# Complexity of Intuitionistic Predicate Logic with One Variable

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**Abstract**

The validity problem for intuitionistic predicate logic with one variable is complete for the class of co–nexptime problems. This result holds for both constant and for expanding domains. The upper bound follows because every refutable formula can be refuted in a Kripke model whose size is exponential in the length of the formula. The lower bound is obtained by an encoding of the square tiling problem.

Prior first investigated intuitionistic predicate logic with one variable. It is known that the validity problem for this logic is decidable (Ono 1977), (Fisher-Servi 1978), whereas the same problem for the logic with two variables is undecidable (Gabbay and Shehtman 1993). Here we consider the computational complexity of this validity problem. It is known that for intuitionistic propositional logic this problem is complete for the class of polynomial space problems (PSPACE). In classical logic the situation is as follows: The validity problem for propositional logic is co-NP complete, the same holds for the one variable fragment of first order logic (this is just modal S5). The validity problem for the two variable fragment is decidable (Scott 1962, Mortimer 1975) and complete for the class of co non deterministic exponential time problems (NEXPTIME)(Grädel et al. 1997). Undecidability strikes only with at least three variables in first order logic.

Thus we consider the question of transfer of complexity of the validity problem when going from a propositional logic to its one variable fragment. In classical logic complexity transfers. Our results show that in intuitionistic logic it does not: it jumps from PSPACE to co-NEXPTIME, both for constant and for expanding domains.

The lower bound is obtained by an interpretation of the square tiling problem with the width given in binary. Fürer (Fürer 1981) showed NEXPTIME completeness of this problem. The upper bound is obtained by an interpretation into the product of modal logics $\mathbf{S4} \times \mathbf{S5}$, just as in (Fisher-Servi 1978) and (Gabbay and Shehtman 1998). Unfortunately the complexity of the validity problem for the bimodal $\mathbf{S4} \times \mathbf{S5}$ is higher than the obtained lower bound, it is hard for exponential space. But the range of the translation is smaller than the bimodal language, and in particular all formulas are "upward closed" (they are of the form $\boxdot\,\phi$ with $\boxdot$ the $\mathbf{S4}$ box). For these formulas we show that satisfiability implies satisfiability on a finite model whose size is single exponential in the lenght of the formula. This gives us the matching upper bound.

$\mathbf{S4} \times \mathbf{S5}$ can be seen as the one variable fragment of modal predicate logic **S4** with constant domains. (Gabbay and Shehtman 1993) shows that for several well known modal logics (eg. $\mathbf{K}, \mathbf{T}, \mathbf{S4}, \mathbf{S5}$), the modal predicate logic with two variables is undecidable. So we again have a question of transfer of complexity for the one variable case. For constant domains, the results are negative: for all of $\mathbf{K}, \mathbf{T}, \mathbf{S4}, \mathbf{S5}$, the validity problem of the one variable modal predicate logic is co-NEXPTIME hard (Marx 1999). For expanding domains things might be better. We conjecture that for **K** the problem remains in PSPACE. We plan a more systematic study in a sequel to this paper. This finishes our introduction. The rest of the paper is devoted to a proof of the following theorem. The logics are defined just after it.

**Theorem 1** *The validity problem for intuitionistic predicate logic with one variable is complete for the class of problems solvable in co–non–deterministic exponential time. This holds both for expanding and for constant domains.*

The language of intuitionistic predicate logic with one variable $x$ is defined by the following recursion:

$$P_i(x) \mid \bot \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \rightarrow \psi \mid \forall x \phi \mid \exists x \phi.$$

As a convention we write $P_i(x)$ as $p_i$.

A Kripke *model* $\mathfrak{M}$ for this language is a structure $(W, \leq, f, I_w)_{w \in W}$ with $(W, \leq)$ a poset, $f : W \longrightarrow \mathcal{P}(D)$ for some set $D$, and $I_w$ a function assigning to each variable $p_i$ a subset of $f(w)$. $I_w$ is such that $w \leq w'$ implies $I_w(p_i) \subseteq I_{w'}(p_i)$. We say that $\mathfrak{M}$ is a model with *constant domains* if $f(w) = f(w')$ for all $w, w' \in W$. We call $\mathfrak{M}$ a model with *expanding domains* if $w \leq w'$ implies $f(w) \subseteq f(w')$.

