IMPLICIT AND EXPLICIT STANCES IN LOGIC

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Abstract We identify a pervasive contrast in modeling styles in logic between what may be called 'implicit' and 'explicit' approaches. In order to accommodate new topics entering the field, implicit approaches change the meaning of existing logical constants and consequence, while explicit approaches merely extend classical logical systems with new vocabulary. We discuss the contrast in intuitionistic vs. epistemic logic, default reasoning, information dynamics and games, and then define the stances more sharply. New issues arise concerning dualities and merges between the two styles of logical analysis. Finally, we discuss the contrast in greater depth, considering what it means for an understanding of logic as a repertoire of natural styles of analysis.

1 Explicit and implicit stances in logical analysis

Logic has had a rich history since Antiquity with themes running a spectrum ranging from description of basic ontological structures in the world to elucidating basic patterns in inferential or communicative behavior. Modern logic added a methodology of formal systems, with formulas evaluated for truth or falsity in matching models, and valid consequence defined as truth of all premises implying truth of the conclusion. The latter methodology is enshrined in classical first-order predicate logic, still the research tool par excellence of the field, and its standard textbook fare. This system deals with the forms of atomic predication, as well as propositional connectives and basic quantifiers, enabling us to make complex assertions about what the world is like.

But modern logic has a much richer agenda of concepts to be studied, including notions of information, knowledge, belief, action, agency, and many others that are staple fare in philosophical logic or computational logic. After all, logic is not a finished field but an advancing quest. How are these new topics to be brought into the scope of the mathematical systems methodology? [1] There are many variations on, and extensions of, classical logic developed for these purposes, and the aim of this paper is to point at two main lines, representing a sort of watershed – at least for most of what is to follow.

One line of enriching classical logic just adds new operators for new notions that are to be analyzed, leaving old explanations of existing logical notions untouched. A paradigm for this is modal logic (cf. van Benthem 2010), where operators are added for modalities, while nothing changes in the propositional base logic, and classical predicate logic can even be used to analyze modalities under translation. Let us call this the *explicit* style of analysis, although other epithets are appropriate, too, such as `conservativeness': since we do not touch the logical analyses that are already in place.

But here is another line, where we use the new concepts to modify or enrich our understanding of what the old logical constants meant, or what the old notion of valid consequence was supposed to do. This second line leads us to non-standard semantics, perhaps re-interpreting truth as some sort of 'forcing', and to alternative logics whose laws differ from those of classical logic on the original vocabulary of propositional connectives and quantifiers. Here the structure of the new richer setting is reflected, not in new

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[1] I leave aside the issue whether a formal systems methodology is always the best way to proceed: cf. the critical discussion of unreflected `systems imprisonment' in van Benthem 1999.

laws for new vocabulary, but in the deviations on reasoning patterns in the old language, and even serious failures of classical laws may be highly significant and informative. A typical paradigm for this approach is intuitionistic logic, though instances keep emerging. Let us call this the *implicit* style of analysis. Of course, this term has no pejorative connotation here. In fact, most world cultures seem to prefer implicitness over explicitness in communication and deliberation, with the possible exception of the Dutch.

Our plan is as follows. We will discuss a number of illustrations displaying the contrast in more detail, and analyze what makes them tick. Throughout, we will take both explicit and implicit approaches seriously, although the discerning reader may see a small personal bias occasionally toward explicit paradigms. Our conclusion will be that the co-existence and interplay of the two stances raises a number of interesting points, and in fact quite a few open problems concerning the architecture of logic. We end with a few attempts at sharpening up our contrast, and seeing what it means in general.

2 Information, knowledge, and epistemic logic

One of the most natural additions to the classical heartland of predicate logic are concepts of knowledge and information, that have been part of logic from ancient times until today (cf. Kneale & Kneale 1962, Barwise & Perry 1983, van Benthem & Martinez 2008). In what follows, we are not going into the intricacies of contemporary logics for epistemology (cf. Holliday 2012, Baltag & Smets 2008 for some samples), interesting and innovative though these are. The contrast in modus operandi that we are after can be demonstrated at much simpler level, dating back to the 1960s. [2]

Here is a major explicit way of taking knowledge and information seriously. We add modal operators for knowledge to classical propositional logic, and study the laws of the resulting *epistemic logics* on top of classical propositional logic. This approach aims for conservative operator extensions of classical logical systems, and these turn out to come with interesting structure and modeling power, also for notions beyond knowledge, such as belief. Thus, epistemic logic is now a family of widely used systems in several disciplines: not just philosophy, but also linguistics, computer science or economics.

In a bit more detail, the epistemic logic of Hintikka 1962 proposes an analysis of the notion of knowledge based on the intuitive conception of information as a current range of possible candidates for the actual situation (or 'world', 'state': what's in a name). [3] This range may be large, in which case we know little, or it may be small (perhaps as a result of many prior information updates eliminating possibilities) and then we know a lot. In this setting, an agent *knows that* φ at a current world s if φ is true in all worlds in the current range of s, the 'epistemically accessible' worlds from s, given by some binary relation $s \sim t$. To express forms of reasoning in a perspicuous matching syntax, we take, say, standard propositional logic as our base language, and add a formation clause for constructing formulas of the form $K\varphi$ – that can be subscripted to $K_i\varphi$ for

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[2] In general, we will make our points to come by referring only to the propositional structure of classical logic whenever we can get away with it. Of course, extensions versus modifications of classical predicate logic are an interesting test case for our claims then.

[3] One can think of this as the notion of 'semantic information' proposed in Bar-Hillel & Carnap 1953, one of the varieties of information in logic discussed in van Benthem & Martinez 2008.

different indices *i* in case we want to distinguish between what different agents *i* know. Then the preceding intuition becomes the following truth definition:

$\boldsymbol{M}, s \models p$	iff	$s \in V(p)$
$\boldsymbol{M}, s \models \neg\varphi$	iff	*not* $\boldsymbol{M}, s \models \varphi$
$\boldsymbol{M}, s \models \varphi\wedge\psi$	iff	$\boldsymbol{M}, s \models \varphi$ *and* $\boldsymbol{M}, s \models \psi$
$\boldsymbol{M}, s \models K\varphi$	iff	$\boldsymbol{M}, t \models \varphi$ *for all t with* $s \sim t$.

This scheme of interpretation is clearly classical logic extended: the three basic clauses are exactly as in standard propositional logic, with a fourth operator clause added. It is often assumed in these models that epistemic accessibility ~ is an equivalence relation – but we can add more finesse if needed (see below for an example). The resulting modal logic is the well-known *S5* for each single agent, without non-trivial bridge axioms relating knowledge of different agents. [4] Thus, basic epistemic logic is a conservative extension of classical logic, and the same is true for variations such as *S4* or *S4.2* that arise as one varies modal axioms following different intuitions for the *K*-operator.

Few people today believe in the original epistemic logic as a viable vehicle for analyzing the philosophical notion of knowledge. But as already suggested in our presentation, this system is a perfect fit for another basic notion, the *semantic information* that an agent has at her disposal. And, the simple perspicuous explicit syntax of epistemic logic is still in wide use as a lingua franca for framing epistemological debates, for instance, for or against such basic principles of reasoning about knowledge as

'omniscience' or 'closure'	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
'introspection'	$K\varphi \to KK\varphi$

But these are debates about intuitively acceptable principles for knowledge, not about the underlying propositional logic. The more sophisticated current accounts of knowledge in the philosophical literature that we mentioned above define knowledge as a much more complex notion, involving structure beyond mere semantic ranges, such as relevance order, plausibility order of alternatives to deal with belief (something that we will discuss at greater length later on), or similarity order to deal with conditionals. (Cf. van Benthem 2016, Baltag, van Benthem & Smets 2016 for current panorama's.) The relevant point to note for us here is that logics for these extended settings tend to be multi-modal systems, that is, they still fall under what we call the explicit approach.

Caveat This is a shamelessly brief exposition of epistemic logic, and as such it may be misleading. In particular, epistemic logic as described here has come into wide use in game theory and computer science because of its potential for describing multi-agent interactions in communication or games. [5] We refer to for Fagin, Halpern, Moses & Vardi 1995, van Ditmarsch, Halpern, van der Hoek & Kooi, eds., 2015 for more.

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[4] More interesting base laws emerge when one adds syntax for *group agents* to the language as well as matching modalities of suitably defined 'common knowledge' or 'distributed knowledge' in such groups. These are essential in many current applications of epistemic logic. But we will not need such extensions for the purposes of this paper.

