SUBORDINATIONS, CLOSED RELATIONS, AND COMPACT HAUSDORFF SPACES

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Abstract. We introduce the concept of a subordination, which is dual to the well-known concept of a precontact on a Boolean algebra. We develop a full categorical duality between Boolean algebras with a subordination and Stone spaces with a closed relation, thus generalizing the results of [14]. We introduce the concept of an irreducible equivalence relation, and that of a Gleason space, which is a pair (X, R), where X is an extremally disconnected compact Hausdorff space and R is an irreducible equivalence relation on X. We prove that the category of Gleason spaces is equivalent to the category of compact Hausdorff spaces, and is dually equivalent to the category of de Vries algebras, thus providing a "modal-like" alternative to de Vries duality.

1. Introduction

By the celebrated Stone duality [26], the category of Boolean algebras and Boolean homomorphisms is dually equivalent to the category of Stone spaces (compact Hausdorff zero-dimensional spaces) and continuous maps. De Vries [13] generalized Stone duality to the category of compact Hausdorff spaces and continuous maps. Objects of the dual category are complete Boolean algebras B with a binary relation $\prec$ (called by de Vries a compingent relation) satisfying certain conditions that resemble the definition of a proximity on a set [23].

Another generalization of Stone duality is central to modal logic. We recall that modal algebras are Boolean algebras B with a unary function $\Box : B \to B$ preserving finite meets, and modal spaces (descriptive frames) are Stone spaces X with a binary relation R satisfying certain conditions. Stone duality then generalizes to a duality between the categories of modal algebras and modal spaces (see, e.g., [12, 21, 11]).

The dual of a modal algebra $(B, \Box)$ is the modal space (X, R), where X is the Stone dual of B (the space of ultrafilters of B), while the binary relation $R \subseteq X \times X$ is the Jónsson-Tarski dual of $\Box$ [20]. Unlike the modal case, in de Vries duality we do not split the dual space of $(B, \prec)$ in two components, the Stone dual of B and the relation R. Instead we work with the space of "$\prec$-closed" filters which are maximal with this property.

The aim of this paper is to develop an alternative "modal-like" duality for de Vries algebras, in which we do split the dual space of a de Vries algebra $(B, \prec)$ in two parts: the Stone dual of B and the dual of $\prec$. If X is the de Vries dual of $(B, \prec)$, then the Stone dual Y of B is the Gleason cover of X [4]. We show that the irreducible

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map $\pi : Y \to X$ gives rise to what we call an irreducible equivalence relation R on Y, which is the dual of $\prec$. It follows that compact Hausdorff spaces are in 1-1 correspondence with pairs (Y, R), where Y is an extremally disconnected compact Hausdorff space and R is an irreducible equivalence relation on Y. We call such pairs Gleason spaces, and introduce the category of Gleason spaces, where morphisms are relations rather than functions, and composition is not relation composition. We prove that the category of Gleason spaces is equivalent to the category of compact Hausdorff spaces and continuous maps, and is dually equivalent to the category of de Vries algebras and de Vries morphisms, thus providing an alternate "modal-like" duality for de Vries algebras.

For this we first introduce a general concept of a subordination $\prec$ on a Boolean algebra B. Examples of subordinations (that satisfy additional conditions) are (i) modal operators $\Box$, (ii) de Vries' compingent relations, (iii) lattice subordinations of [6], etc. We show that subordinations on a Boolean algebra B dually correspond to closed relations on the Stone space X of B.

We note that a subordination can be seen as a generalization of the modal operator $\Box$ (see Section 2). If we generalize the dual modal operator $\Diamond$ the same way, then we arrive at the well-known concept of a precontact relation and a precontact algebra [14, 15]. Since subordinations and precontact relations are definable from each other, the representation of precontact relations can be obtained from the representation of subordinations and vice versa. The representation of precontact algebras is given in [14] (see also [15, 2, 3]), but since there are no proofs given in [14], we include all the proofs here. In addition, we also provide duality for the corresponding morphisms, thus establishing a full categorical duality for the categories of interest.

The paper is organized as follows. In Section 2 we introduce the concept of a subordination on a Boolean algebra, show that subordinations are in 1-1 correspondence with precontact relations, and give a number of useful examples of subordinations. In Section 3 we prove that subordinations on a Boolean algebra B are in 1-1 correspondence with closed relations on the Stone space of B, and develop a full categorical duality for the category of Boolean algebras with subordinations, thus generalizing [14]. In Section 4 we show that on objects the duality of Section 3 can be derived from the generalized Jónsson-Tarski duality. In Section 5 we prove that modally definable subordinations are dually described by means of Esakia relations. As a corollary, we derive the well-known duality between the categories of modal algebras and modal spaces. In Section 6 we characterize those subordinations whose dual relations are reflexive, transitive, and/or symmetric, thus obtaining results similar to [15, 14]. In Section 7 we show that a subordination is a lattice subordination iff its dual relation is a Priestley quasi-order. The duality result of [6] follows as a corollary. Finally, in Section 8 we introduce irreducible equivalence relations, Gleason spaces, and give a "modal-like" alternative to de Vries duality.

2. Subordinations on Boolean algebras

In this section we introduce the concept of a subordination on a Boolean algebra. We show that subordinations are in 1-1 correspondence with precontact relations, and that modal operators, de Vries' compingent relations, and lattice subordinations of [6] are all examples of subordinations.

Definition 2.1. A *subordination* on a Boolean algebra B is a binary relation $\prec$ satisfying:

(S1) $0 \prec 0$ and $1 \prec 1$;
(S2) $a \prec b, c$ implies $a \prec b \wedge c$;
(S3) $a, b \prec c$ implies $a \vee b \prec c$;
(S4) $a \le b \prec c \le d$ implies $a \prec d$.

Remark 2.2. It is an easy consequence of the axioms that $0 \prec a \prec 1$ for each $a \in B$. In fact, (S1) can equivalently be stated this way.

Example 2.3. We recall [15, 14] that a *proximity* or *precontact* on a Boolean algebra B is a binary relation δ satisfying

(P1) $a \ \delta \ b \ \Rightarrow \ a, b \neq 0$.
(P2) $a \ \delta \ b \vee c \ \Leftrightarrow \ a \ \delta \ b$ or $a \ \delta \ c$.
(P3) $a \vee b \ \delta \ c \ \Leftrightarrow \ a \ \delta \ c$ or $b \ \delta \ c$.

Let $\prec$ be a subordination on B. Define a binary relation $\delta_\prec$ by $a\delta_\prec b$ iff $a \not\prec \neg b$. It is routine to verify that $\delta_\prec$ is a precontact relation on B. Conversely, if δ is a precontact relation on B, then define $\prec_\delta$ by $a \prec_\delta b$ iff $a \not\delta \neg b$. Then it is easy to see that $\prec_\delta$ is a subordination on B. Moreover, $a \prec b$ iff $a \prec_{\delta_\prec} b$, and $a\delta b$ iff $a\delta_{\prec_\delta} b$. Thus, subordinations are in 1-1 correspondence with precontact relations on B.

We recall that a *modal operator* on a Boolean algebra B is a unary function $\Box : B \to B$ that preserves finite meets (including 1). A *modal algebra* is a pair $(B, \Box)$, where B is a Boolean algebra and $\Box$ is a modal operator on B. We show that modal operators are in 1-1 correspondence with special subordinations.

Example 2.4. Let B be a Boolean algebra and let $\Box$ be a modal operator on B. Set $a \prec_\Box b$ provided $a \le \Box b$. Since $\Box 1 = 1$, it is clear that $\prec_\Box$ satisfies (S1). As $\Box(b \wedge c) = \Box b \wedge \Box c$, we also have that $\prec_\Box$ satisfies (S2). That $\prec_\Box$ satisfies (S3) is obvious, and since $\Box$ is order-preserving, $\prec_\Box$ satisfies (S4). Therefore, $\prec_\Box$ is a subordination on B. Note that $\prec_\Box$ is a special subordination on B that in addition satisfies the following condition: for each $a \in B$, the element $\Box a$ is the largest element of the set $\{x \in B : x \prec_\Box a\}$.

Definition 2.5. Let B be a Boolean algebra and let $\prec$ be a subordination on B. We call $\prec$ *modally definable* provided the set $\{x \in B : x \prec a\}$ has a largest element for each $a \in B$.

In Example 2.4 we saw that if $\Box$ is a modal operator, then $\prec_\Box$ is a modally definable subordination. The converse is also true.

Example 2.6. Let B be a Boolean algebra and let $\prec$ be a modally definable subordination on B. Define $\Box_\prec : B \to B$ by

$$\Box_\prec a \ = \ \text{the largest element of } \{x \in B : x \prec a\}.$$

By (S1), $\Box_\prec 1 = 1$. In addition, by (S4), $\Box_\prec(a \wedge b) \le \Box_\prec a \wedge \Box_\prec b$, and by (S2) and (S4), $\Box_\prec a \wedge \Box_\prec b \le \Box_\prec(a \wedge b)$. Therefore, $\Box_\prec$ is a modal operator on B. Moreover, $\Box_{\prec_\Box} a = \Box a$ and $a \prec_{\Box_\prec} b$ iff $a \prec b$. Thus, modal operators on B are in 1-1 correspondence with modally definable subordinations on B.

