Inference and Update *

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Abstract

We look at two fundamental logical processes, often intertwined in planning and problem solving: inference and update. Inference is an internal process with which we uncover what is implicit in the information we already have. Update, on the other hand, is produced by external communication, usually in the form of announcements and in general in the form of observations, giving us information that might have been not available (even implicitly) before. Both processes have received attention from the logic community, usually separately. In this work, we develop a logical language that allows us to describe them together. We present syntax and semantics, as well as a complete axiom system. We also discuss similarities and differences with other approaches, and we mention some possible ways the work can be extended.

1 Introduction

Consider the following situation, from van Benthem [2008a]:

> You are in a restaurant with your parents, and you have ordered three dishes: fish, meat, and vegetarian, for you, your father and your mother, respectively. Now a new waiter comes from the kitchen with the three dishes. What can he do to get to know which dish corresponds to which person?

The waiter can ask *"Who has the fish?"*; then he can ask *"Who has the meat?"*. Now he does not have to ask anymore: "two questions plus one inference are all that is needed" (van Benthem [2008a]).

The present work looks at these two fundamental logical processes, often intertwined in planning, problem solving and real-life activities. Inference is an *internal* process: the agent revises her own information in search of what can be derived from it. Update, on the other hand, is produced by *external* communication: the agent gets new information via observations. Both are logical processes, both describe dynamics of information, both are used in every day situations, and still, they have been studied separately.

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Inference has been traditionally taken as the main subject of study of logic, allowing us to extract new information from what we already have. Among the most important branches, we can mention Hilbert-style proof systems, natural deduction and tableaux. Recent works, like Duc [1995, 1997] and Jago [2006b,a] have incorporated modal logics to the field, representing inference as a non-deterministic step-by-step process.

Update, on the other hand, has been the main subject of what have been called *Dynamic Epistemic Logic*. Works like Plaza [1989], Gerbrandy [1999a] and Gerbrandy [1999b] turned attention to the effect of public announcements over the knowledge of an agent. Many works have followed them, including the study of more complex actions (Baltag et al. [1999], van Ditmarsch [2000], Baltag and Moss [2004]) and the effect of announcements over a more wide propositional attitudes (the soft/hard facts of van Benthem [2007], the knowledge/belief of Baltag and Smets [2008]).

In van Benthem [2008c], the author shows how these two phenomena fall directly within the scope of modern logic. As he emphasize, "asking a question and giving an answer is just as 'logical' as drawing a conclusion!". Here, we propose a merging of the two traditions. We consider that both processes are equally important in their own right, but so it is their interaction. In this work, we develop a logical language that allows us to express both inference and update together.

We start in section by 2 providing a modal framework for representing the agent's *implicit* and *explicit* information, and isolate the case of true information. Then, in section 3, we provide a representation of *inference* and we focus on the truth-preserving case. Moreover, we show how dynamics of the inference process itself can be described too. Section 4 introduces the other logical process: *update*. We compare our work with other approaches in section 5, and we conclude in section 6 with a summary and some further work we consider interesting. The present work focuses in the single-agent case, leaving the analysis of group-information concepts like common or distributed knowledge for the future.

2 Implicit and explicit information

Our goal is to represent the agent's information, and how it evolves through the use of inference and update. The *Epistemic Logic* (EL) framework with Kripke models (Hintikka [1962]) is one of the most widely used for representing and reasoning about agents' information. Nevertheless, it is not fine enough to represent the Restaurant example. Agents whose information is represented within such framework suffer from what Hintikka called the *logical omniscience* problem[1]: they are informed of all validities and their information is closed under truth-preserving inference.

Though this feature is useful in some applications, it is too much in some others. More important for us, it *hides* the inference process. If we represent the Restaurant example with Kripke models, the answer to the second question informs the waiter not only that your father will get the meat but also that your mother will get the vegetarian dish. In this case, the inference is very

[1]See Sim [1997] for a survey about the logical omniscience problem.

simple, and some could even say that there is no inference at all. Nevertheless, inferences are in general more complex: proving a theorem basically consists on successive application of deductive inference steps in order to show that the conclusion indeed follows from the premises. Some theorems may be straightforward but, as we know, some are not.

As van Benthem argues in van Benthem [2006], we can give to the modal operator a more *implicit* reading, describing not the agent's current information, but what she may get to know after enough time and inference steps. This work uses that interpretation, reading formulas of the form $\Box\varphi$ as *"the agent is implicitly informed about φ"*. With this idea in mind, we extend the EL framework to represent *explicit* information too. Moreover, we also provide a mechanism with which the agent can increase it. The work of this section resembles those presented in Fagin and Halpern [1988], Duc [1995, 1997], Jago [2006b] and Jago [2006a]; the precise relation will be clarified in section 5.

2.1 Formulas, rules and the explicit/implicit information language

The agent's explicit information will be given by a set of formulas; the mechanism that allows her to improve it will be given by syntactic rules. Our first step is to define the language from which these formulas come from, and what a rule in that language is.

Definition 2.1 (Formulas and rules)**.** Let P be a set of atomic propositions

- A formula γ of the *internal language* $\mathcal{I}$ is given by

$$\gamma ::= p \mid \neg\gamma \mid \gamma \vee \delta$$

  with $p \in \mathrm{P}$.

- A *rule* based on $\mathcal{I}$ is a pair (Γ, γ) (sometimes represented as $\Gamma \Rightarrow \gamma$) where Γ is a finite set of formulas and γ is a formula, all of them in $\mathcal{I}$. Given a rule $\rho = (\Gamma, \gamma)$, we call Γ the *set of premises of* ρ (prem(ρ)) and γ the *conclusion of* ρ (conc(ρ)). We denote by $\mathcal{R}_{\mathcal{I}}$ the set of rules based on formulas of $\mathcal{I}$, omitting the subindex when no confusion arises.

While formulas describe situations about the world, rules describe relations between such situations. Intuitively, the rule (Γ, γ) indicates that if every $\delta \in \Gamma$ is true, so it is γ.

The internal language is used for representing the agent's explicit information. By using just the propositional one, we allow the agent to have explicit information about situations, but not about her own information or the information of other agents. This is indeed a limitation, but it makes possible the update definition of section 4. In section 6 we briefly discuss the reasons for this limitation, leaving a deep analysis for a further work.

Now we can give the definition of our framework. Syntactically, we extend *Epistemic Logic* by adding two new kinds of formulas: one to express the agent's explicit information ($I\,\gamma$) and another to express the rules she can apply ($L\,\rho$).

Definition 2.2 (Explicit/implicit information language $\mathcal{EI}$)**.** Let $\mathtt{P}$ be a set of atomic propositions. The formulas of the *explicit/implicit information language* $\mathcal{EI}$ are given by

$$\varphi ::= \top \mid p \mid I\,\gamma \mid L\,\rho \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\varphi$$

with $p \in \mathtt{P}$, $\gamma \in \mathcal{I}$ and $\rho \in \mathcal{R}$. Formulas of the form $I\,\gamma$ are read as *"the agent is explicitly informed about γ"*), and formulas of the form $L\,\rho$ are read as *"the agent can apply rule ρ"*.

The boolean connectives $\wedge$, $\rightarrow$ and $\leftrightarrow$ as well as the modal operator $\Box$ are defined as usual.

Semantically, each world of our model has three components: a valuation for the truth value of atomic propositions (just as in possible worlds models) and two sets: one indicating the *formulas* the agent is explicitly informed, and other indicating the *rules* she can apply at that world. We have just one relation between worlds, the accessibility one, indicating which ones the agent considers possible from a given one.

Definition 2.3 (Explicit/implicit information model)**.** Let $\mathtt{P}$ be a set of atomic propositions. An *explicit/implicit information model* is a tuple $M = \langle W, R, V, Y, Z\rangle$ where:

- W is a non-empty set of worlds.
- $R \subseteq W \times W$ is the *accessibility relation*, describing the agent's implicit information.
- $V : W \rightarrow \wp(\mathtt{P})$ is an *atomic valuation function*, indicating the atomic propositions that are true at each possible world.
- $Y : W \rightarrow \wp(\mathcal{I})$ is the *information set function*, indicating the explicit information of the agent at each possible world. We ask for the information sets to be preserved by the accessibility relation: if $\gamma \in Y(w)$ and Rwu, then $\gamma \in Y(u)$ (the *coherence* property for formulas).
- $Z : W \rightarrow \wp(\mathcal{R})$ is the *rule set function*, indicating the rules the agent can apply at each possible world. We ask for the rule sets to be preserved by the accessibility relation: if $\rho \in Z(w)$ and Rwu, then $\rho \in Z(u)$ (the *coherence* property for rules).

We will denote with **EI** the class of all explicit/implicit information models.

In the definition of the model we have two restrictions, reflecting our idea of what it represents. Intuitively, the sets $Y(w)$ and $Z(w)$ represent the formulas and rules the agent is explicitly informed about. If while staying in w the agent considers u possible, then it is natural to ask for u to preserve everything the agent is explicitly informed at w.

Formulas of $\mathcal{EI}$ are interpreted in models of **EI** as follows.

Definition 2.4 (Semantics for $\mathcal{EI}$)**.** Given a model $M = \langle W, R, V, Y, Z\rangle$ in **EI** and a world $w \in W$, the *satisfaction* relation $\Vdash$ between the pair (M, w) and $\top$, negations and disjunctions is given as usual. The case for atomic propositions p and implicit information formulas $\Diamond\varphi$ is just like in epistemic logic. For explicit information and rule formulas, we just look at the corresponding sets.

$(M,w) \Vdash I\,\gamma \quad \textit{iff} \quad \gamma \in Y(w)$
$(M,w) \Vdash L\,\rho \quad \textit{iff} \quad \rho \in Z(w)$

As usual, we have the following definitions.

- A formula φ is *true at w in M* whenever $(M,w) \Vdash \varphi$.
- A a formula φ is *valid in M* (notation: $M \Vdash \varphi$) whenever φ is *true at w in M* for *all worlds w* in M.
- A formula φ is *valid in the class of models* $\mathbf{M}$ (notation: $\mathbf{M} \Vdash \varphi$) if φ is valid in M for *all models M* in $\mathbf{M}$.

We provide a syntactic characterization of formulas of $\mathcal{EI}$ that are valid in the class of models **EI**. Non-defined concepts, like a (modal) logic, Λ-consistent (inconsistent) set and maximal Λ-consistent set (for a normal modal logic Λ) are completely standard, and can be found in chapter 4 of Blackburn et al. [2001].

Definition 2.5 (Logic EI)**.** The logic EI is built from the axioms and rules shown in table 1. Axioms *P*, *K* and *Dual* as well as the two rules *MP* and *Gen* are standard for modal logic. Axioms $Coh_{\mathcal{I}}$ and $Coh_{\mathcal{R}}$ describe the particular requirements of the models, the *coherence* property for formulas and rules, respectively.

Axioms and inference rules for EI	
(*P*)	All propositional tautologies
(*K*)	$\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\phi)$
(*Dual*)	$\vdash \Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$
($Coh_{\mathcal{I}}$)	$\vdash I\,\gamma \to \Box I\,\gamma$
($Coh_{\mathcal{R}}$)	$\vdash L\,\rho \to \Box L\,\rho$
(*MP*)	From $\vdash \varphi$ and $\vdash \varphi \to \psi$ infer $\vdash \psi$
(*Gen*)	From $\vdash \varphi$ infer $\vdash \Box\varphi$

Table 1: Axioms and inference rules for EI.

