The Incompleteness Theorems, their content and their meaning

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1 Introduction

In the year 2006, 100 years after Gödel's birth, it is time to think about the lasting values of his achievements. In this paper I will be concerned with his most famous results, his two incompleteness theorems[1] [9]. In Section 2, a rough version of the theorems and the in my opinion main achievements of Gödel by these theorems are given. In Sections 3 and 4, the historical content of the theorems is sketched, in Section 5, I define what a formal system is. In Sections 6 and 7, I describe the content of the first incompleteness theorem somewhat more precisely and give the main ideas contained in the proof. The second incompleteness theorem is treated in section 8. Two developments after Gödel, Provability logic and Feferman's arithmetization of metamathematics are considered in Section 9. I conclude with some remarks on Lucas' and others' attempts to derive philosophical conclusions concerning human beings vs machines from the first incompleteness theorem.

2 Rough statement of the theorems, Gödel's lasting achievements

Let us start with stating the theorems roughly.

1. In formal systems in which statements about natural numbers can be proved and which prove only true statements of that kind there are always true statements that have no proof in the system.
2. One of the true statements that cannot be proved in such a formal system can be interpreted as the consistency of the system itself, the assertion that no contradictory statements can be proven in the system.

[1] The content of this paper is a rewritten version of two different Gödel centennial lectures, the first one on April 26, 2006, at the ILC, Sun Yat Sen University, Guangzhou, the second one on May 26, 2006, for the Nederlandse Verening voor Logica en Wijsbegeerte der Exacte Wetenschappen in Utrecht. I am grateful to Albert Visser for sharing his insights with me for many years. The opinions in this paper are all my own.

And let us continue by stating what in my opinion are Gödel's main lasting achievements by his proof of these incompleteness theorems.

1. The distinction between truth and formal provability.
2. The insight that proving consistency of a formal system cannot in general increase our confidence in the system.
3. The methods of arithmetization, formalization and diagonalization in arithmetic.
4. The insight that by diagonalization paradoxical statements can be integrated in formal systems and can then lead to important insights.

To be able to appreciate the importance of the incompleteness theorems it is essential to first reconstruct the context in which these theorems were discovered and stated. That will be the object of the next section.

3 The state of Philosophy and Mathematics around 1900

In the history of philosophy, from its origins in Greece onwards, the remarkable *certainty of mathematical statements* has played an important part. Why is $2+2=4$ so much more sure than a statement of fact even such a simple one as that that apple is lying in front of me on the table?

Of course, many explanations have been given, but up to modern times it was usually thought that mathematics derived its certainty from its method rather than from its subject matter. Its certainty seemd to originate in *the axiomatic method* as it was developed by Euclid in his postulates for the foundation of geometry. Spinoza still founded his philosophical system on an axiomatic basis with Euclid as his glowing example. Nevertheless, in the last centuries slowly both the certainty and the absolute value of mathematics were prey to erosion. The enormously influential philosopher Immanuel Kant did defend the certainty of mathematics, especially even of euclidean geometry, but he founded his certainty on the way we see and have to see the world, the so-called *synthetic a priori*.

As so often happens his important contributions opened people's eyes to other possibilities, also ones opposite to his own point of view. In mathematics doubts had been slowly accumulating for centuries, Euclid's parallel postulate had long been considered unfortunate in that it was much more complicated than the other postulates; many (unsuccessful) attempts had been made to derive it from the others. In the nineteenth century, the century that shook up many certainties, finally geometries were developed that simply started postulating axioms contradicting the parallel postulate: non-euclidean geometries. Whether the originators were convinced of the truth of the parallel postulate or not, their work in which rather natural other posibilities were envisioned inevitably led to uncertainty about the truth of the parallel postulate. And if one mathematical proposition is not so sure then . . . .

At the same time mathematics was starting to grow at an ever increasing pace. Moreover, this growth was not centered in geometry but in number theory (in a wide sense), in particular in analysis. Already Descartes had shown how geometry can be interpreted in real number theory, and now more and more the attention of the mathematicians themselves was directed towards the calculus: *the dominant place of geometry was taken over by number theory*.

