dots, s_m). \end{aligned}$$

Then for every $l \in [0, k]$, we have

$$u_l = \sum_{j \in \beta^{-1}(l)} s_j = \sum_{j \in \beta^{-1}(l)} \sum_{i \in \alpha^{-1}(j)} t_i = \sum_{i \in (\beta \circ \alpha)^{-1}(l)} t_i,$$

as required. □

As a corollary, we have by Proposition 1.2.9,

Corollary 1.2.11. *There are unique adjoint functors* $|\cdot| \dashv$ **Top** *such that the diagram*

$$\begin{tikzcd} \mathbf{ssSet} \arrow[r, shift left, "|\cdot|"] & \mathbf{Top} \arrow[l, shift left, "S"] \\ \Delta_i \arrow[u, "\Delta_i"] \arrow[ur, "r"'] & \end{tikzcd}$$

commutes.

For X a semisimplicial set, the topological space $|X|$ is called the *geometric realisation* of X.

1.2.3 Geometric product and exponential

Being a presheaf category, **ssSet** of course has products. However, those products do not behave as we would like. For instance, for any semisimplicial set Y, the semisimplicial set $\Delta_i[0] \times Y$ has no higher-dimensional simplices than points, while geometric intuition tells us that for any space its product with a point should be the space itself. For this reason we introduce the notion of the *geometric product* of two semisimplicial sets. It already appears in [RS71, Section 3] and a very concrete formulation can be found in [Sch13].

We define the geometric product $- \otimes - :$ **ssSet** $\times$ **ssSet** in a more abstract way, again making use of Proposition 1.2.9. First, let $N_i : \mathrm{Pos}_i \longrightarrow$ **ssSet** be the functor from the category of posets and injective monotone maps between them, given by

$$(N_i(P))_n \cong \mathrm{Hom}_{\mathrm{Pos}_i}([n], P).$$

Note that this is precisely the nerve functor of Example 1.1.7.

For each $[n]$ in $\Delta_i[n]$, we take the left Kan extension

$$\begin{tikzcd} \mathbf{ssSet} \arrow[r, "\Delta_i[n] \otimes -"] & \mathbf{ssSet} \\ \Delta_i \arrow[u, "\Delta_i"] \arrow[ur, "N_i([n] \times -)"'] & \end{tikzcd}$$

In the above diagram the symbol $\times$ denotes the categorical product in Pos. We get a functor $\Delta_i[n] \otimes - :$ **ssSet** $\longrightarrow$ **ssSet** for every $[n]$ in $\Delta_i[n]$. Next, we take another left Kan extension

$$\begin{tikzcd} \mathbf{ssSet} \arrow[r, "- \otimes X"] & \mathbf{ssSet} \\ \Delta_i \arrow[u, "\Delta_i"] \arrow[ur, "\Delta_i \otimes X"'] & \end{tikzcd}$$

which gives us for each semisimplicial set X a functor $- \otimes X : \mathbf{ssSet} \longrightarrow \mathbf{ssSet}$. To make $\otimes$ functorial in the second argument, we put $Y \otimes f := \varinjlim_{\Delta_i[n] \to Y} (\Delta_i[n] \otimes f)$.

The geometric product inherits its symmetry from the categorical product of posets. Indeed, there is a chain of isomorphisms

$$\begin{aligned} X \otimes Y &\cong \varinjlim_{\Delta_i[n]\to X} (\Delta_i[n] \otimes Y) \cong \varinjlim_{\Delta_i[n]\to X} \varinjlim_{\Delta_i[m]\to Y} N_i([n]\times[m]) \\ &\cong \varinjlim_{\Delta_i[m]\to Y} \varinjlim_{\Delta_i[n]\to X} N_i([m]\times[n]) \cong \varinjlim_{\Delta_i[m]\to Y} \Delta_i[n] \otimes X \cong Y \otimes X, \end{aligned}$$

where the second to last isomorphism exists due to the fact that colimits commute with colimits (see *e.g.* [Rie17, Theorem 3.8.1]) and that the categorical product is symmetric in Pos.

Moreover, we have $\Delta_i[0] \otimes X \cong \varinjlim_{\Delta_i[n]\to X} N_i([0]\times[n]) \cong X$, which is conform the geometric intuition as formulated in the introduction of this section. By construction, the functor $- \otimes X$ has a right adjoint $[X, -] : \mathbf{ssSet} \longrightarrow \mathbf{ssSet}$, which we will call the *geometric exponential*. We have established (modulo some additional verification left to the reader):

Theorem 1.2.12. *The triple* $(\mathbf{ssSet}, \otimes, \Delta_i[0])$ *is a symmetric closed monoidal category.*

1.3 Simplicial sets

In this section we introduce simplicial sets and investigate their relation to semisimplicial sets. Along the way we will see how the former can also be seen as a generalisation of the latter.

Let Δ be the category of finite ordinals and *all* monotone functions between them.

Definition 1.3.1. The presheaf category $[\Delta^{op}, \mathrm{Set}]$ is called the category of *simplicial sets* and is denoted **sSet**.

We adopt notational conventions analogous to the ones we have for semisimplicial sets. For instance, we let $\Delta : \Delta \longrightarrow \mathbf{sSet}$ be the Yoneda functor and we write $x\alpha$ for the n-simplex

$$\Delta[n] \xrightarrow{\Delta\alpha} \Delta[m] \xrightarrow{x} X$$

of a simplicial set X.

Lemma 1.3.2. *Every morphism α of Δ can be uniquely factored $\alpha = \epsilon\eta$ as surjection followed by an injection.*

Proof. Let $q = |\mathrm{im}\ \alpha|$ be the cardinality of the image of the function α. Then there are obvious surjective η and injective ϵ forming a commutative diagram

$$\begin{tikzcd} {[m]} \arrow[r, two heads, "\eta"'] \arrow[rr, bend left, "\alpha"] & {[q]} \arrow[r, hook, "\epsilon"'] & {[n]}. \end{tikzcd}$$

Namely, η is the 'flattening' or 'surjectification' of α in the sense that it increments in the same way as α, yet leaves no gaps, so that it becomes a function onto $[q]$. The injective function ϵ reverses this flattening.

For uniqueness, suppose we have another factorisation $\alpha = \epsilon' \eta'$, then for $\epsilon' : [q'] \longrightarrow [n]$, we have

$$|\text{im } \epsilon'| = |\text{im } \epsilon' \eta'| = |\text{im } \alpha|,$$

where the first equality holds due to the fact that η' is surjective. It follows from the injectivity and order-preservingness of ϵ' that $[q'] = [q]$ and $\epsilon' = \epsilon$. Finally, because ϵ is mono, we have that $\eta' = \eta$. □

Definition 1.3.3. An n-simplex x of a simplicial set X is called *degenerate* whenever there is a surjection $\eta : [n] \longrightarrow [m]$ with $m < n$ and an m-simplex y of X such that $x = y\eta$. ■

Remark 1.3.4. Like semisimplicial sets, a simplicial set X can also be described more combinatorially as a graded set equipped with face maps and additional *degeneracy maps*, the latter corresponding to $X(\eta^k) : X_n \longrightarrow X_{n+1}$ where $\eta^k : [n+1] \longrightarrow [n]$ is the unique surjective order-preserving morphism covering k twice. The face and degeneracy maps are required to satisfy certain relations extending (1.2). ▲

Lemma 1.3.5 (Eilenberg-Zilber). *Let X be a simplicial set. Then every n-simplex x of X can be uniquely written as $x = y\eta$, with η a surjection and y non-degenerate.*

Proof. We start with existence. If x is non-degenerate, then we can take $\eta = id_{[n]}$. Otherwise, we have $x = x_0\eta_0$ for some lower-dimensional simplex x_0 and surjection η_0. We ask the same for x_0: if it is non-degenarate, we are done, and otherwise we have $x = x_0\eta_0 = x_1\eta_1\eta_0$. We continue this process until we reach a non-degenerate simplex, which we at some point will, since all 0-simplices are non-degenerate.

For uniqueness, suppose that $y\eta = x = y'\eta'$. Let ϵ and ϵ' be sections of η and η', respectively. We have

$$y = y\eta\epsilon = x\epsilon = y'\eta'\epsilon \qquad \text{and, symmetrically} \qquad y' = y'\eta'\epsilon' = x\epsilon' = y\eta\epsilon'.$$

By Lemma 1.3.2, there are

$$y = y'\eta'\epsilon = y'\sigma\tau \qquad \text{and} \qquad y' = y\eta\epsilon' = y\sigma'\tau',$$

with σ, σ' injective and τ, τ' surjective. By the non-degeneracy of y and y', we have that $\tau = \tau' = id$. Thus, we also find that $y = y'\sigma = y\sigma'\sigma$ and so $\sigma = \sigma' = id$. It follows that ϵ' is a section of η and ϵ is a section of η' and, for those were chosen arbitrarily, that η and η' have precisely the same sections, which means that $\eta = \eta'$. □

As a result a simplicial morphism $f : X \longrightarrow Y$ is determined by what it does on the non-degenerate simplices. Indeed, if $x = y\eta$ with y non-degenerate, then it follows from naturality that $f(x) = f(y\eta) = f(y)\eta$.

Lemma 1.3.6. *Let $f : X \longrightarrow Y$ be a simplicial morphism. Then:*

- *f takes degenerates to degenerates.*
- *If f is a monomorphism, then f takes non-degenerates to non-degenerates.*

Proof. The first part is simply by the fact that f is a natural transformation:

$$f(x) = f(x_0\eta) = f(x_0)\eta.$$

For the second part, suppose that f is mono and x a non-degenerate simplex of X. Let η be a surjective morphism such that $f(x) = y_0\eta$ for some simplex y_0 of Y and pick any right inverse ϵ of η. Then it holds that:

$$f(x\epsilon\eta) = f(x)\epsilon\eta = y_0\eta\epsilon\eta = y_0\eta.$$

It follows that $x\epsilon\eta = x$ and thus, by the non-degeneracy of x, that $\epsilon = \eta = id$. □

Definition 1.3.7. For X a simplicial set, its *category of simplices* (denoted ΔX) is the comma category $(\Delta \downarrow X)$. ■

Entirely analogous to the semisimplicial case, we have the following two propositions.

Proposition 1.3.8. *For any simplicial set X,*

$$\varinjlim_{\substack{x:\Delta[n]\longrightarrow X \\ in\ \Delta X}} \Delta[n] \cong X.$$

Proposition 1.3.9. *Let $\mathscr{E}$ be a cocomplete category and $f : \Delta \longrightarrow \mathscr{E}$ a functor. Then there is a unique left adjoint $f_! : \mathbf{sSet} \longrightarrow \mathscr{E}$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, "f_!"] & \mathscr{E} \\ \Delta \arrow[u, "\Delta"] \arrow[ur, "f"'] & \end{tikzcd}$$

Let $i : \Delta_i \hookrightarrow \Delta$ be the inclusion functor. We call the induced precomposition functor $i^* : \mathbf{sSet} \longrightarrow \mathbf{ssSet}$ the *forgetful functor*. It sends a simplicial set X to the semisimplicial set with the same underlying set and face maps, 'forgetting' how it acts on the non-injective maps of Δ. The following is again an instance of a basic result of category theory.

Proposition 1.3.10. *The functor $i^* : \mathbf{sSet} \longrightarrow \mathbf{ssSet}$ has both adjoints*

$$i_! \dashv i^* \dashv i_*.$$

Proof. First take the left Kan extension

$$\begin{tikzcd} \mathbf{ssSet} \arrow[r, "i_!"] & \mathbf{sSet} \\ \Delta_i \arrow[u, "\Delta_i"] \arrow[ur, "\Delta \circ i"'] & \end{tikzcd}$$

to obtain a left adjoint $i_! : \mathbf{ssSet} \longrightarrow \mathbf{sSet}$. We claim that its right adjoint is i^*. Indeed, for X a semisimplicial set and Y a simplicial set, we have $Y_n = Y(i[n]) = (i^*Y)_n$ which gives the following chain of natural isomorphisms:

$$\varprojlim_{\Delta_i[n]\to X} \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n], Y) \cong \varprojlim_{\Delta_i[n]\to X} Y(i[n]) \cong \varprojlim_{\Delta_i[n]\to X} \mathrm{Hom}_{\mathbf{ssSet}}(\Delta_i[n], i^*Y).$$

It follows that $\mathrm{Hom}_{\mathbf{sSet}}(i_!X, Y) \cong \mathrm{Hom}_{\mathbf{ssSet}}(X, i^*Y)$. Next we take a left Kan extension in $\mathbf{sSet}$:

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, "(i^* \circ \Delta)_!"] & \mathbf{ssSet} \\ \Delta \arrow[u, "\Delta"] \arrow[ur, "i^* \circ \Delta"'] & \end{tikzcd}$$

We claim that $(i^* \circ \Delta)_! = i^*$. Indeed, for any simplicial set X, we have

$$(i^* \circ \Delta)_!(X)_m = \varinjlim_{\Delta X}(i^*\Delta[n])_m \cong \varinjlim_{\Delta X}((i^*\Delta[n])_m) \cong \varinjlim_{\Delta X}(\Delta[n]_m) \cong X_m.$$

Thus i^* has a right adjoint i_*. □

Example 1.3.11.

- For every representable $\Delta_i[n]$, we have $i_!\Delta_i[n] = \Delta[n]$.
- The right adjoint $i_* : \mathbf{ssSet} \longrightarrow \mathbf{sSet}$ sends any finite semisimplicial set to the empty set (the initial object of $\mathbf{ssSet}$). ▲

Example 1.3.12 (The simplicial set $i_!X$). It will be useful to have a more concrete description of simplicial set $i_!X$, for some semisimplicial set X. Roughly, $i_!X$ is the simplicial set X^* generated by 'freely adding degeneracies' to X. This can be made precise as follows:

$$X^*_n = \{(x, \eta) \mid x \in X_k,\ \eta : [n] \longrightarrow [k] \text{ surjective}\},$$

with $(x, \eta)\alpha = (x\epsilon_0, \eta_0)$, where $\eta\alpha = \epsilon_0\eta_0$ is the unique factorisation from Lemma 1.3.2. Note that $X_0 = X^*_0$ and $X_n \subseteq X^*_n$ for every $n > 0$.

To see that indeed

$$X^* \cong \varinjlim_{\substack{x:\Delta_i[n]\to X \\ \text{in } \Delta_i X}} \Delta[n],$$

define the components $\varphi_x : \Delta[n] \longrightarrow X$, to be the morphisms determined by mapping $id_{[n]}$ to $(x, id_{[n]})$. This is a cocone to the limits diagram, because for $\epsilon : y \longrightarrow x$ in $\Delta_i X$, we have

$$\varphi_x\epsilon(id_{[m]}) = \varphi_x(\epsilon) = (x, id_{[n]})\epsilon = (x\epsilon, id_{[m]}) = (y, id_{[m]}) = \varphi_y(id_{[m]}),$$

as required. Now let $(C, \Delta[n] \xrightarrow{\psi_x} X)$ be another cocone to the same diagram. Then there is a unique cocone morphism $X^* \longrightarrow C$, sending each $(x, id_{[n]})$ to $\psi_x(id_{[n]})$.

Observe that the non-degenerate simplices of $i_!X$ are precisely the (x, id), or, equivalently, the simplices of the form $i_!x : \Delta[n] \longrightarrow i_!X$, for some $x : \Delta_i[n] \longrightarrow X$. The morphism $i_!f : i_!X \longrightarrow i_!Y$ is determined by sending (x, id) to (fx, id). ▲

Lemma 1.3.13. *The functor $i_!$ reflects isomorphisms.*

Proof. Suppose that $i_! f : i_! X \longrightarrow i_! Y$ has a two-sided inverse g in **sSet**. Then g is a monomorphism and thus, by Lemma 1.3.6, it sends non-degenerates to non-degenerates. Therefore, for each simplex y of the semisimplicial set Y, we have $g(y, id) = (x, id)$ for some x in X. Letting $g'(y) = x$ defines a semisimplicial morphism $g' : Y \longrightarrow X$ such that $i_! g' = g$. It follows that g' is the two-sided inverse of f. □

It is also straightforward to show:

Lemma 1.3.14. *The functor $i_!$ preserves and reflects monomorphisms.*

Both the functors $i_!$ and i^* are faithful and injective on objects, and can thus be see as embedding semisimplicial sets in simplicial sets and vice-versa. A disadvantage of the embedding i^* is that it does not respect geometric realisation. For instance, the geometric realisation $|i^*\Delta[0]|$ of the simplicial point gives the strange object that is an infinite-dimensional simplex of which each side is glued to the others.

In contrast, the functor $i_!$ does preserve geometric realisation. Indeed, the function r from Section 1.2.2 can easily be extended to $r_\Delta : \Delta \longrightarrow$ **Top**. Consequently, the Corollary 1.2.11 can be replicated to obtain a commutative diagram

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, shift left, "|\cdot|_\Delta"] & \mathbf{Top} \arrow[l, shift left, "S_\Delta"] \\ \Delta \arrow[u, "\Delta"] \arrow[ur, "r_\Delta"'] & \end{tikzcd}$$

Since both $|\cdot|_\Delta$ and $i_!$ are left adjoints, their composition $|\cdot|_\Delta \circ i_!$ is as well. By the uniqueness of $|\cdot|$, it follows that $|\cdot| \cong |\cdot|_\Delta \circ i_!$.

One way to further describe how the category of semisimplicial sets fits inside category of simplicial sets is by considering the comonad (G, ϵ, δ) induced by the adjunction $i_! \dashv i^*$. That is, the endofunctor G is given by $i_! i^*$, the counit ϵ is the counit of the adjunction, and the comultiplication δ is $i_! \eta_{i^*} : G \longrightarrow G^2$, where η is the unit of the adjunction.

By standard abstract arguments, there is a functor F : **ssSet** $\longrightarrow \mathrm{CoAlg}^G$ into the category of coalgebras over G, defined by mapping a semisimplicial set X to $(i_! X, i_! \eta_X)$. A well-known theorem (or its dual) by Beck implies that if the source category of a left adjoint has equalisers, the left adjoint preserves them and moreover reflects isomorphisms, then such a functor F is an equivalence of categories (and the adjunction is called *comonadic*). In light of Lemma 1.3.13, it remains in our case only to show that $i_!$ preserves equalisers.

Proposition 1.3.15. *The adjunction $i_! \dashv i^*$ is comonadic.*

Proof. Let

$$\begin{tikzcd} E \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

be an equaliser in **ssSet**. For each n, we have $E_n = \{x \in X_n \mid f_n(x) = g_n(x)\}$. It follows that:

$$\begin{aligned}(i_! E)_n &= \{(x,\eta) \mid x \in E_k,\ \eta : [n] \longrightarrow [k] \text{ surjective}\}\\ &= \{(x,\eta) \mid x \in X_k,\ f_k(x) = g_k(x),\ \eta : [n] \longrightarrow [k] \text{ surjective}\}\\ &= \{(x,\eta) \in i_! X_n \mid (i!\, f)_n(x,\eta) = (i_! g)_n(x,\eta)\},\end{aligned}$$

and thus $i_! E$ is the equaliser of $(i_! f, i_! g)$. Since, by Lemma 1.3.13, $i_!$ reflects isomorphisms, the result follows from Beck's comonadicity theorem. □

Now that we have established that **ssSet** $\simeq \mathrm{CoAlg}^G$, it is worthwhile to take a closer look at the coalgebras over G. First note that the unit and counit of $i_! \dashv i^*$ are given by

$$\begin{aligned}\eta &: 1_{\mathbf{ssSet}} \Longrightarrow i^* i_! & \qquad \epsilon &: i_! i^* \Longrightarrow 1_{\mathbf{sSet}}\\ \eta_X &: X \longrightarrow i^* i_! X & \epsilon_Y &: i_! i^* Y \longrightarrow Y\\ &: x \longmapsto (x, id) & &: (y,\eta) \longmapsto y\eta\end{aligned}$$

A G-coalgebra (X,ξ) consists of a simplicial set X together with a morphism $\xi : X \longrightarrow i_! i^* X$ such that

$$\begin{tikzcd} X \arrow[r, "\xi"] \arrow[dr, equal] & i_! i^* X \arrow[d, "\epsilon_X"] \\ & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "\xi"] \arrow[d, "\xi"'] & i_! i^* X \arrow[d, "i_! i^* \xi"] \\ i_! i^* X \arrow[r, "i_! \eta_{i^* X}"'] & i_! i^* i_! i^* X \end{tikzcd} \tag{1.4}$$

commute.

It turns out that if a simplicial set X admits a G-coalgebra structure, then it does so in a unique way.

Proposition 1.3.16. *A simplicial set X admits a G-coalgebra structure if and only if there is a simplicial morphism $X \longrightarrow i_! i^* X$ mapping each non-degenerate $x \longmapsto (x, id)$.*

Proof. $\Longrightarrow$. Let (X,ξ) be a G-coalgebra. For x non-degenerate, the only preimage of x under ϵ_X is (x, id). By the commutativity of the first diagram of (1.3.23), we must have $\xi : x \longmapsto (x, id)$.

