Lattices of DNA-Logics and Algebraic Semantics of Inquisitive Logic

MSc Thesis *(Afstudeerscriptie)*

written by

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under the supervision of **Dr. Nick Bezhanishvili** and **Drs. Gianluca Grilletti**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

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Abstract

This thesis studies algebraic semantics for the inquisitive logic `InqB` and for the related class of `DNA`-logics. `DNA`-logics were previously known in literature as negative variants of intermediate logics and have been studied only in syntactic terms. In this thesis, we show that there is a dual isomorphism between the lattice of `DNA`-logics and the lattice of suitable classes of Heyting algebras that we call `DNA`-varieties. We study several properties of `DNA`-logics and `DNA`-varieties and we prove a version of Tarski and Birkhoff Theorems for `DNA`-varieties. A special attention is then paid to introduce a notion of locally finiteness for this setting and to prove two key results concerning this property, i.e. that the `DNA`-variety of all Heyting algebras is not locally finite and that locally finite `DNA`-logics can be axiomatised by a version of Jankov formulas. Finally, we apply the general theory of `DNA`-logics to the case of inquisitive logic. We show that `InqB` is a `DNA`-logic and we use the method of Jankov formulas to prove that the sublattice $\Lambda(\texttt{InqB})$ of the extensions of `InqB` is dually isomorphic to $\omega + 1$.

Acknowledgements

I would like to spend few words here to thank the people who accompanied me in these two years of experience in the Master of Logic, and in this last endeavour of writing a thesis.

First, I feel I am very indebted to Nick and Gianluca. You have been excellent supervisors and you dedicated to our meetings, not to mention the nights spent for checking and correcting the previous versions of this thesis, way more time than I could have ever wished. I really learned a lot from you and I am very thankful for it. I also wish to thank my committee members – Dick, Floris and Yde – as their comments and questions surely contributed to improve the present work.

Studying at the ILLC would not have been fun alone. Thanks to Giuliano, for his vocal messages. Thanks to Marta, for her five minutes. Thanks to Dimi for his colourful shoes, and thanks to Chase for his white shoes. Thanks to Rachael, because after all the gin-tonics I owe you, you too will forget the difference between Muppets and Sesame Street – in the end it is all about puppets. Thanks to Nuno, who is a great set-theory buddy. Thanks to Gaia, because yes, I really like your memes, but you still owe me 5€ – just sayin'. Thanks to Nikos for his dark humour, and thanks to Robin for her awesome cynical attitude. Thanks to Romain for his magic tricks, and thanks to Hrafn for his foosball skills. And thanks to all those whose daily, but most importantly nightly, presence in the MoL room has been a certainty these years. It was great doing this with you.

And thank you David, for the opera, the concerts, the Cineville card we did not make the best of, the hipster CREA, the Hollywood movies from the Sixties, the Stedelijk museum and all the cultural stuff we surprisingly managed to attend. Thanks for scrapping so many proofs at 4 a.m. and, also, thanks for teaching me so much about mathematics. It was all, I would say, very nice indeed.

Also, I wish to thank my family – Anna, Andrea and Daniele – who have been always supportive through all the years of my quite long education. Thanks to Davide, who is always there. Thanks to Adrien and Maïa, who always pop up at some point, no matter where you are. Thanks to Roberto and Malvina – you might not be great assistant drivers, but for sure you are great friends. Thanks to Vita, because I really liked visiting Palermo with you. And thanks to my friends in Italy who were always so close despite the distance. It meant a lot to me.

Finally – thank you Nachiket, Georgi and Kristoffer. I have heard some people believe the *Answer to the Ultimate Question of Life, the Universe, and Everything* is 42. They are wrong. It is clearly 24.

Contents

Chapter 1

Introduction

This thesis provides an algebraic study of the propositional system `InqB` of inquisitive logic and of the wider class of `DNA`-logics. Inquisitive semantics was introduced one decade ago by Ciardelli, Groenendijk and Roelofsen as a formal framework to analyse questions. More specifically, inquisitive logic originates from the so-called "partition semantics" of Groenendijk and Stokhof [29, 26] and was formally developed in [8, 15, 12, 28, 27]. The first systematic presentation of inquisitive semantics is Ciardelli's MSc Thesis [10], later reworked and published in [14]. Ciardelli's PhD thesis [11] and the recent textbook [13] give the state of the art in the field.

The usual approach in linguistics and philosophy of language has for long been focused on the analysis of contents in terms of truth-conditions. For instance, this was the methodology suggested by Davidson in his seminal paper [18] and has since then become the standard view for the analysis of meaning. A consequence of this attitude is that formal linguists focused mainly on sentences and paid little attention to questions – in fact, questions were often considered in relation to pragmatics rather than semantics. In contrast, inquisitive semantics treats questions as a proper part of semantics and aims at representing questions in formal terms.

Since its first appearance ten years ago, inquisitive semantics has seen a rich development, directed both towards various applications in linguistics and also towards the refinement and development of its logical tools. In particular, inquisitive propositional logic `InqB` has been investigated in [10, 14, 44, 43, 42, 22]. It is only recently, however, that inquisitive logic is also being considered from an algebraic perspective (however, see [45] for a different algebraic approach to inquisitive logic). Bezhanishvili, Grilletti and Holliday have introduced in [3] an algebraic and topological semantics for `InqB`. In this thesis we aim at developing the work done in this paper in a natural way and we present an algebraic semantics which applies both to `InqB` and to the related family of `DNA`-logics. In fact, the introduction of algebraic semantics has been of much importance for intermediate and modal logics [7] and has also been subjected to abstract investigations *per se* [21]. The development of an algebraic semantics for inquisitive logic can thus fill a gap in the current status of the discipline and provide a better understanding of the mathematics behind inquisitive semantics. For instance, the introduction of an algebraic semantics for `InqB` gives new tools to study the relation between inquisitive logic and intermediate logics. In this thesis we also show how this novel algebraic setting can be used to prove results concerning the extensions of $\mathtt{InqB}$.

While inquisitive logic is now widely known and recognized, the related class of $\mathtt{DNA}$-logics has not been object of many investigations. In this thesis we introduce $\mathtt{DNA}$-logics as negative variants of intermediate logics. A $\mathtt{DNA}$-logic Λ is thus a set of propositional formulas such that, for some intermediate logic L, $\varphi \in \Lambda$ if and only if $\varphi[{}^{\neg p}/_{p}] \in L$. Thus the name $\mathtt{DNA}$ stands for *double negation atoms*, since every $\mathtt{DNA}$-logic Λ proves the formula $\neg\neg p \rightarrow p$ for every atomic formula $p \in \mathtt{AT}$. The relation between $\mathtt{InqB}$ and negative variants of intermediate logics was already pointed out in [10], but it has not led to an independent study of this class of logics. The first work dealing with $\mathtt{DNA}$-logics is [41], who also show several properties of these systems, though from a different perspective from the one we take. In this thesis we study in detail this novel class of logics and we investigate the corresponding classes of Heyting algebras that we call $\mathtt{DNA}$-varieties.

The original contributions of this thesis are therefore of two kinds. On the one hand, we develop a general algebraic semantics for $\mathtt{DNA}$-logics and we prove some fundamental results concerning $\mathtt{DNA}$-logics and $\mathtt{DNA}$-varieties. On the other hand, we apply this general algebraic setting to inquisitive logic: we study the sublattice of the extensions of $\mathtt{InqB}$ and we give an axiomatisation of them by using Jankov formulas. More generally, the algebraic setting that we have considered places inquisitive logic in a new mathematical context and makes it possible to raise new questions about it. We will consider some of these issues at the end of our work.

In Chapter 2, we introduce the reader to the mathematical tools and frameworks which are needed to follow the rest of the thesis. First, we present some general notions from universal algebra and we state some results which we will refer to later on. Secondly, we outline the general theory of intermediate logics and we present their algebraic semantics and their relation to varieties of Heyting algebras. Most of the results can be found in classic textbooks [7, 17] and [6].

In Chapter 3, we introduce the class of $\mathtt{DNA}$-logics as the class of negative variants of intermediate logics. Here we provide algebraic semantics for $\mathtt{DNA}$-logics and we introduce a corresponding notion of $\mathtt{DNA}$-varieties, which are classes of Heyting algebras closed under suitable operations. The key result of this chapter is a dual isomorphism between the class of $\mathtt{DNA}$-logics and the class of $\mathtt{DNA}$-varieties. This provides us with algebraic tools to study $\mathtt{DNA}$-logics, which so far have only been considered in syntactic terms.

In Chapter 4, we show some further results pertaining the theory of $\mathtt{DNA}$-logics and $\mathtt{DNA}$-varieties. We study in some details the connection between $\mathtt{DNA}$-logics and intermediate logics, which has previously been considered in [10]. Moreover, we extend to the framework of $\mathtt{DNA}$-logics some theorems and methodologies usually employed in the study of intermediate logics. The key results of these section are: (i) every $\mathtt{DNA}$-logic Λ determines a bounded sublattice of intermediate logics which have Λ as their negative variant, (ii) every $\mathtt{DNA}$-variety $\mathcal{X}$ is generated by its collection $\mathcal{X}_{RSI}$ of regular subdirectly irreducible elements, (iii) the negative variant $\mathtt{IPC}^{\neg}$ of $\mathtt{IPC}$ is not locally finite and (iv) locally finite $\mathtt{DNA}$-varieties are axiomatisable by Jankov formulas.

In Chapter 5, we apply the general setting and methodologies that we have developed in the previous chapters to the study of inquisitive logic. Firstly, we prove that InqB can be seen as a DNA-logic and we show that InqB is the negative variant of any intermediate logic L such that ND $\subseteq L \subseteq$ ML. Although this result has already been proved in [10, 14], we provide an alternative algebraic proof which in turn follows from [3]. This also provides the tools to show that the DNA-variety of InqB is locally finite. Together with the method of Jankov formulas developed in the previous chapter, this gives the key result of this section. We show that the extensions of InqB form a countable descending chain with an extra bottom element. Finally, we provide an axiomatisation of each of these logics and we show that they coincide with the so-called *inquisitive hierarchy* considered in [10]. It thus follows from our result that the inquisitive hierarchy comprises all the possible ways in which InqB can be extended to other DNA-logics.

Chapter 2

Preliminaries

In this chapter we introduce the mathematical tools needed in the rest of the thesis. In Section 2.1 we introduce the necessary concepts and definitions from universal algebra: we provide the basic definitions of algebra, equational classes and varieties and we recall some important results that we will use in this work. In Section 2.2 we outline the general theory of intermediate logics: we define them as sets of formulas which extend intuitionistic logic and are closed under modus ponens and uniform substitution, we introduce in details their relational and algebraic semantics and we recall several properties of such logics. In what follows we assume a basic familiarity with the notions of sets, classes, relations and functions along with the basic theory of ordinals and cardinals. For further reference the reader may refer to [35].

2.1 Universal Algebra

Universal algebra deals with the abstract notion of algebra and the properties that follow in this general setting. To keep this work self-contained, we recall in this section the definitions and results which we will later make use of. For the proofs of these results and related details, the reader may refer to [6], whilst [19] provides more information on Heyting algebras and [49] on Boolean algebras.

2.1.1 Basic Definitions

We introduce some very preliminary definitions. As a first step we define the notion of algebraic language.

Definition 2.1 (Algebraic Language)**.** An *algebraic language* (or *similarity-type*) $\mathcal{L}$ is a set of *function symbols* $f \in \mathcal{L}$ with an associated natural number $n \in \mathbb{N}$ called its *arity* and denoted by $ar(f)$.

When $ar(f) = 0$ then we say that the function symbol f is a *constant symbol.* If $\mathcal{L}$ is a language, we then write $\mathcal{L}_n$ for the set of function symbols $f \in \mathcal{L}$ such that $ar(f) = n$. If a language $\mathcal{L}$ is finite, we often write it as $(f_0, ...f_n)$, where $f_0, ..., f_n$ are its function symbols. In what follows, we use languages to talk about algebras. Intuitively, an algebra can be thought of as a set provided with a suitable interpretation for every function symbol of the language. Let X^n be the n-product of X, then we say that f is an *operation* on X of arity n if $f : X^n \to X$. When $f : X^0 \to X$ has arity zero, then we also say that it is a *constant.* Algebras can be thought of as sets supplemented with some operations.

Definition 2.2 ($\mathcal{L}$-Algebra)**.** Given an algebraic language $\mathcal{L}$, an $\mathcal{L}$-*algebra* or an *algebra of similarity-type* $\mathcal{L}$ is a pair $\mathbf{A} = (A, F)$, where A is a set called the *universe* of the algebra and F is a set containing, for each function symbol $f \in \mathcal{L}_n$, an operation $f_A : A^n \to A$.

In this thesis, we always work with finite similarity types, if not specified otherwise. Also, when the context is clear, we usually refer to $\mathcal{L}$-algebras simply as algebras and we use capital letters as A both for the universe of an algebra and for the algebra itself. A function f_A is also called the *interpretation* of f in A. With a slight abuse of notation, we generally drop the indices and write f both for function symbols and functions. If $\mathcal{L}$ is finite, we often write an $\mathcal{L}$-algebra as $\mathbf{A} = (A, f_0, ..., f_n)$, where A is the universe and $f_0..., f_n$ are the interpretations of the function symbols of its language.

Universal algebra can be seen as a restricted version of model theory. In fact, algebraic languages can be considered as first-order languages without relation symbols and algebras can be seen as first-order-structures without relations. Similarly, whilst in first-order logic we are generally interested in the satisfaction of formulas, in universal algebra we are mostly interested in the validation of identities. We fix some algebraic language $\mathcal{L}$.

Definition 2.3 ($\mathcal{L}$-Terms)**.** Let X be a set of variables, then the set $T(X)$ of $\mathcal{L}$-*terms* over X is the smallest set such that:

(i) $X \cup \mathcal{L}_0 \subseteq T(X)$;

(ii) if $f \in \mathcal{L}_n$ and $x_1, ..., x_n \in T(X)$, then $f(x_1, ..., x_n) \in T(X)$.

When the context is clear, we usually refer to $\mathcal{L}$-terms simply as *terms.* If t is a term, we then write $t(\overline{x})$ or respectively $t(x_0, ..., x_n)$ to designate that the variables of t are among those of $\overline{x}$ or respectively of $x_0, ..., x_n$. We say that a term t has arity n if the number k of variables that it contains is $k \leq n$. Given some $\mathcal{L}$-algebra A, every $\mathcal{L}$-term $t \in T(X)$ determines a corresponding *term function* t_A defined as follows.

Definition 2.4 (Term Function)**.** Let $t(x_1, ..., x_n) \in T(X)$ be an n-ary $\mathcal{L}$-term, then we define its corresponding *term function* $t_A : A^n \to A$ over an $\mathcal{L}$-algebra A as follows.

(i) If $t(x_1, ..., x_n) \in T(X)$ is a variable x_i, then $t_A : (x_1, ..., x_n) \mapsto x_i$ is the i_{th} projection function.

(ii) If $t(x_1, ..., x_n)$ is of the form $f(t^0(x_1, ..., x_n), ..., t^m(x_1, ..., x_n))$, with $f \in \mathcal{L}_m$, then:

$$t_A(x_1, ..., x_n) = f_A(t^0_A(x_1, ..., x_n), ..., t^m_A(x_1, ..., x_n)).$$

We then define algebraic models as follows.

Definition 2.5 (Algebraic Model)**.** An *algebraic model* of similarity-type $\mathcal{L}$ over a set of variables X is a pair $\mathcal{M} = (A, V)$, where A is an $\mathcal{L}$-algebra and $V : X \to A$ a *valuation.*

When the valuation V is clear from the context, then we use a capital letter A both for the algebra and the model (A, V). Given an algebraic model (A, V) and a term $t(\overline{x})$, we say that $t_A(V(\overline{x}))$ is the *interpretation of t in A* and we also write it as $V(t)$. Given two $\mathcal{L}$-terms $p, q \in T(X)$, an *identity* in $\mathcal{L}$ is an expression of the form $p \approx q$. We define the notions of validation and truth in the following way.

Definition 2.6 (Validation)**.** Let $p \approx q$ be an identity in $\mathcal{L}$ over a set of variables X and let (A, V) be an algebraic model of similarity-type $\mathcal{L}$. Then we say that A *validates* the identity $p(x_1, ..., x_n) \approx q(y_0, ..., y_n)$ and we write $A \vDash p \approx q$ if for every valuation $V : X \to A$ we have that $V(p) = V(q)$.

If $A \vDash p \approx q$ then we also say that $p \approx q$ is *true in* A. If Σ is a set of identities, then we write $A \vDash \Sigma$ if $A \vDash p \approx q$ for all identities $p \approx q \in \Sigma$. If $\mathcal{K}$ is a class of $\mathcal{L}$-algebras, then we write $\mathcal{K} \vDash p \approx q$ if for all $A \in \mathcal{K}$ we have that $A \vDash p \approx q$. Similarly, we write $\mathcal{K} \vDash \Sigma$, if for every $A \in \mathcal{K}$ we have that $A \vDash \Sigma$. Given a class of algebras $\mathcal{K}$, we denote by $Id_X(\mathcal{K})$ the class of identities over the set $T(X)$ of terms over X which are true in $\mathcal{K}$. Given a class Σ of identities we denote by $M(\Sigma)$ the class of algebras which validate all the identities in Σ. Now we define equational classes as follows.

Definition 2.7 (Equational Class)**.** We say that a class of algebras $\mathcal{K}$ is an *equational class* if there is some set of identities Σ such that $\mathcal{K} = M(\Sigma)$.

2.1.2 Varieties

After establishing the general notion of algebra, we now turn to the study of maps between algebras and to different ways to construct new algebras starting from already given ones. In particular, we are interested in the notions of subalgebra, homomorphic image and product, as they allow us to define the notion of variety. First, we introduce some mappings between algebras of the same similarity type.

Definition 2.8 (Homomorphism)**.** Let A and B be two $\mathcal{L}$-algebras, then a function $h : A \to B$ is a *homomorphism* if for every $f \in \mathcal{L}$ we have that

$$h(f_A(a_0, ..., a_i)) = f_B(h(a_0), ..., h(a_i)).$$

If a homomorphism $h : A \to A$ is such that its domain and its codomain are the same then we say it is an *endomorphism.* If a homomorphism h from A to B is injective then we say it is an *embedding.* If a homomorphism h from A to B is surjective we also write it as $h : A \twoheadrightarrow B$. If a homomorphism $h : A \to B$ is both surjective and injective then we say it is an *isomorphism.* If there is an isomorphism between two $\mathcal{L}$-algebras A and B then we say that they are *isomorphic* and we write $A \cong B$.

We now define the three key constructions that lead to the notion of variety, i.e. subalgebras, homomorphic images and products.

Definition 2.9 (Subalgebra)**.** Let A and B be two algebras of the same similarity type. Then we say that A is a *subalgebra* of B and we write $A \preceq B$ if $A \subseteq B$ and for every $f \in \mathcal{L}_n$ we have that $f_A = f_B \upharpoonright A^n$.

Alternatively, we can also say that A is a subalgebra of B if the subset relation is itself an embedding. If we have an embedding $h : A \to B$, it then follows that $h[A]$ is a subalgebra of B. Homomorphic images are defined as follows.

Definition 2.10 (Homomorphic Image)**.** Let A and B be two algebras of the same similarity type. Then we say that A is a *homomorphic image* of B if there exists a surjective homomorphism $h : A \twoheadrightarrow B$.

Intuitively, a homomorphic image is obtained by collapsing together the elements of an algebra in a way which is compatible with the algebra operations. This idea will become more precise later when we introduce the notion of congruence of an algebra. To introduce the product of a family of algebras, let us first recall that the set-theoretic product of a family $\{X_i\}_{i\in I}$ of sets is the set $\prod_{i\in I} X_i$ defined as follows:

$$\prod_{i\in I} X_i = \{f : I \to \bigcup_{i\in I} X_i : \forall i \in I(f(i) \in X_i)\}.$$

The product of a family $\{A_i\}_{i\in I}$ of algebras is then defined as follows.

Definition 2.11 (Product)**.** Let $\{A_i\}_{i\in I}$ be a family of algebras of the same similarity-type. Then the *product* $\prod_{i\in I} A_i$ is an algebra whose universe is the set theoretic product $\prod_{i\in I} A_i$ and such that for every n-ary operation $f \in \mathcal{L}_n$ the result of its application to some $a_1, ..., a_n \in \prod_{i\in I} A_i$ is computed pointwise:

$$f_{\prod_{i\in I} A_i}(a_1, ..., a_n)(i) = f_{A_i}(a_1(i), ..., a_n(i)).$$

If I is finite, we also write $A_0 \times ... \times A_n$ for the product $\prod_{i\in I} A_i$. For every $i \in I$ we also have a *projection function* $\pi_i : \prod_{i\in I} A_i \to A_i$ such that $\pi_i : \alpha \mapsto \alpha(i)$. It is easy to show that every such projection function is a surjective homomorphism.

Now, we introduce the following closure maps.

Definition 2.12. Let $\mathcal{K}$ be a set of algebras of the same similarity-type, we then define the following:

$A \in I(\mathcal{K})$ iff A is isomorphic to some algebra in $\mathcal{K}$

$A \in S(\mathcal{K})$ iff A is a subalgebra of some algebra in $\mathcal{K}$

$A \in H(\mathcal{K})$ iff A is homomorphic image of some algebra in $\mathcal{K}$

$A \in P(\mathcal{K})$ iff A is product of a nonempty family of algebras in $\mathcal{K}$.

The following proposition provides a characterisation of how the previous maps interact with one another.

Proposition 2.13. *Let $\mathcal{K}$ be an arbitrary class of algebras, we then have that $SH(\mathcal{K}) \subseteq HS(\mathcal{K})$, $PS(\mathcal{K}) \subseteq SP(\mathcal{K})$ and $PH(\mathcal{K}) \subseteq HP(\mathcal{K})$. Moreover, the operators I, S, H, P are all idempotent, i.e. $I^2(\mathcal{K}) = I(\mathcal{K})$, $S^2(\mathcal{K}) = S(\mathcal{K})$, $H^2(\mathcal{K}) = H(\mathcal{K})$ and $P^2(\mathcal{K}) = P(\mathcal{K})$.*

The notion of variety is then defined as follows.

Definition 2.14 (Variety)**.** We say that a class of algebras $\mathcal{V}$ is a *variety* if it is closed under subalgebras, homomorphic images and products.

If $\mathcal{K}$ is an arbitrary class of $\mathcal{L}$-algebras, then we write $\mathcal{V}(K)$ for the variety generated by $\mathcal{K}$, i.e. for the smallest class of $\mathcal{L}$-algebras containing $\mathcal{K}$ which is closed under subalgebras, homomorphic images and products. Finally, we recall the following important theorems, which turn out to be very useful in the study of varieties. The first theorem, due to Tarski, characterizes the variety generated by a set of algebras in terms of the closure maps defined above. The second theorem is an important result by Birkhoff which establishes that varieties and equational classes actually coincide. For a proof of these results see [6, Thm. 9.5, Thm. 11.9].

Theorem 2.15 (Tarski)**.** *Let $\mathcal{K}$ be a class of algebras of some similarity type, we then have that $\mathcal{V}(\mathcal{K}) = HSP(\mathcal{K})$.*

Theorem 2.16 (Birkhoff)**.** *A class of algebras $\mathcal{K}$ is a variety iff it is an equational class.*

2.1.3 Orders and Lattices

We introduce here orders and lattices. We first define partial orders and recall some order-theoretic concepts we will encounter down the line in in this thesis.

Definition 2.17 (Partial Order)**.** A *partial order*, (or *poset*) is a set A equipped with a relation $\leq$ which is reflexive, transitive and antisymmetric, i.e. for all $a, b, c \in A$:

$$\begin{aligned} \text{Reflexivity: } & a \leq a; \\ \text{Transitivity: } & a \leq b \text{ and } b \leq c \text{ implies } a \leq c; \\ \text{Antisymmetry: } & a \leq b \text{ and } b \leq a \text{ implies } a = b. \end{aligned}$$

If the relation $\leq$ is only reflexive and transitive then we say that $(A, \leq)$ is a *preorder.* If $(A, \leq)$ is a poset and for all $a, b \in A$ we also have $a \leq b$ or $b \leq a$, then we say that A is a *linear order* or also a *total order.*

Definition 2.18. Let $(A, \leq)$ be a poset and $B \subseteq A$ an arbitrary subset, then given some element $a \in A$ we say that:

a is an *upper bound* for B if $\forall x \in B, x \leq a$;
a is a *lower bound* for B if $\forall x \in B, a \leq x$;
a is the *greatest element* of B if $a \in B$ and $\forall x \in B, x \leq a$;
a is the *least element* of B if $a \in B$ and $\forall x \in B, a \leq x$;
a is the *supremum* of B if it is its least upper bound;
a is the *infimum* of B if it is its greatest lower bound.

A *chain* is a subset of a poset which is linearly ordered. If $(A, \leq)$ is a poset and $X \subseteq A$, we denote by $\inf(X)$ the infimum of X in A and by $\sup(X)$ the supremum of X in A. If $(A, \leq)$ is a poset, then we say that a subset $X \subseteq A$ is an *upset* if for all $x, y \in A$ such that $x \leq y$ we have that if $x \in X$ then $y \in X$. If $(A, \leq)$ is a poset, then we say that a subset $X \subseteq A$ is a *downset* if for all $x, y \in A$ such that $x \leq y$ we have that if $y \in X$ then $x \in X$. Given a subset of a poset $X \subseteq A$ we define $\uparrow X = \{x \in A : \exists y \in X(y \leq x)\}$ and $\downarrow X = \{x \in A : \exists y \in X(x \leq y)\}$. For a singleton $\{x\}$ we write simply $\uparrow x$ or $\downarrow x$. If for an upset (or downset) X there is a subset $Y \subseteq X$ such that $X = \uparrow Y$ (or $X = \downarrow Y$), then we say that Y *generates* X and its elements are called the *generators* of X. Finally, if $(A, \leq)$ is a poset, we denote by $Up(A)$ the set of all its upsets and $Dw(A)$ the set of all its downsets.

Let X and Y be two posets. We say that a function $f : X \to Y$ is *order preserving* if $x \leq y$ entails $f(x) \leq f(y)$ for all $x, y \in X$. We say that a function $f : X \to Y$ is *order reversing* if $x \leq y$ entails $f(y) \leq f(x)$ for all $x, y \in X$. If $f : X \to Y$ is both a bijection and order-preserving, then we say it is an *isomorphism*. If $f : X \to Y$ is both a bijection and order-reversing, then we say it is an *dual-isomorphism*. If X and Y are two posets and there is an isomorphism between them then we write $X \cong Y$, if there is a dual isomorphism between them then we write $X \cong^{op} Y$.

Lattices are a key example both of orders and algebras and they also play an important role in universal algebra. On the one hand, many objects in universal algebra have a lattice structure, a common example being congruences or varieties. On the other hand, lattices themselves are a prime example of algebras and they also form the backbone of many other algebraic structures, such as Heyting and Boolean algebras. A first definition of lattices can be given in order-theoretic terms. For more on lattices we refer the reader to the classic textbooks [4] and [24].

Definition 2.19 (Lattice – Order-Theoretic Definition)**.** Let $(A, \leq)$ be a partial order. We say that $(A, \leq)$ is a *lattice* if for all $a, b \in A$, there are elements $\inf\{a, b\}$ and $\sup\{a, b\}$ in $(A, \leq)$.

Alternatively, lattices can be introduced as a prime example of algebras.

Definition 2.20 (Lattice – Algebraic Definition)**.** An algebra $(A, \wedge, \vee)$ of universe A and with two binary operations $\wedge, \vee$ respectively called *meet* and *join* is a *lattice* if A satisfies the following set of equations:

$$
\begin{aligned}
&\text{(L1) } a \vee b \approx b \vee a \\
&\text{(L2) } a \wedge b \approx b \wedge a \\
&\text{(L3) } a \vee (b \vee c) \approx (a \vee b) \vee c \\
&\text{(L4) } a \wedge (b \wedge c) \approx (a \wedge b) \wedge c \\
&\text{(L5) } a \vee a \approx a \\
&\text{(L6) } a \wedge a \approx a \\
&\text{(L7) } a \approx a \vee (a \wedge b) \\
&\text{(L8) } a \approx a \wedge (a \vee b)
\end{aligned}
$$

Here lattices are introduced as a class of algebras satisfying the equations above. Thus, it is immediate that lattices are an algebraic equational class and so by Birkhoff Theorem 2.16 that they are also algebraic varieties. We denote by *Lat* the variety of lattices. The order-theoretic and the algebraic definition of lattices can be related as follows. If $(A, \leq)$ is a poset such that every two elements a, b have both an infimum and a supremum, then we can define the two operations $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$ and verify they obey the laws (L1)-(L8) above. Similarly, given an algebraic structure $(A, \wedge, \vee)$ that validates the laws (L1)-(L8), we can define the corresponding poset $(A, \leq)$ by fixing $a \leq b \Leftrightarrow a \wedge b = a$ and verify that under this order $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$.

We say that a lattice $(A, \leq)$ is *complete* if every subset $B \subseteq A$ has an infimum and a supremum denoted respectively by $\bigwedge B$ and $\bigvee B$. A *distributive lattice* is a lattice $(A, \wedge, \vee)$ which satisfies the equations $a \wedge (b \vee c) \approx (a \wedge b) \vee c$ and $a \vee (b \wedge c) \approx (a \vee b) \wedge c$. A *bounded lattice* is an algebra $(A, \wedge, \vee, 0, 1)$ where $(A, \wedge, \vee)$ is a lattice and $0, 1$ are two elements which satisfy the equations $a \wedge 0 \approx 0$ and $a \vee 1 \approx 1$. Given a lattice A, we say that A^{op} is its *dual lattice* if $a \vee_A b = a \wedge_{A^{op}} b$ and $a \wedge_A b = a \vee_{A^{op}} b$. Since lattices are algebraic structures, the notions and the maps that we have defined in the previous sections apply to lattices as special case. Now we introduce two more definitions. A homomorphism $h : A \to B$ is an *order-reversing homomorphism* if $h(a \vee_A b) = h(a) \wedge_B h(b)$ and $h(a \wedge_A b) = h(a) \vee_B h(b)$. A lattice A is *dually isomorphic* to a lattice B, written $A \cong^{op} B$, if $A \cong B^{op}$. It is then easy to see that two lattices are (dually) isomorphic as algebras if and only if they are (dually) isomorphic as order-structures.

Finally, we can define the notion of filter over a lattice. For more on filters and related concepts the reader may refer to [17, §2].

Definition 2.21 (Filter)**.** Given a lattice L, a *filter* F is a subset $F \subseteq L$ such that:

(i) $F \neq \emptyset$;

(ii) if $x, y \in F$ then $x \wedge y \in F$;

(iii) if $x \in F$ and $x \leq y$, then $y \in F$.

We say that a filter F is *principal* if there is some element $x \in L$ such that $F = \uparrow x$. We denote the set of filters over a lattice L as $Fil(L)$. It can then be verified that $Fil(L)$ is itself a lattice ordered by the set-theoretic inclusion $\subseteq$.

2.1.4 Free Algebras and Subdirectly Irreducible Algebras

Given a variety of algebras, we are often interested in ways in which it can be characterised in terms of a restricted class of generators. Here we provide two ways to generate varieties which are often very useful: *via* free algebras and *via* subdirectly irreducible algebras.

To study these two notions we first need to introduce congruences over an algebra. Let us recall that a relation $R \subseteq A^2$ is called an *equivalence relation* if it is reflexive, transitive and symmetric. The congruences of an algebra are then defined as follows.

Definition 2.22 (Congruence)**.** Let A be an algebra of some similarity-type $\mathcal{L}$, then a relation $\theta \subseteq A^2$ is a *congruence* if it is an equivalence relation and it is also compatible with the algebra operations: i.e. for every n-ary function symbol $f \in \mathcal{L}$ and elements $a_1, ..., a_n, b_1, .., b_n \in A$, if $a_i \theta b_i$ for all $0 \leq i \leq n$, then $f_A(a_0, ..., a_n) \theta f_A(b_0, ..., b_n)$.

It is then easy to show that the *diagonal-relation* $\Delta = \{(a, a) \in A^2 : a \in A\}$ and the *universal-relation* $\nabla = A^2$ are both congruences over A. We denote by $Con(A)$ the set of all congruences of A. It is an important result that $(Con(A), \cup, \cap))$ is a complete bounded lattice ordered by $\subseteq$ with Δ as least element and ∇ as greatest element. Given a congruence θ over an algebra A, we define the quotient algebra A/θ. We recall that if $R \subseteq A^2$ is an equivalence relation and $x \in A$ then we say that x/R is the *equivalence class* of x and $x/R = \{y \in A : yRx\}$. The set-theoretic quotient of a set under an equivalence relation R is then the set X/R of all its equivalence classes.

Definition 2.23 (Quotient Algebra)**.** Let A be an algebra of some similarity type and θ be a congruence over A, then the *quotient algebra* A/θ is the algebra whose universe is the set theoretic quotient A/θ and for every function symbol $f \in \mathcal{L}$ we have:

$$f_{A/\theta}(a_1/\theta, ..., a_n/\theta) = f_A(a_1, ..., a_n)/\theta.$$

Given a congruence θ over A and the quotient algebra A/θ, the *natural map* $\eta_\theta : A \to A/\theta$ which is defined as $\eta_\theta : a \mapsto a/\theta$ can be proved to be a surjective homomorphism between A and A/θ. Homomorphisms are related to congruences also as follows: given a homomorphism $h : A \to B$ its *kernel* $ker(h) = \{(a_1, a_2) \in A^2 : h(a_1) = h(a_2)\}$ is a congruence over A. The following correspondence theorem allows us to relate the congruence lattice of an algebra and the congruence lattice of its quotient algebra under some congruence. Given two elements a, b in a lattice L, we denote by $[a, b]$ its bounded sublattice with $a = 0$ and $b = 1$.

Theorem 2.24 (Correspondence Theorem)**.** *Let A be an algebra and $\theta \in Con(A)$, then $Con(A/\theta) \cong [\theta, \nabla_A]$, where $[\theta, \nabla_A]$ is a sublattice of $Con(A)$.*

One way to characterize varieties is provided by free algebras. To introduce this construction we first introduce term algebras.

Definition 2.25 (Term Algebra)**.** A term algebra $T(X)$ of signature $\mathcal{L}$ and over a set of variables X is an algebra with universe the set of $\mathcal{L}$-terms $T(X)$ and such that for every function symbol $f \in \mathcal{L}_n$ the corresponding function $f_{T(X)} : T(X)^n \to T(X)$ is defined as $f_{T(X)} : (x_0, ..., x_n) \mapsto f(x_0, ..., x_n)$.

Let $\mathcal{K}$ be a class of algebras and X a set of variables, then we define the following set of congruences over the term algebra $T(X)$:

$$\Phi_{\mathcal{K}}(X) = \{\theta \in Con(T(X)) : T(X)/\theta \in IS(\mathcal{K})\}.$$

This allows us to find, for every such class $\mathcal{K}$, a minimal congruence $\theta_{\mathcal{K}}$ defined as $\theta_{\mathcal{K}}(X) = \bigcap \Phi_{\mathcal{K}}(X)$. Let $\overline{X} = X/\theta_{\mathcal{K}}(X)$, free algebras are then defined as follows.

Definition 2.26 (Free Algebra)**.** Let $\mathcal{K}$ be a class of algebras, then the *$\mathcal{K}$-free algebra* is the algebra $F_{\mathcal{K}}(\overline{X}) = T(X)/\theta_{\mathcal{K}}(X)$.

It is a result due to Birkhoff [6, Thm. 10.12] that for any variety $\mathcal{V}$ we have that $F_{\mathcal{V}}(\overline{X}) \in \mathcal{V}$. So free algebras always exist for varieties. Free algebras play an important role in characterizing their varieties, as they have the following universal mapping property.

Theorem 2.27 (Universal Mapping Property)**.** *Let $F_{\mathcal{V}}(\overline{X})$ be the free-algebra of the variety $\mathcal{V}$, then for every $A \in \mathcal{V}$ and every map $h : X \to A$ there is a unique homomorphism $\overline{h} : F_{\mathcal{V}}(\overline{X}) \to A$ which extends h.*

For a proof of this theorem see [6, Thm. 10.10]. Finally, free algebras are also interesting as they exactly satisfy the identities of the variety they belong to. Free algebras are then a sort of "natural representatives" of a variety.

Theorem 2.28. *Let X be any set of variables and let Y be an infinite set of variables, then for every variety $\mathcal{V}$ we have that $Id_{\mathcal{V}}(X) = Id_{F_{\mathcal{V}}(\overline{Y})}(X)$.*

A second method which is often used to characterize a variety is by generating it starting from its subdirectly irreducible elements. We first recall the following definitions. We say that an algebra A is a *subdirect product* of a family of algebras $\{A_i\}_{i\in I}$ if $A \preceq \prod_{i\in I} A_i$ and for every $i \in I$ we also have that $\pi_i(A) = A_i$. We say that an embedding $h : A \to \prod_{i\in I} A_i$ is *subdirect* if $h(A)$ is a subdirect product of $\{A_i\}_{i\in I}$. We then introduce subdirectly irreducible algebras as follows.

Definition 2.29 (Subdirectly Irreducible Algebra)**.** An algebra A is said to be *subdirectly irreducible* if for every family of algebras $\{A_i\}_{i\in I}$ and for every subdirect embedding $h : A \to \prod_{i\in I} A_i$ there is some $i \in I$ such that $\pi_i \circ h : A \to A_i$ is an isomorphism.