[5] *In the remainder of this paper, for convenience, we suppress all agent indices for modalities.*

Finally, our short presentation of epistemic logic stands for something quite common. The explicit operator approach is quite typical for many areas of philosophical logic, such as temporal, deontic, or conditional logic, each adding their own sort of modalities.

3 Intuitionistic logic

Next, we consider a second way of taking knowledge and information seriously, a sort of 'radical chic' revolting against the classical environment that originally nurtured us. We no longer take the old order for granted, but *redefine* the meanings of the logical notions, perhaps also the notion of consequence, to get at crucial aspects of knowledge.

A typical instance of this second approach is *intuitionistic logic* that does not add knowledge operators, but encodes behavior of knowledge precisely in its deviations from the laws of classical consequence. This approach seems more radical, as breaking the classical laws that seemed so evident has an iconoclastic appeal, and more subtly, the absence of explicit expressions for epistemic notions makes the behavior of knowledge now show, not in new laws, but in *absence* of old laws, or modification thereof. [6] For instance, the well-known intuitionistic non-validity of Excluded Middle $\varphi \vee \neg\varphi$ tells us something essential about the incompleteness, in general, of our know-ledge. But on the positive side, the continued intuitionistic validity of $\neg\varphi \leftrightarrow \neg\neg\neg\varphi$ reveals a more delicate form of introspection for knowledge than the simple *S4* law we had above – assuming that negation now talks about knowledge in an implicit manner.

Intuitionistic logic had its origins in the analysis of constructive mathematical proof, with logical constants acquiring their meanings in natural deduction rules via the Brouwer-Heyting-Kolmogorov interpretation ('*BHK*'). In the 1950s, Beth and Kripke proposed models over trees of finite or infinite sequences, and in line with the idea of proof as establishing a conclusion, intuitionistic formulas are true at a node of such a tree when 'verified' there in some strong intuitive sense. A simple version that suffices as an illustration uses *partially ordered* models $\boldsymbol{M} = (W, \leq, V)$ with a valuation V, setting:

$\boldsymbol{M}, s \models p$	iff	$s \in V(p)$
$\boldsymbol{M}, s \models \varphi \wedge \psi$	iff	$\boldsymbol{M}, s \models \varphi$ *and* $\boldsymbol{M}, s \models \psi$
$\boldsymbol{M}, s \models \varphi \vee \psi$	iff	$\boldsymbol{M}, s \models \varphi$ *or* $\boldsymbol{M}, s \models \psi$
$\boldsymbol{M}, s \models \neg\varphi$	iff	*for no* $t \geq s$, $\boldsymbol{M}, t \models \varphi$
$\boldsymbol{M}, s \models \varphi \rightarrow \psi$	iff	*for all* $t \geq s$, *if* $\boldsymbol{M}, t \models \varphi$, *then* $\boldsymbol{M}, t \models \psi$ [7]

When such models are viewed as general partial orders, we can think of their objects s as information stages or information pieces, when we think of models as unraveled to trees, we get an intuitive picture with a temporal record of possible investigations. In line with the idea of accumulating certainty in the process of inquiry envisaged here, the valuation V in these models is *persistent*, i.e.,

if $\boldsymbol{M}, s \models p$ and $s \leq t$, then also $\boldsymbol{M}, t \models p$.

[6] This air of radicalism and mystery may be why implicit logics often have a much better press in philosophy than explicit approaches, for possessing greater 'depth' and significance.

[7] We have to state a separate truth condition for implication since classical inter-definabilities fail in intuitionistic logic. For instance, $\varphi \rightarrow \psi$ is not equivalent to $\neg\varphi \vee \psi$ or to $\neg(\varphi \wedge \neg\psi)$. Or, stated in other terms, intuitionistic logic distinguishes more implications than classical logic. We will discuss this advantage of implicit approaches with generalized semantics later on.

The truth definition as stated here lifts this persistence property to all formulas φ.

Notice what happens with this language and semantics. In contrast with epistemic logic, there is no separate syntactic operator for knowledge or information – but the classical logical constants are *re-interpreted*, making negation and implication sensitive to the information structure of new richer models with an inclusion order that is absent in models for classical logic. In particular, the intuitionistic negation $\neg\varphi$ says that the formula φ is not just 'not true', but that it will never become true at any further stage. [8] Also, we saw how failure of classical definabilities may lead to fine-structure for classical notions like implication, which can now be viewed in several non-equivalent ways.

This '*meaning loading'* of the classical logical operators makes the laws of reasoning for negation and implication different from those for classical logic. Now our earlier points become precise in terms of these models. Famously, this semantics makes the classical law of Excluded Middle $\varphi \vee \neg\varphi$ invalid, as this disjunction fails at states where φ is not yet verified, though it will later become so. These deviations from classical logic are informative in that they implicitly tell us about properties of knowledge for agents. The failure of Excluded Middle tells us that they cannot 'decide' everything a priori. Thus meaning loading makes both the remaining validities informative (since they now say something new), and it packs information in the absence of certain classical laws.

Caveat This is a shamelessly brief exposition of intuitionistic logic, which does not do justice to its connections with proof theory, universal algebra, and category theory, or to the many surprising effects of working in theories with the usual axioms now on top of a weaker base logic. See Troelstra & van Dalen 1988 for an excellent exposition.

4 The explicit/implicit contrast: epistemic logic versus intuitionistic logic

So, now we have two major research paradigms in the field of logic, both meant to take information and knowledge seriously – but doing so in very different ways. Let us highlight the major differences that showed in the above:

epistemic logic	explicit, conservative language extension of classical logic
intuitionistic logic	implicit, meaning change old language, non-classical logic

Summarizing the difference once more, consider the obvious fact that we do not know the answer to every question right now, and maybe never will. This fact showed as follows in intuitionistic logic. Excluded Middle $\varphi \vee_{int} \neg_{int} \varphi$ was not valid – where the indices highlight the fact that this non-validity occurs on a particular understanding of negation and disjunction, though some special cases are valid for which we refer to the literature. In contrast with this, Excluded Middle is unrestrictedly valid in epistemic logic, but it should not be confused with the invalid epistemic formula $K\varphi \vee_{class} K\neg_{class} \varphi$.

Much more can be said about these two approaches to knowledge and information. But for the purposes of the present paper, we will just stipulate that both are based on stable interesting sets of intuitions, both have generated a rich mathematical theory, and both seem worthy instances of what one should call a 'logical' modus operandi.

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[8] The intuitive interpretation of intuitionistic models poses problems that we ignore here. In particular, the original idea of proof steps is not represented in the stages of our models – making for a rather loose connection with the original constructive *BHK* interpretation.

With this single illustration, we assume that the reader has grasped the methodological duality we are after – and we will now feel free to use the terminology 'implicit' versus 'explicit' in other cases, and see what classification or even illumination arises.

A point for reflection Before jumping in, let us address a point that may well occur to the reader at this stage. If epistemic logic and intuitionistic logic are both natural, how should we understand their co-existence, and that of explicit and implicit approaches in logic generally? This theme will recur in this paper, and we will discuss it extensively toward the end. For now, one perhaps helpful way of thinking about the co-existence, admittedly a bit of a play on words, is as follows. Epistemic logic is a form of reasoning *about* knowledge, while intuitionistic logic is a form of reasoning *with* knowledge. In our view, the interplay of "about" and "with" is a general thread throughout logic.

5 Technical connections: translations and merges

But perhaps another point has troubled the reader already. Is the explicit/implicit contrast really that sharp? After all, in the 1930s, Gödel already provided a faithful *translation* from intuitionistic logic into the modal logic *S4*, and even earlier than that, Glivenko gave a converse faithful 'double-negation translation' from classical into intuitio-nistic logic. Indeed, nothing in logic is quite what it seems at the syntactic surface, with its mixture of crucial and irrelevant details, and the real analogies and differences between logical systems tend to lie underneath. Let us look into this issue of technical connections right now, again at the high level that suffices for our purposes.

Translating between formal systems The *Gödel translation* ***t*** recursively turns the ear-lier truth definition into a syntactic recipe. [9] (a) Atoms p are translated as modal formu-las $\Box p$ that enforce the earlier upward persistence on partial orders, (b) conjunctions remain conjunctions, (c) likewise for disjunctions, (d) intuitionistic negations $\neg\varphi$ beco-me modalized classical negations $\Box\neg\varphi$, and (e) intuitionistic implications $\varphi \to \psi$ beco-me modalized classical implications $\Box(\varphi \to \psi)$. Taking the mnemonic *IL* to stand for the standard proof system of basic intuitionistic propositional logic, we then have the following precise connection between an implicit logic and an explicit counterpart:

Fact $IL \vdash \varphi$ iff $S4 \vdash \mathbf{t}(\varphi)$, for all propositional formulas φ.

