Truthmaker Semantics for Epistemic Logic *

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1 Introduction

Can truthmaker semantics offer an improvement on Hintikka-style semantics for epistemic logic?[1] That depends on the answers to various questions:

- What is one's purpose in designing a system for epistemic logic? For instance, is it to capture ordinary knowledge ascription; in-principle knowability relative to a given body of empirical information; or knowledge-level warrant transmission?
- What are the limitations, if any, of the Hintikkan approach relative to the selected goal? In particular, do cogent philosophical arguments bear against the validities that fall out of this approach?
- What are the unequivocal successes of the Hintikkan approach, and can a truthmaker approach emulate them without incurring large costs in complexity?

Our purpose is to briefly explore the scope of a truthmaker approach to epistemic logic and the extent to which it can best the Hintikkan approach. §2 identifies three possible targets of analysis for the epistemic logician. Then, we offer a list of candidate epistemic principles and review the arguments that render some controversial. §5.2 presents the Hintikkan approach and notes - as per its well-known susceptibility to the 'problem of logical omniscience' - that it validates all of the aforementioned principles, controversial or otherwise. §4 lays out a truthmaker framework in the style of Fine (2016, 2017a, forthcoming). §5 presents six different ways of extending this semantics with a (conditional) knowledge operator, drawing on notions of implication and content that are prominent in Fine's work. We demonstrate that different epistemic logics are thereby generated, bearing on the principles from §2. §6 offers preliminary observations about the prospects for each logic, relative to (i) a target of analysis for epistemic logic and (ii) philosophical commitments that bear on the candidate principles in §2. Proofs of propositions are omitted from the main text and presented for reference in a technical appendix.

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[1] Hintikka (1962) kick-started the Hintikkan tradition. See (Fagin et al., 1995), (van Ditmarsch et al., 2008), (van Benthem, 2011) and (Humberstone, 2016) for comprehensive introductions and overviews.

2 Principles of Interest

We introduce an epistemic language $\mathscr{L}_e$ that is sufficient to frame our target (controversial and uncontroversial) epistemic principles. $\mathscr{L}_e$ is defined by the grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid A\varphi \mid \varphi \Rightarrow \psi \mid K_\varphi \psi,$$

where $p \in \mathsf{Prop} = \{p, q, \ldots\}$, the countable set of atoms. We employ the usual abbreviation for disjunction $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, material conditional $\varphi \supset \psi := \neg(\neg\varphi \vee \psi)$, and bi-conditional $\varphi \equiv \psi := (\varphi \supset \psi) \wedge (\psi \supset \varphi)$. Read $A\varphi$ as 'φ is knowable a priori'; $K_\varphi \psi$ as 'knowing φ is sufficient for knowing ψ'; $\varphi \Rightarrow \psi$ as 'ψ is an a priori implication of φ' i.e. 'it is knowable a priori that φ implies ψ'.[2]

Since *a priori knowability* is a relatively standard philosophical notion, we offer little clarification of the intended reading for $A\varphi$ and $\varphi \Rightarrow \psi$, besides standard examples: that 1+1=2 is knowable a priori, as are other mathematical truths; that Jon is a bachelor has the a priori implication that he is unmarried; and so on.

What, however, is it for knowledge of ϕ to be *sufficient* for knowledge of ψ? We focus on three elaborations, relative to three possible goals in designing an epistemic logic. First, one may aim to capture the logic of ordinary (though conditional) knowledge ascription. In this case, read $K_\varphi \psi$ as: 'knowing φ entails knowing ψ'. Second, one may aim for a logic of in-principle knowability, relative to a given body of evidence. That is: the logic of ordinary (though conditional) knowledge ascription as applied to cognitively ideal agents. In this case, read $K_\varphi \psi$ as: 'knowing φ renders ψ knowable in principle, without accrual of further evidence'. Third, one may aim to capture knowledge-level warrant transmission. The term 'epistemic warrant' has labeled various (subtly distinct) notions in the epistemology literature.[3] We use it as follows: an agent has *(strong) warrant* to believe φ exactly when she has all-things-considered propositional justification to believe φ - as opposed to, say, *prima facie* doxastic justification - to the degree necessary for knowing φ. In this case, read $\varphi \Rightarrow \psi$ as: 'knowing φ provides strong warrant for ψ'. Note that it should *not* be read as: 'Knowing φ entails having strong warrant for ψ'. As Wright (1985, 2004) points out, knowing φ may entail having warrant for ψ because coming to know φ *presupposes* having *prior* warrant to believe ψ; in this case, it is one's warrant for ψ that (at least partly) provides warrant for φ, rather than vice versa.

To get an intuitive grip on the target phenomena, consider some examples:

(1) Knowing Jane is a lawyer is sufficient for knowing Jane is a fisherman.

(2) Knowing that Jane is an expert lawyer is sufficient for knowing Jane is a lawyer.

(3) Knowing that Jane is a lawyer is sufficient for knowing Cantor's theorem.

(4) Knowing the conjunction of the ZF axioms (and basic logical principles) is sufficient for knowing Cantor's theorem.

[2] Another approach would include an appropriate conditional $>$ in the language so as to render $\varphi \Rightarrow \psi$ definable as $A(\varphi > \psi)$. The nature of the underlying conditional $>$ is peripheral to our current interests, however - including the statement of our principles of interest.

[3] For instance, Plantinga (1993) uses it to refer to whatever knowledge adds to true belief, Gettier examples in mind; Pryor (2000) uses 'warranted belief' to refer to beliefs that are epistemically appropriate for an agent; Moretti and Piazza (2018) generally takes 'warrant' to be interchangeable with 'justification'.

Claim (1) seems false on any of our three readings of 'sufficient'. An agent can know Jane is a lawyer without knowing Jane is a fisherman. Nor is she thereby *positioned* to know Jane is a fisherman without accrual of further knowledge. Nor does this knowledge provide warrant (strong or otherwise) for believing Jane is a fisherman. Further, only (2) seems true when paraphrased in terms of ordinary knowledge ascriptions. An agent that knows that Jane is an expert lawyer also knows, presumably, that Jane is a lawyer, since the latter is part of knowing the former. However, (2), (3) and (4) all represent inviting claims of relative in-principle knowability: if one's evidence positions one to know that Jane is a lawyer, then one is also positioned, in principle, to know Cantor's theorem. After all, the latter is a priori, so presumably no further (empirical) information is needed to establish it, just requisite conceptual mastery and some hard thinking. In contrast, only (4) is immediately inviting as a claim of knowledge-level warrant transmission. Suppose one is strongly warranted in believing the conjunction of the ZF axioms. Presumably, one *thereby* has strong warrant for believing Cantor's theorem. On the other hand, if one is strongly warranted in believing that Jane is a lawyer, one is not thereby warranted in believing Cantor's theorem. The latter may be knowable in-principle, but the requisite warrant cannot be provided by knowledge about Jane's profession. Claim (2), meanwhile, raises interesting issues: even if it is impossible to know Jane is an expert lawyer without knowing Jane is a lawyer, it is not clear that the knowing the former always provides warrant for believing the latter. It is tempting to sometimes merely claim the partial *converse*: knowing Jane is a lawyer provides one with warrant for believing Jane is an expert *lawyer* (warrant for believing that she is an *expert* lawyer might well have a further source).

We turn to candidate logical principles, organized in three groups. In what follows, φ should be read as $\Vdash \varphi$ (i.e. φ is valid) and $\varphi_1, \ldots, \varphi_n \Vdash \psi$ as logical consequence.

Uncontroversial:

Simplification: $K_{p \wedge q} p$, $K_{p \wedge q} q$

Reflexivity: $K_p p$

Cautious Transitivity: $K_p q, K_{p \wedge q} r \Vdash K_p r$

Cautious Strengthening: $K_p q, K_p r \Vdash K_{p \wedge q} r$

Relatively uncontroversial:

Double Negation: $K_p(\neg\neg p)$

Weak Simplification: $K_{p \wedge q}(p \vee q)$

Weak Omniscience: $K_p(p \vee \neg p)$

Apriority: $Ap \Vdash K_\varphi p$

Controversial:

Negative Addition: $K_\varphi p \Vdash K_\varphi \neg(\neg p \wedge q)$

Agglomeration: $K_\varphi p, K_\varphi q \Vdash K_\varphi(p \wedge q)$

Single-Premise Closure: $K_\varphi p, p \Rightarrow q \Vdash K_\varphi q$

Disjunctive Syllogism: $K_\varphi \neg p, K_\varphi(p \vee q) \Vdash K_\varphi q$

Strengthening: $K_p r \Vdash K_{p\wedge q} r$

Gabbay (1985) advocates **Cautious Transitivity** and **Cautious Strengthening** for any serious conditional logic. Conditional epistemic logic seems no exception.[4]

We proceed on the assumption that an epistemic logician should accept all of the uncontroversial principles. This seems obvious for logics of knowledge ascription and in-principle knowability. The case of warrant transmission, however, requires comment: **Reflexivity** and **Simplification** might here seem questionable. Isn't the claim that knowledge of p provides strong warrant for p an admission of objectionable epistemic circularity? Since knowledge that p can plausibly provide (partial) warrant for $p \wedge q$, shouldn't we resist taking it as a general rule that $p \wedge q$ provides warrant for p? These concerns deserve serious discussion. However, for simplicity, we put them aside for this paper. We will assume that if φ entails ψ then knowledge of the former provides (perhaps degenerate) warrant for believing the latter, *unless* either (i) it is clear that the subject matters of φ and ψ are entirely disjoint or (ii) warrant for the latter is a prerequisite for learning the former i.e. unless one cannot come to know φ without prior warrant for ψ. Note that warrant for p is not a prerequisite for learning p, nor a prerequisite for learning $p \wedge q$: consider coming to know p or $p \wedge q$ directly via testimony, without prior warrant for p.

Our 'relatively uncontroversial' principles are only controversial, or otherwise problematic, relative to certain interpretations of the logic. **Apriority**, for instance, is uncontroversial for a logic of in-principle knowability; uncontroversially wrong for a logic of knowledge ascriptions; and (at least) controversial for a logic of warrant transmission. The other relatively uncontroversial principles are, we take it, only questionable for a logic of knowledge ascription - and here the issue is murky. Suppose that an ordinary agent knows that Jane is a lawyer. Does it follow that she knows that Jane either is a lawyer or not a lawyer? On one hand, our agent might not be familiar with classical formal logic and, in particular, unaware that $p \vee \neg p$ is a tautology. Even if she is familiar with this principle, perhaps her knowing p needn't include her knowing $p \vee \neg p$, with the latter knowledge only available only via a (simple) inference that she has failed to draw. On the other hand, it is difficult to say what coming to know Jane is either a lawyer or not a lawyer *adds* to knowing Jane is a lawyer. Indeed, the former seems to be *vacuous* knowledge about Jane and her profession. Can an agent lack vacuous knowledge?

Similar remarks apply to, say, **Double Negation**. Suppose one knows Jane is friendly. Does one also then know that Jane is not unfriendly? If not, what exactly does one *learn* about Jane, that one did not know before, when one then infers that Jane is not unfriendly?

As for the controversial principles: though the debates are unresolved, arguments exist for rejecting these, for all three interpretations of the logic. In brief, the line is: various (alleged) philosophical paradoxes are best understood, on reflection, as identifying counter-examples to the offered principles.[5]

- **Preface counter-example to Agg:**[6] A historian can know every claim $p_1, \ldots, p_n$ in her new book, but rightly acknowledge that books of this length frequently have at least one error. Hence, she isn't positioned to know, or anyway thereby warranted to believe, $p_1 \wedge \ldots \wedge p_n$.

[4]**Cautious Strengthening** is sometimes called 'Cautious Monotonicity' in the literature.

[5]Going forward, we often abbreviate the names of our candidate principles: **Simp**, **Refl**, **C-Trans**, **C-Strength**, **DN**, **W-Simp**, **W-Omni**, **Apriority**, **Neg Add** (or just **NA**), **Agg**, **SPC**, **DS**, **Strength**.

[6]The preface paradox was introduced by Makinson (1965). It is usually framed and debated as concerning rational belief, rather than knowledge per se. See Douven (2003) and Leitgeb (2013) for recent discussion.

- **Cartesian counter-example to Neg Add and SPC:**[7] Agent A knows that she has hands. Not being a (handless) brain-in-vat is an apriori implication of having hands. Yet our agent is not positioned to know, or anyway thereby warranted to believe, that she is not a (handless) brain-in-vat.

- **Dogmatism counter-example to Neg Add and SPC:**[8] Suppose agent A knows empirical claim p. Let e be any true claim such that: (i) A doesn't know e and (ii) e is evidence against p. Hence, e is, as it happens, misleading on the question of p. However, it seems that A may not be positioned to know (or warranted to believe) that e is misleading on the question of p. That is: she may not be positioned to know, without further ado, that $\neg(e \wedge \neg p)$. For, if A were so positioned, it would presumably be reasonable for her to ignore the usual evidential implications of e were she to come to know e. At least, it would be reasonable for her to *resist* coming to know e, on the grounds that such inquiry threatens her current state of knowledge. Upon generalization, it follows, counter-intuitively, that agents are right to adopt an attitude of extreme dogmatism: it is always epistemically appropriate to ignore or at least actively avoid counter-evidence to claims that one knows (or believes with strong warrant). Further, bearing on **SPC**: let m be the claim that e is misleading with respect to p. Then m is an a priori implication of p. Yet A may not be positioned to know, or anyway warranted in believing, m.

- **Criterion counter-example to Neg Add:**[9] Agent A knows empirical p on the non-deductive basis of knowing e, describing her total empirical evidence. However, she isn't positioned to know, or anyway thereby warranted in believing, that it isn't the case that $e \wedge \neg p$ i.e. that it isn't that e is misleading on the question of p. After all, this is an empirical claim, and it is perfectly consistent with her empirical evidence.