These developments gave rise to a change in the nature of mathematics as well, and this led to considerable uneasiness with a number of mathematicians, uneasiness about the growing abstractness of mathematics, uneasiness about proofs that seemed to construct objects out of thin air. When Kronecker complained that "God made the natural numbers, the rest is human work" he asked for a solid basis of mathematics in the natural numbers and clear ways of construction of other mathematical objects from those. Hilbert's extremely abstract, nonconstructive solution of Gordan's problem in 1895 (each ideal $K[\vec{X}]$ over a field K has a finite basis) shocked a number of mathematicians and Cantor's set theory of the eighteen nineties especially angered people who considered his work metaphysics rather than mathematics. By this time there was an emergence of feeling that a solid foundation of mathematics would be very welcome.

4 The main people on the stage in Gödel's time

- Frege/Russell
- Hilbert
- Brouwer

Frege, Hilbert and Brouwer are the clearest exponents of the three main points of view that were expresed in the foundations of mathematics in the twentieth century:

1. **Platonism**, the view that mathematical objects (as numbers) are objectively existing objects, be it not in the fleeting world of sense experience. Mathematical truths are therefore real truths about real objects. They have to be discovered, obtained by intuition, insight. The fact that mathematical propositions concern this unchangeable, unshakeable world explains their unique status.

2. **Constructivism**, the view that mathematical objects (like numbers) are idealized thought objects created by the mathematician. Mathematical truths are real truths that have to be constructed, created, proved. The certainty of mathematical truths is explained by the fact that the originator of the truth can be certain of it.

3. **Formalism**, the view that mathematics is not about something, it is a manipulation of symbols according to rules, there are no mathematical truths, the formulas of mathematics can sometimes be usefully interpreted. Their certainty is explained by the fact that it is just a matter of checking that the rules have been obeyed.

G. Frege was a clear platonist (as Gödel later). His program was to show that mathematical statements are nothing but logical truths (and therefore *analytic a priori* in

Kant's terminology), in works extending from 1879 [7] to 1903 [8]. In 1903 Russell sent Frege a letter containing a proof in Frege's system of a paradoxical result (using the set of all sets that are not members of themselves).

Frege rather quickly gave up his basic ideas, but Russell continued to push with great force Frege's program to show mathematics to be a part of logic (without having Frege's platonist views). To avoid the paradoxes the system he constructed together with Whitehead in the Principia Mathematica [17] (1910) was very artificial however and not based on grand philosophical ideas. In Gödel's time he had lost most of his supporters.

Actually, similar paradoxes were already known to mathematicians working in Cantor's set theory among which Cantor himself. For them these paradoxical results seemed much less of a problem. Their feeling was that the mathematical results were basically sound and that it was just a technical mathematical matter to find a way to these results without getting involved in the paradoxes. In a philosphical foundation of mathematics this issue lies of course differently. After Russell's paradox all attempts at a foundation should have a clear account how to avoid the paradoxes.

L.E.J. Brouwer, unlike Frege, did not attempt to give existing mathematics a solid foundation. He, like Kronecker and others before him, considered parts of existing mathematics simply wrong. Unlike these predecessors he devised, from his dissertation [2] in 1907 onwards a systematic new way of doing mathematics, called *intuitionistic* mathematics without the suspect methods by considering the objects and proofs of mathematics as constructions of the mind [5]. Clearly this is quite different from the platonistic view, one may see the notion of truth seen as replaced by *informal provability*. By consistently applying his point of view he was lead to deny the validity of reasoning by contradiction:

$$\neg A \text{ is impossible,}$$
$$\text{therefore } A.$$

or more in general the principle of the excluded middle: A *or* $\neg A$ *is always true*, because these principles allow one to prove the existence of objects with certain properties without exhibiting these objects. Denying oneself the use of thses logical laws entailed cutting out substantial parts of classical mathematics, in particular almost all of Cantor's set theory. Most mathematicians didn't follow his lead but some very good ones did, notably Herman Weyl. This was enough to anger