This result is often taken to mean that intuitionistic logic has now been 'embedded' in a richer epistemic logic that explain its key features. For instance, the earlier varieties of implication can now be explained in explicit terms showing their different demands on knowledge. While intuitionistic $\varphi \to \psi$ became the modal $\Box(\varphi \to \psi)$, $\neg\varphi \vee \psi$ becomes the much stronger $\Box\neg\varphi \vee \psi$, and $\neg(\varphi \wedge \neg\psi)$ becomes the weaker $\Box(\varphi \to \Diamond\psi)$. Moreover, the modal setting is richer, as it also supports reasoning about *non-persistent* formulas with existential modal forms that can become false at later stages. Thus, it has a theory of inquiry that allows for revision, not just cumulative update. But more can be said.

System translation is not the final word Translations between implicit and explicit system languages are mathematically interesting, and useful as manuals for relating talk in different logical frameworks, and their communities of practitioners, that help combat philosophical tendencies of seeing huge differences between logics. Even so, they are not the last word in understanding all relevant system relations.

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[9] This is an inversion of the historical order of events, but the perspective is still helpful.

The following may look strange coming from a certified modal logician, but I do think it is relevant to our discussion in this paper. The usual modal enterprise of varying sets of axioms between formal systems *S5, S4, T* or *K*, or even less-known variants, is of course important proof-theoretically, since it forces us to think about minimal deductive power driving standard arguments. Moreover, the enterprise has intuitive meaning through Correspondence Theory (van Benthem 1985, Blackburn, de Rijke & Venema 2000, van Benthem 2010), since we can say precisely what various modal axioms mean in terms matching conditions on the accessibility relation in modal models.

Relating different styles of thinking But granting these benefits, moving from one modal logic to another is not just a matter of fiddling with proof strength, it may involve a switch between quite different conceptual frameworks. In particular, moving from the system *S5* that we used originally for epistemic logic to the modal *S4* of the Gödel translation asks us to make a significant conceptual switch (van Benthem 1996B, 2009).

The reflexive transitive accessibility relation of *S4* does not just represent an epistemic range as before, it also has an essential *temporal* aspect of moving forward through time, motivated by a perspective of *inquiry*. Intuitively, as in our models for intuitionistic logic, each *S4*-model describes an *informational process* where an agent learns progressively about the state of the actual world, encoded in a propositional valuation. At endpoints of the tree, all information is in, and the agent knows the actual world. Here a generalization that we noted already is that we now also allow atomic facts to become false along the way, making *S4*-models also suitable for modeling informational 'retractions', as well as real world change. Thus, while 'translating' intuitionistic logic, we have gone a certain way toward adopting its conceptual mind set.

We will return to the view of modal logics in terms of inquiry below, highlighting the role of time and informational events in the 'static' epistemic logic presented earlier. One technical issue which arises then is whether a change of interpretation for modal models can really be separated from the question what formal language best suits these models. In particular, *S4*-models might be better served by a richer modal language, in which case the usual translations would get entangled with issues of language design. [10]

From translating to merging Another response to the co-existence of explicit and implicit systems has been more pragmatic. Given that epistemic logic and intuitionistic logic both make sense, a weaker, but still significant technical test of compatibility than outright translation is a 'biological' criterion of compatibility. Can the two systems be *merged* consistently in meaningful and useful ways that combine their virtues? As it happens, mixed systems of 'intuitionistic modal' logic have existed since the 1980s (cf. the references in van Benthem, Bezhanishvili & Holliday 2015), and more generally, hybrids of explicit and implicit logics keep emerging in the literature. We will see several examples later on when reflecting on the explicit/implicit contrast.

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[10] This is in fact my current view, and it would entangle modal correspondence theory with translation theory in ways that have not been investigated systematically yet.

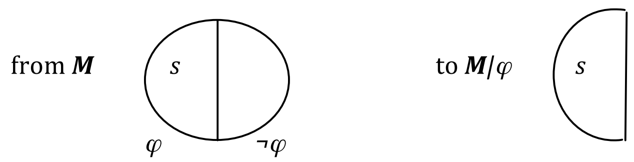
My first response to this merging trend was skeptical. There may be good reasons for pragmatism in concrete cases, but in general I did not see what the juxtaposition solves. [11] However, for a more considered opinion, see the final sections of this paper.

6 Dynamic logic of information change

Let us now move on to test our observations on the explicit/implicit contrast on a more recent development in epistemic logic, broadly conceived. Knowledge and information are one crucial feature of agency, but they are just the static attitudes associated with an ongoing process of action, or in a social setting, *interaction* between different agents.

Statics and dynamics In recent years, work in 'logical dynamics' (Gärdenfors 1988, van Benthem 1996A) has made informational action a focus of logical theory. Key actions that guide behavior of agents come in three kinds occurring together in many scenarios. Acts of *inference* are important ('drawing conclusions'), but equally important are acts of *observation*, and of *communication*. Van Benthem 2011 shows how all these actions, or on an alternative interpretation, the events that embody them, can be dealt with in current 'dynamic-epistemic logics', by adding an explicit vocabulary for the core actions to existing logical systems, and then analyzing the major laws of knowledge change. [12]

Model update To make this program more concrete, here is a pilot system making the dynamic actions behind basic epistemic logic explicit. The key idea is now that informational action is represented by model change. And the simplest case of such model change is a *public announcement* $!\varphi$ of hard information: learning with total reliability that φ is the case eliminates all current worlds with φ false: [13]



As we said earlier when motivating epistemic models, getting information by shrinking a current range of options is a common idea in many fields, that works for information flow by being told, or through observation. We call this *hard information* because of its irrevocable character: in the update step, all counter-examples are eliminated. [14]

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[11] Even the idea of combining virtues is somewhat questionable: some matches combine the worst features of partners, making merging an effective way of having the problems of both.

[12] As we will see later, logical dynamics also extends to other actions, such as belief revision or preference change, and it also works with quantitative probability as its starting point.

[13] The term announcement should be taken with a grain of salt, the information update can also be triggered by a public observation affecting all agents equally, or any other reliable source.

[14] It may be thought that hard information is simple, with actions $!\varphi$ leading to knowledge $K\varphi$. But dynamics typically involves truth value change for complex formulas. While an atom p stays true after update (physical facts do not change, though they can in extended logics of this sort), complex epistemic assertions can change truth value: before the update $!p$, I did not know that p, but afterwards I do. This results in *order dependence*: a sequence $!\neg Kp; !p$ makes sense, but the permuted $!p ; !\neg Kp$ is contradictory. By contrast, 'factual' propositions without epistemic operators have their truth value invariant under public announcements, and a variety of other update actions. For details, see the cited literature, or van Ditmarsch, van der Hoek & Kooi 2007.

Public announcement logic Public announcements are first-class citizens of logical theory in *public announcement logic* (*PAL*), a system extending epistemic logic with a dynamic modality for public announcements, interpreted as follows: [15]

$$\boldsymbol{M}, s \models [!\varphi]\psi \text{ iff } \textit{if } \boldsymbol{M}, s \models \varphi, \textit{ then } \boldsymbol{M}|\varphi, s \models \psi$$

This dynamic modality has a complete logic that records delicate phenomena of truth value change and order dependence, whose precise nature need not concern us here. We just display the crucial 'recursion law' for knowledge obtained after update, the basic equation of the *PAL* dynamics. It occurs in the following complete axiomatization:

The *S5*-laws of epistemic logic, plus obvious axioms for Boolean compounds after update, as well as the following recursion law:

$[!\varphi]K\psi \leftrightarrow (\varphi \rightarrow K(\varphi \rightarrow [!\varphi]\psi))$ [16]

General dynamics There is a method behind this system. One 'dynamifies' a given static logic from the tradition, making its underlying actions explicit and defining them as suitable model transformations. The heart of the dynamic logic then consists in a compositional analysis of post-conditions for the key actions via recursion laws like the one displayed here. This leads to conservative extensions of the base logic, though its expressive requirements sometimes force some redesign of the base language. [17]

Many notions can be treated in the preceding style: such as changes in agents' *beliefs*, *inferences*, current agenda *issues*, or even *preferences* – where model change will often consist in changing the ordering structure of the current model, rather than eliminating worlds. Moreover, more systems like the above can also deal with public and private events in rich multi-agent scenarios such as games (van Benthem 2014).