$\Longleftarrow$. Suppose there is such simplicial morphism ξ. We claim that (X,ξ) is a G-coalgebra. Clearly the first diagram of (1.3.23) commutes. For non-degenerate x, we have,

$$\begin{aligned}(i_! \eta_{i^* X} \circ \xi)(x) &= i_! \eta_{i^* X}(x, id) & \qquad (i_! i^* \xi \circ \xi)(x) &= i_! i^* \xi(x, id)\\ &= (\eta_{i^* X}(x), id) & &= (i^* \xi(x), id)\\ &= ((x, id), id) & &= ((x, id), id),\end{aligned}$$

which means that also the second diagram commutes. □

We now have two characterisations of simplicial sets that are also semisimplicial sets. Namely, as those simplicial sets that lie in the image of $i_!$ and as those for which there is a canonical morphism $X \longrightarrow i_! i^* X$ (*i.e.* those which admit a G-coalgebra structure). We consider one more characterisation due to [RS71].

Definition 1.3.17 ([RS71, p. 323]). A simplicial set X is said to have a *non-degenerate core* if every face of a non-degenerate simplex of X is again non-degenerate. ■

Proposition 1.3.18. *An object lies in the image of $i_!$ precisely when it has a non-degenerate core.*

Proof. If X does *not* have a non-degenerate core, then there are non-degenerate simplices x, y and (respectively) injective and surjective functions $\epsilon \neq id$ and $\eta \neq id$ such that $x\eta = y\epsilon$. But if (X, ξ) were a G-coalgebra, then, by Proposition 1.3.16, we would have

$$\xi(x\eta) = \xi(x)\eta = (x, id)\eta = (x, \eta);$$
$$\xi(y\epsilon) = \xi(y)\epsilon = (y, id)\epsilon = (y\epsilon, id),$$

a contradiction.

For the converse, note that if X has a non-degenerate core, then the assignment of a non-degenerate x to (x, id) is not conflicting and determines a simplicial morphism. □

Proposition 1.3.19. *A simplicial morphism $i_!X \longrightarrow i_!Y$ is in the image of $i_!$ if and only if it maps non-degenerates to non-degenerates.*

Proof. According to the coalgebraic perspective $f : i_!X \longrightarrow i_!Y$ is a semisimplicial morphism precisely when the diagram

$$\begin{tikzcd} i_!X \arrow[r, "i_!\eta_X"] \arrow[d, "f"'] & i_!i^*i_!X \arrow[d, "i_!i^*f"] \\ i_!Y \arrow[r, "i_!\eta_Y"'] & i_!i^*i_!Y \end{tikzcd}$$

commutes. A simple calculation yields that this is the case if and only if each non-degenerate (x, id) in $i_!X$ gets sent to (y, η) such that

$$((y, \eta), id) = i_!i^*f \circ i_!\eta_X(x, id) = (i_!\eta_Y \circ f)(x, id) = ((y, id), \eta),$$

i.e. such that $\eta = id$, as required. □

We again consider a nerve functor, this time from Cat $\longrightarrow$ **sSet**. Unlike Pos_i, which is the codomain of the nerve functor N_i, the category Cat is cocomplete. As a consequence it arises as a right adjoint. Indeed, the inclusion functor $\Delta \longrightarrow$ Cat from the posets of Δ considered as categories, into the category of small categories, yields a left Kan extension with right adjoint $N :$ Cat $\longrightarrow$ **sSet**, such that

$$(NC)_n \cong \mathrm{Hom}_{\mathrm{Cat}}([n], \mathrm{C}).$$

In particular, for a poset P considered a category, $(NP)_n$ consists of the order preserving functions from $[n]$ to P. Since any such function $x : [n] \longrightarrow P$ in *Pos* may be factored

$$[n] \xrightarrow{\eta} \mathrm{im}\, x \xrightarrow{\epsilon} P$$

as a surjection followed by an injection, the non-degenerate simplices of NP are precisely the injective functions. It follows that $i_!N_iP \cong NP$. With this, we can finally show:

Theorem 1.3.20. *For any two semisimplicial sets X and Y it holds that*

$$i_!(X \otimes Y) \cong i_! X \times i_! Y.$$

Proof. Since right adjoints preserve limits, we have $N([n]\times[m]) \cong \Delta[n]\times\Delta[m]$ for any pair $[n],[m]$ of objects of Δ. Thus, we have the following chain of isomorphisms:

$$\begin{aligned} i_!(X \otimes Y) &\cong i_!\big(\varinjlim_{\Delta_i[n]\to X} \varinjlim_{\Delta_i[m]\to Y} N_i([n]\times[m])\big) \\ &\cong \varinjlim_{\Delta_i[n]\to X} \varinjlim_{\Delta_i[m]\to Y} N([n]\times[m]) \\ &\cong \varinjlim_{\Delta_i[n]\to X} \varinjlim_{\Delta_i[m]\to Y} \Delta[n]\times\Delta[n] \cong X \times Y, \end{aligned}$$

as required. □

In particular, for any two semisimplicial sets X and Y, it holds that $i_! X \times i_! Y$ has a non-degenerate core. Thus, in light of Corollary 1.3.18, we could have also defined the geometric product as the preimage of $i_! X \times i_! Y$ under $i_!$, which is how it is done in [RS71]. Note that this also means that we have proven that the category CoAlg^G has products.

We obtain following corollary, and at the same time justification for the name of the geometric product.

Corollary 1.3.21. *For any two semisimplicial sets X and Y, it holds that:*[1]

$$|X \otimes Y| \cong |X|\times|Y|.$$

Proof. It is well-known that $|X\times Y|_\Delta = |X|_\Delta \times |Y|_\Delta$ (see *e.g.* Proposition 2.4 of[GJ99]). We find

$$\begin{aligned} |X \otimes Y| &\cong |i_!(X \otimes Y)|_\Delta \\ &\cong |i_! X \times i_! Y|_\Delta \\ &\cong |i_! X|_\Delta \times |i_! Y|_\Delta \cong |X|\times|Y|, \end{aligned}$$

as desired. □

The following is a standard result on the simplicial nerve functor that will be useful in Chapter 3.

Proposition 1.3.22. *If $\eta : F \Rightarrow G$ is a natural transformation between functors $F, G : \mathrm{C} \longrightarrow \mathrm{D}$, then*

$$|N(\eta)|_\Delta : |N(F)|_\Delta \longrightarrow |N(G)|_\Delta$$

is a homotopy $|N(F)|_\Delta \sim |N(G)|_\Delta$ in Top.

[1] This result actually only holds when geometric realisation is considered as a functor into a more convenient category of topological spaces, such as the category of compact Hausdorff spaces. These details falls beyond the scope of this thesis but can be found in for instance [GJ99, p. 9]

Proof. Let I be the category induced by the poset [1]. It is straightforward to check that the natural transformation η is equivalently a functor

$$\eta : \mathrm{C} \times \mathrm{I} \longrightarrow \mathrm{C}$$

such that the restriction to $C \times \{0\}$ is F and the restriction to $C \times \{1\}$ is G. But then, because N preserves limits, we have $N(\eta) : N(\mathrm{C}) \times \Delta_i[1] \longrightarrow N(\mathrm{D})$. This already what is called a homotopy in sSet (see Section 3.3.2), but under geometric realisation it becomes

$$|N(\eta)|_\Delta : |N(\mathrm{C})|_\Delta \times I \longrightarrow |N(\mathrm{D})|_\Delta,$$

which is a homotopy in Top. □

We end this section with a remark about the monadicity of $i_! \dashv i^*$.

Remark 1.3.23. It is easy to see that $i_! \dashv i^*$ is also monadic. By the dual of the theorem that is used in the proof of Proposition 1.3.15, it suffices to show that i^* reflects isomorphisms and preserves coequalisers. The latter follows directly from the fact that, by Proposition 1.3.10, the functor i^* is a left adjoint. To see that i^* reflects isomorphisms, note that, because any $i^* f : i^* X \longrightarrow i^* Y$ preserves the underlying sets, we only have to show that an inverse g is a simplicial morphism. We make use of the fact that every isomorphism in a functor category is a pointwise isomorphism, and thus both f and g are bijections on each level. For α a morphism of Δ and y some n-simplex of Y, let x be the unique n-simplex of X such that $f_n(x) = y$. Then $f_n(x\alpha) = f_n(x)\alpha = y\alpha$ and therefore,

$$g_n(y\alpha) = x\alpha = g_n(y)\alpha,$$

as required.

It follows by Beck's monadicity theorem that **sSet** is equivalent to the category of algebras over the monad induced by the adjunction $i_! \dashv i^*$. That is, a semisimplicial set X admits a system of degeneracies precisely when there is a morphism $\lambda : i^* i_! X \longrightarrow X$ such that

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & i^* i_! X \arrow[d, "\lambda"] \\ & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} i^* i_! i^* i_! X \arrow[r, "i^* \epsilon_{i_! X}"] \arrow[d, "i^* i_! \lambda"'] & i^* i_! X \arrow[d, "\lambda"] \\ i^* i_! \arrow[r, "\lambda"'] & X \end{tikzcd}$$

commute. ▲

Chapter 2

(Semi)model categories

This chapter introduces the abstract homotopy theory that will be used in this thesis. In particular, we define the notion of a *model structure*, introduced by Quillen in [Qui67]. A model structure imposed on a category gives one a general setting in which to study homotopy theory, with as archetypical example the model structure on the category of topological spaces.

The first two sections build up to this original definition, and further provide a more concise definition in terms of two interacting *weak factorisation systems*. This last definition is very common in the more recent literature.

In Section 2.3, we introduce the weaker notion of a *right semimodel structure*. This weakening first appeared in [Spi], but is therein noted to be inspired by a notion that already appeared in [Hov98]. We will show that, given slightly more data, a right semimodel structure also admits a description in terms of weak factorisation systems.

The rest of this chapter is devoted to developing some general methods for constructing model structures. We will focus on the class of *locally finitely presentable* categories, introduced in Section 2.4. These categories, among which are all presheaf categories, admit two important construction methods. First, there is a useful theorem for constructing weak factorisation systems for a given set generating set, which is the subject of Section 2.5. Second, we will state in Section 2.6 a theorem that for such categories allows weak factorisation systems to be transferred along a left adjoint. This inspired [Hes+17] to define the notion of a *left-induced model structure*, which we adapt to include right semimodel structures.

In the final section we introduce the categorical operations of the *Leibniz product* and *Leibniz exponential*. When dealing with a monoidal category, which, as we have seen in Chapter 1, the semisimplicial sets are, the Leibniz operations can serve as tools for constructing (semi)model structures.

Throughout the chapter we assume to be working with a complete, cocomplete and locally small category C, unless indicated otherwise.

2.1 Lifting problems

Central in (abstract) homotopy theory is the characterization of maps by their lifting properties.

Definition 2.1.1. Let f, g be two morphisms. We say that f *has the left lifting property with respect to* g or, equivalently, that g *has the right lifting property with respect to* f if for every commutative square of the form

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \quad \text{there is a dotted arrow such that} \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd} \quad \text{commutes.}$$

We call that arrow the square's *diagonal filler.* ■

To ease the writing (and reading) about these lifting properties, we will use the following notation. If f has the left lifting property with respect to g, we write $f \boxslash g$. Similarly, given two classes of morphisms F and G, we will take $\mathsf{F} \boxslash$ G to mean that every map in F has the left lifting property with respect to any map in G. Finally, with ${}^{\boxslash}\mathsf{F}$ ($\mathsf{F}^{\boxslash}$) we denote the class of all morphisms f such that $\{f\} \boxslash \mathsf{F}$ ($\mathsf{F} \boxslash \{f\}$).

Definition 2.1.2. For α an ordinal, an *α-sequence* in C is a functor $A : \alpha \longrightarrow \mathrm{C}$ from α (considered as a category) into C, *i.e.* a sequence

$$A_0 \longrightarrow A_1 \longrightarrow \dots \longrightarrow A_\beta \longrightarrow \dots \qquad (\beta < \alpha)$$

such that moreover, for every limit ordinal $\lambda < \alpha$, the canonical morphism,

$$\varinjlim_{\beta<\gamma} A_\beta \longrightarrow A_\gamma$$

is an isomorphism. ■

Definition 2.1.3. The *composition* of an α-sequence $A : \alpha \longrightarrow \mathrm{C}$ is the canonical morphism

$$A_0 \longrightarrow A_\alpha := \varinjlim_{\beta<\alpha} A_\beta. \qquad ■$$

A morphism is said to be an *α-composition* if it can be written as the composition of an α-sequence. If it is an α-composition for some (possibly finite) ordinal α, then it is more generally called a *transfinite composition.*

Definition 2.1.4. Let M be a class of morphisms of C. An α-sequence $A : \alpha \longrightarrow \mathrm{C}$ such that for every successor ordinal $\beta + 1 < \alpha$, the morphism $A_\beta \longrightarrow A_{\beta+1}$ is in M is called an *α-sequence of morphisms in M.* Its composition is an *α-composition of morphisms in M* and morphisms that arise in this way are called *transfinite compositions of morphisms in M.* ■

Definition 2.1.5. A class A of morphisms is said to be *saturated* if it contains all isomorphisms and is closed under pushouts, arbitrary coproducts, transfinite composition and retracts. ■

Proposition 2.1.6. *Any class* L *that can be written as* $\mathsf{L} = {}^{\boxslash}\mathsf{R}$ *for some class* R*, is saturated.*

Proof. Because these arguments are quite routine, we will prove only closure under pushouts. Let $\mathsf{L} = {}^{\boxslash}\mathsf{R}$ and suppose that we have the following lifting problem between the pushout of a morphism in L, and any morphism of R.

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\in \mathsf{L}"'] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d, "\in \mathsf{R}"] \\
\cdot \arrow[r] \arrow[urr, dashed] \arrow[ur, phantom, "\ulcorner", very near end] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

The indicated diagonal filler is part of a cone for the pushout diagram, and the unique corresponding arrow from the initial cone is the desired diagonal filler. It follows that the pushout is in $\mathsf{R}^{\boxslash} = \mathsf{L}$, as required. □

Remark 2.1.7. There is, of course, a dual version of the previous proposition for classes that can be written $\mathsf{R} = \mathsf{L}^{\boxslash}$. In particular, it holds that the right class of R is closed under both pullbacks and retracts. ▲

Lemma 2.1.8. *Let $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ be an adjunction and let f, g be morphisms of C and D, respectively. Then*

$$Ff \boxslash g \iff f \boxslash Gg.$$

Proof. By naturality, the adjunction $\phi : \mathrm{Hom}_{\mathsf{D}}(FA, B) \cong \mathrm{Hom}_{\mathsf{C}}(A, GB)$ gives rise to an adjunction $\mathrm{Hom}_{\mathbb{D}^{\to}}(Ff, g) \cong \mathrm{Hom}_{\mathbb{C}^{\to}}(f, Gg)$ on the corresponding arrow categories. Thus, any lifting problem of the form

$$\begin{tikzcd}
\cdot \arrow[r, "a"] \arrow[d, "Ff"'] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "b"'] \arrow[ur, dashed, "u"] & \cdot
\end{tikzcd} \quad \text{corresponds uniquely to one of the form} \quad \begin{tikzcd}
\cdot \arrow[r, "\phi(a)"] \arrow[d, "f"'] & \cdot \arrow[d, "Gg"] \\
\cdot \arrow[r, "\phi(b)"'] \arrow[ur, dashed, "\phi(u)"] & \cdot
\end{tikzcd}$$

as required. □

2.2 Model structures

We are now ready give the original definition of a model structure.

Definition 2.2.1 (Model structure). A *model structure* on a category C consists of a triple $(\mathsf{Fib}, \mathsf{Cof}, \mathsf{Weq})$ of classes of morphisms of C such that:

(MS1) All of Fib, Cof and Weq are closed under retracts;
(MS2) 2-out-of-3: if any two of the three morphisms f, g, gf are in Weq, then so is the third;
(MS3) $\mathsf{Cof} \cap \mathsf{Weq} \boxslash \mathsf{Fib}$ and $\mathsf{Cof} \boxslash \mathsf{Fib} \cap \mathsf{Weq}$;
(MS4) Every morphism f of C admits a factorisation $f = pi$ in two ways: $i \in \mathsf{Cof} \cap \mathsf{Weq}$ and $p \in \mathsf{Fib}$, or $i \in \mathsf{Cof}$ and $p \in \mathsf{Fib} \cap \mathsf{Weq}$. ■

The maps of Cof, Fib and Weq are called *cofibrations*, *fibrations* and *weak equivalences*, respectively. A category endowed with a model structure is called a *model category*. The following examples will all feature at some point in this thesis.

Example 2.2.2.

1. Every complete and cocomplete category admits a trivial model structure in which all morphisms are both cofibrations and fibrations, and the isomorphisms are the weak equivalences.

2. There is a standard model structure $\mathbf{sSet}_Q$ on the category $\mathbf{sSet}$ of simplicial sets. The cofibrations are the monomorphisms and the weak equivalences are the *geometric homotopy equivalences*: those f of which the geometric realisation $|f|_\Delta$ is a weak homotopy equivalence in **Top**. By Remark 2.2.5 this description uniquely determines the model structure.

3. For C a model category, and D a small category, the *projective model structure* on the functor category $[\mathsf{D}, \mathsf{C}]$, if it exists, is determined by letting a natural transformation be a weak equivalence if it is a componentwise weak equivalence, and a fibration if it is a componentwise fibration. We will later give a sufficient condition on C and D for the projective model structure to exist. ▲

An alternative and more economical definition of a model structure can be given in terms of *weak factorisation systems*.

Definition 2.2.3. A *weak factorisation system* on a category C consists of a pair (L, R) of classes of morphisms of C such that:
1. Every morphism f may be factored as $f = pi$ with $i \in \mathsf{L}$ and $p \in \mathsf{R}$;
2. $\mathsf{L} = {}^{\boxslash}\mathsf{R}$ and $\mathsf{R} = \mathsf{L}^{\boxslash}$. ■

We will prove that a model structure can equivalently be defined as follows.

Proposition 2.2.4. *Let* $(\mathsf{Cof}, \mathsf{TrivFib})$ *and* $(\mathsf{TrivCof}, \mathsf{Fib})$ *be two weak factorisation systems on* C *such that* $\mathsf{TrivCof} \subseteq \mathsf{Cof}$ *and* $\mathsf{TrivFib} \subseteq \mathsf{Fib}$, *and let* Weq *be a class of morphisms satisfying 2-out-of-3. Then these classes define a model structure on* C *if and only if the following four inclusions hold:*

(A1) $\mathsf{TrivCof} \subseteq \mathsf{Cof} \cap \mathsf{Weq}$	(B1) $\mathsf{TrivFib} \subseteq \mathsf{Fib} \cap \mathsf{Weq}$
(A2) $\mathsf{Cof} \cap \mathsf{Weq} \subseteq \mathsf{TrivCof}$	(B2) $\mathsf{Fib} \cap \mathsf{Weq} \subseteq \mathsf{TrivFib}$

Remark 2.2.5. Several elements of the previous definition are determined by the other parts of the definition. By the four inclusions, this is obviously the case for the classes TrivCof and TrivFib. Moreover, given the classes of cofibrations (fibrations) and weak equivalences, the remaining class of fibrations (cofibrations) is uniquely determined.

A slightly less evident redundancy that is important to note is that of the content of Weq. Indeed, given all other conditions (including 2-out-of-3), it is easy to verify that the elements of Weq are determined to be precisely the class of morphisms that may be factored $\cdot \overset{\text{triv}}{\rightarrowtail} \cdot \overset{\text{triv}}{\twoheadrightarrow} \cdot$ as a trivial cofibration followed by a trivial fibration. ▲

The following is a standard lemma.

Lemma 2.2.6. *Suppose that* L *and* R *are two classes of morphisms such that*

1. *Any morphism f may be factored as $f = pi$ with $i \in \mathsf{L}$ and $p \in \mathsf{R}$;*
2. $\mathsf{L} \boxslash \mathsf{R}$.

Then (L, R) *is a weak factorisation system if and only if* L *and* R *are closed under retracts.*

Proof. The implication from right to left follows from Proposition 2.1.6 and its dual. For the implication in the other direction, note that it follows from the second hypothesis that $\mathsf{L} \subseteq {}^{\boxslash}\mathsf{R}$. In order to show the reverse inclusion, let $f \in {}^{\boxslash}\mathsf{R}$. Factoring $f = pi$ with $i \in \mathsf{L}$ and $p \in \mathsf{R}$, we get a commutative square and thus a diagonal filler

$$\begin{tikzcd} \cdot \arrow[r, "i"] \arrow[d, "f"'] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, equal] \arrow[ur, dashed, "u"] & \cdot \end{tikzcd}$$

which gives rise to a commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "f"'] & \cdot \arrow[r, equal] \arrow[d, "i"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "u"'] & \cdot \arrow[r, "p"'] & \cdot \end{tikzcd}$$

that shows that f is a retract of i and therefore that $f \in \mathsf{L}$. Dually, one may show $\mathsf{R} = \mathsf{L}^{\boxslash}$. □

Example 2.2.7 (Weak factorisation system on Δ). We claim that $(\mathsf{Sur}, \mathsf{Inj})$ is a weak factorisation system on Δ, where Sur is the class of surjective, and Inj the class of injective morphisms of Δ.

Factorisation is precisely Lemma 1.3.2. For the lifting property, observe that every commutative square

$$\begin{tikzcd} {[n]} \arrow[r, "\beta"] \arrow[d, "\alpha \in \mathsf{Sur}"'] & {[k]} \arrow[d, "\in \mathsf{Inj}"] \\ {[m]} \arrow[r] & {[l]} \end{tikzcd}$$

admits a lifting $\beta\gamma$, where γ is any section of α. Again we can even show uniqueness, using the injectivity of the right map. Since epi's and mono's are closed under retracts, this, by Lemma 2.2.6, defines a weak factorisation system. ▲

Proof of Proposition 2.2.4. $\Longrightarrow$. The inclusions (A1,2) and (B1,2) are obtained by applying Lemma 2.2.6 twice.