Truth at a pair $w, d$ for $w \in W$, $d \in f(w)$ is defined by the following recursion:

$$\begin{array}{lcl}
\mathfrak{M}, w, d \Vdash p_i & \iff & d \in I_w(p_i) \\
\mathfrak{M}, w, d \nVdash \bot & & \\
\mathfrak{M}, w, d \Vdash \phi \wedge \psi & \iff & \mathfrak{M}, w, d \Vdash \phi \text{ and } \mathfrak{M}, w, d \Vdash \psi \\
\mathfrak{M}, w, d \Vdash \phi \vee \psi & \iff & \mathfrak{M}, w, d \Vdash \phi \text{ or } \mathfrak{M}, w, d \Vdash \psi \\
\mathfrak{M}, w, d \Vdash \phi \rightarrow \psi & \iff & (\forall w' \geq w) : \mathfrak{M}, w', d \Vdash \phi \text{ implies } \mathfrak{M}, w', d \Vdash \psi \\
\mathfrak{M}, w, d \Vdash \exists x \phi & \iff & (\exists d' \in f(w)) : \mathfrak{M}, w, d' \Vdash \phi \\
\mathfrak{M}, w, d \Vdash \forall x \phi & \iff & (\forall w' \geq w)(\forall d' \in f(w')) : \mathfrak{M}, w', d' \Vdash \phi.
\end{array}$$

A formula $\xi$ is true in a model $\mathfrak{M}$ if for all $w$, for all $d \in f(w)$, $\mathfrak{M}, w, d \Vdash \xi$. We define two intuitionistic predicate logics with one variable. **QH1** contains all formulas which are true on all models with expanding domains, and **QH1–C** all formulas which are true on all models with constant domains.

The validity problem for **QH1** (**QH1–C**) is the problem of deciding for given formula $\xi$ whether or not $\xi \in$ **QH1** ($\xi \in$ **QH1–C**). The complement of the validity problem is called the refutability problem: the problem of deciding for arbitrary $\xi$ whether there exist a model $\mathfrak{M}$ and a pair $w, d$ with $d \in f(w)$ such that $\mathfrak{M}, w, d \not\Vdash \xi$.

We are ready for a proof of the theorem. We will show that the refutability problems of the mentioned logics are complete for NEXPTIME. We show the lower bound for the logic with expanding domains, and the upper bound for the logic with constant domains. The following reduction then provides the remaining bounds. Let $\xi$ be a formula, and $e$ an atom not occuring in $\xi$. Define the following recursive translation $(\cdot)^t$: $p^t = p$, $\bot^t = \bot$, $(\cdot)^t$ commutes with the booleans, and

$$(\exists x \phi)^t = \exists x(e \wedge \phi^t) \text{ and } (\forall x \phi)^t = \forall x(e \rightarrow \phi^t).$$

A standard proof shows that for any $\xi$,

$$\xi \text{ is } \mathbf{QH1} \text{ refutable iff } e \rightarrow \xi^t \text{ is } \mathbf{QH1{-}C} \text{ refutable.} \tag{1}$$

**Lower bound.** The lower bound is shown by a reduction to a tiling problem very similar to the one normally used to show undecidability. Now we don't tile the grid $\mathbb{N} \times \mathbb{N}$ but the torus $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$. This means that we have to tile the finite grid $2^n \times 2^n$ with the additional restriction that the bottom colors of the tiles on the bottom row have to match the top colors of the tiles of the top row, and similarly for the border columns. For large enough $n$, this problem is complete for nondeterministic time exponential in $n$, a result of Fürer 1981. For details we refer to Börger et al. 1997.