We use $\mathcal{V}_{SI}$ to denote the collection of subdirectly irreducible algebras of a variety $\mathcal{V}$. The following result due to Birkhoff shows that subdirectly irreducible algebras play an important role as generators of varieties. See [6, Theorem 9.6] for a proof of this theorem.

Theorem 2.30 (Birkhoff)**.** *If $\mathcal{V}$ is a variety, then every member of $\mathcal{V}$ is isomorphic to a subdirect product of subdirectly irreducible members of $\mathcal{V}$.*

Corollary 2.31. *Varieties are generated by their subdirectly irreducible members, i.e. for every variety $\mathcal{V}$, we have $\mathcal{V} = V(\mathcal{V}_{SI})$.*

And finally, the following result shows how congruences can be used to characterize subdirectly irreducible algebras.

Theorem 2.32. *An algebra A is subdirectly irreducible iff A is trivial or there is a minimum congruence in $Con(A) \setminus \{\Delta\}$.*

Where an algebra A is said to be *trivial* if its universe is a singleton, i.e. $|A| = 1$.

2.1.5 Heyting Algebras and Boolean Algebras

We introduce here the classes of Heyting and Boolean algebras, as they will play a central role in this entire thesis. We define both Heyting and Boolean algebras by specifying a set of equations Σ which is valid in these algebras. Thus, it follows that Heyting and Boolean algebras are both equational classes and varieties. We denote by HA the variety/equational class of Heyting algebras and by BA the variety/equational class of Boolean algebras. We refer the reader to [19] and [49] for a more detailed presentation of Heyting and Boolean algebras.

Definition 2.33 (Heyting Algebra)**.** An algebra $(A, \wedge, \vee, \to, 0, 1)$ is a *Heyting algebra* if $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and in addition the binary operation $\to$ called *Heyting implication* satisfies the following equations:

$$\text{(H1) } a \to a \approx 1$$

(H2) $a \wedge (a \to b) \approx a \wedge b$
(H3) $b \wedge (a \to b) \approx b$
(H4) $a \to (b \wedge c) \approx (a \to b) \wedge (a \to c)$

Similar to the case of lattices, we can also give an order-theoretic definition of Heyting algebras. A bounded distributive lattice H is a Heyting algebra if for every $a, b \in H$ there is some element $a \to b \in A$ such that for all $c \in A$ we have that $c \leq a \to b \Leftrightarrow c \wedge a \leq b$. The underlying lattice of a Heyting algebra is always distributive and its order can be also defined by letting, for every $a, b \in H$, $a \leq b \Leftrightarrow a \to b = 1$. Given an element $a \in H$ of a Heyting algebra, we define its *pseudocomplement* $\neg a$ as $\neg a = a \to 0$.

We recall some important results which turn out useful in the study of Heyting algebras.

Proposition 2.34. *Let H be an Heyting algebra and let $a, b \in H$, then the element $a \to b$ can be characterized as follows:*

$$a \to b = \bigvee \{x \in H : x \wedge a \leq b\}.$$

Proposition 2.35. *Let I be an infinite set of indices. A complete lattice is a Heyting algebra iff it satisfies the following infinite distributivity law:*

$$a \wedge \bigvee_{i \in I} b_i \approx \bigvee_{i \in I} (a \wedge b_i).$$

The method of congruences can be used to study the variety of Heyting algebras. First, notice that in the context of Heyting algebras there is the following dual correspondence between congruences and filters:

$$\theta \mapsto \{x \in H : (1, x) \in \theta\}$$
$$F \mapsto \{(x, y) \in H^2 : x \leftrightarrow y \in F\}.$$

It can then be proven that $Con(H) \cong Fil(H)$. Then by using this correspondence and Theorem 2.32 above, one can prove the following characterisation of subdirectly irreducible Heyting algebras.

Theorem 2.36. *Let H be an Heyting algebra. Then H is subdirectly irreducible iff H has a second greatest element s_H.*

Like Heyting algebras, also Boolean algebras are based on lattices.

Definition 2.37 (Boolean Algebra)**.** An algebra $(B, \wedge, \vee, \neg, 0, 1)$ is a *Boolean algebra* if $(B, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\neg$ is an unary operation called *complementation* which satisfies the following equations:

(B1) $a \wedge \neg a \approx 0$
(B2) $a \vee \neg a \approx 1$

Given an element $a \in B$ of a Boolean algebra, the element $\neg a$ is called its *complement*. We thus also say that a Boolean algebra is a bounded distributive lattice such that every element has a complement. Boolean algebras can be also seen as Heyting algebras by defining the Heyting implication as $a \to b \Leftrightarrow \neg a \vee b$, for all $a, b \in B$. Similarly, given a Heyting algebra H we can define $\neg a$ as the pseudocomplement $a \to 0$ of a. It then immediately follows from our definitions that a Boolean algebra is a Heyting algebra where every pseudocomplement satisfies the two equations (B1) and (B2) above. A *power-set algebra* is a Boolean algebra $B = (\wp(X), \cup, \cap, \setminus, \emptyset, X)$, where the universe is a power-set, the algebraic operations of join and meet are the set-theoretic operations of union and intersection and complementation is the set-theoretic complement. We recall the following representation theorem for finite Boolean algebras. For a proof of this result see [17, §5].

Theorem 2.38. *Let B be a finite Boolean algebra, then B is isomorphic to a power-set algebra, i.e. $B \cong \wp(X)$ for some finite set X.*

Thus it follows that finite Boolean algebras are always equivalent up to isomorphism to $\wp(n)$ for some $n \in \mathbb{N}$. Therefore, it is easy to show that if $n \leq m$, then $\wp(n) \preceq \wp(m)$. Then, by identifying every $\wp(n)$ by 2^n, it follows that finite Boolean algebras form an ordered chain of subalgebras:

$$2^0 \preceq 2^1 \preceq 2^2 \preceq 2^3 \preceq 2^4 \preceq \dots$$

We will use this result later in our study of inquisitive logic.

2.2 Intermediate Logics

Intermediate logics are a well-studied class of logics with many applications in mathematics and computer science. The classic text on intermediate logic is [7], to which we refer the reader for the proofs of the results we recall in this section.

2.2.1 Syntactical Definitions

In this section we introduce the abstract notion of logic and outline the general theory of intermediate logics. We always work at the propositional level. Firstly, we fix a countable set $\mathtt{AT}$ of atomic propositional formulas, then we define the set of propositional formulas $\mathcal{L}_P$ inductively as follows.

Definition 2.39. The language $\mathcal{L}_P$ is defined as follows, where p is an arbitrary element of $\mathtt{AT}$:

$$\varphi ::= p \mid \top \mid \bot \mid \psi \wedge \chi \mid \psi \vee \chi \mid \psi \to \chi$$

Negation can be defined as $\neg\varphi := \varphi \to \bot$. If φ is a formula, then we write $\varphi(\overline{x})$ or $\varphi(x_0, ..., x_n)$ to specify that the atomic formulas in φ are among those of $\overline{x}$ or respectively of $x_0, ..., x_n$. A *substitution* is a function $\eta : \mathtt{AT} \to \mathcal{L}_P$ which naturally lifts by induction to formulas by setting, for every connective $\odot$ the map $(\psi \odot \chi) \mapsto \eta(\psi) \odot \eta(\chi)$. If φ is a formula and q occurs in φ, we write $\varphi[{}^{p}/_{q}]$ for the formula obtained by the substitution $\eta : q \mapsto p$. Similarly, if $\overline{q} = q_0, ..., q_n$ are variables in φ, then we write $\varphi[\overline{p}/\overline{q}]$ for the formula obtained by the substitution $\eta : q_i \mapsto p_i$ for all $i \leq n$.

Given a propositional language $\mathcal{L}_P$, we can then give a general definition of logic as a set of formulas of $\mathcal{L}_P$ which satisfies some closure conditions.

Definition 2.40 (Abstract Propositional Logic)**.** An *abstract propositional logic* (or simply *logic*) is a non-empty set L of formulas in $\mathcal{L}_P$ which satisfies the two following conditions:

(i) L is closed under *modus ponens*: if $\varphi \in L$ and $\varphi \to \psi \in L$, then $\psi \in L$.

(ii) L is closed under *uniform substitution*: if $\varphi(p_0, ..., p_n) \in L$ then for every $\psi_0, ..., \psi_n \in \mathcal{L}_P$ we have that $\varphi(\psi_0, ..., \psi_n) \in L$.

Now, we introduce two propositional logics which are of special interest for the rest of our thesis. These are the *intuitionistic propositional calculus* `IPC` and the *classical propositional calculus* `CPC`.

Definition 2.41 (Intuitionistic Logic)**.** The *intuitionistic propositional calculus* `IPC` (also *intuitionistic logic*) is the least set of formulas of $\mathcal{L}_P$ which contains the following axioms:

$$
\begin{aligned}
&\text{(A1) } p \to (q \to p)\\
&\text{(A2) } (p \to (q \to r)) \to (p \to q) \to (p \to r)\\
&\text{(A3) } p \land q \to p\\
&\text{(A4) } p \land q \to q\\
&\text{(A5) } p \to (q \to p \land q)\\
&\text{(A6) } p \to p \lor q\\
&\text{(A7) } q \to p \lor q\\
&\text{(A8) } (p \to r) \to ((q \to r) \to (p \lor q \to r)\\
&\text{(A9) } \bot \to p
\end{aligned}
$$

And, in addition, it is also closed under *modus ponens* and *substitution*, i.e. it is a logic.

Definition 2.42 (Classical Logic)**.** The *classical propositional calculus* `CPC` (also *classical logic*) is the least set of formulas of $\mathcal{L}_P$ which contains the axioms (A1)–(A9) plus the following:

$$\text{(A10) } p \lor \neg p$$

In addition, `CPC` is closed under *modus ponens* and *substitution*, i.e. it is a logic.

In this study, we are particularly interested in those logics that lie between intuitionistic and classical logic.

Definition 2.43 (Intermediate Logic)**.** A *superintuitionistic logic* L is a propositional logic such that $\texttt{IPC} \subseteq L$. An *intermediate logic* is a superintuitionistic logic L which is also consistent, namely $\bot \notin L$.

It can be proven that $\texttt{CPC}$ is the maximal intermediate logic and that intermediate logics are all the logics L such that $\texttt{IPC} \subseteq L \subseteq \texttt{CPC}$. We denote by $L + \varphi$ the closure under substitution and modus ponens of the set of formulas $L \cup \{\varphi\}$ and by $L + \Gamma$ the closure under substitution and modus ponens of the set of formulas $L \cup \Gamma$. If L is an intermediate logic and $\varphi \in L$ then we write $\vdash_L \varphi$ or $L \vdash \varphi$. Moreover, if φ can be obtained by closing the set $L \cup \Gamma$ under modus ponens, then we write $\Gamma \vdash_L \varphi$ and we say that φ is *derivable* from Γ in L. Intermediate logics satisfy the following deduction theorem. See [7, p. 45] for a proof of this result.

Theorem 2.44 (Deduction Theorem)**.** *Let L be any intermediate logic, then if $\Gamma, \varphi \vdash_L \psi$ then $\Gamma \vdash_L \varphi \to \psi$.*

It is a well-know fact [7, ch. 4.1] that intermediate logics form a *bounded lattice* **IL** with $\texttt{IPC} = \bot$ and $\texttt{CPC} = \top$ and where meet and join are defined as follows

$$\begin{aligned} L_0 \wedge L_1 &= L_0 \cap L_1 \\ L_0 \vee L_1 &= L_0 + L_1. \end{aligned}$$

This lattice was shown by Jankov in [34] to have the cardinality of the continuum $2^{\aleph_0}$. We list here some intermediate logics that will be useful for us in this thesis:

$$\begin{aligned} \texttt{KC} &= \texttt{IPC} + \neg p \vee \neg\neg p \\ \texttt{KP} &= \texttt{IPC} + (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r) \\ \texttt{ND}_k &= \texttt{IPC} + (\neg p \to \bigvee_{i \leq k} \neg q_i) \to \bigvee_{i \leq k} (\neg p \to \neg q_i) \\ \texttt{ND} &= \bigcup_{k \geq 2} \texttt{ND}_k \end{aligned}$$

Where each $\texttt{ND}_k$ is defined only for $k \geq 2$. The logic $\texttt{ND}$ was introduced by Maksimova in [39]. The logic $\texttt{KP}$ was introduced by Kreisel and Putnam in [38]. The logic $\texttt{KC}$ is also know as the *logic of the weak excluded middle* and was introduced by Jankov in [32].

Finally, let us see how intermediate logics can be translated into equational theories. We take the set of atomic propositions $\texttt{AT}$ as underlying set of variables X and we fix the signature-type $(\wedge, \vee, \to, \bot, \top)$. Then an intermediate logic L can be seen as an equational theory Σ_L where its elements are identities $\varphi \approx \top$, for every $\varphi \in L$. Notice that this translation allows us to consider logics in the context of universal algebra and to apply for the case of intermediate logics the results that we have developed in that context.

2.2.2 Relational Semantics

We introduce the relational semantics for intuitionistic logic. We refer the reader to [7, Ch. 8] and [5] for a detailed presentation of relational semantics both for intermediate and modal logics. An *intuitionistic relational structure* or *intuitionistic frame* is simply a poset $\mathcal{F} = (X, \leq)$. We often call the elements of X *worlds* and the relation $\leq$ *accessibility relation*. An intuitionistic relational model is then defined as follows.

Definition 2.45 (Intuitionistic Relational Model)**.** An *intuitionistic relational model* (also *Kripke Model*) is a triple $\mathcal{M} = (X, \leq, V)$ where $(X, \leq)$ is an intuitionistic relational structure and $V : \mathtt{AT} \to Up(X)$ is a function called *valuation*.

We then define, by induction, the following notion of *truth in a world*.

Definition 2.46 (Truth in a World)**.** Let $\mathcal{M} = (X, \leq, V)$ be a relational model, φ a formula of $\mathcal{L}_P$ and $x \in X$ a world. Then we say that φ is *true in* x and we write $x \Vdash \varphi$ if the following condition holds:

$$
\begin{aligned}
x \Vdash p &\text{ iff } x \in V(p) \\
x \Vdash \top &\text{ iff } x \in X \\
x \Vdash \bot &\text{ iff } x \in \emptyset \\
x \Vdash \varphi \wedge \psi &\text{ iff } x \Vdash \varphi \text{ and } x \Vdash \psi \\
x \Vdash \varphi \vee \psi &\text{ iff } x \Vdash \varphi \text{ or } x \Vdash \psi \\
x \Vdash \varphi \to \psi &\text{ iff } \forall y \geq x (y \Vdash \varphi \text{ then } y \Vdash \psi).
\end{aligned}
$$

Relational models have the following persistence property.

Theorem 2.47 (Persistence)**.** *Let φ be a formula of $\mathcal{L}_P$ and $\mathcal{M} = (X, \leq, V)$. Then for all $x, y \in X$ we have that if $x \leq y$ and $x \Vdash \varphi$ then $y \Vdash \varphi$.*

We say that a formula φ is *satisfiable in a model* $\mathcal{M} = (X, \leq, V)$ if there is some $x \in X$ such that $x \Vdash \varphi$. We say that a formula φ is *satisfiable in a structure* $\mathcal{F} = (X, \leq)$ if there is some valuation V such that φ is satisfiable in $(\mathcal{F}, V)$. We say that a formula φ is true in a model $\mathcal{M} = (X, \leq, V)$ and we write $\mathcal{M} \Vdash \varphi$ if for all worlds $x \in X$ we have that $x \Vdash \varphi$. We say that a formula φ is valid in a relational structure $\mathcal{F} = (X, \leq)$ and we write $\mathcal{F} \Vdash \varphi$ if for all valuation V we have that $(\mathcal{F}, V) \Vdash \varphi$. If $\mathcal{C}$ is a class of relational structures, we say that a formula φ is valid in $\mathcal{C}$ and we write $\mathcal{C} \Vdash \varphi$ if φ is true in every relational structure $\mathcal{F} \in \mathcal{C}$. We say that a formula φ is *intuitionistically valid* and we write $\Vdash \varphi$ if it is valid for the class of all relational structures.

We say that an intermediate logic L is *Kripke complete* if there is a class of relational structures $\mathcal{C}$ such that $\vdash_L \varphi$ iff $\Vdash_{\mathcal{C}} \varphi$. It was shown by Shehtman in [48] that not every intermediate logic is Kripke complete. The following theorem establishes the completeness of `IPC` with respect to the class of all relational structures.

Theorem 2.48 (Completeness of `IPC`)**.** *Let φ be any formula of $\mathcal{L}_P$, then we have that $\vdash \varphi$ iff $\Vdash \varphi$.*

2.2.3 Algebraic Semantics

Along with relational semantics, intermediate logics also accommodate algebraic semantics. In particular, Heyting algebras can be used to provide algebraic semantics to intermediate logics.

Definition 2.49 (Algebraic Model)**.** An *algebraic model* is a pair $M = (H, V)$ where H is n Heyting algebra and $V : \mathtt{AT} \to H$ is a valuation of propositional atoms over the elements of H.

Given an algebraic model $M = (H, V)$, we define by induction the interpretation of any formula $\varphi \in \mathcal{L}_P$.

Definition 2.50 (Interpretation of Arbitrary Formulas)**.** Given an algebraic model M and a formula $\varphi \in \mathcal{L}$, its *interpretation* $[\![\varphi]\!]^M$ is defined as follows:

1. For $p \in \mathtt{AT}$ we have $[\![p]\!]^M = V(p)$;
2. For $\varphi = \top$ we have $[\![\top]\!]^M = 1_H$;
3. For $\varphi = \bot$ we have $[\![\bot]\!]^M = 0_H$;
4. For $\varphi = \psi \wedge \chi$ we have $[\![\psi \wedge \chi]\!]^M = [\![\psi]\!]^M \wedge_H [\![\chi]\!]^M$;
5. For $\varphi = \psi \vee \chi$ we have $[\![\psi \vee \chi]\!]^M = [\![\psi]\!]^M \vee_H [\![\chi]\!]^M$;
6. For $\varphi = \psi \rightarrow \chi$ we have $[\![\psi \rightarrow \chi]\!]^M = [\![\psi]\!]^M \rightarrow_H [\![\chi]\!]^M$.

When the valuation V is clear from the context, we simply write $[\![\varphi]\!]^H$ for the interpretation of φ in H under V. We say that a formula φ is *true under V in H* or *true in the model $M = (H, V)$* and write $M \vDash \varphi$ if $[\![\varphi]\!]^M = 1$. We say that φ is *valid in H* and write $H \vDash \varphi$ if φ is true in every algebraic model $M = (H, V)$ over H. Given a class of Heyting algebras $\mathcal{C}$, we say that φ is *valid in $\mathcal{C}$* and write $\mathcal{C} \vDash \varphi$ if φ is valid in every Heyting algebra $H \in \mathcal{C}$. Finally, we say that φ is a *validity* if φ is valid in any Heyting algebra H.

The previous algebraic semantics allows us to prove an important duality result relating logics and algebras. Let **HA** be the lattice of Heyting algebras and **IL** the lattice of intermediate logics, we then define the two maps $Var : \mathbf{IL} \rightarrow \mathbf{HA}$ and $Log : \mathbf{HA} \rightarrow \mathbf{IL}$ as follows:

$$Var : L \mapsto \{H \in \mathbf{HA} : H \vDash L\};$$
$$Log : \mathcal{V} \mapsto \{\varphi \in \mathcal{L}_P : \mathcal{V} \vDash \varphi\}.$$

That the two former functions are well defined follows from $Var(L)$ being a variety of Heyting algebras and $Log(\mathcal{V})$ being an intermediate logic. Also, one can prove that both these maps are order-reversing homomorphisms. Since we have already seen that intermediate logics can be considered as equational theories, it is easy to see that these two maps amount to special cases of the maps $Id_X(\mathcal{K})$ and $M(\Sigma)$ which we have defined above. We say that a variety of Heyting algebras $\mathcal{V}$ is *defined* by a set of formulas Γ if $\mathcal{V} = Var(\Gamma)$ and we say $\mathcal{V}$ is definable if there exists one such Γ. We then say that an intermediate logic L is *algebraically complete* with respect to a class of Heyting algebras $\mathcal{C}$ if $L = Log(\mathcal{C})$. The universal algebra tools that we have introduced above give us the two following results.

Theorem 2.51 (Definability Theorem)**.** *Every variety of Heyting algebras $\mathcal{V}$ is defined by its validities, i.e. for every Heyting algebra H,*

$$H \in \mathcal{V} \Leftrightarrow H \vDash Log(\mathcal{V}).$$

Theorem 2.52 (Algebraic Completeness)**.** *Every intermediate logic L is complete with respect to its corresponding variety of Heyting algebras, i.e. for every $\varphi \in \mathcal{L}_P$,*

$$\varphi \in L \Leftrightarrow Var(L) \vDash \varphi.$$

The reader may refer to [7, Section 7] for a full proof of the aforementioned two results and the related constructions. Here let us only remark that the first of these two results is an immediate application of Birkhoff Theorem 2.16, once we consider logics as equational theories. The second result relies essentially on the free-algebra construction, namely on the Lindenbaum-Tarski algebra of intermediate logics. These results together give us the following theorem.

Theorem 2.53 (Dual Isomorphism)**.** *The lattice of intermediate logics is dually isomorphic to the lattice of varieties of Heyting algebras, i.e.* $\mathbf{IL} \cong^{op} \mathbf{HA}$.

Here, the isomorphisms between **IL** and **HA** are the two maps Log and Var. We sometimes refer to the previous theorem as a duality result concerning **IL** and **HA**. Notice that we are implicitly excluding from the lattice **HA** the trivial variety generated by a singleton set, for it would dually correspond to the inconsistent logic containing all the formulas of $\mathcal{L}_P$.

2.2.4 Properties of Intermediate Logics

Here, we introduce some properties of intermediate logics and we characterize the relations between them. We proceed to more complex varieties from simpler ones.

Definition 2.54 (Tabularity)**.** An intermediate logic L is *tabular* if it is the logic of a single finite algebra, i.e. $Var(L) = \mathcal{V}(H)$, where H is a finite Heyting algebra.

Tabular logics are thus exactly those intermediate logics which are complete with respect to a matrix-semantics with a finite number of truth-values. If a logic is not tabular, we can then enquire whether it has the following local tabularity property.

Definition 2.55 (Local Tabularity)**.** Let L be an intermediate logic. Then L is *locally tabular*, if, for any $n \in \mathbb{N}$, there are finitely many non-equivalent formulas in L with at most n variables.

It is then possible to give an algebraic counterpart to the locally finiteness of a logic. Let H be a Heyting algebra and $X \subseteq H$ an arbitrary subset of it. Then we define $\langle X \rangle$ as the least subalgebra of H such that $X \subseteq \langle X \rangle$ and we say that X *generates* $\langle X \rangle$. We say that a Heyting algebra H is *finitely generated* if there is a finite set of elements $x_0, ..., x_n \in H$ such that $\langle x_0, ..., x_n \rangle = H$. We then define locally finite logic in the following way.

Definition 2.56 (Local Finiteness)**.** An intermediate logic L is *locally finite* if every finitely generated $H \in Var(L)$ is also finite.

One can then show that a logic is locally finite if and only if it it is locally tabular. Local finiteness thus provides us with a semantic way to check whether a logic is locally tabular. Local finiteness is often useful as it allows us to work in a finite setting. A related property which also allows us to work with finite algebras is the finite model property.

Definition 2.57 (Finite Model Property)**.** An intermediate logic L has the *finite model property* (the FMP for short) if it is the logic of a class $\mathcal{C}$ of finite Heyting algebras, i.e. $Var(L) = \mathcal{V}(\mathcal{C})$, where for all $H \in \mathcal{C}$ we have that $|H| < \aleph_0$.

And, finally, we recall from the previous section the following notion of Kripke-completeness.

Definition 2.58 (Kripke Completeness)**.** An intermediate logic L is *Kripke complete* if it is the logic of a class $\mathcal{C}$ of relational structures.

It is possible to prove the following relations between these properties:

$$\text{Tabularity} \;\Rightarrow\; \text{Locally Finiteness} \;\Rightarrow\; \text{FMP} \;\Rightarrow\; \text{Kripke Completeness.}$$

Finally, under the algebraic semantics defined above, we have that $Var(\texttt{IPC}) = HA$ and $Var(\texttt{CPC}) = BA$. By using algebraic methods, one can then show two important results concerning intuitionistic and classical logic. First, the example of the Rieger-Nishimura ladder shows that the intuitionistic logic `IPC` is not locally tabular. On the other side of the spectrum, we have that `CPC` is tabular, since one can show that the only subdirectly irreducible Boolean algebra is 2 and thus the variety BA is equal to $\mathcal{V}(2)$. It follows by our previous considerations that `CPC` is locally finite, that it has the finite model property and it is Kripke complete.

Chapter 3

Algebraic Semantics for DNA-Logics

This chapter contains the basics of the theory of DNA-logics and DNA-varieties. In Section 3.1 we define DNA-logics as negative variants of intermediate logics and DNA-varieties as negative closures of varieties of Heyting algebras. In Section 3.2 we introduce a suitable algebraic semantics for DNA-logics and we prove several results connecting standard validity and DNA-validity. Finally, in Section 3.3 we provide two proofs of the dual isomorphism result between the lattice of DNA-logics and the lattice of DNA-varieties. The first proof relies on the standard dual isomorphism between intermediate logics and varieties of Heyting algebras, while the second one follows a suitable construction of Lindenbaum-Tarski algebras for DNA-logics.

3.1 DNA-Logics and DNA-Varieties

3.1.1 DNA-Logics

We proceed by introducing the notion of negative variant of an intermediate logic. Negative variants were first introduced by Miglioli et al. in [41] and later employed by Ciardelli in [10]. If $\varphi \in \mathcal{L}_P$ is an arbitrary formula, we often say that the formula $\varphi[\overline{\neg p}/\overline{p}]$ obtained by replacing all the atomic letters in φ with their negation is its *negative variant.*

Definition 3.1 (Negative Variant of a Logic)**.** For every intermediate logic L, its *negative variant* $L^{\neg}$ is defined as follows:

$$L^{\neg} = \{\varphi \in \mathcal{L}_P : \varphi[\overline{\neg p}/\overline{p}] \in L\}.$$

A DNA-*logic* is then defined as the negative variant of some intermediate logic L. The name DNA stands for *double negation atoms*, which refers to the fact that, as we shall see, DNA-logics prove $\neg\neg p \rightarrow p$ for all atomic formulas $p \in \mathtt{AT}$. We will use the notation $L^{\neg}$ to refer to the negative variant of an intermediate logic L. If not specified otherwise, we reserve uppercase greek letters as Γ and Δ to denote arbitrary sets of formulas and Λ and Π to denote DNA-logics. The following proposition provides us with an axiomatisation for every DNA-logic.

Proposition 3.2. *Let Λ be a* DNA*-logic with $\Lambda = L^{\neg}$, then Λ is the least set of formulas such that:*

1. *$L \subseteq \Lambda$;*
2. *For all atomic propositional formulas $p \in \mathtt{AT}$ we have that $\neg\neg p \to p \in \Lambda$;*
3. *Λ is closed under the modus ponens rule:*

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\ \textit{(MP)}$$

Proof. Firstly we prove that $L^{\neg}$ satisfies these three conditions, then we check that it is the least such set. (1) Suppose $\varphi \in L$, then since L is an intermediate logic it is closed under substitution and thus $\varphi[\overline{\neg p}/\overline{p}] \in L$. By the definition of negative variant, it follows that $\varphi \in L^{\neg} = \Lambda$. (2) Since $\mathtt{IPC} \subseteq L$ and it is a theorem of IPC that $\neg\neg\neg p \to \neg p$ for every atomic letter $p \in \mathtt{AT}$, it follows by the definition of negative variant that $\neg\neg p \to p \in L^{\neg} = \Lambda$. (3) Now, suppose $\varphi, \varphi \to \psi \in L^{\neg}$, then it follows by the definition of negative variant that $\varphi[\overline{\neg p}/\overline{p}], (\varphi \to \psi)[\overline{\neg p}/\overline{p}] \in L$, and thus $\varphi[\overline{\neg p}/\overline{p}], \varphi[\overline{\neg p}/\overline{p}] \to \psi[\overline{\neg p}/\overline{p}] \in L$. Then it follows, by the fact that L is closed under modus ponens, that $\psi[\overline{\neg p}/\overline{p}] \in L$ and so $\psi \in L^{\neg} = \Lambda$.

It remains to be proven that Λ is the least such set. Suppose X also validates the three conditions above, we need to show that $\Lambda \subseteq X$. Consider any $\varphi \in \Lambda = L^{\neg}$, then by the definition of negative variant $\varphi[\overline{\neg p}/\overline{p}] \in L$. Therefore, by uniform substitution, $\varphi[\overline{\neg\neg p}/\overline{p}] \in L$ and therefore since $L \subseteq X$ also $\varphi[\overline{\neg\neg p}/\overline{p}] \in X$. Finally, since for every $p \in \mathtt{AT}$, $\neg\neg p \to p \in X$, it follows that $\varphi[\overline{\neg\neg p}/\overline{p}] \to \varphi \in X$ and thus considering X is closed under modus ponens $\varphi \in X$. □

We can now show that DNA-logics give rise to a lattice structure ordered by the set-theoretic inclusion. The meet of two DNA-logics Λ_0, Λ_1 is just their intersection and their join is the closure of their union under modus ponens. We will thus write $\Lambda \wedge \Lambda_1 := \Lambda_0 \cap \Lambda_1$ and $\Lambda_0 \vee \Lambda_1 := (\Lambda_0 \cup \Lambda_1)^{MP}$, where we denote by $(\Gamma)^{MP}$ the closure under modus ponens of a set Γ of formulas. If φ can be obtained by closing the set Γ of formulas under modus ponens, then we have $\Gamma \vdash \varphi$, i.e. φ is *derivable* from Γ. We prove the following proposition.

Proposition 3.3. *Let Λ_0 and Λ_1 be two* DNA*-logics, then: (i) $\Lambda_0 \wedge \Lambda_1$ is a* DNA*-logic and it is the infimum of Λ_0 and Λ_1, (ii) $\Lambda_0 \vee \Lambda_1$ is a* DNA*-logic and it is the supremum of Λ_0 and Λ_1.*

Proof. (i) By definition $\Lambda_0 \wedge \Lambda_1 = \Lambda_0 \cap \Lambda_1$. We show this is a DNA-logic. Let us suppose, without loss of generality, that $\Lambda_0 = L_0^{\neg}$ and $\Lambda_1 = L_1^{\neg}$, then since $L_0 \subseteq \Lambda_0$ and $L_1 \subseteq \Lambda_1$, we have $L_0 \cap L_1 \subseteq \Lambda_0 \cap \Lambda_1$. For any propositional formula $p \in \mathtt{AT}$ we have $\neg\neg p \to p \in \Lambda_0$ and $\neg\neg p \to p \in \Lambda_1$ so $\neg\neg p \to p \in \Lambda_0 \cap \Lambda_1$. Similarly, closure under modus ponens follows from the fact that both Λ_0 and Λ_1 are closed under modus ponens. Therefore, it follows by Proposition 3.2 that $\Lambda_0 \wedge \Lambda_1 = (L_0 \wedge L_1)^{\neg}$ and so that it is a DNA-logic. Finally, since $\Lambda_0 \wedge \Lambda_1 = \Lambda_0 \cap \Lambda_1$, it is obvious that $\Lambda_0 \wedge \Lambda_1$ is the infimum of Λ_0 and Λ_1.

(ii) By definition $\Lambda_0 \vee \Lambda_1 = (\Lambda_0 \cup \Lambda_1)^{MP}$. Let us suppose, without loss of generality, that $\Lambda_0 = L_0^\neg$ and $\Lambda_1 = L_1^\neg$. Since $L_0 \subseteq \Lambda_0$ and $L_1 \subseteq \Lambda_1$ we have that $L_0 \cup L_1 \subseteq \Lambda_0 \cup \Lambda_1$ and therefore $(L_0 \cup L_1)^{MP} \subseteq (\Lambda_0 \cup \Lambda_1)^{MP}$, which means that $L_0 \vee L_1 \subseteq \Lambda_0 \vee \Lambda_1$. Also, since $\neg\neg p \to p \in \Lambda_0$ and $\neg\neg p \to p \in \Lambda_1$ we have $\neg\neg p \to p \in \Lambda_0 \vee \Lambda_1$. Closure by modus ponens follows by definition. Therefore, it follows from Proposition 3.2 that $\Lambda_0 \vee \Lambda_1 = (L_0 \vee L_1)^\neg$ and hence it is a `DNA`-logic. Finally, we show that $\Lambda_0 \vee \Lambda_1$ is the supremum of Λ_0 and Λ_1. Suppose there is some `DNA`-logic Π such that $\Lambda_0 \subseteq \Pi$ and $\Lambda_1 \subseteq \Pi$, we show that $(\Lambda_0 \cup \Lambda_1)^{MP} \subseteq \Pi$. Consider any $\varphi \in (\Lambda_0 \cup \Lambda_1)^{MP}$, then φ can be derived by modus ponens from some formulas $\psi_0, ..., \psi_n \in \Lambda_0 \cup \Lambda_1$. But then $\psi_0, ..., \psi_n \in \Pi$ and since Π is also closed under modus ponens the same derivation entails that $\varphi \in \Pi$. $\square$

We denote by **DNAL** the lattice of `DNA`-logics. Since intermediate logics also form a lattice **IL**, we can then show that the map $(-)^\neg : \mathbf{IL} \to \mathbf{DNAL}$ which assigns each intermediate logic to its negative variant is a lattice homomorphism.

Proposition 3.4. *The map* $(-)^\neg : \mathbf{IL} \to \mathbf{DNAL}$ *is a lattice homomorphism.*

Proof. Obviously $(-)^\neg$ sends $\bot_{\mathbf{IL}}$ to $\bot_{\mathbf{DNAL}}$ and $\top_{\mathbf{IL}}$ to $\top_{\mathbf{DNAL}}$, so it suffices to check that $(-)^\neg$ preserves meet and join.

(i) Consider two intermediate logics L_0 and L_1, then it is straightforward that:

$$\begin{aligned}(L_0 \wedge L_1)^\neg &= (L_0 \cap L_1)^\neg \\ &= \{\varphi \in \mathcal{L}_P : \varphi[\neg\overline{p}/\overline{p}] \in L_0 \cap L_1\} \\ &= \{\varphi \in \mathcal{L}_P : \varphi[\neg\overline{p}/\overline{p}] \in L_0\} \cap \{\varphi \in \mathcal{L}_P : \varphi[\neg\overline{p}/\overline{p}] \in L_1\} \\ &= L_0^\neg \cap L_1^\neg \\ &= L_0^\neg \wedge L_1^\neg.\end{aligned}$$

which shows that $(-)^\neg$ preserves the meet operator.

(ii) Consider two intermediate logics L_0 and L_1. We have by definition that $(L_0 \vee L_1)^\neg = ((L_0 \cup L_1)^{MP})^\neg$ and $L_0^\neg \vee L_1^\neg = (L_0^\neg \cup L_1^\neg)^{MP}$. It suffices to show that $((L_0 \cup L_1)^{MP})^\neg = (L_0^\neg \cup L_1^\neg)^{MP}$. ($\subseteq$) Suppose $\varphi \in ((L_0 \cup L_1)^{MP})^\neg$, then it follows that $\varphi[\neg\overline{p}/\overline{p}] \in (L_0 \cup L_1)^{MP}$, hence for some formulas $\psi_0, ..., \psi_n \in L_0 \cup L_1$ we have $\psi_0, ..., \psi_n \vdash \varphi[\neg\overline{p}/\overline{p}]$. We immediately obtain that $\psi_0[\neg\overline{p}/\overline{p}], ..., \psi_n[\neg\overline{p}/\overline{p}] \vdash \varphi[\neg\neg\overline{p}/\overline{p}]$. So, since for every $p \in$ `AT` we have $\neg\neg p \to p \in L_0^\neg, L_1^\neg$ it follows that $\psi_0[\neg\overline{p}/\overline{p}], ..., \psi_n[\neg\overline{p}/\overline{p}] \vdash \varphi$ and hence $\varphi \in (L_0^\neg \cup L_1^\neg)^{MP}$. ($\supseteq$) Suppose $\varphi \in (L_0^\neg \cup L_1^\neg)^{MP}$, then it follows that for some formulas $\psi_0, ..., \psi_n \in L_0^\neg \cup L_1^\neg$ we have that $\psi_0, ..., \psi_n \vdash \varphi$. Therefore, for some formulas $\psi_0[\neg\overline{p}/\overline{p}], ..., \psi_n[\neg\overline{p}/\overline{p}] \in L_0 \cup L_1$ we have by the same derivation $\psi_0[\neg\overline{p}/\overline{p}], ..., \psi_n[\neg\overline{p}/\overline{p}] \vdash \varphi[\neg\overline{p}/\overline{p}]$, so $\varphi[\neg\overline{p}/\overline{p}] \in (L_0 \cup L_1)^{MP}$ and finally $\varphi \in ((L_0 \cup L_1)^{MP})^\neg$. So $(-)^\neg$ also preserves the join operator and is thus a lattice homomorphism. $\square$

3.1.2 `DNA`-Varieties

We have introduced `DNA`-logics in purely syntactical terms, as negative variants of intermediate logics. To provide `DNA`-logics with an algebraic semantics we first introduce `DNA`-varieties. In particular, we define `DNA`-varieties as negative closures of varieties of Heyting algebras. Firstly, if H is an Heyting algebra, then we say that an element $x \in H$ is *regular* if $x = \neg\neg x$. For any Heyting algebra H we then denote by $H_\neg$ the set:

$$H_\neg = \{x \in H : x = \neg\neg x\}.$$

So $H_\neg$ consists of all regular elements of the Heyting algebra H. Take note that since in every Heyting algebras we have that $\neg x = \neg\neg\neg x$, the set of regular elements of H can also be specified as $H_\neg = \{y \in H : \exists x \in H(y = \neg x)\}$. We then define the negative closure of a variety of Heyting algebras as follows.