Other examples of subordinations are the lattice subordinations of [6] and the compingent relations of [13].

Definition 2.7. ([6]) A subordination $\prec$ on a Boolean algebra B is a *lattice subordination* if in addition $\prec$ satisfies

$$a \prec b \textit{ implies that there exists } c \in B \text{ with } c \prec c \text{ and } a \leq c \leq b.$$

Definition 2.8. ([13]) A subordination $\prec$ on a Boolean algebra B is a *compingent relation* if in addition it satisfies:

(S5) $a \prec b$ implies $a \leq b$;
(S6) $a \prec b$ implies $\neg b \prec \neg a$;
(S7) $a \prec b$ implies there is $c \in B$ with $a \prec c \prec b$;
(S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

We let Sub be the category whose objects are pairs $(B, \prec)$, where B is a Boolean algebra and $\prec$ is a subordination on B, and whose morphisms are Boolean homomorphisms h satisfying $a \prec b$ implies $h(a) \prec h(b)$.

3. Subordinations and closed relations

In this section we show that subordinations on a Boolean algebra B can be dually described by means of closed relations on the Stone space of B, and work out a full categorical duality between the category of Boolean algebras with subordinations and the category of Stone spaces with closed relations. These results generalize the results of [14].

Definition 3.1. Let X be a topological space and let R be a binary relation on X. We call R a *closed relation* provided R is a closed set in the product topology on $X \times X$.

Let R be a binary relation on X. As usual, for $x \in X$, let

$$R[x] = \{y \in X : xRy\}$$
$$R^{-1}[x] = \{y \in X : yRx\}.$$

Also, for $U \subseteq X$, let

$$R[U] = \{y \in X : \exists x \in U \text{ with } xRy\}$$
$$R^{-1}[U] = \{y \in X : \exists x \in U \text{ with } yRx\}.$$

The next lemma generalizes [10, Prop. 2.3], where a characterization of closed quasi-orders (reflexive and transitive relations) is given. In fact, the proofs of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are the same as in [10], so we only sketch them. For the rest of the implications we provide all details.

Lemma 3.2. Let X be a compact Hausdorff space and let R be a binary relation on X. The following conditions are equivalent.

(1) R is a closed relation.
(2) For each closed subset F of X, both $R[F]$ and $R^{-1}[F]$ are closed.
(3) If A is an arbitrary subset of X, then $\overline{R[A]} \subseteq R[\overline{A}]$ and $\overline{R^{-1}[A]} \subseteq R^{-1}[\overline{A}]$.
(4) If $(x, y) \notin R$, then there is an open neighborhood U of x and an open neighborhood V of y such that $R[U] \cap V = \varnothing$.

Proof. (1) $\Rightarrow$ (2): Suppose that F is a closed subset of X. As X is compact Hausdorff, the projections $\pi_1, \pi_2 : X \times X \to X$ are closed maps. Since R is a closed relation, $R[F] = \pi_2((F \times X) \cap R)$, and $R^{-1}[F] = \pi_1((X \times F) \cap R)$, both $R[F]$ and $R^{-1}[F]$ are closed subsets of X.

(2) ⇒ (3): Let A be an arbitrary subset of X. Since $A \subseteq \overline{A}$, we have $R[A] \subseteq R[\overline{A}]$. As $\overline{A}$ is closed, by (2), $R[\overline{A}]$ is also closed. Therefore, $\overline{R[A]} \subseteq R[\overline{A}]$. A similar argument gives $\overline{R^{-1}[A]} \subseteq R^{-1}[\overline{A}]$.

(3) ⇒ (4): Let $(x,y) \notin R$. Then $y \notin R[x]$. Since X is Hausdorff, $\{x\}$ is closed. By (3), $\overline{R[x]} \subseteq R[x]$, so $R[x]$ is also closed. As X is compact Hausdorff, and hence regular, there exist disjoint open sets W, V such that $R[x] \subseteq W$ and $y \in V$. Set $U = X - R^{-1}[X - W]$. Since $X - W$ is closed, by (3), $\overline{R^{-1}[X - W]} \subseteq R^{-1}[X - W]$. Therefore, $R^{-1}[X - W]$ is closed, hence $U = X - R^{-1}[X - W]$ is open. Moreover, $R[x] \subseteq W$ implies $x \in U$. If $v \in R[U] \cap V$, then there is $u \in U$ with uRv. This yields $v \in R[u] \subseteq W$, so $W \cap V \neq \varnothing$. The obtained contradiction proves that $R[U] \cap V = \varnothing$. Thus, U is an open neighborhood of x, V is an open neighborhood of y, and $R[U] \cap V = \varnothing$.

(4) ⇒ (1): Let $(x,y) \notin R$. By (4), there is an open neighborhood U of x and an open neighborhood V of y such that $R[U] \cap V = \varnothing$. Therefore, there is an open neighborhood $U \times V$ of (x,y) such that $(U \times V) \cap R = \varnothing$. Thus, R is a closed subset of $X \times X$. □

For $i = 1, 2$, let R_i be a relation on X_i. Following [8], we call a map $f : X_1 \to X_2$ *stable* provided xR_1y implies $f(x)R_2f(y)$. The following is straightforward.

Lemma 3.3. The following are equivalent:

(1) $f : X_1 \to X_2$ is stable.
(2) $f(R_1[x]) \subseteq R_2[f(x)]$ for each $x \in X_1$.
(3) $R_1[f^{-1}(y)] \subseteq f^{-1}(R_2[y])$ for each $y \in X_2$.

Proof. Easy. □

We recall that a subset U of a topological space X is *clopen* if it is both closed and open, and that X is *zero-dimensional* if it has a basis of clopen sets. A *Stone space* is a compact, Hausdorff, zero-dimensional space. The celebrated Stone duality yields that the category of Boolean algebras and Boolean homomorphisms is dually equivalent to the category of Stone spaces and continuous maps. We next extend Stone duality to the category Sub.

Let StR be the category whose objects are pairs (X, R), where X is a Stone space and R is a closed relation on X, and whose morphisms are continuous stable morphisms. We will prove that Sub is dually equivalent to StR.

For a Boolean algebra B, let X be the set of ultrafilters of B. For $a \in B$, set $\varphi(a) = \{x \in X : a \in x\}$, and topologize X by letting $\{\varphi(a) : a \in B\}$ be a basis for the topology. The resulting space is called the Stone space of B and is denoted B_*. Let $(B, \prec) \in \mathsf{Sub}$. For $S \subseteq B$, let

$$\begin{aligned} \Uparrow S &= \{b \in B : \exists a \in S \text{ with } a \prec b\} \\ \Downarrow S &= \{b \in B : \exists a \in S \text{ with } b \prec a\}. \end{aligned}$$

Definition 3.4. For $(B, \prec) \in \mathsf{Sub}$, let $(B, \prec)_* = (X, R)$, where X is the Stone space of B and xRy iff $\Uparrow x \subseteq y$.

Lemma 3.5. If $(B, \prec) \in \mathsf{Sub}$, then $(B, \prec)_* \in \mathsf{StR}$.

Proof. It is sufficient to prove that R is a closed relation on X. Let $(x,y) \notin R$. Then $\Uparrow x \nsubseteq y$. Therefore, there are $a \in x$ and $b \notin y$ with $a \prec b$.

Claim. $a \prec b$ implies $R[\varphi(a)] \subseteq \varphi(b)$.

Proof of Claim. Let $v \in R[\varphi(a)]$. Then there is $u \in \varphi(a)$ with uRv. Therefore, $a \in u$ and $\twoheaduparrow u \subseteq v$. Since $a \prec b$, we have $b \in v$, so $v \in \varphi(b)$. Thus, $R[\varphi(a)] \subseteq \varphi(b)$. □

We set $U = \varphi(a)$ and $V = X - \varphi(b)$. Then U is an open neighborhood of x, V is an open neighborhood of y, and $R[U] \cap V = \varnothing$. Thus, by Lemma 3.2, R is a closed relation on X, which completes the proof. □

Definition 3.6. For $i = 1,2$, let $(B_i, \prec_i) \in \mathsf{Sub}$ and let $(X_i, R_i) = (B_i, \prec_i)_*$. For a morphism $h : B_1 \to B_2$ in Sub, let $h_* : X_2 \to X_1$ be given by $h_*(x) = h^{-1}(x)$.

Lemma 3.7. If h is a morphism in Sub, then h_* is a morphism in StR.

Proof. By Stone duality, h_* is a well-defined continuous map. Suppose $x, y \in X_2$ with xR_2y. Then $\twoheaduparrow_2 x \subseteq y$. Let $b \in \twoheaduparrow_1 h^{-1}(x)$. So there is $a \in h^{-1}(x)$ with $a \prec_1 b$. Since h is a morphism in Sub, we have $h(a) \prec_2 h(b)$. Therefore, $h(b) \in \twoheaduparrow_2 x$. This implies $h(b) \in y$. Thus, $b \in h^{-1}(y)$, yielding $\twoheaduparrow_1 h^{-1}(x) \subseteq h^{-1}(y)$. Consequently, h_* is a stable continuous map, hence a morphism in StR. □

Definition 3.8. Define $(-)_* : \mathsf{Sub} \to \mathsf{StR}$ as follows. If $(B, \prec) \in \mathsf{Sub}$, then $(B, \prec)_* = (X, R)$, and if h is a morphism in Sub, then $h_* = h^{-1}$. Applying Lemmas 3.5 and 3.7 it is straightforward to verify that $(-)_*$ is a well-defined contravariant functor.