Theorem 1 (Soundness and completeness of EI w.r.t. **EI**)**.** *The logic* EI *is sound and strongly complete with respect to the class* **EI***.*

Proof. For soundness, we just need to prove that axioms of EI are valid in **EI**, and that its rules preserve validity. We omit the details here.

For completeness, we recall that strong completeness is equivalent to satisfiability of consistent set of formulas (see Proposition 4.12 of Blackburn et al. [2001]). We define the canonical model M^{EI} for the logic EI; with the Lindenbaum's Lemma, the Existence Lemma and the Truth Lemma, we show that every EI-consistent set of formulas is satisfiable in M^{EI}. Finally, we show that M^{EI} is indeed a model in **EI**. See section A.1 for details. □

Note how the agent's implicit information includes all validities (as the validity-preserving of the *Gen* rule shows) and is closed under modus ponens (as the validity of axiom *K* shows). Nevertheless, her explicit information does not have that properties, since the validity of γ does not imply the validity of $I\,\gamma$ and the formula $I\,(\gamma \to \delta) \to (I\,\gamma \to I\,\delta)$ is not valid.

2.2 When information is knowledge

As it is currently defined, an explicit/implicit information model do not impose any restriction on the information of the agent. We do not ask for any property for the accessibility relation, so there are no constrains for implicit information, others than those given by the representation itself, like closure under modus ponens (the K axiom) and the inclusion of validities (the *Gen* rule). In the case of explicit information, we can have information sets with formulas that are not true, or even inconsistent ones. Moreover, the models do not impose any restriction about the rules the agent can apply: they do not have to be truth-preserving.

Among the models in **EI**, we can distinguish those in which both implicit and explicit information are true, and the rules are truth-preserving. We start by considering models with equivalence accessibility relations, as it is usually done in *Epistemic Logic*; this makes implicit information true. For explicit information, we ask for every formula in the information set to be true at the corresponding world. Finally, in the case of rules, we define a translation TR that takes rules in $\mathcal{R}$ into an implication in $\mathcal{EI}$ whose antecedent is the conjunction of the premises and whose consequent is the conclusion.

$$\mathrm{TR}(\rho) := \bigwedge \mathrm{prem}(\rho) \rightarrow \mathrm{conc}(\rho)$$

The formal definition of this particular class of models is as follows.

Definition 2.6 (The class $\mathbf{EI}_K$)**.** We denote by $\mathbf{EI}_K$ the class of explicit/implicit information models satisfying

- ***Equivalence*:** R is an equivalence relation.
- ***Truth for formulas*:** for every world $w \in W$, if $\gamma \in Y(w)$, then $(M, w) \Vdash \gamma$.
- ***Truth for rules*:** for every world $w \in W$, if $\rho \in Z(w)$, then $(M, w) \Vdash \mathrm{TR}(\rho)$.

Models in this class will be also called *knowledge* models, and instead of explicit/implicit information we will talk about explicit/implicit knowledge.

For a syntactic characterization of formulas valid in $\mathbf{EI}_K$ models, we have the logic EI_K.

Definition 2.7 (Logic EI_K)**.** The logic EI_K extends EI (see definition 2.5) by adding the axioms of table 2. Axioms T, 4 and B make the accessibility relation an equivalence relation. Axiom $Tth_{\mathcal{I}}$ guarantees that formulas in the agent's information set are true, and axiom $Tth_{\mathcal{R}}$ guarantees that the rules she can apply are truth-preserving.

Theorem 2 (Soundness and completeness of EI_K w.r.t. $\mathbf{EI}_K$)**.** *The logic EI_K is sound and strongly complete with respect to the models in the class $\mathbf{EI}_K$.*

Proof. The fact that axioms of table 2 are valid in $\mathbf{EI}_K$ proves soundness. The fact that the canonical model for the logic EI_K satisfies the *equivalence*, *truth for formulas* and *truth for rules* properties proves completeness. Details can be found in section A.2. □

Extra axioms for the logic EI_K	
(T)	$\vdash \varphi \to \Diamond\varphi$
(4)	$\vdash \Diamond\Diamond\varphi \to \Diamond\varphi$
(B)	$\vdash \Diamond\Box\varphi \to \varphi$
$(Tth_{\mathcal{I}})$	$\vdash I\,\gamma \to \gamma$
$(Tth_{\mathcal{R}})$	$\vdash L\,\rho \to \mathrm{TR}(\rho)$

Table 2: Extra axioms for the logic EI_K.

We recall the *coherence* property for formulas and rules: if $\gamma \in Y(w)$ ($\rho \in Z(w)$) and Rwu, then we have $\gamma \in Y(u)$ ($\rho \in Z(u)$). When the accessibility relation is an equivalence relation, we get the same information and rule set for all the worlds that belong to the same equivalence class.

Note how in knowledge models, from axiom $Coh_{\mathcal{I}}$ ($I\,\gamma \to \Box I\,\gamma$) and axiom $Tth_{\mathcal{I}}$ ($I\,\gamma \to \gamma$) we get $I\,\gamma \to \Box\gamma$. The same apply for rules: from $Coh_{\mathcal{R}}$ ($L\,\rho \to \Box L\,\rho$) and $Tth_{\mathcal{R}}$ ($L\,\rho \to \mathrm{TR}(\rho)$) we get $L\,\rho \to \Box\mathrm{TR}(\rho)$. These formulas indicate that whatever is part of the agent's explicit information also belongs to her implicit information.

It is now time to turn our attention into the *dynamics* of information. In the following sections, we extend the language to describe two different ways in which the agent can improve her knowledge. Through inferences, she will extract information that is implicit in what she explicitly have; moreover, we will provide her a mechanism to improve her inferential abilities. Through updates, she will get information that may not be available (even implicitly) to her before.

3 Inference

The agent can extend her explicit knowledge by using rules. As mentioned before, a rule (Γ, γ) intuitively indicates that if every $\delta \in \Gamma$ is true, so it is γ. However, in principle, there is no restriction in the way the agent can use a rule. She can use it to get the conclusion without having all the premises, or even deriving the premises whenever she has the conclusion.

In the previous section we distinguished, among all the explicit/implicit information models, those in which the agent's information is true. In the same spirit, in this section we will define an inference operation that preserves truth.

3.1 A particular case: truth-preserving inference

The inference process adds formulas to the information set. Since we want to represent a *truth-preserving* inference, we restrict the way in which the rule can be applied. The *deduction* operation over a model M is defined as follows.

Definition 3.1 (Deduction operation)**.** Let $M = \langle W, R, V, Y, Z\rangle$ be a model in the class **EI**, and let σ be a rule in $\mathcal{R}$. The model $M_\sigma = \langle W', R', V', Y', Z'\rangle$ is given by

- $W' := W$, $R' := R$

and, for every $w \in W'$,

- $V'(w) := V(w),\ Z'(w) := Z(w)$ and
- $Y'(w) := \begin{cases} Y(w) \cup \{\text{conc}(\sigma)\} & \text{if prem}(\sigma) \subseteq Y(w) \text{ and } \sigma \in Z(w) \\ Y(w) & \text{otherwise} \end{cases}$

The operation $(\cdot)_\sigma$ is called the *deduction operation* with rule σ.

Note how the conclusion of the rule is added to a world just when all the premises and the rule are already there. This allows us to prove that, in particular, the deduction operation preserves models in $\mathbf{EI}_K$.

Proposition 1. *Let σ be a rule in $\mathcal{R}$. If M is a model in $\mathbf{EI}_K$, so it is M_σ.*

Proof. See section A.3. □

The language $\mathcal{EID}$ extends $\mathcal{EI}$ by closing it under deduction operations. Take a rule σ: if φ is a formula in $\mathcal{EID}$, so it is $\langle \mathrm{D}_\sigma \rangle \varphi$. These new formulas are read as *"there is a way of deductively applying σ after which φ is the case"*. Define the abbreviation

$$\mathrm{Pre}_\sigma \equiv I\ \mathrm{prem}(\sigma) \wedge L\,\sigma$$

where, for Γ a finite set of formulas of the internal language, we write $I\,\Gamma$ for $\bigwedge_{\gamma \in \Gamma} I\,\gamma$. Then, the semantics for deduction formulas is given as follows.

Definition 3.2. Let M be a model in **EI**, and take a world w in it.

$$(M, w) \Vdash \langle \mathrm{D}_\sigma \rangle \varphi \quad \textit{iff} \quad (M, w) \Vdash \mathrm{Pre}_\sigma \text{ and } (M_\sigma, w) \Vdash \varphi$$

The formula $[\mathrm{D}_\sigma]\varphi$ is defined as the dual of $\langle \mathrm{D}_\sigma \rangle \varphi$, that is, $[\mathrm{D}_\sigma]\varphi \leftrightarrow \neg\langle \mathrm{D}_\sigma \rangle \neg\varphi$. Therefore,

$$(M, w) \Vdash [\mathrm{D}_\sigma]\varphi \quad \textit{iff} \quad (M, w) \Vdash \mathrm{Pre}_\sigma \text{ implies } (M_\sigma, w) \Vdash \varphi$$

We now provide a syntactic characterization of the formulas in $\mathcal{EID}$ that are valid in models in $\mathbf{EI}_K$. By proposition 1, the deduction operation is closed for models in $\mathbf{EI}_K$, so we can rely on the logic EI_K: all we have to do is give a set of reduction axioms for formulas of the form $\langle \mathrm{D}_\sigma \rangle \varphi$

Definition 3.3 (Logic EI_{KD})**.** The logic EI_{KD} is built from axioms and rules of EI_K (see definition 2.7) plus axioms and rules in table 3. Each one of the axioms express how formulas *after* the deduction operation are related with formulas *before* the operation.

Theorem 3 (Soundness and completeness of EI_{KD} w.r.t. $\mathbf{EI}_K$)**.** *The logic EI_{KD} is sound and strongly complete for the class of models $\mathbf{EI}_K$.*

Proof. Soundness follows from the validity of the new axioms and the validity-preserving property of the new rule. Strong completeness follows from the fact that, by a repetitive application of such axioms, any deduction operation formula can be reduced to a formula in $\mathcal{EI}$, for which EI_K is strongly complete with respect to $\mathbf{EI}_K$. □

Axioms and inference rules for the deduction operation		
$D_\top$	$\vdash \langle \mathrm{D}_\sigma \rangle \top \;\leftrightarrow\; \mathrm{Pre}_\sigma$	
D_p	$\vdash \langle \mathrm{D}_\sigma \rangle p \;\leftrightarrow\; (\mathrm{Pre}_\sigma \wedge p)$	
$D_\neg$	$\vdash \langle \mathrm{D}_\sigma \rangle \neg\varphi \;\leftrightarrow\; (\mathrm{Pre}_\sigma \wedge \neg\langle \mathrm{D}_\sigma \rangle \varphi)$	
$D_\vee$	$\vdash \langle \mathrm{D}_\sigma \rangle (\varphi \vee \psi) \;\leftrightarrow\; (\langle \mathrm{D}_\sigma \rangle \varphi \vee \langle \mathrm{D}_\sigma \rangle \psi)$	
$D_\diamond$	$\vdash \langle \mathrm{D}_\sigma \rangle \diamond\varphi \;\leftrightarrow\; (\mathrm{Pre}_\sigma \wedge \diamond\langle \mathrm{D}_\sigma \rangle \varphi)$	
D_I	$\vdash \langle \mathrm{D}_\sigma \rangle I \,\mathrm{conc}(\sigma) \;\leftrightarrow\; \mathrm{Pre}_\sigma$	
D_I	$\vdash \langle \mathrm{D}_\sigma \rangle I \gamma \;\leftrightarrow\; (\mathrm{Pre}_\sigma \wedge I \gamma)$	for $\gamma \neq \mathrm{conc}(\sigma)$
D_L	$\vdash \langle \mathrm{D}_\sigma \rangle L \rho \;\leftrightarrow\; (\mathrm{Pre}_\sigma \wedge L \rho)$	
Gen_{D}	From $\vdash \varphi$, infer $\vdash [\mathrm{D}_\sigma] \varphi$	

Table 3: Axioms and rules for deduction operation formulas.