Definition 3.5 (Negative Closure of a Variety)**.** For every variety of Heyting algebras $\mathcal{V}$, its *negative closure* $\mathcal{V}^\uparrow$ is defined as follows:

$$\mathcal{V}^\uparrow = \{H : \exists A \in \mathcal{V} \text{ such that } A_\neg = H_\neg \text{ and } A \preceq H\}.$$

A `DNA`-*variety* is then defined as the negative closure of some variety $\mathcal{V}$ of Heyting algebras. We use the notation $\mathcal{V}^\uparrow$ to refer to the negative variant of a variety $\mathcal{V}$ and we generally write $\mathcal{X}$ for `DNA`-varieties. If not specified otherwise, we reserve $\mathcal{C}$ to denote arbitrary classes of Heyting algebras, $\mathcal{V}$ or $\mathcal{U}$ to denote standard varieties and $\mathcal{X}$ or $\mathcal{Y}$ to denote `DNA`-varieties.

We now want to show that `DNA`-varieties are also standard varieties, i.e. they are closed under the usual operations of subalgebra, homomorphic image and product. Moreover, we also show they are closed under the following condition.

Definition 3.6. We say that a Heyting algebra K is a *core superalgebra* of H if $H_\neg = K_\neg$ and $H \preceq K$.

A core superalgebra K of a Heyting algebra H is thus an algebra of which H is subalgebra such that they share the same regular elements. The following proposition provides us with a characterisation of `DNA`-varieties.

Proposition 3.7. *A class of Heyting algebras $\mathcal{C}$ is a* `DNA`*-variety if and only if it is closed under subalgebras, homomorphic images, products and core superalgebras.*

Proof. ($\Leftarrow$) If a set of algebras $\mathcal{C}$ is closed under subalgebras, homomorphic images and products then it is a variety. Moreover, since it is also closed under core superalgebras, it is straightforward to see that $\mathcal{C} = \mathcal{C}^\uparrow$, so that we can see $\mathcal{C}$ as the negative variant of itself and thus as a `DNA`-variety.

($\Rightarrow$) Consider now a `DNA`-variety $\mathcal{X}$. By definition it is the negative variant of some standard variety $\mathcal{V}$, so we have $\mathcal{X} = \mathcal{V}^\uparrow$. We need to check that $\mathcal{V}^\uparrow$ is closed under the above four operations.

(1) We check closure under subalgebras. Suppose $H \in \mathcal{V}^\uparrow$ and $K \preceq H$. Then by definition of `DNA`-variety it follows that there is some $H' \in \mathcal{V}$ such that $H'_\neg = H_\neg$ and $H' \preceq H$. Now consider $K' = H' \cap K$, since K' is the intersection of two subalgebras of H, it will also be closed under the Heyting algebra operations. Thus we have that K' is also a Heyting algebra and $K' \preceq H'$ and $K' \preceq K$. Therefore, by the fact that $K \in \mathcal{V}$ and $\mathcal{V}$ is closed under subalgebras, it then follows that $K' \in \mathcal{V}$. Moreover, since $H'_\neg = H_\neg \supseteq K_\neg$, we have that $K'_\neg = H'_\neg \cap K_\neg = K_\neg$. Finally, we showed that for $K' \in \mathcal{V}$ we have $K' \preceq K$ and $K'_\neg = K_\neg$, which entails $K \in \mathcal{V}^\uparrow$.

(2) We check closure under homomorphic images. Suppose $H \in \mathcal{V}^{\uparrow}$ and $f : H \twoheadrightarrow K$, then by the definition of `DNA`-variety we have that for some $H' \in V$ that $H'_{\neg} = H_{\neg}$ and $H' \preceq H$. Consider $K' = f[H']$. Since homomorphic images preserve subalgebras, we have $K' \preceq K$ and, by the closure of standard varieties under homomorphic images we have $K' \in \mathcal{V}$. Moreover, since by assumption $H_{\neg} = H'_{\neg}$ and since homomorphisms preserve the algebra operations, we have $K_{\neg} = f[H_{\neg}] = f[H'_{\neg}] = K'_{\neg}$. Thus for $K' \in \mathcal{V}$ we have $K' \preceq K$ and $K'_{\neg} = K_{\neg}$, which yields $K \in \mathcal{V}^{\uparrow}$.

(3) We check closure under products. We only consider binary products, but it is easy to see that our proof immediately generalizes to arbitrary products. Suppose $H^0, H^1 \in \mathcal{V}^{\uparrow}$, we need to check that also $H^0 \times H^1 \in \mathcal{V}^{\uparrow}$. By the definition of `DNA`-variety it immediately follows that there are $K^0, K^1 \in \mathcal{V}$ such that $H^0_{\neg} = K^0_{\neg}$, $H^1_{\neg} = K^1_{\neg}$ and $K^0 \preceq H^0$, $K^1 \preceq H^1$. Then by the closure under products of $\mathcal{V}$, we have that $K^0 \times K^1 \in \mathcal{V}$. Since $K^0 \preceq H^0$ and $K^1 \preceq H^1$ it is straightforward to lift these two embeddings to the product to get $K^0 \times K^1 \preceq H^0 \times H^1$. Similarly, from the fact that $H^0_{\neg} = K^0_{\neg}$ and $H^1_{\neg} = K^1_{\neg}$ we then obtain, by the definition of products, $(H^0 \times H^1)_{\neg} = (K^0 \times K^1)_{\neg}$. It then follows from the definition of `DNA`-varieties and the fact $K^0 \times K^1 \in \mathcal{V}$, that $H^0 \times H^1 \in \mathcal{V}^{\uparrow}$.

(4) We check closure under core superalgebras. Suppose $H \in \mathcal{V}^{\uparrow}$ and for some K we have that $H_{\neg} = K_{\neg}$ and $H \preceq K$. By the definition of `DNA`-varieties we have that there is some $H' \preceq H$ such that $H' \in \mathcal{V}$ and $H'_{\neg} = H_{\neg}$. Since $H' \preceq H$ and $H \preceq K$ we then have $H' \preceq K$ by the transitivity of subalgebra relation. Moreover, since $H'_{\neg} = H_{\neg} = K_{\neg}$ and $H \in \mathcal{V}$, it finally follows that $K \in \mathcal{V}^{\uparrow}$. □

As in the case of standard varieties, also `DNA`-varieties give rise to a lattice structure ordered by the set-theoretic inclusion. As we have done for standard varieties, we implicitly exclude from the lattice of `DNA`-varieties the trivial `DNA`-variety of one-element algebras. The meet of two `DNA`-varieties $\mathcal{X}_0, \mathcal{X}_1$ is then just their intersection and their join is the smallest class containing their union and closed under the `DNA`-variety operations. For any class $\mathcal{C}$ of Heyting algebras we define $\mathcal{X}(\mathcal{C}) = \mathcal{V}(\mathcal{C})^{\uparrow}$, making $\mathcal{X}(\mathcal{C})$ the smallest `DNA`-variety containing $\mathcal{C}$. We will thus define $\mathcal{X}_0 \wedge \mathcal{X}_1 := \mathcal{X}_0 \cap \mathcal{X}_1$ and $\mathcal{X}_0 \vee \mathcal{X}_1 := \mathcal{X}(\mathcal{X}_0 \cup \mathcal{X}_1)$. We proceed to prove the following proposition.

Proposition 3.8. *Let $\mathcal{X}_0$ and $\mathcal{X}_1$ be two* `DNA`*-varieties, then: (i) $\mathcal{X}_0 \wedge \mathcal{X}_1$ is a* `DNA`*-variety and it is the infimum of $\mathcal{X}_0$ and $\mathcal{X}_1$; (ii) $\mathcal{X}_0 \vee \mathcal{X}_1$ is a* `DNA`*-variety and it is the supremum of $\mathcal{X}_0$ and $\mathcal{X}_1$.*

Proof. (i) By definition $\mathcal{X}_0 \wedge \mathcal{X}_1 := \mathcal{X}_0 \cap \mathcal{X}_1$. That this is a `DNA`-variety follows immediately from the fact that, since both $\mathcal{X}_0$ and $\mathcal{X}_1$ are closed under subalgebras, homomorphic images, products and core superalgebras, then also their intersection is closed under these operations. Moreover, since $\mathcal{X}_0 \wedge \mathcal{X}_1 := \mathcal{X}_0 \cap \mathcal{X}_1$, it follows that $\mathcal{X}_0 \wedge \mathcal{X}_1$ is the infimum of $\mathcal{X}_0$ and $\mathcal{X}_1$.

(ii) By definition $\mathcal{X}_0 \vee \mathcal{X}_1 = \mathcal{X}(\mathcal{X}_0 \cup \mathcal{X}_1) = \mathcal{V}(\mathcal{X}_0 \cup \mathcal{X}_1)^{\uparrow}$, which is a `DNA`-variety. Now suppose $\mathcal{Y}$ is also a `DNA`-variety and $\mathcal{X}_0 \cup \mathcal{X}_1 \subseteq \mathcal{Y}$. Then since $\mathcal{Y}$ is also a variety it follows that $\mathcal{V}(\mathcal{X}_0 \cup \mathcal{X}_1) \subseteq \mathcal{Y}$ and since $\mathcal{Y}$ is also closed under core superalgebras it follows that $\mathcal{V}(\mathcal{X}_0 \cup \mathcal{X}_1)^{\uparrow} = \mathcal{X}(\mathcal{X}_0 \cup \mathcal{X}_1) \subseteq \mathcal{Y}$ and in turn gives us $\mathcal{X}(\mathcal{X}_0 \cup \mathcal{X}_1) = \mathcal{X}_0 \vee \mathcal{X}_1$ is the supremum of $\mathcal{X}_0$ and $\mathcal{X}_1$. □

We denote the lattice of DNA-varieties by **DNAV**. As varieties of Heyting algebras also form a lattice **HA**, one can then show that the map $(-)^{\uparrow} : \mathbf{HA} \to \mathbf{DNAV}$ which assigns every variety of Heyting algebras to its negative closure is a lattice homomorphism.

Proposition 3.9. *The map $(-)^{\uparrow} : \mathbf{HA} \to \mathbf{DNAV}$ is a lattice homomorphism.*

Proof. Obviously $(-)^{\uparrow}$ sends $\bot_{\mathbf{HA}}$ to $\bot_{\mathbf{DNAV}}$ and $\top_{\mathbf{HA}}$ to $\top_{\mathbf{DNAV}}$, so it suffices to check that $\uparrow$ preserves meet and join.

(i) Consider two standard varieties $\mathcal{V}_0$ and $\mathcal{V}_1$, then we have:

$$\begin{aligned}(\mathcal{V}_0 \wedge \mathcal{V}_1)^{\uparrow} &= \{H : \exists A \in \mathcal{V}_0 \cap \mathcal{V}_1 \text{ such that } A_{\neg} = H_{\neg} \text{ and } A \preceq H\} \\ &= \{H : \exists A \in \mathcal{V}_0(A_{\neg} = H_{\neg}, A \preceq H)\} \cap \{H : \exists A \in \mathcal{V}_1(A_{\neg} = H_{\neg}, A \preceq H)\} \\ &= \mathcal{V}_0^{\uparrow} \wedge \mathcal{V}_1^{\uparrow}.\end{aligned}$$

which shows that $(-)^{\uparrow}$ preserves the meet operator.

(ii) Consider two standard varieties $\mathcal{V}_0$ and $\mathcal{V}_1$, then we have by definition that $(\mathcal{V}_0 \vee \mathcal{V}_1)^{\uparrow} = (\mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1))^{\uparrow}$ and $\mathcal{V}_0^{\uparrow} \vee \mathcal{V}_1^{\uparrow} = \mathcal{X}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow}) = \mathcal{V}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow})^{\uparrow}$. It thus suffices to show that $\mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1)^{\uparrow} = \mathcal{V}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow})^{\uparrow}$. ($\subseteq$) Let us suppose $X \in (\mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1))^{\uparrow}$ which implies that there is some $K \in \mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1)$ such that $K_{\neg} = H_{\neg}$ and $K \preceq H$. Then clearly $K \in \mathcal{V}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow})$ and thus $H \in \mathcal{V}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow})^{\uparrow}$. ($\supseteq$) Suppose now $H \in \mathcal{V}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow})^{\uparrow}$, then for some $K \in \mathcal{V}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow})$ we have that $K_{\neg} = H_{\neg}$ and $K \preceq H$. Thus we can assume, without loss of generality, that K is generated by some algebras $A^1, ..., A^n \in \mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow}$ and that there are algebras $B^1, ..., B^n \in \mathcal{V}_0 \cup \mathcal{V}_1$ such that for every $i \leq n$, $B^i \preceq A^i$ and $A^i_{\neg} = B^i_{\neg}$. But then it follows immediately that $A^1, ..., A^n \in (\mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1)^{\uparrow})$ and so since $(\mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1)^{\uparrow})$ is a variety, we have that $K \in (\mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1))^{\uparrow}$. Finally, since $(\mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1))^{\uparrow}$ is also a DNA-variety, we also have that $H \in (\mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1))^{\uparrow}$. ☐

In this section we have provided two parallel characterisations of DNA-logics and DNA-varieties. On the one hand, we have introduced them in terms of negative variants of some intermediate logics or in terms of negative closure of some variety of Heyting algebras. On the other hand, we have also given an independent characterization of DNA-logics and DNA-varieties, as sets of formulas closed under some conditions or as set of algebras closed under some operations. During the course of this thesis we will alternate between the two perspectives, i.e. consider DNA-logics and DNA-varieties in terms of negative variants or consider them as sets satisfying some closure properties.

3.2 Algebraic Semantics for DNA-Logics

3.2.1 DNA-Models

We now introduce the algebraic semantics of DNA-logics. This semantics generalizes the semantics for inquisitive logic InqB given in [3].

Definition 3.10 (DNA-Model)**.** A DNA-*model* is a pair $M = (H, V^{\neg})$ where H is a Heyting algebra and $V^{\neg} : \mathtt{AT} \to H_{\neg}$ is a valuation of propositional atoms over the regular elements of H.

Given a $\mathtt{DNA}$-model $M = (H, V^{\neg})$, we define by induction the interpretation of any formula $\varphi \in \mathcal{L}_P$.

Definition 3.11 (Interpretation of Arbitrary Formulas)**.** Given a $\mathtt{DNA}$-model M and a formula $\varphi \in \mathcal{L}_P$, its *interpretation* $[\![\varphi]\!]^M$ is defined as follows:

1. For $p \in \mathtt{AT}$ we have $[\![p]\!]^M = V^{\neg}(p)$;
2. For $\varphi = \top$ we have $[\![\top]\!]^M = 1_H$;
3. For $\varphi = \bot$ we have $[\![\bot]\!]^M = 0_H$;
4. For $\varphi = \psi \wedge \chi$ we have $[\![\psi \wedge \chi]\!]^M = [\![\psi]\!]^M \wedge_H [\![\chi]\!]^M$;
5. For $\varphi = \psi \vee \chi$ we have $[\![\psi \vee \chi]\!]^M = [\![\psi]\!]^M \vee_H [\![\chi]\!]^M$;
6. For $\varphi = \psi \to \chi$ we have $[\![\psi \to \chi]\!]^M = [\![\psi]\!]^M \to_H [\![\chi]\!]^M$.

When the valuation $V^{\neg}$ is clear from the context, we simply write $[\![\varphi]\!]^H$ for the interpretation of φ in H under $V^{\neg}$. From the former definitions it is straightforward to adapt the usual definitions of truth at a model and validity. We say that a formula φ is *true under* $V^{\neg}$ *in* H or *true in the model* $M = (H, V^{\neg})$ and write $M \vDash^{\neg} \varphi$ if $[\![\varphi]\!]^M = 1$. We say that φ is $\mathtt{DNA}$-*valid in* H and write $H \vDash^{\neg} \varphi$ if φ is true in every model $M = (H, V^{\neg})$ over H. Given a class $\mathcal{C}$ of Heyting algebras, we say that φ is $\mathtt{DNA}$-*valid in* $\mathcal{C}$ and write $\mathcal{C} \vDash^{\neg} \varphi$ if φ is $\mathtt{DNA}$-valid in every Heyting algebra $H \in \mathcal{C}$. In particular, given a $\mathtt{DNA}$-variety $\mathcal{X}$, we say that φ is $\mathtt{DNA}$-*valid in* $\mathcal{X}$ if φ is $\mathtt{DNA}$-valid in every Heyting algebra $H \in \mathcal{X}$. Finally, we say that φ is a $\mathtt{DNA}$-*validity* if φ is valid in any Heyting algebra H. When the context is clear, we drop the qualification $\mathtt{DNA}$ from the definitions above and talk simply of *validity*.

Remark 3.12. It is a well-known fact that the subset of regular elements $H_{\neg}$ of H is a Boolean algebra with respect to the operations $\wedge_H, \to_H, 1_H, 0_H$ and the disjunction $x \vee_B y = \neg(\neg x \wedge_H \neg y)$. The resulting algebra $(H_{\neg}, \wedge_B, \vee_B, \to_B, 1_H, 0_H)$ is a Boolean algebra which is a subset of H and also accords with H with respect to the operations of meet, negation and implication, while dissents with respect to the join operator. One can then have a different look at the carrier of a $\mathtt{DNA}$-model by introducing $\mathtt{DNA}$-structures as pairs (B, H) of a Boolean algebra and a Heyting algebra such that $B \subseteq H$ and B accords with H with respect to meet, negation and implication. From this, it is easy to show that the greatest Boolean algebra which satisfies these requirements is exactly the Boolean algebra of regular elements $H_{\neg}$. Based on this perspective, it is then possible to consider two different join operators between any two formulas of a $\mathtt{DNA}$-logic: the join operator $\vee_H$ of the Heyting algebra and the join operator $\vee_B$ of the Boolean algebra of regular elements. As we shall see later in the next chapter, these two algebraic operators correspond to the two disjunction symbols one usually has in inquisitive logic. The reader may also refer to [3].

3.2.2 Connection between Validity and $\mathtt{DNA}$-Validity

We now prove some important propositions which allow us to relate the usual algebraic semantics of intermediate logics to the $\mathtt{DNA}$-semantics just defined. Firstly, let us introduce the notion of negative variant of a valuation.

Definition 3.13 (Negative Variant of a Valuation)**.** Let H be a Heyting algebra and V an arbitrary valuation. Then we say that $V^{\neg}$ is the *negative variant* of V if for all $p \in \mathtt{AT}$ we have that $V^{\neg}(p) = \neg V(p)$.

The following lemma shows that the set of $\mathtt{DNA}$-valuations and the set of all negative variants of standard valuations actually coincide.

Lemma 3.14. *A valuation $V^{\neg}$ is a $\mathtt{DNA}$-valuation iff it is the negative variant of some valuation V.*

Proof. ($\Rightarrow$) Suppose $V^{\neg}$ is a $\mathtt{DNA}$-valuation, then since for every $p \in \mathtt{AT}$ we have $V^{\neg}(p) \in H_{\neg}$ it follows that $V^{\neg}(p) = \neg\neg V^{\neg}(p)$ and hence $V^{\neg}$ is the negative variant of its negative variant $U : p \mapsto \neg V^{\neg}(p)$. ($\Leftarrow$) Suppose $V^{\neg}$ is the the negative variant of some valuation U, then for all $p \in \mathtt{AT}$, $V^{\neg}(p) = \neg U(p)$ so that for all $p \in \mathtt{AT}$, $V^{\neg}(p)$ is regular and thus $V^{\neg}$ is $\mathtt{DNA}$. $\square$

We can now prove the following important lemma.

Lemma 3.15. *For every Heyting algebra H, for every valuation V and any formula φ, we have*

$$[\![\varphi]\!]^{(H,V^{\neg})} = [\![\varphi[{}^{\neg p}/_{p}]]\!]^{(H,V)}.$$

Proof. By induction on the complexity of φ.

1. For $p \in \mathtt{AT}$ we have:

$$[\![p]\!]^{(H,V^{\neg})} = V^{\neg}(p) = \neg V(p) = \neg[\![p]\!]^{(H,V)} = [\![\neg p]\!]^{(H,V)}.$$

2. For $\varphi = \psi \wedge \chi$ we have:

$$\begin{aligned}[\![\psi \wedge \chi]\!]^{(H,V^{\neg})} &= [\![\psi]\!]^{(H,V^{\neg})} \wedge [\![\chi]\!]^{(H,V^{\neg})} \\ &= [\![\psi[{}^{\neg p}/_{p}]]\!]^{(H,V)} \wedge [\![\chi[{}^{\neg p}/_{p}]]\!]^{(H,V)} \\ &= [\![\psi[{}^{\neg p}/_{p}] \wedge \chi[{}^{\neg p}/_{p}]]\!]^{(H,V)} \\ &= [\![(\psi \wedge \chi)[{}^{\neg p}/_{p}]]\!]^{(H,V)}.\end{aligned}$$

3. For $\varphi = \psi \vee \chi$ we have:

$$\begin{aligned}[\![\psi \vee \chi]\!]^{(H,V^{\neg})} &= [\![\psi]\!]^{(H,V^{\neg})} \vee [\![\chi]\!]^{(H,V^{\neg})} \\ &= [\![\psi[{}^{\neg p}/_{p}]]\!]^{(H,V)} \vee [\![\chi[{}^{\neg p}/_{p}]]\!]^{(H,V)} \\ &= [\![\psi[{}^{\neg p}/_{p}] \vee \chi[{}^{\neg p}/_{p}]]\!]^{(H,V)} \\ &= [\![(\psi \vee \chi)[{}^{\neg p}/_{p}]]\!]^{(H,V)}.\end{aligned}$$

4. For $\varphi = \psi \to \chi$ we have:

$$\begin{aligned}
[\![\psi \to \chi]\!]^{(H,V^{\neg})} &= [\![\psi]\!]^{(H,V^{\neg})} \to [\![\chi]\!]^{(H,V^{\neg})} \\
&= [\![\psi[{}^{\overline{\neg p}}/_{\overline{p}}]]\!]^{(H,V)} \to [\![\chi[{}^{\overline{\neg p}}/_{\overline{p}}]]\!]^{(H,V)} \\
&= [\![\psi[{}^{\overline{\neg p}}/_{\overline{p}}] \to \chi[{}^{\overline{\neg p}}/_{\overline{p}}]]\!]^{(H,V)} \\
&= [\![(\psi \to \chi)[{}^{\overline{\neg p}}/_{\overline{p}}]]\!]^{(H,V)}.
\end{aligned}$$

The cases for $\varphi = \top$ and $\varphi = \bot$ are obvious. Then this establishes that $[\![\varphi]\!]^{(H,V^{\neg})} = [\![\varphi[{}^{\overline{\neg p}}/_{\overline{p}}]]\!]^{(H,V)}$. □

From this we can derive the following result.

Proposition 3.16. *For any Heyting algebra H, $H \vDash^{\neg} \varphi$ iff $H \vDash \varphi[{}^{\overline{\neg p}}/_{\overline{p}}]$.*

Proof. We prove both directions by contraposition. ($\Rightarrow$) Suppose $H \nvDash \varphi[{}^{\overline{\neg p}}/_{\overline{p}}]$, then for some valuation V we have that $(H,V) \nvDash \varphi[{}^{\overline{\neg p}}/_{\overline{p}}]$ and hence $[\![\varphi[{}^{\overline{\neg p}}/_{\overline{p}}]]\!]^{(H,V)} \neq 1$. Then consider the negative variant $V^{\neg}$ of V. It follows by Lemma 3.15 that $[\![\varphi]\!]^{(H,V^{\neg})} \neq 1$, which implies $H \nvDash^{\neg} \varphi$. ($\Leftarrow$) Suppose $H \nvDash^{\neg} \varphi$, then for some negative valuation $V^{\neg}$ we have that $(H,V^{\neg}) \nvDash^{\neg} \varphi$ and hence $[\![\varphi]\!]^{(H,V^{\neg})} \neq 1$. Now, by Lemma 3.14 from above, there is a valuation $V^{\neg\neg}$ such that $V^{\neg}$ is the negative variant of $V^{\neg\neg}$. It then follows by Lemma 3.15 that $[\![\varphi[{}^{\overline{\neg p}}/_{\overline{p}}]]\!]^{(H,V^{\neg\neg})} \neq 1$, which means that $H \nvDash \varphi[{}^{\overline{\neg p}}/_{\overline{p}}]$. □

Thus we end up with the following proposition; if a Heyting algebra validates a logic, then it validates also its negative variant.

Proposition 3.17. *Let H be a Heyting algebra and L an intermediate logic. Then we have that $H \vDash L$ entails $H \vDash^{\neg} L^{\neg}$*

Proof. Suppose $H \vDash L$, then for all $\varphi \in L^{\neg}$ we have that $\varphi[{}^{\overline{\neg p}}/_{\overline{p}}] \in L$ and so $H \vDash \varphi[{}^{\overline{\neg p}}/_{\overline{p}}]$. By Proposition 3.16 above it follows that $H \vDash^{\neg} \varphi$ and hence $H \vDash^{\neg} L^{\neg}$. □

Notice that the converse of the previous proposition does not hold in general, since a formula can be true in a Heyting algebra under all `DNA`-valuations but not under all valuations. However, the next proposition is a weaker version of it which we will need later. Let $\langle H_{\neg} \rangle$ be the subalgebra of H generated by $H_{\neg}$. First we prove the following lemma.

Lemma 3.18. *For any Heyting algebra H we have that $H \vDash^{\neg} \varphi$ iff $\langle H_{\neg} \rangle \vDash^{\neg} \varphi$.*

Proof. Suppose $\langle H_{\neg} \rangle$ is the subalgebra of H generated by $H_{\neg}$, then it follows that $H_{\neg} = \langle H_{\neg} \rangle_{\neg}$. So we have that $V^{\neg}$ is a `DNA`-valuation over H iff it is a `DNA`-valuation over $\langle H_{\neg} \rangle$. Then, since $\langle H_{\neg} \rangle$ is a subalgebra of H, we have that for any formula ψ it holds that $[\![\psi]\!]^{(\langle H_{\neg} \rangle, V)} = [\![\psi]\!]^{(H,V)}$, hence $(H, V^{\neg}) \vDash^{\neg} \varphi$ iff $(\langle H_{\neg} \rangle, V^{\neg}) \vDash^{\neg} \varphi$. Finally, this entails $H \vDash^{\neg} \varphi$ iff $\langle H_{\neg} \rangle \vDash^{\neg} \varphi$. □

Now we prove the following proposition.

Proposition 3.19. *Let H be a Heyting algebra and L an intermediate logic. Then we have that $H \vDash^{\neg} L^{\neg}$ entails $\langle H_{\neg} \rangle \vDash L$.*

Proof. Consider any Heyting algebra H, and suppose that $\langle H_\neg \rangle \nvDash L$, then there is some formula $\varphi \in L$ and some valuation V such that $(\langle H_\neg\rangle, V) \nvDash \varphi$. Now, since $\langle H_\neg\rangle$ is the subalgebra generated by $H_\neg$, we can express every element $x \in \langle H_\neg\rangle$ as a polynomial δ^x_H of elements of $H_\neg$. We thus have $x = \delta^x_H(\overline{y})$, where for each y_i we have that $y_i \in H_\neg$. By writing $\overline{p} = p_1, ..., p_n$ for the variables contained in φ and $\overline{\delta^x_H(\overline{y})}$ for the polynomials of the elements $x_1 = V(p_1), ..., x_n = V(p_n)$, we get that $[\![\varphi(\overline{p})]\!]^{(\langle H_\neg\rangle, V)} = \varphi_H(\overline{\delta^x_H(\overline{y})})$. Since all the elements $\overline{y}$ in the polynomials δ^x_H are regular elements, we can define a DNA-valuation $U^\neg : \mathtt{AT} \to H_\neg$ such that $U^\neg : q_i \mapsto y_i$ for all $i \leq n$. Then it follows immediately that $[\![\varphi[\overline{\delta^x(\overline{q})}/\overline{p}]]\!]^{(\langle H_\neg\rangle, U^\neg)} = \varphi_H(\overline{\delta^x_H(\overline{y})})$. But then, since we also had $[\![\varphi(\overline{p})]\!]^{(\langle H_\neg\rangle, V)} = \varphi_H(\overline{\delta^x_H(\overline{y})})$, it follows that $[\![\varphi[\overline{\delta^x(\overline{q})}/\overline{p}]]\!]^{(\langle H_\neg\rangle, U^\neg)} = [\![\varphi(\overline{p})]\!]^{(\langle H_\neg\rangle, V)}$. So since $(\langle H_\neg\rangle, V) \nvDash \varphi$, we also get that $(\langle H_\neg\rangle, U^\neg) \nvDash^\neg \varphi[\overline{\delta(\overline{y})}/\overline{p}]$. So it then follows by Lemma 3.18 that $(H, U^\neg) \nvDash^\neg \varphi[\overline{\delta^x(\overline{q})}/\overline{p}]$, hence $H \nvDash^\neg \varphi[\overline{\delta^x(\overline{q})}/\overline{p}]$. Now, since L is an intermediate logic, it admits free substitution and so, since $\varphi \in L$, we also get that $\varphi[\overline{\delta^x(\overline{q})}/\overline{p}] \in L$ and therefore as $L \subseteq L^\neg$ also $\varphi[\overline{\delta^x(\overline{q})}/\overline{p}] \in L^\neg$. Finally, this means that $H_\neg \nvDash^\neg L^\neg$, thus proving our claim. $\square$

3.2.3 The maps $Log^\neg$ and $Var^\neg$

Given the DNA-semantics defined in the previous section, there are two obvious ways to relate formulas and algebras. We define the map $Var^\neg$ sending sets of formulas to the class of Heyting algebras in which they are DNA-valid and the map $Log^\neg$ sending classes of Heyting algebras to the set of their DNA-validities. We have:

$$Var^\neg : \Gamma \mapsto \{H \in HA : H \vDash^\neg \Gamma\};$$
$$Log^\neg : \mathcal{C} \mapsto \{\varphi \in \mathcal{L}_P : \mathcal{C} \vDash^\neg \varphi\}.$$

We then say that a DNA-variety of Heyting algebras $\mathcal{X}$ is DNA-*defined* by a set of formulas Γ if $\mathcal{X} = Var^\neg(\Gamma)$. We say that a class of Heyting algebras $\mathcal{C}$ is DNA-*definable* if there is a set Γ of formulas such that $\mathcal{X} = Var^\neg(\Gamma)$. When the context is clear, we often drop the qualification DNA and talk simply of definability. We say that a DNA-logic Λ is *algebraically complete* with respect to a class of Heyting algebras $\mathcal{C}$ if $\Lambda = Log(\mathcal{C})$. We shall prove in the next section a definability theorem and an algebraic completeness theorem for DNA-logics. We shall then establish that every DNA-variety is defined by its validities and that every DNA-logic is complete with respect to its corresponding DNA-variety.

We will next show that $Var^\neg(\Gamma)$ is always a DNA-variety and $Log^\neg(\mathcal{C})$ is always a DNA-logic. First we prove the following important lemma showing that the DNA-validity of a formula is preserved by the key operations of a DNA-variety.

Lemma 3.20 (Preservation of DNA-Validity)**.** *The* DNA*-validity of a formula φ is preserved by the operations of subalgebras, homomorphic images, products and core superalgebras, i.e:*

(i) if $H \vDash^\neg \varphi$ and $K \preceq H$, then $K \vDash^\neg \varphi$;

(ii) if $H \vDash^\neg \varphi$ and $H \twoheadrightarrow K$, then $K \vDash^\neg \varphi$;

(iii) if $A_i \vDash^\neg \varphi$ for all $i \in I$ of a family $\{A_i\}_{i \in I}$ of algebras, then $\prod_{i \in I} A_i \vDash^\neg \varphi$;

(iv) if $H \vDash^{\neg} \varphi$ and for some K such that $K\neg = H\neg$ we have that $H \preceq K$, then $K \vDash^{\neg} \varphi$.

Proof. We check that the DNA-validity of formulas is preserved by the operations of subalgebras, homomorphic images, products and core superalgebras.

(i) We check preservation under subalgebras. Suppose by *reductio* that $K \preceq H$ and $K \nvDash^{\neg} \varphi$, then for some negative valuation $V^{\neg}$ we have $(K, V^{\neg}) \nvDash^{\neg} \varphi$. Now, since $K \preceq H$, we can then consider $V^{\neg}$ also as a valuation over H. It is straightforward to see that for every formula ψ we have $[\![\psi]\!]^{(K,V^{\neg})} = [\![\psi]\!]^{(H,V^{\neg})}$ and thus $(H, V^{\neg}) \nvDash^{\neg} \varphi$ and $H \nvDash^{\neg} \varphi$.

(ii) We check preservation under homomorphic images. Suppose by *reductio* $f : H \twoheadrightarrow K$ and $K \nvDash^{\neg} \varphi$. Then by Proposition 3.16 it follows that $K \nvDash \varphi[\neg p/\bar{p}]$. So since validity is preserved by homomorphic images it follows that $H \nvDash \varphi[\neg p/\bar{p}]$ and therefore by using Proposition 3.16 again we obtain that $H \nvDash^{\neg} \varphi$.

(iii) We check preservation under products. We prove this claim for binary products but it is easy to see how our proof generalizes to arbitrary products. Suppose by *reductio* that $H^0 \times H^1 \nvDash^{\neg} \varphi$. Then we have that for some valuation $V^{\neg}$ it is the case that $1_{H^0 \times H^1} = (1_{H^0}, 1_{H^1}) \neq [\![\varphi]\!]^{(H^0 \times H^1, V^{\neg})}$. Then by defining $V_0^{\neg} := \pi_0 \circ V^{\neg}$ and $V_1^{\neg} := \pi_1 \circ V^{\neg}$ as the two projections of $V^{\neg}$ into the elements of the product and by the properties of the product construction we get that either $1_{H^0} \neq [\![\varphi]\!]^{(H^0, \pi_0 \circ V^{\neg})}$ or $1_{H^0} \neq [\![\varphi]\!]^{(H^1, \pi_1 \circ V^{\neg})}$, which means that either $(H^0, \pi_0 \circ V^{\neg}) \nvDash^{\neg} \varphi$ or $(H^1, \pi_1 \circ V^{\neg}) \nvDash^{\neg} \varphi$, which proves our claim.

(iv) Finally, we check preservation under core superalgebras. Suppose by *reductio* $K \nvDash^{\neg} \varphi$ and $H_{\neg} = K_{\neg}$ and $H \preceq K$. Then for some valuation $V^{\neg}$ we have $(K, V^{\neg}) \nvDash^{\neg} \varphi$. Since $H_{\neg} = K_{\neg}$ we can then immediately consider $V^{\neg}$ as a valuation over H and, since $H \preceq K$, we have that for every formula ψ we have $[\![\psi]\!]^{(K,V^{\neg})} = [\![\psi]\!]^{(H,V^{\neg})}$. It thus follows that $(H, V^{\neg}) \nvDash^{\neg} \varphi$ and hence $H \nvDash^{\neg} \varphi$, which finally proves our claim. □

It follows immediately that for every set of formulas Γ the class of Heyting algebras $Var^{\neg}(\Gamma)$ is a DNA-logic.

Proposition 3.21. *The class of Heyting algebras $Var^{\neg}(\Gamma)$ is a DNA-variety.*

Proof. Consider any set of formulas Γ, then by the previous Lemma 3.20 it follows that the corresponding set $Var^{\neg}(\Gamma)$ is closed under the operations of subalgebra, homomorphic image, product and core superalgebra. Therefore, it follows by Proposition 3.7 that it is a DNA-variety. □

It is a straightforward consequence of the proposition above that every DNA-definable class of Heyting algebras is also a DNA-variety. The next proposition shows that for every class $\mathcal{C}$ of Heyting algebras its set of validities $Log^{\neg}(\mathcal{C})$ is a DNA-logic.

Proposition 3.22. *The class of formulas $Log^{\neg}(\mathcal{C})$ is a DNA-logic.*

Proof. We check that for any class $\mathcal{C}$ of Heyting algebras the corresponding set of formulas $Log^{\neg}(\mathcal{C})$ is a DNA-logic. In particular we show that $Log^{\neg}(\mathcal{C}) = Log(\mathcal{C})^{\neg}$. We have the following:

$$\varphi \notin Log^{\neg}(\mathcal{C}) \Leftrightarrow \exists H \in \mathcal{C} \text{ such that } H \nvDash^{\neg} \varphi$$

$$\begin{aligned}
&\Leftrightarrow \exists H \in \mathcal{C} \text{ such that } H \nvDash \varphi[{}^{\neg p}/_{p}] \qquad \text{(by Proposition 3.16)}\\
&\Leftrightarrow \varphi[{}^{\neg p}/_{p}] \notin Log(\mathcal{C})\\
&\Leftrightarrow \varphi \notin Log(\mathcal{C})^{\neg}.
\end{aligned}$$

Which shows that $Log^{\neg}(\mathcal{C})$ is the negative variant of $Log(\mathcal{C})$. $\square$

3.3 Duality between DNA-Logics and DNA-Varieties

In this section we provide two different proofs of the dual isomorphism between DNA-Logics and DNA-Varieties. In Section 3.3.1 we prove that $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$ by relying on the standard result that $\mathbf{IL} \cong^{op} \mathbf{HA}$. In Section 3.3.2 we introduce Lindenbaum-Tarski algebras for DNA-Logics and we use them to given an alternative proof of the same result. We believe that these two different strategies give us different insights and perspectives on this key result about DNA-Logics and DNA-Varieties. We usually refer to the dual isomorphism $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$ as DNA-duality and to the dual isomorphism $\mathbf{IL} \cong^{op} \mathbf{HA}$ as standard duality.

