On the logical foundations of game theory
(Abstract)

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1. Introduction

During the last 15 years there has been growing interest among game theorists in epistemic conditions for game-theoretic solution concepts. Most of the work in this area has more or less explicitly employed some version of Kripke-style epistemic logic. Actually, most game theorists do not work with the syntactic formulations of epistemic logics, but instead with information-structures. An information structure, however, can be viewed as a Kripke model, and as the relation between Kripke models and normal modal systems are well known to logicians, I do not need to go into that here.

Instead, I will use the weakest normal system K (in fact, the multi-agent version thereof) to explain a problem with this kind of logic which I believe can only be adequately resolved by moving beyond "normal" modal systems. In the second part of the paper I will therefore suggest an epistemic logic which resolves the problem in what I believe to be a satisfactory way.

2. The logic K_Γ

For a given finite extensive game of perfect information (PI) Γ, we define the logic K_Γ as follows: atomic formulas are: move formulas a, b, c, ... one for each move of Γ, and preference formulas $a \succ_i b$ ... where a, b are any moves, and i is a player of Γ. Wffs are made up from these atomic formulas in the usual way by applications of negation, conjunction, and belief operators B_i. The axioms of K_Γ are the usual ones of multi-agent K plus Γ-specific axioms describing the rules of the game Γ and the preferences of the players according to the payoff function of Γ. For the simplest nontrivial example of a PI game, Γ_0, which has just one player 1, who has to choose between moves a and b, whereof he prefers the former, the Γ-specific axioms are $(a \vee b) \wedge \neg(a \wedge b)$ and $a \succ_1 b$. The rules of inference are modus ponens and epistemization (which may be applied to all the axioms including the Γ-specific ones).

3. The problem of self-knowledge of rationality and options

Within K_Γ, we give a sufficient condition for the backward induction play of Γ which can be shown to be weaker than the one of Aumann (1995). In this abstract, we explain our condition only for the one-player example Γ_0 (described above), which suffices to explain the problem we seek to solve in this paper.

As the player may have false beliefs in $K_{\Gamma 0,}$ his choice of *b* – contrary to his preference – may be due to his belief that *a* is not possible. This motivates a condition we call *relative* rationality:

(RR) $\neg B_1 \neg a \;\Rightarrow \neg\, b$

As $a \succ_1 b$ is an axiom of $\Gamma_{0,}$ this says that the player will not take action b if he considers the preferred action possible. Clearly, RR $\Rightarrow$ a does not hold in $K_{\Gamma 0.}$ However, it seems natural to add the assumption that the player does consider *a* possible. We call this assumption Possibility of Backward Induction moves:

(PBI) $\neg B_1 \neg a$

For our simple example, RR $\wedge$ PBI $\Rightarrow$ *a* is trivially a theorem of Γ_0. (For the general case, an analogous, but more elaborate theorem holds.) However, a problem arises from the fact that it seems natural and in line with the usual informal assumptions of game theory to assume of *all* moves that they are considered possible, and that there is mutual (or even common) belief in rationality and the structure of the game. Clearly, $B_1($ RR $\wedge \neg B_1 \neg a \wedge \neg B_1 \neg b)$ is inconsistent in Γ_0: The player can infer from what he believes that what he considers possible will not be the case.

2. The logic L_Γ

To resolve the above problem we suggest an epistemic logic which has a sequence of belief operators B^0, B^1, B^2, … for each player, corresponding to the temporal sequence of the player's states of belief. Limiting ourselves (in this abstract) to the one-player case again, we consider the axiomatic system (for which we also provide a belief-set semantics, similar to the autoepistemic logics of Moore, 1985, and Konolige, 1988) with the following axiom schemes:

(A1) φ , whenever φ is a propositional calculus tautology or a Γ-specific axiom;
(A2) $B^t(\varphi)$, whenever φ is a propositional calculus tautology or a Γ-specific axiom;
(A3) $B^t(\varphi) \wedge B^t(\psi) \Rightarrow B^t(\varphi \wedge \psi)$;
(A4) $B^t(\varphi) \Rightarrow B^{t+1}(\varphi)$;
(A5) $B^t(\varphi) \Rightarrow B^{t+1}(B^t(\varphi))$;
(A6) $\neg B^t(\varphi) \Rightarrow B^{t+1}(\neg B^t(\varphi))$;

the sole rule of inference being modus ponens. Among other properties of this logic L_Γ , we show that a delayed version of the epistemization rule holds.

3. A Solution to the Problem

Within L_Γ, the problem explained above can be easily resolved: Writing (RR^0) for $\neg B^0 \neg a \Rightarrow \neg b$, one can verify that $\neg B^0 \neg a \wedge \neg B^0 \neg b \wedge B^0(RR^0)$ is consistent, and so is $B^1(RR^0 \wedge \neg B^0 \neg a \wedge \neg B^0 \neg b)$. These formulas can be naturally taken to describe a situation where *initially* the player considers both options possible and himself to be rational, *and then*, on reflection, recognizes that he will not take b, while remembering that he initially considered both options possible.

A multi-agent version of L_Γ can be used to reconstruct both the backward induction argument and that it may fail if the players have insufficient knowledge about each other's reasoning processes.

References

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