For a topological space X, let $\mathsf{Clop}(X)$ be the set of clopen subsets of X. Then it is well known and easy to see that $\mathsf{Clop}(X)$ is a Boolean algebra with respect to the set-theoretic operations of union, intersection, and complement.

Definition 3.9. For $(X, R) \in \mathsf{StR}$, let $(X, R)^* = (\mathsf{Clop}(X), \prec)$, where $U \prec V$ iff $R[U] \subseteq V$.

Lemma 3.10. If $(X, R) \in \mathsf{StR}$, then $(X, R)^* \in \mathsf{Sub}$.

Proof. Since $R[\varnothing] = \varnothing$, it is clear that $\prec$ satisfies (S1). That $\prec$ satisfies (S2) is obvious. From $R[U \cup V] = R[U] \cup R[V]$ it follows that $\prec$ satisfies (S3). Finally, as $U \subseteq V$ implies $R[U] \subseteq R[V]$, we obtain that $\prec$ satisfies (S4). Thus, R is a subordination on $\mathsf{Clop}(X)$, and hence $(X, R)^* \in \mathsf{Sub}$. □

Definition 3.11. For $i = 1,2$, let $(X_i, R_i) \in \mathsf{StR}$ and let $(B_i, \prec_i) = (X_i, R_i)^*$. For a morphism $f : X_1 \to X_2$ in StR, let $f^* : \mathsf{Clop}(X_2) \to \mathsf{Clop}(X_1)$ be given by $f^*(U) = f^{-1}(U)$.

Lemma 3.12. If f is a morphism in StR, then f^* is a morphism in Sub.

Proof. It follows from Stone duality that f^* is a Boolean homomorphism. Let $U, V \in \mathsf{Clop}(X_2)$ with $U \prec_2 V$. Then $R_2[U] \subseteq V$. This implies $f^{-1}(R_2[U]) \subseteq f^{-1}(V)$. Since f is a stable map, by Lemma 3.3, $R_1[f^{-1}(U)] \subseteq f^{-1}(R_2[U])$. Therefore, $R_1[f^{-1}(U)] \subseteq f^{-1}(V)$. Thus, $f^{-1}(U) \prec f^{-1}(V)$, and hence f^* is a morphism in Sub. □

Definition 3.13. Define $(-)^* : \mathsf{StR} \to \mathsf{Sub}$ as follows. If $(X, R) \in \mathsf{StR}$, then $(X, R)^* = (\mathsf{Clop}(X), \prec)$, and if f is a morphism in StR, then $f^* = f^{-1}$. Applying Lemmas 3.10 and 3.12 it is straightforward to see that $(-)^*$ is a contravariant functor.

Lemma 3.14. Let $(B, \prec) \in \mathsf{Sub}$ and let $\varphi : B \to (B_*)^*$ be the Stone map. Then $a \prec b$ iff $\varphi(a) \prec \varphi(b)$.

Proof. Let $a, b \in B$ with $a \prec b$. By the Claim in the proof of Lemma 3.5, this implies $R[\varphi(a)] \subseteq \varphi(b)$, so $\varphi(a) \prec \varphi(b)$. Next suppose that $a \not\prec b$. Then $b \notin \twoheaduparrow a$. Since $\prec$ is a subordination, it is easy to see that $\twoheaduparrow a$ is a filter. Therefore, by the ultrafilter theorem, there is an ultrafilter x such that $\twoheaduparrow a \subseteq x$ and $b \notin x$.

Claim. $\twoheaduparrow a \subseteq x$ implies that there is an ultrafilter y such that $a \in y$ and $\twoheaduparrow y \subseteq x$.

Proof of Claim. Let $F = {\uparrow} a$ and $I = B - x$. Then F is a filter containing a and I is an ideal. We show that $\twoheaduparrow F \cap I = \varnothing$. If $c \in \twoheaduparrow F \cap I$, then $c \in I$ and there is $d \in F$ with $d \prec c$. Therefore, $a \le d \prec c$ and $c \notin x$. Thus, $a \prec c$, so $c \in \twoheaduparrow a$ and $c \notin x$. This yields $\twoheaduparrow a \nsubseteq x$, a contradiction. Consequently, the set Z consisting of the filters G satisfying $a \in G$ and $\twoheaduparrow G \subseteq x$ is nonempty because $F \in Z$. It is easy to see that $(Z, \subseteq)$ is an inductive set, hence by Zorn's lemma, Z has a maximal element, say y. We show that y is an ultrafilter. Suppose $c, \neg c \notin y$. Let F_1 be the filter generated by $\{c\} \cup y$ and F_2 be the filter generated by $\{\neg c\} \cup y$. Since F_1 and F_2 properly contain y, they do not belong to Z, so $\twoheaduparrow F_1, \twoheaduparrow F_2 \nsubseteq x$. This gives $d_1, d_2 \in y$ and $e \notin x$ such that $c \wedge d_1, \neg c \wedge d_2 \prec e$. By (S3) and distributivity, $(c \vee \neg c) \wedge (c \vee d_2) \wedge (d_1 \vee \neg c) \wedge (d_1 \vee d_2) \prec e$. But $(c \vee \neg c) \wedge (c \vee d_2) \wedge (d_1 \vee \neg c) \wedge (d_1 \vee d_2) \in y$, so $e \in \twoheaduparrow y \subseteq x$. The obtained contradiction proves that y is an ultrafilter. Since $y \in Z$, we have $a \in y$ and $\twoheaduparrow y \subseteq x$, which completes the proof of the claim. □

It follows from the Claim that there is $y \in B_*$ such that $y \in \varphi(a)$ and yRx. Therefore, $x \in R[\varphi(a)]$. On the other hand, $x \notin \varphi(b)$. Thus, $R[\varphi(a)] \nsubseteq \varphi(b)$, yielding $\varphi(a) \not\prec \varphi(b)$. □

For a Stone space X, define $\psi : X \to (X^*)_*$ by $\psi(x) = \{U \in \mathsf{Clop}(X) : x \in U\}$. It follows from Stone duality that ψ is a homeomorphism.

Lemma 3.15. Let $(X, R) \in \mathsf{StR}$ and let $\psi : X \to (X^*)_*$ be given as above. Then xRy iff $\psi(x) R \psi(y)$.

Proof. First suppose that xRy. To see that $\psi(x) R \psi(y)$ we must show that $\twoheaduparrow \psi(x) \subseteq \psi(y)$. Let $V \in \twoheaduparrow \psi(x)$. Then there is $U \in \psi(x)$ with $U \prec V$. Therefore, $x \in U$ and $R[U] \subseteq V$. Thus, $y \in V$, so $\twoheaduparrow \psi(x) \subseteq \psi(y)$, and hence $\psi(x) R \psi(y)$.

Conversely, suppose that $(x, y) \notin R$. Since X has a basis of clopens and R is a closed relation, by Lemma 3.2, there exist a clopen neighborhood U of x and a clopen neighborhood W of y such that $R[U] \cap W = \varnothing$. Set $V = X - W$. Then $U \in \psi(x)$, $V \notin \psi(y)$, and $R[U] \subseteq V$. Therefore, $U \prec V$, so $V \in \twoheaduparrow \psi(x)$, but $V \notin \psi(y)$. Thus, $(\psi(x), \psi(y)) \notin R$. □

Theorem 3.16. The categories Sub and StR are dually equivalent.

Proof. By Definition 3.8, $(-)_* : \mathsf{Sub} \to \mathsf{StR}$ is a well-defined contravariant functor, and by Definition 3.13, $(-)^* : \mathsf{StR} \to \mathsf{Sub}$ is a well-defined contravariant functor. By Stone duality and Lemmas 3.14 and 3.15, each $(B, \prec) \in \mathsf{Sub}$ is isomorphic in Sub to $((B, \prec)_*)^*$ and each $(X, R) \in \mathsf{StR}$ is isomorphic in StR to $((X, R)^*)_*$. That these isomorphisms are natural is easy to see. Thus, Sub is dually equivalent to StR. □

Remark 3.17. As follows from Example 2.3, there is a 1-1 correspondence between subordinations and precontact relations on a Boolean algebra B. Therefore, each precontact algebra (B, δ) can be represented as $(\mathsf{Clop}(X), \delta_R)$, where (X, R) is the dual of $(B, \prec_\delta)$ and $U \delta_R V$ iff $R[U] \cap V \neq \varnothing$. This yields the representation theorem of [14, Thm. 3]. In fact, this representation theorem can be generalized to a full categorical duality. Let PCon be the category of precontact algebras and Boolean homomorphisms h satisfying $h(a)\ \delta\ h(b)$ implies $a\ \delta\ b$. It follows from Example 2.3 that this condition is equivalent to the more natural condition that $a \prec b$ implies $h(a) \prec h(b)$. This yields that the categories Sub and PCon are isomorphic. Thus, by Theorem 3.16, PCon is dually equivalent to StR.

4. Subordinations, strict implications, and Jónsson-Tarski duality

In this section we show that on objects the duality of the previous section can also be derived from the generalized Jónsson-Tarski duality.