Digression While adding a dynamic component looks like just conservative extension of a given static base, it may affect received views of static modal logics. Consider the earlier-mentioned variety of modal logics over models with special conditions on their accessibility relations. From a dynamic viewpoint, one can ask *why* these conditions came to hold, and one answer is: as a result of action performed on initial models. For instance, transitive relations arise from arbitrary accessibility relations by a transformation ***tc*** of epistemic 'reflection' or 'exploration' replacing a current relation by its transitive closure, either in one fell swoop, or stepwise. But then, a static *K4*-modality is really an ordinary *K*-modality over models that have undergone this transformation:

The single static *K4*-modality $K\varphi$ is a compound dynamic-static $[\boldsymbol{tc}]\Box\varphi$.

Technically, *S4* over pre-order models then becomes embedded in a standard propositional dynamic logic over arbitrary models. Thus, the usual variety of modal logics may dissolve into one logic with different added dynamic operators for model transformation, explaining instead of just postulating the usual special relational properties.

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[15] The antecedent "*if* $\boldsymbol{M}, s \models \varphi$" reflects the assumption that the announcement is truthful.

[16] The reader may also want to check other interesting laws, like the one for iterated updates: $[!\varphi][!\psi]\chi \leftrightarrow [!(\varphi \wedge [!\varphi]\psi)]\chi$. The total system of dynamic axioms reduces formulas with dynamic operators to pure epistemic base formulas. But this is not an essential feature: such a reduction no longer works in the generalized temporal 'protocol models' for *PAL* that we discuss later on.

[17] Interesting language extensions, with independent motivations, occur with common knowledge for groups, or with dynamic operations on evidence (van Benthem & Smets 2015).

7 Implicit dynamics in intuitionistic logic

We have now extended epistemic logic, an explicit approach to knowledge, to a dynamic logic with explicit informational actions. But then, given our awareness of co-existing implicit approaches, it makes sense to ask what an analogous move would mean in the latter realm. Does intuitionistic logic support information dynamics?

One mathematical way of answering this question adds *PAL*-type operations directly to intuitionistic logic, resulting in a mixed implicit-explicit system, *PAL* on an intuitionistic base (Conradie, Ghilardi & Palmigiano 2014). But even without such a move, intuitionistic logic may already have an implicit theory of informational action.

Finding the hidden actions Intuitionistic models were stages of a process of inquiry, with endpoints as final stages where we know the truth value of all proposition letters.

What are the implicit steps in the background of such a process? Looking at some simple examples, we see that moves from one state to a successor come in two kinds.

Example The hidden dynamics of intuitionistic models.
Consider two models $\boldsymbol{M_1}$, $\boldsymbol{M_2}$, where the first refutes the classical double negation law $\neg\neg p \rightarrow p$, and the second the law of 'weak excluded middle' $\neg p \vee \neg\neg p$:

Here is what the annotations say. The two branches of $\boldsymbol{M_2}$ may be viewed as public announcements of which endpoints, viewed as classical valuations, the process can get to. This is like *PAL*-style learning by elimination of worlds. But in other intuitionistic steps, like the one in $\boldsymbol{M_1}$, there is no such elimination of endpoints, and we merely get more proposition letters true at the next stage. Van Benthem 2009 explains this move as a new type of informational action, namely, 'awareness raising' $\#\varphi$ that some fact φ is the case, where awareness involves *syntactic* in addition to semantic information.

Factual and procedural information But, there is one more new feature of interest. Through their tree structure, intuitionistic models register *two* notions of semantic information on a par, a distinction also proposed for epistemic inquiry (Hoshi 2009):

(a) *factual information* about how the world is,
(b) *procedural information* about our current investigative process.

How we can get our knowledge matters, not just what is the case. While endpoints record eventual factual information states, the branching tree structure of intuitionistic models themselves, both its available and its missing intermediate stages, encodes further non-trivial information: viz. agents' knowledge of the latter procedural kind.

This distinction between types of information changes the standard translational view of how intuitionistic and epistemic logic connect. The obvious epistemic logic matching the notion of semantic information seems the system *S5*, and the fact that the Gödel translation rather takes us into *S4* reflects the tree-like nature of intuitionistic models, standing for arbitrary temporal procedures of inquiry. Thus, the proper explicit counterpart to intuitionistic logic may eventually be some temporal version of dynamic epistemic logic: cf. van Benthem 2009, 2011 for details. In fact, temporal *protocol models* with a designated set of 'admissible histories' of inquiry have been used already in dynamic-epistemic logic, for the purpose of modeling the role of procedural information in long-term processes of inquiry or learning beyond local dynamic steps. [18]

Thus, implicit and explicit stances can learn from each other, as their line of view highlights different aspects of the phenomenon that forms their shared interest.

8 Dynamic semantics, meaning as information change potential

However, the currently more relevant 'implicit' counterpart to dynamic epistemic logics like *PAL* is another logical paradigm, that we shall discuss briefly in this section. As with the case of epistemic logic and intuitionistic logic, this juxtaposition raises interesting conceptual issues, some of which we will identify in Section 9 below.

Here is a fundamental idea from the area of the *dynamic semantics* of natural language, Groenendijk & Stokhof 1991, Veltman 1996. The influential guiding intuition of this approach to language involves communication-oriented 'information change potential':

> The meaning of an expression is its potential for changing information
> states of someone who accepts the information conveyed by the expression.

This statement sounds much like a plea for taking informational actions seriously, as we did in the preceding section – but this time, they are treated, not by adding new operators, but by loading the meanings of classical vocabulary with dynamic features.

Dynamic semantics can be implemented in different ways, witness the references. In this section, we will use Veltman's 'update semantics' for propositional logic for a comparison with our earlier explicit approach. In update semantics, given a universe of information states viewed as sets of atomic valuations, each propositional formula φ induces a *state transformation* $[[\varphi]]$ by the following recursion:

$$
\begin{aligned}
[[p]](S) &= S \cap [[p]] \\
[[\varphi \vee \psi]](S) &= [[\varphi]](S) \cup [[\psi]](S) \\
[[\varphi \wedge \psi]](S) &= [[\psi]][[\varphi]](S), \\
[[\Box\varphi]](S) &= S, \text{ if } [[\varphi]](S) \neq \emptyset, \text{ and } \emptyset, \text{ otherwise.}
\end{aligned}
$$

Note how conjunction now stands for a dynamic notion: *sequential composition* of actions, while an existential modality becomes a 'test' on the current information state.

As with intuitionistic logic, these new meanings for old operators result in deviations from classical logic. For instance, conjunction is no longer commutative, because of the typical order dependence of dynamic procedures. [19] Facts about the informational process are now encoded in the logic of the usual logical operators in this system, especially when we take a natural notion of *dynamic consequence* saying that after processing the information in the successive premises, the conclusion has no further effect:

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[18] Cf. van Benthem, Gerbrandy, Hoshi & Pacuit 2009 for a systematic study of connections between dynamic-epistemic and epistemic-temporal logics over branching time models.

[19] However, also as in intuitionistic logic, such classical laws may still hold in special cases. For instance, $p \wedge q$ has the same update effects as $q \wedge p$, since testing facts has no side-effects.

$\varphi_1, ..., \varphi_n \models \psi$ iff for every information state X in any model, $\varphi_n(... (\varphi_1(X))$ is a fixed point for $[[\psi]]$: this set stays the same under an update $[[\psi]]$.

This dynamic notion of consequence behaves differently from classical consequence, and its deviations encode crucial features of the update process, such as its sensitivity to ordering of premises, or to different numbers of repetitions of the same premise. [20]

There are much more sophisticated systems of dynamic semantics for other classes of expressions, with different notions of meaning, state change and dynamic consequence – but the above is a fair description of the basic mechanics.