- **Criterion counter-example to DS:** Agent A knows $\neg p$ - that she is not disembodied - on the basis of her empirical evidence. She also knows $p \vee a$: either she is disembodied or her empirical evidence supports an accurate verdict on the question of p. But she is not thereby positioned to know, or thereby warranted in believing, that her empirical evidence supports an accurate verdict on the question of p. If she were, she would presumably be so positioned on the basis of her empirical evidence, since a is a contingent, empirical claim. However, it is objectionably circular to claim that an agent's total empirical evidence supports knowledge or warranted belief about the accuracy of verdicts drawn from that agent's total empirical evidence.

- **Surprise exam counter-example to DS:**[10] A teacher announces that there will be a surprise exam the following week. The students thereby know $p \vee q$: either the exam is on Friday (p) or earlier in the week (q). They may also conclude $\neg p$: if the exam were left for Friday, then they would

[7] Dretske (1970) and Nozick (1981) pioneered the rejection of 'epistemic closure' as a solution to the skeptical conundrum. See Stine (1976), Lewis (1996) and Hawthorne (2004, 2005) for spirited defenses of epistemic closure. Wright (2004) rejects the necessary transmission of warrant across implication; see Pryor (2004) for push-back.

[8] The dogmatism paradox is due to Kripke, in a work that eventually appeared as (Kripke, 2011). The paradox first appeared in the literature in a discussion by Harman (1973). It is explicitly waged against 'epistemic closure' by Sharon and Spectre (2010, 2017). Recent discussions include Sorensen (1988), Lasonen-Aarnio (2014) and Beddor (forthcoming).

[9] For a classic discussion of the problem of the criterion, see Chisholm (1973). The puzzle has re-emerged in recent discussions on the issue of 'easy knowledge': see, for instance, Cohen (2002) and Sosa (2009).

[10] See Sorensen (2018, Sect. 1) for an overview of the paradox and some responses in the literature. Kripke (2011) offers another recent discussion.

not be surprised when it arrived. But the students cannot pool this knowledge to come to know (or be warranted in believing) q: if they could, they could iterate the reasoning to arrive at the paradoxical conclusion that either there will be no exam after all, or when it arrives it will not be a surprise.

- **Defeasibility counter-example to Strength:**[11] Agent A knows f that there is a fire on the non-deductive basis of knowing e: that there is smoke and smoke typically indicates fire. However, A would not be positioned to know, or anyway thereby warranted in believing, f on the alternative basis of knowing $e \wedge c$, where c is the claim that there is a nearby cabin that can emit smoke through its chimney. For c defeats the prima facie conclusions that follow from e alone.

Of course, one does not need elaborate or controversial arguments to raise doubts about many of our controversial principles for a logic of ordinary knowledge ascription: rather, one just reminds oneself that ordinary agents sometimes fail to draw inferences, even ready ones. **Neg Add** and **SPC**, for instance, might seem obviously incorrect for such a logic, while **DS** seems readily questionable.

To emphasize: we do not claim that the above (alleged) counter-examples are universally accepted by philosophers, nor that rejection of the controversial principles is a popular or cost-free resolution of the associated epistemic paradoxes (indeed, an obvious cost is that the controversial principles tend to have pre-theoretic appeal, at least for logics of knowability and warrant transmission). Rather, our point is that treating the paradoxes as yielding such counter-examples deserves serious discussion: the associated paradoxes are not, it seems, amenable to a cost-free resolution, and it is striking that giving up the controversial principles provide us with one such resolution. Hence, epistemic logics that accommodate the invalidity of these principles can serve as a neutral tool for framing the philosophical debate and, in particular, a tool for epistemologists that accept the force of certain alleged counter-examples.

3 A Classical Approach to Epistemic Logic

It is well-known[12] that the classic approach to epistemic logic - taking it as a normal modal logic, in the spirit of Hintikka (1962) - does not accommodate the nuances of the previous section: it validates every relatively uncontroversial principle and controversial principle. This rules it out as a plausible logic of ordinary knowledge ascription or warrant transmission. Taken as a logic of knowability, it is controversially strong.

We spell out classical epistemic logic - both its semantics and syntax - somewhat unusually to facilitate presentation as a logic of conditional knowledge. We do not depart from its core ideas, however: an agent's total body of knowledge κ at world w is modeled as a set of possible worlds, and knowledge of φ is rightly ascribed to the agent just in case (the proposition expressed by) φ is entailed by κ. A key underlying idea is that content - i.e. a proposition - is well modeled as a set of possible worlds and entailment, therefore, as set containment.

[11] For a recent defense of the possibility of defeasible knowledge, and extensive references to the literature, see Brown (2018).

[12] The relevant set of issues is usually collected, somewhat unhelpfully, under the umbrella of the 'problem of logical omniscience' and used to motivate (i) interpreting the Hintikkan logic, in its basic form, as something like a logic of knowability and (ii) certain refinements that make the logic more 'realistic' i.e. closer to an accurate description of ordinary, non-idealized agents. See, for instance, van Benthem (2011, Chs. 2,5), Fagin et al. (1995, Ch. 9) and Humberstone (2016, Sect. 5.1.). Issues of logical omniscience have been obvious from the start: it is for exactly this reason that Hintikka (1962, Sect. 2) is careful to specify the exact interpretation of his logic.

Definition 1 (Hintikka Model)**.** *A* Hintikka model *is a pair $\mathscr{M} = \langle W, \mathrm{v} \rangle$ where W is a non-empty set of possible worlds and* $\mathrm{v} : \mathsf{Prop} \to 2^W$ *is a valuation function that assigns a subset of W (a proposition) to each atom.*

Definition 2 (Hintikka Semantics)**.** *Given a Hintikka model $\mathscr{M} = \langle W, \mathrm{v} \rangle$ and a possible world $w \in W$, the $\vDash$-sementics for $\mathscr{L}_e$ is recursively defined as:*

$$
\begin{array}{lll}
w \vDash p & \textit{iff} & w \in v(p) \\
w \vDash \neg\varphi & \textit{iff} & w \nvDash \varphi \\
w \vDash \varphi \wedge \psi & \textit{iff} & w \vDash \varphi \textit{ and } w \vDash \psi \\
w \vDash A\varphi & \textit{iff} & u \vDash \varphi \textit{ for all } u \in W \\
w \vDash \varphi \Rightarrow \psi & \textit{iff} & \textit{for all } u \in W (\textit{if } u \vDash \varphi \textit{ then } u \vDash \psi) \\
w \vDash K_\varphi \psi & \textit{iff} & \textit{for all } u \in W (\textit{if } u \vDash \varphi \textit{ then } u \vDash \psi)
\end{array}
$$

Truth in a model, logical consequence and *validity* are defined in standard ways: with $\varphi_1, \ldots, \varphi_n, \psi \in \mathscr{L}_e$:

- ψ is true in $\mathscr{M}$, denoted by $\mathscr{M} \vDash \psi$, iff $w \vDash \psi$ for every $w \in W$,
- ψ is a logical consequence of $\{\varphi_1, \ldots, \varphi_n\}$, denoted by $\varphi_1, \ldots, \varphi_n \vDash \psi$, iff $w \vDash \varphi_1 \wedge \ldots \wedge \varphi_n$ materially implies $w \vDash \psi$ for every w in every $\mathscr{M}$, and
- ψ is logically valid, denoted by $\vDash \psi$, iff $\mathscr{M} \vDash \psi$ for all $\mathscr{M}$.

The above modal components exhibit various redundancies: $\varphi \Rightarrow \psi$ and $K_\varphi \psi$ have exactly the same interpretation; and $A\psi$ and $K_\varphi \psi$ (and, thus, $\varphi \Rightarrow \psi$) are interdefinable via the equivalences $A\psi \equiv K_\top \psi$ and $K_\varphi \psi \equiv A(\varphi \supset \psi)$, where $\top$ is a propositional tautology. However, we prefer using the full language $\mathscr{L}_e$ since: (1) the truthmaker semantics examined in §4 can discern these elements of the language, and (2) we would like to keep the object language fixed throughout.

Thus, the Hintikkan approach formalizes both conditional knowledge and apriori implication as strict implication. Though this gives a relatively easy way of modeling epistemic logics, it is well known that it yields notions of knowledge that are highly idealized.

Proposition 1. *The classical approach validates every principle, controversial or uncontroversial.*

Thus, someone who both adopts certain goals for epistemic logic and is sympathetic to certain philosophical arguments has cause to seek a refinement of the Hintikkan approach. In some cases, this is obvious: for instance, a logic for warrant transmission should not validate **Apriority**.

4 An Exact Truthmaker Semantics

To what extent can a Fine-style truthmaker semantics yield more liberal and flexible epistemic logics - in particular, when focusing on natural generalizations of the Hintikkan treatment and/or building on notions of content and implication that are explicitly discussed by Fine? We proceed in three stages. The current section presents a basic Fine-style truthmaker semantics, with a flourish: expressions $A\varphi$ and

$\varphi \Rightarrow \psi$ are included and given a natural semantics. The proceeding sections introduce six possible treatments of $K_\varphi \psi$, exploiting ideas of particular salience in the recent literature on truthmaker semantics. Along the way, we observe the various outcomes for the principles listed in §2.

We again utilize $\mathscr{L}_e$ and various fragments thereof. The fragment without the conditional knowledge modality $K_\varphi \psi$ is denoted by $\mathscr{L}$, and the fragment also without $\varphi \Rightarrow \psi$ and $A\varphi$ is denoted by $\mathscr{L}_{pl}$ (*pl* for propositional logic). We use the so-called "inclusive" semantics of Fine (2016, Sect. 3) for $\mathscr{L}_{pl}$.

Definition 3 (State Space)**.** *A* state space *is a tuple $\langle S, \leq \rangle$ where S is a non-empty set of states and $\leq$ is a partial order on S. In other words, a state space is a non-empty poset.*

Relation $\leq$ is the *parthood* relation on S: $s \leq t$ reports that state s is a part of ('contained in') state t.

Given a state space $\langle S, \leq \rangle$ and a subset $T \subseteq S$, we say $s \in S$ is an *upper bound* of T iff $t \leq s$ for all $t \in T$. We call $s \in S$ *the least upper bound of T* if s is an upper bound of T and for any upper bound $s' \in S$ of T, $s \leq s'$. We call a state space $\langle S, \leq \rangle$ *complete* if every subset $T \subseteq S$ has a least upper bound. For any $T \subseteq S$ we denote the least upper bound of T by $\bigsqcup T$ and call it the *fusion* of T. In particular, the least upper bound of a two-element set $\{s,t\} \subseteq S$ is denoted by $s \sqcup t$ and called the fusion of s and t. Finally, we call a subset $T \subseteq S$ *downward closed* iff for all $s,t \in S$, $s \in T$ and $t \leq s$ implies $t \in T$.

Definition 4 (Modalized State Space)**.** *A* modalized state space *is a tuple $\langle S, P, \leq \rangle$ where $\langle S, \leq \rangle$ is a complete state space and P is a non-empty, downward closed subset of S.*

Set P is the subspace of *possible* states. States s,t are called *compatible* when $s \sqcup t \in P$. On this picture, a 'world' w (if such exists) is taken to be a 'maximal' possible state, in the following sense: if s is compatible with w then $s \leq w$, for all $s \in S$. We here understand 'possibility' (primarily) in an epistemic sense. In particular, P is understood to contain *basic empirical possibilities*: each requires (non-degenerate) empirical information to be ruled out. States in $S - P$ are thus thought of as ruled out a priori i.e. without any requirement of empirical information. Thus, a claim that is true at every world in P (if such there be) is a priori (an 'epistemic necessity'), knowledge of which does not require empirical information.

A (unilateral) *proposition* is a subset of S that is closed under fusions. A *bilateral proposition* is a pair of propositions. Propositions A,B are incompatible when $s \sqcup t \notin P$ for every $s \in A$ and $t \in B$.[13]

Definition 5 (Model)**.** *A model is a tuple $\mathtt{M} = \langle S, P, \leq, \mathtt{v} \rangle$ where $\langle S, P, \leq \rangle$ is a modalized state space and $\mathtt{v} : \mathsf{Prop} \to (2^S \times 2^S)$ assigns a bilateral proposition $\langle p^+, p^- \rangle$ to each atom $p \in \mathsf{Prop}$, with p^+ and p^- incompatible.*

Definition 6 (Exact verification & falsification)**.** *Given a model $\mathtt{M} = \langle S, P, \leq, \mathtt{v} \rangle$ and a state $s \in S$, exact verification $\vdash$ and exact falsification $\dashv$ for $\mathscr{L}$ is recursively defined as:*

- *$s \vdash p$ iff $s \in p^+$*
- *$s \dashv p$ iff $s \in p^-$*

[13] Fine (2017a, Sect. 2) considers three natural conditions on unilateral propositions in the state-based setting: closure under fusions; non-emptiness; and convexity. A set of states A is convex if $s \leq t \leq u$ and $s \in A$ and $u \in A$ implies $t \in A$. Fine (2016) shows that the class of propositions admit a natural definition of (and associated logic for) content parthood if propositions are assumed to be closed, non-empty and convex. For simplicity, the current paper will not engage with the question as to whether convexity should be imposed on contents. In contrast, §6 will emphasize that the question of non-emptiness is crucial for assessing some of the accounts of $K_\varphi \psi$ we consider.