Definition 4.1. Let B be a Boolean algebra and let $\mathbf{2}$ be the two element Boolean algebra. We call a map $\to : B \times B \to \mathbf{2}$ a *strict implication* if it satisfies

(I1) $0 \to a = a \to 1 = 1$.
(I2) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$.
(I3) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

Example 4.2. Let $\prec$ be a subordination on a Boolean algebra B. Define $\to_\prec : B \times B \to \mathbf{2}$ by

$$a \to_\prec b = \begin{cases} 1 & \text{if } a \prec b, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\to_\prec$ is a strict implication. Conversely, if $\to : B \times B \to \mathbf{2}$ is a strict implication, then define $\prec_\to \subseteq B \times B$ by

$$a \prec_\to b \text{ iff } a \to b = 1.$$

It is straightforward to see that $\prec_\to$ is a subordination on B. Moreover, $a \prec b$ iff $a \prec_{\to_\prec} b$ and $a \to b = a \to_{\prec_\to} b$. Thus, there is a 1-1 correspondence between subordinations and strict implications on B.

This observation opens the door for obtaining the duality for subordinations from Jónsson-Tarski duality [20]. Let A, B, C be Boolean algebras and X, Y, Z be their Stone spaces, respectively. Suppose that $f : A \times B \to C$ is a map. Following the terminology of [25], we call f a *meet-hemiantimorphism in the first coordinate* provided

- $f(0, b) = 1$,
- $f(a \vee b, c) = f(a, c) \wedge f(b, c)$;

and a *meet-hemimorphism in the second coordinate* provided

- $f(a, 1) = 1$,
- $f(a, b \wedge c) = f(a, c) \wedge f(b, c)$.

By the generalized Jónsson-Tarski duality [18, 25], such maps are dually described by special ternary relations $S \subseteq X \times Y \times Z$. For $z \in Z$, let

$$S^{-1}[z] := \{(x, y) \in X \times Y : (x, y, z) \in S\},$$

and for $U \in \mathsf{Clop}(X)$ and $V \in \mathsf{Clop}(Y)$, let

$$\Box_S(U, V) := \{z \in Z : (\forall x \in X)(\forall y \in Y)\,[(x, y, z) \in S \Rightarrow x \notin U \text{ or } y \in V]\}.$$

Definition 4.3. We call $S \subseteq X \times Y \times Z$ a *JT-relation* (*Jónsson-Tarski relation*) provided

(JT1) $S^{-1}[z]$ is closed for each $z \in Z$,
(JT2) $\Box_S(U, V)$ is clopen for each $U \in \mathsf{Clop}(X)$ and $V \in \mathsf{Clop}(Y)$.

By the generalized Jónsson-Tarski duality [18, 25], the dual ternary relation $S \subseteq X \times Y \times Z$ of $f : A \times B \to C$ is given by

(1) $(x, y, z) \in S$ iff $(\forall a \in A)(\forall b \in B)(f(a, b) \in z$ implies $a \notin x$ or $b \in y)$;

and the dual map $f : \mathsf{Clop}(X) \times \mathsf{Clop}(Y) \to \mathsf{Clop}(Z)$ of $S \subseteq X \times Y \times Z$ is given by

(2) $$f(U, V) = \Box_S(U, V).$$

Now let $\to$ be a strict implication on a Boolean algebra B. By Definition 4.1, $\to$ is a meet-hemiantimorphism in the first coordinate and a meet-hemimorphism in the second coordinate. Let X be the Stone space of B. The Stone space of $\mathbf{2}$ is the singleton discrete space $\{z\}$, where $z = \{1\}$ is the only ultrafilter of $\mathbf{2}$. Therefore, the dual ternary relation $S \subseteq X \times X \times \{z\}$ of $\to$ is given by

$$(x, y, z) \in S \text{ iff } (\forall a, b \in B)(a \to b = 1 \text{ implies } a \notin x \text{ or } b \in y).$$

The ternary relation S gives rise to the binary relation$R \subseteq X \times X$ by setting

$$xRy \text{ iff } (x, y, 1) \in S.$$

If $\prec$ is the subordination corresponding to the strict implication $\to$, then $a \prec b$ iff $a \to b = 1$. Therefore, the binary relation R is given by

$$xRy \text{ iff } (\forall a, b \in B)(a \prec b \text{ implies } a \notin x \text{ or } b \in y).$$

Proposition 4.4. Let $\prec$ be a subordination on a Boolean algebra B, and let (X, R) be the dual of $(B, \prec)$. Then $\Uparrow x \subseteq y$ iff $(\forall a, b \in B)(a \prec b$ implies $a \notin x$ or $b \in y)$.

Proof. First suppose that $\Uparrow x \subseteq y$. Let $a \prec b$ and $a \in x$. Then $b \in \Uparrow x$, so $b \in y$. Conversely, suppose $(\forall a, b \in B)(a \prec b$ implies $a \notin x$ or $b \in y)$. If $b \in \Uparrow x$, then there is $a \in x$ with $a \prec b$. Therefore, $y \in b$, and hence $\Uparrow x \subseteq y$. □

Applying Proposition 4.4 then yields

$$xRy \text{ iff } \Uparrow x \subseteq y.$$

Consequently, the dual binary relation R of a subordination $\prec$ can be described from the dual ternary relation S of the corresponding strict implication. In fact, if $S \subseteq X \times X \times \{z\}$ is a JT-relation, then (JT2) is redundant, while (JT1) means that R is a closed relation.

The converse is also true. Given a closed relation R on a Stone space X, define the ternary relation $S \subseteq X \times X \times \{z\}$ by

$$(x, y, z) \in S \text{ iff } xRy.$$

Since R is a closed relation, S satisfies (JT1), and S satisfies (JT2) trivially, hence S is a JT-relation. Let $\to: \mathsf{Clop}(X) \times \mathsf{Clop}(Y) \to \mathbf{2}$ be the corresponding strict implication. Then

$$U \to V = \begin{cases} 1 & \text{if } (\forall x \in X)(\forall y \in Y)(xRy \Rightarrow x \notin U \text{ or } y \in V) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.5. Let X be a Stone space, R be a closed relation on X, and $U, V \in \mathsf{Clop}(X)$. Then $R[U] \subseteq V$ iff $(\forall x, y \in X)(xRy$ implies $x \notin U$ or $y \in V)$.

Proof. First suppose that $R[U] \subseteq V$, xRy, and $x \in U$. Then $y \in R[U]$, so $y \in V$. Conversely, suppose that $(\forall x, y \in X)(xRy$ implies $x \notin U$ or $y \in V)$. If $y \in R[U]$, then there is $x \in U$ with xRy. Therefore, $y \in V$, and hence $R[U] \subseteq V$. □

If $\prec$ is the subordination corresponding to $\rightarrow$, then it follows from Proposition 4.5 that $U \prec V$ iff $R[U] \subseteq V$ iff $U \rightarrow V = 1$. This shows that on objects our duality for subordinations is equivalent to a special case of the generalized Jónsson-Tarski duality.

Remark 4.6. A homomorphism between two Boolean algebras with strict implication $(B_1, \rightarrow_1)$ and $(B_2, \rightarrow_2)$ is a Boolean homomorphism $h : B_1 \rightarrow B_2$ such that $h(a \rightarrow_1 b) = h(a) \rightarrow_2 h(b)$. On the other hand, a morphism between two Boolean algebras with subordination $(B_1, \prec_1)$ and $(B_2, \prec_2)$ is a Boolean homomorphism $h : B_1 \rightarrow B_2$ such that $a \prec_1 b \Rightarrow h(a) \prec_2 h(b)$. It is easy to verify that $a \prec_1 b \Rightarrow h(a) \prec_2 h(b)$ is equivalent to $h(a \rightarrow_1 b) \leq h(a) \rightarrow_2 h(b)$, while $h(a \rightarrow_1 b) = h(a) \rightarrow_2 h(b)$ is equivalent to $a \prec_1 b$ iff $h(a) \prec_2 h(b)$. Thus, continuous stable morphisms dually correspond to $h(a \rightarrow_1 b) \leq h(a) \rightarrow_2 h(b)$, while the equality $h(a \rightarrow_1 b) = h(a) \rightarrow_2 h(b)$ requires an additional condition: If $y, u \in X_2$ and $y R_2 u$, then there exist $x, z \in X_1$ such that $x R_1 z$, $f(x) = y$, and $f(z) = u$. This is equivalent to $f : X_1 \rightarrow X_2$ being a bounded morphism with respect to the ternary relations S_1 on X_1 and S_2 on X_2.

5. Modally definable subordinations and Esakia relations

In this section we show that modally definable subordinations dually correspond to Esakia relations, and derive the well-known duality between the categories of modal algebras and modal spaces from the duality of Section 3.

Definition 5.1. Let X be a Stone space. We call a binary relation R on X an *Esakia relation* provided $R[x]$ is closed for each $x \in X$ and $U \in \mathsf{Clop}(X)$ implies $R^{-1}[U] \in \mathsf{Clop}(X)$.

Remark 5.2.