$\Longleftarrow$. First note that, by Lemma 2.2.6, both the classes Fib and Cof are closed under retracts. Therefore, the only thing left to show is that the class Weq is as well. For this we use an argument from [Rie] that is due to André Joyal.

Suppose we are given a map w in Weq together with a retract f, witnessed by the following diagram.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[r] \arrow[d, "w"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd} \tag{2.1}$$

We first show that f is in Weq whenever f is in Fib, saving the general case for later. Factor $w = vu$ with u in TrivCof and v in Fib. Since TrivCof = Cof ∩ Weq, we have, by 2-out-of-3, that v is in Fib ∩ Weq. This gives us arrows s and t as in the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "s_0"] \arrow[dd, "f"'] \arrow[dr, dashed, "s"'] & \cdot \arrow[r] \arrow[d, "u"] & \cdot \arrow[dd, "f"] \\
& \cdot \arrow[d, "v"] \arrow[ur, dashed, "t"'] & \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd} \tag{2.2}$$

where s is simply us_0 and t is a diagonal filler of the square formed by $u \in$ TrivCof and $f \in$ Fib. The triangles at the top of the diagram commute and thus ts is the identity. It follows that f is a retract of v and therefore, by Lemma 2.2.6, that f is a weak equivalence.

We return to a diagram of the form 2.1 again, but this time f may be any morphism. First factor $f = hg$ with $g \in$ TrivCof and $h \in$ Fib and take the pushout as in the following diagram.

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "g"'] & \cdot \arrow[r, "r_0"] \arrow[d, "c"'] \arrow[dd, bend left, "w"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "x"'] \arrow[d, "h"'] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] & \cdot \arrow[d, "h"] \\
\cdot \arrow[r, "b"'] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

Note that, since $g \in$ TrivCof, it follows from Proposition 2.1.6 that c is as well. We indicate two other cones of this pushout diagram, namely the one formed by w and bh, and the one formed by gr_0 and the identity. Using the universal property, we obtain

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "g"'] & \cdot \arrow[r, "r_0"] \arrow[d, "c"'] \arrow[dd, bend left, "w"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "x"'] \arrow[d, "h"'] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "y"'] \arrow[d, dashed, "d"'] & \cdot \arrow[d, "h"] \\
\cdot \arrow[r, "b"'] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

wherein $dc = w$ and $yx = id$. The former equality implies, by 2-out-of-3, that $d \in$ Weq and the latter that h is a retract of d. Since $h \in$ Fib, we can use the above argument to conclude and $h \in$ Weq. Finally, it follows from 2-out-of-3 that $f = gh \in$ Weq, as required. □

Definition 2.2.8. An object X of a model category is called *fibrant* if the unique morphism $!_X : X \longrightarrow 1$ into the terminal object is a fibration. Dually, an object is *cofibrant* if $*_X : 0 \longrightarrow X$ is a cofibration. ■

Example 2.2.9. Because every monomorphism in $\mathbf{sSet}_Q$ is a cofibration, every object is cofibrant. ▲

Many basic facts of algebraic topology can be proven in the general setting of a Quillen model structure. For instance the following fact, that we state without proof.

Lemma 2.2.10 ([JT08, A.6.2.]). *In any model category, the pushout of a weak equivalence between cofibrant objects, along a cofibration, is again a weak equivalence.*

This is a generalisation of the standard fact that, in **Top** (strictly: in a more convenient category of topological spaces), the pushout of a homotopy equivalence along a cofibration is again a homotopy equivalence. From Lemma 2.2.10 we can deduce a general property of model structures that will be useful in Chapter 3.

Proposition 2.2.11. *In any model category, finite coproducts of weak equivalences between cofibrant objects are again weak equivalences.*

Proof. Since, by 2-out-of-3, weak equivalences are closed under composition, it suffices to show that for $w : A \longrightarrow A'$ a weak equivalence between cofibrant objects and B cofibrant, the morphism $w + 1_B : A + B \longrightarrow A' + B$ is a weak equivalence between cofibrant objects. First note that the square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "w"'] \arrow[dr, phantom, "\ulcorner"] & A + B \arrow[d, "w' + id_B"] \\ A' \arrow[r] & A' + B \end{tikzcd}$$

is a pushout square, and thus by Lemma 2.2.10, if the inclusion $A \longrightarrow A + B$ is a cofibration, then $w' + id_B$ is a weak equivalence. But, like any coproduct, $A + B$ may be written as the pushout

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d] \\ B \arrow[r] & A + B \end{tikzcd}$$

from which it indeed follows, by Proposition 2.1.6, that $A \longrightarrow A + B$ is a cofibration. Finally, both $A + B$ and $A' + B$ are cofibrant, because by the same proposition cofibrations are closed under coproducts. □

We end this section by giving the standard notion for comparing model structures. Just like in the setting of general categories, there is a notion of equivalence which is a strengthening of the notion of adjunction.

Definition 2.2.12. Let $F : \mathrm{C} \rightleftarrows \mathrm{D} : G$ be an adjunction between model categories. We say that:

- F is a *Quillen adjunction* if F preserves cofibrations and trivial cofibrations;
- F is a *Quillen equivalence* if it is a Quillen adjunction such that for every cofibrant object C of C and every fibrant object D of D:

$$f : FC \longrightarrow D \text{ is a weak equivalence if and only if its adjunct } \overline{f} : C \longrightarrow GD \text{ is.}$$

The (left) right adjoint in a Quillen adjunction is called a *(left) right Quillen functor.* ■

Example 2.2.13. The geometric realisation of simplicial sets is by construction part of an adjunction $|\cdot|_\Delta \dashv S_\Delta$. A standard fact of abstract homotopy theory is that this adjunction is a Quillen equivalence between $\mathbf{sSet}_Q$ and the standard model structure on **Top**. ▲

2.3 Semimodel structures

A weakening of the notion of a model structure is given in [Spi]. These categories, which we will call *semimodel categories*, appear in two dual variations, but in each case conditions 3 and 4 of Proposition 2.2.1 are loosened for a certain class of maps. For instance, in a *left* semimodel category, fibrations are only required to have the right lifting property with respect to the morphisms $i : A \longrightarrow B$ of $\mathsf{Cof} \cap \mathsf{Weq}$ such that the domain A is cofibrant. The first factorisation condition, into a morphism of $\mathsf{Cof} \cap \mathsf{Weq}$ followed by one of Fib, is likewise only demanded for morphisms with cofibrant domain. We are interested in the dual, the *right* semimodel structure.

Let us fix some notation: given a class of morphisms F, we write $\mathsf{F}_\bullet$ for the class of morphisms f of F with fibrant codomain, *i.e.* the $f : A \longrightarrow B$ such that $!_B \in \mathsf{Fib}$.

Definition 2.3.1. A *right semimodel structure* on a category C consists of a triple of classes of morphisms $(\mathsf{Fib}, \mathsf{Cof}, \mathsf{Weq})$ of C such that

1. All of Fib, Cof and Weq are closed under retracts;
2. The class Weq satisfies 2-out-of-3;
3. $\mathsf{Cof} \cap \mathsf{Weq} \boxslash \mathsf{Fib}$ and $\mathsf{Cof} \boxslash \mathsf{Fib}_\bullet \cap \mathsf{Weq}_\bullet$;
4. Every morphism f admits a factorisation $f = pi$, where $i \in \mathsf{Cof} \cap \mathsf{Weq}$ and $p \in \mathsf{Fib}$. Furthermore, if f has fibrant codomain, it may also be factored as $f = pi$ with $i \in \mathsf{Cof}$ and $p \in \mathsf{Fib}_\bullet \cap \mathsf{Weq}_\bullet$. ■

A category endowed with a right semimodel structure is called a *right semimodel category*. Note that in a right semimodel category, any morphism $!_X : X \longrightarrow 1$ may be factored $X \overset{\text{triv}}{\rightarrowtail} \overline{X} \twoheadrightarrow 1$, in which case $\overline{X}$ is fibrant. We call such $\overline{X}$, which is a fibrant object connected to X by a trivial cofibration, a *fibrant replacement* of X.

The following is a slightly stronger formulation of this definition in terms of weak factorisation systems. Observe that it is a weakening of the conditions of Proposition 2.2.4 in that the inclusions (B1) and (B2) are replaced by (B1') and (B2'), which concern only the morphisms with fibrant codomain.

Proposition 2.3.2. *Let* $(\mathsf{Cof}, \mathsf{TrivFib})$ *and* $(\mathsf{TrivCof}, \mathsf{Fib})$ *be two weak factorisation systems on* C *such that* $\mathsf{TrivCof} \subseteq \mathsf{Cof}$ *and* $\mathsf{TrivFib} \subseteq \mathsf{Fib}$, *and let* Weq *be a class of morphisms satisfying 2-out-of-3. If the following inclusions hold:*

(A1) $\mathsf{TrivCof} \subseteq \mathsf{Cof} \cap \mathsf{Weq}$ (B1') $\mathsf{TrivFib}_\bullet \subseteq \mathsf{Fib}_\bullet \cap \mathsf{Weq}_\bullet$

(A2) $\mathsf{Cof} \cap \mathsf{Weq} \subseteq \mathsf{TrivCof}$ (B2') $\mathsf{Fib}_\bullet \cap \mathsf{Weq}_\bullet \subseteq \mathsf{TrivFib}_\bullet$

Then these classes impose a right semimodel structure on C.

Proof. This again boils down to showing that Weq is closed under retracts. The crux of the matter is whether the argument in the proof of Proposition 2.2.4 still works. To see that it does, note that nowhere in the proof the weak factorisation system $(\mathsf{Cof}, \mathsf{Fib} \cap \mathsf{Weq})$ is used. □

In Remark 2.2.5 it is noted that in a regular model structure the class of weak equivalences is determined by the other classes. We will show that this is also the case for right semimodel structures. Let $f : X \longrightarrow Y$ be a morphism of a category endowed with such structure. Taking two fibrant replacements $\overline{X}$ and $\overline{Y}$ gives a dotted diagonal filler $\overline{f}$ as in the following commutative square:

$$\begin{tikzcd} X \arrow[r, tail, "\text{triv}"] \arrow[d, "f"'] & \overline{X} \arrow[d, dashed, "\overline{f}"] \\ Y \arrow[r, tail, "\text{triv}"] & \overline{Y} \end{tikzcd} \tag{2.3}$$

Proposition 2.3.3. *In a right semimodel structure the class* Weq *of weak equivalences contains precisely the morphisms f such that any diagonal filler $\overline{f}$ in a square of the form (2.3) admits a factorisation into a trivial cofibration followed by a trivial fibration.*

Proof. $\Longrightarrow$. Let $f : X \xrightarrow{\sim} Y$. Then for any $\overline{f}$ as in (2.3), we find, by applying 2-out-of-3 twice, that $\overline{f}$ is a weak equivalence. Factoring

$$\overline{f} : \overline{X} \rightarrowtail E \xrightarrow{\text{triv}} \!\!\!\!\rightarrow \overline{Y},$$

we have $E \xrightarrow{\text{triv}}\!\!\!\!\rightarrow \overline{Y} \in \mathsf{TrivFib}_{\bullet}$ and thus $E \overset{\text{triv}}{\underset{\sim}{\twoheadrightarrow}} \overline{Y}$. It follows by 2-out-of-3 that $\overline{X} \overset{}{\underset{\sim}{\rightarrowtail}} E$ and thus $\overline{X} \overset{\text{triv}}{\underset{\sim}{\rightarrowtail}} E$, as required.

$\Longleftarrow$. For the converse, we first obtain a square of the form (2.3) by taking fibrant replacements and a diagonal filler. The resulting $\overline{f}$ can be factored as a trivial cofibration followed by a trivial fibration, with the latter in $\mathsf{TrivFib}_{\bullet}$. It follows from applying 2-out-of-3 several times, that $f \in \mathsf{Weq}$. □

Remark 2.3.4. Since every model structure is trivially a right semimodel structure, it holds in that setting that the class of morphisms defined above is the same as the class of morphisms that can be factored as a trivial cofibration followed by a trivial fibration. In contrast, in the setting of a right semimodel structure it is unclear to the author how these classes relate. ▲

We define the notions of *Quillen adjunction* and *Quillen equivalence* between right semimodel structures in precisely the same way as in Definition 2.2.12.

2.4 Locally finitely presentable categories

For the following sections we will restrict our attention to *locally finitely presentable* categories. These are essentially categorical generalisations of algebraic lattices, in the sense that every object of such a category can be generated from objects satisfying a certain compactness property. We shall use a category $(L, \leq)$ induced from a complete lattice as running example. Moreover, we will see that the category *Set* is locally finitely presentable, for every set can be written as the union of its finite subsets. Another important class of examples is given by the presheaf categories, which of course includes both **ssSet** and **sSet**. In sections 2.5 and 2.6 we will show that these categories admit some useful methods for constructing model structures. The treatment in this section is based on [AR94, Chapter 1].

Recall the following definition.

Definition 2.4.1. A poset $(I, \leq)$ is called *directed* if every finite subset of I has an upper bound. A diagram whose index category is induced from a directed poset is a *directed diagram.* The colimit of such diagram is called a *directed colimit.* ■

A directed diagram $D : (I, \leq) \longrightarrow (L, \leq)$ is an order-preserving map between posets. Its (directed) colimit is the join $\bigvee\{D(i) \mid i \in I\}$.

Definition 2.4.2. An object X is called *finitely presentable* whenever $\mathrm{Hom}(X, -)$ preserves directed colimits. That is, if for every directed diagram $D : (I, \leq) \longrightarrow \mathrm{C}$, the canonical morphism:

$$\varinjlim_{i \in I} \mathrm{Hom}_{\mathrm{C}}(X, D_i) \underset{\cong}{\longrightarrow} \mathrm{Hom}_{\mathrm{C}}(X, \varinjlim_{i \in I} D_i).$$

is an isomorphism. ■

We leave it to the reader to compute that an object $a \in L$ is finitely presentable in the category $(L, \leq)$, whenever for each directed subset $I \subseteq L$,

$$\text{there is an } i \in I \text{ such that } a \leq i \iff a \leq \bigvee I.$$

That is, whenever a is what in domain theory is called a *compact* (or *finite*) element of the lattice L. For general categories Definition 2.4.2 may be written out as follows.

Lemma 2.4.3. *An object X is finitely presentable iff for every directed diagram $D : (I, \leq) \longrightarrow \mathrm{C}$ and initial cocone $(Y, (c_i)_{i \in I})$ to D it holds that:*

1. *Every morphism $f : X \longrightarrow Y$ factors through some D_i. That is, there is an $i \in I$ and an arrow $g : X \longrightarrow D_i$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ D_i \arrow[ur, "c_i"'] & \end{tikzcd}$$

   commutes.

2. *This morphism is* essentially unique*, in the sense that for all $i \in I$,*

$$f = c_i g' = c_i g'' \text{ implies that } D(i \leq j)g' = D(i \leq j)g'' \text{ for some } j \geq i.$$

Proof. By definition, an object X is finitely presentable whenever we have the isomorphism

$$u : \varinjlim_{i \in I} \mathrm{Hom}_{\mathrm{C}}(X, D_i) \cong \mathrm{Hom}_{\mathrm{C}}(X, Y).$$

A basic fact about colimits is that the former expression can be computed in terms of a coproduct and coequaliser, as in:

$$\coprod_{i \leq j} \mathrm{Hom}_{\mathrm{C}}(X, D_i) \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{i} \mathrm{Hom}_{\mathrm{C}}(X, D_i) \xrightarrow{\ e\ } \varinjlim_{i} \mathrm{Hom}_{\mathrm{C}}(X, D_i).$$

where $\phi(f) = f$ and $\psi(f) = D(i \leq j)f$. In other words,

$$\varinjlim_i \mathrm{Hom}(X, D_i) = \coprod_i \mathrm{Hom}_{\mathsf{C}}(X, D_i)/\sim,$$

where $\sim$ is the equivalence relation generated by $f \sim D(i \leq j)f$ for all $i \leq j$ in I.

Writing $[g]$ for the $\sim$-equivalence class of g, it follows that u is surjective if and only if for every $f : X \longrightarrow Y$, there is a $g : X \longrightarrow D_i$ such that $u([g]) = f$. Since u is a cocone morphism, we have $uep_i = c_i$, where p_i is the i-th projection into the coproduct. That means that $u([g]) = c_i g$, which give us precisely condition 1.

We claim that the injectivity of u is equivalent to condition 2. The implication from left to right is easy: we have $c_i g' = c_i g''$ precisely when $u([g']) = u([g''])$ and

$$\text{There is a } j \geq i \text{ such that } D(i \leq j)g' = D(i \leq j)g'' \text{ if and only if } [g'] = [g''],$$

as required. For the other direction, suppose that $u([g']) = u([g''])$ for some $g' : X \longrightarrow D_k$ and $g'' : X \longrightarrow D_l$. Because D is directed, we may assume that $k = l$, after which it follows from condition 2 that there is an $r \geq k$ such that $D(k \leq r)g' = D(k \leq r)g''$. □

Example 2.4.4 (Finitely presentable objects).

- An object in the category Set is finitely presentable if and only if it is finite. Indeed, given a finite set $X = \{x_0, \dots, x_n\}$ and a function $f : X \longrightarrow Y$ into an initial cocone $(Y, (c_i)_{i \in I})$ to a directed $D : (I, \leq) \longrightarrow Set$, there are representatives $d_0 \in D_{i_0}, \dots, d_n \in D_{i_n}$ such that

  $$f(x_0) = [d_0], \dots, f(x_n) = [d_n].$$

  Let $r \geq i_0, \dots, i_n$ and define $g : X \longrightarrow D_r$ by $g(x_k) = D(i_k \leq r)(d_k)$. Then we have for every x_k that $c_r g(x_k) = c_k(d_k) = [d_k] = f(x_k)$, as required. Similarly, if $f = c_r g' = c_r g''$, then there is q such that

  $$D(r \leq q)(g'(x_0)) = D(r \leq q)(g''(x_0)), \dots, D(r \leq q)(g'(x_n)) = D(r \leq q)(g''(x_n)),$$

  and so $D(r \leq q)g' = D(r \leq q)g''$. For the converse, we use the fact that every set is the directed colimit of its finite subsets ordered by inclusion. If X is finitely presentable, then the function $1_X : X \longrightarrow \varinjlim X_i$ may be factored though a finite set $X_i \subseteq X$ and thus X is a finite set.

- In the presheaf category $\hat{\mathsf{C}}$, it holds, by the Yoneda lemma, that for object C of C,

  $$\varinjlim_i \mathrm{Hom}_{\hat{\mathsf{C}}}(yC, D_i) \cong \varinjlim_i D_i(C) \cong (\varinjlim_i D_i)(C) \cong \mathrm{Hom}_{\hat{\mathsf{C}}}(yC, \varinjlim_i D_i),$$

  where the second isomorphism holds due to the fact that colimits of presheaves are computed pointwise. It follows that every representable presheaf yC is a finitely presentable. ▲

Definition 2.4.5. A category C is called *locally finitely presentable* if it is cocomplete and there is a set of finitely presentable objects of C such that every object is a directed colimit of objects from that set. ◼

Thus $(L, \leq)$ is locally finitely presentable whenever every $a \in L$ is the join of a directed subset of compact elements of L. But this equivalent to having for each $a \in L$,

$$a = \bigvee \{b \in L \mid b \leq a, \text{ and } b \text{ is compact}\},$$

which in domain theory means precisely that the lattice $(L, \leq)$ is *algebraic*.

Example 2.4.6 (Locally finitely presentable categories).

- *Set* is locally finitely presentable, since every set can be written as a directed colimit of finite sets, and the finite sets form a (countable) set.
- The presheaf category $\hat{\mathrm{C}}$ over a small category is locally finitely presentable. By Example 2.4.4, the set of representables consists of finitely presentable objects and, by the proposition below, so does its closure under finite colimits, let us call that set S. We claim that every presheaf P can be written as a directed colimit of objects in S. By the general version of Proposition 1.2.8, every presheaf can be written as a colimit of representables, *i.e.* as a coequaliser of coproducts of representables:

  $$\coprod_{l \in L} A_l \rightrightarrows \coprod_{m \in M} B_m \longrightarrow P$$

  We can again make use of the fact that every set can be written as a colimit of its finite subsets, to write the above as:

  $$\varinjlim_{S \in \mathscr{P}_{\mathrm{fin}}(L)} \coprod_{s \in S} A_s \rightrightarrows \varinjlim_{T \in \mathscr{P}_{\mathrm{fin}}(M)} \coprod_{t \in T} B_t \longrightarrow P$$

  Since colimits commute with colimits, this can be written

  $$\varinjlim_{(S,T) \in \mathscr{P}_{\mathrm{fin}}(L) \times \mathscr{P}_{\mathrm{fin}}(M)} (\coprod_{s \in S} A_s \rightrightarrows \coprod_{t \in T} B_t \longrightarrow P_{S,T})$$

  as a directed colimit of objects in S. ▲

Proposition 2.4.7. *Any finite colimit of finitely presentable objects is finitely presentable.*

Proof. Let $F : \mathrm{F} \longrightarrow \mathrm{C}$ be a finite diagram with colimiting cocone $(K, (k_n)_{n \in \mathrm{F}})$ such that every F_n is a finitely presentable object of C. We will show that, for every given directed diagram $D : (I, \leq) \longrightarrow \mathrm{C}$ with colimiting cocone $(Y, (c_i)_{i \in I})$, the colimit K satisfies conditions 1 and 2 of Lemma 2.4.3.