- $s \vdash \neg\varphi$ *iff* $s \dashv \varphi$

- $s \dashv \neg\varphi$ *iff* $s \vdash \varphi$

- $s \vdash \varphi \wedge \psi$ *iff there exists* $t, u \in S$ *such that* $s = t \sqcup s$ *and* $t \vdash \varphi$ *and* $u \vdash \psi$

- $s \dashv \varphi \wedge \psi$ *iff* $s \dashv \varphi$ *or* $s \dashv \psi$ *or there exists* $t, u \in S$ *such that* $s = t \sqcup s$ *and* $t \dashv \varphi$ *and* $u \dashv \psi$

- $s \vdash A\varphi$ *iff for all* $t \in P$ *there is* $t' \in S$ *such that* $t' \sqcup t \in P$ *and* $t' \vdash \varphi$

- $s \dashv A\varphi$ *iff there is* $t \in P$ *such that for all* $u \in S$ *either* $t \sqcup u \notin P$ *or* $u \dashv \varphi$

- $s \vdash \varphi \Rightarrow \psi$ *iff for all* $t \in P$ *(if there is* $t' \in S$ *such that* $t' \leq t$ *and* $t' \vdash \varphi$*, then there is* $u \in S$ *such that* $t \sqcup u \in P$ *and* $u \vdash \psi$*)*

- $s \dashv \varphi \Rightarrow \psi$ *iff there is* $t \in P$ *such that there exits* $t' \in S$ *with* $t' \leq t$ *and* $t' \vdash \varphi$ *and there is no* $u \in S$ *such that* $t \sqcup u \in P$ *and* $u \vdash \psi$

We say that s *exactly verifies* φ (or *makes exactly* φ *true*) when $s \vdash \varphi$; that s *exactly falsifies* φ (or *makes exactly* φ *false*) when $s \dashv \varphi$; that s *verifies* φ (or *inexactly* verifies φ, for emphasis) when there exists $t \leq s$ such that $t \vdash \varphi$; and that s *falsifies* φ (or *inexactly* falsifies φ, for emphasis) when there exists $t \leq s$ such that $t \dashv \varphi$.

The $\vdash$-clause for $A\varphi$ is intended to echo the definition of a necessary state in (Fine, forthcoming, Sect. 5) i.e. a state is necessary just in case it is compatible with every possible state. For us, $A\varphi$ is made true just in case every possible state is compatible with a state that makes exactly φ true. In particular, every possible world will be compatible with a φ verifier, and so will inexactly verify φ. The $\vdash$-clause for $\varphi \Rightarrow \psi$, meanwhile, relativizes that for $A\varphi$ to the possible φ verifiers. Claim $\varphi \Rightarrow \psi$ is made true just in case: every possible state that (inexactly) verifies φ can be extended to a possible state that (inexactly) verifies ψ. This is intended to echo the definition of *loose verification* (and so *loose/classical consequence*) in the appendix of (Fine, 2017a): a state s loosely verifies φ just in case any state compatible with s is compatible with a state that verifies φ.

The $\dashv$-clauses for $A\varphi$ and $\varphi \Rightarrow \psi$ will not bear on the central points of our discussion. We regard them as a reasonable first pass.

As we define $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, we obtain the following exact verification and falsification clauses for disjunction:

- $s \vdash \varphi \vee \psi$ iff $s \vdash \varphi$ or $s \vdash \psi$ or $s \vdash \varphi \wedge \psi$

- $s \dashv \varphi \vee \psi$ iff there exists $t, u \in S$ such that $s = t \sqcup u$ and $t \dashv \varphi$ and $u \dashv \psi$

Exact verification in a model, logical consequence and *validity* are defined as follows: with $\varphi_1, \ldots, \varphi_n, \psi \in \mathscr{L}$:

- ψ is exactly verified in $\mathtt{M}$, denoted by $\mathtt{M} \Vdash \psi$, iff $s \vdash \psi$ for every $s \in S$,

- ψ is a logical consequence of $\{\varphi_1, \ldots, \varphi_n\}$, denoted by $\varphi_1, \ldots, \varphi_n \Vdash \psi$, iff $s \vdash \varphi_1, \ldots, s \vdash \varphi_n$ materially implies $s \vdash \psi$ for every s in every $\mathtt{M}$, and

- ψ is logically valid, denoted by $\Vdash \psi$, iff $\mathtt{M} \Vdash \psi$ for all $\mathtt{M}$.[14]

The *exact unilateral content* of φ is the proposition:

$$|\varphi| = \{s \in S : s \text{ exactly verifies } \varphi \text{ i.e. } s \vdash \varphi\}$$

The *inexact unilateral content* of φ is the proposition:

$$||\varphi|| = \{s \in S : s \text{ has a part } t \text{ that exactly verifies } \varphi\}$$

Then, the exact and inexact bilateral contents for φ are, respectively, the bilateral propositions $\langle|\varphi|, |\neg\varphi|\rangle$ and $\langle||\varphi||, ||\neg\varphi||\rangle$. We here ape the introduction of unilateral and bilateral content in (Fine, 2017a).

5 Possible Definitions of Conditional Knowledge

5.1 Bodies of Knowledge

We now aim to capture claims $K_\varphi \psi$ in the truthmaker setting, to the effect that a body of knowledge described by φ is sufficient for knowing ψ.

To start: how to think about a body of knowledge in the truthmaker setting - in particular, that associated with interpreted sentence φ? Echoing the Hintikkan approach, we take a body of knowledge to be a proposition - in particular, a set of *possible* states closed under fusion.

Four unilateral propositions are naturally associated with sentence φ:

$$|\varphi|, |\neg\varphi|, ||\varphi||, ||\neg\varphi||$$

Two bilateral propositions are naturally associated with φ:

$$\langle|\varphi|, |\neg\varphi|\rangle, \langle||\varphi||, ||\neg\varphi||\rangle$$

The second last - $\langle|\varphi|, |\neg\varphi|\rangle$ - is best viewed as *the* proposition associated with φ, since the rest can be derived from it.

In this case, we primarily think of the body of knowledge described by φ - *what* an agent knows when she knows φ - as the set $|\varphi| \cap P$, and derivatively as the set $||\varphi|| \cap P$.

The restriction to possible states is natural for the capturing of knowability-in-principle or strong warrant: these are well understood as describing a cognitively idealized agent that rules out all epistemically impossible states.[15] For such an agent, no body of knowledge includes impossible states i.e. states that require no empirical information for their elimination. The restriction is less natural for a logic of ordinary knowledge ascriptions. Nevertheless, it is interesting to see how far we can get in locating an appropriate logic of this type under this restriction, and a uniform assumption simplifies our discussion. So we proceed.

[14] These definitions extend to the whole language $\mathscr{L}_e$ in the same way.

[15] In the latter case: propositional justification is presumably independent of contingent psychological foibles.

5.2 Hintikka-Style Conditional Knowledge

We now introduce four accounts of conditional knowledge. They are naturally grouped into pairs. Account (1) is a 'ruling in' account: roughly, $K_\varphi \psi$ is (made) true because restriction to the possible φ states amounts to a restriction to the possible ψ states. Account (2) is the corresponding 'ruling out' account: roughly, $K_\varphi \psi$ is (made) true because elimination of the possible $\neg\varphi$ states amounts to elimination of the possible $\neg\psi$ states. Similar remarks apply to accounts (3) and (4).

We label this selection 'Hintikka-style' since, as we see it, each translates one of two (equivalent) conceptions of the classic Hintikkan account of $K_\varphi \psi$ into the truthmaker setting: every φ world is a ψ world; and every $\neg\psi$ world is a $\neg\varphi$ world. In contrast, their analogues come apart in the truthmaker setting: in particular, they generate different logics.

Here follow the accounts:

(1) $\mathtt{M} \Vdash K_\varphi \psi$ iff every possible state that makes φ true can be extended to a possible state that also makes ψ true.

To achieve this effect, we adopt the following definitions for the exact verification and falsification clauses:

- $s \vdash K_\varphi \psi$ iff for all $t \in P$ (if there is $t' \in S$ such that $t' \leq t$ and $t' \vdash \varphi$ then there is $u \in S$ such that $t \sqcup u \in P$ and $u \vdash \psi$)
- $s \dashv K_\varphi \psi$ iff there is $t \in P$ such that there exits $t' \in S$ with $t' \leq t$ and $t' \vdash \varphi$ and there is no $u \in S$ such that $t \sqcup u \in P$ and $u \vdash \psi$

(2) $\mathtt{M} \Vdash K_\varphi \psi$ iff every possible state that makes ψ false can always be extended to a possible state that also makes φ false.

To achieve this effect, we adopt the following definitions for $\vdash$ and $\dashv$:

- $s \vdash K_\varphi \psi$ iff for all $t \in P$ (if there is $t' \in S$ such that $t' \leq t$ and $t' \dashv \psi$, then there is $u \in S$ such that $t \sqcup u \in P$ and $u \dashv \varphi$)
- $s \dashv K_\varphi \psi$ iff there is $t \in P$ such that there exits $t' \in S$ with $t' \leq t$ and $t' \dashv \psi$ and there is no $u \in S$ such that $t \sqcup u \in P$ and $u \dashv \varphi$

(3) $\mathtt{M} \Vdash K_\varphi \psi$ iff every possible truthmaker for φ has a part that makes ψ true.

We use the following definitions for $\vdash$ and $\dashv$:

- $s \vdash K_\varphi \psi$ iff for all $t \in P$ (if $t \vdash \varphi$ then there is $t' \in S$ such that $t' \leq t$ and $t' \vdash \psi$)
- $s \dashv K_\varphi \psi$ iff there is $t \in P$ such that $t \vdash \varphi$ and there is no $t' \in S$ with $t' \leq t$ and $t' \vdash \psi$.

(4) $\mathtt{M} \Vdash K_\varphi \psi$ iff every possible falsemaker for ψ has a part that makes φ false.

We use the following definitions for $\vdash$ and $\dashv$:

- $s \vdash K_\varphi \psi$ iff for all $t \in P$ (if $t \dashv \psi$ then there is $t' \in S$ such that $t' \leq t$ and $t' \dashv \varphi$)
- $s \dashv K_\varphi \psi$ iff there is $t \in P$ such that $t \dashv \psi$ and there is no $t' \in S$ with $t' \leq t$ and $t' \dashv \varphi$.

Proposition 2.

(1) Definition (1) validates everything: uncontroversial, relatively uncontroversial and controversial.

(2) Definition (2) validates everything except ***SPC*** *and* ***DS****.*

(3) Definition (3) validates everything except ***SPC*** *and* ***Apriority****.*

(4) Definition (4) validates everything except ***SPC*** *and* ***DS****.*

5.3 Immanent Conditional Knowledge

We now introduce two final accounts of $K_\varphi \psi$. We label both under the heading of *immanent* accounts of conditional knowledge. Yablo (2014, Sect. 7.3) observes that, for some φ and ψ, knowing ψ is part of knowing φ. In this case, we have an instance of *immanent closure*: knowing φ entails knowing ψ because knowledge is closed under parts. It is natural to elaborate as follows: knowing ψ is part of knowing φ just in case content ψ is part of content φ. To know Jane is a lawyer and a fisherman is to know she is a lawyer, since the proposition that Jane is a lawyer is part of the proposition that Jane is a lawyer and a fisherman. To know Jane is an expert lawyer is to know she is a lawyer, since the proposition that Jane is a lawyer is part of the proposition that Jane is an expert lawyer.

Yablo (2014) and Fine (2016, 2017a) offer similar accounts of content parthood: roughly, content ψ is part of content φ just in case both φ entails ψ and the subject matter of φ includes that of ψ. Fine explicates this sentiment in terms of partial verification. Let A and B be unilateral propositions on Fine's account: each a set of states, thought of as exact verifiers. Then, by Fine's lights, B is part of A just in case (i) every exact verifier for A has an exact verifier for B as a part and (ii) every exact verifier for B is contained in some exact verifier for A. By (i), if A is (made) true then B is (made) true; by (ii), if B is (made) true, then A is partly (made) true. The connection to subject matter is drawn by way of the account in Fine (2017b) of the subject matter of a unilateral proposition: A's subject matter is the fusion of the verifiers for A. It follows that if B is part of A then B's subject matter is contained in A's subject matter.

Fine (2017a, Sect. 5) extends his definition of content parthood to the bilateral case: $\mathbf{B} = \langle B, \overline{B} \rangle$ is part of $\mathbf{A} = \langle A, \overline{A} \rangle$ just in case (i) B is part of A and (ii) $\overline{B} \subseteq \overline{A}$ i.e. every falsifier of $\mathbf{B}$ is a falsifer of $\mathbf{A}$.

All this suggest an account of $K_\varphi \psi$: $K_\varphi \psi$ holds just in case the content (expressed by) ψ is part of the content (expressed by) φ, by Finean lights. This serves, roughly, as our fifth account of $K_\varphi \psi$.

A variation is nearby. Fine (2017b) defines the subject matter of a *bilateral* content $\mathbf{A} = \langle A, \overline{A} \rangle$ as the fusion of the states in $A \cup \overline{A}$ i.e. the fusion of all of $\mathbf{A}$'s exact verifiers and falsifiers. This is naturally explicated further as: the subject matter of $\mathbf{B}$ is contained in that of $\mathbf{A}$ exactly when every exact verifier of $\mathbf{B}$ is contained in either an exact verifier or exact falsifier of $\mathbf{A}$; ditto for $\mathbf{B}$'s exact falsifiers. Then we may develop an 'immanent' account of $K_\varphi \psi$ as: $K_\varphi \psi$ holds exactly when φ implies ψ and the subject matter of ψ is contained in that of φ, by the lights of the forgoing account. This serves, roughly, as our sixth account of $K_\varphi \psi$. Notably, the logic thereby generated is significantly different to that generated by the fifth account.