D. Hilbert, the most powerful mathematician of the day. He was active in all fields of mathematics. He had been concerned with *consistency* (no contradiction is provable) of mathematics for a long time. In 1890 he published his Grundlagen der Geometrie [12] in which the so-called *relative consistency* of geometries was a topic: if system A is consistent, then so is system B. Usually such results are obtained by constructing a model (*interpretation*) for the system B in terms of system A. He noticed that geometry is consistent relative to number theory (by the model of *analytic geometry* discovererd by Descartes). In his famous lecture in 1900, about mathematical problems for the 20th century, he stated the consistency of number theory as the second one and one of the more important. A relative consistency proof seems improbable and anyway one has to stop somewhere. He considered mathematical theories to be formal systems as explained the next section, and considered formal systems to be the basis for a new type of absolute consistency proof.

5 Formal systems

A formal system has three main components.

- An exact description of the *language*, the *symbols* you are going to use. Among these usually there are the connectives $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\rightarrow$ (if. . .then. . .), and the quantifiers $\exists$ (there exists) and $\forall$ (for all).
- An exact description of how symbols may be strung together to form *formulas*.
- An exact description how (finitely) many formulas may be strung together to form *proofs*.

Note that seen in this manner formulas are sequences of symbols, proofs are sequences of formulas.

Hilbert's idea was that all of existing mathematics can be put into this format, and that is what mathematics really is (*formalism*). In this manner there is no question of real truth or falsity. What remains of truth is *formal provability*, a formula is acceptable if a proof can be obtained playing according to the rules. Unlike some predecessors tending to a formalistic point of view he did realize that to understand what a formal system is and does one already needs some (non-formal) basic mathematical methods (including some induction over the natural numbers or something equivalent to it). He called the basic methods he was prepared to use for this: *finitistic methods*. These methods ought to be acceptable to any mathematician (they are certainly less encompassing even than the methods contained in Brouwer's intuitionistic mathematics). To avoid the paradoxes his clinching idea was that a consistency proof of number theory should be given by these basic finitistic methods. This did not seem implausible since *consistency* is from this point of view a very simple combinatorial statement:

- It is not possible in the formal system to obtain proofs of both formulas A and $\neg A$,

or equivalently in systems containing a theory of natural numbers (let us call the formal system we are considering from now on T)

- It is not possible to obtain a proof in T of $1 = 0$.

The program based on these ideas is called *Hilbert's program*. It had very strong support, especially in Hilbert's own department of mathematics in Göttingen which was the strongest department in the world containing among other people the young von Neumann.

6 The First Incompleteness Theorem

It was in this constellation that Gödel proved his theorems. Let us formulate his first theorem somewhat more precisely.

Theorem 1 First Incompleteness Theorem. *Let T prove only true formulas. Then there exists a statement about natural numbers 0,1,2,3, . . . of the form $\forall x Ax$ (for all x, Ax) of the system which is not provable in T, while for each individual number n, An is provable in T.*

The way we have stated it here immediately stresses the role of truth versus formal provability (Gödel's first amin achievement). If one accepts all the formulas An as true, then $\forall x Ax$ is true as well. Of course the infinite jump from the infinitely many formulas An to $\forall x Ax$ can never be part of a formal system. Only in this special case we know that $A0, A1, A2, \ldots$ are all provable, it is obvious that there are many formulas Bx for which $\forall x Bx$ is not provable but Bn is provable for all n without us knowing it. Also clear is that adding $\forall x Ax$ as a new axiom to T is perfectly legitimite but does not solve the problem, Gödel's theorem can be applied to the resulting new formal system again and then produces a new formula not provable in the new system.

We purposely did not state the theorem in the form: there exists a B such that neither B nor $\neg B$ is provable in T. But note that in our formulation it follows that $\neg\forall x Ax$ or equivalently, $\exists x \neg Ax$ is not provable. This is because this statement cannot be true if T proves An for each individual number n, and T proves only true statements. Actually, as we will see, with the proof Rosser later gave the condition on T can be weakened to just require consistency of T.