9 The contrast returns: dynamic semantics vs. dynamic logic of information

PAL and dynamic semantics The attentive reader will have seen where we are heading with the preceding thumbnail sketches. Dynamic epistemic logics like *PAL* and update semantics for propositional logic both take information change seriously, with often analogous scenarios and intuitions. And both systems have a precise account for the dynamics of informational actions. But one does so explicitly, and the other implicitly:

> *Dynamic semantics* keeps the actions implicit, while giving the old language of propositional logic richer 'dynamic meanings' supporting a new notion of consequence, with a technical theory that differs from standard propositional logic,
>
> *Dynamic epistemic logic* makes the actions explicit, provides them with explicit recursion laws, extends the old base language while retaining the old meanings for it, and in all this, it still works with standard consequence. [21]

Co-existence? This contrast raises issues about coexistence of approaches. One might wonder to which extent our distinction in this particular application area is a matter of empirical description level, with dynamic semantics closer to semantics of natural language, and dynamic-epistemic logic closer to pragmatics of speech acts. Also relevant is our earlier distinction between reasoning "with" and "about" a subject matter. Natural language may encompasses both, with many dimensions of meaning for its expressions and stances for its users. We will return to these issues in the final sections below.

Translations Technically, one can ask if systems of dynamic semantics and of dynamic-epistemic logic are inter-translatable, as we saw with epistemic and intuitionistic logic. An early result in this vein is the faithful translation from update semantics for propositional logic into modal *S5* of van Benthem 1989. Van Eijck & de Vries 1992 translate dynamic semantics in the assignment-changing format of Groenendijk & Stokhof 1991 into propositional dynamic logic *PDL*. Finally, van Benthem 1996A axiomatizes the structure theory of dynamic consequence by translating into a simple extension of basic modal logic. But no general theory is known, and in fact, there are delicate technical obstacles to comparing the two paradigms in their full technical versions. [22]

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[20] Van Benthem 1996A has a complete axiomatization of the structural rules governing dynamic consequence. Rothschild & Yalcin 2012 prove that any family of update operators satisfying Permutation and Contraction consists of world eliminations for underlying static propositions.

[21] Still, some systems of dynamic semantics retain a classical base logic for *fragments*, e.g., atomic facts in update semantics. Also, some dynamic semantics, like Yalcin's recent work on epistemic modals, employ ideas close to dynamic-epistemic logic (cf. Holliday & Icard 2012).

[22] A striking difference is that update semantics recurses on the structure of formulas viewed as actions while *PAL* does not recurse on formulas in announcements $!\varphi$, but on post-conditions for

A technical digression The preceding paragraph showed the clear technical interest of comparing implicit and explicit approaches, there being no simple trick that always works. There is no unique way of comparing and connecting, but rather a web of issues. We add a few more illustrations, comparing update semantics (*US*) and public announcement logic. (a) Is there a direct translation from *US* into *PAL*? This seems possible, given the above-mentioned translation of *US* into modal *S5*. It assigns, to each propositional formula φ interpreted as an update instruction, an *S5*-formula $t(\varphi, p)$ with the proposition letter p not occurring in φ, which defines, for each set of worlds X denoted by p, the set of worlds $[[\varphi]](X)$. But no precise translation result has been given yet. (b) Other notions in the *US* framework may require extending *PAL*. Take dynamic consequence, which wants a certain set of worlds to remain the same when an update $[[\psi]]$ is applied for the conclusion. This is a 'fixed-point operator' that is not available in the syntax of *PAL*. Now van Benthem 1996 solved this abstractly by axiomatizing a basic modal logic with additional propositional 'loop constants' (a) saying that $R_a ss$ holds for the current world s. But for *PAL*, a fixed-point operator $(!\varphi)$ is a new notion, saying that all worlds in the model satisfy φ. Thus we need to use *PAL* with a 'universal modality' added, and adjust its set of recursion laws accordingly. (c) Next, consider the opposite direction. Can *PAL* be translated faithfully into *US*? Probably not, since *US* has no explicit epistemic modality, but we are not aware of a formal proof of this impossibility. (d) Finally, we can also ignore the dynamics, as we did when comparing intuitionistic logic with modal *S4,* both static logics without explicit informational events in their syntax. What is the static explicit modal counterpart to *US*? Modal *S5* is a candidate by the result in van Benthem 1989 cited above, but can we do better?

Further linguistic phenomena: questions More practically, once we see the above implicit/explicit contrast, we can predict that it will arise in further areas of dynamic semantics, whether or not a pre-existing dynamic logic exists. A case in point is current *inquisitive semantics* (Cardelli, Groenendijk & Roelofsen 2012), where indicative and interrogative propositional formulas get assigned richer compositionally computed 'inquisitive meanings' having to do with their role in directing discourse or inquiry. The resulting logic of the usual propositional connectives then turns out to be a non-classical system of 'intermediate logic', related to Medvedev's 'logic of problems' from the constructive tradition, whose behavior encodes subtle features of inquiry. One would predict the possibility of an explicit counterpart then, and indeed, one can be found. Recent dynamic-epistemic logics of inquiry (not necessarily tied to natural language) involve explicit acts of *'issue management'*, where questions and related actions modify current 'issue structures' on top of epistemic models (van Benthem & Minica 2012).

We hope to have shown that dynamic-epistemic logics and dynamic semantics provide a new instance of the explicit/implicit contrast, raising interesting new issues of their own on translation and merging. [23] Many of these issues are yet to be settled. [24]

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modalities $[!\varphi]$. A *PAL*-counterpart to the update semantics recursion would add a repertoire of *conversational programs* built from atomic actions $!\varphi$ by sequential composition, guarded choice and iteration. But program logics over *PAL* are highly complex: cf. Miller & Moss 2005.

[23] E.g., Hamami & Roelofsen, eds., 2013 presents modern logics of questions in both styles.

[24] As an issue of some historical interest to this author: Is the modern Amsterdam trademark paradigm of dynamic semantics perhaps the true inheritor of Dutch intuitionism?

10 Dynamic logics of soft information

Our discussion so far has centered on knowledge. However, the points that we made are not confined to this special cognitive attitude. In particular, the implicit/explicit distinction applies just as well to logics of *belief*. These also show interesting features of their own, so we develop this theme a bit further. In particular, in Section 11, we will take this case study as our point of entry into another area where the explicit/implicit contrast is prominent, namely, the study of alternative consequence relations.

Belief and conditional belief 'Epistemic-doxastic' models for belief order the earlier bare epistemic ranges by a relation of *relative plausibility* $\leq xy$ between worlds x, y. These models interpret operators of absolute and conditional belief:

$\boldsymbol{M}, s \models B\varphi$ iff $\boldsymbol{M}, t \models \varphi$ for all $t \sim s$ *maximal* in the order $\leq$ on $\{u \mid u \sim s\}$

$\boldsymbol{M}, s \models B^{\psi}\varphi$ iff $\boldsymbol{M}, t \models \varphi$ for all $\leq$-maximal t in $\{u \mid s \sim u$ and $\boldsymbol{M}, u \models \psi\}$

But there is a richer repertoire of epistemic notions available on this models. For instance, on binary world-independent orderings, a good addition is 'safe belief', a standard modality intermediate in strength between knowledge and belief:

$\boldsymbol{M}, s \models [\leq]\varphi$ iff $\boldsymbol{M}, t \models \varphi$ for all t with $s \leq t$ [25]

Logics for conditional belief are the conditional logics of Lewis 1973, Burgess 1981. For a more general picture of natural modalities on these models, see Baltag & Smets 2008.

Belief change under hard information Beliefs guide our decisions and actions, and they are fed by default inferences going beyond what we know that. But beliefs can be wrong, and may be refuted by new information leading to acts of belief change, learning, and *correction*. One trigger for belief revision are the earlier public announcements. Here is the dynamic recursion law that governs the matching model changes:

$$[!\varphi]B\psi \quad \leftrightarrow \quad \varphi \rightarrow B^{\varphi}[!\varphi]\psi$$

A similar principle for updating conditional beliefs axiomatizes the system completely. There is also a recursion law for safe belief under public announcement, which is even simpler. The following equivalence holds on plausibility models:

$$[!\varphi][\leq]\psi \leftrightarrow (\varphi \rightarrow [\leq][!\varphi]\psi)$$

Belief change under soft information But our richer belief models also support new transformations. In addition to hard information, there is *soft information*, when we take a signal as serious, but not infallible. Its mechanism is not eliminating worlds, but *changing plausibility order*. A widely used soft update is the *radical upgrade*

> $\Uparrow\varphi$ changes a current model $\boldsymbol{M}$ to $\boldsymbol{M}\Uparrow\varphi$, where all φ-worlds become better than all $\neg\varphi$-worlds; within these zones, the old order remains.

The dynamic modality for radical upgrade is interpreted as follows:

$$\boldsymbol{M}, s \models [\Uparrow\varphi]\psi \text{ iff } \boldsymbol{M}\Uparrow\varphi, s \models \psi$$

and its dynamic logic can again be axiomatized completely using recursion laws.