3.3.1 DNA-Duality by Standard Duality

So far, we have considered $Var^{\neg}$ as a map defined over arbitrary classes of Heyting algebras and $Log^{\neg}$ as a map defined over arbitrary sets of propositional formulas. Now we restrict our attention to the case in which the domain of $Var^{\neg}$ is the lattice of DNA-logics $\mathbf{DNAL}$ and the domain of $Log^{\neg}$ is the lattice of DNA-varieties $\mathbf{DNAV}$. Since we have shown above that $Var^{\neg}(\Gamma)$ is always a DNA-variety and $Log^{\neg}(\mathcal{C})$ is always a DNA-logic it follows that we have two maps:

$$\begin{aligned}
Var^{\neg} &: \mathbf{DNAL} \to \mathbf{DNAV};\\
Log^{\neg} &: \mathbf{DNAV} \to \mathbf{DNAL}.
\end{aligned}$$

In this section we provide a first proof that these two maps describe a dual isomorphism between the lattice of DNA-logics and the lattice of DNA-varieties. This proof essentially relies on the standard isomorphism between the lattice of intermediate logics and the lattice of varieties of Heyting algebras. Let us introduce the following diagram:

$$\begin{tikzcd}
\mathbf{IL} \arrow[r, "\neg"] \arrow[d, leftrightarrow, "\cong^{op}"'] & \mathbf{DNAL} \arrow[d, leftrightarrow, "\cong^{op}"] \\
\mathbf{HA} \arrow[r, "\uparrow"] & \mathbf{DNAV}
\end{tikzcd}$$

Where the four objects in the diagram are the following:

IL is the lattice of intermediate logics;
HA is the lattice of varieties of Heyting algebras;
DNAL is the lattice of DNA-logics;

DNAV is the lattice of DNA-varieties.

And the arrows are the following. Firstly, $(-)^{\neg} : \mathbf{IL} \to \mathbf{DNAL}$ is the map we introduced above that assigns to every intermediate logic L its negative variant $L^{\neg}$. Secondly, $(-)^{\uparrow} : \mathbf{HA} \to \mathbf{DNAV}$ is the map we introduced above that assigns to each variety of Heyting algebras $\mathcal{V}$ its negative closure $\mathcal{V}^{\uparrow}$. The isomorphism $\mathbf{IL} \cong^{op} \mathbf{HA}$ is given by the standard duality for intermediate logics and varieties of Heyting algebras. The two maps of this bijection are $Log : \mathbf{HA} \to \mathbf{IL}$ and $Var : \mathbf{IL} \to \mathbf{HA}$, which we have defined above in the preliminaries. By using the fact that $\mathbf{IL} \cong^{op} \mathbf{HA}$ we show in this section that also $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$ holds. We proceed as follows. First we show that the diagram that we have described commutes, then we show that $Var^{\neg}$ and $Log^{\neg}$ are inverse maps of each other and finally we prove they are order-reversing homomorphisms between **DNAL** and **DNAV**. Thus we will obtain a dual isomorphism $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$.

Commutativity of the Diagram

We first prove the two following propositions, thereby establishing that our diagram commutes.

Proposition 3.23. *For every intermediate logic L, $Var^{\neg}(L^{\neg}) = Var(L)^{\uparrow}$.*

$$\begin{tikzcd} \mathbf{IL} \arrow[r, "\neg"] \arrow[d, "Var"'] & \mathbf{DNAL} \arrow[d, "Var^{\neg}"] \\ \mathbf{HA} \arrow[r, "\uparrow"] & \mathbf{DNAV} \end{tikzcd}$$

Proof. ($\subseteq$) Consider any Heyting algebra $H \in Var^{\neg}(L^{\neg})$. Then we have $H \vDash^{\neg} L^{\neg}$ and so by Proposition 3.19 it follows that $\langle H_{\neg} \rangle \vDash L$. So we clearly have that $\langle H_{\neg} \rangle \in Var(L)$ and since $\langle H_{\neg} \rangle_{\neg} = H_{\neg}$ and $\langle H_{\neg} \rangle \preceq H$ also $H \in Var(L)^{\uparrow}$. ($\supseteq$) Consider any Heyting algebra $H \in Var(L)^{\uparrow}$, then there is some $K \in Var(L)$ such that $K \preceq H$ and $H_{\neg} = K_{\neg}$. Then we have that $K \vDash L$, so by Lemma 3.17 above we obtain that $K \vDash^{\neg} L^{\neg}$ which entails $K \in Var^{\neg}(L^{\neg})$. Finally, since DNA-varieties are closed under core superalgebra, it follows that $H \in Var^{\neg}(L^{\neg})$. $\square$

Proposition 3.24. *For every variety $\mathcal{V}$ of Heyting algebras $Log^{\neg}(\mathcal{V}^{\uparrow}) = Log(\mathcal{V})^{\neg}$.*

$$\begin{tikzcd} \mathbf{IL} \arrow[r, "\neg"] & \mathbf{DNAL} \\ \mathbf{HA} \arrow[r, "\uparrow"] \arrow[u, "Log"] & \mathbf{DNAV} \arrow[u, "Log^{\neg}"'] \end{tikzcd}$$

Proof. We prove both directions by contraposition. ($\subseteq$) Suppose first $\varphi \notin Log(\mathcal{V})^{\neg}$, then we have that $\varphi[{}^{\neg p}/_{p}] \notin Log(\mathcal{V})$ and hence there is some Heyting algebra $H \in \mathcal{V}$ such that $H \nvDash \varphi[{}^{\neg p}/_{p}]$. By Proposition 3.16 this means that $H \nvDash^{\neg} \varphi$ and so since $H \in \mathcal{V} \subseteq \mathcal{V}^{\uparrow}$ we also have $\varphi \notin Log^{\neg}(\mathcal{V}^{\uparrow})$. ($\supseteq$) Suppose now that $\varphi \notin Log^{\neg}(\mathcal{V}^{\uparrow})$.

It follows that there is some Heyting algebra $H \in \mathcal{V}^{\uparrow}$ such that $H \nvDash^{\neg} \varphi$, hence by Lemma 3.18 we have that $\langle H_{\neg} \rangle \nvDash^{\neg} \varphi$. It thus follows by Proposition 3.16 that $\langle H_{\neg} \rangle \nvDash \varphi[{}^{\neg p}/_{p}]$. Now, since $H \in \mathcal{V}^{\uparrow}$, we have for some $K \in \mathcal{V}$ that $K \preceq H$ and $K_{\neg} = H_{\neg}$. Thus, by the fact that $\langle H_{\neg} \rangle$ is the algebra generated by $H_{\neg}$, it follows that $\langle H_{\neg} \rangle \preceq K$ and therefore $\langle H_{\neg} \rangle \in \mathcal{V}$. Finally, since $\langle H_{\neg} \rangle \nvDash \varphi[{}^{\neg p}/_{p}]$ we get that $\varphi[{}^{\neg p}/_{p}] \notin Log(\mathcal{V})$ and hence $\varphi \notin Log(\mathcal{V})^{\neg}$. □

In particular, when $\mathcal{V}$ is itself a DNA-variety we obtain the following corollary.

Corollary 3.25. *For every* DNA*-variety* $\mathcal{X}$ *we have* $Log^{\neg}(\mathcal{X}) = Log(\mathcal{X})^{\neg}$*.*

Definability Theorem and Algebraic Completeness

By relying on the commutativity result described above, we can now prove that the two maps $Var^{\neg}$ and $Log^{\neg}$ are inverse of one another. It is then easy to see that suitable versions of the definability theorem and algebraic completeness follow from this result.

Proposition 3.26. $Var^{\neg} \circ Log^{\neg} = 1_{\boldsymbol{DNAV}}$.

Proof. For any DNA-variety $\mathcal{X}$ we have:

$$\begin{aligned}
Var^{\neg}(Log^{\neg}(\mathcal{X})) &= Var^{\neg}(Log(\mathcal{X})^{\neg}) && \text{(by Corollary 3.25)}\\
&= Var(Log(\mathcal{X}))^{\uparrow} && \text{(by Proposition 3.23)}\\
&= \mathcal{X}^{\uparrow} && \text{(by standard duality)}\\
&= \mathcal{X}.
\end{aligned}$$

And thus $Var^{\neg} \circ Log^{\neg} = 1_{\mathbf{DNAV}}$. □

Theorem 3.27 (Definability Theorem)**.** *Every* DNA*-variety* $\mathcal{X}$ *is defined by its* DNA*-validities, i.e. for every Heyting algebra* H*,*

$$H \in \mathcal{X} \Leftrightarrow H \vDash^{\neg} Log(\mathcal{X}).$$

We then have that every DNA-variety is DNA-definable. Moreover, by Proposition 3.21 we have that every DNA-definable class is also a DNA-variety, the following corollary also follows.

Corollary 3.28 (Birkhoff Theorem for DNA-Varieties)**.** *A class of Heyting algebras* $\mathcal{C}$ *is a* DNA*-variety if and only if it is* DNA*-definable by some set of formulas.*

The algebraic completeness of DNA-logics is proved as follows.

Proposition 3.29. $Log^{\neg} \circ Var^{\neg} = 1_{\mathbf{DNAL}}$.

Proof. For any DNA-logic Λ such that $\Lambda = L^{\neg}$ we have:

$$\begin{aligned}
Log^{\neg}(Var^{\neg}(\Lambda)) &= Log^{\neg}(Var^{\neg}(L^{\neg}))\\
&= Log^{\neg}(Var(L)^{\uparrow}) && \text{(by Proposition 3.23)}\\
&= Log(Var(L))^{\neg} && \text{(by Proposition 3.24)}
\end{aligned}$$

$$\begin{aligned} &= L^{\neg} && \text{(by standard duality)} \\ &= \Lambda. \end{aligned}$$

And thus $Log^{\neg} \circ Var^{\neg} = 1_{\mathbf{DNAL}}$. □

Theorem 3.30 (Algebraic Completeness)**.** *Every* DNA*-logic* Λ *is complete with respect to its corresponding* DNA*-variety, i.e. for every* $\varphi \in \mathcal{L}_P$*,*

$$\varphi \in \Lambda \Leftrightarrow Var^{\neg}(\Lambda) \vDash^{\neg} \varphi.$$

Dual Isomorphism

Finally, by relying on the standard dual isomorphism $\mathbf{HA} \cong^{op} \mathbf{IL}$ and the commutative square above, it is easy to show that $Var^{\neg}$ and $Log^{\neg}$ are order-reversing homomorphisms that invert the lattice structure of **DNAL** and **DNAV**.

Proposition 3.31. $Var^{\neg}$ *is an order-reversing homomorphism.*

Proof. It suffices to check that $Var^{\neg}$ inverts meet and join. Let Λ_0, Λ_1 be two DNA-logics such that $\Lambda_0 = L_0^{\neg}$ and $\Lambda_1 = L_1^{\neg}$. The case for $\wedge$ is as follows:

$$\begin{aligned} Var^{\neg}(\Lambda_0 \wedge \Lambda_1) &= Var^{\neg}(L_0^{\neg} \wedge L_1^{\neg}) \\ &= Var^{\neg}((L_0 \wedge L_1)^{\neg}) && \text{(by Proposition 3.4)} \\ &= Var(L_0 \wedge L_1)^{\uparrow} && \text{(by Proposition 3.23)} \\ &= (Var(L_0) \vee Var(L_1))^{\uparrow} && \text{(by standard duality)} \\ &= Var(L_0)^{\uparrow} \vee Var(L_1)^{\uparrow} && \text{(by Proposition 3.9)} \\ &= Var^{\neg}(L_0^{\neg}) \vee Var^{\neg}(L_1^{\neg}) && \text{(by Proposition 3.23)} \\ &= Var^{\neg}(\Lambda_0) \vee Var^{\neg}(\Lambda_1). \end{aligned}$$

The case for $\vee$ is analogous. □

Proposition 3.32. $Log^{\neg}$ *is an order-reversing homomorphism.*

Proof. It suffices to check that $Log^{\neg}$ inverts meet and join. Let $\mathcal{X}_0, \mathcal{X}_1$ be two DNA-varieties such that $\mathcal{X}_0 = \mathcal{V}_0^{\uparrow}$ and $\mathcal{X}_1 = \mathcal{V}_1^{\uparrow}$. The case for $\wedge$ is as follows:

$$\begin{aligned} Log^{\neg}(\mathcal{X}_0 \wedge \mathcal{X}_0) &= Log^{\neg}(\mathcal{V}_0^{\uparrow} \wedge \mathcal{V}_1^{\uparrow}) \\ &= Log^{\neg}((\mathcal{V}_0 \wedge \mathcal{V}_1)^{\uparrow}) && \text{(by Proposition 3.9)} \\ &= Log(\mathcal{V}_0 \wedge \mathcal{V}_1)^{\neg} && \text{(by Proposition 3.24)} \\ &= (Log(\mathcal{V}_0) \vee Log(\mathcal{V}_1))^{\neg} && \text{(by standard duality)} \\ &= Log(\mathcal{V}_0)^{\neg} \vee Log(\mathcal{V}_1)^{\neg} && \text{(by Proposition 3.4)} \\ &= Log^{\neg}(\mathcal{V}_0^{\uparrow}) \vee Log^{\neg}(\mathcal{V}_1^{\uparrow}) && \text{(by Proposition 3.24)} \\ &= Log^{\neg}(\mathcal{X}_0) \vee Log^{\neg}(\mathcal{X}_1). \end{aligned}$$

The case for $\vee$ is analogous. □

It is a consequence of the previous results that $Var^{\neg}$ and $Log^{\neg}$ are two order-reversing homomorphism between **DNAL** and **DNAV** which are inverse of one another. The following duality theorem follows.

Theorem 3.33 (Duality)**.** *The lattice of* `DNA`*-logics is dually isomorphic to the lattice of* `DNA`*-varieties of Heyting algebras, i.e.* **DNAL** $\cong^{op}$ **DNAV***.*

3.3.2 `DNA`-Duality by Lindenbaum-Tarski Algebras

In this section we provide a different proof of the duality result between **DNAL** and **DNAV** shown above. Whilst the former proof relies in an essential way on the diagram we have constructed, we now want to give a proof which is autonomous and does not use the standard duality result between intermediate logics and varieties of Heyting algebras. To this end, we introduce a suitable Lindenbaum-Tarski construction for `DNA`-logics. For a general introduction to Lindenbaum-Tarski constructions we refer the reader to [7, §.7.2] and [21, Chp. 1]. By using the Lindenbaum-Tarski method we then prove both a suitable version of the definability theorem and algebraic completeness of `DNA`-logics and we also show that $Var^{\neg}$ and $Log^{\neg}$ are order-reversing homomorphisms. Finally, we obtain the result that **DNAL** and **DNAV** are dually isomorphic.

Lindenbaum-Tarski Algebras

Firstly we introduce a suitable form of the Lindenbaum-Tarski construction. Let $\mathcal{P}$ be a set of atomic propositional formulas and let $\mathcal{L}_P$ be the set of propositional formulas over $\mathcal{P}$. If not specified otherwise, we generally assume that the set $\mathcal{P}$ has cardinality $\aleph_0$ and we denote it by `AT`. We can also consider $\mathcal{L}_P$ as the term algebra $\mathcal{T} = T(\mathcal{P})$ of all the terms over $\mathcal{P}$ in the signature $(\top, \bot, \land, \lor, \to)$. In the case of an intermediate logic L, a Lindenbaum-Tarski algebra is obtained by taking the quotient of the term algebra $\mathcal{T}$ by the equivalence $\equiv_L$ induced by the logic L. We adopt a similar approach in the case of `DNA`-logics. First, we define for every `DNA`-logic Λ the following equivalence relation:

Definition 3.34. For any `DNA`-logic Λ the equivalence relation $\equiv_\Lambda$ is defined as follows:

$$\varphi \equiv_\Lambda \psi \Leftrightarrow \varphi \leftrightarrow \psi \in \Lambda.$$

It is easy to check that this is indeed an equivalence relation. By the closure of `DNA`-logics under modus ponens it is also easy to verify that $\equiv_\Lambda$ is a congruence over the term algebra $\mathcal{T}$. A Lindenbaum-Tarski algebra for Λ is thus obtained by quotienting the term algebra $\mathcal{T}$ by the congruence relation $\equiv_\Lambda$.

Definition 3.35 (Lindenbaum-Tarski Algebra)**.** The *Lindenbaum-Tarski Algebra* $\mathcal{F}_\Lambda$ of a `DNA`-logic Λ is defined as $\mathcal{F}_\Lambda = \mathcal{T}/\equiv_\Lambda$.

Since $\mathcal{F}_\Lambda$ is a quotient of the term-algebra $\mathcal{T}$, we can regard its elements as equivalence classes of terms in $\mathcal{T}$. If $\varphi \in \mathcal{T}$ is a term, then we denote the corresponding equivalence class in $\mathcal{F}_\Lambda$ by $[\varphi]$. Moreover, since $\equiv_\Lambda$ is a congruence, all the operations on the formulas in $\mathcal{T}$ immediately lift to the equivalence classes of formulas of $\mathcal{F}_\Lambda$. We now prove that for every $\mathtt{DNA}$-logic Λ its Lindenbaum-Tarski algebra $\mathcal{F}_\Lambda$ is always a Λ-algebra. As we shall need this fact in the proof, we introduce the notion of negative substitution and show that $\mathtt{DNA}$-logics are closed under negative substitution. The proof of the following lemma was originally given by Ciardelli in [10, p. 29-30], though the notion of negative substitution was first considered in [41].

Definition 3.36 (Negative Substitution)**.** We say that a substitution $(-)^*$ is *negative* with respect to the logic Λ if for every atomic proposition $p \in \mathtt{AT}$ we have that $(p)^* \equiv_{L^\neg} \neg\neg(p^*)$.

Lemma 3.37. *Every* $\mathtt{DNA}$*-logic is closed under negative substitution.*

Proof. Let Λ be a $\mathtt{DNA}$-logic and $\Lambda = L^\neg$ without loss of generality. Suppose $\varphi \in \Lambda$, then we need to show that also $(\varphi)^* \in \Lambda$, where $(-)^*$ is a negative substitution. We denote by $(-)^\neg$ the negative substitution $p \mapsto \neg p$. Now, since $\varphi \in \Lambda = L^\neg$, it follows immediately by the definition of negative variant that $\varphi^\neg \in L$. By uniform substitution we get that $\varphi^{\neg\neg*\neg} \in L$ and thus by the definition of negative variant that $\varphi^{\neg\neg*} \in L^\neg$. Then, by the fact that $(-)^*$ is negative, we have that $(p)^* \equiv_{L^\neg} \neg\neg(p^*)$ and so $\varphi^* \equiv_{L^\neg} \varphi^{\neg\neg*}$. Finally, this means that $\varphi^* \in L^\neg = \Lambda$. □

We can now prove the following.

Proposition 3.38. *For every* $\mathtt{DNA}$*-logic* Λ*, we have that* $\mathcal{F}_\Lambda \in Var^\neg(\Lambda)$*.*

Proof. Firstly, let us remark that for any $\mathtt{DNA}$-logic Λ we have that $\mathtt{IPC} \subseteq \Lambda$, which means that $\equiv_{\mathtt{IPC}} \subseteq \equiv_\Lambda$. It then immediately follows from the fact that $\mathcal{T}/\equiv_{\mathtt{IPC}}$ is the Lindenbaum-Tarski algebra for $\mathtt{IPC}$ that also $\mathcal{F}_\Lambda$ is a Heyting algebra. The notion of $\mathtt{DNA}$-valuation and of $\mathtt{DNA}$-validity are thus well-defined on $\mathcal{F}_\Lambda$. Then, to show that $\mathcal{F}_\Lambda \in Var^\neg(\Lambda)$ we need to check for any formula $\varphi \in \Lambda$ that $\mathcal{F}_\Lambda \vDash^\neg \varphi$. But this is equivalent to showing that under any $\mathtt{DNA}$-valuation $V^\neg$ we have $1_{\mathcal{F}_\Lambda} = [\![\varphi]\!]^{(\mathcal{F}_\Lambda, V^\neg)}$. Let $p_0, ..., p_n$ be the atomic letters in φ and $x_0, ..., x_n \in \mathcal{F}_\Lambda$ their interpretation under $V^\neg$, then what we need to prove is the following:

$$1_{\mathcal{F}_\Lambda} = \varphi_{\mathcal{F}_\Lambda}(x_0, ..., x_n),$$

where φ is a polynomial over the elements $x_0, ..., x_n$ of $\mathcal{F}_\Lambda$. Now, since the elements of $\mathcal{F}_\Lambda$ are equivalence classes of formulas, there are some formulas $\psi_0, ..., \psi_n$ such that $x_0 = [\psi_0], ..., x_n = [\psi_n]$ where for every $i \leq n$ we have $V(p_i) = x_i = [\psi_i]$. We then have that $\varphi_{\mathcal{F}_\Lambda}(x_0, ..., x_n) = \varphi_{\mathcal{F}_\Lambda}([\psi_0], ..., [\psi_n])$. Then by the fact that $\equiv_\Lambda$ is a congruence it follows that we have $\varphi_{\mathcal{F}_\Lambda}([\psi_0], ..., [\psi_n]) = [\varphi(\psi_0, ..., \psi_n)]$. Notice that, since $V^\neg$ is a $\mathtt{DNA}$-valuations, each of the elements $[\psi_0], ..., [\psi_n]$ is regular. So it follows from the fact that $\equiv_\Lambda$ is a congruence together with the property of regular elements that for each ψ_i with $i \leq n$ we have $\psi_i \equiv_\Lambda \neg\neg\psi_i$. This implies that the substitution $(p_i)^* = \psi_i$ is negative and thus, since we assumed that $\varphi \in \Lambda$, it then follows by the previous lemma that $\varphi(\psi_0, ..., \psi_n) \in \Lambda$. But this means that $\top \equiv_\Lambda \varphi(\psi_0, ..., \psi_n)$ and therefore by the definition of the Lindenbaum-Tarski algebra we have $1_{\mathcal{F}_\Lambda} = [\varphi(\psi_0, ..., \psi_n)] = \varphi_{\mathcal{F}_\Lambda}([\psi_0], ..., [\psi_n]) = \varphi_{\mathcal{F}_\Lambda}(x_0, ..., x_n)$, which completes our proof. □

For every DNA-logic Λ one can easily see that $\equiv_\Lambda$ is the least congruence over $Var^\neg(\Lambda)$ and thus the Lindenbaum-Tarski algebra $\mathcal{F}_\Lambda$ is exactly the free algebra in $Var^\neg(\Lambda)$ over a given set $\mathcal{P}$ of generators. Thus it immediately follows that Lindenbaum-Tarski algebras have the following universal mapping property.

Proposition 3.39 (Universal Mapping Property)**.** *Let $\mathcal{F}_\Lambda$ be the Lindenbaum-Tarski algebra of Λ and $H \in Var^\neg(\Lambda)$. Then for every (surjective) homomorphism $h : \mathtt{AT} \to H$ there is a unique (surjective) homomorphism $\overline{h} : \mathcal{F}_\Lambda \to H$ such that $\overline{h} \upharpoonright \mathtt{AT} = h$.*

To prove this proposition it suffices to extend h to arbitrary equivalence classes of terms in $\mathcal{F}_\Lambda$ by defining inductively $\overline{h} : [\varphi \odot \psi] \mapsto \overline{h}[\varphi] \odot \overline{h}[\psi]$. That this works is then guaranteed by the properties of congruences. For more details on free algebras and their properties we refer the reader to [6, Sec. 10]. Now notice that, if we generate the Lindenbaum-Tarski algebra $\mathcal{F}_\Lambda$ from a suitable set AT of atomic terms of cardinality $|\mathtt{AT}| \geq |H|$, we can take as homomorphism any surjection $h : \mathtt{AT} \to H$ so that we get the following result.

Proposition 3.40. *Let $\mathcal{F}_\Lambda$ be the Lindenbaum-Tarski algebra of Λ and $H \in Var^\neg(\Lambda)$ then there is a surjective homomorphism $\overline{h} : \mathcal{F}_\Lambda \twoheadrightarrow H$.*

Namely every algebra in $Var^\neg(\Lambda)$ is homomorphic image of the Lindenbaum-Tarski algebra $\mathcal{F}_\Lambda$. It follows immediately from the proposition above that for every DNA-logic Λ we have $Var^\neg(\Lambda) = Var^\neg(\mathcal{F}_\Lambda)$, namely that every Lindenbaum-Tarski algebra $\mathcal{F}_\Lambda$ generates the variety $Var^\neg(\Lambda)$.

Remark 3.41. Here we have defined the Lindenbaum-Tarski algebra of a DNA-logic $L^\neg$ as the quotient algebra $\mathcal{T}/\equiv_\Lambda$. However, it is also possible to introduce Lindenbaum-Tarski algebras differently, by using the Lindenbaum-Tarski construction for intermediate logics exclusively. Let Λ be a DNA-logic such that $\Lambda = L^\neg$, then we construct its Lindenbaum-Tarski algebra as follows. Consider the two algebras $\mathcal{F}_{\mathtt{CPC}}$ and $\mathcal{F}_L$. We can show that the map $h : \mathcal{F}_{\mathtt{CPC}} \to (\mathcal{F}_L)_\neg$ such that $h : [\varphi] \mapsto [\neg\neg\varphi]$ is an isomorphism. This also shows that $h : \mathcal{F}_{\mathtt{CPC}} \to \mathcal{F}_L$ is an embedding with respect to the operations $1, 0, \wedge, \to$. Now let $\langle \mathcal{F}_{\mathtt{CPC}} \rangle$ be the Heyting algebra generated in $\mathcal{F}_L$ by $h[\mathcal{F}_{\mathtt{CPC}}]$, i.e. $\langle \mathcal{F}_{\mathtt{CPC}} \rangle = \langle (\mathcal{F}_L)_\neg \rangle$. One can then show that $\langle \mathcal{F}_{\mathtt{CPC}} \rangle \cong \mathcal{F}_\Lambda$. That these two algebras are indeed isomorphic follows immediately from the fact that every element of $\langle \mathcal{F}_{\mathtt{CPC}} \rangle$ is a polynomial over elements in $h[\mathcal{F}_{\mathtt{CPC}}]$, i.e. it is the equivalence class of a formula over Boolean atoms. The map between $\langle \mathcal{F}_{\mathtt{CPC}} \rangle$ and $\mathcal{F}_\Lambda$ is then defined by $\varphi[\overline{\neg\neg p}/\overline{p}] \mapsto \varphi$. This provides a way to construct the Lindenbaum-Tarski algebra of Λ by just using the standard Lindenbaum-Tarski construction for intermediate logics. We leave to the reader to check the details of this construction.

Definability Theorem and Algebraic Completeness

We can now use Lindenbaum-Tarski algebras to prove our two results. We first provide an alternative proof of the definability theorem for DNA-varieties. We copy the statements of these results from the previous sections and give here an alternative proofs of them.

Theorem 3.27. *Every* DNA*-variety $\mathcal{X}$ is defined by its* DNA*-validities, i.e. for every Heyting algebra H,*

$$H \in \mathcal{X} \Leftrightarrow H \vDash^{\neg} Log(\mathcal{X}).$$

Proof. ($\Rightarrow$) Follows immediately by the definition of $Log^{\neg}$. ($\Leftarrow$) Suppose now that $H \vDash^{\neg} Log^{\neg}(\mathcal{X})$. Let $\mathcal{Y} = Var^{\neg}(Log^{\neg}(\mathcal{X}))$, then $H \in \mathcal{Y}$. From the definitions of $Var^{\neg}$ and $Log^{\neg}$ it then follows that $Log^{\neg}(\mathcal{X}) = Log^{\neg}(\mathcal{Y})$. From this latter fact, we then immediately have that the Lindenbaum-Tarski algebras of these two logics are equivalent, namely that $\mathcal{F}_{Log^{\neg}(\mathcal{X})} = \mathcal{F}_{Log^{\neg}(\mathcal{Y})}$. Since we already had that $H \in \mathcal{Y}$, it then follows by the universal mapping property of Lindenbaum-Tarski algebras that $\mathcal{F}_{Log^{\neg}(\mathcal{Y})} \twoheadrightarrow H$ and so $\mathcal{F}_{Log^{\neg}(\mathcal{X})} \twoheadrightarrow H$. Since varieties are closed under homomorphic images, we finally have $H \in \mathcal{X}$. $\square$

It is thus a consequence of the previous theorem that every DNA-variety is definable by its validities. Moreover, since we have already established in Proposition 3.21 that DNA-definable classes are DNA-varieties we immediately obtain the following corollary.

Corollary 3.28. *A class of Heyting algebras $\mathcal{C}$ is a* DNA*-variety if and only if it is* DNA*-definable by some set of formulas.*

To prove the algebraic completeness of DNA-logics we first show that every logic is the logic of a Lindenbaum-Tarski algebra.

Proposition 3.42. *Let Λ be any* DNA*-logic, then we have that $\Lambda = Log^{\neg}(\mathcal{F}_{\Lambda})$.*

Proof. ($\subseteq$) By Proposition 3.38 we have $\mathcal{F}_{\Lambda} \in Var^{\neg}(\Lambda)$ so $\mathcal{F}_{\Lambda} \vDash^{\neg} \Lambda$, which means $\Lambda \subseteq Log^{\neg}(\mathcal{F}_{\Lambda})$. ($\supseteq$) Suppose by contraposition that $\varphi \notin \Lambda$, thus $\varphi \not\equiv_{\Lambda} \top$ and therefore $1_{\mathcal{F}_{L^{\neg}}} = [\top] \neq [\varphi]$. So $\mathcal{F}_{\Lambda} \nvDash^{\neg} \varphi$ and thus $\varphi \notin Log^{\neg}(\mathcal{F}_{\Lambda})$. $\square$

Lindenbaum-Tarski algebras are thus witnesses of the validity of formulas. The algebraic completeness of every DNA-logic follows immediately.

Theorem 3.30. *Every* DNA*-logic Λ is complete with respect to its corresponding* DNA*-variety, i.e. for every $\varphi \in \mathcal{L}_P$:*

$$\varphi \in \Lambda \Leftrightarrow Var^{\neg}(\Lambda) \vDash^{\neg} \varphi.$$

Proof. ($\Rightarrow$) Follows immediately by the definition of $Var^{\neg}$. ($\Leftarrow$) Suppose by contraposition that $\varphi \notin \Lambda$, then by Proposition 3.42 we have that $\varphi \notin Log^{\neg}(\mathcal{F}_{\Lambda})$ and then, since by Proposition 3.38 we have that $\mathcal{F}_{\Lambda} \in Var^{\neg}(\Lambda)$, it follows immediately that $Var^{\neg}(\Lambda) \nvDash^{\neg} \varphi$. $\square$

Dual Isomorphism

Finally, we give a proof that $Var^{\neg}$ and $Log^{\neg}$ are order-reversing homomorphisms which does not use the standard duality but relies implicitly on the Lindenbaum-Tarski construction introduced above.

Proposition 3.31. *$Var^{\neg}$ is an order-reversing homomorphism.*

Proof. (i) We first check the case for $\vee$. We notice that the following equalities easily follow from the algebraic completeness of DNA-logics 3.30 and our definitions:

$$\begin{aligned}&\Lambda_0 \vee \Lambda_1\\ =&Log^{\neg}(Var^{\neg}(\Lambda_0)) \vee Log^{\neg}(Var^{\neg}(\Lambda_1^{\neg}))\\ =&(\{\varphi \in \mathcal{L}_P : Var^{\neg}(\Lambda_0) \vDash^{\neg} \varphi\} \cup \{\varphi \in \mathcal{L}_P : Var^{\neg}(\Lambda_1) \vDash^{\neg} \varphi\})^{MP}\\ =&\{\varphi \in \mathcal{L}_P : Var^{\neg}(\Lambda_0) \wedge Var^{\neg}(\Lambda_1) \vDash^{\neg} \varphi\}\\ =&Log^{\neg}[Var^{\neg}(\Lambda_0) \wedge Var^{\neg}(\Lambda_1)].\end{aligned}$$

And so it follows by the previous observation and the Definability Theorem 3.27 that:

$$\begin{aligned}Var^{\neg}(\Lambda_0 \vee \Lambda_1) &= Var^{\neg}(Log^{\neg}[Var^{\neg}(\Lambda_0) \wedge Var^{\neg}(\Lambda_1)])\\ &= Var^{\neg}(\Lambda_0) \wedge Var^{\neg}(\Lambda_1).\end{aligned}$$

(ii) We now check the case for $\wedge$. It follows immediately by our definitions that:

$$\begin{aligned}Var^{\neg}(\Lambda_0 \wedge \Lambda_1) &= \{H : H \vDash^{\neg} \Lambda_0 \wedge \Lambda_1\}\\ &= \{H : H \vDash^{\neg} \Lambda_0\} \cup \{H : H \vDash^{\neg} \Lambda_1\}\\ &= Var^{\neg}(\Lambda_0) \vee Var^{\neg}(\Lambda_1).\end{aligned}$$

And so we have that $Var^{\neg}$ sends the join of two logics to the meet of their varieties and the meet of two logics to the join of their varieties. □

Proposition 3.32. *$Log^{\neg}$ is an order-reversing homomorphism.*

Proof. (i) We first check the case for $\vee$. It follows by our definitions that:

$$\begin{aligned}Log^{\neg}(\mathcal{X}_0 \vee \mathcal{X}_1) &= \{\varphi \in \mathcal{L}_P : \mathcal{X}_0 \vee \mathcal{X}_1 \vDash^{\neg} \varphi\}\\ &= \{\varphi \in \mathcal{L}_P : \mathcal{X}_0 \vDash^{\neg} \varphi\} \cap \{\varphi \in \mathcal{L}_P : \mathcal{X}_1 \vDash^{\neg} \varphi\}\\ &= Log^{\neg}(\mathcal{X}_0) \wedge Log^{\neg}(\mathcal{X}_1).\end{aligned}$$

(ii) We now check the case for $\wedge$. We notice that the following equalities easily follow from the definability theorem of DNA-varieties 3.27 and our definitions:

$$\begin{aligned}\mathcal{X}_0 \wedge \mathcal{X}_0 &= Var^{\neg}(Log^{\neg}(\mathcal{X}_0)) \wedge Var^{\neg}(Log^{\neg}(\mathcal{X}_1))\\ &= \{H : H \vDash^{\neg} Log^{\neg}(\mathcal{X}_0)\} \cap \{H : H \vDash^{\neg} Log^{\neg}(\mathcal{X}_1)\}\\ &= \{H : H \vDash^{\neg} Log^{\neg}(\mathcal{X}_0) \vee Log^{\neg}(\mathcal{X}_1)\}\\ &= Var^{\neg}(Log^{\neg}(\mathcal{X}_0) \vee Log^{\neg}(\mathcal{X}_1).\end{aligned}$$

And so it follows by the previous observation and the algebraic completeness of DNA-logics that:

$$\begin{aligned}Log^{\neg}(\mathcal{X}_0 \wedge \mathcal{X}_0) &= Log^{\neg}(Var^{\neg}(Log^{\neg}(\mathcal{X}_0) \vee Log^{\neg}(\mathcal{X}_1)))\\ &= Log^{\neg}(\mathcal{X}_0) \vee Log^{\neg}(\mathcal{X}_1)).\end{aligned}$$

Which shows that $Log^{\neg}$ sends the join of two DNA-varieties to the meet of their logics and the meet of two DNA-varieties to the join of their logics. □

Thus we have by our version of the Definability Theorem that $\mathcal{X} = Var^{\neg}(Log^{\neg}(\mathcal{X}))$ and by the algebraic completeness of `DNA`-logics that $\Lambda = Log^{\neg}(Var^{\neg}(\Lambda))$. Therefore the two maps $Var^{\neg}$ and $Log^{\neg}$ are inverse of one another and since they are also order-reversing homomorphisms we then get that $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$. The duality between **DNAL** and **DNAV** is thus an immediate consequence of the results above.

Theorem 3.33. *The lattice of* `DNA`*-logics is dually isomorphic to the lattice of* `DNA`*-varieties, i.e.* $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$.

We have seen how Lindenbaum-Tarski algebras also allows us to prove the existence of a dual isomorphism $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$. We will see in the next chapter some applications of this result.

Chapter 4

Locally Finite DNA-Varieties and Jankov Formulas

In this chapter we apply the dual isomorphism $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$ introduced in the previous chapter to study properties of DNA-logics and DNA-varieties. In Section 4.1 we study the lattice $\mathcal{I}(\Lambda)$ of intermediate logics which have the same DNA-logic Λ as their negative variant and we provide a characterisation of its least and greatest elements. In Section 4.2 we prove suitable versions of Tarski and Birkhoff theorems for DNA-varieties and we also show that the DNA-variety of all Heyting algebras is not locally finite. Finally, in Section 4.3 we adapt the method of Jankov formula to the context of DNA-varieties and we show how Jankov formulas can be used to axiomatise locally finite DNA-logics.