(5) $\mathtt{M} \Vdash K_\varphi \psi$ iff, for all s, (i) if s is a possible exact verifier for φ then s verifies ψ, (ii) if s is a possible exact falsifier for ψ then it is a possible exact falsifier for φ, (iii) if s is an exact verifier for ψ then s is contained in an exact verifier for φ, .

Corresponding exact verification and falsification clauses are obtained by strengthening and weakening, respectively, the third truthmaker semantics for $K_\varphi\psi$ (given as the first items above) as follows[16]:

$s \vdash K_\varphi\psi$ iff (1) for all $t \in P$ (if $t \vdash \varphi$ then there is $t' \in S$ such that $t' \leq t$ and $t' \vdash \psi$), and
(2) for all $t \in P$ (if $t \dashv \psi$ then $t \dashv \varphi$), and
(3) for all $u \in S$ (if $u \vdash \psi$ then there is $u' \in S$ such that $u \leq u'$ and $u' \vdash \varphi$).

$s \dashv K_\varphi\psi$ iff (1) there is $t \in P$ such that $t \vdash \varphi$ and there is no $t' \in S$ with $t' \leq t$ and $t' \vdash \psi$, or
(2) there is $t \in P$ ($t \dashv \psi$ and $t \not\dashv \varphi$), or
(3) there is $u \in S$ ($u \vdash \psi$ and there is no $u' \in S$ such that $u \leq u'$ and $u' \vdash \varphi$).

(6) $\mathtt{M} \Vdash K_\varphi\psi$ iff (i) for every possible truthmaker for φ there is a compatible state that exactly verifies ψ and (ii) the subject matter of ψ is contained in the overall subject matter of φ.

Corresponding exact verification and falsification clauses are obtained by strengthening and weakening, respectively, the first truthmaker semantics for $K_\varphi\psi$ (given as the first items above) as follows:

$s \vdash K_\varphi\psi$ iff (1) for all $t \in P$ (if there is $t' \in S$ such that $t' \leq t$ and $t' \vdash \varphi$ then there is $u \in S$ such that $t \sqcup u \in P$ and $u \vdash \psi$)), and
(2) for all $u \in S$ (if $u \vdash \psi$ then there is $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$), and
(3) for all $u \in S$ (if $u \dashv \psi$ then there is $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$)

$s \dashv K_\varphi\psi$ iff (1) there is $t \in P$ such that there exits $t' \in S$ with $t' \leq t$ and $t' \vdash \varphi$, and there is no $u \in S$ such that $t \sqcup u \in P$ and $u \vdash \psi$, or
(2) there is $u \in S$ ($u \vdash \psi$ and there is no $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$), or
(3) there is $u \in S$ ($u \dashv \psi$ and there is no $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$)

Now that all six accounts of $K_\varphi\psi$ are on the table, we can state a result that partially vindicates our choices for the exact falsification conditions proposed for $A\varphi$, $\varphi \Rightarrow \psi$ and $K_\varphi\psi$.

Proposition 3. *Given a model* $\mathtt{M} = \langle S, P, \leq, \mathtt{v} \rangle$ *and* $\varphi \in \mathscr{L}_e$: $|\varphi|$ *and* $|\neg\varphi|$ *are propositions - in particular, closed under fusions, for all six interpretations of* $K_\varphi\psi$.

Proposition 4.

*(1) Definition (5) invalidates everything except **Refl**, **C-Trans**, **DN**, and **Agg**.*

*(2) Definition (6) invalidates everything except **Refl**, **C-Trans**, **DN**, **W-Omni**, and **Agg**.*

[16] We use $s \not\vdash \varphi$ ($s \not\dashv \varphi$) as an abbreviation for "it is not the case that $s \vdash \varphi$ ($s \dashv \varphi$)".

	1	2	3	4	5	6
Simplification	✓	✓	✓	✓	X	X
Reflexivity	✓	✓	✓	✓	✓	✓
Cautious Transitivity	✓	✓	✓	✓	✓	✓
Cautious Strengthening	✓	✓	✓	✓	X	X
Double Negation	✓	✓	✓	✓	✓	✓
Weak Simplification	✓	✓	✓	✓	X	X
Weak Omniscience.	✓	✓	✓	✓	X	✓
Apriority	✓	✓	X	✓	X	X
Negative Addition	✓	✓	✓	✓	X	X
Agglomeration	✓	✓	✓	✓	✓	✓
Single-Premise Closure	✓	X	X	X	X	X
Disjunctive Syllogism	✓	X	✓	X	X	X
Strengthening	✓	✓	✓	✓	X	X

Table 1: Validities (✓) and invalidities (X) in models given in Definition 5. Numbers in the top row refer to the proposed truthmaker accounts of $K_\varphi \psi$ in §5.

Table 1 summarizes Propositions 2 and 4. It turns out that our 'immanent' definitions of conditional knowledge (unlike the previous 'Hintikka-style' ones) are sensitive to whether or not an important constraint is imposed on our models.

Definition 7 (Atomic Verfiability)**.** *A proposition is* verifiable *if it is non-empty. A model* $\mathtt{M} = \langle S, P, \leq, \mathtt{v} \rangle$ *has* atomic verifiability *if for each atom* $p \in \mathsf{Prop}$*,* p^+ *and* p^- *are verifiable.*

Proposition 5. *In models with atomic verifiability,*

(1) Definitions (1)-(4) (in)validate exactly the same principles given in Proposition 2,

*(2) Definition (5) validates everything except **W-Omni**, **Apriority**, **NA**, **SPS**, and **DS**,*

*(3) Definition (6) validates everything except **Apriority**, **NA**, **SPS**.*

(Moreover, a close inspection of the proof of Proposition 5 shows that having p^+ and p^- closed under fusions affects the list of validities and invalidities only for definitions (5) and (6).) Table 2 provides a summary of the results in Proposition 5.

This is foreshadowed in the literature. Fine (forthcoming, Sect.4) notes the sensitivity of his account of partial content to whether or not empty sets of verifiers are admitted, observing that the content of $\varphi \wedge \psi$ need not contain that of φ if $\varphi \wedge \psi$ has no verifiers.

6 Discussion

With the summaries in Tables 1 and 2 at hand, we offer preliminary remarks on the capacity of truthmaker semantics to help epistemic logic escape Hintikkan confines.

First, remarks on the 'positive' side. The flexibility and power of truthmaker semantics is in full evidence: we already have six distinct accounts of $K_\varphi \psi$ on the table that deserve serious attention. If we restrict attention to models with atomic verifiability, then the corresponding logics all yield the

	1	2	3	4	5	6
Simplification	✓	✓	✓	✓	✓	✓
Reflexivity	✓	✓	✓	✓	✓	✓
Cautious Transitivity	✓	✓	✓	✓	✓	✓
Cautious Strengthening	✓	✓	✓	✓	✓	✓
Double Negation	✓	✓	✓	✓	✓	✓
Weak Simplification	✓	✓	✓	✓	✓	✓
Weak Omniscience	✓	✓	✓	✓	X	✓
Apriority	✓	✓	X	✓	X	X
Negative Addition	✓	✓	✓	✓	X	X
Agglomeration	✓	✓	✓	✓	✓	✓
Single-Premise Closure	✓	X	X	X	X	X
Disjunctive Syllogism	✓	X	✓	X	X	✓
Strengthening	✓	✓	✓	✓	✓	✓

Table 2: Validities (✓) and invalidities (X) in models with atomic verifiability.

uncontroversial validities of §2. Further, those suspicious of **Neg Add**, **SPC** and **DS** will mark progress: various natural set-ups in the truthmaker setting invalidate these principles. What's more, we have located a number of set-ups that reject **Apriority**: if validating **Apriority** is a dividing line between logics that are candidates for a logic of knowability, on one hand, and those that are candidates for a logic of knowledge ascription or warrant transmission, on the other, then the truthmaker setting opens the door to serious investigation of the latter. Indeed, our fifth account of $K_\varphi \psi$ holds attraction as an account of conditional knowledge ascription: it violates all of **Neg Add**, **SPC** and **DS**. Good news, some might think: ordinary knowledge ascription should not be closed under inference (even simple ones), only content parthood. On the other hand, the sixth account of $K_\varphi \psi$ will appeal to some as an account of strong warrant transmission: **Neg Add** and **SPC** fail, as one might hope (if one is sympathetic to, say, Cartesian and Criterion counter-examples). Meanwhile, that the sixth account validates **DS** holds attraction for one who accepts certain general principles marking warrant transmission failure: conclusion q neither (apparently) introduces new subject matter relative to premises $\neg p$ and $p \vee q$, nor (apparently) is knowledge of q a prerequisite for learning $\neg p$ or $p \vee q$.

Now for remarks on the 'negative' side. None of our candidate systems invalidate **Agg** or **Strength**. Those sympathetic to alleged counter-examples to these principles will see little progress here. Of course, we did not expend serious effort in designing systems with such effects, so the present concern is best understood as impetus for further investigation. (Indeed, if **Agg** or **Strength** are rejected, a natural thought is that these effects are products of probabilistic features of knowledge ascription, as has received recent discussion in, for instance, (Brown, 2018).)

Further, the *combination* of validities/invalidities generated by some of our systems might give one pause. The second, third and fourth accounts of $K_\varphi \psi$ all invalidate **SPC** without invalidating **Neg Add**. Some will find this objectionable: as per §2, the alleged counter-examples **SPC** and **Neg Add** mostly go hand-in-hand. Conjoined with the view that the logic of knowability observes **Apriority**, this problematizes our candidates for $K_\varphi \psi$ that depart significantly from the classic Hintikkan approach. (Though see (Roush, 2010) for an intriguing case for accepting **Neg Add** even if one rejects **SPC**.)

Finally, one might worry about the fact that accounts five and six only accept certain uncontroversial principles when restricted to models with atomic verifiability. These accounts are apparently at odds with an appealing view: atomic mathematical claims like '3 is prime' have no falsemakers. Compare $p \vee \neg p$, where p is contingent: as Fine (forthcoming) points out, we can make sense of a falsemaker for this claim as a (virtual) fusion of two possible states: a truthmaker for p and a truthmaker for $\neg p$. In contrast, situations that make '3 is prime' false seem hard to get an intuitive grip on, even if granted as impossible. (Though see Fine (forthcoming, Sect. 5) for a proposed method for constructing such states out of possible states). Resistance to admitting such situations amounts to a resistance to accepting accounts five and six. Hence, the most promising candidates we've canvassed for the logic of knowledge ascription or warrant transmission are subject to serious dispute.

7 Conclusion

As they stand, the considerations of the last section are obviously not decisive. Hence, the firmest conclusions we can draw are as follows. First, truthmaker semantics allows for the development of various logical systems that are worth taking seriously as candidate epistemic logics. Second, the sample we consider in this paper establishes that such logics can depart from a normal modal logic (and each other), yielding patterns of validities and invalidities that bear on longstanding epistemic paradoxes.

Much work remains. We have barely scratched the surface in assessing the relative merits of our candidate logics. Further, we have said nothing on the subject of meta-logical results. Finally, subtle variations of our candidate systems can no doubt be produced by tweaking various technical parameters. The costs and gains of such tweaks remain to be seen.

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Proofs

Proof of Proposition 2

Lemma 1. *Given a model* $\langle S, P, \leq \mathtt{v}\rangle$ *and* $s, t \in S$*: if* $s \notin P$ *then* $s \sqcup t \notin P$.

Proof. Let $s, t \in S$ such that $s \notin P$ and suppose $s \sqcup t \in P$. This implies, since P is downward closed and $s \leq s \sqcup t$, that $s \in P$, contradicting the assumption. □

Proof of Proposition 2:
Counter-models are given in figures immediately below the corresponding explanations. In figures of models, white diamonds represent impossible states and black dots represent possible states. Exact verification and falsification are given by labelling nodes together with symbols $\vdash$ and $\dashv$, respectively.

Let $\mathtt{M} = \langle S, P, \leq, \mathtt{v}\rangle$ be a model and $s \in S$ be a state.

(1) Recall the exact verification and falsification clauses:

- $s \vdash K_\varphi \psi$ iff for all $t \in P$ (if there is $t' \in S$ such that $t' \leq t$ and $t' \vdash \varphi$ then there is $u \in S$ such that $t \sqcup u \in P$ and $u \vdash \psi$)
- $s \dashv K_\varphi \psi$ iff there is $t \in P$ such that there exits $t' \in S$ with $t' \leq t$ and $t' \vdash \varphi$ and there is no $u \in S$ such that $t \sqcup u \in P$ and $u \vdash \psi$

Note that none of the (in)validities depends on p^+ or p^- being closed under fusion.

Simplification: $K_{p\wedge q} p, K_{p\wedge q} q$:
We prove only the former, the latter follows similarly: let $t \in P$ such that there is $t' \in S$ with $t' \leq t$ and $t' \vdash p \wedge q$. Thus, by the exact verification clause for $\wedge$, there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \vdash p$, and $u' \vdash q$. Since $u \leq t' \leq t$, we have $t \sqcup u = t \in P$. As $u \vdash p$ as well, we obtain that $s \vdash K_{p\wedge q} p$.

Reflexivity: $K_p p$
Let $t \in P$ such that there is $t' \in S$ with $t' \leq t$ and $t' \vdash p$. Since $t \sqcup t' = t \in P$ as well, we obtain that $s \vdash K_p p$.