For the first condition, suppose we have a morphism $f : K \longrightarrow Y$. We will construct a factorisation $g : K \longrightarrow D_i$ such that for every $n \in \mathrm{F}$, it holds that $f k_n = c_i g k_n$. It then follows from the fact that K is a colimit that $f = c_i g$.

Because the F_n are finitely presentable, we have for each n that fk_n factors through c_{i_n} for some $i_n \in I$. Let j be an upper bound of the (finitely many) i_n. Then for every $n \in \mathsf{F}$, there is a g_n such that $fk_n = c_j g_n$.

Now let $\alpha : n \longrightarrow m$ be some arrow in F. We have $k_n = k_m F\alpha$, which gives us the following two factorisations of fk_n:

$$(1)\ fk_n = c_j g_n, \text{ and } (2)\ fk_n = fk_m F\alpha = c_j g_m F\alpha.$$

It follows that, because F_n is finitely presentable, there is some $j_\delta \geq j$ in I such that

$$D(j \leq j_\delta) g_n = D(j \leq j_\delta) g_m F\alpha.$$

Since F has finitely many morphisms, there is an upper bound l of the j_δ and it can be easily verified that $(D_l, (D(j \leq l)g_n)_{n \in \mathsf{F}})$ forms a cocone to F. Thus, there is a cocone morphism $g : K \longrightarrow D_l$, from which we find that for every $n \in \mathsf{F}$,

$$fk_n = c_j g_n = c_l D(j \leq l) g_n = c_l g k_n,$$

as desired.

For the second condition, note that whenever $f = c_i g' = c_i g''$, we have for every $n \in \mathsf{F}$,

$$fk_n = c_i g' k_n = c_i g'' k_n.$$

By the fact that F_n is finitely presentable, it follows that there is a $i_n \geq i$ such that

$$D(i \leq i_n) g' k_n = D(i \leq i_n) g'' k_n.$$

Let j be an upper bound of all of the i_n. Then $D(i \leq j) g' k_n = D(i \leq j) g'' k_n$ for every n, which, by the fact that K is a colimit, means that $D(i \leq j) g' = D(i \leq j) g''$. $\square$

2.5 Quillen's small object argument

In this section we prove a useful theorem that allows us to generate a weak factorisation system from any suitable set of morphisms. Since its introduction by Quillen, this argument has been much studied, leading to many different generalisations. A particularly strong generalisation is in [Gar09]. Below, however, we proof a relatively weak version that is still sufficient for our purpose.

Definition 2.5.1. For I a set of morphisms, we say that a weak factorisation system (L, R) is *cofibrantly generated* by I if $\mathsf{R} = I^{\boxslash}$.

Theorem 2.5.2 (Quillen's small object argument). *Let I be a set of morphisms with finitely presentable domains. Then any morphism $f : X \longrightarrow Y$ may be factored as*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & E \arrow[d, "p"] \\ & Y \end{tikzcd}$$

with $p \in I^{\boxslash}$ and $i \in {}^{\boxslash}(I^{\boxslash})$. It follows that I cofibrantly generates a weak factorisation system.

Proof. By induction, we will define an ω-sequence $X : \omega \longrightarrow \mathrm{C}$,

$$X = X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} X_2 \xrightarrow{i_3} \dots$$

such that $i_k \in {}^{\boxslash}(I^{\boxslash})$ for each k, together with morphisms $f_k : X_k \longrightarrow Y$ such that it holds that $f_{k+1} \circ i_{k+1} = f_k$. It will then follow that Y is a cocone to X. For $E = \varinjlim X$, the morphism $i : X \longrightarrow E$ will be the ω-composition of the chain and we will take $p : E \longrightarrow Y$ to be the unique morphism determined by the colimits universal property. It follows by Proposition 2.1.6 that $i \in {}^{\boxslash}(I^{\boxslash})$.

We let $f_0 = f$. Having defined the i_k up to $k = n$, we define i_{n+1} by first considering the set L_n of all commutative diagrams of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\in I"'] & X_n \arrow[d, "f_n"] \\ B \arrow[r] & Y \end{tikzcd}$$

and then taking the coproduct over L_n to get a diagram

$$\begin{tikzcd} \coprod_{L_n} A \arrow[r] \arrow[d] & X \arrow[d, "f_n"] \\ \coprod_{L_n} B \arrow[r] & Y \end{tikzcd}$$

Then i_{n+1} is the pushout

$$\begin{tikzcd} \coprod_{L_n} A \arrow[r] \arrow[d] & X_n \arrow[d, "i_{n+1}"] \arrow[ddr, bend left, "f_n"] & \\ \coprod_{L_n} B \arrow[r] \arrow[drr, bend right] & X_{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "f_{n+1}"'] & \\ & & Y \end{tikzcd}$$

and f_{n+1} is the induced morphism. Note that i_{n+1} is a pushout of a coproduct of morphisms in I and thus, again by Proposition 2.1.6, we have $i \in {}^{\boxslash}(I^{\boxslash})$.

It remains to show that $p \in I^{\boxslash}$. To that end, observe that in any lifting problem

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\in I"'] & E \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

the morphism h admits a factorisation

$$\begin{tikzcd} A \arrow[r, "\tilde{h}"] \arrow[d, "\in I"'] & X^m \arrow[r, "g^m"] \arrow[d, "f^m"] & E \arrow[dl, "p"] \\ B \arrow[r] & Y & \end{tikzcd}$$

Indeed, because A is finitely presentable, and the fact that the ordinal ω considered as a category is directed, this is given by Lemma 2.4.3. But the square in the second diagram is in L_m, so there is a pushout square

$$\begin{tikzcd} A \arrow[r, "\tilde{h}"] \arrow[d] & X_m \arrow[d, "i^m"] \arrow[ddr, bend left, "f_m"] & \\ B \arrow[r] \arrow[drr, bend right] & X_{m+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "f_{m+1}"'] & \\ & & Y \end{tikzcd}$$

This means that we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i^m \tilde{h}"] \arrow[d] & X^{m+1} \arrow[d, "f^{m+1}"] \\ B \arrow[r] \arrow[ur] & Y \end{tikzcd} \quad \text{and thus have a lifting} \quad \begin{tikzcd} A \arrow[r, "h"] \arrow[d] & E \arrow[d, "p"] \\ B \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

as required. ☐

As a corollary we find that the left class of a weak factorisation system that is cofibrantly generated by such I admits some convenient characterisations.

Corollary 2.5.3. *If I is a set of morphisms with finitely presentable domains, then the following four classes of morphisms are the same:*

(1) The saturated class generated *by I,* i.e. *the least saturated class containing I;*
(2) The class ${}^{\boxslash}(I^{\boxslash})$;
(3) The retracts of ω-compositions of pushouts of coproducts of morphisms in I;
(4) The retracts of transfinite compositions of pushouts of morphisms in I.

Proof. The inclusion (1) ⊆ (2) is Proposition 2.1.6. For (2) ⊆ (3), let $f \in {}^{\boxslash}(I^{\boxslash})$ and factor $f = pi$ using the method in the above proof. By construction, the morphism i is an ω-composition of pushouts of coproducts of I, and by the lifting

$$\begin{tikzcd} \cdot \arrow[r, "i"] \arrow[d, "f"'] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, equal] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

we find that f is a retract of i. To establish (3) ⊆ (4) it suffices to show that any pushout of coproducts of morphisms in I can be written as a transfinite composition of pushouts of morphisms in I. To that end, suppose that for each $l \in L$ we have a morphism $i_l : A_l \longrightarrow B_l$ in I and a pushout

$$\begin{tikzcd} \coprod_{l \in L} A_l \arrow[r] \arrow[d, "\coprod_{l\in L} i_l"'] & X \arrow[d, "f"] \\ \coprod_{l \in L} B_l \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y \end{tikzcd}$$

We claim that f can be written as a transfinite composition of pushouts of morphisms in I. Indeed, by the axiom of choice we may assume that L is an ordinal and construct the following L-sequence $C : L \longrightarrow \mathsf{C}$ by transfinite induction. Set $C_0 = \coprod_{l \in L} A_l$, and for each successor ordinal $\beta + 1 \in L$, let $C_\beta \longrightarrow C_{\beta+1}$ be the following pushout:

$$\begin{tikzcd} A_\beta \arrow[r] \arrow[d] & C_\beta \arrow[d] \\ B_\beta \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C_{\beta+1} \end{tikzcd}$$

At limit stages, we set $C_\lambda = \varinjlim_{\beta<\lambda} C_\beta$. Then the composition $C_0 \longrightarrow C_L$ of C is isomorphic to $\coprod_{l\in L} f_l$ and it is a transfinite composition of pushouts of morphisms in I, as desired. We finish the proof by noting that the inclusion (4) $\subseteq$ (1) holds by definition. □

For I a set of morphisms with finitely presentable domains, the class of morphisms described in the above corollary will be denoted $\mathrm{cof}(I)$.

Remark 2.5.4. One of the many ways in which the small object argument can be strengthened is that it can be used to show that in a locally finitely presentable category *any* set I of morphisms cofibrantly generates a weak factorisation system, even though of course not every object needs to be finitely presentable. The idea is to show that, since every object is a directed colimit of a finitely presentable object, there is some cardinal κ such that every morphism of I has a domain which is a κ-small (*i.e.* of a diagram of size less than κ) directed colimit of finitely presentable objects. One can then show that every domain X is *κ-presentable*, which means that $\mathrm{Hom}(X, -)$ preserves *κ-directed colimits*, *i.e.* colimits of posets in which every set of cardinality less than κ has an upper bound. Finally, a transfinite version of the above argument can be used to obtain the desired result.

This stronger version of the small object argument is, however, not needed for any of the constructions in this thesis and we will not give it any more treatment than we already have in this remark. ▲

The following easy but important fact follows immediately from characterisation (4) of Corollary 2.5.3.

Lemma 2.5.5. *Let A be a set of arrows of C with finitely presentable domains and $F : \mathrm{C} \longrightarrow \mathrm{D}$ be a functor that preserves colimits and finitely presentable objects. Then*

$$F(I) = \{Ff : FX \longrightarrow FY \mid f : X \longrightarrow Y \in A\}$$

is a set of morphisms of D with finitely presentable domains, and $F(\mathrm{cof}(I)) \subseteq \mathrm{cof}(F(I))$.

Definition 2.5.6. A model structure (Cof, Fib, Weq) is said to be *cofibrantly generated* whenever there are two sets of morphisms I, J such that I cofibrantly generates the weak factorisation system (Cof, TrivFib) and J the system (TrivCof, Fib). Then I (resp. J) is called the set of *generating (trivial) cofibrations.* ■

Example 2.5.7. The model structure $\mathbf{sSet}_Q$ is cofibrantly generated by the so-called *boundary inclusions* and *horn inclusions*, which will be defined in Chapter 3. ▲

The following proposition is stated without proof.

Proposition 2.5.8 ([Lur09, Prop. A.2.8.2]). *If C is a category equipped with a cofibrantly generated model structure and D a small category, then the projective model category exists on* $[\mathrm{D}, \mathrm{C}]$.

The following is a nice conceptual proof of a proposition that will come in handy later in this thesis.

Proposition 2.5.9 ([RR15, Prop. 4.1]). *Let* C *be a cofibrantly generated model category such that the class of cofibrations is cofibrantly generated by a set I of morphisms between finitely presentable objects. Then the weak equivalences are closed under directed colimits.*

Proof. Let D $\longrightarrow$ C be a directed diagram. By Proposition 2.5.8, the projective model structure exists on [D, C] and we claim that the adjunction

$$\varinjlim \dashv \Delta$$

between the colimit functor [D, C] $\longrightarrow$ C and the diagonal functor is a Quillen adjunction. Indeed, by definition Δ preserves fibrations and trivial fibrations and thus, by Lemma 2.1.8, the left adjoint $\varinjlim$ preserves cofibrations and trivial cofibrations. We must show that $\varinjlim$ preserves weak equivalences. To that end, take a weak equivalence in [D, C] and factor it (using Remark 2.2.5) into a trivial cofibration followed by a trivial fibration:

$$\cdot \overset{\text{triv}}{\rightarrowtail} \cdot \overset{\text{triv}}{\twoheadrightarrow} \cdot$$

Then, by the above, $\varinjlim$ preserves the first factor, and we claim that it also preserves trivial fibrations. Indeed, because the morphisms in I are between finitely presentable objects, any lifting problem

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\in I"'] & \varinjlim X_d \arrow[d, "\varinjlim f_d"] \\ \cdot \arrow[r] & \varinjlim Y_d \end{tikzcd} \quad \text{may be factored} \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d, "\in I"'] & X_d \arrow[r] \arrow[d, "f_d"] & \varinjlim X_d \arrow[d, "\varinjlim f_d"] \\ \cdot \arrow[r] & Y_d \arrow[r] & \varinjlim Y_d \end{tikzcd} \quad \text{for some } d \in \mathrm{D}.$$

Note that we use the fact that D is directed to find a common index d. It follows that the colimit is in $I^{\boxslash}$, *i.e.* a trivial fibration. To obtain the result, we again use the fact that the weak equivalences in a model category are precisely the morphisms that admit a factorisation into a trivial cofibration followed by a trivial fibration. □

2.6 Left-induced right semimodel structures

Another way to obtain a model structure on a category C, is by starting with a model structure on another category, connected to C by a pair of adjoint functors, and *transferring* it along one of the adjoints. In this section we will focus on transferring a model structure along a *left* adjoint, to obtain a *right* semimodel structure.

The following is another important lemma for locally finitely presentable categories, the proof of which falls beyond the scope of this thesis.

Lemma 2.6.1. *Let F : C $\longrightarrow$ D be a colimit preserving functor between locally finitely presentable categories and suppose that* (L, R) *is a cofibrantly generated weak factorisation system on* D*. Then*

$$(F^{-1}(\mathsf{L}), F^{-1}(\mathsf{L})^{\boxslash})$$

is a weak factorisation system on C*, cofibrantly generated by*

$$\{f : A \longrightarrow B \mid Ff \in \mathsf{L} \text{ and } A, B \text{ are finitely presentable}\}.$$

Proof. This is a special case of [MR14, Theorem 3.2]. □

In [Hes+17] the above lemma is used to prove a recognition theorem for left-induced model structures. We will not need this theorem, because in Chapter 3 we will establish a right semimodel structure directly. However, it is useful to state the following 'right-semi'-analogue of their definition of a left-induced model structure.

Definition 2.6.2 (cf. [Hes+17, 2.1.3]). Let $(\mathsf{Cof}, \mathsf{Fib}, \mathsf{Weq})$ be a model structure on a category C and suppose we are given an adjunction $L : \mathsf{D} \rightleftarrows \mathsf{C} : R$. Then, if it exists, the *left-induced right semimodel structure* on D is given by the weak factorisation systems

$$(L^{-1}(\mathsf{Cof}),\ L^{-1}(\mathsf{Cof})^{\boxslash}) \text{ and } (L^{-1}(\mathsf{TrivCof}),\ L^{-1}(\mathsf{TrivCof})^{\boxslash}),$$

together with the class $L^{-1}(\mathsf{Weq})$ of weak equivalences. ■

2.7 Leibniz product and exponential

This final section is concerned with one more tool for constructing model structures, now under the assumption that our category C is equipped with a tensor product $\otimes$ and a unit object $\top$ such that $(\mathsf{C}, \otimes, \top)$ is a symmetric closed monoidal category. Examples of course include every Cartesian closed category with the standard categorical product, but also **ssSet** with the geometric product. We denote the internal hom by $[-,-] : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \longrightarrow \mathsf{C}$.

Let $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ be morphisms. Note that for every object C, the square

$$\begin{tikzcd} \mathrm{Hom}(C \otimes B, X) \arrow[r, "\mathrm{Hom}(C \otimes f, X)"] \arrow[d, "\mathrm{Hom}(C \otimes B, g)"'] & \mathrm{Hom}(C \otimes A, X) \arrow[d, "\mathrm{Hom}(C \otimes A, g)"] \\ \mathrm{Hom}(C \otimes B, Y) \arrow[r, "\mathrm{Hom}(C \otimes f, Y)"'] & \mathrm{Hom}(C \otimes A, Y) \end{tikzcd}$$

commutes. By naturality of the tensor-internal hom adjunction, this square is isomorphic to the exponential transpose

$$\begin{tikzcd} \mathrm{Hom}(C, [B, X]) \arrow[r, "\mathrm{Hom}(C, [f, X])"] \arrow[d, "\mathrm{Hom}(C, [B, g])"'] & \mathrm{Hom}(C, [A, X]) \arrow[d, "\mathrm{Hom}(C, [A, g])"] \\ \mathrm{Hom}(C, [B, Y]) \arrow[r, "\mathrm{Hom}(C, [f, Y])"'] & \mathrm{Hom}(C, [A, Y]) \end{tikzcd}$$

The special case where $C = [B, X]$, and the naturality of $- \otimes -$ show respectively that

$$\begin{tikzcd} {[B, X]} \arrow[r, "{[f, X]}"] \arrow[d, "{[B, g]}"'] & {[A, X]} \arrow[d, "{[A, g]}"] \\ {[B, Y]} \arrow[r, "{[f, Y]}"'] & {[A, Y]} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes X \arrow[r, "A \otimes g"] \arrow[d, "f \otimes X"'] & A \otimes Y \arrow[d, "f \otimes Y"] \\ B \otimes X \arrow[r, "B \otimes g"'] & B \otimes Y \end{tikzcd}$$

commute.

Definition 2.7.1. Let $f : A \longrightarrow B$ and $g : X \longrightarrow Y$. We define:

- The *Leibniz product* $f \mathbin{\hat{\otimes}} g : A \otimes Y +_{A \otimes X} B \otimes X \longrightarrow B \otimes Y$ is the unique map

$$\begin{tikzcd}
A \otimes X \arrow[r, "A \otimes g"] \arrow[d, "f \otimes X"'] & A \otimes Y \arrow[d] \arrow[ddr, bend left, "f \otimes Y"] \arrow[dl, phantom, "\ulcorner"] & \\
B \otimes X \arrow[r] \arrow[drr, bend right, "B \otimes g"'] & A \otimes Y +_{A \otimes X} B \otimes X \arrow[dr, dashed, "f \mathbin{\hat{\otimes}} g"] & \\
 & & B \otimes Y
\end{tikzcd}$$

- The *Leibniz exponential* $f \mathbin{\widehat{\exp}} g : [B, X] \longrightarrow [A, X] \times_{[A,Y]} [B, Y]$ is the unique map

$$\begin{tikzcd}
{[B, X]} \arrow[drr, bend left, "{[f, X]}"] \arrow[dr, dashed, "f \mathbin{\widehat{\exp}} g"] \arrow[ddr, bend right, "{[B, g]}"'] & & \\
 & {[A, X] \times_{[A,Y]} [B, Y]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & {[A, X]} \arrow[d, "{[A, g]}"] \\
 & {[B, Y]} \arrow[r, "{[f, Y]}"'] & {[A, Y]}
\end{tikzcd}$$

These morphisms are related in the following way.

Theorem 2.7.2. *The operations* $\hat{\otimes}$ *and* $\widehat{\exp}$ *define bifunctors on the arrow category* $\mathrm{C}^{\rightarrow}$ *and give rise to an adjunction*

$$Hom_{\mathrm{C}^{\rightarrow}}(f \mathbin{\hat{\otimes}} g, h) \cong Hom_{\mathrm{C}^{\rightarrow}}(f, g \mathbin{\widehat{\exp}} h).$$

Proof. Let us begin by showing the bifunctoriality of $\hat{\otimes}$. It suffices to show that for each morphism g, we have a functor $- \mathbin{\hat{\otimes}} g : \mathrm{C}^{\rightarrow} \longrightarrow \mathrm{C}^{\rightarrow}$, because it follows from the symmetry of both the pushout and $\otimes$ that $- \mathbin{\hat{\otimes}} g \cong g \mathbin{\hat{\otimes}} -$. On arrows, we define

$$- \mathbin{\hat{\otimes}} g : \begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\
B \arrow[r, "b"'] & B'
\end{tikzcd} \longmapsto \begin{tikzcd}
A \otimes Y +_{A \otimes X} B \otimes X \arrow[r, "\tilde{a}"] \arrow[d, "f \mathbin{\hat{\otimes}} g"'] & A' \otimes Y +_{A' \otimes X} B' \otimes X \arrow[d, "f' \mathbin{\hat{\otimes}} g"] \\
B \otimes Y \arrow[r, "\tilde{b}"'] & B' \otimes Y
\end{tikzcd}$$

Here $\tilde{b} = b \otimes Y$ and $\tilde{a}$ is the unique map indicated in the following diagram

$$
\begin{tikzcd}
A \otimes X \arrow[r, "A \otimes g"] \arrow[d, "f \otimes X"'] & A \otimes Y \arrow[d] \arrow[ddrr, bend left, "a \otimes Y"] & & & \\
B \otimes X \arrow[r] \arrow[ddrr, bend right, "b \otimes X"'] & A \otimes Y +_{A \otimes X} B \otimes X \arrow[ddrr, dashed, "\tilde{a}"] & & & \\
& & A' \otimes X \arrow[r, "A' \otimes g"] \arrow[d, "f' \otimes X"'] & A' \otimes Y \arrow[d, "p_0"] \arrow[ddr, bend left, "f' \otimes Y"] & \\
& & B' \otimes X \arrow[r, "p_1"'] \arrow[drr, bend right, "B' \otimes g"'] & A' \otimes Y +_{A' \otimes X} B' \otimes X \arrow[dr, "f' \hat{\otimes} g"] & \\
& & & & B' \otimes Y
\end{tikzcd}
$$

which exists because

$$
\begin{aligned}
p_0(a \otimes Y)(A \otimes g) &= p_0(A' \otimes g)(a \otimes X) && \text{(Naturality of } - \otimes -\text{)} \\
&= p_1(f' \otimes X)(a \otimes X) = p_1(b \otimes X)(f \otimes X). && \text{(Pushout square; } f'a = bf\text{)}
\end{aligned}
$$

By uniqueness the resulting square is an arrow in $\mathsf{C}^{\to}$. We leave the verification of functoriality to the reader. Very similar arguments can be used to show that $- \hat{\exp}\, g$ and $f \hat{\exp} -$ are contra- and covariant functors, respectively.