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[25] Safe belief can define absolute belief $B_i\varphi$ and conditional belief $B^{\psi}\varphi$ (Boutilier 1994).

Logics of belief change Recursion laws have been found for belief changes under a wide variety of soft events representing different levels of trust or acceptance for the newly received information (van Benthem & Liu 2007, Baltag & Smets 2008, Girard, Liu & Seligman 2012). An area where this variety makes special sense is Learning Theory (Gierasimczuk 2010): different update rules induce different policies for reaching true belief in the limit. [26] Van Benthem & Smets 2015 has details on the landscape of modal logics for belief change, and connections with *AGM*-style postulational approaches.

We think of the material presented here as explicit logic in a double sense. Not only are events and acts that usually figure in the background motivation of logical systems made an object of study as first-class citizens of the realm of logic, but also, the dynamic logics considered here have explicit syntax and explicit laws for these actions. The new structure is not assimilated into the meanings of the original language, and hence we have conservative extensions of earlier static logics, unless – as we have seen earlier – the dynamics brings to light a serious expressive deficiency of the old vocabulary.

11 Non-monotonic consequence relations as implicit devices

Implicit accounts of belief revision Now, where there is explicit logic, there is implicit logic. How to give an account of belief revision in implicit style? One line is dynamic semantics as before, using ordering changes in current models to provide new meanings for belief-related linguistic expressions such as epistemic modals. Veltman 1996, Yalcin 2007, and Rodenhaüser 2014 have state of the art samples of this approach. Many of our earlier points will be found to apply, but we will not pursue these here.

Varieties of consequence Instead, we switch gears and consider a new arena where our explicit/implicit contrast makes sense: the study of different *consequence relations* that started in the 1980s in Artificial Intelligence, and has blossomed since. We will look at this phenomenon, too, in terms of explicit and implicit approaches. Our emphasis will be on beliefs – but 'explicitizing' consequence relations may have many faces.

Non-monotonic logic Here is our running example. Around 1980, the study of common sense-based AI modeled problem solving as a process of *non-standard inference*. In particular, the method of 'circumscription' (McCarthy 1980, Shoham 1986) says that, when reasoning in problem solving or related tasks, the following inferences are allowed:

> A conclusion need not be true in all models of the premises,
> but only in the most preferred, or most plausible models.

Thus, problem solving involves only inspection of currently most relevant cases. This style of reasoning deviates from classical logic. In particular, it is 'non-monotonic': a conclusion C may follow from a premise P in this sense, but fail to follow from the extended set of premises P, Q. The reason is that the maximal models within the set of models for the conjunction $P \wedge Q$ need not be maximal among the models of P.

Many other forms of defeasible inference have been studied, since there seems to be a large natural repertoire of human reasoning styles – and complete structural rules or proof systems have been found for them. [27] The subtle differences of these systems with

[26] There is a connection here with our earlier theme of 'procedural information' about inquiry.
[27] Van Benthem 1996A calls systematic charting varieties of reasoning the 'Bolzano Program'.

classical logic: usually they drop some classical rules, while retaining subtly weaker variants, are typically taken to encode basic features of such styles of reasoning.

Non-standard consequence as implicit analysis Now this looks much like the implicit approaches discussed earlier. What is new about a proposed notion of consequence is not explicitly on the table, but it shows in its differences and analogies with classical consequence for propositions in an unchanged classical language, often that of propositional logic. More generally, we see today's vast area of non-standard consequence relations as an implicit study of underlying processes of inference and information.

While this way of casting things is attractive as a challenge to standard consequence, a 'second opinion' makes sense. Non-standard consequence relations usually have a concrete motivation, they do not just arise by tinkering with classical structural rules. [28]

Making it explicit Can we provide alternative accounts leaving the notion of consequence standard, while adding vocabulary to make the origins of the new consequence notions clear? Of course, we need a guiding semantic perspective for doing so, and this will be different depending on the precise motivation for the new consequence relation. In the following case study, we concentrate on the role of belief in circumscription.

From inference to belief change When reading the original text of McCarthy's paper, one could also construe what is said about problem solving quite differently. We have some beliefs about the problem and where its solution might go, based on the scenarios that seem most plausible or most convenient for us to consider – and as we take in new information, this set of scenarios may change, and the beliefs may be modified. Now, the way the semantics of non-monotonic logic has gone, this fits precisely with our earlier treatment of beliefs. For instance, a circumscriptive consequence $\varphi_1, ..., \varphi_n \Rightarrow \psi$ is easily translated into our earlier dynamic logic for belief change, using the formula

$$[!\varphi_1]...[!\varphi_n]B\psi$$

This translation explain many of the usual 'deviations' of non-monotonic logic from classical consequence, since the structural rules of circumscription can now be derived from our earlier logic of belief revision. [29] The underlying explanation of the deviant inferential behavior is two-fold: the key attitude is fallible belief rather than knowledge, and we now have explicit dynamic events.

This juxtaposition of perspectives raises interesting issues. Again we see a trade-off, similar to the one we have discussed earlier for implicit and explicit logics. In a schema:

nonstandard consequence	old classical language, deviant rules of reasoning
explicit dynamic reanalysis	new language with belief and action modalities, consequence is just classical consequence.

On the second approach, non-standard reasoning is a mixture of classical reasoning and further features, not a family of radical alternatives. Dynamic logics of belief change enrich the original language with informational events and attitude changes, but they

[28] In the terminology of van Benthem 2011, structural rules are symptoms: a diagnosis is some positive semantic account of what motivates the new consequence relation and makes it tick.

[29] For instance, our earlier logic of belief change now *explains* the non-monotonicity precisely, for instance, in showing how $[!\varphi]B\psi$ does not imply $[!\varphi][!\alpha]B\psi$ for all formulas α.

then work conservatively with a classical consequence relation, explaining 'deviant' features of non-standard consequence defined as a mix of attitude change and information handling through the recursion laws for the new dynamic operators.

In the following section, we will elaborate on this difference in approaches.

12 Comparisons, translations, and merges

One can see advantages to one approach over the other here. For instance, our explicit approach involves an 'emancipation of informational events' that may be of interest per se, since it adds new events over acts of inference. But van Benthem 2008 proposes a neutral two-way dynamics of switching between two natural perspectives.

In one direction, given a non-standard notion of consequence, one can tease out informational or other events motivating it intuitively, and write their explicit dynamic logic. This dissolves non-standard deviation from classical logic into dynamic extension of classical logic. Explicit events behind non-standard notions of consequence are often easy to find, but there are also some unresolved challenges. [30]

But vice versa, given a dynamic logic of informational events, one can package some of its basic structure in the form of new consequence relations, and study those per se. The latter move can even add to the existing fund of styles of reasoning. Here is an illustration. Our earlier logics of belief change predict the existence of *new styles of default inference* based on their large repertoire of different informational events and attitudes, of which we saw a few. For instance, problem solving may involve different attitudes, such as both knowledge and belief, and also, it may take some new information as hard, and some in the earlier soft sense, leading to variants of circumscription such as

'soft-weak circumscription' $[\Uparrow\varphi_1]...[\Uparrow\varphi_n]B\psi$
'soft-strong circumscription' $[\Uparrow\varphi_1]...[\Uparrow\varphi_n]K\psi$

where the premises are just taken as soft upgrades, not as public announcements. [31] These notions all have different structural rules reflecting the differences in the underlying process of drawing a conclusion (van Benthem 2011). Thus, an explicit perspective may give us clusters of related but not quite the same notions of consequence.

Thus, if we think of comparing consequence relations only, the implicit and explicit approaches of this paper may well live in harmony, and we can sometimes perform a Gestalt switch from one to the other. There are also more technical aspects to this. Van Benthem 2009 relates the switching view to generating implicit dynamics via systematic abstraction maps, and of finding explicit dynamics via representation theorems. And such analyses can also lead to formal system translations in our earlier sense.

Digression Nevertheless, not all is quiet on this front. For instance, non-standard consequence relations are usually chosen to reflect prior intuitions about what follows from what in the initial base language, usually, one for describing empirical facts. But

[30] For instance, there is no dynamic reanalysis so far of *linear logic*, whose non-classical notion of consequence is based on proof-theoretic resource intuitions, rather than semantic ones.