Cautious Transitivity: $K_p q, K_{p\wedge q} r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_{p\wedge q} r$, and let $t \in P$ such that there is $t' \in S$ with $t' \leq t$ and $t' \vdash p$. Then, by (a), there is $u \in S$ such that $t \sqcup u \in P$ and $u \vdash q$. Thus, by the exact verification clause for $\wedge$, we obtain $t' \sqcup u \vdash p \wedge q$. Then, as $t' \sqcup u \leq t \sqcup u \in P$, by (b), we have that there is $u' \in S$ such that $(t \sqcup u) \sqcup u' \in P$ and $u' \vdash r$. Now consider $t \sqcup u'$. Since $t \sqcup u' \leq (t \sqcup u) \sqcup u' \in P$, we have $t \sqcup u' \in P$. Since $u' \vdash r$ as well, we conclude that $s \vdash K_p r$.

Cautious Strengthening: $K_p q, K_p r \Vdash K_{p\wedge q} r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_p r$, and let $t \in P$ such that there is $t' \in S$ with $t' \leq t$ and $t' \vdash p \wedge q$. Hence, by the exact verification clause for $\wedge$, there are $u, u' \in S$ with $u \sqcup u' = t'$, $u \vdash p$, and $u' \vdash q$. Then, since $u \leq t' \leq t$, $u \vdash p$, by (b), we obtain that there is $s' \in S$ such that $s' \sqcup t \in P$ and $s' \vdash r$. Therefore, $s \vdash K_{p\wedge q} r$.

Double Negation: $K_p(\neg\neg p)$
Similar to the proof of Reflexivity, note that $|p| = |\neg\neg p|$.

Weak Simplification: $K_{p\wedge q}(p\vee q)$
Let $t\in P$ such that there is $t'\in S$ with $t'\leq t$ and $t'\vdash p\wedge q$. This implies, by the exact verification clause for $\vee$, that $t'\vdash p\vee q$. Since $t'\sqcup t=t\in P$, we obtain that $s\vdash K_{p\wedge q}(p\vee q)$.

Weak Omniscience: $K_p(p\vee\neg p)$
Let $t\in P$ such that there is $t'\in S$ with $t'\leq t$ and $t'\vdash p$. This implies, by the exact verification clause for $\vee$, that $t'\vdash p\vee\neg p$. Since $t'\sqcup t=t\in P$, we obtain that $s\vdash K_p(p\vee\neg p)$.

Apriority: $Ap\Vdash K_\varphi p$.
Suppose $s\vdash Ap$. This means that for all $t\in P$ there is $t'\in S$ such that $t'\sqcup t\in P$ and $t'\vdash p$. Therefore, $s\vdash K_\varphi p$.

Negative Addition: $K_\varphi p\Vdash K_\varphi\neg(\neg p\wedge q)$
Suppose $s\vdash K_\varphi p$ and let $t\in P$ such that there is $t'\in S$ with $t'\leq t$ and $t'\vdash\varphi$. Then, by the assumption that $s\vdash K_\varphi p$, there is $u\in S$ such that $t\sqcup u\in P$ and $u\vdash p$. This means that $u\dashv\neg p$, and that $u\dashv\neg p\wedge q$. Therefore, $u\vdash\neg(\neg p\wedge q)$. As $t\sqcup u\in P$ as well, we conclude that $s\vdash K_\varphi\neg(\neg p\wedge q)$.

Agglomeration: $K_\varphi p, K_\varphi q\Vdash K_\varphi(p\wedge q)$
Suppose (a) $s\vdash K_\varphi p$ and (b) $s\vdash K_\varphi q$ and let $t\in P$ such that there is $t'\in S$ with $t'\leq t$ and $t'\vdash\varphi$. Then, by (a), there is $u\in S$ such that $t\sqcup u\in P$ and $u\vdash p$. Since $t'\leq t\leq t\sqcup u$ we obtain by (b) that there is $u'\in S$ such that $(t\sqcup u)\sqcup u'\in P$ and $u'\vdash q$. Then, $u\sqcup u'\vdash p\wedge q$. As $t\sqcup(u\sqcup u')=(t\sqcup u)\sqcup u'\in P$, we obtain that $s\vdash K_\varphi(p\wedge q)$.

Single-Premise Closure: $K_\varphi p, p\Rightarrow q\Vdash K_\varphi q$
Suppose (a) $s\vdash K_\varphi p$ and (b) $s\vdash p\Rightarrow q$ and let $t\in P$ such that there is $t'\in S$ with $t'\leq t$ and $t'\vdash\varphi$. Then, by (a), there is $u\in S$ such that $t\sqcup u\in P$ and $u\vdash p$. Since $u\leq t\sqcup u\in P$, we obtain by (b) that there is $u'\in S$ such that $(t\sqcup u)\sqcup u'\in P$ and $u'\vdash q$. Since $t\sqcup u'\leq(t\sqcup u)\sqcup u'\in P$, we have $t\sqcup u'\in P$. As $u'\vdash q$ as well, we conclude that $s\vdash K_\varphi q$.

Disjunctive Syllogism: $K_\varphi\neg p, K_\varphi(p\vee q)\Vdash K_\varphi q$
Suppose (a) $s\vdash K_\varphi\neg p$ and (b) $s\vdash K_\varphi(p\vee q)$ and let $t\in P$ such that there is $t'\in S$ with $t'\leq t$ and $t'\vdash\varphi$. Then, by (b), there is $u\in S$ such that $t\sqcup u\in P$ and $u\vdash p\vee q$. Since $t'\leq t\sqcup u\in P$, we have by (a) that there is $u'\in S$ such that $(t\sqcup u)\sqcup u'\in P$ and $u'\vdash\neg p$. Therefore, there is no $s'\leq u$ such that $s'\vdash p$: otherwise $(t\sqcup u)\sqcup u'\notin P$, by Lemma 1. Then, since $u\vdash p\vee q$, we have that $u\vdash q$. Since $t\sqcup u\in P$ as well, we conclude that $s\vdash K_\varphi q$.

Strengthening: $K_p r\Vdash K_{p\wedge q}r$
Same as the proof for Cautious Strengthening.

(2) Recall the exact verification and falsification clauses:

- $s\vdash K_\varphi\psi$ iff for all $t\in P$ (if there is $t'\in S$ such that $t'\leq t$ and $t'\dashv\psi$, then there is $u\in S$ such that $t\sqcup u\in P$ and $u\dashv\varphi$)
- $s\dashv K_\varphi\psi$ iff there is $t\in P$ such that there exits $t'\in S$ with $t'\leq t$ and $t'\dashv\psi$ and there is no $u\in S$ such that $t\sqcup u\in P$ and $u\dashv\varphi$

Note that none of the (in)validities depends on p^+ or p^- being closed under fusion.

Simplification: $K_{p\wedge q}p, K_{p\wedge q}q$
We prove only the former, the latter follows similarly: let $t\in P$ such that there is $t'\in S$ with $t'\leq t$

and $t' \dashv p$. Thus, by the exact falsification clause for $\wedge$, we have $t' \dashv p \wedge q$. Since $t \sqcup t' = t \in P$ as well, we conclude that $s \vdash K_{p\wedge q}p$.

Reflexivity: $K_p p$
Let $t \in P$ such that there is $t' \in S$ with $t' \leq t$ and $t' \dashv p$. Since $t \sqcup t' = t \in P$ as well, we have that $s \vdash K_p p$.

Cautious Transitivity: $K_p q, K_{p\wedge q} r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_{p\wedge q} r$, and let $t \in P$ such that there is $t' \in S$ with $t' \leq t$ and $t' \dashv r$. Then, by (b), there is $u \in S$ such that $t \sqcup u \in P$ such that $u \dashv p \wedge q$. This means that, either (1) $u \dashv p$, or (2) $u \dashv q$, or (3) there are t', s' such that $u = t' \sqcup s'$, $t' \dashv p$ and $s' \dashv q$. If (1) is the case, we are done. If (2) is the case: since $u \leq u \sqcup t$ we obtain by (a) that there is a $t'' \in S$ such that $(u \sqcup t) \sqcup t'' \in P$ and $t'' \dashv p$. Since $t \sqcup t'' \leq (u \sqcup t) \sqcup t'' \in P$, we have $t \sqcup t'' \in P$. If (3) is the case: since $t' \leq u$, we have $t \sqcup t' \leq t \sqcup u \in P$. We therefore have $t \sqcup t' \in P$ and $t' \dashv p$. Therefore, we can conclude that $s \vdash K_p r$.

Cautious Strengthening: $K_p q, K_p r \Vdash K_{p\wedge q} r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_p r$, and let $t \in P$ such that there is $t' \in S$ with $t' \leq t$ and $t' \dashv r$. Then, by (b), there is $u \in S$ such that $t \sqcup u \in P$ and $u \dashv p$. Then, by the exact falsification clause of $\wedge$, we have $u \dashv p \wedge q$. This implies that $s \vdash K_{p\wedge q} r$.

Double Negation: $K_p(\neg\neg p)$
Similar to the proof of Reflexivity, note that $|\neg p| = |\neg\neg\neg p|$.

Weak Simplification: $K_{p\wedge q}(p \vee q)$
Let $t \in P$ such that there is $t' \in S$ with $t' \leq t$ and $t' \dashv p \vee q$. Thus, by the exact falsification clause for $\vee$, there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \dashv p$, and $u' \dashv q$. This implies that $t' \dashv p \wedge q$. Since $t \sqcup t' = t \in P$, conclude that $s \vdash K_{p\wedge q}(p \vee q)$.

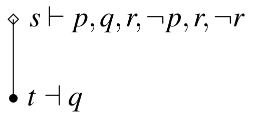
Weak Omniscience: $K_p(p \vee \neg p)$
Since no possible state exactly falsifies $p \vee \neg p$, weak omniscience is vacuously valid.

Apriority: $Ap \Vdash K_\varphi p$
Suppose $s \vdash Ap$. This means that for all $t \in P$ there is $t' \in S$ such that $t' \sqcup t \in P$ and $t' \vdash p$. This implies that there is no $t \in P$ such that $t \dashv p$, thus, $s \vdash K_\varphi p$ is vacuously the case.

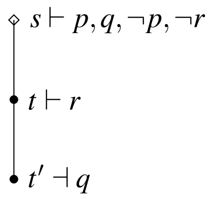
Negative Addition: $K_\varphi p \Vdash K_\varphi \neg(\neg p \wedge q)$
Suppose $s \vdash K_\varphi p$ and let $t \in P$ such that there is $t' \in S$ with $t' \leq t$ and $t' \dashv \neg(\neg p \wedge q)$. The latter means, by the exact falsification of $\neg$, that $t' \vdash \neg p \wedge q$. Thus, there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \vdash \neg p$, and $u' \vdash q$. Since $u \leq t' \leq t$ and $u \dashv p$, we obtain by the first assumption that there is $s' \in S$ such that $t \sqcup s' \in P$ and $s' \dashv \varphi$. We then conclude that $s \vdash K_\varphi \neg(\neg p \wedge q)$.

Agglomeration: $K_\varphi p, K_\varphi q \Vdash K_\varphi(p \wedge q)$
Suppose (a) $s \vdash K_\varphi p$ and (b) $s \vdash K_\varphi q$ and let $t \in P$ such that there is $t' \in S$ with $t' \leq t$ and $t' \dashv p \wedge q$. Therefore, either $t' \dashv p$, or $t' \dashv q$, or there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \dashv p$, and $u' \dashv q$. If $t' \dashv p$, then by (a) there is $s' \in S$ such that $t \sqcup s'$ and $s' \dashv \varphi$. If $t' \dashv q$, then by (b) there is $s' \in S$ such that $t \sqcup s'$ and $s' \dashv \varphi$. If there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \dashv p$, and $u' \dashv q$, we obtain the same results by (a) and (b) since $u, u' \leq t' \leq t$. Therefore, $s \vdash K_\varphi(p \wedge q)$.

Single-Premise Closure: $K_\varphi p, p \Rightarrow q \Vdash K_\varphi q$
Counterexample: Since none of the possible states exactly falsifies or verifies p, we have $K_r p$ and $p \Rightarrow q$ exactly verified at every state. However, $t \dashv q$ and $t \leq t$ but there is no state t' such that $t \sqcup t' \in P$ and $t' \dashv r$. Therefore, no state exactly verifies $K_r q$.



Disjunctive Syllogism: $K_\varphi \neg p, K_\varphi (p \vee q) \Vdash K_\varphi q$
Counterexample: Since none of the possible states exactly falsifies $\neg p$ or $p \vee q$, we have $K_r \neg p, K_r(p \vee q)$ exactly verified at every state. Moreover, $t' \dashv q$ and $t' \leq t'$ but there is no state s' such that $s' \sqcup t' \in P$ and $s' \dashv r$. Therefore, no state exactly verifies $K_r q$.



Strengthening: $K_p r \Vdash K_{p \wedge q} r$
Same as the proof for Cautious Strengthening.

(3) Recall the exact verification and falsification clauses:

- $s \vdash K_\varphi \psi$ iff for all $t \in P$ (if $t \vdash \varphi$ then there is $t' \in S$ such that $t' \leq t$ and $t' \vdash \psi$)
- $s \dashv K_\varphi \psi$ iff there is $t \in P$ such that $t \vdash \varphi$ and there is no $t' \in S$ with $t' \leq t$ and $t' \vdash \psi$.

Note that none of the (in)validities depends on p^+ or p^- being closed under fusion.

Simplification: $K_{p \wedge q} p, K_{p \wedge q} q$
Let $t \in P$ such that $t \vdash p \wedge q$. This means that there are $u, u' \in S$ such that $u \sqcup u' = t$, $u \vdash p$, and $u' \vdash q$. Since $u \leq t$ and $u' \leq t$, we conclude that $s \vdash K_{p \wedge q} p$ and $s \vdash K_{p \wedge q} q$.

Reflexivity: $K_p p$
Follows from the fact that for all $t \in P$, $t \leq t$.