(1) Let $\mathcal{V}(X)$ be the Vietoris space of X. It is well known (see, e.g., [16]) that R is an Esakia relation iff the map $\rho_R : X \rightarrow \mathcal{V}(X)$ given by $\rho(x) = R[x]$ is a well-defined continuous map. Because of this, Esakia relations are also called *continuous relations*.

(2) It is easy to see that Esakia relations are exactly the inverses of binary JT-relations with the same source and target (see, e.g., [18]). Inverses of binary JT-relations with not necessarily the same source and target were first studied by Halmos [19].

It is a standard argument that each Esakia relation is closed, but there exist closed relations that are not Esakia relations. In fact, for a closed relation R on a Stone space X, the following are equivalent:

(1) R is an Esakia relation.
(2) $U \in \mathsf{Clop}(X)$ implies $R^{-1}[U] \in \mathsf{Clop}(X)$.
(3) U open implies $R^{-1}[U]$ is open.

Therefore, Esakia relations are special closed relations. We show that they dually correspond to modally definable subordinations. Our proof is a generalization of [6, Lem. 5.6].

Lemma 5.3.

(1) Suppose that $(B, \prec) \in \mathsf{Sub}$ and $(X, R) = (B, \prec)_*$. If $\prec$ is modally definable, then R is an Esakia relation.

(2) Suppose that R is an Esakia relation on a Stone space X and $(B, \prec) = (X, R)^*$. Then $\prec$ is modally definable.

Proof. (1) Suppose that $\prec$ is modally definable and $\Box_\prec$ is the largest element of $\{b \in B : b \prec a\}$.

Claim. $\varphi(\Box_\prec a) = X - R^{-1}[X - \varphi(a)]$.

Proof of Claim. We have $x \in X - R^{-1}[X - \varphi(a)]$ iff $R[x] \subseteq \varphi(a)$. This is equivalent to $(\forall y \in X)(\Uparrow x \subseteq y \Rightarrow a \in y)$. Since $\Uparrow x$ is a filter, by the ultrafilter theorem, it is the intersection of the ultrafilters containing it. Therefore, the last condition is equivalent to $a \in \Uparrow x$. Because $\Box_\prec$ is the largest element of $\{b \in B : b \prec a\}$, this is equivalent to $\Box_\prec a \in x$, which means that $x \in \varphi(\Box_\prec a)$. Thus, $\varphi(\Box_\prec a) = X - R^{-1}[X - \varphi(a)]$. $\Box$

Now, let $U \in \mathsf{Clop}(X)$. Then $X - U \in \mathsf{Clop}(X)$, so there is $a \in B$ with $\varphi(a) = X - U$. Therefore, $\varphi(\Box_\prec a) = X - R^{-1}[X - \varphi(a)] = X - R^{-1}[U]$. This yields $X - R^{-1}[U] \in \mathsf{Clop}(X)$, so $R^{-1}[U] \in \mathsf{Clop}(X)$. Since R is also a closed relation, we conclude that R is an Esakia relation.

(2) Let $U \in \mathsf{Clop}(X)$. We show that $X - R^{-1}[X - U]$ is the largest element of $\{V \in \mathsf{Clop}(X) : V \prec U\}$. Let $y \in R[X - R^{-1}[X - U]]$. Then there is $x \in X - R^{-1}[X - U]$ with xRy. From $x \in X - R^{-1}[X - U]$ it follows that $R[x] \subseteq U$. Therefore, $y \in U$, yielding $X - R^{-1}[X - U] \prec U$. Suppose that $V \in \mathsf{Clop}(X)$ with $V \prec U$. Then $R[V] \subseteq U$, so $V \subseteq X - R^{-1}[X - U]$. Thus, $X - R^{-1}[X - U]$ is the largest element of $\{V \in \mathsf{Clop}(X) : V \prec U\}$, and hence $\prec$ is modally definable. $\Box$

We recall that a *modal space* is a pair (X, R), where X is a Stone space and R is an Esakia relation on X. Modal spaces are also known as *descriptive frames*. They are fundamental objects in the study of modal logic as they serve as dual spaces of modal algebras (see, e.g., [12, 21, 11]).

Let $\mathsf{MS}^{\mathsf{st}}$ be the category whose objects are modal spaces and whose morphisms are continuous stable morphisms. Let also MSub be the full subcategory of Sub consisting of the objects $(B, \prec)$ of Sub in which $\prec$ is modally definable. It is an immediate consequence of Theorem 3.16 and Lemma 5.3 that MSub is dually equivalent to $\mathsf{MS}^{\mathsf{st}}$.

But modal logicians are more interested in bounded morphisms rather than stable morphisms since they dually correspond to modal algebra homomorphisms. We recall that a *modal homomorphism* is a Boolean homomorphism $h : B_1 \to B_2$ such that $h(\Box_1 a) = \Box_2 h(a)$. We also recall that a *bounded morphism* is a stable morphism $f : X_1 \to X_2$ such that $f(x) R_2 y$ implies the existence of $z \in X_1$ with $x R_1 z$ and $f(z) = y$. Let MA be the category whose objects are modal algebras and whose morphisms are modal homomorphisms, and let MS be the category whose objects are modal spaces and whose morphisms are continuous bounded morphisms. (Note that MS is not a full subcategory of $\mathsf{MS}^{\mathsf{st}}$.) It is a standard result in modal logic that MA is dually equivalent to MS. We next show how to obtain this dual equivalence from our results.

Let $h : B_1 \to B_2$ be a morphism in MSub. For $a \in B_1$, let $\Box_1 a$ be the largest element of $\{x \in B_1 : x \prec_1 a\}$, and for $b \in B_2$, let $\Box_2 b$ be the largest element of $\{y \in B_2 : y \prec_2 b\}$. Since $\Box_1 a \prec_1 a$, we have $h(\Box_1 a) \prec_2 h(a)$. Therefore, $h(\Box_1 a) \leq_2 \Box_2 h(a)$. Conversely, suppose that h is a Boolean homomorphism satisfying $h(\Box_1 a) \leq_2 \Box_2 h(a)$ for each $a \in B_1$. Let $a, b \in B_1$ with $a \prec_1 b$. Then $a \leq_1 \Box_1 b$.

Therefore, $h(a) \leq_2 h(\Box_1 b) \leq_2 \Box_2 h(b)$. Thus, $h(a) \prec_2 h(b)$, and h is a morphism in MSub.

We call a morphism h in MSub a *modal homomorphism* if $h(\Box_1 a) = \Box_2 h(a)$. Let $\mathsf{MSub^m}$ be the category whose objects are the objects of MSub and whose morphisms are modal homomorphisms. Then $\mathsf{MSub^m}$ is a non-full subcategory of MSub. From Examples 2.4 and 2.6 and the discussion above it is evident that $\mathsf{MSub^m}$ is isomorphic to MA.

We show that $\mathsf{MSub^m}$ is dually equivalent to MS. For this, taking into account the dual equivalence of MSub and $\mathsf{MS^{st}}$, it is sufficient to see that if h is a morphism in $\mathsf{MSub^m}$, then h_* is a morphism in MS, and that if f is a morphism in MS, then f^* is a morphism in $\mathsf{MSub^m}$. This is proved in the next lemma, which generalizes [6, Lem. 5.7].

Lemma 5.4.
(1) Let $(B_1, \prec_1), (B_2, \prec_2) \in \mathsf{MSub^m}$ and $h : B_1 \to B_2$ be a morphism in $\mathsf{MSub^m}$. Then h_* is a morphism in MS.
(2) Let $(X_1, R_1), (X_2, R_2) \in \mathsf{MS}$ and $f : X_1 \to X_2$ be a morphism in MS. Then f^* is a morphism in $\mathsf{MSub^m}$.

Proof. (1) From the dual equivalence of MSub and $\mathsf{MS^{st}}$ we know that h_* is continuous and stable. Suppose that $h_*(x) R_1 y$. Then $\twoheaduparrow_1 h^{-1}(x) \subseteq y$. Let F be the filter generated by $\twoheaduparrow_2 x \cup h(y)$ and let I be the ideal generated by $h(B_1 - y)$. If $F \cap I \neq \varnothing$, then there exist $a \in \twoheaduparrow_2 x$, $b \in y$, and $c \notin y$ such that $a \wedge_2 h(b) \leq_2 h(c)$. Therefore, $a \leq_2 h(b \to_1 c)$. From $a \in \twoheaduparrow_2 x$ it follows that there is $d \in x$ with $d \prec_2 a$. So $d \leq_2 \Box_2 a$. But $a \leq_2 h(b \to_1 c)$ implies $\Box_2 a \leq_2 \Box_2 h(b \to_1 c) = h(\Box_1 (b \to_1 c))$. This yields $\Box_1 (b \to_1 c) \in h^{-1}(x)$, so $b \to_1 c \in \twoheaduparrow_1 h^{-1}(x) \subseteq y$, which is a contradiction since $b \in y$ and $c \notin y$. Thus, $F \cap I = \varnothing$, and by the ultrafilter theorem, there is an ultrafilter z containing F and missing I. From $\twoheaduparrow_2 x \subseteq z$ it follows that $x R_2 z$, and from $h(y) \subseteq z$ and $h(B_1 - y) \cap z = \varnothing$ it follows that $h^{-1}(z) = y$. Consequently, there is z such that $x R_2 z$ and $h_*(z) = y$, yielding that h_* is a morphism in MS.