In previous versions of the present work (Velazquez-Quesada [2008]) inference was represented as a modal relation between worlds. Formulas of the form $\langle \sigma \rangle \varphi$, read as *"there is a way to apply the rule σ after which φ is the case"*, were interpreted with a relation D_σ for every rule σ. In order to preserve the intuitive meaning, some properties for the relation were required. Inference relations should relate worlds with the same valuation and, for the particular case of deduction, the following four properties were stated, indicating properties that the sets of formulas should have to properly represent a truth-preserving inference:

1. to apply a rule we need the premises and the rule,
2. after applying it, we preserve the explicit information we had before,
3. the explicit information is increased by the conclusion of the rule, and
4. there is no other difference between the explicit information before and after the rule application.

In the present work, inference is represented not as a modal relation but as a model operation. From its definition we can see that deduction preserves world-valuation. But not only that: the four previous properties still hold, as the validity of the following formulas shows.

1. $\langle \mathrm{D}_\sigma \rangle \top \rightarrow \mathrm{Pre}_\sigma$	2. $I \gamma \rightarrow [\mathrm{D}_\sigma] I \gamma$
3. $[\mathrm{D}_\sigma] I \,\mathrm{conc}(\sigma)$	4. $\langle \mathrm{D}_\sigma \rangle I \gamma \rightarrow I \gamma$ (for $\gamma \neq \mathrm{conc}(\sigma)$)

Moreover, in this representation, inference is functional: we can apply a rule deductively *every time* we have it together with its premises, and the result of the application is unique. The first property was a problem for update operations in the previous setting, and the second one was only expressible with the use of nominals.

The deduction operation adds a formula to the information sets only if it is the conclusion of an applicable rule (i.e., we already have the rule and its premises). Note how in such cases, the conclusion of the rule is actually part of the agent's implicit information: if a rule is applicable in the current world, then we have it as well as its premises. Axioms $Coh_{\mathcal{I}}$ and $Coh_{\mathcal{R}}$ put both premises and rule in every world of the equivalence class, and $Tth_{\mathcal{I}}$ and $Tth_{\mathcal{R}}$ make the premises and the implication that results of translating the rule true at them. Then, the K axiom makes the conclusion true in all these worlds. The validity of $(I\ \text{prem}(\rho) \wedge L\,\rho) \rightarrow \Box\text{conc}(\rho)$ proves our point, but the converse is not true. In general there are formulas belonging to the agent's implicit information that are not reachable via deduction, since her rule set may not have a rule to derive each one of them. And even if there are such rules, the agent may not have all the needed premises.

3.2 Dynamics of deduction

Just as the agent's explicit information changes, her inferential abilities can also change. This may be because he is informed about another rule (as we will describe when we work with *updates* in section 4), but it may be also because she *builds* new rules from the ones she already has. For example, from the rules $(\{p\}, q)$ and $(\{q\}, r)$, it makes sense to derive the rule $(\{p\}, r)$. It may take one step to the agent to add it to her rule set, but it will save intermediate steps later by allowing the agent to go directly from having p to having r.

In fact the example, a kind of *transitivity*, represents the application of *Cut* over the rules the agent has available. In general, inference relations can be characterized by the *structural rules* they satisfy. These structural rules indicate how to derive new rules from the ones we already have. In the case of deduction the rules are the following:

Reflexivity: $\dfrac{}{\varphi \Rightarrow \varphi}$ *Contraction:* $\dfrac{\psi, \chi, \xi, \chi, \phi \Rightarrow \varphi}{\psi, \chi, \xi, \phi \Rightarrow \varphi}$

Permutation: $\dfrac{\psi, \chi, \xi, \phi \Rightarrow \varphi}{\psi, \xi, \chi, \phi \Rightarrow \varphi}$ *Monotonicity:* $\dfrac{\psi, \phi \Rightarrow \varphi}{\psi, \chi, \phi \Rightarrow \varphi}$

Cut: $\dfrac{\chi \Rightarrow \xi \quad \psi, \xi, \phi \Rightarrow \varphi}{\psi, \chi, \phi \Rightarrow \varphi}$

Each time a structural rule is applied, we get a new rule. In our framework, the application of an structural rule modifies the model by adding the new rule to the rule set. Note that *Contraction* and *Permutation* are not so interesting for us, since we are already considering the premises of a rule as a set, and hence their application does not yield a new rule. On the other hand, *Reflexivity*, *Monotonicity* and *Cut* can increase the agent's inferential abilities.

Definition 3.4 (Structural operations)**.** Let $M = \langle W, R, V, Y, Z\rangle$ be a model in **EI**.

Reflexivity. Let δ be a formula of the internal language. Consider the rule $\varsigma_\delta = (\{\delta\}, \delta)$. The model $M_{\mathbf{Ref}(\delta)} = \langle W', R', V', Y', Z'\rangle$ is the result of adding ς_δ to every rule set. Formally,

- $W' := W$, $R' := R$,
- $V'(w) := V(w)$, $Y'(w) := Y(w)$, for every $w \in W'$,
- $Z'(w) := Z(w) \cup \{\varsigma_\delta\}$, for every $w \in W'$.

The operation $(\cdot)_{\mathbf{Ref}(\delta)}$ is called the *reflexivity operation* with δ.

Monotonicity. Let δ be a formula in $\mathcal{I}$, and let ς be a rule over $\mathcal{I}$. Consider the rule $\varsigma' = (\text{prem}(\varsigma) \cup \{\delta\}, \text{conc}(\varsigma))$, extending ς by adding δ to its premises. The model $M_{\mathbf{Mon}(\delta,\sigma)} = \langle W', R', V', Y', Z' \rangle$ is the result of adding ς' to every rule set that already contains ς. Formally,

- $W' := W, \; R' := R$

and, for every $w \in W'$,

- $V'(w) := V(w), \; Y'(w) := Y(w)$ and
- $Z'(w) := \begin{cases} Z(w) \cup \{\varsigma'\} & \text{if } \varsigma \in Z(w) \\ Z(w) & \text{otherwise} \end{cases}$

The operation $(\cdot)_{\mathbf{Mon}(\delta,\varsigma)}$ is called the *monotonicity operation* with δ and ς.

Cut. Let ς_1, ς_2 be rules over the internal language, such that the conclusion of ς_1 is contained in the premises of ς_2 (that is, $\text{conc}(\varsigma_1) \in \text{prem}(\varsigma_2)$). Consider the rule $\varsigma' = ((\text{prem}(\varsigma_2) - \{\text{conc}(\varsigma_1)\}) \cup \text{prem}(\varsigma_1)\,,\, \text{conc}(\varsigma_2))$, combining ς_1 and ς_2. The model $M_{\mathbf{Cut}(\varsigma_1,\varsigma_2)} = \langle W', R', V', Y', Z' \rangle$ is the result of adding ς' to every rule set that already has ς_1 and ς_2. Formally,

- $W' := W, \; R' := R$

and, for every $w \in W'$,

- $V'(w) := V(w), \; Y'(w) := Y(w)$ and
- $Z'(w) := \begin{cases} Z(w) \cup \{\varsigma'\} & \text{if } \{\varsigma_1, \varsigma_2\} \subseteq Z(w) \\ Z(w) & \text{otherwise} \end{cases}$

The operation $(\cdot)_{\mathbf{Cut}(\varsigma_1,\varsigma_2)}$ is called the *cut operation* with ς_1 and ς_2.

Just as the deduction operation, the three structural operations preserve models in $\mathbf{EI}_K$.

Proposition 2 (Closure of structural operations)**.** *Let M be a model in $\mathbf{EI}_K$, and let $M_{\mathbf{STR}}$ stand for either $M_{\mathbf{Ref}(\delta)}$, $M_{\mathbf{Mon}(\delta,\varsigma)}$ or $M_{\mathbf{Cut}(\varsigma_1,\varsigma_2)}$. If M is in $\mathbf{EI}_K$, so it is $M_{\mathbf{STR}}$.*

Proof. See section A.4. □

We enrich the language to express how the structural operations change the agent's inferential abilities. The language $\mathcal{EID}^*$ extends $\mathcal{EID}$ by making it closed under formulas representing structural operations: if φ is in $\mathcal{EID}^*$, so they are $\langle Ref_\delta \rangle \varphi$, $\langle Mon_{\delta,\varsigma} \rangle \varphi$ and $\langle Cut_{\varsigma_1,\varsigma_2} \rangle \varphi$. Each one of the formulas are read as *"there is a way of applying the structural operation after which φ is the case"*. With the following abbreviations

$$\begin{aligned} \text{Pre}_{\mathbf{Mon}(\delta,\varsigma)} &\equiv L\,\varsigma \\ \text{Pre}_{\mathbf{Cut}(\varsigma_1,\varsigma_2)} &\equiv L\,\varsigma_1 \wedge L\,\varsigma_2 \wedge (I\ \text{prem}(\varsigma_2) \to I\ \text{conc}(\varsigma_1)) \end{aligned}$$

the semantics of the new formulas is given as follows.

Definition 3.5. Let M be a model in $\mathbf{EI}$, and let $w \in W$ be a world in it. Then,

$$\begin{array}{lll}(M,w) \Vdash \langle Ref_{\delta}\rangle\,\varphi & \textit{iff} & (M_{\mathbf{Ref}(\delta)}, w) \Vdash \varphi \\ (M,w) \Vdash \langle Mon_{\delta,\varsigma}\rangle\,\varphi & \textit{iff} & (M,w) \Vdash \mathrm{Pre}_{\mathbf{Mon}(\delta,\varsigma)} \text{ and } (M_{\mathbf{Mon}(\delta,\varsigma)}, w) \Vdash \varphi \\ (M,w) \Vdash \langle Cut_{\varsigma_1,\varsigma_2}\rangle\,\varphi & \textit{iff} & (M,w) \Vdash \mathrm{Pre}_{\mathbf{Cut}(\varsigma_1,\varsigma_2)} \text{ and } (M_{\mathbf{Cut}(\varsigma_1,\varsigma_2)}, w) \Vdash \varphi\end{array}$$

Just as before, the *boxed* versions of the structural operation formulas is defined as the dual of their correspondent *diamond* versions:

$$\begin{array}{lll}(M,w) \Vdash [Ref_{\delta}]\,\varphi & \textit{iff} & (M_{\mathbf{Ref}(\delta)}, w) \Vdash \varphi \\ (M,w) \Vdash [Mon_{\delta,\varsigma}]\,\varphi & \textit{iff} & (M,w) \Vdash \mathrm{Pre}_{\mathbf{Mon}(\delta,\varsigma)} \text{ implies } (M_{\mathbf{Mon}(\delta,\varsigma)}, w) \Vdash \varphi \\ (M,w) \Vdash [Cut_{\varsigma_1,\varsigma_2}]\,\varphi & \textit{iff} & (M,w) \Vdash \mathrm{Pre}_{\mathbf{Cut}(\varsigma_1,\varsigma_2)} \text{ implies } (M_{\mathbf{Cut}(\varsigma_1,\varsigma_2)}, w) \Vdash \varphi\end{array}$$

It is now time to provide reduction axioms for the new formulas. The fact that structural operations are closed for models in $\mathbf{EI}_K$ (Proposition 2) allows us to rely on the logic EI_K once again. Tables 4, 5 and 6 provide axioms indicating how the truth value of formulas *after* the structural operations depends on the truth value of formulas *before* them.