4.1 Connections to Intermediate Logics

In the previous sections we have introduced DNA-logics as negative variants of intermediate logics under the map $(-)^{\neg} : \mathbf{IL} \to \mathbf{DNAL}$. In this section we investigate the relation between intermediate logics and DNA-logics more in detail. In particular, given a DNA-logic Λ, we will be interested in studying the lattice $\mathcal{I}(\Lambda)$ of those intermediate logics that have Λ as their negative variant. In particular, we show that this sublattices is always bounded and we provide a characterisation of its greatest and least element. Whilst the former had already been considered in the literature, see in particular [41] and [10], the latter follows from the algebraic perspective that we are taking here.

4.1.1 The Sublattice $\mathcal{I}(\Lambda)$

We first want to show that the map $(-)^{\neg}$ which sends every intermediate logic to its negative variant is not injective. The following proposition was proved by Ciardelli in [10, p. 75] and exemplifies how different intermediate logics can share the same negative variant. We recall that KC is the logic of the weak excluded middle, i.e. $\mathtt{KC} = \mathtt{IPC} + \neg\varphi \vee \neg\neg\varphi$.

Lemma 4.1. *Let L be any intermediate logic such that* $\mathtt{KC} \subseteq L$*, then* $L^{\neg} = \mathtt{CPC}$*.*

Proof. Suppose L is an intermediate logic such that $\mathtt{KC} \subseteq L$, then one can show by induction that for every formula φ we have $\varphi \vee \neg\varphi \in L^{\neg}$. Suppose for the base case that $p \in \mathtt{AT}$, then since $\mathtt{KC} \subseteq L$ we have for all $p \in \mathtt{AT}$ that $\neg p \vee \neg\neg p \in L$ and therefore that $p \vee \neg p \in L^{\neg}$. We leave it to the reader to verify the induction steps. Finally, this shows that $L^{\neg} = \mathtt{IPC} + \varphi \vee \neg\varphi = \mathtt{CPC}$. $\square$

Therefore, for intermediate logics L_0, L_1 such that $\mathtt{KC} \subseteq L_0, L_1$ and $L_0 \neq L_1$ we have that $L_0^{\neg} = L_1^{\neg} = \mathtt{CPC}$ and thus that $(-)^{\neg}$ is not injective.

Proposition 4.2. *The map* $(-)^{\neg} : \mathbf{IL} \to \mathbf{DNAL}$ *is not injective.*

Therefore, since $(-)^{\neg}$ is not injective, we have that every $\mathtt{DNA}$-logic Λ determines a subset of the lattice $\mathbf{IL}$ which consists of all those logics which have Λ as their negative variant. Moreover, it is easy to see that this subset is also a sublattice, since the map $(-)^{\neg}$ is a homomorphism. Similarly, since also $(-)^{\uparrow}$ is a homomorphism, we can also consider the sublattice $\mathcal{I}(\mathcal{X})$ of all varieties $\mathcal{V}$ in $\mathbf{HA}$ whose negative closure is $\mathcal{X}$. We then define as follows the preimage of a $\mathtt{DNA}$-logic and the preimage of a $\mathtt{DNA}$-variety.

Definition 4.3. Let Λ be a $\mathtt{DNA}$-logic and $\mathcal{X}$ be a $\mathtt{DNA}$-variety. The *preimage* of Λ is the sublattice $\mathcal{I}(\Lambda)$ of all intermediate logics L such that $L^{\neg} = \Lambda$. The *preimage* of $\mathcal{X}$ is the sublattice $\mathcal{I}(\mathcal{X})$ of all varieties $\mathcal{V}$ such that $\mathcal{V}^{\uparrow} = \mathcal{X}$.

By the duality $\mathbf{IL} \cong^{op} \mathbf{HA}$ and the fact that the square introduced in Section 3.3.1 commutes, we then immediately have the following proposition.

Proposition 4.4. *For every* $\mathtt{DNA}$*-logic* Λ *and every* $\mathtt{DNA}$*-variety* $\mathcal{X}$*, we have that if* $\mathcal{X} = Var^{\neg}(\Lambda)$ *and* $\Lambda = Log^{\neg}(\mathcal{X})$ *then* $\mathcal{I}(\Lambda) \cong^{op} \mathcal{I}(\mathcal{X})$.

Where the isomorphism $\mathcal{I}(\Lambda) \cong^{op} \mathcal{I}(\mathcal{X})$ is the restriction of the dual isomorphism $\mathbf{IL} \cong \mathbf{HA}$. We use this duality to characterize the two lattices $\mathcal{I}(\Lambda)$ and $\mathcal{I}(\mathcal{X})$. We focus in the next sections the greatest and the least elements of these lattices.

4.1.2 Greatest Logic in $\mathcal{I}(\Lambda)$

In this section we prove that the preimage $\mathcal{I}(\Lambda)$ of some $\mathtt{DNA}$-logic Λ has a greatest element and we provide a characterisation of it. First, we introduce a map which associates to every $\mathtt{DNA}$-logic Λ the maximal intermediate logic in $\mathcal{I}(\Lambda)$. The following notion of schematic fragment of a $\mathtt{DNA}$-logic was first introduced under the name of *standardization* in [41, p. 545] and later considered by Ciardelli in [10, p. 45]. That this operation on $\mathtt{DNA}$-logics provides us with a maximal intermediate logic in $\mathcal{I}(\Lambda)$ was first proved in [41].

Definition 4.5 (Schematic Fragment)**.** Let Λ be a $\mathtt{DNA}$-logic, then we define its schematic fragment $Schm(\Lambda)$ as:

$$Schm(\Lambda) = \{\varphi \in \Lambda : \forall \overline{\psi} \in \mathcal{L}_P, \varphi[\overline{\psi}/\overline{p}]\}.$$

So $Schm(\Lambda)$ is the set of all schematic formulas in Λ, namely those formulas for which Λ is closed under substitution. We show that $Schm(\Lambda)$ is an intermediate logic.

Proposition 4.6. *$Schm(\Lambda)$ is an intermediate logic.*

Proof. We check the conditions for intermediate logics. (i) First, we have that $\mathtt{IPC} \subseteq \Lambda$ and so since every substitution instance of an intuitionistic validity is still in $\mathtt{IPC}$, we have that $\mathtt{IPC} \subseteq Schm(\Lambda)$. (ii) Obviously $Schm(\Lambda)$ is closed under substitution, by the definition of *Schm*. (iii) Finally, we check closure under modus ponens. Suppose $\varphi \in Schm(\Lambda)$ and $\varphi \to \psi \in Schm(\Lambda)$ and consider any substitution instance $\psi[\overline{\chi}/\overline{p}]$ of ψ. Then by the fact that φ and $\varphi \to \psi$ are schematic, it follows $\varphi[\overline{\chi}/\overline{p}] \in \Lambda$ and $(\varphi \to \psi)[\overline{\chi}/\overline{p}] = \varphi[\overline{\chi}/\overline{p}] \to \psi[\overline{\chi}/\overline{p}] \in \Lambda$. So by the closure of $\mathtt{DNA}$-logics under *modus ponens* it follows that $\psi[\overline{\chi}/\overline{p}] \in \Lambda$. Therefore, since this was an arbitrary substitution instance, we then have that $\psi \in Schm(\Lambda)$. ☐

The following two propositions show that $Schm(\Lambda)$ is the maximal intermediate logic whose negative variant is Λ.

Proposition 4.7. *Let Λ be any $\mathtt{DNA}$-logic. Then, for every intermediate logic L such that $L^{\neg} = \Lambda$ we have that $L \subseteq Schm(\Lambda)$.*

Proof. Suppose that $\varphi \in L$. We denote by $\overline{p} = p_0, ..., p_n$ the atomic letters in φ. We need to check that for any sequence of formulas $\overline{\chi} = \chi_0(\overline{q}), ..., \chi_n(\overline{q}) \in \mathcal{L}_P$ with atomic letters $\overline{q}$, it is the case that $\varphi^{\chi} = \varphi[\overline{\chi}/\overline{p}] \in \Lambda$. Now, since $\varphi \in L$, it follows by uniform substitution that $\varphi^{\chi} \in L$. Then, again by uniform substitution, we have that $\varphi^{\chi}[\neg q/\overline{q}] \in L$ and therefore $\varphi^{\chi} \in \Lambda$, which means that $\varphi \in Schm(\Lambda)$ and thus proves our claim. ☐

Proposition 4.8. *For every $\mathtt{DNA}$-logic Λ, $Schm(\Lambda)^{\neg} = \Lambda$.*

Proof. ($\subseteq$) If $\varphi \in Schm(\Lambda)^{\neg}$, then $\varphi[\neg p/\overline{p}] \in Schm(\Lambda)$, so by uniform substitution $\varphi[\neg\neg p/\overline{p}] \in Schm(\Lambda)$. Then, since $Schm(\Lambda) \subseteq \Lambda$, it follows $\varphi[\neg\neg p/\overline{p}] \in \Lambda$ and so since $p \equiv_{\Lambda} \neg\neg p$ it follows that $\varphi \in \Lambda$. ($\supseteq$) Suppose $\varphi \in \Lambda$ and without loss of generality that $\Lambda = L^{\neg}$. It follows that $\varphi[\neg p/\overline{p}] \in L$. By Proposition 4.7 above, we have that $L \subseteq Schm(\Lambda)$, hence $\varphi[\neg p/\overline{p}] \in Schm(\Lambda)$ and thus $\varphi \in Schm(\Lambda)^{\neg}$. ☐

The following theorem immediately follows by the previous propositions.

Theorem 4.9. *Let Λ be a $\mathtt{DNA}$-logic. The schematic fragment $Schm(\Lambda)$ is the greatest intermediate logic whose negative variant is Λ.*

Therefore, the preimage $\mathcal{I}(\Lambda)$ of a $\mathtt{DNA}$-logic Λ has always a greatest element. By Theorem 3.33 we also obtain a dual characterisation of the corresponding $\mathtt{DNA}$-varieties. In fact, we have that $Var(Schm(\Lambda))$ is the least variety whose negative closure is $Var^{\neg}(\Lambda)$. We define the map $least_V : \mathbf{DNAV} \to \mathbf{HA}$ as follows:

$$least_V : \mathcal{X} \mapsto Var(Schm(Log^{\neg}(\mathcal{X}))).$$

The following proposition follows easily.

Proposition 4.10. *The following diagram commutes in both directions, i.e. $Var \circ Schm = least_V \circ Var^{\neg}$ and $Log \circ least_V = Schm \circ Log^{\neg}$.*

$$\begin{tikzcd} \mathbf{IL} & \mathbf{DNAL} \arrow[l, "Schm"'] \arrow[d, leftrightarrow, "\cong^{op}"] \\ \mathbf{HA} \arrow[u, leftrightarrow, "\cong^{op}"] & \mathbf{DNAV} \arrow[l, "least_V"] \end{tikzcd}$$

Proof. By the definition of $least_V$ and the dual isomorphism $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$ we have $least_V \circ Var^{\neg} = Var \circ Schm \circ Log^{\neg} \circ Var^{\neg} = Var \circ Schm$ and $Log \circ least_V = Log \circ Var \circ Schm \circ Log^{\neg} = Schm \circ Log^{\neg}$. □

Therefore for every DNA-logic Λ we have that $Schm(\Lambda)$ is the greatest logic in $\mathcal{I}(\Lambda)$ and $least_V(Var^{\neg}(\Lambda))$ is the least variety in $\mathcal{I}(Var^{\neg}(\Lambda))$.

4.1.3 Smallest Logic in $\mathcal{I}(\Lambda)$

Similarly to what we have done above, we now want to show that the lattice $\mathcal{I}(\Lambda)$ has always a least element. That this holds follows directly from the fact that for every DNA-variety $\mathcal{X}$, there is a greatest variety whose negative closure is exactly $\mathcal{X}$.

Proposition 4.11. *For every* DNA*-variety $\mathcal{X}$, there is a greatest variety $\mathcal{V}$ such that $\mathcal{V}^{\uparrow} = \mathcal{X}$.*

Proof. Firstly notice that by Proposition 3.7 we have that DNA-varieties are also varieties. Moreover, by the closure under core superalgebra of DNA-varieties we also have that for every DNA-variety $\mathcal{X}$, $\mathcal{X}^{\uparrow} = \mathcal{X}$. It is then obvious that for any variety $\mathcal{U}$ such that $\mathcal{U}^{\uparrow} = \mathcal{X}$, we have $\mathcal{U} \subseteq \mathcal{X}$ and hence $\mathcal{X}$ is the greatest variety $\mathcal{V}$ such that $\mathcal{V}^{\uparrow} = \mathcal{X}$. □

The following theorem immediately follows by the previous propositions and DNA-duality.

Theorem 4.12. *Let $\mathcal{X}$ be a* DNA*-variety. The logic $Log(\mathcal{X})$ is the least among the intermediate logics whose negative variant is $Log^{\neg}(\mathcal{X})$.*

We thus define a map $least_L : \mathbf{DNAL} \to \mathbf{IL}$ as follows:

$$least_L : \Lambda \mapsto Log(Var^{\neg}(\Lambda)).$$

The following proposition follows easily.

Proposition 4.13. *The following diagram commutes in both directions, i.e. $Var \circ least_L = id \circ Var^{\neg}$ and $Log \circ id = least_L \circ Log^{\neg}$.*

$$\begin{tikzcd} \mathbf{IL} & \mathbf{DNAL} \arrow[l, "least_L"'] \arrow[d, leftrightarrow, "\cong^{op}"] \\ \mathbf{HA} \arrow[u, leftrightarrow, "\cong^{op}"] & \mathbf{DNAV} \arrow[l, "id"] \end{tikzcd}$$

Proof. By the definition of $least_L$ and the dual isomorphism $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$ we have $Var \circ least_L = Var \circ Log \circ Var^{\neg} = id \circ Var^{\neg}$ and $least_L \circ Log^{\neg} = Log \circ Var^{\neg} \circ Log^{\neg} = Log \circ id$. $\square$

Therefore, it is the case that for every DNA-logic Λ we have that $least_L(\Lambda)$ is the smallest logic in $\mathcal{I}(\Lambda)$ and $Var^{\neg}(\Lambda)$ is the greatest variety in $\mathcal{I}(Var^{\neg}(\Lambda))$.

4.1.4 Further Characterisations

By the results above it thus follows that the sublattices $\mathcal{I}(\Lambda)$ and $\mathcal{I}(\mathcal{X})$ are bounded sublattices of **IL** and **HA**. We introduce the following definitions.

Definition 4.14 (DNA-*maximality* and DNA-*minimality*)**.** Let L be an intermediate logic. (i) We say that L is DNA-*maximal* if it is the greatest logic in $\mathcal{I}(L^{\neg})$. (ii) We say that L is DNA-*minimal* if it is the least logic in $\mathcal{I}(L^{\neg})$.

In [41, p. 546] and [10, §5.2] intermediate logics L such that $L = Schm(L^{\neg})$ are called *stable*. The following proposition thus establishes that a logic is DNA-maximal iff it is stable. However, we will not use here this terminology, as the notion of stable logic has been employed e.g. in [31] with a rather different meaning. The following proposition is an immediate consequence of our definition and the previous results.

Proposition 4.15. *Let L be an intermediate logic, then:*

(i) L is DNA*-maximal iff $L = Schm(L^{\neg})$;*

(ii) L is DNA*-minimal iff $Var(L) = Var^{\neg}(L^{\neg})$.*

And so the lattice $\mathcal{I}(\Lambda)$ looks as follows:

$$\begin{tikzcd}
Schm(\Lambda) \arrow[drr, mapsto, "\neg"] & & \\
& & \Lambda \\
Log(Var^{\neg}(\Lambda)) \arrow[urr, mapsto, "\neg"'] & &
\end{tikzcd}$$

Notice now that there is some kind of asymmetry in our characterisation of DNA-maximal logics and DNA-minimal logics. In fact, we have a syntactic characterisation of DNA-maximal logics and a semantical characterisation of DNA-minimal logics. We know want to provide also a semantic criterion to establish whether an intermediate logic is DNA-maximal. In [10, p. 65] a criterion for DNA-maximality was given in the context of Kripke frames. We propose a criterion in terms of regular algebras.

Definition 4.16 (Regular Heyting Algebras)**.** An Heyting algebra H is said to be *regular* if $H = \langle H_{\neg} \rangle$.

These algebras have been introduced in [3] to provide an algebraic semantics to propositional inquisitive logic. A regular Heyting algebra is an algebra generated by its set $H_{\neg}$ of regular elements. For this reason we call regular Heyting algebras also *regularly generated.* The following theorem gives us a semantic criterion to determine if a logic is DNA-maximal.

Theorem 4.17. *If an intermediate logic L is the logic of a class of regularly generated Heyting algebras, then it is* DNA*-maximal.*

Proof. By the previous proposition this is equivalent to the statement that if an intermediate logic L is such that $L = Log(\mathcal{C})$, where $\mathcal{C}$ is a class of regularly generated Heyting algebras, then $L = Schm(L^{\neg})$. So, suppose that $\mathcal{C}$ is a class of regularly generated Heyting algebras, we need to show that $Log(\mathcal{C}) = Schm(Log(\mathcal{C})^{\neg})$. Since $Schm(Log(\mathcal{C})^{\neg})$ is DNA-maximal it follows that $Log(\mathcal{C}) \subseteq Schm(Log(\mathcal{C})^{\neg})$, so that we only need to show that $Schm(Log(\mathcal{C})^{\neg}) \subseteq Log(\mathcal{C})$. Now suppose by contraposition that $\varphi \notin Log(\mathcal{C})$, then we have that for some $H \in \mathcal{C}$ and for some valuation V, we have that $(H, V) \nvDash \varphi$. Now, since H is regularly generated, every element $x_i \in H$ can be written out as a polynomial $\delta_i(\overline{y_i^k})$ of regular elements of H. Then we define the DNA-valuation $V^{\neg} : p_i^k \mapsto y_i^k$ so that we then get $[\![\delta_i(\overline{p_i^k})]\!]^{(H,V^{\neg})} = \delta_i(\overline{y_i^k})$, so that we have, for some appropriate choice of polynomials, that $[\![\varphi]\!]^{(H,V)} = [\![\varphi[\overline{\delta_i(\overline{p_i^k})}/\overline{q}]]\!]^{(H,V^{\neg})}$. We then immediately get that $(H, V^{\neg}) \nvDash \varphi[\overline{\delta_i(\overline{p_i^k})}/\overline{q}]$ and, since $H \in \mathcal{C} \subseteq \mathcal{C}^{\uparrow}$, it follows $\varphi[\overline{\delta_i(\overline{p_i^k})}/\overline{q}] \notin Log^{\neg}(\mathcal{C}^{\uparrow}) = Log(\mathcal{C})^{\neg}$. Finally, since $\varphi[\overline{\delta_i(\overline{p_i^k})}/\overline{q}]$ is a substitution instance of φ, it follows that $\varphi \notin Schm(Log(\mathcal{C})^{\neg})$. □

4.2 Properties of DNA-Logic and DNA-Varieties

In this section we introduce several properties of DNA-logics and DNA-varieties and we show their connection to their counterparts for intermediate logics and varieties of Heyting algebras.

4.2.1 Universal Algebra of DNA-Varieties

We have already encountered in the previous sections the subalgebra $\langle H_{\neg} \rangle$ generated by the regular elements $H_{\neg}$ of an Heyting algebra H. We shall now prove some further results on this kind of regular Heyting algebras and show their central role for the theory of DNA-logics and DNA-varieties. Already in Chapter 3 we have seen two important properties of such regular algebras:

Lemma 3.18. *$H \vDash^{\neg} \varphi$ iff $\langle H_{\neg} \rangle \vDash^{\neg} \varphi$.*

Proposition 3.19. *If $H \vDash^{\neg} L^{\neg}$ then $\langle H_{\neg} \rangle \vDash L$.*

We now prove two further results showing that varieties $\mathcal{V}$ with the same negative closure $\mathcal{X}$ have the same collection of regular Heyting algebras. We first show the following proposition.

Proposition 4.18. *Let H be a regularly generated Heyting algebra such that for some* DNA*-logic Λ we have that $H \vDash^{\neg} \Lambda$. Then, for every intermediate logic L such that $L^{\neg} = \Lambda$ we have that $H \vDash L$.*

Proof. Suppose that $H \vDash^{\neg} \Lambda$, then by Proposition 4.8 it follows that $H \vDash^{\neg} Schm(\Lambda)^{\neg}$ and so by Proposition 3.19 $H \vDash Schm(\Lambda)$. Finally, by Proposition 4.7 we have that $L \subseteq Schm(\Lambda)$ and hence $H \vDash L$. □

By the dual isomorphism **DNAL** $\cong^{op}$ **DNAV**, the following proposition follows immediately.

Proposition 4.19. *Let H be a regular Heyting algebra. If $H \in \mathcal{X}$, then for every variety $\mathcal{V}$ such that $\mathcal{V}^{\uparrow} = \mathcal{X}$ we have that $H \in \mathcal{V}$.*

Proof. Suppose $H \in \mathcal{X}$, then $H \vDash^{\neg} Log^{\neg}(\mathcal{X})$. Then, since $\mathcal{V}^{\uparrow} = \mathcal{X}$, it follows by Proposition 3.24 that $Log(\mathcal{V})^{\neg} = Log^{\neg}(\mathcal{X})$. So $H \vDash^{\neg} Log(\mathcal{V})^{\neg}$ and by the previous Proposition 4.18, $H \vDash Log(\mathcal{V})$ which entails $H \in \mathcal{V}$. $\square$

We thereby have that all standard varieties whose negative closure is the same `DNA`-variety contain exactly the same regularly generated Heyting algebras. Interestingly, the previous proposition also provides us with a test to check whether a regular Heyting algebra validates a logic, as it now suffices to check whether it `DNA`-validates its negative variant.

We now recall what it means for a class to generate a `DNA`-variety. Let $\mathcal{X}$ be a `DNA`-variety, then we say that $\mathcal{X}$ is *generated* by the class $\mathcal{C} \subseteq \mathcal{X}$ and we write $\mathcal{X} = \mathcal{X}(\mathcal{C})$ if $\mathcal{X}$ is the least class of Heyting algebras such that $\mathcal{C} \subseteq \mathcal{X}$ and $\mathcal{X}$ is closed under the operations of subalgebra, homomorphic image, product and core superalgebra. It is easy to see that for any class $\mathcal{C}$ we have that $\mathcal{X}(\mathcal{C}) = \mathcal{V}(\mathcal{C})^{\uparrow}$, where $\mathcal{V}(\mathcal{C})$ is the least variety containing $\mathcal{C}$. We first adapt Tarski's theorem about varieties to the case of `DNA`-varieties.

Theorem 4.20 (`DNA`-Tarski)**.** *Let $\mathcal{C}$ be a class of Heyting algebras, then we have that $\mathcal{X}(\mathcal{C}) = HSP(\mathcal{C})^{\uparrow}$*

Proof. By definition we have that $\mathcal{X}(\mathcal{C}) = \mathcal{V}(\mathcal{C})^{\uparrow}$ and by Tarski's theorem 2.15 we have that $\mathcal{V}(\mathcal{C}) = HSP(\mathcal{C})$. Therefore $\mathcal{X}(\mathcal{C}) = HSP(\mathcal{C})^{\uparrow}$. $\square$

We then have the following theorem.

Theorem 4.21. *Let $\mathcal{X}$ be a `DNA`-variety, then*

$$\mathcal{X} = \mathcal{X}(\mathcal{C}) \Leftrightarrow Log^{\neg}(\mathcal{X}) = Log^{\neg}(\mathcal{C}).$$

Proof. ($\Rightarrow$) Since $\mathcal{C} \subseteq \mathcal{X}$, the inclusion from right to left is straightforward. Suppose now that $\mathcal{X} \nvDash^{\neg} \varphi$ then there is some Heyting algebra $H \in \mathcal{X}$ such that $H \nvDash^{\neg} \varphi$. Then since $\mathcal{X} = \mathcal{X}(\mathcal{C})$, it follow by Theorem 4.20 that $H \in HSP(\mathcal{C})^{\uparrow}$. Hence, there are $A_0, ..., A_n \in \mathcal{C}$ such that there is some subalgebra $B \preceq \prod_{i \leq n} A_i$ such that $B \twoheadrightarrow H$. Then, since `DNA`-validities are preserved under homomorphisms, subalgebras and products it immediately follows that for some $i \leq n$ we have $A_i \nvDash^{\neg} \varphi$.

($\Leftarrow$) Suppose now that $Log^{\neg}(\mathcal{X}) = Log^{\neg}(\mathcal{C})$. Then it follows by the Duality Theorem 3.33 that $Var^{\neg}(Log^{\neg}(\mathcal{X})) = Var^{\neg}(Log^{\neg}(\mathcal{C}))$. Finally, this means exactly that $\mathcal{X} = \mathcal{X}(\mathcal{C})$. $\square$

We now prove the following result stating that every `DNA`-variety $\mathcal{X}$ is generated by its collection of regular Heyting algebras. If $\mathcal{X}$ is a `DNA`-variety, then we denote by $\mathcal{X}_R$ its subclass of regular Heyting algebras.

Proposition 4.22. *Every `DNA`-variety is generated by its collection of regular elements, i.e. $\mathcal{X} = \mathcal{X}(\mathcal{X}_R)$.*

Proof. Let $\mathcal{X}$ be a `DNA`-variety, then for any non-regular $H \in \mathcal{X}$ we have that $\langle H_\neg \rangle \preceq H$ and $H_\neg = \langle H_\neg \rangle_\neg$. So since $\langle H_\neg \rangle \in \mathcal{X}_R$ it follows $H \in \mathcal{X}(\mathcal{X}_R)$. □

We thus have, by Birkhoff theorem, that every `DNA`-variety is generated by its subdirectly irreducible elements and, by the previous proposition, that every `DNA`-variety is generated by its regular elements. We now want to show that we actually have something more, namely that `DNA`-varieties are generated by their regular, subdirectly irreducible elements. Now if $\mathcal{X}$ is a `DNA`-variety, we denote by $\mathcal{X}_{RSI}$ its subset of regular subdirectly irreducible Heyting algebras. We thus want to show that for every `DNA`-variety we have $\mathcal{X} = \mathcal{X}(\mathcal{X}_{RSI})$. We now prove some lemmas which will turn out useful later. First we recall that if L is a lattice, then an *ideal* is a nonempty subset $I \subseteq L$ such that (i) for any $x, y \in I$ we have $x \vee y \in I$ and (ii) if $x \in I$ and $y \leq x$ then $y \in I$. Also, let us recall the statement of Zorn's Lemma, which we will use in the next proof. For Zorn's Lemma and its equivalent formulations we refer the reader to [17] and [35].

Zorn's Lemma. *Let P be a non-empty partially ordered set. Then if every nonempty ordered chain in P has an upper bound then P has a maximal element.*

The following theorem is an important result about maximal filters which follows from Zorn's Lemma and which we will use in the proof of the next lemma. We refer the reader to [36, p. 14] which proves a statement which is dual to the one we are considering here.

Theorem 4.23. *Let H be a Heyting algebra, I an ideal in L and F a filter in L such that $I \cap F = \emptyset$. Then there is a maximal filter F' in L such that $F \subseteq F'$ and $F \cap I = \emptyset$.*

Proof. Consider the family P of all filters G such that $F \subseteq G$ and $G \cap I = \emptyset$. It is straightforward to see that this family is a poset under the inclusion relation. Now consider any nonempty chain $C = \langle G_\gamma : \gamma < \alpha \rangle$, where α is the ordinal equal to the length of C. Then we have that for every every $G_\gamma \in C$, $G_\gamma \subseteq \bigcup_{\gamma \leq \alpha} G_\gamma$ and also that $\bigcup_{\gamma \leq \alpha} G_\gamma$ is a filter, as it is easy to see that the union of a family of filters is again a filter. Hence it follows that $\bigcup_{\gamma \leq \alpha} G_\gamma$ is an upper bound of the chain C. Therefore, we have by Zorn's lemma that the family of all filters G such that $F \subseteq G$ and $G \cap I = \emptyset$ contains a maximal element, which means that there is a maximal filter F' in L such that $F \subseteq F'$ and $F \cap I = \emptyset$. □

The next result is a well-known fact in the literature and was proved in [50]. The proof of this lemma relies on the previous theorem.

Lemma 4.24. *Let $B \in HA$. Then if $b \neq 1_B$ there is a subdirectly irreducible algebra C and a surjective homomorphism $h : B \twoheadrightarrow C$ such that $f(b) = s_C$, where s_C is the second greatest element of C.*

Proof. Consider any Heyting algebra B and any element $b \in B$ such that $b \neq 1_B$. Consider now the ideal $\downarrow b$ and the filter $\{1_B\}$, it is obvious that $\downarrow b \cap \{1_B\} = \emptyset$. Therefore, it follows immediately from Theorem 4.23 above that we can extend $\{1_B\}$ to a maximal filter F in B such that $F \cap \downarrow b = \emptyset$. By the correspondence of filters and congruences, we consider now the quotient algebra $C = B/\theta$, where θ is the congruence defined by F. It is clear that C is a Heyting algebra and that the natural map $f = \eta_\theta$ is a surjective homomorphism. We then only need to show that C is subdirectly irreducible and that $f(b) = \eta_\theta(b) = s_c$.

To prove that C is subdirectly irreducible it suffices by Theorem 2.32 to show that $Con(C) \setminus \Delta_C$ has a least element. By Theorem 2.24 we have that $Con(C) = Con(B/\theta) \cong [\theta, \nabla_B]$, hence $Con(C) \setminus \Delta_C \cong [\theta, \nabla_B] \setminus \theta$. Therefore, by the correspondence between filters and congruences, it suffices to prove that $[\theta, \nabla_B] \setminus \theta \cong [F, \uparrow 1_B] \setminus F$ has a least element. We now claim that the filter $\uparrow b$ is the least filter in $[F, \uparrow 1_B] \setminus F$. Consider any filter $G \in [F, \uparrow 1_B] \setminus F$, then we have that $F \subset G$. By assumption, we have that F is the maximal filter which does not intersect $\downarrow b$, therefore F is the maximal filter which does not contain b. But then, by the fact that $F \subset G$, it follows immediately that $b \in G$ and therefore that $\uparrow b \subseteq G$. So $\uparrow b$ is the least filter in $[F, \uparrow 1_B] \setminus F$, which immediately entails that C is subdirectly irreducible.

Finally, we also have that since $\uparrow b$ is the least filter in $[F, \uparrow 1_B] \setminus F$, then $\uparrow [b]$ is the least non-trivial filter in $Fil(C)$. Hence we also have that $[b] = \eta_\theta(b) = f(b)$ is the second greatest element of C. □

The following two lemmas ensure that homomorphisms preserve regularity both for elements and Heyting algebras.

Lemma 4.25. *Suppose $h : A \twoheadrightarrow B$ is a homomorphism of Heyting algebras. If $a \in A_\neg$ then $h(a) \in B_\neg$.*

Proof. Since $a \in A_\neg$ is regular, we have that $a = \neg\neg a$ and thus by the property of homomorphisms $h(a) = h(\neg\neg a) = \neg\neg h(a)$, which proves that $h(a)$ is regular and hence $h(a) \in B_\neg$. □

Lemma 4.26. *The homomorphic image of a regular Heyting algebra is regular.*

Proof. Let $h : A \twoheadrightarrow B$ be a surjective homomorphism, then if A is regular we have that $A = \langle A_\neg \rangle$. Therefore, since h is surjective, it follows that $B = h[A] = h[\langle A_\neg \rangle]$. But then, by the property of homomorphisms, $B = h[\langle A_\neg \rangle] = \langle h[A_\neg] \rangle$. By Lemma 4.25 it follows that $h[A_\neg] \subseteq B_\neg$, and so that $B = \langle B_\neg \rangle$, which shows that B is regular. □

Finally, we can prove a `DNA`-version of Birkhoff theorem for `DNA`-varieties.

Theorem 4.27 (`DNA`-Birkhoff)**.** *Every `DNA`-variety is generated by its collection of regular subdirectly irreducible elements: $\mathcal{X} = \mathcal{X}(\mathcal{X}_{RSI})$.*

Proof. By the dual isomorphism between `DNA`-logics and `DNA`-varieties it suffices to show that $Log^\neg(\mathcal{X}) = Log^\neg(\mathcal{X}(\mathcal{X}_{RSI}))$, which is equivalent by Theorem 4.21 to $Log^\neg(\mathcal{X}) = Log^\neg(\mathcal{X}_{RSI})$. The direction $Log^\neg(\mathcal{X}) \subseteq Log^\neg(\mathcal{X}_{RSI})$ follows immediately from the inclusion $\mathcal{X}_{RSI} \subseteq \mathcal{X}$. It thus suffices to show that $Log^\neg(\mathcal{X}_{RSI}) \subseteq Log^\neg(\mathcal{X})$.

Suppose by contraposition that $\varphi \notin Log^\neg(\mathcal{X})$, then for some $H \in \mathcal{X}$ and some `DNA`-valuation $V^\neg$, we have that $(H, V^\neg) \nvDash^\neg \varphi$ and so by Proposition 3.16 that

$(\langle H_\neg\rangle, V^\neg) \nvDash^\neg \varphi$. Then, since $x = [\![\varphi]\!]^{(\langle H\rangle,V^\neg)} \neq 1_H$ it follows by Lemma 4.24 that there is a subdirectly irreducible algebra C such that there is surjective homomorphism $h : \langle H\rangle \twoheadrightarrow C$ with $h(x) = s_C$. Then, consider the valuation $U^\neg = h \circ V^\neg$, then it follows by Lemma 4.25 that $U^\neg$ is a DNA valuation. Now let $p_0, .., p_n$ be the variables in φ, it follows by the properties of homomorphisms that:

$$\begin{aligned}
[\![\varphi(p_0, .., p_n)]\!]^{(C,U^\neg)} &= \varphi_C[U^\neg(p_0), ..., U^\neg(p_n)] \\
&= \varphi_C[h(V^\neg(p_0)), ..., h(V^\neg(p_n))] \\
&= h[\![\varphi(p_0, .., p_n)]\!]^{(\langle H\rangle,V^\neg)} \\
&= s_C.
\end{aligned}$$

From which it immediately follows that $(C, U^\neg) \nvDash \varphi$ and so that $C \nvDash \varphi$. Now, since $H \in \mathcal{X}$, we have that $\langle H\rangle \in \mathcal{X}$ and so since $h : \langle H\rangle \twoheadrightarrow C$ also that $C \in \mathcal{X}$. Moreover, we have that C is subdirectly irreducible and by Lemma 4.26 also that C is regular, since it is homomorphic image of $\langle H_\neg\rangle$. Finally, this means that $C \in \mathcal{X}_{RSI}$ and so that $\varphi \notin Log^\neg(\mathcal{X}_{RSI})$, which proves our claim. □

4.2.2 Locally Finite DNA-Logics and DNA-Varieties

The notion of local finiteness plays an important role in the theory of intermediate logics, as we have already mention in Chapter 2. Here we introduce a suitable notion of local finiteness for DNA-varieties and DNA-logics, which we will also later employ in our study of inquisitive logic.

Locally Finiteness and Finite Model Property

We say that an Heyting algebra H is DNA-*finitely generated* if there are finitely many elements $x_0, ..., x_n \in H_\neg$ such that $\langle x_0, ..., x_n\rangle = H$. We then define locally finite DNA-varieties and locally finite DNA-logics.

Definition 4.28. An DNA-variety $\mathcal{X}$ is DNA-*locally finite* if every DNA-finitely generated $H \in \mathcal{X}$ is also finite. A DNA-logic Λ is DNA-*locally finite* if its corresponding DNA-variety $Var^\neg(\Lambda)$ is locally finite.

When the context make it clear we then drop the prefix DNA and talk simply of locally finiteness. If not specified otherwise, every time we talk of locally finiteness of a DNA-variety or a DNA-logic we actually refer to the property of DNA-local finiteness. The following proposition follows straightforwardly and allows us to relate the local finiteness of intermediate logics to the local finiteness of DNA-logics.

Proposition 4.29. *Let L be any intermediate logic, then if L is locally finite, also $L^\neg$ is locally finite.*

Proof. If L is locally finite, then every finitely generated $H \in Var(L)$ is also finite. Now consider any $H \in Var^\neg(L^\neg)$ and suppose for some $x_0, ..., x_n \in H_\neg$ we have $\langle x_0, ..., x_n\rangle = H$. Then it follows that $H = \langle H_\neg\rangle$ and so that H is regular. Then, we have by Proposition 4.19 that $H \in Var(L)$ and so since H is finitely generated by $x_0, ..., x_n$ it also follows that H is finite. This shows that $L^\neg$ is locally finite. □

A property of DNA-logics which is closely connected to the local finiteness is the finite model property (also FMP). We introduce it as follows.

Definition 4.30 (Finite Model Property)**.** A DNA-variety $\mathcal{X}$ has the DNA-*finite model property* (FMP) if $\mathcal{X} = \mathcal{X}(\mathcal{C})$ where $\mathcal{C}$ is a collection of finite Heyting algebras. A DNA-logic Λ has the DNA-*finite model property* if if its corresponding DNA-variety $Var^{\neg}(\Lambda)$ has the finite model property.

When the context make it clear we then drop the prefix DNA and talk simply of finite model property. If not specified otherwise, every time we talk of the finite model property of a DNA-variety or a DNA-logic we actually refer to the DNA-finite model property. The finite model property allows, for every formula $\varphi \notin \Lambda$, to find a finite algebra H which validates Λ and refutes φ. Similarly to what happens in the case of local finiteness, the finite model property of an intermediate logic entails the finite model property of its negative variant.