(2) From the dual equivalence of MSub and $\mathsf{MS^{st}}$ we know that f^* is a Boolean homomorphism satisfying $U \prec_2 V$ implies $f^*(U) \prec_1 f^*(V)$ for each $U, V \in \mathsf{Clop}(X_2)$. Therefore, $f^*(\Box_2 U) \leq_1 \Box_1 f^*(U)$ for each $U \in \mathsf{Clop}(X_2)$. Suppose that $x \in \Box_1 f^*(U)$. Then $R_1[x] \subseteq f^{-1}(U)$, so $f(R_1[x]) \subseteq U$. Since f is a bounded morphism, $f(R_1[x]) = R_2[f(x)]$. Therefore, $R_2[f(x)] \subseteq U$, yielding $f(x) \in \Box_2 U$. Thus, $x \in f^{-1}(\Box_2 U)$. This implies that $f^*(\Box_2 U) = \Box_1 f^*(U)$ for each $U \in \mathsf{Clop}(X_2)$, hence f^* is a morphism in $\mathsf{MSub^m}$. $\Box$

As a consequence, we obtain that $\mathsf{MSub^m}$ is dually equivalent to MS, and since $\mathsf{MSub^m}$ is isomorphic to MA, as a corollary, we obtain the well-known dual equivalence of MA and MS. Below we summarize the results of this section.

Theorem 5.5.
(1) MSub *is dually equivalent to* $\mathsf{MS^{st}}$.
(2) $\mathsf{MSub^m}$ *is isomorphic to* MA.
(3) $\mathsf{MSub^m}$ *is dually equivalent to* MS*, hence* MA *is dually equivalent to* MS.

6. Further duality results

In modal logic, modal algebras corresponding to reflexive, transitive, and/or symmetric modal spaces play an important role. In this section we characterize those $(B, \prec) \in \mathsf{Sub}$ which correspond to $(X, R) \in \mathsf{StR}$ with R reflexive, transitive, and/or symmetric. Since there is a 1-1 correspondence between subordinations and precontact relations, these results are similar to the ones given in [15, 14], but our proofs are different.

Lemma 6.1. Let $(B, \prec) \in \mathsf{Sub}$ and let (X, R) be the dual of $(B, \prec)$.

(1) R is reflexive iff $\prec$ satisfies (S5).
(2) R is symmetric iff $\prec$ satisfies (S6).
(3) R is transitive iff $\prec$ satisfies (S7).

Proof. (1) First suppose that R is reflexive. Let $a, b \in B$ with $a \prec b$. By Lemma 3.14, $\varphi(a) \prec \varphi(b)$, so $R[\varphi(a)] \subseteq \varphi(b)$. Since R is reflexive, $\varphi(a) \subseteq R[\varphi(a)]$. Therefore, $R[\varphi(a)] \subseteq \varphi(b)$ implies $\varphi(a) \subseteq \varphi(b)$. Thus, $a \le b$, and hence $\prec$ satisfies (S5). Next suppose that $\prec$ satisfies (S5). Let $x \in X$ and $a \in \twoheaduparrow x$. Then there is $b \in x$ with $b \prec a$. This, by (S5), yields $b \le a$, so $a \in x$. Therefore, $\twoheaduparrow x \subseteq x$, which means that xRx. Thus, R is reflexive.

(2) Suppose that R is symmetric. Let $a, b \in B$ with $a \prec b$. By Lemma 3.14, $R[\varphi(a)] \subseteq \varphi(b)$. We show that $R[X - \varphi(b)] \subseteq X - \varphi(a)$. Let $x \in R[X - \varphi(b)]$. Then there is $y \notin \varphi(b)$ with yRx. Since R is symmetric, xRy. If $x \in \varphi(a)$, then $y \in R[\varphi(a)]$, so $y \in \varphi(b)$, a contradiction. Therefore, $x \notin \varphi(a)$, and hence $R[X - \varphi(b)] \subseteq X - \varphi(a)$. This implies $R[\varphi(\neg b)] \subseteq \varphi(\neg a)$. Applying Lemma 3.14 again yields $\neg b \prec \neg a$. Thus, $\prec$ satisfies (S6). Conversely, suppose that $\prec$ satisfies (S6). Let $x, y \in X$ with xRy. Then $\twoheaduparrow x \subseteq y$. Suppose $a \in \twoheaduparrow y$. So there is $b \in y$ with $b \prec a$. By (S6), $\neg a \prec \neg b$. Since $\twoheaduparrow x \subseteq y$ and $\neg b \notin y$, we see that $\neg a \notin x$. Therefore, as x is an ultrafilter, $a \in x$, yielding $\twoheaduparrow y \subseteq x$. Thus, yRx, and hence R is symmetric.

(3) Suppose that R is transitive. Let $a, b \in B$ with $a \prec b$. Then $R[\varphi(a)] \subseteq \varphi(b)$. Therefore, $\varphi(a) \subseteq X - R^{-1}[X - \varphi(b)]$. Denoting $X - R^{-1}[X - \varphi(b)]$ by $\Box_R \varphi(b)$, we obtain $\varphi(a) \subseteq \Box_R \varphi(b)$. Since R is transitive, $\Box_R \varphi(b) \subseteq \Box_R \Box_R \varphi(b)$. This implies $\varphi(a) \subseteq \Box_R \Box_R \varphi(b)$, so $R[\varphi(a)] \subseteq \Box_R \varphi(b)$. Because R is a closed relation, $R[\varphi(a)]$ is closed and $\Box_R \varphi(b)$ is open. Thus, as X is a Stone space, there is clopen U with $R[\varphi(a)] \subseteq U \subseteq \Box_R \varphi(b)$. But $U = \varphi(c)$ for some $c \in B$. The first inclusion gives $\varphi(a) \prec \varphi(c)$ and the second yields $\varphi(c) \prec \varphi(b)$. Consequently, there is $c \in B$ with $a \prec c \prec b$, and $\prec$ satisfies (S7). Conversely, suppose that $\prec$ satisfies (S7). Let $x, y, z \in X$ with xRy and yRz. Then $\twoheaduparrow x \subseteq y$ and $\twoheaduparrow y \subseteq z$. Suppose $a \in \twoheaduparrow x$. Then there is $b \in x$ with $b \prec a$. By (S7), there is $c \in B$ with $b \prec c \prec a$. From $b \prec c$ and $b \in x$, we have $c \in \twoheaduparrow x$, hence $c \in y$. But then $c \prec a$ yields $a \in \twoheaduparrow y$, so $a \in z$. Thus, xRz, and hence R is transitive. □

Remark 6.2. Let $(B, \prec) \in \mathsf{Sub}$ and let (X, R) be the dual of $(B, \prec)$. Lemma 6.1 shows that axioms (S5), (S6), and (S7) correspond to elementary conditions on R. Developing a general theory which characterizes the class of axioms for subordinations corresponding to elementary conditions on R is closely related to the field of Sahlqvist theory in modal logic [12, 11]. In fact, by the perspective of Section 4, the results of Lemma 6.1 can be seen as instances of the standard Sahlqvist theory, applied to a binary modality. A Sahlqvist correspondence for logics corresponding to precontact algebras is developed in [1].

Definition 6.3.

(1) Let $\mathsf{SubK4}$ be the full subcategory of Sub consisting of the $(B, \prec) \in \mathsf{Sub}$ that satisfy (S7).

(2) Let SubS4 be the full subcategory of Sub consisting of the $(B, \prec) \in$ Sub that satisfy (S5) and (S7).
(3) Let SubS5 be the full subcategory of Sub consisting of the $(B, \prec) \in$ Sub that satisfy (S5), (S6), and (S7).

Clearly SubS5 is a full subcategory of SubS4, and SubS4 is a full subcategory of SubK4.

Definition 6.4.
(1) Let $\mathsf{StR}^{\mathsf{tr}}$ be the full subcategory of StR consisting of the $(X, R) \in$ StR, where R is transitive.
(2) Let $\mathsf{StR}^{\mathsf{qo}}$ be the full subcategory of StR consisting of the $(X, R) \in$ StR, where R is a quasi-order (that is, R is reflexive and transitive).
(3) Let $\mathsf{StR}^{\mathsf{eq}}$ be the full subcategory of StR consisting of the $(X, R) \in$ StR, where R is an equivalence relation.

Clearly $\mathsf{StR}^{\mathsf{eq}}$ is a full subcategory of $\mathsf{StR}^{\mathsf{qo}}$, and $\mathsf{StR}^{\mathsf{qo}}$ is a full subcategory of $\mathsf{StR}^{\mathsf{tr}}$. The next theorem is an immediate consequence of Theorem 3.16 and Lemma 6.1.

Theorem 6.5.
(1) SubK4 is dually equivalent to $\mathsf{StR}^{\mathsf{tr}}$.
(2) SubS4 is dually equivalent to $\mathsf{StR}^{\mathsf{qo}}$.
(3) SubS5 is dually equivalent to $\mathsf{StR}^{\mathsf{eq}}$.