Already in the *completeness theorem* in his dissertation the year before had Gödel shown his insight in truth versus (formal) provability: if a formula of the predicate calculus is true under any interpretation, it is formally provable in the predicate calculus. The proof of the completeness theorem was far less difficult than that of the first incompleteness theorem but could not have been given by a person without deep insight in the distinction. It is noteworthy that in the interpretation of the first incompleteness theorem truth may be read in Brouwer's sense: *informal provability*. Gödel's argument is a mathematically convincing argument for $\forall x Ax$, an informal proof of Ax, even if in the formal system only An is formally provable for each n.

The first incompleteness theorem is not easy to accomodate for a formalist. Of course, a present day formalist will not recognize that the distinction between truth and provability is essential for its interpretation. A possible reaction could be that the Gödel sentence is always provable in the formal system plus its reflection principles (see for that notion Section 9).

Before we go into the content of the second Incompleteness Theorem it is necessary to discuss some of the ideas in the proof of the First Incompleteness Theorem.

7 On the proof of the First Incompleteness Theorem

The three main insights in Gödel's proof are

1. The insight that number theory can discuss all kinds of discrete objects (specifically syntactic objects like symbols, formulas, proofs), and simple manipulations of them, by coding (*gödel numbering*) them by means of natural numbers (*arithmetization*), plus the insight that the relevant true properties can then be expressed and proved in the formal system T (*formalization*).

2. Combining this with a diagonal procedure to mimick self-reference in T, obtaining statements that in an indirect manner express properties of themselves.

3. Applying this self-reference to seemingly paradoxical cases.

(1) and (2) form what, in Section 2 we called Gödel's third main achievement, (3) the fourth. The basic insight of (1) is that sequences of natural numbers can be coded by one natural number in a systematic manner so that

- For each finite sequence of natural numbers one can calculate the unique number coding that sequence.

- For each natural number one can calculate the unique finite sequence of natural numbers that it codes (if any).

Therefore, by coding the basic symbols by natural numbers, formulas and proofs can be coded by natural numbers. These codes are called *gödel numbers*. If B is a formula, we will write b for its gödel number, c will be C's gödel number, etc. (We will not distinguish in notation between a natural number and its notation in the formal system, often called a *numeral*.) "being a proof" becomes a number-theoretic property of a number; "being provable" another number-theoretic property of a number. And one number "can be a proof of another one". Statements concerning the provability of number-theoretic formulas become number-theoretic statements. And actually these number-theoretic statements are simple enough to be formulated in T.

Another insight Gödel used in his proof was the insight (2) that for any property expressible in T there exists a sentence (of the formal system) for which it can be proved that it is equivalent to itself having this particular property.

Lemma 2 Diagonalization lemma. *For any expressible number-theoretic property Ax, a formula B exists such that Ab is true if and only if B and this is actually* provable *in* T.

Gödel applied this lemma to a variant of the *Liar paradox* ("This statement is false"), the number-theoretic property "x is not provable", written here $\neg Prov(x)$. The lemma then produces a sentence G (*the Gödel sentence*) such that $G \leftrightarrow \neg Prov(g)$ is provable and true; G seems to say: "I am not provable".

Both assuming G is provable and $\neg G$ is provable lead to unsurmountable difficulties:

- G *is provable* quickly leads to an inconsistency, because the provability of $Prov(g)$ follows easily from the provability of G, and, of course, $\neg Prov(g)$ does so as well. So: [If G is provable, then $1 = 0$ is provable].

- $\neg G$ *is provable* leads to the fact that $Prov(g)$ is provable, and $Prov(g)$ is an obvious falsity if G is not provable.

This leads to the version of the first theorem now most stated:
If the system of arithmetic one studies proves no *simple* false statements, then the statement $G(= \neg Prov(g))$ is true but not provable: $Prov(g)$ is a simple statement in the intended (exactly defined) sense.

Note that $\neg Prov(g)$ says that no number is a proof of G, $\forall x \neg Proof(x, g)$ (if one writes $Proof(x, g)$ for the formal expression of *x is a proof of* G). For each particular number n it is easy to show that it is not a proof of G: $\neg Proof(n, g)$. We have arrived at the phenomenon discussed just after Theorem 1 by which we can conclude immediately that G is true.