Proposition 4.31. *Let L be any intermediate logic, then if L has the finite model property also $L^{\neg}$ has the finite model property.*

Proof. Suppose L has the finite model property, then $Var(L) = \mathcal{V}(\mathcal{C})$ for some class $\mathcal{C}$ of finite Heyting algebras. Then, we have that $Var^{\neg}(L^{\neg}) = Var^{\neg}(L^{\neg})^{\uparrow} = \mathcal{V}(\mathcal{C})^{\uparrow} = \mathcal{X}(\mathcal{C})$, which shows that $L^{\neg}$ also has the finite model property. □

Now, if a DNA-variety has the finite model property we can then refine as follows our version of Birkhoff theorem. We denote by $\mathcal{X}_{RFSI}$ the collection of finite, regular, subdirectly irreducible elements in $\mathcal{X}$.

Theorem 4.32. *If a DNA-variety $\mathcal{X}$ has the finite model property, then it is generated by its finite, regular subdirectly irreducible elements, i.e. $\mathcal{X}$, $\mathcal{X} = \mathcal{X}(\mathcal{X}_{RFSI})$.*

Proof. By Theorem 4.21 it suffices to check that $Log^{\neg}(\mathcal{X}_{RFSI}) = Log^{\neg}(\mathcal{X}(\mathcal{X}_{RFSI}))$. The direction $Log^{\neg}(\mathcal{X}) \subseteq Log^{\neg}(\mathcal{X}_{RFSI})$ is obvious, for if φ is true in every algebra in $\mathcal{X}$ it is also true in $\mathcal{X}_{RFSI}$. Now, consider the direction $Log^{\neg}(\mathcal{X}_{RFSI}) \subseteq Log^{\neg}(\mathcal{X})$. First notice that if a DNA-variety $\mathcal{X}$ has the finite model property, then for some class of finite Heyting algebras $\mathcal{C}$, we have that $\mathcal{X} = \mathcal{X}(\mathcal{C})$. Suppose now by contradiction that $\varphi \notin Log^{\neg}(\mathcal{X})$, then by Theorem 4.21 there is some finite $H \in \mathcal{C}$ such that $H \nvDash^{\neg} \varphi$. Therefore, it follows immediately by Lemma 3.16 that $\langle H_{\neg} \rangle \nvDash^{\neg} \varphi$. Then, by the same argument of the proof of DNA-Birkhoff Theorem 4.27, we obtain a regular subdirectly irreducible algebra C such that $h : \langle H_{\neg} \rangle \twoheadrightarrow C$ and $C \nvDash \varphi$. Moreover, by the fact that C is homomorphic image of $\langle H_{\neg} \rangle$ it also follows that C is finite. We thus obtain that $C \in \mathcal{X}_{RFSI}$ and since $C \nvDash^{\neg} \varphi$ that $\varphi \notin Log^{\neg}(\mathcal{X}_{RFSI})$, which proves our claim. □

Moreover, we can also show that if a DNA-variety $\mathcal{X}$ is locally finite, then it has the finite model property. We denote by $\mathcal{X}_F$ the subcollection of finite Heyting algebras in $\mathcal{X}$.

Theorem 4.33. *Let $\mathcal{X}$ be a DNA-variety. If $\mathcal{X}$ is locally finite, then it has the finite model property.*

Proof. By Theorem 4.21 it suffices to show that $Log^{\neg}(\mathcal{X}) = Log^{\neg}(\mathcal{X}_F)$. The inclusion $Log^{\neg}(\mathcal{X}) \subseteq Log^{\neg}(\mathcal{X}_F)$ is obvious, so we show that $Log^{\neg}(\mathcal{X}_F) \subseteq Log^{\neg}(\mathcal{X})$. Suppose $\varphi \notin Log^{\neg}(\mathcal{X})$, then there is some $H \in \mathcal{X}$ such that for some DNA-valuation $V^{\neg}$ we have that $(H, V^{\neg}) \nvDash^{\neg} \varphi$. Now let Let $\overline{p}$ be the variables in φ and $V^{\neg}(\overline{p})$ their interpretation in H. Then, since $\mathcal{X}$ is locally finite we have that the generated subalgebra $\langle V^{\neg}(\overline{p})\rangle$ is also finite. Moreover, since $(H, V^{\neg}) \nvDash \varphi$ and by the fact that the interpretation of φ lies inside $\langle V^{\neg}(\overline{p})\rangle$, it immediately follows that $(\langle V(\overline{p})\rangle, V^{\neg}) \nvDash \varphi$. So, since $\langle V^{\neg}(\overline{p})\rangle \in \mathcal{X}_F$, it follows that $\varphi \notin Log^{\neg}(\mathcal{X}_F)$, which proves our claim. $\square$

Similarly to what happens in the case of intermediate logics, it is possible to find DNA-logics which have the finite model property but are not locally finite. For instance, it is an important result that IPC has the finite model property and it is not locally finite. Now, since IPC has the finite model property, it follows immediately from Proposition 4.31 that $\texttt{IPC}^{\neg}$ has the finite model property as well. However, similarly to the case of IPC, we can show that $\texttt{IPC}^{\neg}$ is not locally finite. We do this in the next section by adapting the method of the Rieger-Nishimura lattice to the context of DNA-logics.

$\texttt{IPC}^{\neg}$ is not Locally Finite

We now show that the locally finiteness of a DNA-variety of Heyting algebras is not a trivial property. We do this by showing that the DNA-variety of all Heyting algebras $Var(\texttt{IPC}^{\neg})$ is not locally finite. Our argument consists in an adaptation of the Rieger-Nishimura lattice construction. First, we prove the following lemma.

Lemma 4.34. *There are infinitely many intuitionistic formulas φ_i over the negations of two variables p, q such that for every $i, j \in \mathbb{N}$ with $i \neq j$ we have that $\varphi_i \not\equiv \varphi_j$.*

Proof. We prove our claim by presenting a Kripke model showing that for every $i, j \in \mathbb{N}$ with $i \neq j$ we have that $\varphi_i \not\equiv \varphi_j$. Consider the adaptation of the Rieger-Nishimura ladder in Figure 4.1. We call the points in the left column $\alpha_0, \alpha_1, ...$ starting from the top-most element, and the points in the right column $\beta_0, \beta_1, ...$ starting from the top-most element. We call ω the extra point to the left such that $\alpha_1 \leq \omega$ and $\alpha_0 \nleq \omega$.

Over this frame, we define the valuation V over the two propositional variables p, q, such that $V : p \mapsto \{\alpha_0\}$ and $V : q \mapsto \{\beta_0\}$. Then under the valuation V one can see that:

$$\begin{aligned}
(\alpha_0, V) &\Vdash p \equiv \neg\neg p;\\
(\alpha_1, V) &\Vdash q \equiv \neg\neg q;\\
(\omega, V) &\Vdash \neg(\neg\neg p \vee \neg\neg q);\\
(\alpha_1, V) &\Vdash \neg q;\\
(\beta_1, V) &\Vdash \neg\neg(\neg\neg p \vee \neg\neg q).
\end{aligned}$$

Then, for every $n \geq 2$ we define two formulas:

$$\varphi_n := \psi_{n-1} \to (\varphi_{n-2} \vee \psi_{n-2})$$

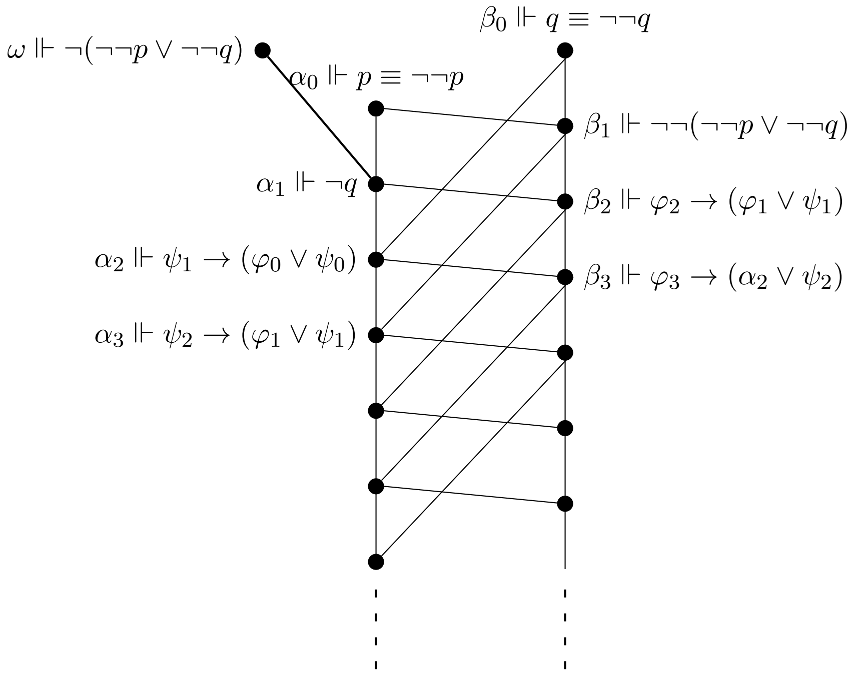

Figure 4.1: `DNA`-Rieger-Nishimura Ladder

$$\psi_n := \varphi_n \to (\varphi_{n-1} \vee \psi_{n-1}).$$

One can now show that the following claim holds.

Claim. *Let $n \in \mathbb{N}$ such that $2 \leq n$, then:*

(i) $(w, V) \Vdash \varphi_n$ *iff* $\alpha_n \leq w$*;*

(ii) $(w, V) \Vdash \psi_n$ *iff* $\beta_n \leq w$*.*

Proof. We prove (i) and (ii) simultaneously by induction on $n \geq 2$. The base case for $n = 2$ follows immediately from the definition of the valuation V and the interpretation of the formulas above. Now suppose $n > 2$, then we have:

$$\begin{aligned}
&(w, V) \Vdash \varphi_n \\
\Leftrightarrow\ &(w, V) \Vdash \psi_{n-1} \to (\varphi_{n-2} \vee \psi_{n-2}) \\
\Leftrightarrow\ &\forall y \geq w (y \Vdash \psi_{n-1} \text{ then } y \Vdash \varphi_{n-2} \vee \psi_{n-2}) \\
\Leftrightarrow\ &\forall y \geq w (\beta_{n-1} \leq y \text{ then } \alpha_{n-2} \leq y \text{ or } \beta_{n-2} \leq y) \\
\Leftrightarrow\ &\alpha_n \leq w.
\end{aligned}$$

The induction step for (ii) follows analogously. $\square$

Therefore, for $i, j, k, l \in \mathbb{N}$ we have that $\varphi_i, \varphi_j, \psi_k, \psi_l$ are pairwise non-equivalent, since each of these formulas is valid in a different upset of the Kripke frame that we are considering. Finally, it follows that there are infinitely many formulas in $\mathcal{L}_P$ over the negation of two variables p, q which are not equivalent modulo IPC. $\square$

From the former lemma it is easy to show that the DNA-variety of all Heyting algebras is not locally finite.

Proposition 4.35. $Var(\texttt{IPC}^\neg)$ *is not locally finite.*

Proof. Consider the Rieger-Nishimura lattice generated by two elements, namely the free Heyting algebras with two generators, and let this be denoted by $\mathcal{RN}^2$. Then it follows immediately by Lemma 4.34 that the finitely generated subalgebra $\langle \neg\neg p, \neg\neg q\rangle$ of $\mathcal{RN}^2$ is not finite, since there are countably many formulas over the negations of the two variables p, q which are not equivalent modulo IPC. Therefore, since $\mathcal{RN}^2 \in Var^\neg(\texttt{IPC}^\neg)$ it follows directly from our definition of locally finiteness that $Var^\neg(\texttt{IPC}^\neg)$ is not locally finite. $\square$

4.3 Jankov Formulas for DNA-Models

Jankov formulas (or Jankov-de Jongh formulas) have played an important role in the study of intermediate logics. These formulas are a sort of counterpart in algebraic logic of what diagrams are in model theory: they are formulas which express in syntactic terms some key semantic properties of the corresponding algebra. Jankov introduced these formulas in [33, 34], where he used them to show that the lattice of intermediate logic has the same cardinality of the continuum. Formulas having similar properties have also been introduced in the same years by de Jongh [37] and later by Fine in the context of modal logics [20]. We refer the interested reader to [16, 1, 7] for more information on Jankov formulas and their history. In this section we adapt Jankov formulas to the setting of DNA-logics and we show how they can be used to axiomatise locally finite DNA-logics.

4.3.1 Jankov Theorem

In this section we introduce a version of Jankov formulas which suits our setting of DNA-logics. We adapt the approach originally presented by Wronski in [50].

Let $\mathcal{X}_{RFSI}$ be the class of regular, finite, subdirectly irreducible algebras of a DNA-variety $\mathcal{X}$. First, we show how to decorate a Heyting algebra $H \in HA_{RFSI}$ with what we will call Jankov representants. Consider any $H \in HA_{RFSI}$, then since H is regular we have that $H = \langle H_\neg \rangle$ and since H is finite we have that $H_\neg$ is also finite. We can thus assume without loss of generality that H is generated by a finite set of elements $a_0, ..., a_n$ and that every element $x \in H$ can be expressed as a polynomial $\delta_H(a_0, ..., a_n)$ over the regular elements of H. We then associate every element $x \in H$ to a formula ψ_x called its *Jankov representant.*

Definition 4.36 (Jankov Representant)**.** Let $H \in HA_{RFSI}$ and $x \in H$, then the *Jankov Representant* of x is a formula ψ_x defined as follows:

(i) If $x \in H_\neg$, then $\psi_x = p_x$, where $p_x \in \texttt{AT}$;

(ii) If $x = \delta_X(a_0, ..., a_n)$ with $a_0, ..., a_n \in H_\neg$, then $\psi_x = \delta(p_{a_0}, ..., p_{a_n})$.

Notice that when we decorate an Heyting algebra H with Jankov representants we are making a fundamental use of the fact that H is regular. Notice also that the Jankov representant of an element $x \in H$ does not need to be unique, as there can be different polynomials over regular elements characterizing the same non-regular element of a regular Heyting algebra. The Jankov representant is thus the formulas corresponding to any of those polynomials.

Once we have the notion of Jankov representant, we can define Jankov formulas for the setting of DNA-logics as follows.

Definition 4.37 (Jankov DNA-Formula)**.** Let $H \in HA_{RFSI}$, let 0 be the least element of H and s its second greatest element. Then the *Jankov* DNA*-Formula* $\chi^{\texttt{DNA}}(H)$ is defined as follows:

$$\chi^{\texttt{DNA}}(H) = \alpha \to \beta.$$

Where α and β are the following formulas:

$$\begin{aligned}\alpha = (\psi_0 \leftrightarrow \bot) \wedge & \bigwedge\{(\psi_a \wedge \psi_b) \leftrightarrow \psi_{a\wedge b} : a, b \in H\} \wedge \\ & \bigwedge\{(\psi_a \vee \psi_b) \leftrightarrow \psi_{a\vee b} : a, b \in H\} \wedge \\ & \bigwedge\{(\psi_a \to \psi_b) \leftrightarrow \psi_{a\to b} : a, b \in H\} \\ \beta = \psi_s. &\end{aligned}$$

When its clear from the context that we are working with Jankov DNA-formulas and not with the standard Jankov formulas, we drop the apex and write just $\chi(H)$ for the Jankov DNA-formula of H. We now prove a lemma which plays an important role in the proof of out Jankov theorem.

Lemma 4.38. *Let $H \in HA_{RFSI}$, then $H \nvDash^{\neg} \chi(H)$.*

Proof. Suppose $H \in HA_{RFSI}$ and $\chi(H)$ is its DNA-Jankov formula. Then we define the DNA-valuation $V^{\neg}$ such that for all atomic Jankov representant we have that $V : p_a \mapsto a$, for all $a \in H_{\neg}$. Moreover, if an element $x \in H \setminus H_{\neg}$ is described by a polynomial $\delta_H(a_0, ..., a_n)$ over regular element of H, it follows by the definition of Jankov representant that $[\![\delta(p_a, ..., p_a)]\!]^{(H,V^{\neg})} = \delta_H(a_0, ..., a_n)$. We then have that for every element $x \in H$ it is the case that $[\![\psi_x]\!]^{(H,V^{\neg})} = x$. But then it follows straightforwardly that for all $a, b \in H$ and for any connective $\odot$ we have $[\![\psi_a \odot \psi_b]\!]^{(H,V^{\neg})} = [\![\psi_{a\odot b}]\!]^{(H,V^{\neg})}$ so that the antecedent of the DNA-Jankov formula is $[\![\alpha]\!]^{(H,V^{\neg})} = 1_A$ and its consequent is $[\![\beta]\!]^{(H,V^{\neg})} = [\![\psi_x]\!]^{(H,V^{\neg})} = s_c$. Therefore, we have that:

$$[\![\chi(H)]\!]^{(H,V^{\neg})} = [\![\alpha \to \beta]\!]^{(H,V^{\neg})} = [\![\alpha]\!]^{(H,V^{\neg})} \to [\![\beta]\!]^{(H,V^{\neg})} = 1_A \to s_A = s_A \neq 1_A.$$

And, therefore, we have that $(H, V^{\neg}) \nvDash^{\neg} \chi(H)$ and so that $H \nvDash^{\neg} \chi(H)$. □

If A and B are two Heyting algebras, then we define $A \leq B$ iff $A \in HS(B)$. It is easy to show that this is indeed a partial order. We now prove a suitable version of Jankov Theorem for our setting. We adapt to our setting a similar proof given in [1].

Theorem 4.39 (Jankov Theorem for DNA-Models)**.** *Let $A \in HA_{RFSI}$ and $B \in HA$ then:*

$$B \nvDash^{\neg} \chi(A) \text{ iff } A \leq B.$$

Proof. ($\Rightarrow$) Suppose that $B \nvDash^{\neg} \chi(A)$, then for some DNA-valuation $V^{\neg}$ we have $[\![\chi(A)]\!]^{(B,V^{\neg})} = b \neq 1_B$. It follows from the previous Lemma 4.24 that there is a subdirectly irreducible Heyting algebra C such that there is a surjective homomorphism $f : B \twoheadrightarrow C$ such that $f(b) = s_C$. By the previous Lemma 4.25 we know that f sends regular elements to regular elements so that we can define the DNA-valuation $U^{\neg} = f \circ V^{\neg}$. It thus follows from the definition of our valuation that we have $[\![\chi(A)]\!]^{(C,U^{\neg})} = [\![\alpha \to \psi_s]\!]^{(C,U^{\neg})} = f(b) = s_C$. Now, since $[\![\alpha \to \psi_s]\!]^{(C,U^{\neg})} \neq 1_C$ we immediately have that $[\![\psi_s]\!]^{(C,U^{\neg})} \neq 1_C$ and so since s_C is the second greatest element of C we also have that $[\![\psi_s]\!]^{(C,U^{\neg})} \leq s_C$. Also, since $s_C \leq [\![\alpha \to \psi_s]\!]^{(C,U^{\neg})}$, it follows that $s_C \wedge [\![\alpha]\!]^{(C,U^{\neg})} \leq [\![\psi_s]\!]^{(C,U^{\neg})}$, so that $s_C \leq [\![\psi_s]\!]^{(C,U^{\neg})}$ and thus $[\![\psi_s]\!]^{(C,U^{\neg})} = s_C$. Moreover, since $1_C \nleq [\![\alpha \to \psi_s]\!]^{(C,U^{\neg})}$, it follows that $1_C \wedge [\![\alpha]\!]^{(C,U^{\neg})} \nleq [\![\psi_s]\!]^{(C,U^{\neg})}$ hence $[\![\alpha]\!]^{(C,U^{\neg})} \nleq s_C$ and therefore $[\![\alpha]\!]^{(C,U^{\neg})} = 1_C$.

We now prove that the map $h : A \to C$ such that $h : x \mapsto [\![\psi_x]\!]^{(C,U^{\neg})}$ is an embedding of A into C. First, we show that h is a homomorphism. Since $[\![\alpha]\!]^{(C,U^{\neg})} = 1_C$ it follows immediately that $[\![\psi_0 \leftrightarrow \bot]\!]^{(C,U^{\neg})} = 1_C$ and for every connective $\odot$ and every elements $a, b \in C$, we have $[\![(\psi_a \odot \psi_b) \leftrightarrow \psi_{a\odot b}]\!]^{(C,U^{\neg})} = 1_C$. From this we immediately get that $[\![\psi_0]\!]^{(C,U^{\neg})} = 0_C$ and $[\![\psi_a \odot \psi_b]\!]^{(C,U^{\neg})} = [\![\psi_{a\odot b}]\!]^{(C,U^{\neg})}$. That h is a homomorphism follows then immediately from the two following equalities.

$$h(0_A) = [\![\psi_0]\!]^{(C,U^{\neg})} = 0_C;$$
$$h(a \odot b) = [\![\psi_{a\odot b}]\!]^{(C,U^{\neg})} = [\![\psi_a \odot \psi_b]\!]^{(C,U^{\neg})} = [\![\psi_a]\!]^{(C,U^{\neg})} \odot [\![\psi_b]\!]^{(C,U^{\neg})} = h(a) \odot h(b).$$

We show that h is also injective. Suppose that $a \neq b$, then we can assume without loss of generality that $a \nleq b$ which is equivalent to $a \to b \neq 1_A$. To show that $h(a) \neq h(b)$ it then suffices to show $h(a) \to h(b) \neq 1_C$. Now, since $a \to b \neq 1_A$, it follows that $a \to b \leq s_A$, where s_A is the second greatest element of A, and therefore that $(a \to b) \to s_A = 1_A$. Since h is a homomorphism we have that $h(1_A) = 1_C$. Moreover, by the property of homomorphisms and the fact that $[\![\psi_s]\!]^{(C,U^{\neg})} = s_C$:

$$\begin{aligned} h((a \to b) \to s_A) &= h(a \to b) \to h(s_A) \\ &= (h(a) \to h(b)) \to h(s_A) \\ &= (h(a) \to h(b)) \to [\![\psi_s]\!]^{(C,U^{\neg})} \\ &= (h(a) \to h(b)) \to s_C. \end{aligned}$$

So that $(h(a) \to h(b)) \to s_C = 1_C$ and therefore $(h(a) \to h(b)) \leq s_C$ which means $(h(a) \to h(b)) \neq 1_C$ and so $h(a) \nleq h(b)$, thus proving the injectivity of h.

Therefore, we have that h is an embedding and thus $h[A] \preceq C$, showing that A is a subalgebra of C up to isomorphism . Then since $B \twoheadrightarrow C$ it follows that $A \in SH(B)$ and then since by Proposition 2.13 we have $SH(B) \subseteq HS(B)$, we obtain that $A \in HS(B)$ which proves $A \leq B$.

($\Leftarrow$) Suppose that $A \leq B$, namely that $A \in HS(B)$, then we know there is some subalgebra $B' \preceq B$ such that there is a surjective homomorphism $h : B' \twoheadrightarrow A$.

Moreover, by the previous Lemma 4.38 we have that $A \nvDash^{\neg} \chi(A)$. Then, since $h : B' \twoheadrightarrow A$ it follows immediately by the fact that the DNA-validity of a formula is preserved by homomorphic images that $B' \nvDash^{\neg} \chi(A)$. Moreover, since $B' \preceq B$ it follows by the preservation of DNA-validity under subalgebra that $B \nvDash^{\neg} \chi(A)$, which proves our claim. □

4.3.2 Axiomatisation of Locally-Finite DNA-logics by Jankov Formulas

Once we have shown that Jankov Theorem holds for our setting, we can use Jankov's machinery to characterize the lattice of subvarieties of locally finite DNA-varieties. We denote by $\Lambda^{\neg}(\mathcal{X})$ the lattice of subvarieties of some DNA-variety $\mathcal{X}$ and we first prove the following useful proposition.

Definition 4.40 (Hereditary FMP)**.** We say that a DNA-variety $\mathcal{X}$ has the *hereditary* DNA-*finite model property* if every DNA-variety $\mathcal{Y} \in \Lambda^{\neg}(\mathcal{X})$ has the finite model property.

As we always do, when the context is clear we drop the prefix DNA and talk simply of the hereditary finite model property.

Proposition 4.41. *If a* DNA-*variety $\mathcal{X}$ is locally finite, then $\mathcal{X}$ has the hereditary finite model property.*

Proof. Suppose that $\mathcal{X}$ is locally finite and consider any subvariety $\mathcal{Y} \in \Lambda^{\neg}(\mathcal{X})$. Since $\mathcal{X}$ is locally finite we have that every DNA-finitely generated $H \in \mathcal{X}$ is also finite and thus since $\mathcal{Y} \subseteq \mathcal{X}$ also that DNA-finitely generated $H \in \mathcal{X}_{RSI}$ is finite. Hence we have that $\mathcal{Y}$ is locally finite and therefore, by Proposition 4.33 above, it follows that $\mathcal{Y}$ also has the finite model property. □

We now prove the following theorem characterizing the sublattice of locally finite DNA-varieties. We denote by $Dw(\mathcal{X}_{RFSI})$ the downsets of $\mathcal{X}_{RFSI}$ under the partial order $\leq$ defined above.

Theorem 4.42. *Let $\mathcal{X}$ be a* DNA-*variety which is locally finite. Then the lattice of negative subvarieties of $\mathcal{X}$ is isomorphic to the lattice of downsets over $\mathcal{X}_{RFSI}$, i.e:*

$$\Lambda^{\neg}(\mathcal{X}) \cong Dw(\mathcal{X}_{RFSI}).$$

Proof. Consider the map $\alpha : \mathcal{Y} \mapsto \mathcal{Y}_{RFSI}$ which sends every subvariety $\mathcal{Y} \subseteq \mathcal{X}$ to its subclass of finite regular subdirectly irreducible elements. We claim that α is welldefined and also it is an isomorphism between $\Lambda^{\neg}(\mathcal{V})$ and $Dw(\mathcal{X}_{RFSI})$. (i) First, we show that $\mathcal{Y}_{RFSI} \in Dw(\mathcal{X}_{RFSI})$. Suppose $B \in \mathcal{Y}_{RFSI}$, $A \in HS(B)$ and $A \in \mathcal{X}_{RFSI}$. As varieties are closed under homomorphic image and subalgebra, we have that $A \in \mathcal{Y}$ and so since $A \in \mathcal{X}_{RFSI}$ also that $A \in \mathcal{Y}_{RFSI}$. (ii) To show injectivity, consider two subvarieties $\mathcal{Y}, \mathcal{W} \in \Lambda(\mathcal{V})$ such that $\mathcal{Y} \neq \mathcal{W}$. By Proposition 4.41 we have that since $\mathcal{X}$ is DNA-locally finite then it has the hereditary finite model property. Therefore, it follows from Theorem 4.32 that every subvariety of $\mathcal{X}$ is generated by its finite regular subdirectly irreducible elements. So we have that

$\mathcal{Y} = \mathcal{Y}_{RFSI}$ and $\mathcal{W} = \mathcal{W}_{RFSI}$ and so it follows that $\mathcal{Y}_{RFSI} \neq \mathcal{W}_{RFSI}$. (iii) For surjectivity, consider any downset $D \in Dw(\mathcal{X}_{RFSI})$. Then this defines a DNA-variety $\mathcal{Y} = \mathcal{X}(D)$. We now claim that $D = \mathcal{Y}_{RFSI}$. For the left-to-right inclusion suppose $A \in D$, then we also have that $A \in \mathcal{X}_{RFSI}$ and $A \in \mathcal{X}(D) = \mathcal{Y}$, which together imply $A \in \mathcal{Y}_{RFSI}$. For the other direction, suppose that $A \in \mathcal{Y}_{RFSI}$, then we have by Lemma 4.38 that $A \nvDash^{\neg} \chi^{\neg}(A)$. Then since $A \in \mathcal{Y} = \mathcal{X}(D)$ it follows that there is some $B \in D$ such that $B \nvDash^{\neg} \chi^{\neg}(A)$. Finally, it follows by the Jankov Theorem for DNA-varieties 4.39 that $A \leq B$ and thus since D is a downset that $A \in D$. $\square$

Moreover, we can also show how one can use Jankov formulas to axiomatise subvarieties of a DNA-variety $\mathcal{X}$ which is locally finite. To this end, we notice that for every proper subvariety $\mathcal{Y} \in \Lambda^{\neg}(\mathcal{X})$ we have that $\mathcal{Y}_{RFSI}$ is a downset and $\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}$ is a nonempty upset over $\mathcal{X}_{RFSI}$. Now, since every algebra in $H \in \mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}$ is finite, we cannot have infinite descending chains of the form $H_0 \geq H_1 \geq H_2 ...$, for $|H_n| \geq |H_{n+1}|$ and $|H_n|$ is finite. It follows that every set of the form $\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}$ has some minimal element. We thus define the following notion of minimal counterexamples of a subvariety of $\mathcal{X}$.

Definition 4.43 (Minimal Counterexample)**.** Let $\mathcal{Y} \in \Lambda^{\neg}(\mathcal{X})$ be a subvariety of $\mathcal{X}$ such that $\mathcal{Y} \neq \mathcal{X}$. A *minimal counterexample* to $\mathcal{Y}$ is a Heyting algebra $H \in \mathcal{X} \setminus \mathcal{Y}$ such that for all $K \leq H$, if $K \ncong H$ then $K \in \mathcal{Y}$.

For every $\mathcal{Y} \in \Lambda^{\neg}(\mathcal{X})$, we denote by $min(\mathcal{X} \setminus \mathcal{Y})$ its collection of minimal counterexamples in $\mathcal{X}$. It follows from our previous reasoning that this collection is always nonempty when $\mathcal{Y}$ is a proper subvariety of $\mathcal{X}$. We prove the following theorem.

Theorem 4.44. *Let $\mathcal{X}$ be a locally finite* DNA*-variety, then for every subvariety $\mathcal{Y} \in \Lambda^{\neg}(\mathcal{X})$ such that $\mathcal{Y} \neq \mathcal{X}$ we have that:*

$$\mathcal{Y} = \mathcal{X}\{H \in \mathcal{X}_{RFSI} : H \vDash^{\neg} \chi(A) \text{ for all } A \in min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})\}.$$

Proof. It suffices to show that $\mathcal{Y}_{RFSI} = \{H \in \mathcal{X}_{RFSI} : H \vDash^{\neg} \chi(A) \text{ for all } A \in min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})\}$. ($\subseteq$) Suppose $H \in \mathcal{Y}_{RFSI}$, then since $\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}$ is a nonempty upset it follows that $min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}) \neq \emptyset$. But then, for all $A \in min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})$ we have that $A \nleq H$. Therefore, it follows by Jankov theorem for DNA-varieties 4.39 that $H \vDash^{\neg} \chi(A)$ and so $H \in \{H \in \mathcal{X}_{RFSI} : H \vDash^{\neg} \chi(A) \text{ for all } A \in min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})\}$. ($\supseteq$) Suppose now that $H \in \{H \in \mathcal{X}_{RFSI} : H \vDash^{\neg} \chi(A) \text{ for all } A \in min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})\}$, then for all $A \in min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})$ it follows that $H \vDash^{\neg} \chi(A)$, hence by Jankov theorem for DNA-varieties 4.39 we have that $A \nleq H$. But then, since $min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})$ is the set of minimal elements in $\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}$, it follows that $H \notin \mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}$ and so since $H \in \mathcal{X}_{RFSI}$ that $H \in \mathcal{Y}_{RFSI}$. $\square$

The previous theorem provides a set of formulas which axiomatise the subvarieties of a locally finite variety. By the dual isomorphism **DNAL** $\cong^{op}$ **DNAV** we can extend the previous result to the corresponding DNA-logics. We say that a DNA-logic Π is an *extension* of a DNA-logic Λ if $\Lambda \subseteq \Pi$. Theorem 4.44 thus immediately allows us to axiomatise the extensions of a logic Λ which is locally finite. We denote by $Var^{\neg}_{RFSI}(\Lambda)$ the collection of finite, regular, subdirectly irreducible elements of the

DNA-variety $Var^{\neg}(\Lambda)$ and by $\Lambda + \Gamma$ the closure under modus ponens of the set of formulas $\Lambda \cup \Gamma$.

Corollary 4.45. *Let Λ be a locally finite* DNA*-logic. Then every* DNA*-logic Π such that $\Lambda \subseteq \Pi$ can be axiomatised as follows:*

$$\Pi = \Lambda + \{\chi(A) : A \in min(Var^{\neg}_{RFSI}(\Lambda) \setminus Var^{\neg}_{RFSI}(\Pi)\}.$$

Proof. Since Λ is locally finite we have that $Var^{\neg}(\Lambda)$ is locally finite. Moreover, since $\Lambda \subseteq \Pi$ it follows by DNA-duality that $Var^{\neg}(\Pi) \subseteq Var^{\neg}(\Lambda)$. Let $K = \{H \in Var^{\neg}_{RFSI}(\Lambda) : H \vDash^{\neg} \chi(A)$ for all $A \in min(Var^{\neg}_{RFSI}(\Lambda) \setminus Var^{\neg}_{RFSI}(\Pi))\}$, then by Theorem 3.30 above it follows that $Var^{\neg}(\Pi) = \mathcal{X}(K)$. Moreover, we have by Theorem 4.21 that $Log^{\neg}(\mathcal{X}(K)) = Log^{\neg}(K)$. By DNA-duality we then have:

$$\Pi = Log^{\neg}(Var^{\neg}(\Pi)) = Log^{\neg}(\mathcal{X}(K)) = Log^{\neg}(K).$$

Hence, since it is easy to see that $Log^{\neg}(K) = \Lambda + \{\chi(A) : A \in min(Var^{\neg}_{RFSI}(\Lambda) \setminus Var^{\neg}_{RFSI}(\Pi)\}$, we finally obtain that $\Pi = \Lambda + \{\chi(A) : A \in min(Var^{\neg}_{RFSI}(\Lambda) \setminus Var^{\neg}_{RFSI}(\Pi)\}$, which proves our claim □

We will apply Corollary 4.45 and the method of Jankov formulas in next chapter to axiomatise the extensions of the system InqB of inquisitive logic.

Chapter 5

Applications to Inquisitive Logic

In this chapter we put to work the general theory of `DNA`-logics that we have developed in the previous chapter. We will show that the system `InqB` of inquisitive logic is a `DNA`-logic and we will use the method of Jankov formulas to characterise its lattice of extensions. In Section 5.1 we introduce inquisitive logic `InqB` in semantic terms, as the set of formulas valid in all evaluation states. Also, we show in this section that `InqB` can be expressed as a `DNA`-logic, by proving that `InqB` is the negative variant of all those intermediate logics L such that $\mathtt{ND} \subseteq L \subseteq \mathtt{ML}$. In Section 5.2 we then use the algebraic semantics of `DNA`-logics to show that `InqB` is locally finite and that can therefore be studied by using the method of Jankov formulas. We thus prove that the sublattice of `DNA`-logics which extend `InqB` is linearly ordered and that it also coincides with the *inquisitive hierarchy* considered by Ciardelli in [10].

5.1 The Inquisitive Logic `InqB`

In this section we introduce inquisitive logic in its standard semantic formulation and we prove `InqB` is the negative variant of all intermediate logics L such that $\mathtt{ND} \subseteq L \subseteq \mathtt{ML}$. This result was already shown by Ciardelli in [10, Thm. 3.4.9]. Our proof that $\mathtt{InqB} = \mathtt{ML}^{\neg}$ is partially similar to Ciardelli's though we do not introduce negative saturated frames. Moreover, to prove that $\mathtt{InqB} = \mathtt{ND}^{\neg}$ we proceed using the approach of [3] and we reason in a more algebraic fashion. Finally, we shall see how from these two results we obtain a syntactical axiomatisation of `InqB` which is well-know in the literature.

5.1.1 Inquisitive Semantics

We recall inquisitive logic and its standard semantic formulation. We refer the reader to Ciardelli's original presentation in [10] and to [13] for more details about inquisitive semantics and its applications in linguistics.

We formulate inquisitive logic `InqB` in the language $\mathcal{L}_P$ introduced in Section 2.2.1. We recall that the set of propositional formulas $\mathcal{L}_P$ is defined inductively as follows:

$$\varphi ::= p \mid \top \mid \bot \mid \psi \wedge \chi \mid \psi \vee \chi \mid \psi \to \chi$$

where p is an arbitrary element of a countable set $\mathtt{AT}$ of atomic propositional formulas. Negation can be defined as $\neg\varphi := \varphi \to \bot$.

We shall see in this section that $\mathtt{InqB}$ is a $\mathtt{DNA}$-logic and admits the algebraic semantics defined in Chapter 3. As we have already mentioned in Remark 3.12, the disjunction of inquisitive logic can be seen as the syntactic counterpart of the join operator of a Heyting algebra. Moreover, since regular elements of a Heyting algebra form a Boolean algebra, one can also introduce a further connective which mirrors the join operator of this Boolean algebra. This fact is often employed in inquisitive logic, where one can work both with an *inquisitive disjunction* and a *classical disjunction*. Here however we will follow [10] and present $\mathtt{InqB}$ in the same language $\mathcal{L}_P$ of intermediate logics.

Inquisitive logic in defined as the logic of all evaluation states. Given a set of atomic formulas in $\mathcal{L}_P$, a *classical valuation* (or simply *valuation*) is a function $w : \mathtt{AT} \to \{0, 1\}$. When the set $\mathtt{AT}$ is fixed, we refer to the set $2^{\mathtt{AT}}$ of all classical valuations over $\mathtt{AT}$ as the *evaluation space* over $\mathtt{AT}$. An *evaluation state* (or simply *state*) is a set s of valuations $s \in \wp(2^{\mathtt{AT}})$. We introduce as follows the notion of support in a state.