For the adjunction, suppose we are given an element

$$
\begin{tikzcd}
A \otimes Y +_{A \otimes X} B \otimes X \arrow[r, "c"] \arrow[d, "f \hat{\otimes} g"'] & Z \arrow[d, "h"] \\
B \otimes Y \arrow[r, "d"'] & W
\end{tikzcd}
$$

of $\mathrm{Hom}_{\mathsf{C}^{\to}}(f \hat{\otimes} g, h)$. First note that we have, by the universal mapping property, that the morphism $c : A \otimes Y +_{A \otimes X} B \otimes X \longrightarrow Z$ corresponds uniquely to a commutative square

$$
\begin{tikzcd}
A \otimes X \arrow[r, "A \otimes g"] \arrow[d, "f \otimes X"'] \arrow[dr, phantom, "(1)"] & A \otimes Y \arrow[d, "z_0"] \\
B \otimes X \arrow[r, "z_1"'] & Z
\end{tikzcd}
\quad \text{which gives} \quad
\begin{tikzcd}
A \otimes Y \arrow[r, "z_0"] \arrow[d, "f \otimes Y"'] \arrow[dr, phantom, "(2)"] & Z \arrow[d, "h"] \\
B \otimes Y \arrow[r, "d"'] & W
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
B \otimes X \arrow[r, "z_1"] \arrow[d, "B \otimes g"'] \arrow[dr, phantom, "(3)"] & Z \arrow[d, "h"] \\
B \otimes Y \arrow[r, "d"'] & W
\end{tikzcd}
$$

These three commuting squares have corresponding adjoint transposes

$$
\begin{tikzcd}
A \arrow[r, "\overline{z_0}"] \arrow[d, "f"'] \arrow[dr, phantom, "(1')"] & {[Y, Z]} \arrow[d, "{[g, Z]}"] \\
B \arrow[r, "\overline{z_1}"'] & {[X, Z]}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "\overline{z_0}"] \arrow[d, "f"'] \arrow[dr, phantom, "(2')"] & {[Y, Z]} \arrow[d, "{[Y, h]}"] \\
B \arrow[r, "\overline{d}"'] & {[Y, W]}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
B \arrow[r, "\overline{z_1}"] \arrow[d, "\overline{d}"'] \arrow[dr, phantom, "(3')"] & {[X, Z]} \arrow[d, "{[X, h]}"] \\
{[Y, W]} \arrow[r, "{[g, W]}"'] & {[X, W]}
\end{tikzcd}
$$

Now by (3') there is a unique map $\tilde{d} : B \longrightarrow [X,Z]\times_{[X,W]}[Y,W]$ and (1') and (2') imply, by the universal mapping property, that the square

$$\begin{tikzcd} A \arrow[r, "\overline{z_0}"] \arrow[d, "f"'] & {[Y,Z]} \arrow[d, "g \hat{\exp} h"] \\ B \arrow[r, "b_1"'] & {[X,Z]\times_{[X,W]}[Y,W]} \end{tikzcd}$$

commutes, as required. To see that the function described above is not only injective but also surjective, note that any square of the above form determines corresponding squares of the forms (3'), (2') and, most importantly, (1'). The verification of naturality is left to the reader. □

The following standard definition describes model categories which are monoidal in a compatible way. We extend it to include right semimodel structures.

Definition 2.7.3. A (right semi)model model structure on a closed monoidal category $(\mathrm{C}, \otimes, \top)$ is *monoidal* if:

1. It satisfies the *Leibniz axiom*: for any two cofibrations f, g, the Leibniz product $f \hat{\otimes} g$ is a cofibration, which is trivial if either f or g is.
2. For every factorisation of the unique morphism from the initial object into the unit object
$$0 \longrightarrow C \longrightarrow \top,$$
with C cofibrant and $C \longrightarrow \top$ a weak equivalence, the induced morphism
$$C \otimes X \longrightarrow \top \otimes X \underset{\cong}{\longrightarrow} X,$$
is a weak equivalence. ■

It turns out that if every object is cofibrant, then the second condition of Definition 2.7.3 follows from the first.

Proposition 2.7.4. *Let* $(\mathrm{M}, \otimes, \top)$ *be a closed monoidal category equipped with a right semimodel structure such that every object is cofibrant. Then this structure is monoidal if it satisfies the Leibniz axiom.*

Proof. Because M is a *closed* monoidal category, we have for every two objects A, B,

$$\mathrm{Hom}_{\mathrm{M}}(0 \otimes A, B) \cong \mathrm{Hom}_{\mathrm{M}}(0, [B, A]).$$

It follows that $A \otimes 0 \cong 0$. Thus, for X cofibrant and $i : A \longrightarrow B$ a cofibration, the Leibniz product $i \hat{\otimes} *_X$ may be identified with $i \otimes X$. By the Leibniz axiom, the functor $- \otimes X$ preserves (trivial) cofibrations. In particular, it sends trivial cofibrations between cofibrant objects to weak equivalences, which by Ken Brown's Lemma ([JT08, Lem. 2.3.1.]) means that it sends weak equivalences between cofibrant objects to weak equivalences. Therefore, if $C \longrightarrow \top$ is a weak equivalence, then so is $C \otimes X \longrightarrow \top \otimes X$, which suffices. □

Example 2.7.5. Again $\mathbf{sSet}_Q$ provides an example: it is known to be a monoidal model category with respect to the monoidal structure $(\mathbf{sSet}, \times, \Delta[0])$, where $\times$ is the categorical product. Since every object in $\mathbf{sSet}_Q$ is cofibrant, this already follows from the fact that it satisfies the Leibniz axiom. ▲

We close this chapter by stating a useful lemma that is satisfied by model structures that verify the Leibniz axiom.

Lemma 2.7.6 ([JT08, Thm. 3.2.1]). *Let* $(\mathsf{Cof}, \mathsf{TrivFib})$ *and* $(\mathsf{TrivCof}, \mathsf{Fib})$ *be two weak factorisation systems on a monoidal category* $(\mathsf{C}, \otimes, \top)$ *satisfying the Leibniz axiom. Then, for* k *a cofibration and* p *a fibration, their Leibniz exponent* $k \mathbin{\hat{\exp}} p$ *is a fibration, which is trivial if either* k *or* p *is.*

Proof. Let $i : A \to B$ be a cofibration and consider the lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & {[Z,E]} \arrow[d, "k \mathbin{\hat{\exp}} p"] \\ B \arrow[r] & {[Y,E] \times_{[Y,X]} [Z,X]} \end{tikzcd}$$

By Theorem 2.7.2, this square is equivalent to the adjoint transpose

$$\begin{tikzcd} B \otimes Y +_{A \otimes Y} X \otimes Z \arrow[r] \arrow[d, "i \mathbin{\hat{\otimes}} k"'] & E \arrow[d, "p"] \\ B \otimes Z \arrow[r] & X \end{tikzcd}$$

But by the Leibniz axiom, the left arrow is a trivial cofibration whenever i or k is, which gives us our result. □

As a corollary, we have:

Corollary 2.7.7. *Under the hypotheses of the previous lemma, if* $p : E \to X$ *is a fibration, then so is* $[Z, p] : [Z, E] \to [Z, X]$. *In particular, if* X *is fibrant, then so is* $[Z, X]$.

Proof. Note that $[Z, p] = *_Z \mathbin{\hat{\exp}} p$. □

Chapter 3

Left-induced model structures on semisimplicial sets

In this chapter we investigate the possibility of imposing a (weakened) left-induced model structure on **ssSet**. Constructing model structure step-by-step gives intermediate points where classes of morphisms which are supposed to become equal, are not yet shown to be. For instance, we may find ourselves in a situation where we have specified the classes of cofibrations, weak equivalences, and trivial cofibrations, but have not yet shown that the morphisms which are both cofibrations and weak equivalences, are trivial cofibrations. Moreover, in a right semimodel structure, fibrations which are weak equivalences need not even be trivial. To distinguish between these classes, we introduce the following terminology: (co)fibrations which are also weak equivalences are called *acyclic* (co)fibrations.

In Section 3.1 we prove some negative results, showing among other things that there is no model structure in which the weak equivalences are precisely the geometric homotopy equivalences.

The rest of the chapter is devoted to showing that there *is* such a right semimodel structure. We begin in Section 3.2 by showing that the weak factorisation system $(\mathsf{Cof}, \mathsf{TrivFib})$ transfers to semisimplicial sets without difficulty. Choosing between possible classes of trivial cofibrations is harder, and in Section 3.3 we postpone that choice by first proving a general theorem that takes a suitable class of trivial cofibrations as argument. Given such a class, the theorem imposes a structure on **ssSet** that is almost a right semimodel structure, with the only thing left to show that every acyclic cofibration is trivial, instead of just those with fibrant codomain.

In Section 3.4 this theorem is applied to obtain a right semimodel structure $\mathbf{ssSet}_Q$. This structure will be shown to be left induced in the sense of Definition 2.6.2.

In Section 3.5 a couple of further properties of $\mathbf{ssSet}_Q$ are proved. Specifically, that it is Quillen equivalent to the right semimodel structure underlying $\mathbf{sSet}_Q$, that its fibrant objects and fibrations between fibrant objects are characterized by liftings against semisimplicial horn inclusions and that it has strictly more trivial cofibrations than the saturated class generated by the semisimplicial horn inclusions.

3.1 Negative results for a left-induced model structure

In this section we show that it is impossible to construct a left-induced model structure on semisimplicial sets. We will even see that there is no model structures in which only the weak equivalences are left-induced, in the sense that they are precisely the weak equivalences under the left adjoint $i_!$.

It is relatively easy to show that there is no model structure in which both the weak equivalences and the cofibrations are preserved by $i_!$.

Proposition 3.1.1. *There is no model structure on* **ssSet** *in which the cofibrations are monomorphisms and the weak equivalences are geometric homotopy equivalences.*

Proof. Suppose there is, then the morphism $\Delta_i[0] + \Delta_i[0] \longrightarrow \Delta_i[0]$ that sends two points to the same point admits a factorisation

$$\Delta_i[0] + \Delta_i[0] \rightarrowtail E \xrightarrow{\sim} \Delta_i[0]$$

as a cofibration followed by a weak equivalence. It follows that E contains only 0-simplices and thus that its geometric realisation is a discrete space. However, the weak equivalence implies that the geometric realisation of E is contractible, *i.e.* E is a single point, which contradicts the monicity of the cofibration. □

The proof of the following proposition is based on the MathOverflow answer [May].

Proposition 3.1.2. *There is no model structure on* **ssSet** *in which the weak equivalences are precisely the geometric homotopy equivalences.*

Proof. Suppose there is such model structure. We claim that every trivial fibration $p : E \xrightarrow{\text{triv}} X$ is an isomorphism. To show this, we will prove by induction on n that every $p_n : E_n \longrightarrow X_n$ is a bijection. Suppose it holds for all $k < n$ and let $x : \Delta_i[n] \longrightarrow X$ be an n-simplex of X. Consider the pullback

$$\begin{tikzcd} x^*E \arrow[r] \arrow[d, "x^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ \Delta_i[n] \arrow[r, "x"'] & X \end{tikzcd}$$

By preservation under pullback (given by the dual of Lemma 2.1.6), we have that x^*p is a trivial fibration. Thus x^*p is a weak equivalence and its geometric realisation

$$|x^*p| : |x^*E| \longrightarrow \Delta^n$$

is a homotopy equivalence. By the induction hypothesis we have for each $m < n$ that p_m, and thus $(x^*p)_m$, is an isomorphism. We separate two cases:

- $(x^*E)_n$ is empty. Then x^*E is isomorphic to $\delta\Delta_i[n]$, which means its geometric realisation is an $n-2$-sphere, hence not contractible.

- $(x^*E)_n$ has more than one element. Then $|x^*E|$ is a collection of $n-1$-spheres with common boundaries, which is also not contractible.

However, the geometric realisation of $\Delta_i[n]$ *is* contractible, for it is an $n-1$-ball. It follows that $(x^*E)_n$ has precisely one element, and thus $x^*E \cong \Delta_i[n]$. This gives us a unique $e : \Delta_i[n] \longrightarrow E$ such that $pe = x$, as desired.

We finish the argument by noting that, since every trivial fibration is an isomorphism, all maps are cofibrations. This would mean that the class of geometric homotopy equivalences is precisely the class of trivial cofibrations, and therefore closed under pushouts. Though, as implied by Lemma 2.2.10, this is true for pushouts along cofibrations, it is not true in general. Take for instance the following counterexample. Let A be the semisimplicial set from Section 1.2, *i.e.* A has precisely one 0-simplex and one 1-simplex. Moreover, let D be the extension of A that further contains precisely one 2-simplex. Consider the pushout

$$\begin{tikzcd} \Delta_i[1] \arrow[r] \arrow[d] & A \arrow[d] \\ D \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & D \end{tikzcd} \tag{3.1}$$

Note that all maps in (3.1) are uniquely determined. Under geometric realisation, the morphism $\Delta_i[1] \longrightarrow D$ is a continuous map between contractible spaces, and therefore a homotopy equivalence. In contrast, the geometric realisation of A is a circle, which is not contractible. □

Remark 3.1.3. Of course above propositions still leave room for a model structure in which the weak equivalences are a *subset* of the geometric homotopy equivalences. But for any such model structure, the argument in the proof of Proposition 3.1.2 can still be used to show that every trivial fibration is an isomorphism and, therefore, that every morphism is a cofibration. Such a model structure would be highly degenerated and not interesting. ▲

3.2 Semisimplicial cofibrations and trivial fibrations

In this section we will fix the weak factorisation system of cofibrations and trivial fibrations that will be used in the rest of this chapter.

Definition 3.2.1. For every $n \geq 0$, we define the following semisimplicial subsets of $\Delta_i[n]$.

- The *boundary* $\delta\Delta_i[n]$ of $\Delta_i[n]$ is obtained by removing the top face of $\Delta_i[n]$, *i.e.*

$$(\delta\Delta_i[n])_m = \begin{cases} \Delta_i[n]_m & \text{if } m < n; \\ \emptyset & \text{otherwise.} \end{cases} \boxtimes$$

- For $0 \leq k \leq n$, the *k-th horn* $\Lambda^k[n]$ of $\Delta_i[n]$ is the union of all but the k-th face of $\Delta_i[n]$:

$$(\Lambda_i^k[n])_m = \begin{cases} \Delta_i[n]_m & \text{if } m < n-1; \\ \boxtimes \bigcup \{\epsilon^i \mid i \neq k\} & \text{if } m = n-1; \\ \emptyset & \text{otherwise.} \end{cases}$$ ■

The following two classes of semisimplicial morphisms are called the *semisimplicial boundary inclusions* and *semisimplicial horn inclusions*:

$$\begin{aligned} \mathsf{I} &= \{i_n \,:\, \delta\Delta_i[n] \hookrightarrow \Delta_i[n] \mid n \geq 0\}; \\ \mathsf{J} &= \{j_n^k \,:\, \Lambda_i^k[n] \hookrightarrow \Delta_i[n] \mid n \geq 0,\ 0 \leq k \leq n\}. \end{aligned}$$

Note that the morphisms in both I and J are between objects that have only finitely many simplices. The category of simplices of such an object is finite, and thus the object can be written as a finite colimit of representables, which means, by Lemma 2.4.7, that it is finitely presentable. We use the small object argument on the set I to obtain a to obtain a weak factorisation system (Cof, TrivFib) of *cofibrations* and *trivial fibrations*.

We will give a characterisation of Cof later in this section, but first let us say something about the sets I and J under the functor $i_!$. The functor $i_!$ sends the semisimplicial boundary and horn inclusions to the simplicial morphisms that are standardly called the simplicial boundary and horn inclusions. By the definition of $i_!$, it is easy to see that these morphisms are also between finitely presentable objects and, as mentioned in Example 2.5.7, they cofibrantly generate two weak factorisations systems $(\mathsf{Cof}_\Delta, \mathsf{TrivFib}_\Delta)$ and $(\mathsf{TrivCof}_\Delta, \mathsf{Fib}_\Delta)$ that determine the model structure $\mathbf{sSet}_Q$. Note that it follows that $\mathbf{sSet}_Q$ verifies the hypotheses of Proposition 2.5.9.

The proof of the following proposition is based on an argument in [JT08, p. 38] for simplicial sets.

Proposition 3.2.2. *The class* Cof *of cofibrations is precisely the class of monomorphisms.*

Proof. In this proof we will use the functors $\mathrm{tr}^n \,:\, \mathbf{ssSet} \longrightarrow \mathbf{ssSet}$. For each $n \geq -1$, it is given by

$$\mathrm{tr}^n(X)_m = \begin{cases} X_m & \text{if } m \leq n; \\ \varnothing & \text{otherwise.} \end{cases} ⊠$$

Let $m \,:\, A \longrightarrow X$ be an arbitrary monomorphism and assume without loss of generality that m is an inclusion. Let x be an n-simplex of $X - A$, then the square

$$\begin{tikzcd} \delta\Delta_i[n] \arrow[r] \arrow[d] & \mathrm{tr}^{n-1}(X) \cup A \arrow[d] \\ \Delta_i[n] \arrow[r, "x"'] & \mathrm{tr}^n(X) \cup A \end{tikzcd}$$

commutes. Summing over all n-simplices in $X - A$ yields the square

$$\begin{tikzcd} \coprod_{x\in(X-A)_n} \delta\Delta_i[n] \arrow[r] \arrow[d] & \mathrm{tr}^{n-1}(X) \cup A \arrow[d] \\ \coprod_{x\in(X-A)_n} \Delta_i[n] \arrow[r, "x"'] & \mathrm{tr}^n(X) \cup A \end{tikzcd} \tag{3.2}$$

of which the left side is of course in Cof. We will show, by considering each dimension m, that the above square is a pushout square. For $m < n$, both the left and the right sides are isomorphisms and a simple diagram chase shows that $(\mathrm{tr}^n(X) \cup A)_m$ is indeed a pushout. At the level $m = n$, we are looking for a pushout from the initial object, *i.e.* a coproduct. And indeed,

$$\coprod_{x \in (X-A)_n} (\Delta_i[n])_n + (\mathrm{tr}^{n-1}(X)_n \cup A_n) \cong \coprod_{x \in (X-A)_n} \{*\} + A_n \cong (X-A)_n + A_n \cong X_n \cong (\mathrm{tr}^n(X))_n \cup A_n.$$

Finally, at the level $m > n$ the objects on the left are both initial, and the objects on the right are equal, which also gives us a pushout square.

It follows that the ω-composition of the right side of 3.2 for all $n \geq 0$, *i.e.*

$$A \cong \mathrm{tr}^{-1}(X) \cup A \longrightarrow \varinjlim_{n \geq -1} \mathrm{tr}^n(X) \cup A \cong X$$

is in Cof. The result follows from Corollary 2.5.3 and the fact that saturation preserves monomorphisms. □

By Lemma 1.3.14, we have:

Corollary 3.2.3. *The functor* $i_!$: $\mathbf{ssSet} \longrightarrow \mathbf{sSet}_Q$ *preserves and reflects cofibrations.*

It follows that (Cof, TrivFib) is what in [Hes+17] is called the left-induced weak factorisation system of $(\mathrm{Cof}_\Delta, \mathrm{TrivFib}_\Delta)$.

3.3 A structure generating theorem

In this section we prove a general theorem that, for a given candidate weak factorisation system of trivial cofibrations and fibrations, provides a structure on **ssSet** that is almost a right semimodel structure. The proof relies on some subtheorems, most of which are inspired by the treatment in [JT08] of simplicial sets. If some result can directly be copied over to the semisimplicial case, we will refer to the corresponding statement in [JT08] (but for the sake of completeness we will still include the proof). If the semisimplicial case requires a different approach, then that will also be made explicit.

Consider the set

$$\mathsf{I}_\otimes := \{\epsilon^k \hat{\otimes} i_n \mid k \in \{0,1\},\ i_n \in \mathsf{I}\},$$

where

$$\epsilon^k \hat{\otimes} i_n : \Delta_i[0] \otimes \Delta_i[n] +_{\Delta_i[0] \otimes \delta\Delta_i[n]} \Delta_i[1] \otimes \delta\Delta_i[n] \longrightarrow \Delta_i[1] \otimes \Delta_i[n]$$

is the Leibniz product of the endpoint inclusion ϵ^k and the boundary inclusion i_n. This section is devoted to proving the following theorem.