Cautious Transitivity: $K_p q, K_{p \wedge q} r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_{p \wedge q} r$, and let $t \in P$ such that $t \vdash p$. Then, by (a), there is $u \in S$ such that $u \leq t$ such that $u \vdash q$. This means that $u \sqcup t = t \vdash p \wedge q$. Thus, by (b), there is $t' \in S$ such that $t' \leq t$ and $t' \vdash r$. Therefore, we conclude that $s \vdash K_p r$.

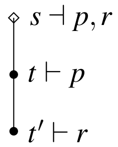
Cautious Strengthening: $K_p q, K_p r \Vdash K_{p \wedge q} r$
Suppose (a) $s \vdash K_p q$ and (b) $K_p r$, and let $t \in P$ such that $t \vdash p \wedge q$. The latter means that there are $u, u' \in S$ such that $u \sqcup u' = t$, $u \vdash p$, and $u' \vdash q$. Then, by (b), there is a $t' \leq u$ such that $t' \vdash r$. As $t' \leq u \leq t$, we conclude that $s \vdash K_{p \wedge q} r$.

Double Negation: $K_p(\neg\neg p)$
Similar to the proof of Reflexivity, note that $|p| = |\neg\neg p|$.

Weak Simplification: $K_{p\wedge q}(p \vee q)$
Let $t \in P$ such that $t \vdash p \wedge q$. This implies, by the exact verification clause of $\vee$, that $t \vdash p \vee q$. Since $t \leq t$, we obtain that $s \vdash K_{p\wedge q}(p \vee q)$.

Weak Omniscience: $K_p(p \vee \neg p)$
Let $t \in P$ such that $t \vdash p$. This means, by the exact verification clause of $\vee$, that $t \vdash p \vee \neg p$. Since $t \leq t$, we obtain that $s \vdash K_p(p \vee \neg p)$.

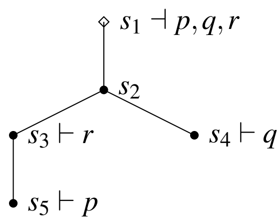
Apriority: $Ap \Vdash K_\varphi p$
Counterexample: Ap is exactly verified everywhere in the model since the only possible states are t and t', and $t \vdash p$, $t' \sqcup t = t \sqcup t = t \in P$. However, although $t' \vdash r$ there is no $u \leq t'$ such that $u \vdash p$. Therefore, no state exactly verifies $K_r p$.



Negative Addition: $K_\varphi p \Vdash K_\varphi \neg(\neg p \wedge q)$
Suppose $s \vdash K_\varphi p$ and let $t \in P$ such that $t \vdash \varphi$. Then, by the assumption, there is $t' \leq t$ such that $t' \vdash p$. Then, following the exact verification and falsification clauses for $\neg$ and $\wedge$, we obtain that $t' \vdash \neg(\neg p \wedge q)$. Therefore, as $t' \leq t$, we obtain that $s \vdash K_\varphi \neg(\neg p \wedge q)$.

Agglomeration: $K_\varphi p, K_\varphi q \Vdash K_\varphi(p \wedge q)$
Suppose (a) $s \vdash K_\varphi p$ and (b) $s \vdash K_\varphi q$ and let $t \in P$ such that $t \vdash \varphi$. Then, by (a), there is $u \in S$ such that $u \leq t$ and $u \vdash p$. And, by (b), there is $u' \in S$ such that $u' \leq t$ and $u' \vdash q$. Therefore, $u \sqcup u' \vdash p \wedge q$. Moreover, $u \sqcup u' \leq t$ since both $u \leq t$ and $u' \leq t$. Therefore, $s \vdash K_\varphi(p \wedge q)$.

Single-Premise Closure: $K_\varphi p, p \Rightarrow q \Vdash K_\varphi q$
Counterexample: The only possible state that exactly verifies r is s_3 and $s_5 \leq s_3$ such that $s_5 \vdash p$. Thus, $K_r p$ is exactly verified everywhere on the model. Moreover, only the possible states s_2, s_3 and s_5 are such that $s_5 \leq s_2, s_5 \leq s_3$, $s_5 \leq s_5$ and $s_5 \vdash p$, and $s_4 \vdash q$, $s_2 \sqcup s_4, s_3 \sqcup s_4, s_5 \sqcup s_4 \in P$. Therefore, $p \Rightarrow q$ is exactly verified everywhere in the model. However, $s_3 \vdash r$ but there is no $u \in S$ such that $u \leq s_3$ and $u \vdash q$. Therefore, no state exactly verifies $K_r q$.



Disjunctive Syllogism: $K_\varphi \neg p, K_\varphi(p \vee q) \Vdash K_\varphi q$
Suppose (a) $s \vdash K_\varphi \neg p$ and (b) $s \vdash K_\varphi(p \vee q)$ and let $t \in P$ such that $t \vdash \varphi$. Then, by (a), there is

$u \in S$ such that $u \leq t$ and $u \vdash \neg p$. This implies, since $t \in P$, that there is no $t' \leq t$ such that $t' \vdash p$ (by Lemma 1). Moreover, (b) and the fact that $t \vdash \varphi$ imply that $t \vdash p \vee q$. Therefore, we obtain that $t \vdash q$. Hence, as $t \leq t$, we conclude that $s \vdash K_{\varphi} q$.

Strengthening: $K_p r \Vdash K_{p \wedge q} r$
Same as the proof for Cautious Strengthening.

(4) Recall the exact verification and falsification clauses:

- $s \vdash K_{\varphi}\psi$ iff for all $t \in P$ (if $t \dashv \psi$ then there is $t' \in S$ such that $t' \leq t$ and $t' \dashv \varphi$)
- $s \dashv K_{\varphi}\psi$ iff there is $t \in P$ such that $t \dashv \psi$ and there is no $t' \in S$ with $t' \leq t$ and $t' \dashv \varphi$.

Note that none of the (in)validities depends on p^+ or p^- being closed under fusion.

Simplification: $K_{p \wedge q} p, K_{p \wedge q} q$
We prove only the former, the latter follows similarly: let $t \in P$ such that $t \dashv p$. This implies, by the exact falsification clause of $\wedge$, that $t \dashv p \wedge q$. Since $t \leq t$, we conclude that $s \vdash K_{p \wedge q} p$.

Reflexivity: $K_p p$
Follows from the fact that for all $t \in P$, $t \leq t$.

Cautious Transitivity: $K_p q, K_{p \wedge q} r \Vdash K_p r$

Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_{p \wedge q} r$, and let $t \in P$ such that $t \dashv r$. Then, by (b), there is $t' \in S$ such that $t' \leq t$ and $t' \dashv p \wedge q$. This means that either (1) $t' \dashv p$, or (2) $t' \dashv q$, or (3) there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \dashv p$ and $u' \dashv q$. Since $u, u' \leq t' \leq t \in P$, we have that $t', u, u' \in P$. If (1) is the case, since $t' \leq t$, we have the desired result. If (2) is the case, by (a), there is $s' \in S$ such that $s' \leq t'$ and $s' \dashv p$. As $s' \leq t' \leq t$ and $\leq$ is transitive, we have that $s' \leq t$. If (3) is the case: since $u \leq t' \leq t$, we have $u \leq t$. Then, as $u \dashv p$, we obtain that the desired conclusion. Therefore, $s \vdash K_p r$.

Cautious Strengthening: $K_p q, K_p r \Vdash K_{p \wedge q} r$
Suppose (a) $s \vdash K_p q$ and (b) $K_p r$, and let $t \in P$ such that $t \dashv r$. Then, by (b), there is $t' \in S$ such that $t' \leq t \in P$ and $t' \dashv p$. Then, by the exact falsification clause for $\wedge$, we have $t' \dashv p \wedge q$. Therefore, $s \vdash K_{p \wedge q} r$.

Double Negation: $K_p(\neg\neg p)$
Similar to the proof of Reflexivity, note that $|\neg p| = |\neg\neg\neg p|$.

Weak Simplification: $K_{p \wedge q}(p \vee q)$
Let $t \in P$ such that $t \dashv p \vee q$. This means that there are $u, u' \in S$ such that $u \sqcup u' = t$, $u \dashv p$, and $u' \dashv q$. This implies that $t \dashv p \wedge q$. Since $t \leq t \in$, we obtain that $s \vdash K_{p \wedge q}(p \vee q)$.

Weak Omniscience: $K_p(p \vee \neg p)$
Since no possible state exactly falsifies $(p \vee \neg p)$, weak omniscience is vacuously valid.

Apriority: $Ap \Vdash K_{\varphi} p$
Suppose $s \vdash Ap$. This means that for all $t \in P$ there is $t' \in S$ such that $t' \sqcup t \in P$ and $t' \vdash p$. This implies that there is no $t \in P$ such that $t \dashv p$, thus, $s \vdash K_{\varphi} p$ is vacuously the case.

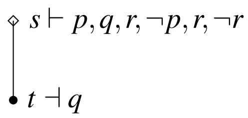
Negative Addition: $K_{\varphi} p \Vdash K_{\varphi} \neg(\neg p \wedge q)$
Suppose $s \vdash K_{\varphi} p$ and let $t \in P$ such that $t \dashv \neg(\neg p \wedge q)$. The latter means that $t \vdash \neg p \wedge q$. Thus,

there are $u, u' \in S$ such that $u \sqcup u' = t$, $u \vdash \neg p$, and $u' \vdash q$. Since $u \leq t \in P$, $u \in P$. Therefore, by the first assumption, we have that there is $u'' \leq u$ such that $u'' \dashv \varphi$. Since $u'' \leq u \leq t$, we also obtain that $s \vdash K_\varphi \neg(\neg p \wedge q)$.

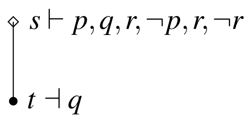
Agglomeration: $K_\varphi p, K_\varphi q \Vdash K_\varphi(p \wedge q)$
Suppose (a) $s \vdash K_\varphi p$ and (b) $s \vdash K_\varphi q$ and let $t \in P$ such that $t \dashv p \wedge q$. Therefore, either $t \dashv p$, or $t \dashv q$, or there are $u, u' \in S$ such that $u \sqcup u' = t$, $u \dashv p$, and $u' \dashv q$. If $t \dashv p$, then, by (a), there is $t' \in S$ such that $t' \leq t$ and $t' \dashv \varphi$. If $t \dashv q$, then by (b), there is $t' \in S$ such that $t' \leq t$ and $t' \dashv \varphi$. If there are $u, u' \in S$ such that $u \sqcup u' = t$, $u \dashv p$, and $u' \dashv q$, we obtain the same results by (a) and (b) since $u, u' \leq t$. Therefore, $s \vdash K_\varphi(p \wedge q)$.

Single-Premise Closure: $K_\varphi p, p \Rightarrow q \Vdash K_\varphi q$

Counterexample: Since none of the possible states exactly verifies or falsifies p, we have $K_r p$ and $p \Rightarrow q$ exactly verified at every state. However, $t \dashv q$ but there is no state t' such that $t' \leq t \in P$ and $t' \dashv r$. Therefore, no state exactly verifies $K_r q$.



Disjunctive Syllogism: $K_\varphi \neg p, K_\varphi(p \vee q) \Vdash K_\varphi q$
Counterexample: Since none of the possible states exactly falsifies $\neg p$, we have $K_r \neg p$ exactly verified at every state. Similarly, since none of the possible states exactly falsifies $p \vee q$ (since there is no possible state exactly falsifying p), we have $K_r(p \vee q)$ exactly verified at every state. However, $t \dashv q$ but there is no state t' such that $t' \leq t \in P$ and $t' \dashv r$. Therefore, no state exactly verifies $K_r q$.



Strengthening: $K_p r \Vdash K_{p \wedge q} r$
Same as the proof for Cautious Strengthening.

$\square$

Proof of Proposition 3

Lemma 2. *Given a model* $\mathtt{M} = \langle S, P, \leq, \mathtt{v} \rangle$ *and* $\varphi \in \mathscr{L}_e$: $|\neg\varphi| = \{s \in S : s \dashv \varphi\}$.

Proof. Follows immediately by the exact verification and falsification clauses for $\neg$ and the definition of $|\varphi|$. $\square$

Lemma 3. *Given a model* $\mathtt{M} = \langle S, P, \leq, \mathtt{v}\rangle$ *and* $\varphi, \psi \in \mathscr{L}_e$*, the following hold for all six interpretations of* $K_\varphi \psi$*:*

(1) if $|A\varphi| \neq \emptyset$ *then* $|A\varphi| = S$*,*

(2) if $|\neg A\varphi| \neq \emptyset$ *then* $|\neg A\varphi| = S$*,*

(3) if $|\varphi \Rightarrow \psi| \neq \emptyset$ *then* $|\varphi \Rightarrow \psi| = S$*,*

(4) if $|\neg(\varphi \Rightarrow \psi)| \neq \emptyset$ *then* $|\neg(\varphi \Rightarrow \psi)| = S$*,*

(5) if $|K_\varphi \psi| \neq \emptyset$ *then* $|K_\varphi \psi| = S$*, and*

(6) if $|\neg K_\varphi \psi| \neq \emptyset$ *then* $|\neg K_\varphi \psi| = S$*.*

Proof. Easy consequence of the corresponding exact truthmaker semantics since none of the exact verification and falsification clauses for $A\varphi$, $\varphi \Rightarrow \psi$, and $K_\varphi \psi$ is state dependent. □

Proof of Proposition 3: The proof follows by induction on the structure of φ.
Case for $p \in \mathsf{Prop}$: holds by the definition of a model (Definition 5).