Definition 3.6 (Logic EI_{KDS})**.** The logic EI_{KDS} extends EI_{KD} (definition 3.3) with axioms and rules of tables 4, 5 and 6. Just as the axioms for deduction formulas, those of the mentioned tables express how formulas *after* the structural operations are related with formulas *before* them.

Axioms and inference rules for the reflexivity operation with ς_δ the rule $(\{\delta\}, \delta)$		
$Ref_{\top}$	$\vdash \langle Ref_{\delta}\rangle \top$	
Ref_{p}	$\vdash \langle Ref_{\delta}\rangle p \;\leftrightarrow\; p$	
$Ref_{\neg}$	$\vdash \langle Ref_{\delta}\rangle \neg\varphi \;\leftrightarrow\; \neg\langle Ref_{\delta}\rangle \varphi$	
$Ref_{\vee}$	$\vdash \langle Ref_{\delta}\rangle (\varphi \vee \psi) \;\leftrightarrow\; (\langle Ref_{\delta}\rangle \varphi \vee \langle Ref_{\delta}\rangle \psi)$	
$Ref_{\Diamond}$	$\vdash \langle Ref_{\delta}\rangle \Diamond\varphi \;\leftrightarrow\; \Diamond\langle Ref_{\delta}\rangle \varphi$	
Ref_{I}	$\vdash \langle Ref_{\delta}\rangle I\,\gamma \;\leftrightarrow\; I\,\gamma$	
Ref_{L}	$\vdash \langle Ref_{\delta}\rangle L\,\varsigma_\delta$	
Ref_{L}	$\vdash \langle Ref_{\delta}\rangle L\,\rho \;\leftrightarrow\; L\,\rho$	for $\rho \neq \varsigma_\delta$
Gen_{Ref}	From $\vdash \varphi$, infer $\vdash [Ref_{\delta}]\,\varphi$	

Table 4: Axioms and rules for *reflexivity* formulas.

Theorem 4 (Soundness and completeness of EI_{KDS} w.r.t. $\mathbf{EI}_K$)**.** *The logic* EI_{KDS}*is sound and strongly complete with respect to the models in the class* $\mathbf{EI}_K$*.*

Proof. Just as for the logic EI_{KD}, soundness follows from the validity of the new axioms and the validity-preserving property of the new rules. Strong completeness follows from the fact that, by a repetitive application of such axioms, any structural operation formula can be reduced to a formula in $\mathcal{EID}$, for which EI_{KD} is strongly complete with respect to $\mathbf{EI}_K$. □

Strictly speaking, we do not need axioms relating deduction and structural operations. We can focus on the deepest occurrence of them, apply the correspondent reduction axioms to eliminate it and then proceed with the next until we remove all the operation formulas. Nevertheless, it is interesting to

Axioms and inference rules for the monotonicity operation with ς' the rule $(\text{prem}(\varsigma) \cup \{\delta\}, \text{conc}(\varsigma))$		
$Mon_\top$	$\vdash \langle Mon_{\delta,\varsigma} \rangle \top \ \leftrightarrow\ \text{Pre}_{\mathbf{Mon}(\delta,\varsigma)}$	
Mon_p	$\vdash \langle Mon_{\delta,\varsigma} \rangle p \ \leftrightarrow\ (\text{Pre}_{\mathbf{Mon}(\delta,\varsigma)} \wedge\ p)$	
$Mon_\neg$	$\vdash \langle Mon_{\delta,\varsigma} \rangle \neg\varphi \ \leftrightarrow\ (\text{Pre}_{\mathbf{Mon}(\delta,\varsigma)} \wedge\ \neg\langle Mon_{\delta,\sigma} \rangle \varphi)$	
$Mon_\vee$	$\vdash \langle Mon_{\delta,\varsigma} \rangle (\varphi \vee \psi) \ \leftrightarrow\ (\langle Mon_{\delta,\varsigma} \rangle \varphi \vee \langle Mon_{\delta,\varsigma} \rangle \psi)$	
$Mon_\Diamond$	$\vdash \langle Mon_{\delta,\varsigma} \rangle \Diamond\varphi \ \leftrightarrow\ (\text{Pre}_{\mathbf{Mon}(\delta,\varsigma)} \wedge\ \Diamond\langle Mon_{\delta,\varsigma} \rangle \varphi)$	
Mon_I	$\vdash \langle Mon_{\delta,\varsigma} \rangle I\,\gamma \ \leftrightarrow\ (\text{Pre}_{\mathbf{Mon}(\delta,\varsigma)} \wedge I\,\gamma)$	
Mon_L	$\vdash \langle Mon_{\delta,\varsigma} \rangle L\,\varsigma' \ \leftrightarrow\ \text{Pre}_{\mathbf{Mon}(\delta,\varsigma)}$	
Mon_L	$\vdash \langle Mon_{\delta,\varsigma} \rangle L\,\rho \ \leftrightarrow\ (\text{Pre}_{\mathbf{Mon}(\delta,\varsigma)} \wedge\ L\,\rho)$	for $\rho \neq \varsigma'$
Gen_{Mon}	From $\vdash \varphi$, infer $\vdash [Mon_{\delta,\varsigma}]\,\varphi$	

Table 5: Axioms and rules for *monotonicity* formulas.

Axioms and inference rules for the cut operation with ς' the rule $(\,(\text{prem}(\varsigma_2) - \{\text{conc}(\varsigma_1)\}) \cup \text{prem}(\varsigma_1),\ \text{conc}(\varsigma_2)\,)$		
$Cut_\top$	$\vdash \langle Cut_{\varsigma_1,\varsigma_2} \rangle \top \ \leftrightarrow\ \text{Pre}_{\mathbf{Cut}(\varsigma_1,\varsigma_2)}$	
Cut_p	$\vdash \langle Cut_{\varsigma_1,\varsigma_2} \rangle p \ \leftrightarrow\ (\text{Pre}_{\mathbf{Cut}(\varsigma_1,\varsigma_2)} \wedge\ p)$	
$Cut_\neg$	$\vdash \langle Cut_{\varsigma_1,\varsigma_2} \rangle \neg\varphi \ \leftrightarrow\ (\text{Pre}_{\mathbf{Cut}(\varsigma_1,\varsigma_2)} \wedge\ \neg\langle Cut_{\varsigma_1,\varsigma_2} \rangle \varphi)$	
$Cut_\vee$	$\vdash \langle Cut_{\varsigma_1,\varsigma_2} \rangle (\varphi \vee \psi) \ \leftrightarrow\ (\langle Cut_{\varsigma_1,\varsigma_2} \rangle \varphi \vee \langle Cut_{\varsigma_1,\varsigma_2} \rangle \psi)$	
$Cut_\Diamond$	$\vdash \langle Cut_{\varsigma_1,\varsigma_2} \rangle \Diamond\varphi \ \leftrightarrow\ (\text{Pre}_{\mathbf{Cut}(\varsigma_1,\varsigma_2)} \wedge\ \Diamond\langle Cut_{\varsigma_1,\varsigma_2} \rangle \varphi)$	
Cut_I	$\vdash \langle Cut_{\varsigma_1,\varsigma_2} \rangle I\,\gamma \ \leftrightarrow\ (\text{Pre}_{\mathbf{Cut}(\varsigma_1,\varsigma_2)} \wedge I\,\gamma)$	
Cut_L	$\vdash \langle Cut_{\varsigma_1,\varsigma_2} \rangle L\,\varsigma' \ \leftrightarrow\ \text{Pre}_{\mathbf{Cut}(\varsigma_1,\varsigma_2)}$	
Cut_L	$\vdash \langle Cut_{\varsigma_1,\varsigma_2} \rangle L\,\rho \ \leftrightarrow\ (\text{Pre}_{\mathbf{Cut}(\varsigma_1,\varsigma_2)} \wedge\ L\,\rho)$	for $\rho \neq \varsigma'$
Gen_{Cut}	From $\vdash \varphi$, infer $\vdash [Cut_{\varsigma_1,\varsigma_2}]\,\varphi$	

Table 6: Axioms and rules for *cut* formulas.

see the relation between the different operations and, in particular, it is interesting to see how deduction is affected by structural operations. We finish this section presenting some validities, expressing how deduction after structural operations is related with deduction before them.

Theorem 5. *The formulas in table 7 are valid in models of the class* $\mathbf{EI}_K$.

Reflexivity with ς_δ the rule $(\{\delta\}, \delta)$	
• $\langle Ref_\delta \rangle \langle \mathrm{D}_\sigma \rangle \varphi \ \leftrightarrow\ \langle \mathrm{D}_\sigma \rangle \langle Ref_\delta \rangle \varphi$	for $\sigma \neq \varsigma_\delta$
• $\langle Ref_\delta \rangle \langle \mathrm{D}_{\varsigma_\delta} \rangle \varphi \ \leftrightarrow\ (\langle \mathrm{D}_{\varsigma_\delta} \rangle \varphi \vee (I\,\delta \wedge \langle Ref_\delta \rangle \varphi))$	
Monotonicity with ς' the rule $(\text{prem}(\varsigma) \cup \{\delta\}, \text{conc}(\varsigma))$	
• $\langle Mon_{\delta,\varsigma} \rangle \langle \mathrm{D}_\sigma \rangle \varphi \ \leftrightarrow\ \langle \mathrm{D}_\sigma \rangle \langle Mon_{\delta,\varsigma} \rangle \varphi$	for $\sigma \neq \varsigma'$
• $\langle Mon_{\delta,\varsigma} \rangle \langle \mathrm{D}_{\varsigma'} \rangle \varphi \ \leftrightarrow\ (\langle \mathrm{D}_{\varsigma'} \rangle \varphi \vee (I\,\delta \wedge L\,\varsigma \wedge \langle \mathrm{D}_\varsigma \rangle \langle Mon_{\delta,\varsigma} \rangle \varphi))$	
Cut with ς' the rule $(\,(\text{prem}(\varsigma_2) - \{\text{conc}(\varsigma_1)\}) \cup \text{prem}(\varsigma_1),\ \text{conc}(\varsigma_2)\,)$	
• $\langle Cut_{\varsigma_1,\varsigma_2} \rangle \langle \mathrm{D}_\sigma \rangle \varphi \ \leftrightarrow\ \langle \mathrm{D}_\sigma \rangle \langle Cut_{\varsigma_1,\varsigma_2} \rangle \varphi$	for $\sigma \neq \varsigma'$
• $\langle Cut_{\varsigma_1,\varsigma_2} \rangle \langle \mathrm{D}_{\varsigma'} \rangle \varphi \ \leftrightarrow$ $(\langle \mathrm{D}_{\varsigma'} \rangle \varphi \vee (I\ \text{prem}(\varsigma_1) \wedge L\,\varsigma_1 \wedge (I\ \text{conc}(\varsigma_1) \rightarrow \langle \mathrm{D}_{\varsigma_2} \rangle \langle Cut_{\varsigma_1,\varsigma_2} \rangle \varphi)))$	

Table 7: Relation between structural operations and deduction

Proof. See section A.5. □

4 Update

So far, our language can express the agent's *internal* dynamics, but it cannot express *external* ones. We can express how deductive steps modify explicit knowledge, and even how structural operations extends the rules the agent can apply, but we cannot express how both explicit and implicit knowledge are affected by external observations. Here we add the other fundamental source of information; in this section, we extend the language to express updates.