[31] Using both knowledge and belief in problem solving also affects our earlier analysis. In this setting, the earlier 'update-to-test' consequence relation of implicit update semantics would be expressed by an explicit *PAL* validity making the conclusion *knowledge*: $[!P_1]...[!P_k]\ K\varphi$. This is 'strong-strong' in the above terminology, while the original circumscription is 'strong-weak'.

once we add epistemic attitudes and dynamic modalities, the issue arises whether those initial intuitions are still valid for all propositions in this richer language. This is a recurrent issue in many areas of logic: how do intuitions fare under language change? And also, some philosophical positions invested in intuition-based non-standard reasoning may come under pressure by our reanalysis, not matter how modestly presented. The existence of many different consequence relations on a par has been taken to be an argument for 'Logical Pluralism', that is, the view that logic can and should acknowledge genuinely competing views on the nature of logical consequence, and perhaps also on other core notions of the field (Beall & Restal 2006). But on the dynamic explicit analysis, this competition disappears, and instead, we get entirely compatible different extensions of classical logic, without any pluralism in a competitive sense.

13 Yet another flavor: logic and games

Our final instance of the implicit/explicit contrast, with again special flavors of its own, occurs at the currently thriving interfaces of logic and games. This is a huge area with many strands, and we merely mention what we see as the crucial point.

Logic of games One conspicuous research direction in recent decades has been the logic *of games:* the study of the logical structure of games in terms of players' actions, information, and preferences. This is the standard way of using logical tools to analyze concepts, as well as the structure of reasoning patterns involving these, either in common sense or in scientific theories. Logic of games is prominent in current 'epistemic game theory', as well as many other logical systems that analyze reasoning with notions of rationality, the structure of players' strategies, the existence of game-theoretic equilibria, and many other key features of interactive social behavior.

There is a flourishing literature at this interface involving standard combined modal logics of action, knowledge, belief, and preference (cf. van der Hoek & Pauly 2006, Perea 2012, Brandenburger 2014, Pacuit & Roy 2015), as well as richer formalisms. Such logics explicitly identify major structures in games, giving them matching notions in syntactic formalisms that extend classical logical languages, and then describe the laws of reasoning for such combined systems. Typical applications analyze various assumptions that go into game-theoretic solution concepts, types of rational agent that would play these, and the equilibria that result. An appreciable body of theory has accumulated in this vein, that we cannot survey here (cf. van Benthem 2014).

In our view, logic of games reflects the classical mode of foundational research. And in its repertoire of new syntax related to games and extensions of classical logic, it represents what we have called the explicit approach to wielding tools from logic.

Logic as games But there is also another connection between the two fields, perhaps an even more intimate one, often called logic *as games*. Starting from the 1950s, basic logical notions such as truth, valid consequence, or invariance between models have been cast as two-player games by a variety of authors (Ehrenfeucht 1961, Lorenzen 1955, Hintikka 1973). For instance, in many areas of logic today, truth of a formula φ in a model $\boldsymbol{M}$ is analyzed as existence of a *winning strategy* for the 'Verifier' player in a natural evaluation game for φ played over $\boldsymbol{M}$. Likewise, there are logic games for model construction, finding proofs, measuring expressive power, and many other purposes.

While logic games are often viewed as mere didactic devices for studying standard logical systems whose main notions and properties have already been written in stone, they have become essential recently in providing semantics for computational logics, such as fixed-point languages, or in the form of 'game semantics', for linear logic. Indeed, in such semantics, the meaning of a formula is a generic game that can be played over arbitrary models. And so, basic logic becomes basic game algebra for the relevant games, a system that may well deviate in its validities from ordinary logics. [32]

And thus, with logic games, we see the typical features of an implicit approach once more – perhaps most clearly in the case of evaluation games. The classical truth-value-based meanings of predicate logic are replaced by game-based meanings. And also, strategic powers of players in games are reflected not in explicit logics describing these powers, but in the game-theoretic laws of predicate logic newly understood.

A current challenge 'Logic of games' versus 'logic as games' is a still ill-understood duality in how to approach the interface of logic and games, being two valid perspectives that show natural affinities. The manifold interactions between these perspectives are the grand theme of van Benthem 2014, which presents many partial connection results, but no definitive view of their coexistence and technical reductions. This may be all to the good. It may well be the creative tension between implicit and explicit approaches that seems a powerful driving force in the formation of a new field.

14 Two stances: implicit versus explicit

Stances Having worked through a number of examples, the reader may feel entitled to a precise definition of the 'implicit/explicit' contrast. But perhaps disappointingly, we do not have any tight description to offer – and if challenged, the best we could offer is provide yet more examples. [33] Still, we did give general features that should be helpful. Implicit approaches enrich old meanings, and locate subtle information in deviant notions of consequence – explicit approaches identify new items, introduce new vocabulary for them, and conservatively extend classical logic. In many concrete cases, that seems enough to attach the two labels to existing logics in a significant way. Indeed, we think of these two ways of going about things as natural *stances* throughout logic.

Practical consequences But then, even in the absence of an iron-clad definition, there are other ways of showing that a proposed distinction is real. In particular, it is easy to show that the contrast pointed out in this paper has at least interesting *consequences*. These are all around. Here are just a few concrete issues that arise once you see it.

Finding a counterpart If we see one stance on a topic, we can usually find a complementary one, whether or not this existed already. In this way, the explicit/implicit contrast of this paper becomes a force for logical system building. We saw this with dynamic semantics of questions, which suggested an explicit companion logic of issue modi-

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[32] For instance, van Benthem 2003 shows how the basic sequential game algebra underlying first-order predicate logic is decidable, and so considerably different from predicate logic itself.

[33] For instance, this paper has concentrated on examples from propositional modal logic on its usual relational semantics. Two further sources, omitted here purely for reasons of length, are logics with first-order and generalized *quantifiers*, and general modal *neighborhood models*.

fying events. Also, explicit logics of belief change suggested new notions of consequence in the area of non-monotonic logic. And it is easy to find further examples. [34]

Borrowing ideas Seeing two systems as different stances on the same thing may lead to creative borrowing of ideas. An example is epistemic logic, which developed a rich analysis of varieties of multi-agent knowledge and group knowledge. Intuitionistic logic could profit from the same ideas, creating accounts of mathematics closer to the undeniable reality of mathematical research as a highly successful social activity. [35]

One can also borrow ideas between systems inside the same stance. Here is an example that has long intrigued me. One attractive feature of meanings in intuitionistic logic is the proof-theoretic *BHK* interpretation of the basic logical constants. This makes intuitionistic logic an intriguing meeting place of proof theory and model theory. Could the same sort of analysis be applied to dynamic semantics, the other major implicit paradigm for highlighting information dynamics? Is there an as yet unknown proof-theoretic, or at least combinatorial side to dynamic semantics as a view of language use that would mirror the power and elegance of the proof-theoretic interpretation?

Translations? Could we define the explicit/implicit contrast technically, using a mathematical notion of translation or reduction between logical systems? We have seen that much more can be said along these lines in specific cases, using relative interpretations between non-classical logical systems and classical ones with extended vocabulary. [36] But as things currently stand, I have not been able to distil a general definition out of these case studies. [37] Indeed, I have expressed doubts whether technical reductions of this sort are intuitively faithful to the spirit of the two frameworks being connected.

Merging If we cannot translate different stances into each other in an impeccable technical manner, another test for a meaningful relationship is existence of natural *merges*. Indeed, many joint systems exist in the literature with features of both: intuitionistic modal logics (Dosen 1985), merges of logic games and game logics (van Benthem 2005, Agotnes & van Ditmarsch 2011), dynamic-epistemic inquisitive logics (Roelofsen 2011), joint systems of linear game logic and temporal logic (van Benthem 2013), and so on. In earlier versions of this paper, I expressed doubts on what creating such merges really establishes, but sometimes, the merged system really feels natural. A recent example is the intuitionistic semantics for classical logic developed in van Benthem, Bezhanishvili & Holliday 2015 as a 'possibility semantics' for classical logic. [38]

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[34] Even so, this system building need not always be an automatic switch between stances. For instance, we mentioned linear logic already – and a more general challenge has been the area of non-classical substructural 'resource logics' (Restal 2000). What would be an explicit dynamic counterpart, with appropriate notions of resources and, presumably, proof dynamics?

[35] Some multi-agent aspects have been introduced into *justification logics* (Artemov 2007) that derive from merging implicit and explicit aspects of proof-based constructive logics.