Now suppose inductively that the statement holds for $\psi, \chi \in \mathscr{L}_e$.
Case for $\neg\psi$: By the induction hypothesis (IH), we already have that $|\neg\psi|$ is closed under fusion. Moreover, observe that $|\neg\neg\psi| = |\psi|$, by the exact verification and falsification clauses for $\neg$. Therefore, again by IH, $|\neg\neg\psi|$ is a proposition.
Case for $\psi \wedge \chi$: Let $\emptyset \neq T \subseteq |\psi \wedge \chi|$. This means, by the exact verification clause for $\wedge$, that for all $t \in T$, there are $u, u' \in S$ such that $u \sqcup u' = t$, $u \vdash \psi$, and $u' \vdash \chi$. Denote $T_\psi = \{u \in S : u \vdash \psi$ and there is $u' \in S$ such that $u \sqcup u' \in T$ and $u' \vdash \chi\}$, and similarly,
$T_\chi = \{u' \in S : u' \vdash \chi$ and there is $u \in S$ such that $u \sqcup u' \in T$ and $u \vdash \psi\}$. Since $T \neq \emptyset$, we have $T_\psi \neq \emptyset$ and $T_\chi \neq \emptyset$. Moreover, by IH, $\bigsqcup T_\psi \in |\psi|$ and $\bigsqcup T_\chi \in |\chi|$. Finally, since $T \subseteq |\psi \wedge \chi|$, we also have that $\bigsqcup T = \bigsqcup T_\psi \sqcup \bigsqcup T_\chi$. Therefore, by the exact verification clauses for $\wedge$, we obtain that $\bigsqcup T \in |\psi \wedge \chi|$. For $|\neg(\psi \wedge \chi)|$: let $\emptyset \neq T \subseteq |\neg(\psi \wedge \chi)|$. This means, by Lemma 2 and the exact falsification clause for $\wedge$, that for all $t \in T$, either $t \dashv \psi$ or $t \dashv \chi$, or there are $u, u' \in S$ such that $u \sqcup u' = t$, $u \dashv \psi$, and $u' \dashv \chi$. Denote $T_\psi = \{u \in S : u \vdash \psi$ and either $u \in T$ or there is $u' \in S$ such that $u \sqcup u' \in T$ and $u' \vdash \chi\}$, and, similarly,
$T_\chi = \{u' \in S : u' \vdash \chi$ and either $u' \in T$ or there is $u \in S$ such that $u \sqcup u' \in T$ and $u \vdash \psi\}$. The rest follows similarly to the case for $|\psi \wedge \chi|$.
Case for $A\psi$, $\psi \Rightarrow \chi$, and $K_\psi \chi$: Follows from Lemma 3 and the fact that $\langle S, \leq\rangle$ is a complete state space. □

Proof of Proposition 4

In figures of models, white diamonds represent impossible states and black dots represent possible states. Exact verification and falsification are given by labelling nodes together with symbols $\vdash$ and $\dashv$, respectively.

Let $\mathtt{M} = \langle S, P, \leq, \mathtt{v}\rangle$ be a model and $s \in S$ be a state.

(1) Recall the exact verification and falsification clauses:

$$s \vdash K_\varphi \psi \text{ iff } (1) \text{ for all } t \in P \text{ (if } t \vdash \varphi \text{ then there is } t' \in S \text{ such that } t' \leq t \text{ and } t' \vdash \psi\text{), and}$$
$$(2) \text{ for all } t \in P \text{ (if } t \dashv \psi \text{ then } t \dashv \varphi\text{), and}$$
$$(3) \text{ for all } u \in S \text{ (if } u \vdash \psi \text{ then there is } u' \in S \text{ such that } u \leq u' \text{ and } u' \vdash \varphi\text{).}$$

$$s \dashv K_\varphi \psi \text{ iff } (1) \text{ there is } t \in P \text{ such that } t \vdash \varphi \text{ and there is no } t' \in S \text{ with } t' \leq t \text{ and } t' \vdash \psi\text{, or}$$
$$(2) \text{ there is } t \in P \text{ (} t \dashv \psi \text{ and } t \not\dashv \varphi\text{), or}$$
$$(3) \text{ there is } u \in S \text{ (} u \vdash \psi \text{ and there is no } u' \in S \text{ such that } u \leq u' \text{ and } u' \vdash \varphi\text{).}$$

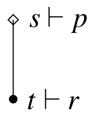
Simplification: $K_{p \wedge q} p, K_{p \wedge q} q$
Counterexample: Consider the one-state model in the following figure. $t \vdash p$ but there is no $u \in S$ such that $t \leq u$ and $u \vdash p \wedge q$. This violates item (3) in the above exact verification clause. Therefore, $K_{p \wedge q} p$ is not exactly verified by t. If we take $t \vdash q$ (instead of $t \vdash p$) in the same structure, we obtain that $K_{p \wedge q} q$ is not exactly verified.

$$\bullet\ t \vdash p$$

Reflexivity: $K_p p$
Item (1) is proven for the third definition of knowledge. Item (2) is vacuously true. Item (3) follows from the fact that $u \leq u$ for all $u \in S$.

Cautious Transitivity: $K_p q, K_{p \wedge q} r \Vdash K_p r$:
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_{p \wedge q} r$. Item (1) is proven for the third definition of knowledge. For (2): let $t \in P$ such that $t \dashv r$. Then, by (b), we have that $t \dashv p \wedge q$. Then, either (a*) $t \dashv p$, or (b*) $t \dashv q$, or (c*) there are u_1, u_2 such that $t = u_1 \sqcup u_2$, $u_1 \dashv p$, and $u_2 \dashv q$. If (a*) is the case, we are done. If (b*) is the case, by (a), we have that $t \dashv p$. If (c*) is the case, by (a), we have that $u_2 \dashv p$. Then, since $|\neg p|$ is closed under fusion, we obtain by Lemme 2, that $t = u_1 \sqcup u_2 \dashv p$. For (3): let $t \in S$ such that $t \vdash r$. Then, by (b), there is $t' \geq t$ such that $t' \vdash p \wedge q$. This means that there are $u, u' \in S$ such that $t' = u \sqcup u'$, $u \vdash p$, and $u' \vdash q$. Then, by (a), there is $s' \in S$ such that $s' \geq u'$ and $s' \vdash p$. Observe that, since $t' = u \sqcup u'$ and $s' \geq u'$, we have that $u \sqcup s' \geq t' \geq t$. And, since $|p|$ is closed under fusion, $u \sqcup s' \vdash p$. Therefore, $s \vdash K_p r$.

Cautious Strengthening: $K_p q, K_p r \Vdash K_{p \wedge q} r$
Counterexample: It is easy to check that both $K_p q$ and $K_p r$ are exactly verified in the model given below. However, no state exactly verifies $K_{p \wedge q} r$, since $t \vdash r$ but there is no u such that $t \leq u$ and $u \vdash p \wedge q$. This violates item (3) in the above exact verification clause.

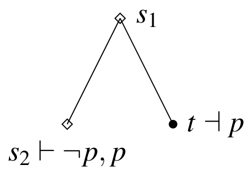


Double Negation: $K_p(\neg\neg p)$
Similar to the proof of Reflexivity.

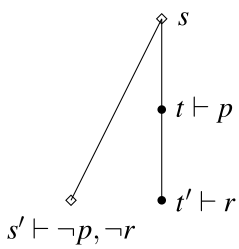
Weak Simplification: $K_{p\wedge q}(p \vee q)$
Counterexample: Consider the one-state model in the following figure: $t \vdash p$, therefore, $t \vdash p \vee q$. However, but there is no $u \in S$ such that $t \leq u$ and $u \vdash p \wedge q$. This violates item (3) in the above exact verification clause. Therefore, $K_{p\wedge q}p$ is not exactly verified by t.

$$\bullet\ t \vdash p$$

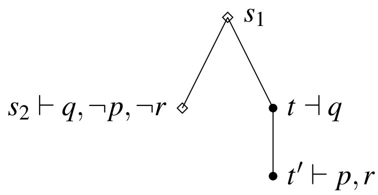
Weak Omniscience: $K_p(p \vee \neg p)$
Counterexample: For $t \vdash \neg p$, we have $t \vdash p \vee \neg p$. However, there is no $t' \in S$ such that $t' \geq t$ and $t' \vdash p$. This violates item (3) in the above exact verification clause. Therefore, no state in the model exactly verifies $K_p(p \vee \neg p)$.



Apriority: $Ap \Vdash K_\varphi p$:
Counterexample: Ap is exactly verified everywhere in the model since the only possible states are t and t', and $t \vdash p$, $t' \sqcup t = t \sqcup t = t \in P$. However, t is the only state that exactly verifies p but there is no u such that $u \geq t$ and $u \vdash r$. This violates item (3) in the above exact verification clause. Therefore, no state in the model exactly verifies $K_r p$.

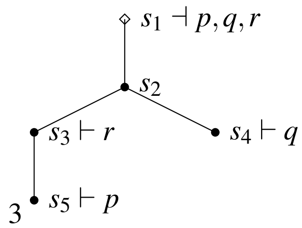


Negative Addition: $K_\varphi p \Vdash K_\varphi \neg(\neg p \wedge q)$
Counterexample: $K_r p$ is exactly verified at every state of the model. However, for example, $t \vdash \neg(\neg p \wedge q)$ but there is no $u \geq t$ such that $u \vdash r$. This violates item (3) in the above exact verification clause. Thus, no state of the model exactly verifies $K_\varphi \neg(\neg p \wedge q)$.

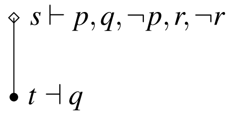


Agglomeration: $K_\varphi p, K_\varphi q \Vdash K_\varphi(p \wedge q)$
Suppose (a) $s \vdash K_\varphi p$ and (b) $s \vdash K_\varphi q$. Item (1) is proven for the third definition of knowledge. To prove item (2), let $t \in P$ such that $t \dashv p \wedge q$. Then, either (a*) $t \dashv p$, or (b*) $t \dashv q$, or (c*) there are u_1, u_2 such that $t = u_1 \sqcup u_2$, $u_1 \dashv p$, and $u_2 \dashv q$. If (a*) is the case, by (a), we have $t \dashv \varphi$. If (b*) is the case, by (b), we have that $t \dashv \varphi$. If (c*) is the case, by (a) and (b), we have that $u_1 \dashv \varphi$ and $u_2 \dashv \varphi$. Then, by Proposition 3, we obtain that $t = u_1 \sqcup u_2 \dashv \varphi$. To prove (3) suppose $t \in S$ such that $t \vdash p \wedge q$. Then, there are $u, u' \in S$ such that $t = u \sqcup u'$, $u \vdash p$, and $u' \vdash q$. Then, by (a) and (b), there are $s', t' \in S$ such that $u \leq s'$ and $s' \vdash \varphi$, and $u' \leq t'$ such that $t' \vdash \varphi$. Then, by Proposition 3, $s' \sqcup t' \vdash \varphi$. As $t = u \sqcup u'$, $u \leq s'$, and $u' \leq t'$, we also have $t \leq s' \sqcup t'$, which proves item (3).

Single-Premise Closure: $K_\varphi p, p \Rightarrow q \Vdash K_\varphi q$
Counterexample: It is easy to check that $K_r p$ and $p \Rightarrow q$ are exactly verified everywhere on the model. However, $s_3 \vdash r$ but there is no $u \in S$ such that $u \leq s_3$ and $u \vdash q$. This violates item (1) in the above exact verification clause. Therefore, no state exactly verifies $K_r q$.



Disjunctive Syllogism: $K_\varphi \neg p, K_\varphi(p \vee q) \Vdash K_\varphi q$
Counterexample: It is easy to check that both $K_r \neg p$ and $K_r(p \vee q)$ are exactly verified at every state of the model. However, $t \in P$ and $t \dashv q$ but $t \not\dashv r$. This violates item (2) in the above exact verification clause. Therefore, no state exactly verifies $K_r q$.



Strengthening: $K_p r \Vdash K_{p \wedge q} r$
See the counterexample for Cautious Strengthening.

(2) Recall the exact verification and falsification clauses:

$s \vdash K_\varphi \psi$ iff (1) for all $t \in P$ (if there is $t' \in S$ such that $t' \leq t$ and $t' \vdash \varphi$ then there is $u \in S$ such that $t \sqcup u \in P$ and $u \vdash \psi$)), and
(2) for all $u \in S$ (if $u \vdash \psi$ then there is $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$), and
(3) for all $u \in S$ (if $u \dashv \psi$ then there is $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$)

$s \dashv K_\varphi \psi$ iff (1) there is $t \in P$ such that there exits $t' \in S$ with $t' \leq t$ and $t' \vdash \varphi$, and there is no $u \in S$ such that $t \sqcup u \in P$ and $u \vdash \psi$, or
(2) there is $u \in S$ ($u \vdash \psi$ and there is no $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$), or
(3) there is $u \in S$ ($u \dashv \psi$ and there is no $u' \in S$ s.t. $u \leq u'$ and $u' \vdash \varphi \vee \neg\varphi$)

Simplification: $K_{p \wedge q} p, K_{p \wedge q} q$
Counterexample: see the counterexample for Simplification for definition (5). It violates item (2) for $K_{p\wedge q} p$.

Reflexivity: $K_p p$
Item (1) is proven for the first definition of knowledge. (2) and (3) follows from the facts that for all $t \in S$: $t \leq t$, and if $t \vdash p$ or $t \vdash \neg p$, we have $t \vdash p \vee \neg p$.