Remark 6.6. We recall (see, e.g., [14]) that a precontact algebra (B, δ) is a *contact algebra* if it satisfies the following two axioms:
(P4) $a \neq 0$ implies $a \ \delta \ a$.
(P5) $a \ \delta \ b$ implies $b \ \delta \ a$.
Let Con be the full subcategory of PCon consisting of contact algebras. By Example 2.3, it is straightforward to see that (P4) is the δ-analogue of axiom (S5), while (P5) is the δ-analogue of axiom (S6). Therefore, Con is isomorphic to the full subcategory of Sub whose objects satisfy axioms (S5) and (S6). Thus, by Theorem 3.16 and Lemma 6.1, Con is dually equivalent to the full subcategory of StR consisting of such $(X, R) \in$ StR, where R is reflexive and symmetric.

Remark 6.7. We recall that a modal algebra $(B, \Box)$ is a **K4**-*algebra* if $\Box a \leq \Box\Box a$ for each $a \in B$; a **K4**-algebra is an **S4**-*algebra* if $\Box a \leq a$ for each $a \in B$; and an **S4**-algebra is an **S5**-*algebra* if $a \leq \Box\Diamond a$ for each $a \in B$. Let K4, S4, and S5 denote the categories of **K4**-algebras, **S4**-algebras, and **S5**-algebras, respectively.

We also let TRS be the category of transitive modal spaces, QOS be the category of quasi-ordered modal spaces, and EQS be the category of modal spaces, where the relation is an equivalence relation. Then it is a well-known fact in modal logic that K4 is dually equivalent to TRS, S4 is dually equivalent to QOS, and S5 is dually equivalent to EQS. These results can be obtain as corollaries of our results as follows.

Let $\mathsf{SubK4}^{\mathsf{m}}$, $\mathsf{SubS4}^{\mathsf{m}}$, and $\mathsf{SubS5}^{\mathsf{m}}$ be the subcategories of SubK4, SubS4, and SubS5, respectively, where morphisms are modal morphisms. It is then clear that $\mathsf{SubK4}^{\mathsf{m}}$ is isomorphic to K4, $\mathsf{SubS4}^{\mathsf{m}}$ is isomorphic to S4, and $\mathsf{SubS5}^{\mathsf{m}}$ is isomorphic to S5. It is also obvious that $\mathsf{SubK4}^{\mathsf{m}}$ is dually equivalent to TRS, $\mathsf{SubS4}^{\mathsf{m}}$ is dually equivalent to QOS, and $\mathsf{SubS5}^{\mathsf{m}}$ is dually equivalent to EQS. The duality results for K4, S4, and S5 follow.

7. Lattice subordinations and the Priestley separation axiom

An interesting class of subordinations is that of lattice subordinations of [6]. In this section we show that a subordination $\prec$ on a Boolean algebra B is a lattice subordination iff in the dual space (X,R) of $(B,\prec)$, the relation R is a Priestley quasi-order. The duality result of [6, Cor. 5.3] follows as a corollary.

We recall that a lattice subordination is a subordination $\prec$ that in addition satisfies

(S9) $a \prec b \Rightarrow (\exists c \in B)(c \prec c \ \& \ a \le c \le b)$.

By [6, Lem. 2.2], a lattice subordination satisfies (S5) and (S7). In addition, since c is reflexive, in the above condition, $a \le c \le b$ can be replaced by $a \prec c \prec b$. Therefore, a lattice subordination is a subordination that satisfies (S5) and a stronger form of (S7), where it is required that the existing c is reflexive.

If $\prec$ is a lattice subordination on B, then as follows from the previous section, in the dual space (X,R), we have that R is a quasi-order. But more is true. Let (X,R) be a quasi-ordered set. We call a subset U of X an *R-upset* provided $x \in U$ and xRy imply $y \in U$. Similarly U is an *R-downset* if $x \in U$ and yRx imply $y \in U$. We recall (see, e.g., [24, 10]) that a quasi-order R on a compact Hausdorff space X satisfies the *Priestley separation axiom* if $(x,y) \notin R$ implies that there is a clopen R-upset U such that $x \in U$ and $y \notin U$. If R satisfies the Priestley separation axiom, then we call R a *Priestley quasi-order*. Each Priestley quasi-order is closed, but the converse is not true in general [27, 10]. A *quasi-ordered Priestley space* is a pair (X,R), where X is a Stone space and R is a Priestley quasi-order on X. As was proved in [6, Cor. 5.3], lattice subordinations dually correspond to Priestley quasi-orders. To see how to derive this result from our results, we will use freely the following well-known fact about quasi-ordered Priestley spaces:

If A,B are disjoint closed subsets of a quasi-ordered Priestley space (X,R), with A an R-upset and B an R-downset, then there is a clopen R-upset U containing A and disjoint from B.

Lemma 7.1. Let $\prec$ be a subordination on B and let (X,R) be the dual of $(B,\prec)$. Then R is a Priestley quasi-order iff $\prec$ satisfies (S9).

Proof. First suppose that R is a Priestley quasi-order. Let $a,b \in B$ with $a \prec b$. By Lemma 3.14, $R[\varphi(a)] \subseteq \varphi(b)$. Therefore, $R[\varphi(a)] \cap (X - \varphi(b)) = \varnothing$. Since $R[\varphi(a)]$ is an R-upset, this yields $R[\varphi(a)] \cap R^{-1}[X - \varphi(b)] = \varnothing$. As $R[\varphi(a)]$ and $R^{-1}[X-\varphi(b)]$ are disjoint closed sets with $R[\varphi(a)]$ an R-upset and $R^{-1}[X-\varphi(b)]$ an R-downset, there is a clopen R-upset U containing $R[\varphi(a)]$ and disjoint from $R^{-1}[X-\varphi(b)]$. But $U = \varphi(c)$ for some $c \in B$. Since U is an R-upset, $R[\varphi(c)] \subseteq \varphi(c)$, so $c \prec c$. As $\varphi(a) \subseteq R[\varphi(a)] \subseteq \varphi(c)$, we have $a \le c$. Finally, since $\varphi(c)$ is disjoint from $R^{-1}[X-\varphi(b)]$, we also have $\varphi(c) \cap (X - \varphi(b)) = \varnothing$, so $\varphi(c) \subseteq \varphi(b)$, and hence $c \le b$. Thus, $\prec$ satisfies (S9).

Next suppose that $\prec$ satisfies (S9). Then $\prec$ satisfies (S5) and (S7), hence R is a quasi-order. Let $x,y \in X$ with $(x,y) \notin R$. Then ${\Uparrow}x \nsubseteq y$. Therefore, there are $a,b \in B$ with $a \in x$, $a \prec b$, and $b \notin y$. By (S9), there is $c \in B$ with $c \prec c$ and $a \le c \le b$. From $c \prec c$ it follows that $R[\varphi(c)] \subseteq \varphi(c)$, so $\varphi(c)$ is a clopen R-upset of X. Since $a \in x$ and $a \le c$, we have $c \in x$, so $x \in \varphi(c)$. As $c \le b$ and $b \notin y$, we also have $c \notin y$, hence $y \notin \varphi(c)$. Thus, there is a clopen R-upset $\varphi(c)$ such that $x \in \varphi(c)$ and $y \notin \varphi(c)$, yielding that R is a Priestley quasi-order. $\square$

Let LSub be the full subcategory of Sub consisting of the $(B, \prec) \in \mathsf{Sub}$, where $\prec$ is a lattice subordination. Let also QPS be the full subcategory of StR consisting of quasi-ordered Priestley spaces. It is an immediate consequence of our results that the dual equivalence of Sub and StR restricts to a dual equivalence of LSub and QPS. Thus, we arrive at the following result of [6, Cor. 5.3].

Theorem 7.2. LSub *is dually equivalent to* QPS.

8. Irreducible equivalence relations, compact Hausdorff spaces, and de Vries duality

In this final section we introduce irreducible equivalence relations, Gleason spaces, and provide a "modal-like" alternative to de Vries duality. We recall [13] that a *compingent algebra* is a pair $(B, \prec)$, where B is a Boolean algebra and $\prec$ is a binary relation on B satisfying (S1)–(S8). In other words, a compingent algebra is an object of $\mathsf{SubS5}$ that in addition satisfies (S8). It follows from our duality results that the dual of $(B, \prec) \in \mathsf{SubS5}$ is a pair (X, R), where X is a Stone space and R is a closed equivalence relation on X. Since X is compact Hausdorff and R is a closed equivalence relation on X, the factor-space X/R is also compact Hausdorff. In order to give the dual description of (S8), we recall that an onto continuous map $f : X \to Y$ between compact Hausdorff spaces is *irreducible* provided the f-image of each proper closed subset of X is a proper subset of Y.

Definition 8.1. We call a closed equivalence relation R on a compact Hausdorff space X *irreducible* if the factor-map $\pi : X \to X/R$ is irreducible.

Remark 8.2. Clearly a closed equivalence relation R on a compact Hausdorff space X is irreducible iff for each proper closed subset F of X, we have $R[F]$ is a proper subset of X. If X is a Stone space, then an immediate application of Esakia's lemma ([16, 7]) yields that we can restrict the condition to proper clopen subsets of X.

Lemma 8.3. Let $(B, \prec) \in \mathsf{SubS5}$ and let (X, R) be the dual of $(B, \prec)$. Then the closed equivalence relation R is irreducible iff $\prec$ satisfies (S8).