[36] Implicit approaches, too, may invite extended vocabulary, witness the case of intuitionistic logic, whose models suggest a richer repertoire of stage-dependent logical constants than those found in classical logic. Thus, rethinking language design makes sense on both stances.

[37] One relevant perspective here is the interplay between translations and nonstandard semantics for logical systems discussed in van Benthem, Bezhanishvili & Holliday 2015.

[38] Such merged implicit/explicit systems need not be taken at face value. The cited paper also presents translations into 'doubly explicit' bimodal logics of functions over partial orders.

Views on co-existence In all, I am led to think that explicit and implicit stances co-exist in a fruitful manner. They may arise from different origins. For instance, in the case of epistemic versus intuitionistic logic, one might locate the contrast between model-theoretic origins (a semantic view of information) versus proof-theoretic origins (the *BHK* interpretation). [39] But they can be compared and merged. It also seems important to note here that there need not be a unique implicit or explicit approach to any given phenomenon. Intuitionism is one way of assigning constructive meanings to classical vocabulary, but not the only one. Epistemic logic is one way of dealing with knowledge explicitly, but not the only one. And in this variety, our contrast may gets blurred a bit. Perhaps we should admit a sliding scale of stances, from less to more explicit.

Instead of saying more on the positive side, let me also delineate our topic by briefly mentioning some other distinctions that seem related, but not quite the same.

First and third-person It has been suggested that implicit and explicit approaches are an instance of the well-known philosophical distinction between 'first-person' and 'third-person' perspectives. For instance, epistemic logic would be a third-person view describing knowledge of agents from the standpoint of an external modeler. By contrast, intuitionistic logic might represent the first person view of someone inside an informational situation, talking about what the world is like with the usual linguistic conventions about communication. [40] I find this a serious proposal for harmony, but it does not quite fit what we have seen. Agents inside informational scenarios, too, can explicitly reflect on and talk about their knowledge. Still there is a point here about the functioning of natural language versus that of logical systems that we will return to.

Object- and meta-level Another well-known distinction that comes to mind is that between object language and meta-language. We can formalize, often in a useful way, the meta-language in which we give the semantics for a given logical system in some other logic – with the 'standard translation' for modal logic as a paradigmatic example (Blackburn, de Rijke & Venema 2000). Note in particular that the meta-language need not be totally different from the object language. Say, for a given modal object language, we could also use modal fragments of a first-order language to describe its semantics, perhaps with additional modalities for structure in the models. [41] Is the meta-logic then the explicit version of the object logic? Again, this does not quite fit our contrast pairs. It is not clear, for instance, in which sense dynamic-epistemic logic is the meta-language of dynamic semantics. We conclude that this distinction, too, is after something else. Maybe this is all to the good, since formalizing a meta-theory is always possible – and hence we might trivialize the implicit/explicit contrast if it just amounted to this.

Still, the object/meta distinction has a value in suggesting connections between implicit and explicit logical systems. For instance, consider the interface of logic and games discussed in Section 13. Van Benthem 2014 shows how implicit logic games generate

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[39] But notice that our explicit/implicit contrast between dynamic-epistemic logic and dynamic semantics did not have such an origin, since both are semantic approaches.

[40] This distinction seems to fit our earlier suggestion that explicit approaches are more concerned with reasoning "about", while implicit approaches study reasoning "with". But again, this does not fit precisely – and the reader might also find this just preposition mongering.

[41] A case in point is bimodal relational semantics for modal neighborhood logic, which introduces two explicit modalities, for the neighborhood relation and elementhood in the models.

explicit game logics that describe their structure. But vice versa, game logics also have associated logic games that may be viewed as implicit practices for dealing with these game logics, and the games that they describe. And the circle of these operations – call them ***L*** and ***G*** – need not even close: ***G(L****(G))* need not be the original system *G*, and ***L(G****(L))* need not be the original system *L*. As a result, we generate many new logical systems having motivations and technical features of both implicit and explicit logics. [42]

I will leave matters of definition at this stage. The explicit/implicit contrast may be one of those things that one recognizes when one sees it, and that can guide useful actions, even if one is unable to define the contrast in general. Geach 1972 suggested that this may be quite enough for moral practice. Perhaps the same is true for logic?

15 Conclusion

We have drawn attention to a significant contrast that we see running through much of the logical literature, between (more) *implicit* and (more) *explicit* stances. Seeing the contrast reveals broad patterns in the landscape of research, while suggesting new questions that have not been asked before. For lack of a better word, one might call this empirical 'philosophy of logic', maybe with more justice than abstruse philosophical studies of issues about logic that have no basis in actual things that logicians do.

But I am sure that this view will not satisfy many philosophers interested in right and wrong. Can we tighten the screws? Should we 'fight it out': claiming that one stance, implicit or explicit, is better, or at least more powerful than another? [43] Indeed, some technical version of this may be true. I am confident that there is some sort of Universal Defining Power result for classical logic, in the sense that any reasonable semantics for any deviant logic can always be formalized in an explicit classical meta-theory. But the problem is: I do not find *that* a very decisive victory for the explicit stance, since it has no guarantee that the explicit logics generated in this way are interesting in any way.

What may also be involved here is one's view of logic as a discipline. Some people striving for methodological unity seek that in a preferred position for certain logical systems, be they intuitionistic or classical logic. To me that is a mistaken focus, coming from an identification of a discipline with some particular products that it creates, in the case of logic: formal systems. In my view, logic, like any scientific discipline, is an evolving practice, and the richer that practice is, the better. In particular, rich practices come with an array of stances one can take toward important themes, [44] and I see the explicit/implicit contrast as one of those stances – and there are more (cf. van Benthem xyzu, or the above discussion of first- and third-person, or object-meta perspectives). The unity of a field is not in one particular stance, but in their sum total including their connections, and the resulting modus operandi of the practitioners. [45]

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[42] One successful example is the game logic of Parikh 1985, which formalizes part of the meta-theory of first-order evaluation games in a modal-dynamic style, as well as several of its strategy-based variations and extensions discussed in van Benthem 2014.

[43] I have been urged by several audiences of lectures on this topic to finally 'come out', declare my allegiance to the explicit stance, and then engage in battle with 'implicitists'.

[44] Being stance-rich is particularly important to a methodological discipline like logic.

[45] Of course, this raises the further interesting question of a deeper *explanation* of the naturalness and utility of the stances composing a field, but I will not enter into that here.

As a final perspective on our stances, ignored in this paper, consider the confrontation of logic with reality. One might say that a choice between stances may depend on their fit with empirical phenomena. [46] Think of current interfaces between logic and cognitive science, with *natural language* as a key example. It has been claimed that natural language *is* implicit, in that information, knowledge, or belief are implicitly conveyed by saying things, the way dynamic semantics has it. Explicit dynamic-epistemic logics for communication are then descriptions external to this process. Through its evolutionary history, natural language might have come to use a meaning-laden syntax for many purposes in an implicit multi-dimensional style, whereas explicit logics of information and agency would be our theorists' attempts at creating analytical order behind this.

I like this view for offering a nice division of labor, but I do not think that it fits the facts. The empirical practice of language is much richer than syntax, including also varieties of behavior that might be captured more adequately by explicit logics. [47] In fact, the striking empirical observation to me is that natural language is a medium where our methodological implicit/explicit contrast is artificial. Of course, there are 'participating' versus 'observing' stances in language use, but the key point is precisely the *universality* of natural language. We can, and do, switch effortlessly between these stances all the time, inside the very same medium of communication. In fact, natural language does not seem to be a good comparison for logical systems in many ways. [48] Thus, I do not think that empirical observations will adjudicate between explicit and implicit in a sweeping manner, even though, of course, in concrete cases, one or the other may fit better.

So, in the end, I am just offering a contrast between explicit and implicit stances in logic, based on some observations and promises about where it might lead. I leave it to the reader to see, or rather feel, if it resonates.

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[46] In what follows, I will ignore the notorious problem of theory-laden observation when looking for 'facts of reasoning', and just grant that a confrontation of logic with reality makes sense. I will also ignore the fact here that logical notions and theories themselves interact with human reasoning, creating the hybrid mixture of theory and practice of today's information society.

[47] Van Benthem xuzy has an extensive discussion of natural language as general agency.

[48] Another instance discussed in van Benthem xyzu is the notion of *translation* between natural languages. This is really correlation of behavior, and the striking thing happening in empirical reality is that languages change under the influence of translating other languages. A good case in point is the development of modern Chinese into a universal global language practice just as rich as English, through a century of cultural encounters including translation projects.

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