Theorem 3.3.1. *Let* (TrivCof, Fib) *be a weak factorisation system on* **ssSet** *that satisfies the Leibniz axiom with respect to* (Cof, TrivFib) *and such that* $\mathsf{I}_\otimes \subseteq \mathrm{TrivCof} \subseteq i_!^{-1}(\mathrm{TrivCof}_\Delta)$*, then the class* Weq *as defined in Proposition 2.3.3 satisfies* 2*-out-of-*3*, and we have*

(A1) $\mathrm{TrivCof} \subseteq \mathrm{Cof} \cap \mathrm{Weq}$ (B1') $\mathrm{TrivFib}_\bullet \subseteq \mathrm{Fib}_\bullet \cap \mathrm{Weq}_\bullet$

(A2') $\mathrm{Cof}_\bullet \cap \mathrm{Weq}_\bullet \subseteq \mathrm{TrivCof}_\bullet$ (B2') $\mathrm{Fib}_\bullet \cap \mathrm{Weq}_\bullet \subseteq \mathrm{TrivFib}_\bullet$

Moreover, the weak equivalences are preserved by $i_!$.

Note that then the only thing still needed for the two weak factorisation systems to define a right semimodel structure, is that the inclusion (A2') should hold not only for the acyclic cofibrations with fibrant codomain, but for *all* acyclic cofibrations, *i.e.* it should become inclusion (A2) of Proposition 2.3.2.

For the remainder of this section, let (TrivCof, Fib) be a weak factorisation system of *trivial cofibrations* and *fibrations* as in the hypothesis of Theorem 3.3.1.

The following lemma strengthens the lower bound on the class of trivial cofibrations to include the Leibniz product of the endpoint inclusions with any monomorphism.

Lemma 3.3.2 ([JT08, Thm. 3.2.3]). *If i is a cofibration, then $\epsilon^k \hat{\otimes} i$ is a trivial cofibration.*

Proof. Let $p : E \to X$ be a fibration. We wish to show that every square of the form

$$\begin{tikzcd}
(\Delta_i[0] \otimes B) +_{\Delta_i[0]\otimes A} (\Delta_i[1] \otimes A) \arrow[r] \arrow[d, "\epsilon^k \hat{\otimes} m"'] & E \arrow[d, "p"] \\
\Delta_i[1] \otimes B \arrow[r] & X
\end{tikzcd} \tag{3.3}$$

with $i : A \to B$ a monomorphism, has a diagonal filler. By adjointess, this is the same as solving the lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & {[\Delta_i[1], E]} \arrow[d, "\epsilon^k \,\hat{\exp}\, p"] \\
B \arrow[r] & {[\Delta_i[0], E] \times_{[\Delta_i[0], X]} [\Delta_i[1], X]}
\end{tikzcd} \tag{3.4}$$

First suppose that i is of the form i_n, *i.e.* a boundary inclusion. Then, since $\mathsf{I}_\otimes \subseteq \mathsf{TrivCof}$, the square (3.3) has a lifting and, by adjointness, so does (3.4). Having the left lifting property is preserved under saturation and thus, by Proposition 3.2.2, it follows that the square (3.4) has a lifting for every monomorphism. Again by adjointness, the same holds for square (3.3), as required. □

3.3.1 Connectedness

We will now start to develop the homotopy theory of semisimplicial sets relative to the given weak factorisation system. In analogy with the topological spaces, a connected object will be one in which the set of connected components is the terminal object. We write $\{*\}$ for the semisimplicial set $\Delta_i[0]$, which may be though of as a point, and I for the the semisimplicial set $\Delta_i[1]$, which one should think of as a line segment.

Definition 3.3.3. The set $\pi_0(X)$ of *connected components* of a semisimplicial set X is the following coequaliser in *Set*

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & X_0 \arrow[r, "q"] & \pi_0(X).
\end{tikzcd}$$

If $\pi_0(X) = 1$, we say that X is *connected*. ■

Ideally, we would like $\pi_0(X)$ to be the quotient of the class of arrows $\{*\} \longrightarrow X$ by the relation "there is a path from x to y", or more formally: there is a morphism $p : I \longrightarrow X$ such that $p\epsilon^1 = x$ and $p\epsilon^0 = y$. However, if $q(x) = q(y)$ then we merely know that there is a sequence of paths

$$x \longrightarrow u_0 \longleftarrow u_1 \dots \longrightarrow u_n \longleftarrow y.$$

The following proposition shows that, if X is fibrant, the relation "there is a path from x to y" becomes an equivalence relation, and consequently $q(x) = q(y)$ implies that there is a path $x \longrightarrow y$.

The proof is based on [JT08, p. 44], but requires a different treatment for two reasons. First, we obtain diagonal fillers by the lifting property against the set $\mathsf{I}_\otimes$, instead of the boundary inclusions. Secondly, reflexivity is trivial in the simplicial case, since for every 0-simplex x, the degenerate 1-simplex $x\eta$ is a path $x \sim x$.

Proposition 3.3.4. *If X is fibrant, then the relation $x \sim y$ on the arrows $\{*\} \longrightarrow X$, given by*

$$x \sim y \textit{ iff there is a } p : I \longrightarrow X \textit{ such that } p\epsilon^1 = x \textit{ and } p\epsilon^0 = y,$$

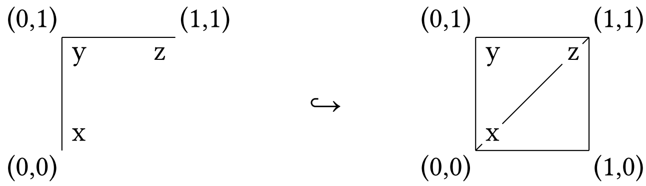
is an equivalence relation.

Proof. We write $p : x \sim y$ for a map $p : I \longrightarrow X$ such that $p\epsilon^1 = x$ and $p\epsilon^0 = y$, and verify the conditions one-by-one.

Transitivity. Suppose that $p : x \sim y$ and $q : y \sim z$. We take the filler

$$\begin{tikzcd}
(\{*\} \otimes I) +_{\{*\}\otimes\{*\}} (I \otimes \{*\}) \arrow[r, "{[p,q]}"] \arrow[d, "\epsilon^1 \hat{\otimes} \epsilon^0"'] & X \\
I \otimes I \arrow[ur, dashed, "f"']
\end{tikzcd}$$

Geometrically, this filling can be pictured as



Let $\alpha : [1] \longrightarrow [1] \times [1]$ be the morphism in Pos_i that sends 0 to $(0,0)$ and 1 to $(1,1)$. Then we have $fN_i(\alpha)\epsilon^1 = p\epsilon^1 = x$ and $fN_i(\alpha)\epsilon^0 = q\epsilon^0 = z$, and thus $fN_i(\alpha) : x \sim z$, as required.

Reflexivity. Let $x : \{*\} \longrightarrow X$ be a 0-simplex of X. Note that $i_0 : 0 \longrightarrow \{*\}$ is the unique morphism from the initial object into $\{*\}$. Since X is fibrant, we have a diagonal filler

$$\begin{tikzcd}
\{*\} \arrow[r, "x"] \arrow[d, "\epsilon^0 \hat{\otimes} i_0"'] & X \\
I \arrow[ur, dashed, "p"']
\end{tikzcd}$$

such that $p : y \sim x$. By the same argument, there is a point z with a path $q : z \sim y$ and, by transitivity, a path $r : z \sim x$.

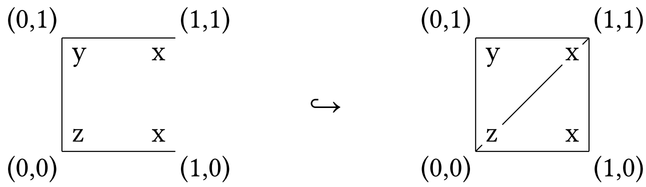
Now consider the monomorphism $[\epsilon^0, \epsilon^1] : \{*\} + \{*\} \longrightarrow I$ and the Leibniz product

$$\epsilon^1 \hat{\otimes} [\epsilon^1, \epsilon^0] : (\{*\} \otimes I) +_{\{*\}\otimes(\{*\}+\{*\})} (I \otimes (\{*\} + \{*\})) \longrightarrow I \otimes I.$$

Take the diagonal filler

$$\begin{tikzcd}
(\{*\} \otimes I) +_{\{*\}\otimes(\{*\}+\{*\})} (I \otimes (\{*\} + \{*\})) \arrow[r, "{[q,[r,p]]}"] \arrow[d, "{\epsilon^1 \hat{\otimes} [\epsilon^1, \epsilon^0]}"'] & X \\
I \otimes I \arrow[ur, dashed] &
\end{tikzcd}$$

which we again picture geometrically:



This gives us a path $s : x \sim x$.

Symmetry. Finally, suppose that $p : x \sim y$. By reflexivity, there is a path $id_x : x \sim x$. A path $p^{-1} : y \sim x$ can be obtained from the diagonal filler indicated in the following diagram.

$$\begin{tikzcd}
(\{*\} \otimes I) +_{\{*\}\otimes(\{*\}+\{*\})} (I \otimes (\{*\} + \{*\})) \arrow[r, "{[id_x,[p,id_x]]}"] \arrow[d, "{\epsilon^1 \hat{\otimes} [\epsilon^1, \epsilon^0]}"'] & X \\
I \otimes I \arrow[ur, dashed, "q"'] &
\end{tikzcd}$$

□

3.3.2 Homotopy

In this section we further develop the homotopy theory of the given structure on **ssSet**.

Definition 3.3.5. Let $f, g : X \longrightarrow Y$ be semisimplicial maps. Then $h : X \otimes I \longrightarrow Y$ is a *homotopy* from f to g if the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "X \otimes \epsilon^1"] \arrow[dr, "f"'] & X \otimes I \arrow[d, "h"] & X \arrow[l, "X \otimes \epsilon^0"'] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

We say that f and g are *homotopic* and write $h : f \sim g$. ■

Given a homotopy $h : X \otimes I \longrightarrow Y$, we write h_k for the restriction $h(X \otimes \epsilon^k) : X \longrightarrow Y$ for $k \in \{0, 1\}$.

Proposition 3.3.6 ([JT08, p. 45]). *On the class of morphisms $f : X \longrightarrow K$ with fibrant codomain, homotopy is an equivalence relation.*

Proof. By adjunction, a homotopy $h : X \otimes I \longrightarrow K$ is equivalent to a path $\overline{h} : I \longrightarrow [X, K]$. The result follows from the fact that, by Corollary 2.7.7, the object $[X, K]$ is fibrant and thus, by Proposition 3.3.4, path-connectedness on $[X, K]$ is an equivalence relation. □

Definition 3.3.7. A morphism $f : X \longrightarrow Y$ is called a *homotopy equivalence* if there is a map $g : Y \longrightarrow X$ together with homotopies $h : X \otimes I \longrightarrow X$ and $h' : Y \otimes I \longrightarrow Y$ such that $h : gf \sim id_X$ and $h' : fg \sim id_Y$. ■

The following lemma records some properties of homotopy (equivalence) on morphisms between fibrant objects that will be useful later on.

Lemma 3.3.8. *For morphisms between fibrant objects:*
(i) Homotopy is stable under composition.
(ii) Homotopy equivalence is stable under homotopy.
(iii) Homotopy equivalences are closed under composition.

Proof. (i) Let $f, g : X \longrightarrow Y$ and $p, q : Y \longrightarrow Z$. Given $h : f \sim g$ and $k : p \sim q$, we have $ph : pf \sim pg$ and (by naturality) also $k(I \otimes g) : pg \sim qg$. Transitivity gives $pf \sim qg$.

(ii) Let f be a homotopy equivalence with homotopy inverse f^{-1}, and let $g \sim f$. By part (i), we have $gf^{-1} \sim ff^{-1} \sim id$ and dually for $f^{-1}g$.

(iii). Let f and g be homotopy equivalences with homotopy inverses f^{-1} and g^{-1}. Then, using part (i) and the fact that homotopy is an equivalence relation, we find

$$f^{-1}g^{-1}gf \sim f^{-1}f \sim id.$$

It can be dually shown that $gff^{-1}g^{-1} \sim id$. □

In **sSet** the above homotopic notions are defined in the same way, but with $\Delta[0]$ and $\Delta[1]$ instead of $\Delta_i[0]$ and $\Delta_i[1]$ as $\{*\}$ and I, and with the categorical product instead of $\otimes$. It can be easily seen that, since $i_!$ preserves all three, homotopies and homotopy equivalences are preserved by $i_!$.

Definition 3.3.9. A homotopy equivalence f is called a *strong* if there are homotopies h and h' witnessing that f is a homotopy equivalence such that the diagram

$$\begin{tikzcd} X \otimes I \arrow[r, "f \otimes I"] \arrow[d, "h"'] & Y \otimes I \arrow[d, "h'"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. ■

The proof of the next proposition is based on the proofs of propositions 3.2.5 and 3.2.6 of [JT08], but with some required adaptations because, unlike in the simplicial case, we do not have projections $K \otimes I \longrightarrow K$ and $K \otimes I \longrightarrow I$.

Proposition 3.3.10. *For p a fibration with fibrant codomain, the following are equivalent:*
(i) p is a homotopy equivalence;
(ii) p is a strong homotopy equivalence;
(iii) p is a trivial fibration.

Proof. Let $p : E \longrightarrow K$ be a fibration with a fibrant codomain. (i) $\Longrightarrow$ (ii). Let $s : K \longrightarrow E$, $h : sp \sim id_E$ and $h' : ps \sim id_K$ be the data that witness that p is a homotopy equivalence. Then the filler of the diagram

$$\begin{tikzcd} K \otimes \{*\} \arrow[r, "s"] \arrow[d, "*_K \hat{\otimes} \epsilon^1"'] & E \arrow[d, "p"] \\ K \otimes I \arrow[r, "h'"'] \arrow[ur, dashed, "t"] & K \end{tikzcd}$$

is such that $pt_0 = h'_0 = id_K$ and $t_1 = s$. By Lemma 3.3.8, we have $t_0 p \sim t_1 p \sim sp \sim id_E$, witnessed by, say $k : t_0 p \sim id_E$. Furthermore, by reflexivity, there is a homotopy $k' : pt_0 \sim id_K$ and so the fact that p is a homotopy equivalence is additionally witnessed by the section t_0 and the homotopies k, k'. We will show that there is a $k^* : t_0 p \sim id_E$ giving the required commutativity.

Let $\alpha : I \otimes I \longrightarrow [E, K]$ be the diagonal filler

$$\begin{tikzcd} (\{*\} \otimes I) +_{\{*\} \otimes (\{*\}+\{*\})} (I \otimes (\{*\} + \{*\})) \arrow[r, "{[\overline{k'(p \otimes I)}, [\overline{pk}, \overline{pk}]]}"] \arrow[d, "{\epsilon^0 \hat{\otimes} [\epsilon^1, \epsilon^0]}"'] & {[E, K]} \\ I \otimes I \arrow[ur, dashed, "\alpha"'] & \end{tikzcd}$$

Then the diagram

$$\begin{tikzcd} (\{*\} \otimes I) +_{\{*\} \otimes (\{*\}+\{*\})} (I \otimes (\{*\} + \{*\})) \arrow[r, "{[[E, t_0]\alpha(\epsilon^1 \otimes I), [\overline{t_0 pk}, \overline{k}]]}"] \arrow[d, "{\epsilon^1 \hat{\otimes} [\epsilon^1, \epsilon^0]}"'] & {[E, E]} \arrow[d, "{[E, p]}"] \\ I \otimes I \arrow[r, "\alpha"'] \arrow[ur, dashed, "\beta"'] & {[E, K]} \end{tikzcd}$$

commutes. Let $\beta : I \otimes I \longrightarrow E^E$ be the induced filler and consider the homotopy $k^* = \overline{\beta(\epsilon^0 \otimes I)}$. We have $k^*_1 = (t_0 pk)_0 = t_0 pt_0 p = t_0 p$ and $k^*_0 = k_0 = id_E$, as well as $pk^* = k'(I \otimes p)$, as required.

(ii) $\Longrightarrow$ (iii). Let $h : sp \sim id_E$ and $h' : ps \sim id_K$ be the homotopies that witness that p is a strong homotopy equivalence. Suppose then that we are given a lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \arrow[r, "b"'] & K \end{tikzcd}$$

with $i : A \longrightarrow B$ a cofibration. It follows that both the squares

$$\begin{tikzcd} A \otimes I \arrow[r, "h(a \otimes I)"] \arrow[d, "i \otimes I"'] & E \arrow[d, "p"] \\ B \otimes I \arrow[r, "h'(b \otimes I)"'] & K \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \otimes \{*\} \arrow[r, "sb"] \arrow[d, "B \otimes \epsilon^1"'] & E \arrow[d, "p"] \\ B \otimes I \arrow[r, "h'(b \otimes I)"'] & K \end{tikzcd}$$

commute. Combining the two squares, we get a commuting square

$$\begin{tikzcd} (A \otimes I) +_{A \otimes \{*\}} (B \otimes \{*\}) \arrow[r] \arrow[d, "i \hat{\otimes} \epsilon^1"'] & E \arrow[d, "p"] \\ B \otimes I \arrow[r, "h'(b \otimes I)"'] \arrow[ur, dashed, "\overline{h}"] & K, \end{tikzcd}$$

which has a lifting $\overline{h} : B \otimes I \longrightarrow E$ such that $p\overline{h} = h'(b \otimes I)$ and $\overline{h}(i \otimes I) = h(a \otimes I)$. But then $p\overline{h}_0 = b$ and $\overline{h}_0 i = a$, which means that $\overline{h}_1$ provides the desired lifting.

(iii) $\Rightarrow$ (i). Suppose that p is trivial. Then it has a section $s : K \longrightarrow E$ obtained as the diagonal filler

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ K \arrow[r, equal] \arrow[ur, dashed, "s"] & K \end{tikzcd}$$

Let $h : ps \sim id_K$, a suitable homotopy $h' : sp \sim id_E$ arises as the diagonal filler

$$\begin{tikzcd} (E \otimes \{*\}) + (E \otimes \{*\}) \arrow[r, "{[sp, id_E]}"] \arrow[d, "{[\epsilon^1, \epsilon^0]}"'] & E \arrow[d, "p"] \\ E \otimes I \arrow[r, "h(p \otimes I)"'] \arrow[ur, dashed, "h'"] & K \end{tikzcd}$$

□

Proposition 3.3.11 ([JT08, Prop. 3.2.3]). *A trivial cofibration between fibrant objects is a strong homotopy equivalence.*

Proof. Let $i : A \longrightarrow B$ be a trivial cofibration between fibrant objects. We obtain a retract r of i as diagonal filler in the following diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "i"'] & A \\ B \arrow[ur, dashed, "r"'] & \end{tikzcd}$$

Let $h : ri \sim ri$, then a homotopy $ir \sim id_B$ is obtained as the indicated filler

$$\begin{tikzcd} (A \otimes I) +_{A \otimes (\{*\}+\{*\})} (B \otimes (\{*\} + \{*\})) \arrow[r, "{[ih, [ir, id_B]]}"] \arrow[d, "{i \hat{\otimes} [\epsilon^1, \epsilon^0]}"'] & B \\ B \otimes I \arrow[ur, dashed] & \end{tikzcd}$$

□

3.3.3 Weak equivalences

In this section we will use the following often-used (*e.g.* on p. 59 of [JT08]) characterisation for the class of weak equivalences.

Proposition 3.3.12. *A map f is a weak equivalence if and only if for any two trivial cofibrations x and y with fibrant codomains, any filler $\overline{f}$ of the diagram*

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] & \overline{X} \arrow[d, dashed, "\overline{f}"] \\ Y \arrow[r, "y"'] & \overline{Y} \end{tikzcd} \tag{3.5}$$

is a homotopy equivalence.

Proof. $\Rightarrow$. It suffices to show that, of a morphism between fibrant objects, a factorisation

$$\overline{X} \overset{\text{triv}}{\rightarrowtail} E \overset{\text{triv}}{\twoheadrightarrow} \overline{Y} \tag{3.6}$$

into a trivial cofibration followed by a trivial fibration, is a homotopy equivalence. To see that it is, note that, since $\overline{Y}$ is fibrant, we have that E is as well. Thus, by propositions 3.3.10 and 3.3.11, both morphisms in (3.6) are homotopy equivalences. The result follows from Lemma 3.3.8.

$\Leftarrow$. Given a square of the form (3.5), we must show that $\overline{f}$ can be factored as a trivial cofibration followed by a trivial fibration. To this end, factor $\overline{f}$ using the weak factorisation system (TrivCof, Fib) to obtain a diagram

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] & \overline{X} \arrow[r, "i"] \arrow[d, "\overline{f}"'] & E \arrow[dl, "p"] \\ Y \arrow[r, "y"'] & \overline{Y} & \end{tikzcd}$$

Because i is a trivial cofibration, the composition ix is as well. Thus, by the hypothesis, p is a homotopy equivalence between fibrant objects, which means that, by Proposition 3.3.10, it is a trivial fibration, as required. □

Note that it follows immediately that a weak equivalence between fibrant objects is a homotopy equivalence: simply let x and y be id_X and id_Y, respectively. We have the following useful lemma about diagrams of this form.