Updates are the result of the agent's social nature. We get new information because of our interaction with our environment, information that may be completely new in the sense that it does not follows from what we explicitly know. Updates are usually represented as operations, modifying the semantic model. In *Public Announcement Logic* (PAL), for example, an announcement is defined by an operation that removes the worlds where the announced formula does not hold, restricting the epistemic relation to those that are not deleted. In our semantic model, we have a finer representation of the agent's knowledge: we have explicit knowledge (her information sets) but we also have implicit one (given by the accessibility relation). We can extend PAL by defining different kinds of model operations, affecting explicit and implicit knowledge in different forms, and therefore expressing different ways the agent processes the incoming information. Here, we present one of the possible definitions, what we have called *explicit observations*.

4.1 Explicit observations

The previously defined operations do not modify the accessibility relation, and therefore do not affect implicit knowledge. With respect to explicit information, they add formulas or rules to the correspondent sets whenever they follow from what is already there. The deduction operation adds the conclusion of the rule whenever the premises and the rule are already present, and the structural operations add a rule whenever it logically follows from the ones that are currently available.

Observations, on the other hand, do modify the accessibility relation because it removes worlds where the observation does not holds. With respect to explicit information, they add arbitrary *true* information (formulas or rules), no matter if it was implicitly available (i.e., it follows from the explicit information) or not.

Definition 4.1 (Explicit observation operation)**.** Let $M = \langle W, R, V, Y, Z\rangle$ be a model in **EI**, and let χ be a formula of $\mathcal{I}$ (a rule based on $\mathcal{I}$). The model $M_{\chi!} = \langle W', R', V', Y', Z'\rangle$ is the result of removing the worlds that are not compatible with χ, restricting the accessibility relation accordingly, and adding χ to the agent's *information* (*rule*) set. Formally,

- $W' := \{ w \in W \mid (M, w) \Vdash \chi \}$, $\quad (W' := \{ w \in W \mid (M, w) \Vdash \mathrm{TR}(\chi) \})$
- $R' := R \cap (W' \times W')$,
- $V'(w) := V(w)$ for every $w \in W'$,

- $Y'(w) := Y(w) \cup \{\chi\}$ $\quad (Y'(w) := Y(w))$ $\quad$ for every $w \in W'$,
- $Z'(w) := Z(w)$ $\quad (Z'(w) := Z(w) \cup \{\chi\})$ $\quad$ for every $w \in W'$.

The operation $(\cdot)_{\chi!}$ is called the *explicit observation operation* with χ.

The operation preserves models in $\mathbf{EI}_K$ too.

Proposition 3 (Closure of structural operations)**.** *Let M be a model in $\mathbf{EI}_K$ and χ a formula in $\mathcal{I}$ (a rule based on $\mathcal{I}$); if M is in $\mathbf{EI}_K$, so it is $M_{\chi!}$.*

Proof. See section A.6. □

The language $\mathcal{EID}^{*!}$ extends $\mathcal{EID}^{*}$ by closing it under explicit observations: if φ is in $\mathcal{EID}^{*!}$, so it is $\langle \chi! \rangle \varphi$. These formulas are read as *"there is a way of explicitly observing χ after which φ is the case"*. In case χ is a formula, define $\mathrm{Pre}_{\chi!} \equiv \chi$; in case χ is a rule, define $\mathrm{Pre}_{\chi!} \equiv \mathrm{TR}(\chi)$. The semantics of explicit observation formulas is given as follows.

Definition 4.2. Let M be a model in **EI** and let $w \in W$ be a world in it. Then,

$$(M, w) \Vdash \langle \chi! \rangle \varphi \quad \textit{iff} \quad (M, w) \Vdash \mathrm{Pre}_{\chi!} \text{ and } (M_{\chi!}, w) \Vdash \varphi$$

The formula $[\chi!]\,\varphi$ is defined as the dual of $\langle \chi! \rangle \varphi$, as usual.

Definition 4.3 (Logic EI_{KDSO})**.** The logic EI_{KDSO} is built from axioms and rules of EI_{KDS} (definition 3.6) plus axioms and rules in table 8.

Axioms and rules for the explicit observation operation		
$EO_{\top}$	$\vdash \langle \chi! \rangle \top \;\leftrightarrow\; \mathrm{Pre}_{\chi!}$	
EO_p	$\vdash \langle \chi! \rangle p \;\leftrightarrow\; (\mathrm{Pre}_{\chi!} \wedge\; p)$	
$EO_{\neg}$	$\vdash \langle \chi! \rangle \neg\varphi \;\leftrightarrow\; (\mathrm{Pre}_{\chi!} \wedge \neg\langle \chi! \rangle \varphi)$	
$EO_{\vee}$	$\vdash \langle \chi! \rangle (\varphi \vee \psi) \;\leftrightarrow\; (\langle \chi! \rangle \varphi \vee \langle \chi! \rangle \psi)$	
$EO_{\Diamond}$	$\vdash \langle \chi! \rangle \Diamond\varphi \;\leftrightarrow\; (\mathrm{Pre}_{\chi!} \wedge \Diamond\langle \chi! \rangle \varphi)$	
If χ is a formula:		
EO_I	$\vdash \langle \chi! \rangle I\,\chi \;\leftrightarrow\; \mathrm{Pre}_{\chi!}$	
EO_I	$\vdash \langle \chi! \rangle I\,\gamma \;\leftrightarrow\; (\mathrm{Pre}_{\chi!} \wedge I\,\gamma)$	for $\gamma \neq \chi$
EO_L	$\vdash \langle \chi! \rangle L\,\rho \;\leftrightarrow\; (\mathrm{Pre}_{\chi!} \wedge L\,\rho)$	
If χ is a rule:		
EO_I	$\vdash \langle \chi! \rangle I\,\gamma \;\leftrightarrow\; (\mathrm{Pre}_{\chi!} \wedge I\,\gamma)$	
EO_L	$\vdash \langle \chi! \rangle L\,\chi \;\leftrightarrow\; \mathrm{Pre}_{\chi!}$	
EO_L	$\vdash \langle \chi! \rangle L\,\rho \;\leftrightarrow\; (\mathrm{Pre}_{\chi!} \wedge L\,\rho)$	for $\rho \neq \chi$
Gen_{EO}	From $\vdash \varphi$, infer $\vdash [\chi!]\,\varphi$	

Table 8: Axioms and rules for *explicit observation* formulas.

Theorem 6 (Soundness and completeness of EI_{KDSO} w.r.t. $\mathbf{EI}_K$)**.** *The logic EI_{KDSO} is sound and strongly complete for the class of models $\mathbf{EI}_K$.*

5 Comparison with other works

The present work explores a representation of explicit/implicit information that allows us to describe the way different process affects both kinds of information. Several other works have proposed similar frameworks; in this section we present a brief comparison between some of those approaches and our proposal.

5.1 Fagin-Halpern's logics of awareness

Fagin and Halpern presented in Fagin and Halpern [1988] what they called *logic of general awareness* ($\mathfrak{L}_A$). Given a set of agents, formulas of the language are given by a set of atomic propositions P closed under negation, conjunction and the modal operators A_i and L_i (for an agent i). Formulas of the form $A_i\varphi$ are read as *"the agent i is aware of φ"*, and formulas of the form $L_i\varphi$ are read as *"the agent i implicitly believes that φ"*. The operator B_i, which expresses explicit beliefs, is defined as $B_i\varphi := A_i\varphi \wedge L_i\varphi$.

A *Kripke structure for general awareness* is defined as a tuple $M = (W, \mathfrak{A}_i, \mathfrak{L}_i, V)$, where $W \neq \emptyset$ is the set of possible worlds, $\mathfrak{A}_i : W \rightarrow \wp(\mathfrak{L}_A)$ is a function that assigns a set of formulas of $\mathfrak{L}_A$ to the agent i in each world (her awareness set), the relation $\mathfrak{L}_i \subseteq (W \times W)$ is a serial, transitive and Euclidean relation over W for each agent i ($\mathfrak{L}_A$ deals with beliefs rather than knowledge) and $V : \mathtt{P} \rightarrow \wp(W)$ is a valuation function.

Given a Kripke structure for general awareness $M = (W, \mathfrak{A}_i, \mathfrak{L}_i, V)$, semantics for atomic propositions, negations and conjunctions are given in the standard way. For formulas of the form $A_i\,\varphi$ and $L_i\,\varphi$, we have

$M, w \Vdash A_i\varphi$	*iff*	$\varphi \in \mathfrak{A}_i(w)$
$M, w \Vdash L_i\varphi$	*iff*	for all $u \in W$, $\mathfrak{L}_i wu$ implies $M, u \Vdash \varphi$

It follows that $M, w \Vdash B_i\varphi$ *iff* $\varphi \in A_i(w)$ and, for all $u \in W$, $\mathfrak{L}_i wu$ implies $M, u \Vdash \varphi$.

Note how the *explicit beliefs implies implicit beliefs* property holds because of the definition of explicit beliefs: an agent explicit believes that φ ($B_i\varphi$) if and only if she is aware of it ($A_i\varphi$) and she implicitly believes it ($L_i\varphi$). In our approach, explicit and implicit information are defined separately, and it is because of the *coherence* and the *truth* properties that explicit information implies explicit information.

Given the similarities between the functions $\mathfrak{A}_i$ and Y and between the relations $\mathfrak{L}_i$ and R, formulas $A_i\varphi$ and $L_i\varphi$ in $\mathfrak{L}_A$ behaves exactly like $I\,\varphi$ and $\Box\varphi$ in $\mathcal{EI}$ (plus the subindexes). The difference between the approaches is in the rules we use to perform inference and, therefore, in the dynamic part.

The rule set function Z represents explicitly the processes that the agent can use at w to improve her explicit information about formulas. The information of the agent consists not only of formulas about the world, but also of rules that allow her to infer new formulas. It is not that the agent knows that after a rule application her information set will change; it is that she knows the *process* that leads the change. We interpret a rule as an object that can be part of the agent's information, and whose presence is needed for the agent to be able to apply it.

For the internal dynamics, the language $\mathfrak{L}_A$ does not express changes in the agent's awareness sets, though later in the same paper the authors explore the incorporation of time to the language by adding a deterministic serial binary relation $\mathfrak{T}$ over W representing steps in time. Still, they do not indicate what the process(es) that change the awareness sets is (are).

Our approach does indicate the process that transform the explicit information: inference. Different from their relational approach, in our work this process is represented not as relation between worlds, but as a model operation that modifies the content of the information set at each world. They represent *steps in the agent's reasoning process*, increasing her explicit information.

Finally, the language of formulas belonging to the awareness sets is more powerful that our internal language $\mathcal{I}$. As we mentioned before, that limitation allows us to define the update operation for representing external dynamics (observations), a process that is not considered in $\mathcal{L}_A$.

5.2 Duc's dynamic epistemic logic

In Duc [1995, 1997] and Duc [2001], Ho Ngoc Duc proposed a dynamic epistemic logic to reason about agents that are neither logically omniscient nor logically ignorant.