For every $n$, for every set of tiles $T = \{t_1, \ldots, t_k\}$ with $k < n$ we define a formula $\xi(n)$ with the properties that

- the length of $\xi(n)$ is quadratic in $n$, and
- $\xi(n)$ is refutable if and only if $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n})$ can be tiled with $T$.

Let *tree* be the intuitionistic propositional formula $\beta_{2n}$ from section 18.2 in (Chagrov and Zakharyaschev 1997). When this formula is refuted at a state $w$ in a model $\mathfrak{M}$, $\mathfrak{M}$ contains as a substructure the binary branching tree of depth $2n$ of which $w$ is the root. Let us assume for clarity that $\mathfrak{M}$ simply is the binary branching tree of depth $2n$. The formula contains variables $q_1, \ldots, q_{2n}$ which count the depth in the tree: $q_i$ is true at $x$ iff $x$ is at least $i+1$ steps removed from the root $w$. Thus at all leaves $q_{2n}$ is refuted. The formula also contains variables $p_1, \ldots, p_{2n}$ and is such that every possible valuation occurs in one of the leaves of the tree. We think of such a valuation as a pair $(k, l)$, with $0 \leq k, l \leq 2^n - 1$, in binary notation, with $p_1, \ldots, p_n$ giving the value of $k$ and $p_{n+1}, \ldots, p_{2n}$ the value of $l$.

We now define $\xi(n)$. The formule *tree* has the form $A \rightarrow C$. $\xi(n)$ is the formula $A \wedge \forall x(\mathit{succ} \wedge \mathit{store} \wedge \mathit{tile}) \rightarrow C$. The formulas *succ* and *store* are evaluated at points at which $q_{2n}$ is refuted. They ensure branching in a state where $\forall x \mathit{tree}$ holds and one in which $\forall x \mathit{tree}$ fails. The formula *tile* will ensure that a correct tiling exists.

*succ* is the formula

$$\forall x\, tree \to q_{2n}, \tag{2}$$

and guarantees that from each state refuting $q_{2n}$ there exists a state with a point in its domain which is (again) the root of a binary branching tree of depth $2n$.

The formula *store* is

$$[(\delta \wedge \forall x(tree \wedge D \wedge U \wedge R)) \to \bot] \to q_{2n}. \tag{3}$$

It states that each state refuting $q_{2n}$ also has a successor in which $\forall x\, tree$ and a variable $\delta$ holds. The formulas $D, U, R$ are defined as follows. $D$ says that all elements which make $\delta$ true have the same valuation of the $p_i$ variables:

$$D := \bigwedge_{1 \leq i \leq 2n} (\exists x(\delta \wedge p_i) \leftrightarrow \forall x(\delta \to p_i)). \tag{4}$$

$U$ and $R$ define two variables *up* and *right*. Their intended meaning is as follows: let $a, b$ be two elements in a domain $D_w$ for some $w$. Let $a$ encode the pair $(k, l)$ (as explained above) and let $b$ make $\delta$ true. Now we want that *up* holds at $a$ precisely when $b$ encodes the pair $(k, m+1)$. *right* should hold at $a$ if $b$ encodes $(k+1, m)$. Thus *up* holds at $a$ iff $a$ and $b$ make the same $p_i$ true for $1 \leq i \leq n$ and the values of $a$ and $b$ described by the variables $p_{n+1}, \ldots, p_{2n}$ differs by one (modulo $2^n$). We can axiomatize this using the well–known characterization of binary addition (e.g., Börger et al. 1997). The formula $U$ which axiomatises *up* is then

$$\begin{array}{rcll} U & := & up & \leftrightarrow \;\; [\bigwedge_{1 \leq i \leq n}(p_i \leftrightarrow \exists x(\delta \wedge p_i))] \wedge \\ & & & \quad\;\; ((\bigwedge_{n < i \leq 2n}(p_i \wedge \exists x(\delta \wedge \neg p_i))) \vee \bigvee_{n < k \leq 2n} U_k). \end{array}$$

The first conjunct states that the $x$–coordinates are the same. The second conjunct lists all cases of adding one in binary modulo $2^n$. The first disjunct covers the case when we start counting at 0 again. The disjunct $U_k$ is the formula describing adding one in binary when the $k$-th bit is the least significant 0:

$$U_k := \neg p_k \wedge \exists x(\delta \wedge p_k) \wedge \bigwedge_{j<k} (p_j \wedge \exists x(\delta \wedge \neg p_j)) \wedge \bigwedge_{i>k} (p_i \leftrightarrow \exists x(\delta \wedge p_i)).$$

A similar formula axiomatises *right*.