Cautious Transitivity: $K_p q, K_{p\wedge q} r \Vdash K_p r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_{p\wedge q} r$. Item (1) is proven for the first definition of knowledge. For (2): let $t \in S$ such that $t \vdash r$. Then, by (b), there is $t' \in S$ such that $t' \geq t$ and $t' \vdash (p \wedge q) \vee \neg(p \wedge q)$. We then have three cases:

Case $t' \vdash p \wedge q$: This means that there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \vdash p$, and $u' \vdash q$. The former implies that $u \vdash p \vee \neg p$. Moreover, $u' \vdash q$ implies by (a) there is $s' \in S$ such that $s' \geq u'$ and $s' \vdash p \vee \neg p$. Since $s' \geq u'$, we obtain that $s' \sqcup u \geq u \sqcup u' = t' \geq t$. Since $|p \vee \neg p|$ is closed under fusion (by Proposition 3), we also have that $s' \sqcup u \vdash p \vee \neg p$.

Case $t' \vdash \neg(p \wedge q)$: Then, we have three subcases.
If $t' \vdash \neg p$, then $t' \vdash p \vee \neg p$. Since $t' \geq t$, we obtain the desired result.
If $t' \vdash \neg q$, then by (a), there is $s' \geq t'$ such that $s' \vdash p \vee \neg p$. Since $s' \geq t' \geq t$, we obtain the desired result.
If there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \vdash \neg p$, and $u' \vdash \neg q$, it follows similar to the first case.

Case $t' \vdash (p \wedge q) \wedge \neg(p \wedge q)$: Then, there are $u, u' \in S$ such that $u \sqcup u' = t'$, $u \vdash (p \wedge q)$, and $u' \vdash \neg(p \wedge q)$. By a similar argument as in the first case, there is $s' \in S$ such that $s' \geq u$ and $s' \vdash p \vee \neg p$. Again, by a similar argument as in the second case, there is $s'' \in S$ such that $s'' \geq u'$ and $s'' \vdash p \vee \neg p$. Since $s' \geq u$ and $s'' \geq u'$, we have $s'' \sqcup s' \geq u \sqcup u' = t' \geq t$. Moreover, since $|p \vee \neg p|$ is closed under fusion, we also have that $s'' \sqcup s' \vdash p \vee \neg p$. Therefore, we have proven (2).

Item (3) follows similarly.

Cautious Strengthening: $K_p q, K_p r \Vdash K_{p\wedge q} r$
Counterexample: see the counterexample for Cautious Strengthening for definition (5). It violates

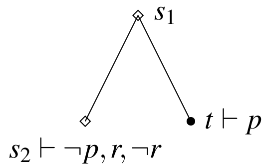
item (2) for $K_{p\wedge q}r$.

Double Negation: $K_p(\neg\neg p)$
Similar to the proof of Reflexivity.

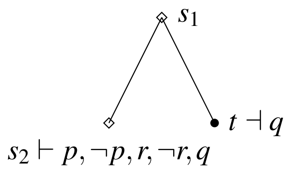
Weak Simplification: $K_{p\wedge q}(p\vee q)$
Counterexample: see the counterexample for Weak Simplification for definition (5). It violates item (2) for $K_{p\wedge q}(p\vee q)$.

Weak Omniscience: $K_p(p\vee\neg p)$
Item (1) is proven for the first definition of knowledge. For (2), let $t\in S$ such that $t\vdash p\vee\neg p$. We then obtain (2) since $t\geq t$. For (3), let $t\in S$ such that $t\dashv p\vee\neg p$. This implies that $t\vdash p\wedge\neg p$. Then, by the exact verification clause for $\vee$, we obtain that $t\vdash p\vee\neg p$. Since $t\geq t$, we have (3).

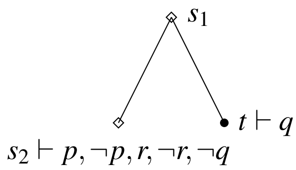
Apriority: $Ap\Vdash K_\varphi p$
Counterexample: Ap is exactly verified everywhere in the model since the only possible state is t and $t\vdash p$, $t\sqcup t=t\in P$. However, $t\vdash p$ but there is no u such that $u\geq t$ and $u\vdash r\vee\neg r$. This violates item (2). Therefore, no state in the model exactly verifies $K_r p$.



Negative Addition: $K_\varphi p\Vdash K_\varphi\neg(\neg p\wedge q)$
Counterexample: It is easy to check that $K_r p$ is exactly verified at every state. However, $t\vdash\neg(\neg p\wedge q)$ but there is no u such that $u\geq t$ and $u\vdash r\vee\neg r$. This violates item (2). Therefore, no state in the model exactly verifies $K_r\neg(\neg p\wedge q)$.



Single-Premise Closure: $K_\varphi p, p\Rightarrow q\Vdash K_\varphi q$
Counterexample: It is easy to check that $K_r p$ and $p\Rightarrow q$ are exactly verified at every state. However, $t\vdash q$ but there is no u such that $u\geq t$ and $u\vdash r\vee\neg r$. This violates item (2). Therefore, no state in the model exactly verifies $K_r q$.



Agglomeration: $K_\varphi p, K_\varphi q \Vdash K_\varphi(p \wedge q)$
Follows similarly to the proof of Agglomeration for definition (5).

Disjunctive Syllogism: $K_\varphi \neg p, K_\varphi(p \vee q) \Vdash K_\varphi q$
Counterexample: Consider the one-state model in the following figure. It is easy to see that $K_r \neg p$ and $K_r(p \vee q)$ are vacuously exactly verified in this model, since no possible state exactly verifies r and no state exactly verifies or falsifies $\neg p$ and $p \vee q$. However, $t \dashv q$ and there is no $u \in S$ such that $t \leq u$ and $u \vdash r \vee \neg r$ This violates item (3) in the above exact verification clause. Therefore, no state exactly verifies $K_r q$.

$$\bullet\ t \dashv q$$

Strengthening: $K_p r \Vdash K_{p \wedge q} r$
See the counterexample for Cautious Strengthening.

Proof of Proposition 5

Clearly, every principle that is valid in models of Definition 5 is also valid in models with atomic verifiability. Moreover, the countermodels - given in the proofs of Propositions 2 and 4 - for the common invalidities of Propositions 2, 4, and 5 satisfy atomic verifiability. Here we provide the validity proofs with respect to models with atomic verifiability for those principles that are listed as invalidities in Proposition 4. Let $\mathtt{M} = \langle S, P, \leq, \mathtt{v} \rangle$ be a model with atomic verifiability and $s \in S$ be a state.

(1) **Simplification:** $K_{p \wedge q} p, K_{p \wedge q} q$
Item (1) is proved for the third definition of knowledge. For (2), let $t \in P$ such that $t \dashv p$. This implies, by the falsification clause for $\wedge$, that $t \dashv p \wedge q$. For (3) let $t \in S$ such that $t \vdash p$. Since $q^+ \neq \emptyset$, there is $s' \in S$ such that $s' \vdash q$. Therefore, $t \sqcup s' \vdash p \wedge q$. Moreover, $t \sqcup s' \geq t$. Hence, we have (3).

Cautious Strengthening: $K_p q, K_p r \Vdash K_{p \wedge q} r$
Suppose (a) $s \vdash K_p q$ and (b) $K_p r$. Item (1) is proven for the third definition of knowledge. To prove (2) let $t \in P$ such that $t \dashv r$. Then, by (b), $t \dashv p$. This immediately implies that $t \dashv p \wedge q$. To prove (3), suppose $t \in S$ such that $t \vdash r$. Then, by (b), there is $t' \in S$ such that $t' \geq t$ and $t' \vdash p$. We know that $q^+ \neq \emptyset$ so there is $s' \in S$ such that $s' \vdash q$. This means that $t' \sqcup s' \vdash p \wedge q$. Since $t' \geq t$, we have $t' \sqcup s' \geq t$, therefore, we obtain (3).

Weak Simplification: $K_{p \wedge q}(p \vee q)$
Item (1) is proved for the third definition of knowledge. To prove (2) let $t \in P$ such that $t \dashv p \vee q$. This implies, by the exact falsification clauses of $\vee$ and $\wedge$, that $t \dashv p \wedge q$. To prove (3), suppose $t \in S$ such that $t \vdash p \vee q$. Then, either (a) $t \vdash p$, or (b) $t \vdash q$, or (c) there are $u, u' \in S$ such that $t = u \sqcup u'$, $u \vdash p$, and $u' \vdash q$. If (a) is the case: we know that $q^+ \neq \emptyset$ so there is $s' \in S$ such that $s' \vdash q$. This implies that $t \sqcup s' \vdash p \wedge q$. Since $t \sqcup s' \geq t$, we obtain the desired result. If (b) is the case: we know that $p^+ \neq \emptyset$ so there is $s' \in S$ such that $s' \vdash p$. This means that $t \sqcup s' \vdash p \wedge q$. Since $t \sqcup s' \geq t$, we obtain the desired result. If (c) is the case, we have that $t \vdash p \wedge q$. As $t \geq t$, we obtain (3).

Strengthening: $K_p r \Vdash K_{p\wedge q} r$
Same as the proof for Cautious Strengthening.

(2) **Simplification:** $K_{p\wedge q} p, K_{p\wedge q} q$
Item (1) is proven for the first definition of knowledge. For (2) let $t \in S$ such that $t \vdash p$. Since $q^+ \neq \emptyset$, there is $s' \in S$ such that $s' \vdash q$. Then, $t \sqcup s' \vdash p \wedge q$. This implies that $t \sqcup s' \vdash (p \wedge q) \vee \neg(p \wedge q)$. As $t \sqcup s' \geq t$, we obtain (2). Item (3) follows easily since $t \dashv p$ implies that $t \dashv p \wedge q$, i.e., $t \vdash \neg(p \wedge q)$. Therefore, $t \vdash (p \wedge q) \vee \neg(p \wedge q)$. The result then follows since $t \leq t$.

Cautious Strengthening: $K_p q, K_p r \Vdash K_{p\wedge q} r$
Suppose (a) $s \vdash K_p q$ and (b) $s \vdash K_p r$. Item (1) is proven for the first definition of knowledge. For (2): let $t \in S$ such that $t \vdash r$. Then, by (b), there is $t' \geq t$ such that $t' \vdash p \vee \neg p$. We then have three cases:

Case $t' \vdash p$: Then, since $q^+ \neq \emptyset$, there is $s' \in S$ such that $s' \vdash q$. Therefore, $t' \sqcup s' \vdash p \wedge q$. This implies that $t' \sqcup s' \vdash (p \wedge q) \vee \neg(p \wedge q)$. Since $t' \sqcup s' \geq t' \geq t$, we obtain the desired result.

Case $t' \vdash \neg p$: This implies that $t' \dashv p \wedge q$, i.e., that $t' \vdash \neg(p \wedge q)$. Therefore, $t' \vdash (p \wedge q) \vee \neg(p \wedge q)$. Since $t' \geq t$, we obtain the desired result.

Case $t' \vdash p \wedge \neg p$: this means that there are $u, u' \in S$ such that $t' = u \sqcup u'$, $u \vdash p$, and $u' \vdash \neg p$. Similar to the first case, there is $s'' \in S$ such that $s'' \geq u$ and $s'' \vdash (p \wedge q) \vee \neg(p \wedge q)$. Moreover, similar to the second case, $u' \vdash (p \wedge q) \vee \neg(p \wedge q)$. Since $s'' \geq u$, we have that $s'' \sqcup u' \geq u \sqcup u' = t' \geq t$. Moreover, since $|(p \wedge q) \vee \neg(p \wedge q)|$ is closed under fusion (by Proposition 3), we have $s'' \sqcup u' \vdash (p \wedge q) \vee \neg(p \wedge q)$, We can then conclude that (2) is the case. Item (3) follows in a similar way.

Weak Simplification: $K_{p\wedge q}(p \vee q)$
Item (1) is proven for the first definition of knowledge. For (2), let $t \in S$ such that $t \vdash p \vee q$. We then have three cases:

Case $t \vdash p$: Then, since $q^+ \neq \emptyset$, there is $s' \in S$ such that $s' \vdash q$. Therefore, $t \sqcup s' \vdash p \wedge q$. This implies that $t \sqcup s' \vdash (p \wedge q) \vee \neg(p \wedge q)$. Since $t \sqcup s' \geq t$, we obtain the desired result.

Case $t \vdash q$: Similar to the above case, use $p^+ \neq \emptyset$.

Case $t \vdash p \wedge q$: this implies that $t \vdash (p \wedge q) \vee \neg(p \wedge q)$. Since $t \geq t$, we conclude that (2) is the case.

For (3), let $t \in S$ such that $t \dashv p \vee q$. This means that there are $u, u' \in S$ such that $t = u \sqcup u'$, $u \dashv p$ and $u' \dashv q$. Therefore $t \dashv p \wedge q$, i.e., $t \vdash \neg(p \wedge q)$. This implies that $t \vdash (p \wedge q) \vee \neg(p \wedge q)$. Since $t \geq t$, we conclude that (3) is the case.

Disjunctive Syllogism: $K_\varphi \neg p, K_\varphi(p \vee q) \Vdash K_\varphi q$
Suppose (a) $s \vdash K_\varphi \neg p$ and (b) $s \vdash K_\varphi(p \vee q)$. Item (1) is proven for the first definition of knowledge. For (2): let $t \in S$ such that $t \vdash q$. This implies that $t \vdash p \vee q$. Then, by (b), we conclude that there is $t' \geq t$ such that $t' \vdash \varphi \vee \neg\varphi$. For (3): let $t \in S$ such that $t \dashv q$. Since $p^- \neq \emptyset$, there is $s' \in S$ such that $s' \dashv p$. This means that $t \sqcup s' \dashv p \vee q$. Then, by (b), there is $u' \geq t \sqcup s'$ such that $u' \vdash \varphi \vee \neg\varphi$. Since $t \leq t \sqcup s' \leq u'$, we conclude that (3) is the case.

Strengthening: $K_p r \Vdash K_{p\wedge q} r$
Same as the proof for Cautious Strengthening.

□