The syntax of the language is very similar to the inference part of our language. There is an internal language, the classic propositional one (PL), to express agent's information. There is also another language to talk about how this information evolves. Formally, At denotes the set of formulas of the form $K\gamma$, for γ in PL. The language $\mathcal{L}_{BDE}$ contains At and is closed under negation, conjunction and the modal operator $\langle F \rangle$. Formulas of the form $K\gamma$ are read as *"γ is known"*; formulas of the form $\langle F \rangle \varphi$ are read as *"φ is true after some course of thought"*. Note how the language just focus on the way the agent's information evolves and does not provide formulas to talk about the real world.

A model M is a tuple (W, R, Y), where $W \neq \emptyset$ is the set of *possible worlds*, $R \subseteq (W \times W)$ is a transitive binary relation and $Y : W \to \wp(At)$ associates a set of formulas of At to each possible world. A *BDE*-model is a model M such that: (1) for all $w \in W$, if $K\gamma \in Y(w)$ and Rwu, then $K\gamma \in Y(u)$; (2) for all $w \in W$, if $K\gamma$ and $K(\gamma \to \delta)$ are in $Y(w)$, then $K\delta$ is in $Y(u)$ for some u such that Rwu; (3) if γ is a propositional tautology, then for all $w \in W$ there is a world u such that Rwu and $K\gamma \in Y(u)$. Such restrictions guarantees that the set of formulas will grow as the agent reasons, and that her information will be closed under modus ponens and will contain all tautologies at some point in the future.

Given a *BDE*-model, the semantics for negation and conjunctions are standard. The semantics of atomic and reasoning-steps formulas are given by:

$M, w \Vdash K\gamma$	*iff*	$K\gamma \in Y(w)$
$M, w \Vdash \langle F \rangle \varphi$	*iff*	there is some $u \in W$ such that Rwu and $M, u \Vdash \varphi$

The main difference between the approaches is the treatment of the mechanism that increases explicit information. While in Duc's approach it is represented as a relation between worlds (the relation R), in our approach it is represented as a model operation. Moreover, Duc's approach do not specify what this mechanism is (he just call it *"course of though"*) while our framework considers a concrete interpretation: inference. Finally, the language is restricted to express what the agent can infer through some *"course of though"*, but it does not express external dynamics, as explicit observations in our language do.

5.3 Jago's logic for resource-bounded agents

In Jago [2006b,a], Jago presented a logic for resource-bounded agents. He considers a semantic model similar to ours, based in Kripke models and with a set of formulas of some internal language assigned to every agent in each possible world to describe explicit information. Moreover, he also considers rule-based inference as the mechanism through which the agent can improve her information. Similar to Duc and different from us, inference is represented as a relation between worlds: there is an arrow labelled with a rule ρ from world w to world u if and only if at w the agent has the rule and all its premises, and at u the agent extends what she has at w by adding the conclusion of ρ.

There are two main differences in the approaches. The first one is again the treatment of the mechanism to improve explicit information, but here we go further in the comparison since Jago's work is the origin of our proposal. As mentioned before, our model-operation representation facilitates the work by giving us a functional treatment of inference for free, while the modal representation forces us to ask for properties of the relation in order for inference to behave in a functional way. Those properties may need a more powerful language to be expressed (the uniqueness of the result of a rule application can only be expressed with nominals, in order to really differentiate worlds) and some of them may be not preserved after updates (the existence of a world resulting from an available rule application is not preserved since new information may turn applicable rules that were not before). And that is precisely the second difference: our approach considers not only internal dynamics (inference) but also external ones (updates).

5.4 van Benthem's acts of realization

In van Benthem [2008b], the author considers a language extending the propositional one with the operators K and I for implicit and explicit information, respectively. Semantically, we have tuples $(W, W^{acc}, \sim, V)$ where $(W, \sim, V)$ is a Kripke model and W^{acc} is the set of *access worlds*: pairs (w, X) with w a standard world and X a set of formulas of the propositional language (the access set). There are three restrictions: accessibility relations are equivalence relations, for each access world (w, X), all formulas in X should be true at (w, X), and epistemically indistinguishable worlds should have the same access set. These models are similar to our *knowledge models* of definition 2.6, the only difference being our rule set function and their properties. Given a model and an access world (w, X), atomic propositions are interpreted according to the valuation at w, boolean connectives and K are interpreted as usual, and $I\varphi$ is true at (w, X) if and only if $\varphi \in X$.

Then, implicit and explicit observations are defined. The first restricts the worlds to those that satisfy the observation, just as public announcements do; the second performs the same operation but also modifies the access sets by adding the observation to it. This explicit observation is exactly like our explicit observation of section 4, and the implicit observation can be defined in our framework.

The main difference in the approaches is that, after the previous definitions, van Benthem notices that *"the preceding is not yet a right division of labour, as events of explicit seeing and implicit seeing 'overlap' in their effects on a model"*. Then he goes further and proposes two more "orthogonal" acts: one simply delimiting the implicit information in the style of standard public announcements (what he calls *"bare observation"*, syntactically represented with $[!\varphi]$) and another one simply adding true formulas to the access sets (what he calls *"act of realization"*, syntactically represented with $[\#\varphi]$). The previous implicit observation becomes a *bare observation*, while the explicit one becomes a composition of a *bare observation* and an *act of realization*.

An *act of realization* is more general than our deduction. As we mentioned before, formulas that can be added by a deduction operation are part of the agent's implicit information (as the validity of $(I\ \mathrm{prem}(\rho) \wedge L\,\rho) \rightarrow \Box\mathrm{conc}(\rho)$ shows) but the converse is not true. On the other hand, *any* formula that is part of the implicit information can be added to the access sets by an act of realization (as the validity of $\langle\#\varphi\rangle I\varphi \leftrightarrow K\varphi$ shows). Validities, for example, can be added without any further consideration.

6 Final remarks and further work

Let us describe the restaurant example with our framework. The initial setting can be given by a model M with six possible worlds, each one of them indicating a possible distribution of the dishes from the new waiter's point of view. For him, all of them are indistinguishable from each other, that is, the accessibility relation is an equivalence relation.

For explicit information, consider a set of atomic propositions of the form $\mathsf{p_d}$ where p stands for a person (**f**ather, **m**other or **y**ou) and d stands for some dish (**m**eat, **f**ish or **v**egetarian). The waiter explicitly knows each person will get only one dish, so we can put the rules

$$\rho_1 : \{\mathsf{y_f}\} \Rightarrow \neg\mathsf{y_v} \qquad\qquad \rho_2 : \{\mathsf{f_m}\} \Rightarrow \neg\mathsf{f_v}$$

and similar ones in each world. Moreover, he explicitly knows that each dish corresponds to one person, so the rule

$$\sigma : \{\neg\mathsf{y_v}, \neg\mathsf{f_v}\} \Rightarrow \mathsf{m_v}$$

can be also added, among many others. Let w be the real world, where $\mathsf{y_f}$, $\mathsf{f_m}$ and $\mathsf{m_v}$ are true. The formula $\neg I\,\mathsf{m_v} \wedge \neg\Box\mathsf{m_v}$, indicating that the waiter does not know yet (neither explicitly nor implicitly) that your mother has the vegetarian, is true at (M, w).

While approaching to the table, the waiter can improve the rules he knows: he can, for example, apply the *Cut* rule over ρ_1 and σ, since he has both rules and the conclusion of the first is in the premises of the second. This does not give him new explicit information, but will allow him to infer faster later. From the application, he gets the rule

$$\varsigma_1 : \{\mathsf{y_f}, \neg\mathsf{f_v}\} \Rightarrow \mathsf{m_v}$$

Then, the formulas

- $\langle Cut_{\rho_1,\sigma}\rangle \neg I\,\mathsf{m_v}$
- $\langle Cut_{\rho_1,\sigma}\rangle L\,\varsigma_1$

are also true at (M, w). Moreover, he can apply *Cut* again, this time with ρ_2 and ς_1, obtaining the rule

$$\varsigma_2 : \{\mathsf{y_f}, \mathsf{f_m}\} \Rightarrow \mathsf{m_v}$$

and making the formulas

- $\langle Cut_{\rho_1,\sigma}\rangle \langle Cut_{\rho_2,\varsigma_1}\rangle \neg I\, \mathsf{m_v}$
- $\langle Cut_{\rho_1,\sigma}\rangle \langle Cut_{\rho_2,\varsigma_1}\rangle L\, \varsigma_2$

true at (M, w).

After the answer to the question *"Who has the fish?"*, the waiter explicitly knows that you have the fish. Four possible worlds are removed from the model and the rule ς_2 is preserved, but he still does not know (neither explicitly nor implicitly) that your mother has the vegetarian. Then,

- $\langle Cut_{\rho_1,\sigma}\rangle \langle Cut_{\rho_2,\varsigma_1}\rangle \langle \mathsf{y_f}!\rangle (I\, \mathsf{y_f} \wedge L\, \varsigma_2)$
- $\langle Cut_{\rho_1,\sigma}\rangle \langle Cut_{\rho_2,\varsigma_1}\rangle \langle \mathsf{y_f}!\rangle (\neg I\, \mathsf{m_v} \wedge \neg\Box \mathsf{m_v})$

are true at (M, w).

Then he asks *"Who has the meat?"*, and the answer removes one of the remaining worlds. Not only he knows implicitly that your mother has the vegetarian dish: he is also able to infer it, adding it to his explicit information:

$$\langle Cut_{\rho_1,\sigma}\rangle \langle Cut_{\rho_2,\varsigma_1}\rangle \langle \mathsf{y_f}!\rangle \langle \mathsf{f_m}!\rangle (\Box \mathsf{m_v} \wedge \langle \mathrm{D}_{\varsigma_2}\rangle I\, \mathsf{m_v})$$

Two structural operations, two explicit observations and one inference is all the waiter needs.

The work can be extended in several ways. The first one is by extending the internal language beyond the propositional one. As we mentioned, we choose the propositional language because it allows us the definition of updates given in section 4. If we extend the language to the full explicit/implicit information one, we may face Moore-type-observations, like $p \wedge \neg\Diamond p$ indicating that p is the case and the agent did not know it explicitly. Formulas like these cannot be simply added to the information sets since, although they are true at the moment they are observed, they become false immediately after. A simple solution is to keep in the information sets those formulas that are true in the new model, but we still have difficulties because of circularity: we define the new information set as those formulas that were there before and are still true in the new model, but in order to check whether an explicit information formula is true or not we already need the information set. A further analysis providing a solution to this limitation will greatly increase the expressivity of the framework.

We have analyzed the particular case where the agent has true information, that is, when the accessibility relation is an equivalence relation and formulas as well as the translation of the rules are true at the correspondent worlds. This is not the general case: by removing such restrictions we can talk not only about knowledge but also about other kinds of information, like *beliefs*. Some recent works combine these two notions, giving us a nice way of studying these two propositional attitudes together.

Moreover, we have analyzed the case where inference preserves truth, but there are other interesting inference processes, like *default reasoning*, *abduction* or *belief revision*. They are not deductive, but they are important and widely used, with particular relevance on incomplete information situations. Within the proposed framework, we can represent different inference processes, and we can study how all of them work together.

For the external dynamics, this finer representation of knowledge allows us to define different kinds of observations. Given the already discussed restriction of the internal language, we can represent observations that do not affect explicit information (like van Benthem's *bare observations* previously discussed) and our already defined explicit observations. If we lift the restriction, we will be able to represent more kinds of observations, all differing between them in how introspective is the agent about the observed fact.