Definition 5.1 (Support at a State)**.** Let φ be a formula of $\mathcal{L}_P$ and $s \in \wp(2^{\mathtt{AT}})$ a state. We say that s *supports* φ and we define $s \vDash \varphi$ inductively as follows:

$$\begin{aligned}
s \vDash p &\text{ iff } \forall w \in s(w(p) = 1)\\
s \vDash \top &\text{ iff } s \subseteq 2^{\mathtt{AT}}\\
s \vDash \bot &\text{ iff } s = \emptyset\\
s \vDash \psi \wedge \chi &\text{ iff } s \vDash \psi \text{ and } s \vDash \chi\\
s \vDash \psi \vee \chi &\text{ iff } s \vDash \psi \text{ or } s \vDash \chi\\
s \vDash \psi \to \chi &\text{ iff } \forall t (\text{ if } t \subseteq s \text{ and } t \vDash \psi \text{ then } t \vDash \chi).
\end{aligned}$$

If $p \in \mathtt{AT}$ is an atomic formula and s a state, we denote by $[\![p]\!]^s$ the set $[\![p]\!]^s = \{w \in s : w(p) = 1\}$ of classical valuations in s that make p true. Since $\neg\varphi = \varphi \to \bot$, the semantic clause of negation is then the following:

$$s \vDash \neg\varphi \text{ iff } \forall t (\text{ if } t \subseteq s \text{ then } t \nvDash \varphi).$$

The system of inquisitive logic $\mathtt{InqB}$ is then defined semantically as follows.

Definition 5.2 (Inquisitive Logic)**.** The valid formulas of *inquisitive logic* $\mathtt{InqB}$ are the formulas $\varphi \in \mathcal{L}_P$ which are supported in every evaluation state:

$$\mathtt{InqB} = \{\varphi \in \mathcal{L}_P : \forall s \in \wp(2^{\mathtt{AT}}), s \vDash \varphi\}.$$

Inquisitive logic can thus be seen as the logic of all evaluation states.

5.1.2 $\mathtt{InqB} = \mathtt{ML}^{\neg}$

We have introduced the system $\mathtt{InqB}$ of inquisitive logic in semantic terms, as the set of propositional formulas that are supported at all evaluation states. Now we want to relate the system $\mathtt{InqB}$ to the setting of $\mathtt{DNA}$-logics that we have studied in the previous chapter of this thesis. We show in this section that `InqB` is a `DNA`-logic by proving that it is the negative variant of Medvedev logic `ML`. This fact was originally shown in [10, pp. 45-46] in a slightly different way, by using so-called negative saturated models.

Let us recall that `ML` is the logic of so-called Medvedev frames and it was introduced by Medvedev in [40]. A relational structure $\mathcal{F}$ is a *Medvedev frame* if $\mathcal{F} \cong (\wp_0(W), \supseteq)$, where W is a finite set and $\wp_0(W) = \{X \subseteq W : X \neq \emptyset\}$. A *Medvedev model* is then defined as a relational model over a Medvedev frame. Let $\mathcal{C}$ be the class of all Medvedev frames, then we have that $\mathtt{ML} = \{\varphi \in \mathcal{L}_P : \mathcal{C} \Vdash \varphi\}$, i.e. `ML` is the set of formulas valid in all Medvedev frames. We now prove two propositions relating state-structures and Medvedev models. On the one hand, we can associate in the following way a finite state structure to a corresponding Medvedev model. We proceed as follows.

Proposition 5.3. *Let $s \in \wp(2^{\mathtt{AT}})$ be a finite non-empty state and $(\wp_0(s), \supseteq, V^{\neg})$ a Medvedev model where $V^{\neg} : \mathtt{AT} \to \wp_0(s)$ is such that $V^{\neg} : p \mapsto (\wp_0(s) \setminus \wp_0([\![p]\!]^s))$, then for any formula $\varphi \in \mathcal{L}_P$ we have that:*

$$s \vDash \varphi \Leftrightarrow (s, V^{\neg}) \Vdash \varphi[{}^{\neg p}/_{p}].$$

Proof. By induction on the complexity of φ. The cases for $\top$ and $\bot$ are trivial and we omit them.

(i) If $\varphi = p$ with $p \in \mathtt{AT}$, then:

$$\begin{aligned} s \vDash p &\Leftrightarrow s \subseteq [\![p]\!]^s \\ &\Leftrightarrow \forall t \subseteq s (t \subseteq [\![p]\!]^s) \\ &\Leftrightarrow \forall t \subseteq s (t \notin V^{\neg}(p)) \\ &\Leftrightarrow \forall t \subseteq s (t \nvDash p) \\ &\Leftrightarrow (s, V^{\neg}) \Vdash \neg p. \end{aligned}$$

(ii) If $\varphi = \chi \wedge \sigma$, then:

$$\begin{aligned} s \vDash \chi \wedge \psi &\Leftrightarrow s \vDash \chi \text{ and } s \vDash \sigma \\ &\Leftrightarrow (s, V) \Vdash \chi[{}^{\neg p}/_{p}] \text{ and } (s, V) \Vdash \sigma[{}^{\neg p}/_{p}] \\ &\Leftrightarrow (s, V) \Vdash \chi[{}^{\neg p}/_{p}] \wedge \sigma[{}^{\neg p}/_{p}] \\ &\Leftrightarrow (s, V) \Vdash (\chi \wedge \sigma)[{}^{\neg p}/_{p}]. \end{aligned}$$

(iii) If $\varphi = \chi \vee \sigma$, then:

$$\begin{aligned} s \vDash \chi \vee \psi &\Leftrightarrow s \vDash \chi \text{ or } s \vDash \sigma \\ &\Leftrightarrow (s, V) \Vdash \chi[{}^{\neg p}/_{p}] \text{ or } (s, V) \Vdash \sigma[{}^{\neg p}/_{p}] \\ &\Leftrightarrow (s, V) \Vdash \chi[{}^{\neg p}/_{p}] \vee \sigma[{}^{\neg p}/_{p}] \\ &\Leftrightarrow (s, V) \Vdash (\chi \vee \sigma)[{}^{\neg p}/_{p}]. \end{aligned}$$

(iv) If $\varphi = \chi \to \sigma$, then:

$$\begin{aligned}
s \vDash \chi \to \sigma &\Leftrightarrow \forall t \subseteq s (t \vDash \chi \Rightarrow t \vDash \sigma) \\
&\Leftrightarrow \forall t (t \subseteq s, t \Vdash \chi[\overline{\neg p}/\overline{p}] \Rightarrow t \Vdash \sigma[\overline{\neg p}/\overline{p}]) \\
&\Leftrightarrow (s, V) \Vdash \chi[\overline{\neg p}/\overline{p}] \to \sigma[\overline{\neg p}/\overline{p}] \\
&\Leftrightarrow (s, V) \Vdash (\chi \to \sigma)[\overline{\neg p}/\overline{p}].
\end{aligned}$$

And this establishes our claim. □

Conversely, we can also relate a Medvedev model $\mathcal{M}$ to a corresponding state structure. Suppose $\mathcal{M} = (\wp_0(W), \supseteq, V)$ is a Medvedev model, then for every singleton $\{x\} \in \wp_0(W)$ we define a corresponding classical valuation $w_x : \mathtt{AT} \to 2$ such that $w_x(p) = 0 \Leftrightarrow \{x\} \Vdash p$. Then we say that X_s is the *negative Medvedev state* of s if $X_s = \{w_x \in 2^{\mathtt{AT}} : \{x\} \in s\}$.

Proposition 5.4. *Let $\mathcal{M} = (\wp_0(W), \supseteq, V)$ be a Medvedev model, then for any formula $\varphi \in \mathcal{L}_P$ and any $s \in \wp_0(W)$ we have that:*

$$X_s \vDash \varphi \Leftrightarrow (s, V) \Vdash \varphi[\overline{\neg p}/\overline{p}].$$

Proof. By induction on the complexity of φ. The cases for $\top$ and $\bot$ are trivial and we omit them.

(i) If $\varphi = p$ with $p \in \mathtt{AT}$, then:

$$\begin{aligned}
X_s \vDash p &\Leftrightarrow X_s \subseteq [\![p]\!]^{X_s} \\
&\Leftrightarrow \forall w_x \in X_s (w_x(p) = 1) \\
&\Leftrightarrow \forall x \in s (\{x\} \nVdash p) \\
&\Leftrightarrow \forall t \subseteq W (t \nVdash p) \\
&\Leftrightarrow (s, V^{\neg}) \Vdash \neg p.
\end{aligned}$$

(ii) If $\varphi = \chi \wedge \sigma$, then:

$$\begin{aligned}
X_s \vDash \chi \wedge \sigma &\Leftrightarrow X_s \vDash \chi \text{ and } X_s \vDash \sigma \\
&\Leftrightarrow (s, V) \Vdash \chi[\overline{\neg p}/\overline{p}] \text{ and } (s, V) \Vdash \sigma[\overline{\neg p}/\overline{p}] \\
&\Leftrightarrow (s, V) \Vdash \chi[\overline{\neg p}/\overline{p}] \wedge \sigma[\overline{\neg p}/\overline{p}] \\
&\Leftrightarrow (s, V) \Vdash (\chi \wedge \sigma)[\overline{\neg p}/\overline{p}].
\end{aligned}$$

(iii) If $\varphi = \chi \vee \sigma$, then:

$$\begin{aligned}
X_s \vDash \chi \vee \psi &\Leftrightarrow X_s \vDash \chi \text{ or } X_s \vDash \sigma \\
&\Leftrightarrow (s, V) \Vdash \chi[\overline{\neg p}/\overline{p}] \text{ or } (s, V) \Vdash \sigma[\overline{\neg p}/\overline{p}] \\
&\Leftrightarrow (s, V) \Vdash \chi[\overline{\neg p}/\overline{p}] \vee \sigma[\overline{\neg p}/\overline{p}] \\
&\Leftrightarrow (s, V) \Vdash (\chi \vee \sigma)[\overline{\neg p}/\overline{p}].
\end{aligned}$$

(iv) If $\varphi = \chi \to \sigma$, then:

$$\begin{aligned} X_s \vDash \chi \to \sigma &\Leftrightarrow \forall X_t \subseteq X_s(X_t \vDash \chi \Rightarrow X_t \vDash \sigma) \\ &\Leftrightarrow \forall t(t \subseteq s, t \Vdash \chi[\neg p/\overline{p}] \Rightarrow t \Vdash \sigma[\neg p/\overline{p}]) \\ &\Leftrightarrow (s, V) \Vdash \chi[\neg p/\overline{p}] \to \sigma[\neg p/\overline{p}] \\ &\Leftrightarrow (s, V) \Vdash (\chi \to \sigma)[\neg p/\overline{p}]. \end{aligned}$$

And this establishes our claim. □

The following proposition establishes a version of the finite model property for `InqB`. We will need this result in the proof that `InqB` is the negative variant of `ML`.

Proposition 5.5 (Finite Model Property of `InqB`)**.** *Suppose $\varphi \notin$ `InqB`. Then there is a finite state $s \in \wp(2^{\mathtt{AT}})$ such that $s \nvDash \varphi$.*

Proof. Suppose $\varphi \notin$ `InqB` then it follows that there is some evaluation state s such that $s \nvDash \varphi$. We now show by induction on the complexity of φ how we can obtain a finite state $s' \subseteq s$ which also refutes φ. The cases for $\top$ and $\bot$ are trivial and we omit them.

(i) If $s \nvDash p$ then there is some $w \in s$ such that $w(p) = 0$. Consider $s' = \{w\}$, then we clearly have that s' is finite and $s' \nvDash p$.

(ii) If $s \nvDash \psi \wedge \chi$ then $s \nvDash \psi$ or $s \nvDash \chi$. It follows by induction hypothesis that for some finite evaluation state $s' \subseteq s$ we have $s' \nvDash \psi$ or $s' \nvDash \chi$ and therefore $s' \nvDash \psi \wedge \chi$.

(iii) If $s \nvDash \psi \vee \chi$ then $s \nvDash \psi$ and $s \nvDash \chi$. It follows by induction hypothesis that for some finite evaluation state $s', t' \subseteq s$ we have $s' \nvDash \psi$ and $t' \nvDash \chi$ and therefore $s' \cup t' \nvDash \psi \vee \chi$.

(iv) If $s \nvDash \psi \to \chi$ then for some $t \subseteq s$ we have that $t \vDash \psi$ and $s \nvDash \chi$. It follows by induction hypothesis that for some finite evaluation state $t' \subseteq t \subseteq t$ we have $t' \vDash \psi$ and $t' \nvDash \chi$ and therefore $t' \nvDash \psi \to \chi$.

And this establishes our claim. □

Finally, the following theorem establishes that `InqB` is the negative variant of `ML` and shows therefore that `InqB` is a `DNA`-logic.

Theorem 5.6. *`InqB` $=$ `ML`$^\neg$.*

Proof. ($\subseteq$) Suppose $\varphi \notin$ `ML`$^\neg$, then by the definition of negative variant $\varphi[\neg p/\overline{p}] \notin$ `ML`. Then, since `ML` is the logic of Medvedev frames, it follows that for some Medvedev model $(\wp_0(W), \supseteq, V)$ there is some $s \in \wp_0(W)$ such that $(s, V) \nVdash \varphi[\neg p/\overline{p}]$. Therefore, it follows by Proposition 5.4 that for the corresponding negative Medvedev state we have $(X_s, V) \nvDash \varphi$ and so since `InqB` is the logic of all evaluation states, we obtain that $\varphi \notin$ `InqB`.

($\supseteq$) Suppose $\varphi \notin$ `InqB`, then since `InqB` is the logic of all evaluation states there is an s such that $s \nvDash \varphi$. Moreover, by Proposition 5.5 we can assume without loss of generality that s is finite. Then by Proposition 5.3 it follows that for the Medvedev Model $(\wp_0(s), \supseteq, V^\neg)$ we have that $(s, V^\neg) \nVdash \varphi[\overline{\neg p}/\overline{p}]$. Therefore, it follows that $\varphi[\overline{\neg p}/\overline{p}] \notin \mathtt{ML}$ and so that $\varphi \notin \mathtt{ML}^\neg$. $\square$

We have thus shown that inquisitive logic InqB is a DNA-logic. This will later allow us to employ the general theory of DNA-logics that we have developed in the previous chapters to study inquisitive logic. Now we want to strengthen this result and show that ML is exactly the schematic fragment of InqB. This was also originally proved in [10] in a similar way, but employing saturated models.

Proposition 5.7. $\mathtt{ML} = Schm(\mathtt{InqB})$.

Proof. ($\subseteq$) It follows immediately from Proposition 4.7 and the fact that $\mathtt{InqB} = \mathtt{ML}^\neg$. ($\supseteq$) Suppose now $\varphi \notin \mathtt{ML}$, then for some Medvedev model $(\wp_0(W), \supseteq, V)$ and some $s \in \wp_0(W)$ we have $(s, V) \nVdash \varphi$. Now, let $p_0, ..., p_n$ be the variables contained in φ and notice that by the definition of Medvedev frames we have that for any p_i we have $V(p_i) \in \wp_0(W)$. Notice then that we can consider every $V(p_i)$ as a union of singletons, namely $V(p_i) = \bigcup\{\{x_i^k\} \in \wp_0(W) : x_i^k \in V(p_i)\}$. Now for each of these singletons we introduce a new variable q_i^k and we define a valuation $U : \mathtt{AT} \to (\wp_0(W), \supseteq)$ such that $U : q_i^k \mapsto \{\{y\} \in \wp_0(W) : \{y\} \neq \{x_i^k\}\}$. Namely U sends q_i^k to the set containing all singletons over W besides for $\{x_i^k\}$. It is then easy to check that $[\![\neg q_i^k]\!]^{(\wp_0(W),\supseteq,U)} = \{x_i^k\}$ and therefore that:

$$
\begin{aligned}
V(p_i) =& \bigcup\{\{x_i^k\} \in \wp_0(W) : x_i^k \in V(p_i)\} \\
=& \bigcup\{[\![\neg q_i^k]\!]^{(\wp_0(W),\supseteq,U)} \in \wp_0(W) : x_i^k \in V(p_i)\} \\
=& \bigvee_{k\leq n} \neg U(q_i^k).
\end{aligned}
$$

Therefore we have:

$$
[\![\varphi(p_0, ..., p_n)]\!]^{(\wp_0(W),\supseteq,V)} = [\![\varphi(\bigvee_{k\leq n} \neg q_0^k, ..., \bigvee_{k\leq n} \neg q_n^k)]\!]^{(\wp_0(W),\supseteq,U)}
$$

and so since $(s, V) \nVdash \varphi$, it follows that $(s, U) \nVdash \varphi(\bigvee_{k\leq n} \neg q_0^k, ..., \bigvee_{k\leq n} \neg q_n^k)$, which by Proposition 5.4 entails $X_s \nvDash \varphi(\bigvee_{k\leq n} q_0^k, ..., \bigvee_{k\leq n} q_n^k)$. Now, since X_s is an evaluation state, it follows that $\varphi(\bigvee_{k\leq n} q_0^k, ..., \bigvee_{k\leq n} q_n^k) \notin \mathtt{InqB}$ and since $\varphi(\bigvee_{k\leq n} q_0^k, ..., \bigvee_{k\leq n} q_n^k)$ is a substitution instance of φ also that $\varphi \notin Schm(\mathtt{InqB})$, which finally proves our claim. $\square$

By Proposition 4.15 it then follows that ML is DNA-maximal, namely that it is the maximal intermediate logic to have InqB as its negative variant. In the next section we identify also the least element in the lattice $\mathcal{I}(\mathtt{InqB})$.

5.1.3 $\mathtt{InqB} = \mathtt{ND}^\neg$

In this section, we show that InqB is the negative variant of the intermediate logic ND and we also prove that ND is DNA-minimal, i.e. that it is the least intermediate logic whose negative variant is InqB. Together with the previous results we obtain a characterization of the sublattice $\mathcal{I}(\mathtt{InqB})$ of intermediate logics whose negative variant is InqB. These results were originally proved by Ciardelli in [10, pp. 46-48]. Our proof however is different and adapts [3], which shows similar results for the logic KP.

Firstly, let us recall that ND is the intermediate logic that contains, for all $n \in \mathbb{N}$, the following axiom:

$$\mathtt{ND}_n \ = (\neg p \to \bigvee_{i\leq n} \neg q_i) \to \bigvee_{i\leq n} (\neg p \to \neg q_i).$$

Whilst the converse of this axiom holds already in IPC. We now prove the following result about the intermediate logics ND and ML.

Proposition 5.8. $\mathtt{ND} \subseteq \mathtt{ML}$

Proof. Suppose this is not the case, then for some Medvedev model $(\wp_0(W), \supseteq, V)$ we have that $(\wp_0(W), \supseteq, V) \nVdash \mathtt{ND}$, which means that for some $n \in \mathbb{N}$ there is some $X \subseteq W$ such that:

$$(1) \quad X \Vdash (\neg p \to \bigvee_{i\leq n} \neg q_i)$$
$$(2) \quad X \nVdash \bigvee_{i\leq n} (\neg p \to \neg q_i).$$

By (1), it follows that for all $Y \subseteq X$, $Y \Vdash \neg p$ entails $Y \Vdash \bigvee_{i\leq n} \neg q_i$. By (2), it follows that for all $i \leq n$ there is some $Z \subseteq X$ such that $Z \nVdash \neg p \to \neg q_i$. Therefore, there is some $K_i \subseteq Z$ such that $K_i \subseteq W \setminus V(p)$ and $K_i \cap V(q_i) \neq \emptyset$. Consider the union $K = \bigcup_{j\leq n} K_i$, then we have $K \subseteq W \setminus V(p)$ and for all $j \leq n$, $K \cap V(q_j) \neq \emptyset$. Then, since $K \subseteq X$ and $K \Vdash \neg p$ we have $K \Vdash \bigvee_{i\leq n} \neg q_i$ which yields that for some $l \leq n$, $K \Vdash \neg q_l$ and so $K \cap V(q_l) = \emptyset$. But this contradicts our former claim that for all $j \leq n$, $K_i \cap V(q_j) \neq \emptyset$. Finally, this means that every Medvedev frame satisfies ND and thus $\mathtt{ND} \subseteq \mathtt{ML}$. ☐

To show that $\mathtt{ND}^{\neg} = \mathtt{InqB}$ we proceed by giving a characterization of the regularly generated ND-Heyting algebras. The following results are proved in [3] for the logic KP. We show here that they hold already for ND.

ND-*extension* of a Boolean Algebra

We introduce the ND-*extension* of a Boolean algebra as done in [3]. Let B be any Boolean algebra and consider the term algebra $T(B)$ over the signature $(\dot\wedge, \dot\vee, \dot\to, \dot 1, \dot 0)$. The algebra $T(B)$ thus consists of all propositional formulas built in this signature from the set of atomic letters $\mathtt{AT} = B$:

$$T(B) = \{\varphi(b_0, ..., b_n) : b_i \in B \text{ and } \varphi \text{ is a formula in } (\dot\wedge, \dot\vee, \dot\to, \dot 1, \dot 0)\}.$$

Then since $T(B)$ is a term algebra, we have that its algebraic operations are exactly the signature operations, i.e. we have that $\varphi \wedge_{T(B)} \psi = \varphi \dot\wedge \psi$ etc. We now quotient the term algebra $T(B)$ to obtain an ND-algebra. We define the congruence $\equiv^e_{\mathtt{ND}}$.

Definition 5.9. Let B be an arbitrary Boolean algebra, then the congruence $\equiv^e_{\mathtt{ND}}$ is the least congruence containing $\equiv_{\mathtt{ND}}$ and such that for all $p, q \in B$ we have that:

$$1_B \equiv^e_{\mathtt{ND}} \dot{1}$$
$$0_B \equiv^e_{\mathtt{ND}} \dot{0}$$
$$p \wedge_B q \equiv^e_{\mathtt{ND}} p \dot{\wedge} q$$
$$p \rightarrow_B q \equiv^e_{\mathtt{ND}} p \dot{\rightarrow} q.$$

That $\equiv^e_{\mathtt{ND}}$ is a congruence follows immediately, since $\equiv_{\mathtt{ND}}$ is a congruence and since the extra-clauses we added are compatible with the operations of B. The ND-*extension* $H^{\mathtt{ND}}(B)$ of B is then defined as the quotient algebra $T(B)/\equiv^e_{\mathtt{ND}}$. Hereafter we will drop the apex and denote the ND-*extension* of B just by $H(B)$. Now notice that since $\equiv_{\mathtt{IPC}} \subseteq \equiv^e_{\mathtt{ND}}$ we have that $H(B)$ validates all the validities of IPC and is thus a Heyting algebra. We then prove the following universal mapping property.

Proposition 5.10 (Universal Mapping Property)**.** *Let B be a Boolean algebra and $H(B)$ its ND-extension, then for every Heyting algebra K such that $K \vDash \mathtt{ND}$ and $K_\neg = B$ there is a unique homomorphism $h : H(B) \rightarrow K$ such that $h \upharpoonright B = id_B$. Moreover, if K is regular then h is also surjective.*

Proof. Let $id_B : B \rightarrow K_\neg$ be the identity map, then we define its extension $h : H(B) \twoheadrightarrow K$ as follows. Let $x \in H(B)$, then it follows by the definition of *ND-extension* that every element of $H(B)$ is an equivalence class of terms in $T(B)$ under $\equiv^e_{\mathtt{ND}}$. So we have that $x = [\delta(a_0, ..., a_n)]$ where for all $i \leq n$, $a_i \in B$. Define for all $[\delta(a_0, ..., a_n)] \in H(B)$ the map $h : [\delta(a_0, ..., a_n)] \mapsto \delta_K(a_0, ..., a_n)$. Since both $H(B)$ and K are ND-algebras, it follows that this map is well defined. Moreover, we have by the property of congruences that $h(\delta([\psi_0], ..., [\psi_n])) = h([\delta(\psi_0, ..., \psi_n)]) = \delta_K(\psi_0, ..., \psi_n)$, implying that h is a homomorphism. Uniqueness follows immediately by the fact that every element in $H(B)$ is a polynomial over B.

Finally, we show that h is also surjective in case K is regular. Since K is regular, every element $x \in K$ can be written as a polynomial over elements of $K_\neg$, i.e. $x = \delta_K(y_0, ..., y_n)$ with every $y_0, ..., y_n \in K_\neg$. Then, since $K_\neg = B$ and by the fact that h is a homomorphism, it follows that $h[\delta(y_0, ..., y_n)] = \delta_K[h(y_0), ..., h(y_n)]$ and therefore, as $h \upharpoonright B = id_B$, we obtain that $\delta_K[h(y_0), ..., h(y_n)] = \delta_K(y_0, ..., y_n)$. ☐

The following two propositions give us a description of the structure of the *ND-extension* $H(B)$ of a Boolean algebra B. In particular, we show that every element of $H(B)$ can be written in a unique way as a disjunction of elements of B. Following [3] we say that every $x \in H(B)$ has a *non-redundant representation.* With a slight abuse of notation we henceforth drop the square brackets and refer to elements of $H(B)$ as formulas rather than equivalence classes thereof. Also, since the algebra operations of $H(B)$ agree with the connectives in $(\dot{\wedge}, \dot{\vee}, \dot{\rightarrow}, \dot{1}, \dot{0})$, we drop the dots and use the same symbols both for connectives and operations.

Proposition 5.11. *For every $x \in H(B)$ we have that $x = a_1 \vee ... \vee a_n$ where $a_1, ..., a_n \in B$ and $a_i \nleq a_j$ for $i \neq j$.*

Proof. First, we show that every $x \in H(B)$ can be expressed as a disjunction $x = a_1 \vee ... \vee a_i$ with $a_1, ..., a_i \in B$. Since $H(B) = T(B)/\equiv^e_{\mathtt{ND}}$ we proceed by induction on the complexity of formulas in $T(B)$.

(i) If $x = a \in B$, then x is already in disjunctive form.

(ii) If $x = p \wedge q$, then by induction $p = \bigvee_{i \leq n} a_i$ and $q = \bigvee_{j \leq m} b_j$, therefore by the distributivity law of Heyting algebras:

$$x = p \wedge q = \bigvee_{i \leq n} a_i \wedge \bigvee_{j \leq m} b_j = \bigvee_{i \leq n} \bigvee_{j \leq m} (a_i \wedge b_j).$$

Where for every i, j we have that $a_i, b_j \in B$ and so that $a_i \wedge b_j \in B$.

(iii) If $x = p \vee q$, then by induction $p = \bigvee_{i \leq n} a_i$ and $q = \bigvee_{j \leq m} b_j$ and therefore $x = p \vee q = \bigvee_{i \leq n} a_i \vee \bigvee_{j \leq m} b_j$.

(iv) If $x = p \rightarrow q$, then by induction $p = \bigvee_{i \leq n} a_i$ and $q = \bigvee_{j \leq m} b_j$, hence we have:

$$\begin{aligned}
x = p \rightarrow q &= \bigvee_{i \leq n} a_i \rightarrow \bigvee_{j \leq m} b_j \\
&= \bigwedge_{i \leq n} [a_i \rightarrow \bigvee_{j \leq m} b_j] \\
&= \bigwedge_{i \leq n} [\neg\neg a_i \rightarrow \bigvee_{j \leq m} \neg\neg b_j] && \text{(by } a_i, b_j \in B\text{)} \\
&= \bigwedge_{i \leq n} [\bigvee_{j \leq m} (\neg\neg a_i \rightarrow \neg\neg b_j)] && \text{(by } \mathtt{ND}\text{)} \\
&= \bigwedge_{i \leq n} [\bigvee_{j \leq m} (a_i \rightarrow b_j)] && \text{(by } a_i, b_j \in B\text{)} \\
&= \bigvee_{f:[n] \rightarrow [m]} [\bigwedge_{i \leq n} (a_i \rightarrow b_{f(i)})].
\end{aligned}$$

Where for every i, j we have that $a_i, b_j \in B$. So since B agrees with H with respect to conjunction and implication, it follows each $\bigwedge_{i \leq n}(a_i \rightarrow b_j) \in B$. Therefore every $x \in H(B)$ has a disjunctive representation $x = \bigvee_{i \leq n} a_i$ with $a_i \in B$ for all $i \leq n$. Now let $A = \{a_0, ..., a_n\}$, then to obtain a non-redundant representation of x it suffices to take the set $I = \{m \leq n : a_m \text{ is minimal in } A\}$. Then clearly $x = \bigvee_{i \leq n} a_i = \bigvee_{i \in I} a_i$ and by construction $a_i \nleq a_j$ for $i, j \in I$ such that $i \neq j$. Therefore every $x \in H(B)$ has also a non-redundant disjunctive representation. $\square$

Downsets Algebras

We introduce Heyting algebras of finitely generated downsets over Boolean algebras and we show that every $H(B)$ is isomorphic to the Heyting algebra of finitely generated downsets of B. Recall that a downset D over a poset $(P, \leq)$ is *finitely generated* if there is a finite set of elements $x_0, ..., x_n$ such that $D = \downarrow \{x_0, ..., x_n\}$. We say that $\{x_0, ..., x_n\}$ is the set of *generators* of D. A downset D over a poset $(P, \leq)$ is *principal* if it is generated by a singleton and we write $D = \downarrow \{x\}$ or just $D = \downarrow x$. Now let B be an arbitrary Boolean algebra, then we define:

$$Dw_{fg}(B) = \{X \subseteq B : X \text{ is a finitely generated downset of } B\}.$$

It is easy to see that $(Dw_{fg}(B), \cap, \cup)$ is a complete lattice which satisfies the infinite distributivity law $a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I}(a \wedge b_i)$. So by Proposition 2.35 it follows that $(Dw_{fg}(B), \cap, \cup)$ is also a Heyting algebra. We now want to prove that $(Dw_{fg}(B)$ is also an `ND`-algebra. We first show the following lemma.

Lemma 5.12. *For every $D \in Dw_{fg}(B)$ we have that $D = \downarrow \{a_1, ..., a_n\}$ with $a_i \not\leq a_j$ for $i \neq j$.*

Proof. For any $D \in Dw_{fg}(B)$ let D_m be its subset of maximal points. Then D_m is clearly finite and $D = \downarrow D_m$. Moreover, it follows immediately by the definition of maximal points that for any $a_i, a_j \in D_m$ we have that $a_i \not\leq a_j$. □

The following theorem provides a characterisation of the `ND`-*extension* $H(B)$ of a Boolean algebra B in terms of the algebra of finitely generated downsets of B.

Theorem 5.13. *Let B be a Boolean algebra, then $H(B) \cong Dw_{fg}(B)$.*

Proof. We define a function $f : H(B) \to Dw_{fg}(B)$ and we show it is an isomorphism. Consider any element $x \in H(B)$, then by Proposition 5.11 it has a unique nonredundant disjunctive representation such that $x = \bigvee_{i \leq n} a_i$. Then we define $f : H(B) \to Dw_{fg}(B)$ by fixing $f : (\bigvee_{i \leq n} a_i) \mapsto \downarrow \{a_1, ..., a_n\}$. Now suppose $x \neq y$. Putting them in disjunctive form: $x = \bigvee_{i \leq n} a_i$ and $y = \bigvee_{j \leq m} b_j$, so $f(x) = \downarrow \{a_1, ..., a_n\}$ and $f(y) = \downarrow \{b_1, ..., b_m\}$. But then we clearly have $\{a_1, ..., a_n\} \neq \{b_1, ..., b_m\}$ and so by the uniqueness of the representation of Proposition 5.11 $\downarrow \{a_1, ..., a_n\} \neq \downarrow \{b_1, ..., b_n\}$ which shows that f is injective. Now let $D \in Dw_{fg}(B)$, then by Lemma 5.12, $D = \downarrow \{a_1, ..., a_n\} = f(\bigvee_{i \leq n} a_i)$ so that f is also surjective.

Finally, we show that f is also an homomorphism. Let $p, q \in H(B)$ and suppose without loss of generality $p = \bigvee_{i \leq n} a_i$ and $q = \bigvee_{j \leq m} b_j$. Then notice that by inspecting the proof of Proposition 5.11 we have the following translation in disjunctive normal form:

$$\begin{aligned} p \wedge q &= \bigvee_{i \leq n} \bigvee_{j \leq m} (a_i \wedge b_j) \\ p \vee q &= \bigvee_{i \leq n} a_i \vee \bigvee_{j \leq m} b_j \\ p \to q &= \bigvee_{f:[n] \to [m]} [\bigwedge_{i \leq n} (a_i \to b_{f(i)})]. \end{aligned}$$

It is important to notice that the disjunctive forms above are not necessarily non-redundant. However, it follows immediately by the construction in Proposition 5.11 that the non-redundant representation consists in the disjunction of the minimal elements among the disjuncts in the formulas above. It thus follows that the disjuncts of the non-redundant representation and those of the disjunctive forms above determine the same downsets. We then prove that f is a homomorphism as follows:

$$\begin{aligned} f(p \wedge q) &= f(\bigvee_{i \leq n} \bigvee_{j \leq m} (a_i \wedge b_j)) \\ &= \downarrow \{a_i \wedge b_j \in B : i \leq n, j \leq m\} \end{aligned}$$

$$
\begin{aligned}
&= \downarrow \{a_i \in B : i \leq n\} \cap \downarrow \{b_j \in B : j \leq m\} \\
&= f(p) \wedge f(q);
\end{aligned}
$$

$$
\begin{aligned}
f(p \vee q) &= f(\bigvee_{i \leq n} a_i \vee \bigvee_{j \leq m} b_j) \\
&= \downarrow [\{a_i \in B : i \leq n\} \cup \downarrow \{b_j \in B : j \leq m\}] \\
&= \downarrow \{a_i \in B : i \leq n\} \cup \downarrow \{b_j \in B : j \leq m\} \\
&= f(p) \vee f(q);
\end{aligned}
$$

$$
\begin{aligned}
f(p \to q) &= f(\bigvee_{f:[n]\to[m]} [\bigwedge_{i \leq n} (a_i \to b_{f(i)})]) \\
&= \downarrow \{a_i \to b_{f(i)} \in B : i \leq n \text{ and } f : [n] \to [m]\} \\
&= \downarrow \{x \in B : x \wedge a_i \leq b_i \text{ for all } i \leq n \text{ and } f : [n] \to [m]\} \\
&= (\downarrow \{a_i \in B : i \leq n\}) \to (\downarrow \{b_j \in B : j \leq m\}) \\
&= f(p) \to f(q).
\end{aligned}
$$

And so we have that $H(B) \cong Dw_{fg}(B)$. □

It follows by the previous theorem that every $H(B)$ is *well-connected*, i.e. that for any $x, y \in H(B)$ it is the case that $x \vee y = 1$ entails $x = 1$ or $y = 1$.

Corollary 5.14. *For any Boolean algebra B, its* `ND`*-extension $H(B)$ is well-connected.*

Proof. By the previous Theorem 5.13 we have that $H(B) \cong Dw_{fg}(B)$, so it suffices to show that $Dw_{fg}(B)$ is well-connected. Suppose by contraposition that D_x and D_y are two finitely generated downset such that $\downarrow 1_B \neq D_x$ and $\downarrow 1_B \neq D_y$. Then it follows that $1_b \notin D_x$ and $1_b \notin D_y$, so $1_B \notin D_x \cup D_y = D_x \vee D_y$ and thus $D_x \vee D_y \neq \downarrow 1_B$, which proves our claim. □

Finally, we show here also a connection between Medvedev frames and downset algebras that we will use consequently. Let $\mathcal{F} = (\wp_0(W), \supseteq)$ be a Medvedev frames and $Dw(\mathcal{F})$ the set of downsets in $(\wp_0(W), \supseteq)$. That this is a Heyting algebra follows immediately from the fact that $(Dw(\mathcal{F}), \subseteq)$ is a finite bounded lattice which satisfies the distributivity law. Now notice that for every $p \in$ `AT` we have that a valuation over a Medvedev frame is such that $V(p) \in Up(\wp_0(W), \supseteq) = Dw(\mathcal{F})$ so that V is a valuation over $(\wp_0(W), \supseteq)$ iff it is a valuation over $Dw(\mathcal{F})$. Let $\mathcal{F} = (\wp_0(W), \supseteq)$ be a Medvedev frame and $Dw(\mathcal{F})$ its corresponding algebra of downsets. We prove the following lemma.

Lemma 5.15. *For all $\varphi \in \mathcal{L}_P$, $[\![\varphi]\!]^{(\mathcal{F},V)} = [\![\varphi]\!]^{(Dw(\mathcal{F}),V)}$.*

Proof. By induction on the complexity of formulas. The cases for $\varphi = \top$ and $\varphi = \bot$ are trivial and we omit them.

(i) For $p \in$ `AT` we have that $[\![p]\!]^{(\mathcal{F},V)} = V(p) = [\![p]\!]^{(Dw(\mathcal{F}),V)}$.