Lemma 3.3.13. *Any two diagonal fillers $\overline{f}_0, \overline{f}_1$ in a square of the form (3.5) are homotopic.*

Proof. Let $h : yf \sim yf$. The required homotopy is obtained as the following diagonal filler.

$$\begin{tikzcd} (X \otimes I) +_{X \otimes (\{*\}+\{*\})} (\overline{X} \otimes (\{*\} + \{*\})) \arrow[rr, "{[h, [\overline{f}_0, \overline{f}_1]]}"] \arrow[d, "{x \hat{\otimes} [\epsilon^1, \epsilon^0]}"'] & & \overline{Y} \\ \overline{X} \otimes I \arrow[urr, dashed, "k"'] & & \end{tikzcd}$$

□

By part (ii) of Lemma 3.3.8, this means that to show that a map is a weak equivalence, it suffices to check the property for a single diagonal filler per square. The following proposition implies that it even suffices to verify a single such square.

Proposition 3.3.14. *In the following diagram with $\overline{X}, \widetilde{X}, \overline{Y}, \widetilde{Y}$ all fibrant replacements,*

$$\begin{tikzcd} \widetilde{X} \arrow[d, dashed, "\widetilde{f}"'] & X \arrow[l] \arrow[r] \arrow[d, "f"] & \overline{X} \arrow[d, dashed, "\overline{f}"] \\ \widetilde{Y} & Y \arrow[l] \arrow[r] & \overline{Y} \end{tikzcd}$$

the filler $\widetilde{f}$ is a homotopy equivalence whenever $\overline{f}$ is.

Proof. We have the following two diagonal fillers:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widetilde{X} \\ \overline{X} \arrow[ur, dashed, "u"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r] \arrow[d] & \overline{X} \\ \widetilde{X} \arrow[ur, dashed, "u^{-1}"'] & \end{tikzcd} \quad \text{which induce squares} \quad \begin{tikzcd} X \arrow[r] \arrow[d] & \overline{X} \arrow[d, bend right, "u^{-1}u"'] \arrow[d, bend left, "id"] \\ X \arrow[r] & \overline{X} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, bend right, "uu^{-1}"'] \arrow[d, bend left, "id"] \\ X \arrow[r] & \widetilde{X} \end{tikzcd}$$

By Lemma 3.3.13, the morphisms u, u^{-1} form a homotopy equivalence. Using the same method, we can find $v : \overline{Y} \longrightarrow \widetilde{Y}$ and a homotopy inverse $v^{-1} : \widetilde{Y} \longrightarrow \overline{Y}$. It follows that the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "v\overline{f}u^{-1}"] \\ Y \arrow[r] & \widetilde{Y} \end{tikzcd}$$

commutes and thus, again by Lemma 3.3.13, we have $\widetilde{f} \sim v\overline{f}u^{-1}$. We claim that $u\overline{f}^{-1}v^{-1}$ is a homotopy inverse of $\widetilde{f}$, where $\overline{f}^{-1}$ is the homotopy inverse of $\overline{f}$. Indeed, using Lemma 3.3.8,

$$\begin{aligned} u\overline{f}^{-1}v^{-1}\widetilde{f} &\sim u\overline{f}^{-1}v^{-1}v\overline{f}u^{-1} \sim id_{\widetilde{X}}, \quad \text{and} \\ \widetilde{f}u\overline{f}^{-1}v^{-1} &\sim v\overline{f}u^{-1}u\overline{f}^{-1}v^{-1} \sim id_{\widetilde{Y}}, \end{aligned}$$

as required. □

Proposition 3.3.15. *The weak equivalences satisfy 2-out-of-3.*

Proof. gf, f w.e $\Rightarrow g$ w.e. Suppose we are given a diagram of the form

$$\begin{tikzcd} Y \arrow[r, "y"] \arrow[d, "g"'] & \overline{Y} \arrow[d, "\overline{g}"] \\ Z \arrow[r, "z"'] & \overline{Z} \end{tikzcd}$$

By factoring $X \overset{\text{triv}}{\rightarrowtail} \overline{X} \twoheadrightarrow 1$, we can extend the above diagram as follows

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "f"'] & \overline{X} \arrow[d, dashed, "\overline{f}"] \\
Y \arrow[r, "y"] \arrow[d, "g"'] & \overline{Y} \arrow[d, "\overline{g}"] \\
Z \arrow[r, "z"'] & \overline{Z}
\end{tikzcd}$$

Now by the hypothesis, there are morphisms $u : \overline{Z} \to \overline{X}$ and $v : \overline{Y} \to \overline{X}$ such that

$$u\overline{g}\overline{f} \sim id_{\overline{X}} \text{ and } v\overline{f} \sim id_{\overline{X}}, \text{ and}$$
$$\overline{g}\overline{f}u \sim id_{\overline{Z}} \text{ and } \overline{f}v \sim id_{\overline{Y}}.$$

Thus we have $\overline{g}\overline{f}u \sim id_Z$ and, using Lemma 3.3.8,

$$\overline{f}u\overline{g} \sim \overline{f}u\overline{g}\overline{f}v \sim \overline{f}v \sim id_{\overline{Y}},$$

as desired.

We finish with f w.e., g w.e. $\Rightarrow gf$ w.e., since the other case is dual to the previous one. Given a square

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "f"'] & \overline{X} \arrow[dd, "\overline{gf}"] \\
Y \arrow[d, "g"'] & \\
Z \arrow[r, "z"'] & \overline{Z}
\end{tikzcd}$$

we take a fibrant replacement of Y to get

$$\begin{tikzcd}
X \arrow[rr, "x"] \arrow[d, "f"'] & & \overline{X} \arrow[dl, "\overline{f}"] \arrow[dd, "\overline{gf}"] \\
Y \arrow[r, "y"] \arrow[d, "g"'] & \overline{Y} \arrow[dr, "\overline{g}"] & \\
Z \arrow[rr, "z"'] & & \overline{Z}
\end{tikzcd}$$

and by Lemma 3.3.8 $\overline{g}\overline{f}$ is a homotopy equivalence. The result follows from Lemma 3.3.13. $\square$

Lemma 3.3.16. *Any homotopy equivalence $f : X \to Y$ is a weak equivalence.*

Proof. Suppose we are given $g : Y \to X$ such $h : gf \sim id_X$ and $h' : fg \sim id_Y$, together with a square of the form of 3.5. This gives us another filler as indicated in the diagram below.

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "f"] & \overline{X} \arrow[d, "\overline{f}"'] \\
Y \arrow[r, "y"'] \arrow[u, bend left, "g"] & \overline{Y} \arrow[u, dashed, bend right, "\overline{g}"']
\end{tikzcd}$$

We claim that this filler $\overline{g}$ is a homotopy inverse of $\overline{f}$. Indeed, we obtain $k : \overline{g}\overline{f} \sim id_{\overline{X}}$ from

$$\begin{tikzcd}
(X \otimes I) +_{X \otimes (\{*\}+\{*\})} (\overline{X} \otimes (\{*\}+\{*\})) \arrow[rr, two heads, "{[xh, [\overline{g}\overline{f}, id_{\overline{X}}]]}"] \arrow[d, "x \hat{\otimes} [\epsilon^1, \epsilon^0]"'] & & \overline{X} \\
\overline{X} \otimes I \arrow[urr, dashed, "k"'] & &
\end{tikzcd}$$

and $k' : \overline{f}\overline{g} \sim id_{\overline{Y}}$ can be found analogously. □

3.3.4 Proving the theorem

The proof of the following proposition is a standard trick.

Proposition 3.3.17. *Every trivial cofibration is acyclic.*

Proof. Let $A \overset{\text{triv}}{\rightarrowtail} B$ be a trivial cofibration and take a fibrant replacement $\overline{A}$ of A. Then of the resulting two morphisms take the pushout B'. Finally take a fibrant replacement $\overline{B}$ of B', to get the following diagram.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \overline{A} \arrow[d] \arrow[dr, dashed] & \\
B \arrow[r] \arrow[ur, phantom, "\ulcorner"] & B' \arrow[r] & \overline{B}
\end{tikzcd}$$

Since trivial cofibrations are preserved under pushout and composition, the morphism $\overline{A} \longrightarrow \overline{B}$ is a trivial cofibration between fibrant objects and thus, by Proposition 3.3.11 a homotopy equivalence. □

The next proposition follows readily from the results obtained so far.

Proposition 3.3.18. *A fibration with fibrant codomain is trivial if and only if it is acyclic.*

Proof. Let p be a fibration.

$\Rightarrow$. If p is trivial, then, by Proposition 3.3.10, p is a homotopy equivalence, which, by Lemma 3.3.16, means that p is a weak equivalence.

$\Leftarrow$. As noted above, a weak equivalence p between fibrant objects is a homotopy equivalence. If p further is a fibration, then we find, again by Proposition 3.3.10, that p is a trivial fibration. □

The proof of the following proposition is a standard retract argument.

Proposition 3.3.19. *Every acyclic cofibration with fibrant codomain is trivial.*

Proof. Given such acyclic cofibration $A \underset{\sim}{\rightarrowtail} K$, factor into $A \overset{\text{triv}}{\rightarrowtail} E \twoheadrightarrow K$ to get a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & E \arrow[d] \\
K \arrow[r, equal] \arrow[ur, dashed] & K
\end{tikzcd}$$

The right side is a weak equivalence by 2-out-of-3. Thus, by Proposition 3.3.18, it is a a trivial fibration and we have a diagonal filler as indicated in the diagram. It follows that $A \longrightarrow K$ is a retract of a trivial cofibration and therefore a trivial cofibration itself. □

Proposition 3.3.20. *Weak equivalences are preserved by $i_!$.*

Proof. Let f be a weak equivalence in the structure on **ssSet**. Taking fibrant replacements, we obtain a square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & \overline{A} \arrow[d, "\overline{f}"] \\ B \arrow[r] & \overline{B} \end{tikzcd}$$

in which the arrow on the right is a homotopy equivalence. Applying $i_!$ and taking fibrant replacements in in **sSet**, we obtain a diagram

$$\begin{tikzcd} i_! A \arrow[r] \arrow[d, "i_! f"'] & i_! \overline{A} \arrow[r] \arrow[d, "i_! \overline{f}"] & A^* \arrow[d, "f^*"] \\ i_! B \arrow[r] & i_! \overline{B} \arrow[r] & B^* \end{tikzcd}$$

The horizontal arrows are trivial cofibrations because, by the assumption that $(\mathsf{TrivCof}, \mathsf{Fib})$ satisfies the hypothesis of Theorem 3.3.1, they are preserved by $i_!$. Since $i_!$ also preserves homotopy equivalences, we have that $i_! \overline{f}$ is a homotopy equivalence. As noted before, it is proved in [Joy08] that the weak equivalences in $\mathbf{sSet}_Q$ can be characterised as in Proposition 3.3.12. It follows that $i_! f$ is a weak equivalence and by 2-out-of-3, so is f^*. But then f^* a weak equivalence between fibrant objects, *i.e.* a homotopy equivalence. It follows that $i_! f$ is a weak equivalence, as desired. □

Proof of Theorem 3.3.1. 2-out-of-3 follows from propositions 3.3.12 and 3.3.15. The four inclusions are proved in Proposition 3.3.17 (A1), Proposition 3.3.19 (A2'), and Proposition 3.3.18 (B1' and B2'). Finally, Proposition 3.3.20 proves the preservation requirement.

3.4 Applying Theorem 3.3.1

3.4.1 Three saturated classes of morphisms

In this section we compare three classes of morphisms that may be used as trivial cofibrations in an application of Theorem 3.3.1. Let $(\mathsf{TrivCof}_0, \mathsf{Fib}_0)$ be the weak factorisation system generated by $\mathsf{I}_\otimes$, which clearly is a set of morphisms with finitely presentable domains, and let $(\mathsf{TrivCof}_1, \mathsf{Fib}_1)$ be generated by the semisimplicial horn inclusions.

The following well-known lemma for simplicial sets adapts to semisimplicial sets without problems.

Lemma 3.4.1 ([JT08, Thm. 3.2.3]). $\mathsf{I}_\otimes \subseteq \mathsf{TrivCof}_1$.

Proof. We begin with the case $k = 1$. That is, we will show that every morphisms of the form

$$\epsilon^1 \hat{\otimes} i_n : (\Delta_i[0] \otimes \Delta_i[n]) +_{\Delta_i[0]\otimes\delta\Delta_i[n]} (\Delta_i[1] \otimes \delta\Delta_i[n]) \longrightarrow \Delta_i[1] \otimes \Delta_i[n]$$

is in $\mathsf{TrivCof}_1$. We will do so by showing that $\epsilon^1 \hat{\otimes} i_n$ can be written as a composition of inclusions obtained as certain pushouts of the horn inclusion $\Lambda_i^k[n+1] \longrightarrow \Delta_i[n+1]$. We have seen that $\Delta_i[1] \otimes \Delta_i[n] \cong N_i([1] \times [n])$, which means that its $n+1$-simplices can be described by monotone injections $\sigma_j : [n+1] \longrightarrow [1] \times [n]$ of the form

$$(0,0) \longrightarrow (0,1) \longrightarrow \cdots \longrightarrow (0,j) \longrightarrow (1,j) \longrightarrow \cdots \longrightarrow (1,n),$$

for $0 \leq j \leq n$. Note that $\sigma_0\epsilon^0 : [n] \longrightarrow \{1\} \times [n]$, so $\sigma_0\epsilon^0 \in \Delta_i[0] \otimes \Delta_i[n]$. Moreover, it holds for $r \geq 2$ that the simplex $\sigma_0\epsilon^r : [n] \longrightarrow [1] \times [n]$ (which skips the $(r-1)$-th element of $[n]$) is in $\Delta_i[1] \otimes \delta\Delta_i[n]$. Indeed, we have

$$\Delta_i[1] \otimes \Delta_i[n] \cong \varinjlim_{\substack{\epsilon : \Delta_i[m] \longrightarrow \Delta_i[n] \\ \text{in } \Delta_i \downarrow \Delta_i[n]}} N_i([1] \times [m])$$

and $\Delta_i[1] \otimes \delta\Delta_i[n]$ is the colimit of the restriction of this diagram to $\Delta_i \downarrow \delta\Delta_i[n]$. It follows that

$$(id_{[n]}, \sigma_0\epsilon^r) \sim (\epsilon^{r-1}, \tau) \in (\Delta_i[1] \otimes \delta\Delta_i[n])_n$$

where $\tau : (0,0) \longrightarrow (1,0) \longrightarrow \cdots \longrightarrow (1, n-1)$.

Thus, we have the pushout diagram

$$\begin{tikzcd}
\Lambda_i^1[n+1] \arrow[r, "\langle \sigma_0\epsilon^0, -, \sigma_0\epsilon^2, \dots, \sigma_0\epsilon^{n+1} \rangle"] \arrow[d] & (\Delta_i[0] \otimes \Delta_i[n]) +_{\Delta_i[0]\otimes\delta\Delta_i[n]} (\Delta_i[1] \otimes \delta\Delta_i[n]) \arrow[d] \\
\Delta_i[n+1] \arrow[r, "\sigma_0"'] & (\Delta_i[1] \otimes \Delta_i[n])^0
\end{tikzcd}$$

where $(\Delta_i[1] \otimes \delta\Delta_i[n])^0$ is the smallest subcomplex of $\Delta_i[1] \otimes \Delta_i[n]$ that contains both the image of $\epsilon^1 \hat{\otimes} i_n$ and the $n+1$-simplex σ_0, together with its faces. Next we consider the simplex σ_1. We have that $\sigma_1\epsilon^1 = \sigma_0\epsilon^0 \in (\Delta_i[1] \otimes \delta\Delta_i[n])^0$ and, by the same reasoning as above, it holds for every $r \neq 1, 2$ that $\sigma_1\epsilon^r \in \Delta_i[1] \otimes \delta\Delta_i[n]$. Thus, letting $(\Delta_i[1] \otimes \delta\Delta_i[n])^1$ be the least subcomplex of $(\Delta_i[1] \otimes \delta\Delta_i[n])^0$ containing σ_1, the following is a pushout square:

$$\begin{tikzcd}
\Lambda_i^2[n+1] \arrow[r, "\langle \sigma_1\epsilon^0, \sigma_1\epsilon^1, -, \sigma_1\epsilon^3, \dots, \sigma_1\epsilon^{n+1} \rangle"] \arrow[d] & (\Delta_i[1] \otimes \Delta_i[n])^0 \arrow[d] \\
\Delta_i[n+1] \arrow[r, "\sigma_1"'] & (\Delta_i[1] \otimes \Delta_i[n])^1
\end{tikzcd}$$

Repeating this process until we reach σ_n, we get a chain

$$\begin{aligned}
(\Delta_i[0] \otimes \Delta_i[n]) +_{\Delta_i[0]\otimes\delta\Delta_i[n]} (\Delta_i[1] \otimes \delta\Delta_i[n]) &\longrightarrow (\Delta_i[1] \otimes \Delta_i[n])^0 \\
&\longrightarrow (\Delta_i[1] \otimes \Delta_i[n])^1 \\
&\longrightarrow \cdots \\
&\longrightarrow (\Delta_i[1] \otimes \Delta_i[n])^n = \Delta_i[1] \otimes \Delta_i[n],
\end{aligned}$$

as required. When $k = 0$, the same proof can be performed backwards, starting with σ_n. □

It obviously follows that $\mathsf{TrivCof}_0 \subseteq \mathsf{TrivCof}_1$. Furthermore, because $i_!$ sends the semisimplicial horn inclusions to the simplicial horn inclusions, it follows from Lemma 2.5.5 that $\mathsf{TrivCof}_1 \subseteq i_!^{-1}(\mathsf{TrivCof}_\Delta)$ and thus both $\mathsf{TrivCof}_0$ and $\mathsf{TrivCof}_1$ lay within the bounds set in the hypothesis of Theorem 3.3.1. A third class of morphisms within these bounds is obtained by applying Lemma 2.6.1 to the weak factorisation system $(\mathsf{TrivCof}_\Delta, \mathsf{Fib}_\Delta)$. we obtain a cofibrantly generated weak factorisation systems on **ssSet**, say $(\mathsf{TrivCof}_{\Delta_i}, \mathsf{Fib}_{\Delta_i})$. Note that $\mathsf{TrivCof}_0$ is the minimal, and $\mathsf{TrivCof}_{\Delta_i}$ the maximal class within the bounds.

Remark 3.4.2. In [JT08, Theorem 3.2.3], it is also proven that, in **sSet**, it holds that the saturated class generated by $i_!(\mathsf{I}_\otimes)$ contains $\mathsf{TrivCof}_\Delta$. It follows that both $i_!(\mathsf{I}_\otimes)$ and the simplicial horn inclusions have $\mathsf{TrivCof}_\Delta$ as the least saturation. Thus, taking similar constraints on a saturated class A as in the hypothesis of Theorem 3.3.1, but in **sSet**, *i.e.*

$$i_!(\mathsf{I}_\otimes) \subseteq \mathsf{A} \subseteq \mathsf{TrivCof}_\Delta,$$

uniquely determines A. In contrast, with Proposition 3.5.7 we will show that $\mathsf{TrivCof}_1$ is strictly contained in $i^{-1}(\mathsf{TrivCof}_\Delta)$, showing that in **ssSet** distinct saturated classes lie between these constraints. It remains an open question whether, in **ssSet**, the class $\mathsf{I}_\otimes$ generates the same saturated class as the semisimplicial horn inclusions. ▲

3.4.2 The left-induced right semimodel structure $\mathbf{ssSet}_Q$

In this section we use Theorem 3.3.1 to obtain a right semimodel structure. Before we do that, let us first show that Theorem 3.3.1 can be used with the minimal class $\mathsf{TrivCof}_0$.

Proposition 3.4.3. $\mathsf{TrivCof}_0$ *satisfies the Leibniz axiom with respect to* Cof.

Proof. Let k be a trivial cofibration and let D be the class of all cofibrations $m : A \longrightarrow B$ such that $m \mathbin{\hat{\otimes}} k$ is a trivial cofibration. It is easily verified that D is saturated and thus it suffices to show that $\mathsf{I}_\otimes \subseteq \mathsf{D}$. To that end, let $\epsilon^k \mathbin{\hat{\otimes}} i_n \in \mathsf{I}_\otimes$. Then, by the associativity of the Leibniz product, we have

$$(\epsilon^k \mathbin{\hat{\otimes}} i_n) \mathbin{\hat{\otimes}} k = \epsilon^k \mathbin{\hat{\otimes}} (i_n \mathbin{\hat{\otimes}} k) \in \mathsf{TrivCof}_0$$

and thus, by Lemma 3.3.2, $\epsilon^k \mathbin{\hat{\otimes}} i_n \in \mathsf{D}$, as required. The full result follows from the symmetry of the Leibniz product. □

We invoke Theorem 3.3.1 to obtain a structure on **ssSet**, say $\mathbf{ssSet}_0$, that is almost a right semimodel structure.

The next application of Theorem 3.3.1 will provide a right semimodel structure, which we will call $\mathbf{ssSet}_Q$.

Proposition 3.4.4. $\mathsf{TrivCof}_{\Delta_i}$ *satisfies the Leibniz axiom with respect to* Cof.