Actually, already Rosser showed how to improve on Gödel's conditions by using a slightly more complicated paradoxical statement: consistency of the formal system is sufficient to insure that his formula and its negation are not provable. One can formulate Rosser's construction as follows: Instead of looking at $Prov(x)$ we look at $Prov_R(x)$ which expresses that the formula coded by x has a proof such that for no number $y < x$, y is a proof of the negation of the formula coded by x. (Informally, this comes down to: a proof of A is recognized to be a Rosser-proof of A only if no smaller proof of $\neg A$ exists.) Instead of G one now considers a formula R such that $R \leftrightarrow \neg Prov_R(r)$ is the case.

8 The Second Incompleteness Theorem

The second incompleteness theorem derives from the observation that *if* G *is provable, then* $1 = 0$ *is provable* (shorter: $Prov(g) \rightarrow Prov(1 = 0)$) is a true number-theoretic statement, and that this statement can, in fact, be proved in arithmetic itself by a long, cumbersome, but basically simple argument[2]. Rewriting gives $\neg Prov(1 = 0) \rightarrow \neg Prov(g)$. The other direction of that implication is much easier to see, so that one in fact gets an equivalence:

$$\neg Prov(1 = 0) \leftrightarrow \neg Prov(g),$$

and therefore G is equivalent to the consistency of T ($1 = 0$ *is not provable*)

$$\neg Prov(1 = 0) \leftrightarrow G.$$

Since G is not provable, neither is $\neg Prov(1 = 0)$. We have arrived at the second incompleteness theorem. In stating it we explicitly mention the preconditions for the theorem to hold (see also Section 9).

Theorem 3 Second Incompleteness Theorem. *Let* T *prove only true formulas, and let* T *and* $Prov$ *satisfy the so-called Hilbert-Bernays-Löb conditions:*

[2]Actually, G ̈odel himself never did this. He just announced that he was going to do it, but when it turned out that everybody believed it without him fully executing the proof, he had more interesting things to do. The fi rst full version of the proof appeared in [13].

1. *For each A, if T proves A, then T proves $prov(a)$,*
2. *For each A, B, T proves $Prov(a \to b) \to (Prov(a) \to Prov(b))$,*
3. *For each A, T proves $Prov(a) \to Prov(Prov(a))$.*

Then T does not prove $\neg Prov(1=0)$.

We have arrived at another of Gödel's 'main achievements'.

- A proof of consistency in certain respect useless, i.e. it can be valuable but cannot in normal cases increase our confidence that the system is consistent.

That is, because to prove consistency of T one needs methods not contained in T. This makes consistency proofs by no means useless in all respects. The beautiful consistency proofs by Gentzen of Peano arithmetic which provide deep insights show the contrary. But the second incompleteness theorem was a death blow to Hilbert's program whatever e.g. is said to the contrary in the introduction to [13].
By formalizing one step further one can obtain that

$$T \text{ proves } \quad Prov(\neg Prov(1=0)) \to Prov(1=0) :$$

T proves: If it is provable that T is consistent, then T is inconsistent.

9 Some developments after Gödel

In this section we consider two more recent developments: provability logic, and Feferman's results on the second incompleteness theorem.
In 1955 Löb proved a strengthening of the second incompleteness theorem:

- For each A, if T proves $Prov(a) \to A$, then T proves A.

Note that Löb's theorem generalizes the second incompleteness theorem. If one replaces A by $1=0$ (and a by $1=0$'s gödel number), then we get the second incompleteness theorem. Löb's theorem expresses that the truth *if A is provable, then A is true* (called a *reflection principle*) can only be proved in the trivial case that A itself is provable. The formalized form of the theorem reads:

- For each A, T proves $Prov(Prov(a) \to a) \to Prov(a)$.

Note that this is not only a negative result, it is also a positive expression of what T can know (prove) about its own provability predicate. It turns out that this is in a sense all that T can prove about its own provability. Solovay [16] proved in 1976 that T can prove such a general statement built up from the connectives and $Prov$ if and only if the statement is derivable from the three Hilbert-Bernays-Löb conditions and Löb's Theorem (by a translation into modal logic). The resulting field of study is called *provability logic* (see e.g. [1]).