(ii) For $\varphi = \psi \wedge \chi$ we have:

$$\begin{aligned}
[\![\psi \wedge \chi]\!]^{(\mathcal{F},V)} &= [\![\psi]\!]^{(\mathcal{F},V)} \cap [\![\chi]\!]^{(\mathcal{F},V)} \\
&= [\![\psi]\!]^{(Dw(\mathcal{F}),V)} \cap [\![\chi]\!]^{(Dw(\mathcal{F}),V)} \\
&= [\![\psi \wedge \chi]\!]^{(Dw(\mathcal{F}),V)}.
\end{aligned}$$

(iii) For $\varphi = \psi \vee \chi$ we have:

$$\begin{aligned}
[\![\psi \vee \chi]\!]^{(\mathcal{F},V)} &= [\![\psi]\!]^{(\mathcal{F},V)} \cup [\![\chi]\!]^{(\mathcal{F},V)} \\
&= [\![\psi]\!]^{(Dw(\mathcal{F}),V)} \cup [\![\chi]\!]^{(Dw(\mathcal{F}),V)} \\
&= [\![\psi \vee \chi]\!]^{(Dw(\mathcal{F}),V)}.
\end{aligned}$$

(iv) For $\varphi = \psi \to \chi$ we have:

$$\begin{aligned}
[\![\psi \to \chi]\!]^{(\mathcal{F},V)} &= \{X \in \wp_0(W) : \forall Y \subseteq X(Y \in [\![\psi]\!]^{(\mathcal{F},V)} \Rightarrow Y \in [\![\chi]\!]^{(\mathcal{F},V)})\} \\
&= \{X \in \wp_0(W) : \forall Y \subseteq X(Y \in [\![\psi]\!]^{(Dw(\mathcal{F}),V)} \Rightarrow Y \in [\![\chi]\!]^{(Dw(\mathcal{F}),V)})\} \\
&= \bigcup\{X \in Dw(\mathcal{F}) : X \cap [\![\psi]\!]^{(Dw(\mathcal{F}),V)} \subseteq [\![\chi]\!]^{(Dw(\mathcal{F}),V)}\} \\
&= [\![\psi]\!]^{(Dw(\mathcal{F}),V)} \to [\![\chi]\!]^{(Dw(\mathcal{F}),V)} \\
&= [\![\psi \to \chi]\!]^{(Dw(\mathcal{F}),V)}.
\end{aligned}$$

This establishes that$[\![\varphi]\!]^{(\mathcal{F},V)} = [\![\varphi]\!]^{(Dw(\mathcal{F}),V)}$. ☐

The following proposition follows quite directly.

Proposition 5.16. *For every Medvedev frame we have that $\mathcal{F} \Vdash \varphi$ iff $Dw(\mathcal{F}) \vDash \varphi$.*

Proof. ($\Rightarrow$) Suppose $Dw(\mathcal{F}) \nvDash \varphi$, so that for some valuation V we have that $(Dw(\mathcal{F}), V) \nvDash \varphi$ and thus since $1_{Dw(\mathcal{F})} =\downarrow \{W\}$ also $[\![\varphi]\!]^{Dw(\wp_0(W),V)} \neq\downarrow \{W\}$. Then by Lemma 5.15 we have that $[\![\varphi]\!]^{(\mathcal{F},V)} \neq\downarrow \{W\}$ and hence $W \notin [\![\varphi]\!]^{(\mathcal{F},V)}$, which implies that $(W, V) \nVdash \varphi$ and thus $\mathcal{F} \nVdash \varphi$. ($\Leftarrow$) Suppose $(\wp_0(W), \supseteq) \nVdash \varphi$, hence for some valuation V and some $X \subseteq W$, we have $(X, V) \nVdash \varphi$. Therefore, $X \notin [\![\varphi]\!]^{(\mathcal{F},V)}$ and so $[\![\varphi]\!]^{(\mathcal{F},V)} \neq\downarrow \{W\}$. Finally, since $1_{Dw(\mathcal{F})} =\downarrow \{W\}$, it follows that $[\![\varphi]\!]^{Dw(\wp_0(W),V)} \neq 1_{Dw(\mathcal{F})}$ and so $Dw(\mathcal{F}) \nvDash \varphi$. ☐

It is a straightforward consequence of the propositions proved in this section that the Heyting algebra $Dw_{fg}(B)$ is always `ND`-algebra and that when a relational frame $\mathcal{F}$ is `ML` then the downset algebra $Dw(\mathcal{F})$ is an `ML`-algebra, i.e. $Dw(\mathcal{F}) \in Var(\mathtt{ML})$. We will use these facts in the next section to show that $\mathtt{InqB} = \mathtt{ND}^{\neg}$.

Equivalence of $\mathtt{ND}^{\neg}$ and `InqB`

We prove in this section that `InqB` is the negative variant of `ND`. The next proposition establishes that every regular `ND`-algebra is also an `ML`-algebra.

Proposition 5.17. *Suppose H is a regular Heyting algebra such that $H \vDash \mathtt{ND}$, then $H \vDash \mathtt{ML}$.*

Proof. Let H be a regular Heyting algebra such that $H \vDash \texttt{ND}$ and let $B = H_\neg$. Then by Proposition 5.10 it follows that $H(B) \twoheadrightarrow H$. By the fact that the validity of formulas is preserved by homomorphic images, to show that $H \vDash \texttt{ML}$ it is sufficient to prove that $H(B) \vDash \texttt{ML}$. Suppose for the sake of a contradiction that this is not the case, then for some $\varphi \in \texttt{ML}$ we have that for some valuation V, $(H(B), V) \nvDash \varphi$. Then let $\overline{p} = p_0, ..., p_n$ be the atomic variables in φ and $V(\overline{p}) = \{V(p_i) \in H : i \leq n\}$, then we clearly have that $(\langle V(\overline{p})\rangle, V) \nvDash \varphi$. Moreover, since for every p_i we have that $V(p_i) \in H(B)$, it follows that there is some polynomial δ_i such that $\delta_i(x_i^0, ..., x_i^m) = V(p_i)$ and every $x_i \in B$. Then since polynomials have a finite number of variables, it follows immediately that the set $A = \bigcup_{i \leq n}\{x_i^k : i \leq n, k \leq m\}$ is finite. Let B' be the Boolean algebra generated by A in B, then since Boolean algebras are locally finite we have $|B'| < \aleph_0$. Now consider the ND-*extension* $H(B')$ of B'. Since $A \subseteq B'$, it follows that $\langle V(\overline{p})\rangle \preceq H(B')$ and thus $H(B') \nvDash \varphi$. Now, by Proposition 5.13 we have that $H(B') \cong Dw_{fg}(B')$ and by Proposition 2.38 that $B' \cong \wp(W)$ for some finite set W. Therefore, we have:

$$H(B') \cong Dw_{fg}(B') \cong Dw_{fg}(\wp(W)) \cong Dw(\wp_0(W)),$$

where the isomorphism $Dw_{fg}(\wp(W)) \cong Dw(\wp_0(W))$ holds by the fact that $\wp_0(W)$ is finite and so every non-empty downset over $\wp_0(W)$ is finitely generated. One can then see that $\wp_0(W)$ is by definition a Medvedev frame and thus $\wp_0(W) \Vdash \texttt{ML}$, which by Proposition 5.16 entails $Dw(\wp_0(W)) \vDash \texttt{ML}$. But then, from the fact $H(B') \cong Dw(\wp_0(W))$ it follows that $H(B') \vDash \texttt{ML}$, which contradicts our former assumption that $(H(B), V) \nvDash \varphi$ for some $\varphi \in \texttt{ML}$. Therefore it follows that H is an ML-algebra. $\square$

Using the proposition above we can finally show that $\texttt{InqB} = \texttt{ND}^\neg$.

Theorem 5.18. $\texttt{InqB} = \texttt{ND}^\neg$.

Proof. ($\subseteq$) Suppose $\varphi \notin \texttt{ND}^\neg$, then by Proposition 4.22 there is some $H \in Var^\neg(\texttt{ND}^\neg)$ such that $H \nvDash^\neg \varphi$ and H is regular. Then, since $H \vDash^\neg \texttt{ND}^\neg$ and H is regular, we have by Proposition 4.18 that $H \vDash \texttt{ND}$ and by Proposition 5.17 that $H \vDash \texttt{ML}$. Therefore, we have that $H \in Var(\texttt{ML}) \subseteq Var(\texttt{ML})^\uparrow = Var^\neg(\texttt{InqB})$. Finally, since $H \nvDash^\neg \varphi$ it follows that $\varphi \notin \texttt{InqB}$. ($\supseteq$) By Proposition 5.8 we have that $\texttt{ND} \subseteq \texttt{ML}$, therefore by the fact that $(-)^\neg$ is a homomorphism it follows $\texttt{ND}^\neg \subseteq \texttt{ML}^\neg$ and thus by Theorem 5.6 we obtain that $\texttt{ND}^\neg \subseteq \texttt{InqB}$. $\square$

Moreover, we show that the variety of ND-algebras is actually a DNA-variety.

Proposition 5.19. *$Var(\texttt{ND})$ is a* DNA*-variety.*

Proof. It suffices to show that $Var(\texttt{ND})$ is closed under core superalgebra. Suppose $H \in Var(\texttt{ND})$, $H \preceq K$ and $H_\neg = K_\neg$, we need to show that $K \in Var(\texttt{ND})$. Suppose by reductio that $K \nvDash \texttt{ND}$, then there is some valuation V and some $n \geq 2$ such that:

$$(K, V) \nvDash (\neg p \to \bigvee_{i \leq n} \neg q_i) \to \bigvee_{i \leq n} (\neg p \to \neg q_i).$$

Then by defining the valuation $V^{\neg\neg}$ such that $V^{\neg\neg} : z \mapsto \neg\neg V(z)$ and by the fact that for every $x \in K$ we have $\neg x = \neg\neg\neg x$, we have that $[\![\neg z]\!]^{(K,V)} = [\![\neg\neg\neg z]\!]^{(K,V^{\neg\neg})} = [\![\neg z]\!]^{(K,V^{\neg\neg})}$ and therefore:

$$(K, V^{\neg\neg}) \nvDash (\neg p \to \bigvee_{i \leq n} \neg q_i) \to \bigvee_{i \leq n} (\neg p \to \neg q_i).$$

But $V^{\neg\neg}$ is clearly a `DNA`-valuation, i.e. $V^{\neg\neg} : \mathtt{AT} \to K_\neg$. Therefore, since $H_\neg = K_\neg$ and $H \preceq K$ it follows immediately that for any ψ we have $[\![\psi]\!]^{(K,V^{\neg\neg})} = [\![\psi]\!]^{(H,V^{\neg\neg})}$. Therefore,

$$(H, V^{\neg\neg}) \nvDash (\neg p \to \bigvee_{i \leq n} \neg q_i) \to \bigvee_{i \leq n} (\neg p \to \neg q_i).$$

Which contradicts the fact that $H \in Var(\mathtt{ND})$. So $Var(\mathtt{ND})$ is a `DNA`-variety. ☐

Now we want to strengthen this result and show that `ND` is exactly the intermediate logic defined by the class of `InqB`-algebras.

Proposition 5.20. $\mathtt{ND} = Log(Var^\neg(\mathtt{InqB}))$

Proof. Since $\mathtt{ND}^\neg = \mathtt{InqB}$ and $Var(\mathtt{ND})$ is a `DNA`-variety we have $\mathtt{ND} = Log(Var(\mathtt{ND})) = Log(\uparrow Var(\mathtt{ND})) = Log(Var^\neg(\mathtt{InqB}))$. ☐

By Proposition 4.15 it then follows that `ND` is `DNA`-minimal, namely that it is the minimal intermediate logic to have `InqB` as its negative variant. This fact had already been proved by Ciardelli in [10], but Proposition 5.20 gives us a novel algebraic interpretation of why `ND` is `DNA`-minimal.

5.1.4 Axiomatisation of `InqB`

The results of the previous sections provide us with a characterisation of the greatest and the least element in the sublattice $\mathcal{I}(\mathtt{InqB})$. In fact, we have shown that `ND` is the smallest logic whose negative variant is `InqB` and `ML` is the greatest whose negative variant is `InqB`. This characterisation of $\mathcal{I}(\mathtt{InqB})$ immediately gives us an axiomatisation of `InqB`. For every intermediate logic L such that $\mathtt{ND} \subseteq L \subseteq \mathtt{ML}$, we have that $L^\neg = \mathtt{InqB}$ and so by Proposition 3.2 we obtain the following characterisation.

Proposition 5.21. *Let L be an intermediate logic such that $\mathtt{ND} \subseteq L \subseteq \mathtt{ML}$, then* `InqB` *is the least set of formulas such that:*

1. $L \subseteq \mathtt{InqB}$*;*
2. *For all atomic propositional formulas $p \in \mathtt{AT}$ we have that $\neg\neg p \to p \in \mathtt{InqB}$;*
3. `InqB` *is closed under the modus ponens rule.*

Now consider the intermediate logic $\mathtt{KP} = \mathtt{IPC} + (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$. It is easy to show that $\mathtt{ND} \subseteq \mathtt{KP}$ and, by adapting the proof of proposition 5.8, it is also possible to prove that $\mathtt{KP} \subseteq \mathtt{ML}$. So we have that $\mathtt{ND} \subseteq \mathtt{KP} \subseteq \mathtt{ML}$ and we can axiomatise `InqB` as follows.

Theorem 5.22 (Axiomatisation of $\mathtt{InqB}$)**.** *The following system of axioms and rules axiomatises* $\mathtt{InqB}$*:*

$$\begin{array}{rl} \textbf{Axioms} & \mathtt{IPC} \\ & (\neg\varphi \to \psi \vee \chi) \to (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi) \textit{ for all } \varphi, \psi, \chi \in \mathcal{L}_P \\ & \neg\neg p \to p \textit{ for all } p \in \mathtt{AT} \\ \textbf{Rule} & \varphi, \varphi \to \psi \Rightarrow \psi. \end{array}$$

We thus get a fairly intuitive axiomatisation of $\mathtt{InqB}$. This proof system is presented in [3] and was first formulated in [10].

5.2 Extensions of $\mathtt{InqB}$

In this section we use the fact that $\mathtt{InqB}$ is the negative variant of $\mathtt{ND}$ and $\mathtt{ML}$ and the method of Jankov formulas introduced in Section 4.3 to characterise the sublattice of extensions of $\mathtt{InqB}$. First, we use the previous results concerning the $\mathtt{ND}$-*extension* of a Boolean algebra to show that $\mathtt{InqB}$ is locally finite.

Theorem 5.23. $\mathtt{InqB}$ *is locally finite.*

Proof. We need to show that every $\mathtt{DNA}$-finitely generated $\mathtt{InqB}$-algebra is finite. Consider any $H \in Var^{\neg}(\mathtt{InqB})$ and suppose H is $\mathtt{DNA}$-finitely generated, then there are elements $x_0, ..., x_n \in H_{\neg}$ such that $\langle x_0, ..., x_n \rangle = H$. So it immediately follows that H is regular. Moreover, by the fact that $\mathtt{ND} = Log(Var^{\neg}(\mathtt{InqB}))$ we also have that $Var^{\neg}(\mathtt{InqB}) = Var(\mathtt{ND})$ and so $H \in Var(\mathtt{ND})$. Then, it follows by Proposition 5.10 that $f : H(H_{\neg}) \twoheadrightarrow \langle x_0, ..., x_n \rangle$. Suppose now without loss of generality that $y_0, ..., y_n \in H(H_{\neg})$ are such that $f(y_0) = x_0, ..., f(y_n) = x_n$ and thus $\langle x_0, ..., x_n \rangle = \langle f(y_0), ..., f(y_n) \rangle$. Now consider the subalgebra $\langle y_0, ..., y_n \rangle$ of $H(H_{\neg})$, it follows by Proposition 5.11 that each element in $\langle y_0, ..., y_n \rangle$ has a unique disjunctive representation, which means without loss of generality that each y_i can be written as $y_i = \bigvee_{j \leq k_i}(a^i_j)$, where k_i is the number of disjuncts in the disjunctive normal form of y_i, and every $a^i_j \in H_{\neg}$. Now let A be $A = \{a^0_0, ..., a^0_{k_0}, ..., a^n_0, ..., a^n_{k_n}\}$, then we clearly have that $\langle y_0, ..., y_n \rangle \preceq \langle A \rangle$ and by the disjunctive normal form given in Proposition 5.11 also that every $x \in \langle A \rangle$ can be written as a disjunction of elements which belong to the Boolean algebra B generated by A. Now, since the cardinality of A is finite, it follows from the local finiteness of Boolean algebras that B is finite as well. Also, since every elements in $\langle A \rangle$ can be written as a disjunctions of elements of B, it follows that $\langle A \rangle \leq 2^{|B|} < \aleph_0$. Finally, we have that:

$$|H| = |\langle x_0, ..., x_n \rangle| = |\langle f(y_0), ..., f(y_n) \rangle| \leq |\langle y_0, ..., y_n \rangle| \leq |\langle A \rangle| \leq 2^{|B|} < \aleph_0.$$

Therefore, it follows that H is finite and so that $\mathtt{InqB}$ is locally finite. $\square$

Since $\mathtt{InqB}$ is locally finite, we have by Theorem 4.32 that it is generated by its collection of finite, regular, subdirectly irreducible elements. The next theorem provides a characterisation of this class of $\mathtt{InqB}$-algebras. Our proof adapts [3, Theorem 4.2].

Theorem 5.24. *Let H be an Heyting algebra. Then $H \in Var^{\neg}_{RFSI}(\mathtt{InqB})$ iff there is some finite Boolean algebra B such that $H \cong H(B)$.*

Proof. ($\Leftarrow$) Suppose $H \cong H(B)$ for some finite Boolean algebra B, then we need to show that H is finite, regular and subdirectly irreducible. First, it follows immediately by construction that $H(B)$ is regular and so that H is regular as well. By Theorem 5.13 we have that $H(B) \cong Dw_{fg}(B)$. Consider the downset $D_s := \{x \in B : x < 1_B\}$, then since B is finite it follows that D_s is finitely generated. Moreover, it is easy to see that for any finitely generated downset $X \neq B$, $X \subseteq D_s$ and so D_s is the second greatest element in $Dw_{fg}(B)$. Since $Dw_{fg}(B) \cong H(B)$ it follows that $H(B)$ has a second greatest element as well hence by Theorem 2.36 it is subdirectly irreducible. Finally, since $|B| < \aleph_0$, we have that $|H(B)| = |Dw_{fg}(B)| \leq |\wp(B)| < \aleph_0$ and hence $H(B)$ is finite.

($\Rightarrow$) Let $H \in Var^{\neg}_{RFSI}(\mathtt{InqB})$, then since H is regular and $Var^{\neg}(\mathtt{InqB}) = Var(\mathtt{ND})^{\uparrow}$, it follows by Proposition 4.19 that $H \in Var(\mathtt{ND})$. From the universal mapping property of Proposition 5.10 there is a surjective homomorphism $h : H(H_{\neg}) \twoheadrightarrow H$, where $H_{\neg}$ is clearly a finite Boolean algebra. We prove now that this homomorphism is also injective. Consider $x, y \in H$ such that $h(x) = h(y)$, then it follows Proposition 5.11 that we have non-redundant representations $x = \bigvee_{i \leq n} a_i$ and $y = \bigvee_{j \leq m} b_j$. Since for all $i \leq n, j \leq m$ we have $a_i, b_j \in H_{\neg}$, it follows by Proposition 4.25 that $h \upharpoonright H_{\neg} = id_{H_{\neg}}$ and so that $h(a_i) = a_i$ and $h(b_j) = b_j$, which means that $h(a_j), h(b_j) \in H_{\neg}$. Now, since $h(x) = h(y)$, we have that $h(\bigvee_{i \leq n} a_i) \leq h(\bigvee_{j \leq m} b_j)$ and $h(\bigvee_{j \leq m} b_j) \leq h(\bigvee_{i \leq n} a_i)$. From the former of these claims we have:

$$h(\bigvee_{i \leq n} a_i) \leq h(\bigvee_{j \leq m} b_j)$$

$$\Rightarrow h(\bigvee_{i \leq n} a_i) \rightarrow h(\bigvee_{j \leq m} b_j) = 1$$

$$\Rightarrow \bigvee_{i \leq n} (h(a_i)) \rightarrow \bigvee_{j \leq m} (h(b_j)) = 1 \qquad \text{(since } h \text{ is a homomorphism)}$$

$$\Rightarrow \bigwedge_{i \leq n} [h(a_i) \rightarrow \bigvee_{j \leq m} h(b_j)] = 1 \qquad \text{(by } \mathtt{IPC}\text{)}$$

$$\Rightarrow \bigwedge_{i \leq n} [\neg\neg h(a_i) \rightarrow \bigvee_{j \leq m} h(b_j)] = 1 \qquad \text{(by } h(a_i) \in H_{\neg}\text{)}$$

$$\Rightarrow \bigwedge_{i \leq n} \bigvee_{j \leq m} [\neg\neg h(a_i) \rightarrow h(b_j)] = 1 \qquad \text{(by } \mathtt{KP}\text{)}$$

$$\Rightarrow \bigwedge_{i \leq n} \bigvee_{j \leq m} [h(a_i) \rightarrow h(b_j)] = 1 \qquad \text{(by } h(a_i) \in H_{\neg}\text{)}$$

$$\Rightarrow \forall i \leq n, \text{ we have } \bigvee_{j \leq m} [h(a_i) \rightarrow h(b_j)] = 1$$

$$\Rightarrow \forall i \leq n, \exists j \leq m \text{ such that } h(a_i) \rightarrow h(b_j) = 1 \quad \text{(by well-connectedness of } H\text{)}$$

$$\Rightarrow \forall i \leq n, \exists j \leq m \text{ such that } h(a_i) \leq h(b_i)$$

$$\Rightarrow \forall i \leq n, \exists j \leq m \text{ such that } a_i \leq b_i \qquad \text{(by } h \restriction H_\neg = id_{H_\neg}\text{)}$$

$$\Rightarrow \bigvee_{i\leq n} a_i \leq \bigvee_{j\leq m} b_j$$

$$\Rightarrow x \leq y.$$

Similarly, starting from $h(\bigvee_{j\leq m} b_j) \leq h(\bigvee_{i\leq n} a_i)$ we then get that $y \leq x$ and so that $x = y$. Finally, this means that the surjective homomorphism $h : H(H_\neg) \twoheadrightarrow H$ is also injective and so that $H \cong H(H_\neg)$. □

From the former theorem it is then easy to prove the following important lemma. We recall from Section 4.3 that if A an B are two Heyting algebras, the order $\leq$ between them is defined as $A \leq B$ iff $A \in HS(B)$. The next lemma shows that under this order the collection of regular, finite, subdirectly irreducible `InqB`-algebra is isomorphic to ω.

Lemma 5.25. *Let $Var^\neg_{RFSI}(\texttt{InqB})$ be the collection of finite, regular, subdirectly irreducible* `InqB`*-algebras and $\leq$ the order defined by $A \leq B \Leftrightarrow A \in HS(B)$. Then $(Var^\neg_{RFSI}(\texttt{InqB}), \leq) \cong \omega$.*

Proof. We show that $Var^\neg_{RFSI}(\texttt{InqB})$ is isomorphic to ω under the order $A \leq B$ iff $A \in HS(B)$. First, consider any algebra $H \in Var^\neg_{RFSI}(\texttt{InqB})$, then it follows by Theorem 5.24 that there is some finite Boolean algebra B such that $H = H(B)$. We have already seen in Section 2.1.5 that the representation Theorem 2.38 of the finite Boolean algebras entails that finite Boolean algebras form the following chain of length ω:

$$2^0 \preceq 2^1 \preceq 2^2 \preceq 2^3 \preceq 2^4 \preceq \dots$$

Now, we have by the definition of the `ND`-*extension* of a Boolean Algebra 2^n that $H(2^n)$ is regular and $H(2^n) = \langle 2^n \rangle$. Therefore, since we have that for all $n \in \mathbb{N}$, $2^n \preceq 2^{n+1}$, it follows that $H(2^n) \preceq H(2^{n+1})$. Finally, since every $H \in Var^\neg_{RFSI}(\texttt{InqB})$ is of the form $H(2^n)$ for some $n \in \mathbb{N}$, it follows that:

$$H(2^0) \preceq H(2^1) \preceq H(2^2) \preceq H(2^3) \preceq H(2^4) \preceq \dots$$

is a chain of length ω ordered by $A \leq B \Leftrightarrow A \in HS(B)$ which contains every element $H \in Var^\neg_{RFSI}(\texttt{InqB})$. Finally, this means that the poset $(Var^\neg_{RFSI}(\texttt{InqB}), \leq)$ is isomorphic to ω. □

Once we have the previous lemma, we can use use the method of Jankov formulas for `DNA`-logics developed in Section 4.3 to show that the lattice of extensions of the system of inquisitive logic `InqB` is linearly ordered and dually isomorphic to $\omega + 1$.

Theorem 5.26. *Let $\Lambda^\neg(\texttt{InqB})$ be the lattice of extensions of* `InqB`*. Then there is a dual isomorphism $\Lambda^\neg(\texttt{InqB}) \cong^{op} \omega + 1$.*

Proof. By the dual isomorphism **DNAL** $\cong^{op}$ **DNAV** we immediately have that $\Lambda^\neg(\texttt{InqB}) \cong^{op} \Lambda^\neg(Var^\neg(\texttt{InqB}))$, where $\Lambda^\neg(Var^\neg(\texttt{InqB}))$ is the lattice of subvarieties of $Var^\neg(\texttt{InqB})$. Therefore, to show that $\Lambda^\neg(\texttt{InqB}) \cong^{op} \omega + 1$ it suffices to show that $\Lambda^{\neg}(Var^{\neg}(\texttt{InqB})) \cong \omega + 1$. Now, by Proposition 5.23 we have that InqB is locally finite and therefore it follows by Theorem 4.42 that $\Lambda^{\neg}(Var^{\neg}(\texttt{InqB})) \cong Dw(Var^{\neg}_{RFSI}(\texttt{InqB}))$. But then, we have by Lemma 5.25 that $Var^{\neg}_{RFSI}(\texttt{InqB}) \cong \omega$ and therefore that $Dw(Var^{\neg}_{RFSI}(\texttt{InqB})) \cong Dw(\omega) = \omega + 1$. To sum up, we have:

$$\Lambda^{\neg}(\texttt{InqB}) \cong^{op} \Lambda^{\neg}(Var^{\neg}(\texttt{InqB})) \cong Dw(Var^{\neg}_{RFSI}(\texttt{InqB})) \cong Dw(\omega) = \omega + 1,$$

which proves our claim. □

The method of Jankov formulas allows us also to provide an axiomatisation for all the extensions Λ of InqB. By DNA-duality and Theorem 4.42 we have that $\Lambda^{\neg}(\texttt{InqB}) \cong^{op} Dw(Var^{\neg}_{RFSI}(\texttt{InqB}))$. Therefore we have that extensions Λ of InqB are uniquely identified by specifying a downset of elements of $Var^{\neg}_{RFSI}(\texttt{InqB})$. For any $n \in \mathbb{N}$, we define by $\texttt{InqB}_n$ the logic $\texttt{InqB}_n = Log^{\neg}(\downarrow H(2^n))$. We now prove the following proposition.

Proposition 5.27. *Let Λ be a proper extension of* InqB, *i.e. Λ is a* DNA*-logic and* $\texttt{InqB} \subset \Lambda$. *Then there is some $n \in \mathbb{N}$ such that*

$$\Lambda = \texttt{InqB}_n = \texttt{InqB} + \chi(H(2^{n+1})).$$

Proof. Suppose that Λ is a DNA-logic and $\texttt{InqB} \subset \Lambda$, then it follows by Theorem 4.42 that $Var^{\neg}(\Lambda) = \mathcal{X}(D)$, where $D \in Dw(Var^{\neg}_{RFSI}(\texttt{InqB}))$. Now, since $\Lambda \neq \texttt{InqB}$, it follows that $D \neq Var^{\neg}_{RFSI}(\texttt{InqB})$. Therefore, it follows immediately from Lemma 5.25 that $D = \downarrow H(2^n)$ for some $n \in \mathbb{N}$ and hence $\Lambda = \texttt{InqB}_n$. Moreover, it is easy to see that the only minimal counterexample in $Var^{\neg}(\texttt{InqB}) \setminus Var^{\neg}(\texttt{InqB}_n)$ is $H(2^{n+1})$. Therefore, we have by Theorem 4.45 that $\texttt{InqB}_n$ is equivalent to $\texttt{InqB} + \chi(H(2^{n+1}))$. □

The previous result allows us to introduce in an alternative way the inquisitive hierarchy originally introduced by Ciardelli [10, Ch. 4]. We define, for every $n \in \mathbb{N}$, the system $\texttt{InqL}_n$ as follows:

$$\texttt{InqL}_n = \{\varphi \in \mathcal{L}_P : \forall s \in \wp(2^{\texttt{AT}}), \text{ such that } |s| \leq n, s \vDash \varphi\}.$$

We can now show that the inquisitive hierarchy is exactly the sublattice of all the proper extensions of InqB. Firstly, we say that a DNA-logic is *tabular* if it is the logic of a finite regular Heyting algebra. Then, since for all $H \in \downarrow H(2^n)$ we have that $H \preceq H(2^n)$, it follows immediately that $\texttt{InqB}_n = Log^{\neg}(\downarrow H(2^n)) = Log^{\neg}(H(2^n))$, i.e. $\texttt{InqB}_n$ is the logic of $H(2^n)$ and is thus tabular. Then we obtain the following theorem.

Theorem 5.28. *For any $n \in \mathbb{N}$, we have that* $\texttt{InqB}_n = \texttt{InqL}_n$.

Proof. For any $n \in \mathbb{N}$, we have the following equalities:

$$\begin{aligned} \texttt{InqB}_n &= Log^{\neg}(\downarrow H(2^n)) \\ &= Log^{\neg}(H(2^n)) \\ &= Log(H(2^n))^{\neg} && \text{(by Proposition 3.24)} \end{aligned}$$

$$
\begin{aligned}
&= Log(Dw_{fg}(2^n))^{\neg} && \text{(by Theorem 5.13)}\\
&= \{\varphi \in \mathcal{L}_P : \wp_0(n) \Vdash \varphi[{}^{\neg p}/_{p}]\} && \text{(by Proposition 5.16)}\\
&= \{\varphi \in \mathcal{L}_P : n \vDash \varphi\} && \text{(by Proposition 5.4)}\\
&= \{\varphi \in \mathcal{L}_P : \forall s \in \wp(2^{\mathtt{AT}}),\ \text{such that } |s| \leq n, s \vDash \varphi\}\\
&= \mathtt{InqL}_n.
\end{aligned}
$$

Which proves our claim. □

Therefore, by defining for every $n \in \mathbb{N}$ the logic $\mathtt{ML}_n$ as the set of formulas valid in all Medvedev frames $\mathcal{F}$ whose cardinality is $|\mathcal{F}| \leq n$, it follows from the previous theorem that $(\mathtt{ML}_n)^{\neg} = \mathtt{InqB}_n = \mathtt{InqL}_n$. The following corollary follows directly from Theorem 5.26 and Theorem 5.28 and is an extension of [10, p. 4.1.6].

Corollary 5.29.

$$
\mathtt{InqB} = \bigcap_{n \in \mathbb{N}} \mathtt{InqB}_n = \bigcap_{n \in \mathbb{N}} \mathtt{InqL}_n = \bigcap_{n \in \mathbb{N}} (\mathtt{ML}_n)^{\neg}.
$$

The results in this section thus provide a characterisation of the extensions of InqB and show that they coincide precisely with the inquisitive hierarchy already studied in the literature. We take this as a key example of the fact that algebraic semantics can be useful and play an important role in the study of inquisitive logic.

Chapter 6

Conclusions and Future Work

In this thesis we developed algebraic semantics for `DNA`-logics and we applied this general setting to inquisitive logic. This semantics allows to apply methods of universal algebra to study `DNA`-logics and inquisitive logic from a novel perspective. Let us briefly summarize our main results. In Chapter 3 we introduced `DNA`-logics and their algebraic semantics and we gave two different proofs of the dual isomorphism $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$ between `DNA`-logics and `DNA`-varieties. In Chapter 4 we studied closer the relation between `DNA`-logics and intermediate logics and we proved a suitable version of some classical results for the setting of `DNA`-varieties. In particular, we showed that every `DNA`-variety is generated by its regular subdirectly irreducible members and that the `DNA`-logic of all Heyitng algebras $\mathtt{IPC}^{\neg}$ is not locally finite. We introduced a suitable version of Jankov formulas and we showed that this provides an axiomatisation of locally finite `DNA`-varieties. Finally, in Chapter 5 we used the algebraic semantics of `DNA`-logics to study the inquisitive logic `InqB`. In particular, we showed that the sublattice of its extensions is dually isomorphic to $\omega + 1$ and that it actually coincides with the inquisitive hierarchy studied in [10].

In addition to these results, we think that one of the main contributions of this thesis is that it provides a new setting for the study of inquisitive logic. The system `InqB` had so far been considered as the logic of the evaluation states or as the negative variant of the logics between `ND` and `ML` – here we showed that one can also consider `InqB` as the logic of a specific class of Heyting algebras, under a suitable semantics. Most importantly, this new perspective at the propositional system of inquisitive logic allows us to raise new questions and consider new issues. We mention here some possible directions for future work, both concerning `InqB` and the general theory of `DNA`-logics.

From Negative Variants to Propositional Variants In this thesis we introduced `DNA`-logics as the negative variant of some intermediate logic L. Every `DNA`-logic Λ is thus such that $\Lambda = \{\varphi \in \mathcal{L}_P : \varphi[{}^{\neg p}/_{p}] \in L\}$ for some intermediate logic L. A possible direction of future work is to study what happens if, instead of the negative substitution $p \mapsto \neg p$, we consider the substitution $p \mapsto \chi(p)$ for an arbitrary polynomial $\chi \in \mathcal{L}_P$. In fact, it seems possible to extend at least part of the theory of `DNA`-logics to this extended framework. In the case of negative variants we rely on the fact that in intuitionistic logic $\neg\neg\neg p = \neg p$. This property however is shared in a more general form by every polynomial

χ. Ruitenberg's Theorem [46, 47, 23] states that for any polynomial χ we can find a number $n \in \mathbb{N}$ such that $\chi^n = \chi^{n+2}$. This allows to introduce the χ-variant of an intermediate logic L as $L^\chi = \{\varphi \in \mathcal{L}_P : \varphi[\overline{\chi^n(p)}/\overline{p}] \in L\}$ and to generalize our study of `DNA`-logics to arbitrary χ-variants. See for instance the upcoming [25].

From Inquisitive Logic to Dependence Logic It was noticed recently that there is a close relation between inquisitive logic and dependence logic. This connection has been studied e.g. in [9, 11] and suggests further directions of research. Similarly to inquisitive logic, the semantics of propositional dependence logic [51] consists of a set of possible valuations instead of a single valuation. Is it possible to adapt the algebraic semantics of `DNA`-logics to obtain an algebraic semantics for propositional dependence logic? A related question which is considered in [30] is what happens, both in inquisitive and dependence logic, if instead of starting with classical valuations we start with intuitionistic valuations. Is it possible to adapt the algebraic semantics developed in this thesis to this alternative setting?

From Jankov Formulas to Canonical Formulas In Section 4.3 we introduced Jankov formulas for `DNA`-models and we showed that locally finite `DNA`-logics are axiomatised by these formulas. Is it possible to extend to the setting of `DNA`-logics other applications of Jankov formulas? For example, can we prove using Jankov formulas that the lattice of `DNA`-logic has the cardinality of the continuum? Or can we extend Jankov formulas to subframe formulas, and in general to canonical formulas, as it is the case both for intermediate [1] and modal logics [2]? There are many ways in which one can use Jankov formulas to study Heyting algebras and it seems natural to extend them to the setting of `DNA`-logics.

From Algebraic to Topological Semantics It is a well-known fact [19] that Heyting algebras are dual to order-topological spaces known as Esakia spaces. This allows us to have both an algebraic and topological semantics for intermediate logics. In this thesis we did not look at possible connections to topology and we restricted our analysis to the algebraic setting. However, already in [3] a topological semantics for `InqB` is provided *via* UV-spaces. Is it possible to generalize this semantics to arbitrary `DNA`-logics? Similarly, can we define a suitable class of topological models for `DNA`-logics to obtain a general duality between `DNA`-models based on Heyting algebras and `DNA`-models based on Esakia spaces? A related issue concerns the characterisation of finite regular subdirectly irreducible Heyting algebras. We know by duality that a finite subdirectly irreducible Heyting algebra is the upset algebra of a finite rooted frame. Can we obtain a similar characterisation for regular finite subdirectly irreducible Heyting algebras? What properties should a rooted frame satisfy in order for its dual Heyting algebra to be regular?

From `InqB` to a Theory of `DNA`-Logics Finally, it is worth mentioning that there are still many open questions concerning `DNA`-logics and their relations to intermediate logics that one should look at. We will mention here some of them. First, in Section 4.1 we have studied the connections between DNA-logics and intermediate logics which they are negative variants of. It is an important result proved in [10] that IPC is a DNA-maximal logic. Therefore, since IPC is obviously also a DNA-minimal logic it follows that it is both DNA-maximal and DNA-minimal. Is IPC the only intermediate logic to be DNA-maximal and DNA-minimal? Can we find other logics with this property? Secondly, we have seen in Section 5.2 that the extensions of InqB are linearly ordered. Is this a feature shared by other DNA-logics or is this a property which is specific for InqB? Finally, the example of InqB also shows that a DNA-logic Λ can be locally finite even if all the intermediate logics in $\mathcal{I}(\Lambda)$ are not. The locally finiteness and the finite model property of DNA-logics are thus interesting properties that can be investigated further. For instance, one could try to define suitable notions of filtrations for these logics and to introduce suitable classes of stable logics [31]. We leave these and possibly other interesting questions for future work.

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