Proof. We use the fact that the statement holds for the standard model structure on the closed monoidal category $(\mathbf{sSet}, \times, \Delta[0])$, where $\times$ is the categorical product. We write $\hat{\times}$ for the corresponding Leibniz product in **sSet**. The strategy is to show that $i_!(j \hat{\otimes} k) = i_!(j) \hat{\times} i_!(k)$, from which the result can be easily seen to follow. Because $i_!$ is a left adjoint, we have a pushout square

$$\begin{tikzcd} i_!A \times i_!X \arrow[r, "i_!A \times i_!k"] \arrow[d, "i_!j \times i_!X"'] \arrow[dr, phantom, "\ulcorner"] & i_!A \times i_!Y \arrow[d, "i_!r_0"] \\ i_!B \times i_!X \arrow[r, "i_!r_1"'] & i_!(A \otimes Y +_{A\otimes X} B \otimes X) \end{tikzcd}$$

Now $i_!B \times i_!Y$ forms a cocone to the pushout diagram in the obvious way, and $i_!j \hat{\times} i_!k$ is the corresponding unique mediating morphism. However, it follows from the functoriality of $i_!$ that $i_!(j \hat{\otimes} k)$ also fits this role, hence we have found our equality. □

Invoking Theorem 3.3.1 gives the structure $\mathbf{ssSet}_Q$.

Theorem 3.4.5. $\mathbf{ssSet}_Q$ *is a right semimodel structure.*

Proof. We only have to show that every acyclic cofibration is trivial, which follows directly from the fact that the weak equivalences are preserved by $i_!$. Indeed, for f an acyclic cofibration of $\mathbf{ssSet}_Q$, we have that $i_!f$ is an acyclic cofibration of the model structure on **sSet**, and thus a trivial cofibration. It follows that f is a trivial cofibration. □

Proposition 3.4.6. *The functor* $i_! : \mathbf{ssSet}_Q \to \mathbf{sSet}_Q$ *reflects weak equivalences.*

Proof. Let f be such that its image under $i_!$ is a weak equivalence in **sSet**. Take fibrant replacements and factor the obtained diagonal filler using $(\mathsf{Cof}_{\Delta_i}, \mathsf{TrivFib}_{\Delta_i})$. Finally, apply $i_!$ to get a diagram

$$\begin{tikzcd} i_!X \arrow[r] \arrow[d, "i_!f"'] & i_!\overline{X} \arrow[r] \arrow[d, "i_!\overline{f}"'] & i_!E \arrow[dl] \\ i_!Y \arrow[r] & i_!\overline{Y} & \end{tikzcd}$$

Since $i_!f$ is a weak equivalence in **sSet**, we have that $i_!\overline{f}$ is as well. But $E \longrightarrow \overline{Y}$ is a weak equivalence, as is $i_!E \longrightarrow i_!\overline{Y}$. It follows that $i_!\overline{X} \longrightarrow i_!E$ is an acyclic cofibration, hence trivial, which means that $\overline{X} \overset{\text{triv}}{\rightarrowtail} E \overset{\text{triv}}{\twoheadrightarrow} \overline{Y}$ is, in fact, a factorisation into a trivial cofibration followed by a trivial fibration. □

Since the left adjoint $i_!$ preserves and reflects cofibrations, trivial cofibrations and weak equivalences, $\mathbf{ssSet}_Q$ is a left-induced right semimodel structure.

3.5 Some properties of $\mathbf{ssSet}_Q$

3.5.1 Quillen equivalence

The following proof is from [KS17, Lemma 6.2], but our formulation is much more explicit.

Proposition 3.5.1. *The counit $\epsilon : i_! i^* \Rightarrow id$ is valued in weak equivalences.*

Proof. We will first show that this holds on the standard simplices and thereafter generalize to arbitrary simplicial sets. We claim that $i_! i^* \Delta[n]$ is contractible, from which it directly follows that $\epsilon_{\Delta[n]}$ is a weak equivalence. The m-simplices of $i_! i^*(\Delta[n])_m$ are given by:

$$(i_! i^* \Delta[n])_m = \{(x, \eta) \mid x : [k] \longrightarrow [n],\ \eta : [m] \longrightarrow [k] \text{ surjective}\}.$$

A simplex (x, η) might thus be degenerate in two ways: either by having a surjective part in x, or by having a surjective part in η. Consequently, the simplicial set $i_! i^* \Delta[n]$ is precisely the nerve of the category $[n]'$, given by the poset $[n]$, regarded as a category, with for each object a an idempotent endomorphism e_a such that for every $0 \leq a, b \leq n$

$$e_b(a \leq b) = \begin{cases} a \leq b & \text{if } a \neq b; \\ e_b & \text{if } a = b. \end{cases} \quad \text{and,} \quad (a \leq b)e_a = \begin{cases} a \leq b & \text{if } a \neq b; \\ e_a & \text{if } a = b. \end{cases}$$

This means that we have $|\mathrm{Hom}_{[n]'}(a, a)| = 2$ and $|\mathrm{Hom}_{[n]'}(a, b)| = 1$ for $a \neq b$. We claim that there is a natural transformation $\gamma : 0 \Rightarrow id_{[n]'}$ from the constant functor to the identity functor on $[n]'$. Indeed, setting $\gamma_0 = e_0$, we have that the naturality square

$$\begin{tikzcd} 0 \arrow[r, "\gamma_0"] \arrow[d, "0(e_0) = id_0"'] & 0 \arrow[d, "id(e_0) = e_0"] \\ 0 \arrow[r, "\gamma_0"'] & 0 \end{tikzcd}$$

commutes. Because for every $a \neq 0$, we have $|\mathrm{Hom}_{[n]'}(0, a)| = 1$, the component γ_a is uniquely determined and for the same reason every other naturality square commutes.

Under N this gives a natural transformation between a constant functor and the identity functor on $i_! i^* \Delta[n]$, which under $|\cdot|$ lifts to the geometric realisation. By Proposition 1.3.22, this corresponds to a homotopy between an identity morphism and a constant morphism, which means that the geometric realisation of $i_! i^* \Delta[n]$ is contractible. Finally, we deduce that $|\epsilon_{\Delta[n]}|_\Delta$, as a map between contractible spaces, is a homotopy equivalence, and thus $\epsilon_{\Delta[n]}$ is a weak equivalence.

The generalisation to arbitrary simplicial sets proceeds in a similar fashion to the proof of Proposition 3.2.2. Define the functors $\mathrm{tr}^n : \mathbf{sSet} \longrightarrow \mathbf{ssSet}$ in the same way, and let $Sk^n = i_! \mathrm{tr}^n$. The simplicial set $Sk^n X$ is called the *n-skeleton* of X and a simplicial set arising in this way is called *n-skeletal.* We will prove by induction on n that ϵ is a weak equivalence on all n-skeletal simplicial sets. The case $n = -1$ is trivially true because then ϵ is an isomorphism. Suppose the thesis holds for every $k \leq n$ and let $e(X)_n$ denote the set of non-degenerate n-simplices of an

n-skeletal simplicial set X. For every n, there is a cube

$$\begin{tikzcd}
& \cdot \arrow[rr] \arrow[dl, "\epsilon"'] \arrow[dd] & & i_! i^* Sk^{n-1} \arrow[dl, "\epsilon_{Sk^{n-1}}"'] \arrow[dd] \\
\coprod_{e(X)_n} \delta\Delta[n] \arrow[rr] \arrow[dd] & & Sk^{n-1}X \arrow[dd] & \\
& \cdot \arrow[rr] \arrow[dl] & & i_! i^* Sk^n X \arrow[dl, "\epsilon_{Sk^n X}"'] \\
\coprod_{e(X)_n} \Delta[n] \arrow[rr] & & Sk^n X &
\end{tikzcd}$$

By an argument similar to the proof of Proposition 3.2.2, the square on the front is a pushout square and since both $i_!$ and i^* preserve colimits, so is the square on the back. The left vertical arrow on the front is a cofibration, for it is a coproduct of cofibrations, as is the corresponding morphism on the back. The top arrows from the back to the front are weak equivalences by the induction hypothesis, and, by the fact that the thesis holds on the standard simplices, the bottom left arrow is the coproduct of weak equivalences and thus, by Proposition 2.2.11, itself a weak equivalence. Hence, we may apply the gluing lemma ([JT08, Theorem 2.3.3]) to deduce that ϵ is a weak equivalence on $Sk^n X \cong X$.

Finally, for X a simplicial set with non-degenerate simplices in infinitely many dimensions, let P be the set of simplicial subsets of X that *are* n-skeletal for some n. Then in the square

$$\begin{tikzcd}
\varinjlim_{X' \in P} i_! i^* X' \arrow[r, "\varinjlim_{X' \in P} \epsilon_{X'}"] \arrow[d] & \varinjlim_{X' \in P} X' \arrow[d] \\
i_! i^* X \arrow[r, "\epsilon_X"'] & X
\end{tikzcd}$$

the arrow on the right is an isomorphism, as is the arrow on the left, again by the fact that $i_!$ and i^* both preserve colimits. The arrow on the top is a directed colimit of weak equivalences, and thus, by Proposition 2.5.9, itself a weak equivalence. Using 2-out-of-3 we find that ϵ_X is a weak equivalence. □

Corollary 3.5.2. *The unit id $\Rightarrow i^* i_!$ is valued in weak equivalences.*

Proof. By a triangle identity, we have for any semisimplicial set X that

$$i_! X \xrightarrow{i_! \eta_X} i_! i^* i_! X \xrightarrow{\epsilon_{i_! X}} i_! X$$

is a weak equivalence. By the previous proposition and 2-out-of-3, $i_! \eta_X$ is as well, and the result follows from the fact that $i_!$ reflects weak equivalences. □

Theorem 3.5.3. *The pair $(i_!, i^*)$ is a Quillen equivalence between right semimodel structures.*

Proof. Let $f : i_! X \longrightarrow Y$ be such that its adjunct $\phi f : X \longrightarrow i^* Y$ is a weak equivalence. Then f is equal to

$$i_! X \xrightarrow{i_! \phi f} i_! i^* Y \xrightarrow{\epsilon_X} Y,$$

which is a composition of weak equivalences. Conversely, if f is a weak equivalence, then ϕf is equal to

$$X \xrightarrow{\eta_X} i^* i_! X \xrightarrow{i^* f} i^* Y.$$

By the fact that $i_! : \mathrm{ssSet}_Q \longrightarrow \mathrm{sSet}_Q$ preserves and reflects weak equivalences, this is a weak equivalence if and only if

$$i_! X \xrightarrow{i_! \eta_X} i_! i^* i_! X \xrightarrow{i_! i^* f} i_! i^* Y$$

is. But, since ϵ is valued in weak equivalences and by 2-out-of-3, this is a weak equivalence precisely when the following composition is:

$$i_! X \xrightarrow{i_! \eta_X} i_! i^* i_! X \xrightarrow{i_! i^* f} i_! i^* Y \xrightarrow{\epsilon_Y} Y.$$

And this composition is just $\phi^{-1} \phi f = f$ and thus a weak equivalence. □

3.5.2 Characterisation of the fibrations between fibrant objects

A third way to apply Theorem 3.3.1 is by using the semisimplicial horn inclusions as generating trivial cofibrations. The following theorem is proved in both [Hen18, p. 74] and [Sat18, Lemma 3.7], with the latter proof following [Joy08, Theorem H0.20].

Proposition 3.5.4. $\mathsf{TrivCof}_1$ *satisfies the Leibniz axiom with respect to* Cof.

Again, applying Theorem 3.3.1 yields a structure on **ssSet**, say $\mathbf{ssSet}_1$, which is almost a right semimodel structure. It is proved in [Sat18] that $\mathbf{ssSet}_1$ has the same weak equivalences as $\mathbf{ssSet}_Q$:

Proposition 3.5.5 ([Sat18, Cor. 3.72]). $i_! : \mathbf{ssSet}_1 \longrightarrow \mathbf{ssSet}_Q$ *reflects weak equivalences.*

This gives a convenient way to recognize the fibrant objects, as well as the fibrations between fibrant objects in $\mathbf{ssSet}_Q$.

Proposition 3.5.6. *In* $\mathbf{ssSet}_Q$*:*

(i) An object is fibrant if only if it has the right lifting property with respect to the semisimplicial horn inclusions.

(ii) A morphism between fibrant objects is a fibration if and only if it has the right lifting property with respect to the semisimplicial horn inclusions.

Proof. Clearly only the implications from right to left are non-trivial. To show that they hold, first let X as in (i). Given a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathsf{TrivCof}_{\Delta_i} \ni"'] & X \\ B & \end{tikzcd}$$

Use the weak factorisation system $(\mathsf{TrivCof}_1, \mathsf{Fib}_1)$ to obtain a fibrant replacement $\overline{B}$ of B. Because, by Proposition 3.5.5, the functor $i_!$ reflects weak equivalences onto $\mathbf{ssSet}_1$, we have that $A \longrightarrow B$ is a weak equivalence in $\mathbf{ssSet}_1$. Thus, $A \longrightarrow \overline{B}$ is as well, which means by the inclusion (A2) of Theorem 3.3.1 that it is trivial. Hence the required lifting exists.

The proof of part (ii) is very similar and is left to the reader. □

3.5.3 Simplicial versus semisimplicial anodyne extension

In this final section we take a closer look at the difference between $\mathsf{TrivCof}_1$ and $\mathsf{TrivCof}_{\Delta_i}$. The simplicial morphisms in $\mathsf{TrivCof}_\Delta$ are usually called the *simplicial anodyne extensions.* Note that by Corollary 2.5.3, these are the retracts of transfinite compositions of pushouts of simplicial horn inclusions. Analogously, we will call the retracts of transfinite compositions of semisimplicial horn inclusions, *i.e.* $\mathsf{TrivCof}_1$, the *semisimplicial anodyne extensions.*

We have seen that $i_!$ sends semisimplicial anodyne extensions to simplicial anodyne extensions. The following proposition shows that $i_!$ does not reflect anodyne extensions.

Proposition 3.5.7. $\mathsf{TrivCof}_1 \subsetneq \mathsf{TrivCof}_{\Delta_i}$

Proof. We adapt a counterexample from [Mos15], which is originally used to separate two classes of *simplicial* morphisms. First consider the following pushout in $\mathbf{ssSet}$ (note that there is only one choice for the top arrow):

$$\begin{tikzcd} \Lambda^0_i[2] \arrow[r] \arrow[d] & \Delta_i[1] \arrow[d] \\ \Delta_i[2] \arrow[r, "x"'] & X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The semisimplicial set X may be pictured:

$$\begin{tikzcd} & b & \\ a \arrow[ur, "\alpha"] \arrow[rr, "\alpha"'] & & b \arrow[ul, "\beta"'] \end{tikzcd}$$

Now let $h : \Lambda^1_i[2] \hookrightarrow \Delta_i[2]$ and take the pushout

$$\begin{tikzcd} \Lambda^1_i[2] \arrow[r, "xh"] \arrow[d] & X \arrow[d] \\ \Delta_i[2] \arrow[r, "y"'] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The resulting Y may be pictured as follows, where edges with the same label should be glued together:

$$\begin{tikzcd} & b & & & b & \\ a \arrow[ur, "\alpha"] \arrow[rr, "\alpha"'] & & b \arrow[ul, "\beta"'] & a \arrow[ur, "\gamma"] \arrow[rr, "\alpha"'] & & b \arrow[ul, "\beta"'] \end{tikzcd} \tag{3.7}$$

Now the composition of pushouts $\Delta_i[1] \longrightarrow Y$ is in $\mathsf{TrivCof}_1$, so its image $\Delta[1] \longrightarrow i_! Y$ under $i_!$ is in $\mathsf{TrivCof}_\Delta$. This means that it is a geometric homotopy equivalence, and, since the unit line is contractible, so is the following morphism into the terminal object:

$$\Delta[1] \to i_! Y \to \Delta[0]$$

It follows from by the 2-out-of-3 property that $i_! Y \longrightarrow \Delta[0]$ is a geometric homotopy equivalence and thus $|i_! Y|_\Delta$ is contractible. Consequently, any map $\Delta[1] \longrightarrow i_! Y$ is weak equivalence, and in particular the morphism corresponding to the non-degenerate 1-simplex γ is in $\mathsf{TrivCof}_\Delta$.

Now let us show that the preimage of the above morphism under $i_!$, *i.e.* the morphism $\Delta_i[1] \longrightarrow Y$ corresponding to γ, is *not* in $\mathsf{TrivCof}_1$. Suppose towards a contradiction that it is and let $A_0 \longrightarrow A_\alpha$ be a transfinite composition of coproducts of pushouts of semisimplicial horn inclusions of least length such that

$$\begin{tikzcd} \Delta_i[1] \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[d] & \Delta_i[1] \arrow[d] \\ Y \arrow[r] & A_\alpha \arrow[r] & Y \end{tikzcd} \tag{3.8}$$

is a retract diagram. Since Y is finite, it holds that α is and we may consider the final pushout of the chain:

$$\begin{tikzcd} \Lambda_i^k[n] \arrow[r] \arrow[d] & A_{\alpha-1} \arrow[d] \\ \Delta_i[n] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & A_\alpha \end{tikzcd} \tag{3.9}$$

The image of Y in A_α is of shape (3.7) and is such that every edge is the face of a 2-simplex. It follows that we must have $n = 2$ in the diagram (3.9), for otherwise $\Delta_i[1] \longrightarrow Y$ is a retract of the shorter composition $A_0 \longrightarrow A_{\alpha-1}$. In particular, the semisimplicial set $A_{\alpha-1}$ must contain only one of the 2-simplices in the image of Y. However, the 2-simplex created by the pushout (3.9) can be neither of the two 2-simplices in the image of Y, since it comes with a new edge and the edges α, β, γ appear already in $A_{\alpha-1}$. □

Note that it follows that $\mathsf{TrivCof}_1 \subsetneq \mathsf{Cof} \cap \mathsf{Weq}_1$ and thus $\mathbf{ssSet}_1$ is *not* a right semimodel structure, for it does not satisfy inclusion (A2). The following terminology is from [Mos15], but this class is considered often in the literature under different names.

Definition 3.5.8. The *strong (semi)simplicial anodyne extensions* are the transfinite compositions of pushouts of (semi)simplicial horn inclusions. ■

Proposition 3.5.9. *The preimage of the strong simplicial anodyne extensions under $i_!$ is precisely the class of strong semisimplicial anodyne extensions.*

Proof. Let $i_! f : i_! A_0 \longrightarrow i_! B$ be a strong simplicial anodyne extension. Now suppose, towards a contradiction, that somewhere in the chain there is a pushout

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "p"] \arrow[d] & A_\beta \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & A_{\beta+1} \end{tikzcd}$$

such that p is not in the image of $i_!$. Then p sends a non-degenerate simplex of $\Lambda^k[n]$ to a degenerate simplex of A_β. Consequently, the image of $id_{[n]}$ in $A_{\beta+1}$ is a non-degenerate simplex with a degenerate face. Since there is a monomorphism $A_{\beta+1} \longrightarrow i_! B$, Lemma 1.3.6 implies that $i_! B$ too has a non-degenerate simplex with a degenerate face, a contradiction.

Thus, the morphism $i_! f$ may be written as a chain

$$i_! A_0 \longrightarrow i_! A_1 \longrightarrow \dots \longrightarrow i!\, A_\beta \longrightarrow \dots \longrightarrow \varinjlim_{\beta<\alpha} i_! A_\beta$$

in which each $A_\beta \longrightarrow A_{\beta+1}$ is a pushout of a semisimplicial horn inclusion in **ssSet**. The result follows from the fact that $i_!$ preserves colimits. $\square$

Conclusion and future work

In this thesis we have studied the homotopy theory of semisimplicial sets, attempting to transfer the standard model structure sSet_Q on simplicial sets along the left adjoint $i_!$. We have seen that the full model structure cannot be transferred, but we were able to transfer the 'right semi'-fragment in such a way that the adjunction becomes a Quillen equivalence between right semimodel structures. There are some interesting avenues for further research.

- One question is whether ssSet_0 is a right semimodel structure. It might be the case that $i_! : \mathrm{ssSet}_0 \longrightarrow \mathrm{sSet}_Q$ reflects weak equivalences, from which it would follow that this question must be answered negatively, like we did with ssSet_1. On the other hand, if ssSet_0 turns out to have strictly less weak equivalences, it might just as well be a right semimodel structure.

- Another pressing question is whether the construction of the right semimodel structure ssSet_Q can also be carried out in a constructive metatheory. Seeing that results in both [Sat18] and [Hen18] are established in this way, this question hinges on whether the left-induced weak factorisation system $(\mathsf{TrivCof}_{\Delta_i}, \mathsf{Fib}_{\Delta_i})$ can still be shown to exist. One way to do this would be to show that the argument in [MR14] is valid, but perhaps a weaker argument can be made that is tailored to this specific setting.

- Another line of research would be to investigate whether right semimodel structures in general, and ssSet_Q in particular, can serve as a model for some type theory.

- It is known that the fibrant objects in ssSet_1, *i.e.* those with the right lifting property with respect to the semisimplicial horn inclusions admit a system of degeneracies (or: lie in the image of i^*). This fact is proven using topological arguments in [RS71] and using combinatorial arguments in [McC13], but it would be nice to have a more abstract proof using categorical arguments. As is pointed out in [Sat18], for any such fibrant X, there is, since η is valued in weak equivalences, a diagonal filler

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\eta_X"'] & X \\ i^* i_! X \arrow[ur, dashed] & \end{tikzcd}$$

  which only leaves to verify the commutativity of the second diagram of (1.3.23).

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