The final part of the formula states that at every point which encodes a pair $(k, l)$ precisely one tile–variable holds, and using the variables *up*, *right* and $\delta$ that colors match. The formula *tile* is the conjunction of

$$\begin{array}{lcl} q_{2n-1} & \to & t_1 \vee \ldots t_k \\ q_{2n-1} & \to & \bigwedge_{i \neq j} \neg(t_i \wedge t_j) \\ t_i \wedge up & \to & \forall x(\delta \to \bigvee\{t_j \mid top(t_i) = bottom(t_j)\}) \\ t_i \wedge right & \to & \forall x(\delta \to \bigvee\{t_j \mid right(t_i) = left(t_j)\}). \end{array}$$

*Claim: if $T$ tiles $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ then $\xi(n)$ is refutable.*
*Proofsketch.* Define the following countable tree structure $(W, \leq)$. From its root grows a binary branching tree of depth $2n+1$. Each point on depth $2n$ splits into a leaf and a point which is again the root of a binary branching tree of depth $2n+1$ and so on. The domain associated with each $w \in W$ is the set of natural numbers smaller than the depth of $w$ divided by $2n+1$ plus 1. The valuations of the $q_i$ and $p_i$ from the formula *tree* are set so that in each pair $(w, a)$ where $w$ is the root of a binary branching tree of depth $2n+1$ and $a \in D_w$ is the new element, the formula *tree* is refuted. This ensures that *tree* is refuted at the root of the model, and that the formula *succ* is universally true. The valuations of the tile variables is set such that whenever a pair $(w, a)$ encodes $(k, l)$, and $(k, l)$ is tiled by $T_i$, then $t_i$ is true at $(k, l)$. The valuation of $\delta$ is set such that for every $w \in W$, for every $d \in D_w$, $(w, d) \Vdash \delta$ iff $w$ is a leaf and $d$ equals the depth of $w$. Let $w$ be a leaf and $(w, d)$ validate $\delta$; for $d' \in D_w$, we let $(w, d') \Vdash up$ just in case when $(w, d')$ encodes $(k, l)$ and $(w, d)$ encodes $(k, m+1)$ (modulo $2^n - 1$), for some $k, l$. We similarly define *right*.

Now it is simply verified that the valuation is correct and that *store* and *tile* hold universally in the model.

*Claim: if $\xi(n)$ is refuted, then $T$ tiles $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$.*
*Proof.* Let $\mathfrak{M} = (W, \leq, f, I)$ be a model and let $(w, a_0)$ refute $\xi(n)$. Then, since $(w, a_0)$ refutes *tree*, for every $(k, l)$ with $k, l \leq 2^n - 1$ there exists $(w', a_0)$ with $w \leq w'$ such that

- $(w', a_0)$ encodes $(k, l)$ and $\mathfrak{M}, (w', a_0) \Vdash t_i$ for exactly one tile–variable $t_i$, and

by virtue of the conjunct *succ* which is true everywhere in $\mathfrak{M}$,

- there exists $w'' \geq w$ and $a_1 \in D_{w''}$ such that $(w'', a_1)$ again refutes *tree*.