In the context of agent diversity (Liu [2006, 2008]), this finer representation of the processes that affect information allows us to make a distinction between agents with different reasoning abilities. The rules an agent has may be very different from those of another, and they will not be able to perform the same inference steps. Moreover, some of them may be able to perform several inference steps at once instead of a single one. The idea works also for external dynamics: agents may have different observational power. It will be interesting to explore how agents that differs in their reasoning and observational abilities interact with each other.

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A Technical appendix

A.1 Completeness for basic models

The key observation is that a logic Λ is strongly complete with respect to a class of structures if and only if every Λ-consistent set of formulas is satisfiable on some structure of the given class (Proposition 4.12 of Blackburn et al. [2001]). Using the the canonical model technique, we show that every EI-consistent set of formulas is satisfiable in a model in **EI**. Proofs of Lindenbaum's Lemma, Existence Lemma and Truth Lemma are standard.

Lemma 1 (Lindenbaum's Lemma)**.** *For any* EI*-consistent set of formulas* Σ*, there is a maximal* EI*-consistent set* Σ^+ *such that* $\Sigma \subseteq \Sigma^+$*.*

Definition A.1 (Canonical model for EI)**.** The canonical model of the logic EI is the epistemic inference model $M^{\mathsf{EI}} = \langle W^{\mathsf{EI}}, R^{\mathsf{EI}}, V^{\mathsf{EI}}, Y^{\mathsf{EI}}, Z^{\mathsf{EI}} \rangle$, where:

- W^{EI} is the set of all maximal EI-consistent set of formulas.
- $R^{\mathsf{EI}} := \{ (w,u) \mid \text{for all } \varphi \text{ in } \mathcal{EI}, \varphi \in u \text{ implies } \Diamond\varphi \in w \}$ (or, equivalently, $R^{\mathsf{EI}} := \{ (w,u) \mid \text{for all } \varphi \text{ in } \mathcal{EI}, \Box\varphi \in w \text{ implies } \varphi \in u \}$).
- $V^{\mathsf{EI}}(w) := \{ p \in \mathtt{P} \mid p \in w \}$.

- $Y^{\mathsf{EI}}(w) := \{\gamma \in \mathcal{I} \mid I\gamma \in w\}$.
- $Z^{\mathsf{EI}}(w) := \{\rho \in \mathcal{R} \mid L\rho \in w\}$.

Lemma 2 (Existence Lemma)**.** *For every world $w \in W^{\mathsf{EI}}$, if $\Diamond\varphi \in w$, then there is a world $u \in W^{\mathsf{EI}}$ such that $R^{\mathsf{EI}}wu$ and $\varphi \in u$.*

Lemma 3 (Truth Lemma)**.** *For all $w \in W^{\mathsf{EI}}$, we have $(M^{\mathsf{EI}}, w) \Vdash \varphi$ iff $\varphi \in w$.*

By the mentioned Proposition of Blackburn et al. [2001], all we have to show is that every EI-consistent set of formulas is satisfiable, so take any such set Σ. By Lindenbaum's Lemma, we can extend it to a maximal EI-consistent set of formulas Σ^+; by the Truth Lemma, we have $(M^{\mathsf{EI}}, \Sigma^+) \Vdash \Sigma$, so Σ is satisfiable in the canonical model of EI at Σ^+. It is only left to show that M^{EI} is indeed a model in **EI**, that is, we have to show that it satisfies *coherence* for formulas and rules.

Remember that any maximal EI-consistent set Φ is closed under modus ponens, that is, if φ and $\varphi \to \psi$ are in Φ, so it is ψ.

- ***Coherence for formulas.*** Suppose $\gamma \in Y^{\mathsf{EI}}(w)$; we want to show that for all u such that $R^{\mathsf{EI}}wu$ we have $\gamma \in Y^{\mathsf{EI}}(u)$. Note that $I\gamma \to \Box I\gamma$ (axiom $Coh_{\mathcal{I}}$) is in w.

  By definition, $\gamma \in Y^{\mathsf{EI}}(w)$ implies $I\gamma \in w$; by the modus ponens closure, we have $\Box I\gamma \in w$. Take any u such that $R^{\mathsf{EI}}wu$; then $I\gamma \in u$ and therefore $\gamma \in Y^{\mathsf{EI}}(u)$.

- ***Coherence for rules.*** Similar to the case of formulas, with the $Coh_{\mathcal{R}}$ axiom.

A.2 Completeness for knowledge models

We know already that EI is complete with respect to models in **EI** (section A.1). In order to show that EI_K is complete with respect to $\mathbf{EI}_K$, we just have to show that the canonical model for EI_K satisfy *equivalence*, *truth for formulas* and *truth for rules*.

Definition A.2 (Canonical model for EI_K)**.** The canonical model for EI_K, $M^{\mathsf{EI}_K} = (W^{\mathsf{EI}_K}, R^{\mathsf{EI}_K}, V^{\mathsf{EI}_K}, Y^{\mathsf{EI}_K}, Z^{\mathsf{EI}_K})$, is defined just as the canonical model for EI, but the worlds are maximal EI_K-consistent sets of formulas instead of maximal EI-consistent ones.

Here is the proof for the three properties.

- ***Equivalence.*** Axioms *T*, *4* and *B* are canonical for reflexivity, transitivity and symmetry, respectively, so R^{EI_K} is an equivalence relation.

- ***Truth for formulas.*** We want to show that $\gamma \in Y^{\mathsf{EI}_K}(w)$ implies $(M^{\mathsf{EI}_K}, w) \Vdash \gamma$. Suppose $\gamma \in Y^{\mathsf{EI}_K}(w)$; then we get $I\gamma \in w$. By axiom $Tth_{\mathcal{I}}$ we have $\gamma \in w$; by the Truth Lemma, $(M^{\mathsf{EI}_K}, w) \Vdash \gamma$.

- ***Truth for rules.*** Similar to the case of formulas, with axiom $Tth_{\mathcal{R}}$.

A.3 Closure of deduction operation

Let M be a model in $\mathbf{EI}_K$. To show that M_ρ (definition 3.1) is also in $\mathbf{EI}_K$, we will show that it satisfes coherence and truth for formulas, coherence and truth for rules and equivalence. Equivalence and both properties of rules are immediate since neither the accessibility relation nor the rule set function are modified. For the properties of formulas, we have the following.

- ***Coherence for formulas.*** Suppose $\gamma \in Y'(w)$ and pick any $u \in W'$ such that $R'wu$; we will show that $\gamma \in Y'(u)$. From $R'wu$ we get Rwu.

  From the definition of Y', we know that either γ was added by the operation or else was already in $Y(w)$. In the first case, γ should be $\text{conc}(\sigma)$ and therefore $\text{prem}(\sigma) \subseteq Y(w)$ and $\sigma \in Z(w)$. But then, by coherence for formulas and rules of M and Rwu, we have $\text{prem}(\sigma) \subseteq Y(u)$ and $\sigma \in Z(u)$; therefore $\text{conc}(\sigma) \in Y'(u)$, that is, $\gamma \in Y'(u)$. In the second case, by coherence for formulas of M and Rwu, we have $\gamma \in Y(u)$ and therefore $\gamma \in Y'(u)$.

- ***Truth for formulas.*** Suppose $\gamma \in Y'(w)$; we will show that $(M_\sigma, w) \Vdash \gamma$.

  Again, from the definition of Y', we know that either γ was added by the operation or else was already in $Y(w)$. In the first case, γ should be $\text{conc}(\sigma)$ and therefore $\text{prem}(\sigma) \subseteq Y(w)$ and $\sigma \in Z(w)$. By truth for formulas of M we have $(M, w) \Vdash \bigwedge \text{prem}(\sigma)$; by truth for rules of M we have $(M, w) \Vdash \bigwedge \text{prem}(\sigma) \rightarrow \text{conc}(\sigma)$. Therefore, we have $(M, w) \Vdash \text{conc}(\sigma)$, i.e., $(M, w) \Vdash \gamma$. In the second case, by truth for formulas of M we get also $(M, w) \Vdash \gamma$.

  Now, γ is a propositional formula, whose truth value depends only on the valuation at w. Since $V'(w) = V(w)$, we get $(M_\sigma, w) \Vdash \gamma$, as required.

A.4 Closure of structural operations

Let $M = \langle W, R, V, Y, Z\rangle$ be a model in $\mathbf{EI}_K$. To show that $M_{\mathbf{Ref}(\delta)}$, $M_{\mathbf{Mon}(\delta,\varsigma)}$ and $M_{\mathbf{Cut}(\varsigma_1,\varsigma_2)}$ are also in $\mathbf{EI}_K$, we will show that they satisfy coherence and truth for formulas, coherence and truth for rules and equivalence. Equivalence and both properties of formulas are immediate since neither the accessibility relation nor the information set function are modified. For the properties of rules, we have the following.

Let $M_{\mathbf{Ref}(\delta)}$ be given by $\langle W', R', V', Y', Z'\rangle$, as in definition 3.4. Recall that $\varsigma_\delta = (\{\delta\}, \delta)$.

- ***Coherence for rules.*** Suppose $\rho \in Z'(w)$ and pick any $u \in W'$ such that $R'wu$; we will show that $\rho \in Z'(u)$. From $R'wu$ we get Rwu.

  From the definition of Z', we know that either ρ was added by the operation or it was already in $Y(w)$. In the first case, ρ should be ς_δ, and we have $\varsigma_\delta \in Z'(u)$ for every $u \in W'$; in the second case, coherence for rules of M and Rwu give us $\rho \in Z(u)$, and therefore $\rho \in Z'(u)$.

- ***Truth for rules.*** Suppose $\rho \in Y'(w)$; we will show that $(M_{\mathbf{Ref}(\delta)}, w) \Vdash \text{TR}(\rho)$.

  From the definition of Z', we know that either ρ was added by the operation or it was already in $Y(w)$. The first case is simple since ρ should be

ς_δ and we clearly have $(M_{\mathbf{Ref}(\delta)}, w) \Vdash \delta \to \delta$; the second case follows from the truth for rules of M ($(M, w) \Vdash \mathrm{TR}(\rho)$), the fact that the truth value of $\mathrm{TR}(\rho)$ depends just on the valuation of the world, and the definition of V' ($V'(w) = V(w)$).

Let $M_{\mathbf{Mon}(\delta,\varsigma)}$ be given by $\langle W', R', V', Y', Z' \rangle$, as in definition 3.4. Recall that $\varsigma' = (\mathrm{prem}(\varsigma) \cup \{\delta\}, \mathrm{conc}(\varsigma))$.

- ***Coherence for rules.*** Suppose $\rho \in Z'(w)$ and pick any $u \in W'$ such that $R'wu$; we will show that $\rho \in Z'(u)$. From $R'wu$ we get Rwu.

  From the definition of Z', we know that either ρ was added by the operation or it was already in $Y(w)$. In the first case, ρ should be ς' and we should have $\varsigma \in Z(w)$. From coherence for rules of M and Rwu we get $\varsigma \in Z(u)$ and therefore $\varsigma' \in Z'(u)$. The second case follows immediately from coherence for rules of M and the definition of Z'.

- ***Truth for rules.*** Suppose $\rho \in Y'(w)$; we will show that $(M_{\mathbf{Mon}(\delta,\rho)}, w) \Vdash \mathrm{TR}(\rho)$.