Feferman noted that in the formulation of the second incompleteness theorem it is necessary to be more precise than one usually is. The definition of the provability predicate $Prov(x)$ (based on the definition of the proof predicate) needs to be *intensionally correct* in the sense that in the formal definition of the proof predicate one has to follow in the formalization the steps of the arithmetization. Only then the Hilbert-Bernays-Löb conditions mentioned in Theorem 3 can be shown to apply. It is not sufficient to require that it is just *extensionally correct* in the sense that A has a proof iff $Prov(a)$ is true.

That may already be plausible to the reader on the basis of the Rosser proof predicate that we introduced at the end of Section 7. Since T is consistent, a situation that a proof of A exists and a smaller proof of $\neg A$ as well simply cannot occur. Therefore, the Rosser provability predicate and the usual provability predicate are equivalent: the Rosser provability predicate is an extensionally correct formalization of provability. Nevertheless, at least under the interpretation of consistency that $1 = 0$ is not provable, T can prove its own consistency with regard to the Rosser proof predicate. A proof of $\neg 1 = 0$ such that no smaller proof of $1 = 0$ exists can easily be constructed. This then readily implies that a Rosser proof of $1 = 0$ cannot exist. Such a simple proof as this one can easily be formalized in T. Hence T proves its own (Rosser) consistency.[3] In any case, the conclusion has to be that T cannot prove the equivalence of ordinary and Rosser provability.

Feferman himself considered a similar, somewhat more complicated provability predicate. Informally: Standard proofs are enumerated one after the other. Each time an inconsistency arises (a formula A such that $\neg A$ has been proved before, or a formula $\neg A$ such that A has been proved before), from the axiom system of T the largest axiom used in the offending proof is removed as well as all larger axioms. All previous 'theorems' using these axioms are removed as well ('large' in the sense of larger gödel numbers). A formula A is considered Feferman provable if A *gets a proof which is never removed afterwards*. As in the case of Rosser provability, since no contradiction ever occurs, Feferman provability is extensionally equivalent to ordinary provability, the same formulas remain provable in T. But again, T can prove its own Feferman consistency.

10 Lucas' Argument

Lucas [14] has argued, and his argument has been taken up later by a number of philosophers (see e.g. [15]) that Gödel's first incompleteness theorem can be used to show that humans are superior to machines. His argument ran as follows:

- Each machine is an instantiation of a formal system.
- Hence, given a machine that is consistent and able to do arithmetic, there is a true arithmetic formula, the Gödel sentence of that system, which the machine is not able to reproduce.

[3] One might object that for a Rosser provability predicate it is more reasonable to define consistency to mean that for no A, A and $\neg A$ are both provable. It may be necessary to consider a variant of the Rosser proof predicate if one wants to show that T proves its own Rosser consistency in this sense.

- Humans can recognize the truth of that Gödel sentence.
- Therefore, humans are superior to machines.

Generally, the gut feeling of logicians is that Gödel's theorems are too abstract to have such applications in the real world (see e.g. [6]). I share this feeling. Let me give some quick arguments.

That each machine is the embodiment of a formal system is at least doubtful. If one looks e.g. at the description of the Feferman provability predicate in Section 9 one will see that its definition is completely mechanical but the predicate is certainly too complicated to be definable as a formal system. In general machines that in reality or in the imagination use time in the way described cannot in general be expressed as a formal system.

For the human to conclude that the Gödel sentence of a system is true the human will need to know that the system is consistent. This is of course not decidable. Reasonably, the human can only conclude that, if the system is consistent its Gödel sentence is true, which does not seem really satisfactory. Moreover, there is another serious objection to this part of the argument. Reproducing the Gödel sentence of any formal system can equally well be executed by a machine in Lucas' sense, it can be stated as an ordinary mechanical procedure. Is only the (already shaky) recognition of the truth of the Gödel sentence the difference between that machine and a human? Then it seems as if Gödel's theorem hardly plays a role in the argument: the essential difference would be that the human can recognize the truth of some propositions and the machine never can (maybe not such an unreasonable argument). It is worth wile to note in passing that this particular machine applied to itself can reproduce its own Gödel sentence, but, by Gödel's second incompleteness theorem, it will not be able to prove its own consistency.

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