Now choose for every $(k, l)$ a pair $(w_{(k,l)}, a_{(k,l)})$ as above in such a way that if $k \leq k'$ (modulo $2^n$) or $l \leq l'$ (modulo $2^n$), then $w_{(k,l)} \leq w_{(k',l')}$, and define a tiling of $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ accordingly. Clearly the tiling is well–defined (by virtue of the conjunct *tile*) but do colors match? To see this, take arbitrary $(k, m)$ and $(k, m+1)$ with colors $t_i$ and $t_j$ respectively. By our choice of pairs we can assume that $w_{(k,m)} \leq w_{(k,m+1)}$. By *store* there exists a $w'_{(k,m+1)} \geq w_{(k,m+1)}$ and $(w'_{(k,m+1)}, a_{(k,m+1)})$ validates $\delta$ and the formulas $\forall x D, \forall x U$ and $\forall x R$. Thus also $w_{(k,m)} \leq w'_{(k,m+1)}$ and by expanding domains, $a_{(k,m)} \in D_{w'_{(k,m+1)}}$. Now $(w'_{(k,m+1)}, a_{(k,m)})$ also encodes $(k, m)$ and validates $t_i$, and similarly $(w'_{(k,m+1)}, a_{(k,m+1)})$ encodes $(k, m+1)$ and validates $t_j$. This means that the formulas $D$ and $U$ enforce that $(w'_{(k,m+1)}, a_{(k,m)})$ must validate $up$, and finally that the colors $t_i$ and $t_j$ match by the formula *tile*.

**Upper bound.** The upper bound follows from the translation into $\mathbf{S4} \times \mathbf{S5}$ and Theorem 2. The product $\mathbf{S4} \times \mathbf{S5}$ of modal logics (cf. (Gabbay and Shehtman 1998)) is the bimodal logic of the class which consists of all frames $\mathfrak{F} \times \mathfrak{G}$ defined as follows. For $\mathfrak{F} = (W, \leq)$ an $\mathbf{S4}$ frame and $\mathfrak{G} = (D, \equiv)$ an $\mathbf{S5}$ frame, let

$$\mathfrak{F} \times \mathfrak{G} = (W \times D, \leq^+, \equiv^+), \text{ with}$$

$$
\begin{array}{lcl}
(w,d) \leq^{+} (w',d') & \iff & w \leq w' \text{ and } d = d' \\
(w,d) \equiv^{+} (w',d') & \iff & w = w' \text{ and } d \equiv d'.
\end{array}
$$

We use Segerberg's geometric diamonds, with $\diamondsuit$ for the (vertical) **S4** and $\ominus$ for the (horizontal) **S5** diamond. The corresponding boxes are defined as usual.

There exists a complete axiomatization of $\mathbf{S4} \times \mathbf{S5}$. The class of $\mathbf{S4} \times \mathbf{S5}$ frames generates the same logic as the class of frames $(W, \leq, \equiv)$ in which

1. $\leq$ is transitive and reflexive,
2. $\equiv$ is an equivalence relation, and
3. $\leq \circ \equiv \; = \; \equiv \circ \leq$.

The Gödel–Tarski translation interprets propositional intuitionistic logic into modal **S4**:

$$
\begin{array}{lcl}
p^t & = & \Box p \\
(\cdot)^t \text{ commutes with } \wedge \text{ and } \vee & & \\
(\phi \to \psi)^t & = & \Box(\phi^t \to \psi^t).
\end{array}
$$

Fisher-Servi 1978 extended this translation for the one–variable fragment to $\mathbf{S4} \times \mathbf{S5}$ by

$$
\begin{array}{lcl}
(\exists x \phi)^t & = & \ominus \phi^t \\
(\forall x \phi)^t & = & \Box \boxminus \phi^t.
\end{array}
$$

Fisher–Servi showed that for every formula $\phi$,

1. $\phi^t$ is equivalent to $\Box \phi^t$, and
2. $\phi \in \mathbf{QH1{-}C}$ iff $\phi^t \in \mathbf{S4} \times \mathbf{S5}$.

This implies that the **QH1−C** validity problem can be reduced to the refutability problem for upward closed $\mathbf{S4} \times \mathbf{S5}$ formulas.