  From the definition of Z', we know that either ρ is ς' or it was already in $Z(w)$. From the definition of Z', we know that either ρ was added by the operation or it was already in $Y(w)$. In the first case, ρ should be ς' and we should have $\varsigma \in Z(w)$; then, truth for rules of M gives us $(M, w) \Vdash \bigwedge \mathrm{prem}(\varsigma) \to \mathrm{conc}(\varsigma)$. Since $V'(w) = V(w)$, we get

  $$(M_{\mathbf{Mon}(\delta,\varsigma)}, w) \Vdash \bigwedge \mathrm{prem}(\varsigma) \to \mathrm{conc}(\varsigma)$$

  and therefore

  $$(M_{\mathbf{Mon}(\delta,\varsigma)}, w) \Vdash (\bigwedge \mathrm{prem}(\varsigma) \wedge \delta) \to \mathrm{conc}(\varsigma)$$

  i.e., $(M_{\mathbf{Mon}(\delta,\varsigma)}, w) \Vdash \mathrm{TR}(\varsigma')$. The second case follows from the truth for rules of M ($(M, w) \Vdash \mathrm{TR}(\varsigma)$), the fact that the truth value of $\mathrm{TR}(\rho)$ depends just on the valuation of the world, and the definition of V' ($V'(w) = V(w)$).

Let $M_{\mathbf{Cut}(\varsigma_1,\varsigma_2)}$ be given by $\langle W', R', V', Y', Z' \rangle$, as in definition 3.4. Recall that $\varsigma' = ((\mathrm{prem}(\varsigma_2) - \{\mathrm{conc}(\varsigma_1)\}) \cup \mathrm{prem}(\varsigma_1)\,,\, \mathrm{conc}(\varsigma_2))$.

- ***Coherence for rules.*** Suppose $\rho \in Z'(w)$ and pick any $u \in W'$ such that $R'wu$; we will show that $\rho \in Z'(u)$. From $R'wu$ we get Rwu.

  From the definition of Z', we know that either ρ was added by the operation or it was already in $Y(w)$. In the first case, ρ should be ς' and we have $\{\varsigma_1, \varsigma_2\} \subseteq Z(w)$; from coherence for rules of M and Rwu we get $\{\varsigma_1, \varsigma_2\} \subseteq\in Z(u)$ and therefore $\varsigma' \in Z'(u)$. The second case follows immediately from coherence for rules of M and the definition of Z'.

- ***Truth for rules.*** Suppose $\rho \in Y'(w)$; we will show that $(M_{\mathbf{Cut}(\varsigma_1,\varsigma_2)}, w) \Vdash \mathrm{TR}(\rho)$.

  From the definition of Z', we know that either ρ was added by the operation or it was already in $Y(w)$. In the first case, ρ should be $\varsigma'$and we have $\{\varsigma_1, \varsigma_2\} \subseteq Z(w)$. Now, suppose $(M, w) \nVdash \mathrm{TR}(\varsigma')$; then,

  $$(M, w) \Vdash \bigwedge \mathrm{prem}(\varsigma') \qquad \text{and} \qquad (M, w) \nVdash \mathrm{conc}(\varsigma')$$

The first part implies $(M, w) \Vdash \gamma$ for every $\gamma \in (\text{prem}(\varsigma_2) - \{\text{conc}(\varsigma_1)\})$ and every $\gamma \in \text{prem}(\varsigma_1)$. Since every $\gamma \in \text{prem}(\varsigma_1)$ is true at w, so should it be $\text{conc}(\varsigma_1)$ by truth for rules of M, because we have $\varsigma_1 \in Z(w)$. Then, we have $(M, w) \Vdash \gamma$ for every $\gamma \in \text{prem}(\varsigma_2)$. But the second part says that $\text{conc}(\varsigma_2)$ is false at w, and we get $(M, w) \nVdash \text{TR}(\varsigma_2)$, contradicting truth for rules of M since $\varsigma_2 \in Z(w)$. Hence, we should have $(M, w) \Vdash \text{TR}(\varsigma')$, from which it follows that $(M_{\mathbf{Cut}(\varsigma_1,\varsigma_2)}, w) \Vdash \text{TR}(\varsigma')$ because of the definition of $V'(w)$.

The second case follows immediately from truth for rules of M and the definition of Z'.

A.5 Structural operations and deduction

The validity of the formulas follows from the bisimilarities between models stated below. In our case, the bisimulation concept extends the standard one by asking for related worlds to have the same information and rule set: given two models $M_1 = \langle W_1, R_1, V_1, Y_1, Z_1 \rangle$ and $M_2 = \langle W_2, R_2, V_2, Y_2, Z_2 \rangle$, a non empty relation B between W_1 and W_2 is a bisimulation if and only if B is a standard bisimulation between $\langle W_1, R_1, V_1 \rangle$ and $\langle W_2, R_2, V_2 \rangle$ and, if Bw_1w_2, then $Y_1(w_1) = Y_2(w_2)$ and $Z_1(w_1) = Z_2(w_2)$.

Let M be the model $\langle W, R, V, Y, Z \rangle$ and let w be a world in W; in all cases the bisimulation is the identity relation over worlds reachable from w.

Reflexivity. Let ς_δ be the rule $(\{\delta\}, \delta)$:

- If $\sigma \neq \varsigma_\delta$, then $(M_{\mathbf{Ref}(\delta)_\sigma}, w) \leftrightarrows (M_{\sigma\mathbf{Ref}(\delta)}, w)$.
- If $\delta \in Y(w)$ and $\varsigma_\delta \in Z(w)$, then $(M_{\mathbf{Ref}(\delta)_{\varsigma_\delta}}, w) \leftrightarrows (M_{\varsigma_\delta}, w)$.
- If $\delta \in Y(w)$, then $(M_{\mathbf{Ref}(\delta)_{\varsigma_\delta}}, w) \leftrightarrows (M_{\varsigma_\delta\mathbf{Ref}(\delta)}, w)$.

Monotonicity. Let ς' be the rule $(\text{prem}(\varsigma) \cup \{\delta\}, \text{conc}(\varsigma))$:

- If $\sigma \neq \varsigma'$, then $(M_{\mathbf{Mon}(\delta,\varsigma)_\sigma}, w) \leftrightarrows (M_{\sigma\mathbf{Mon}(\delta,\varsigma)}, w)$.
- If $\varsigma' \in Z(w)$, then $(M_{\mathbf{Mon}(\delta,\varsigma)_{\varsigma'}}, w) \leftrightarrows (M_{\varsigma'}, w)$.
- If $\delta \in Y(w)$ and $\varsigma \in Z(w)$, then $(M_{\mathbf{Mon}(\delta,\varsigma)_{\varsigma'}}, w) \leftrightarrows (M_{\varsigma\mathbf{Mon}(\delta,\varsigma)}, w)$.

Cut. Let ς' be the rule $(\,(\text{prem}(\varsigma_2) - \{\text{conc}(\varsigma_1)\}) \cup \text{prem}(\varsigma_1),\ \text{conc}(\varsigma_2)\,)$:

- If $\sigma \neq \varsigma'$, then $(M_{\mathbf{Cut}(\varsigma_1,\varsigma_2)_\sigma}, w) \leftrightarrows (M_{\sigma\mathbf{Cut}(,\mathbf{Cut}(M,))}\varsigma_1\varsigma_2, w)$.
- If $\varsigma' \in Z(w)$, then $(M_{\mathbf{Cut}(\varsigma_1,\varsigma_2)_{\varsigma'}}, w) \leftrightarrows (M_{\varsigma'}, w)$.
- If $(\text{prem}(\varsigma_1) \cup \{\text{conc}(\varsigma_1)\}) \in Y(w)$ and $\varsigma_1 \in Z(w)$, then $(M_{\mathbf{Cut}(\varsigma_1,\varsigma_2)_{\varsigma'}}, w) \leftrightarrows (M_{\varsigma_2\mathbf{Cut}(\varsigma_1,\varsigma_2)}, w)$.

The involved model operations (structural ones and deduction) preserve worlds, accessibility relations and valuations. Then, in order to show that the identity relation over worlds reachable from w is a bisimulation, we just need to show that w and all such worlds have the same information and rule set in both models. The proofs follow from the definitions of the structural and the deduction operations.

Consider as an example the third bisimilarity for monotonicity. For information sets, take any γ in the information set of w at $M_{\mathbf{Mon}(\delta,\varsigma)_{\varsigma'}}$; by definition of $(\cdot)_{\varsigma'}$, either it was already in that w at $M_{\mathbf{Mon}(\delta,\varsigma)}$ or else it was added by the deduction operation. In the first case, it is also in the information set of w at M, since structural operations do not modify information sets; then it is also in w at M_ς and finally it is in w at $M_{\varsigma_{\mathbf{Mon}(\delta,\varsigma)}}$. In the second case, if it was added by the deduction operation, then it should be conc(ς') and, moreover, we should have the premises of ς' (and hence those of ς) in w at $M_{\mathbf{Mon}(\delta,\varsigma)}$; then they are already in w at M. But by hypothesis we also have the rule ς in w at M, so conc(ς) = conc(ς') is in w at M_ς and hence it is in w at $M_{\varsigma_{\mathbf{Mon}(\delta,\varsigma)}}$.

For the other direction, take γ in w at $M_{\varsigma_{\mathbf{Mon}(\delta,\varsigma)}}$. Then it is in w at M_ς and therefore either it was already in w at M or else it was added by the deduction operation. In the first case, γ is preserved through the monotonicity and the deduction operations, and therefore it is in w at $M_{\mathbf{Mon}(\delta,\varsigma)_{\varsigma'}}$. In the second case, γ should be conc(ς), and then we should have prem(ς) and ς in the correspondent sets of w at M. Since by hypothesis we have δ in w at M, we have all the premises of ς' in w at M and therefore they are also in w at $M_{\mathbf{Mon}(\delta,\varsigma)}$. Since we have ς in w at M, we have ς' in w at $M_{\mathbf{Mon}(\delta,\varsigma)}$ too. Hence, we have conc(ς') = conc(ς) in w at $M_{\mathbf{Mon}(\delta,\varsigma)_{\varsigma'}}$, as required. The case for rules is similar.

Now suppose a world u is reachable from w through the accessibility relation at $M_{\mathbf{Mon}(\delta,\varsigma)_{\varsigma'}}$. Since neither the relations nor the worlds are not modified by the operations, u is reachable from w at M and therefore u is reachable from w at $M_{\varsigma_{\mathbf{Mon}(\delta,\varsigma)}}$ too. Here we make use of the coherence properties, preserving formulas and rules across accessibility relations. Since $\delta \in Y(w)$ and $\varsigma \in Z(w)$, we have δ and ς in the correspondent sets of u, and then we can apply again the argument we used for w to show that u has the same information and rule set on both models.

A.6 Closure of explicit observation operation

Let $M = \langle W, R, V, Y, Z\rangle$ be a model in $\mathbf{EI}_K$. To show that $M_{\chi!}$ is also in $\mathbf{EI}_K$, we will show that it satisfies coherence and truth for formulas, coherence and truth for rules and equivalence.

If χ is a formula. Equivalence follows immediately, as well as the properties for rules (coherence and truth) since Z is not affected in the remaining worlds. Coherence for formulas holds because χ is added uniformly, and truth for formulas holds because of the definition of W', keeping worlds where χ is true. If χ is a rule. Equivalence is just as before, and the properties for formulas (coherence and truth) hold because Y is not affected in the remaining worlds. Coherence for rules holds because χ is added uniformly, and truth for rules holds because of the definition of W', keeping worlds where TR(χ) holds.

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