A bimodal formula is called V–boxed if it is generated by the following recursion:

$$
\Box p_i \mid \Box \bot \mid \Box(\phi \wedge \psi) \mid \Box(\phi \vee \psi) \mid \Box(\phi \to \psi) \mid \Box \boxminus \phi \mid \Box \ominus \phi.
$$

We say that a $V$–boxed formula is refuted in an $\mathbf{S4} \times \mathbf{S5}$ model $\mathfrak{M}$ if there exists a state $s$ such that $\mathfrak{M}, s \not\Vdash \xi$. For $\xi$ a formula, $Sub(\xi)$ denotes the set of subformulas of $\xi$, and $|\xi|$ the length of $\xi$ measured as a string of symbols.

**Theorem 2** *Every refutable V–boxed formula $\xi$ is refutable in a* $\mathbf{S4} \times \mathbf{S5}$ *model whose size is bounded by* $2^{O(|\xi|^2)}$.

The proof is a mixture of a filtration and a selective generation argument.

Proof. Let $\xi$ be refuted at $\mathfrak{M}, w_0$, with $\mathfrak{M}$ an $\mathbf{S4} \times \mathbf{S5}$ model based on a frame $(W, \leq, \equiv)$. Define for $w \in W$, $\bar{w} = \{\phi \in Sub(\xi) \mid \mathfrak{M}, w \Vdash \phi\}$. Let $Row(w) = \{\bar{w'} \mid w \equiv w'\}$. We can view $Row(w)$ as the filtration through $Sub(\xi)$ of the $\equiv$ equivalence class of $w$. Let $W' \subseteq W$ be a minimal subset satisfying $w_0 \in W'$ and closed under the following rule: if $w \in W'$, and if there is an $\bar{x} \in Row(w)$ and a $\boxdot(\phi \rightarrow \psi) \in Sub(\xi)$ but $\boxdot(\phi \rightarrow \psi) \notin \bar{x}$, then either $\phi \in \bar{x}$ or there exists a $w' \in W'$ and a $\bar{y} \in Row(w')$ such that $\bar{x} \subseteq \bar{y}$ and $\phi \in \bar{y}$ and $\psi \notin \bar{y}$.

We claim that the size of $W'$ can be bounded by $2^{O(|\xi|^2)}$. Indeed, the maximal length of the $\leq$ paths in $W'$ can be bounded by $b$, the number of $\boxdot(\phi \rightarrow \psi)$ subformulas of $\xi$. This holds because $\mathfrak{M}, w \not\Vdash \boxdot(\phi \rightarrow \psi)$ and $\mathfrak{M}, w \Vdash \phi$ implies that $\mathfrak{M}, w \not\Vdash \psi$. (Recall that all formulas are V–boxed.). Thus the size of $W'$ is bounded by $b \cdot (2^{|\xi|})^b \leq 2^{O(|\xi|^2)}$.

Create a model $\mathfrak{M}^* = (W^*, \leq^*, \equiv^*, V^*)$ as follows. $W^*$ is the disjoint union of all $Row(w)$ for $w \in W'$. For $x, y \in W^* : x \equiv^* y$ iff $x, y \in Row(w)$ for some $w$. For $x, y \in W^* : x \leq^* y$, iff there exists $w, w' \in W'$, $w \leq w'$, $x \in Row(w)$, $y \in Row(w')$, and $x \subseteq y$. Finally set $\bar{x} \in V^*(p)$ iff $\boxdot p \in \bar{x}$. Note that $\bar{x} \in V^*(p)$ and $\bar{x} \subseteq \bar{y}$ implies that $\bar{y} \in V^*(p)$.

The size of $W^*$ is less than $2^{|\xi|}$ times the size of $W'$, thus less than $2^{|\xi|^2 + |\xi| + log|\xi|}$. We claim that $\mathfrak{M}^*$ is an $\mathbf{S4} \times \mathbf{S5}$ model and that $\xi$ is refuted at $\bar{w_0}$.

Clearly $\equiv^*$ is an equivalence and $\leq^*$ a reflexive and transitive relation. (For transitivity we use the fact that $W^*$ is defined as a *disjoint union* of rows.) We show one side of commutativity of the relations, the other is shown similarly. Let $\bar{x} \equiv^* \bar{z} \leq^* \bar{y}$. Then for some $w, w'$, it holds that $w \leq w'$ and $x \equiv z \equiv w$ and $y \equiv w'$. Thus, by commutativity in $\mathfrak{M}$, there exists a $u \in W$, such that $x \leq u$ and $u \equiv w'$. But then $\bar{u} \in Row(w')$ and $\bar{x} \leq^* \bar{u}$ and $\bar{u} \equiv^* \bar{y}$. Thus $\mathfrak{M}^*$ is an $\mathbf{S4} \times \mathbf{S5}$ model.

To show that $\xi$ is stil refuted at $\mathfrak{M}^*, \bar{w_0}$, we show that for all $\phi \in Sub(\xi)$, for all $w \in W^*$, $\mathfrak{M}^*, w \Vdash \phi \iff \phi \in w$. We show this by induction on the structure of the (V–boxed) formula. For the $V$–boxed variables it holds by definition. For $\wedge, \vee$ the proof is immediate. For $\boxdot \Diamond \phi$ it holds because we included complete rows. For $\boxdot \boxminus \phi$, $\boxdot \boxminus \phi \in \bar{w}$ implies $\mathfrak{M}, w \Vdash \boxdot \boxminus \phi$, which in turn implies that for all $w' \geq w$ and all $w'' \equiv w'$, $\mathfrak{M}, w'' \Vdash \phi$. Thus by induction hypothesis $\phi \in \bar{w''}$, for all $\bar{w''}$ such that for any $x$, $\bar{w} \leq^* x \equiv^* \bar{w''}$. For the other direction: if $\boxdot \boxminus \phi \notin \bar{w}$, then $\mathfrak{M}, w \not\Vdash \boxdot \boxminus \phi$. Thus –by the constant domain assumption– there exists a $w' \equiv w$ such that $\mathfrak{M}, w' \not\Vdash \phi$. But then $\bar{w'}$ is in $Row(w)$, thus included in $W^*$ and $\bar{w'} \equiv^* \bar{w}$. By inductive hypothesis, $\mathfrak{M}^*, \bar{w'} \not\Vdash \phi$, whence $\mathfrak{M}^*, \bar{w} \not\Vdash \boxdot \boxminus \phi$. Finally, if $\boxdot(\phi \rightarrow \psi) \in \bar{w}$, and $\bar{w} \leq^* \bar{x}$, then $\boxdot(\phi \rightarrow \psi) \in \bar{x}$. Suppose $\mathfrak{M}, x \Vdash \phi$. Then by inductive hypothesis, $\phi \in \bar{x}$. Then $\mathfrak{M}, x \Vdash \boxdot(\phi \rightarrow \psi)$ and $\mathfrak{M}, x \Vdash \phi$, whence $\mathfrak{M}, x \Vdash \psi$ and $\psi \in \bar{x}$. Thus $\mathfrak{M}, \bar{w} \Vdash \boxminus(\phi \rightarrow \psi)$, by the inductive hypothesis.

We arrived to the final and most interesting case. This is the formula for which we carefully crafted the set $W'$. Let $\boxdot(\phi \rightarrow \psi) \notin \bar{x}$ for $\bar{x} \in Row(w)$ for some $w \in W'$. If $\phi \in \bar{x}$, then $\psi \notin \bar{x}$, and by inductive hypothesis indeed $\mathfrak{M}^*, \bar{x} \not\Vdash \boxdot(\phi \rightarrow \psi)$. If $\phi \notin \bar{x}$ then —by construction of the model— there exists a $w' \geq w$ in $W'$, with a $\bar{y} \in Row(w')$ such that $\bar{y} \supseteq \bar{x}$, $\phi \in \bar{y}$ and $\psi \notin \bar{y}$. Then

$\bar{y} \in W^*$, $\bar{x} \leq^* \bar{y}$ and by inductive hypothesis $\mathfrak{M}^* \not\Vdash \boxdot(\phi \to \psi)$. QED


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