Proof. First suppose that R is irreducible. Let $a \in B$ with $a \neq 0$. Then $\varphi(a) \neq \varnothing$, so $X - \varphi(a)$ is a proper closed subset of X. Since R is irreducible, $R[X - \varphi(a)]$ is a proper subset of X. Therefore, $X - R[X - \varphi(a)] \neq \varnothing$, and as $R[X - \varphi(a)]$ is closed, $X - R[X - \varphi(a)]$ is open. As X is a Stone space, there is a nonempty clopen subset U of X contained in $X - R[X - \varphi(a)]$. But $U = \varphi(b)$ for some $b \in B$. Since $U \neq \varnothing$, we have $b \neq 0$. As $\varphi(b) \subseteq X - R[X - \varphi(a)]$ and R is an equivalence relation, $R[\varphi(b)] \subseteq \varphi(a)$. Thus, there is $b \neq 0$ with $b \prec a$, and so $\prec$ satisfies (S8).

Next suppose that $\prec$ satisfies (S8). Let F be a proper closed subset of X. Then $X - F$ is a nonempty open subset of X. Since X is a Stone space, there is a nonempty clopen set contained in $X - F$. Therefore, there is $a \in B - \{0\}$ with $\varphi(a) \subseteq X - F$. By (S8), there is $b \in B - \{0\}$ with $b \prec a$. Thus, $R[\varphi(b)] \subseteq \varphi(a)$. As R is an equivalence relation, this yields $\varphi(b) \subseteq X - R[X - \varphi(a)] \subseteq X - R[F]$. So $R[F] \subseteq X - \varphi(b)$. Since $b \neq 0$, we see that $X - \varphi(b)$ is a proper subset of X, hence $R[F]$ is a proper subset of X. Consequently, R is irreducible. □

Let Com be the full subcategory of $\mathsf{SubS5}$ consisting of compingent algebras; that is, Com consists of the objects of $\mathsf{SubS5}$ that in addition satisfy (S8). Let also

$\mathsf{StR}^{\mathsf{ieq}}$ be the full subcategory of $\mathsf{StR}^{\mathsf{eq}}$ consisting of the pairs (X, R), where R is an irreducible equivalence relation on a Stone space X. The above results yield:

Theorem 8.4. Com is dually equivalent to $\mathsf{StR}^{\mathsf{ieq}}$.

Definition 8.5 ([13, 4])**.** A *de Vries algebra* is a complete compingent algebra.

We recall that a space X is *extremally disconnected* if the closure of every open set is open. We call an extremally disconnected Stone space an *ED-space.* (Equivalently, ED-spaces are extremally disconnected compact Hausdorff spaces.) It is well known that a Boolean algebra B is complete iff its Stone space X is an ED-space. Therefore, the duals of de Vries algebras are pairs (X, R), where X is an ED-space and R is an irreducible equivalence relation on X.

Definition 8.6. We call a pair (X, R) a *Gleason space* if X is an ED-space and R is an irreducible equivalence relation on X.

Our choice of the name is motivated by the fact that Gleason spaces arise naturally by taking Gleason covers [17] of compact Hausdorff spaces. We recall that the Gleason cover of a compact Hausdorff space X is a pair (Y, π), where Y is an ED-space and $\pi : Y \to X$ is an irreducible map. It is well known that Gleason covers are unique up to homeomorphism. Suppose X is compact Hausdorff and (Y, π) is the Gleason cover of X. Define R on Y by xRy iff $\pi(x) = \pi(y)$. Since π is an irreducible map, it is easy to see that R is an irreducible equivalence relation on Y, hence (Y, R) is a Gleason space. In fact, each Gleason space arises this way because if (Y, R) is a Gleason space, then as R is a closed equivalence relation, the factor-space $X := Y/R$ is compact Hausdorff. Moreover, since R is irreducible, the factor-map $\pi : Y \to X$ is an irreducible map, yielding that (Y, π) is (homeomorphic to) the Gleason cover of X [17]. Thus, we have a convenient 1-1 correspondence between compact Hausdorff spaces and Gleason spaces, and both dually correspond to de Vries algebras.

Definition 8.7 ([13, 4])**.** A map $h : A \to B$ between two de Vries algebras is a *de Vries morphism* if it satisfies the following conditions:

(M1) $h(0) = 0$.
(M2) $h(a \wedge b) = h(a) \wedge h(b)$.
(M3) $a \prec b$ implies $\neg h(\neg a) \prec h(b)$.
(M4) $h(a) = \bigvee\{h(b) : b \prec a\}$.

Remark 8.8. Condition (M3) entails a more standard condition $a \prec b$ implies $h(a) \prec h(b)$ (see [5, Lem. 2.2]) and is equivalent to $a \prec c$ and $b \prec d$ imply $h(a \vee b) \prec h(c) \vee h(d)$ (see [9, Prop. 7.4]).

It is an easy consequence of (M1) and (M3) that a de Vries morphism h also satisfies $h(1) = 1$. Therefore, each de Vries morphism is a meet-hemimorphism [19]. Let X be the Stone space of A and Y be the Stone space of B. As follows from [19], meet-hemimorphisms $h : A \to B$ are dually characterized by relations $r \subseteq Y \times X$ satisfying $r[y]$ is closed for each $y \in Y$ and $r^{-1}[U]$ is clopen for each clopen $U \subseteq X$. In [19] such relations are called *Boolean relations.*

Remark 8.9.

(1) In [19] Halmos worked with join-hemimorphisms, which generalize the modal operator $\Diamond$, while meet-hemimorphisms generalize the modal operator $\Box$.

(2) Boolean relations are exactly the inverses of binary JT-relations, and if $X = Y$, then Boolean relations are Esakia relations (see Remark 5.2(2)).

We recall that the dual correspondence between $h : A \to B$ and $r \subseteq Y \times X$ is obtained as follows. Given $h : A \to B$, define $r \subseteq Y \times X$ by setting

$$(y, x) \in r \text{ iff } (\forall a \in A)(h(a) \in y \Rightarrow a \in x).$$

Conversely, given $r : Y \times X$, define $h : \mathsf{Clop}(X) \to \mathsf{Clop}(Y)$ by setting

$$h(U) = Y - r^{-1}[X - U].$$

In order to simplify notation, instead of $(y, x) \in r$, we will often write yrx. We also set

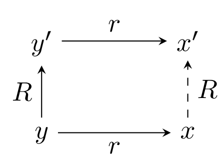
$$\Box_r U := Y - r^{-1}[X - U].$$

Thus, $h(U) = \Box_r U$.

Definition 8.10. Suppose $r \subseteq Y \times X$.

(1) We say that r is *cofinal* provided $(\forall y \in Y)(\exists x \in X)(yrx)$.
(2) We say that r satisfies the *forth condition* provided

$$(\forall y, y' \in Y)(\forall x, x' \in X)(yRy' \ \& \ yrx \ \& \ y'rx' \Rightarrow xRx').$$



(3) We say that r satisfies the *de Vries condition* provided

$$(\forall U \in \mathsf{Clop}(X))(r^{-1}(U) = \mathrm{int}(r^{-1}R^{-1}[U])).$$

Lemma 8.11. Let $(A, \prec)$ and $(B, \prec)$ be de Vris algebras, (X, R) be the dual of $(A, \prec)$, and (Y, R) be the dual of $(B, \prec)$. Suppose $h : A \to B$ is a meet-hemimorphism and $r \subseteq Y \times X$ is its dual.

(1) h satisfies (M1) iff r is cofinal.
(2) h satisfies (M3) iff r satisfies the forth condition.
(3) h satisfies (M4) iff r satisfies the de Vries condition.

Proof. (1) We have $h(0) = 0$ iff $\Box_r(\varnothing) = \varnothing$, which happens iff $r^{-1}[X] = Y$. This in turn is equivalent to $(\forall y \in Y)(\exists x \in X)(yrx)$. Thus, h satisfies (M1) iff r is cofinal.

(2) First suppose that h satisfies (M3). Let $y, y' \in Y$ and $x, x' \in X$ with yRy', yrx, and $y'rx'$. To see that xRx' we must show that $\twoheaduparrow x \subseteq x'$. Let $b \in \twoheaduparrow x$. Then there is $a \in x$ with $a \prec b$. By (M3), $\neg h(\neg a) \prec h(b)$. Since $a \in x$, we have $\neg a \notin x$. As yrx, this yields $h(\neg a) \notin y$. Because y is an ultrafilter, $\neg h(\neg a) \in y$. Therefore, $h(b) \in \twoheaduparrow y$. Since yRy', this gives $h(b) \in y'$. Thus, by $y'rx'$, we obtain $b \in x'$, so xRx'. Consequently, r satisfies the forth condition.

Next suppose that r satisfies the forth condition. Let $a, b \in A$ with $a \prec b$. Then $R[\varphi(a)] \subseteq \varphi(b)$. We have $\varphi(\neg h(\neg a)) = r^{-1}[\varphi(a)]$ and $\varphi(h(b)) = \Box_r \varphi(b)$. Therefore, to see that $\neg h(\neg a) \prec h(b)$, it is sufficient to show that $R[r^{-1}[\varphi(a)]] \subseteq \Box_r \varphi(b)$. Let $y' \in R[r^{-1}[\varphi(a)]]$. Then there are $x \in \varphi(a)$ and $y \in Y$ with yRy' and yrx. Suppose $x' \in X$ with $y'rx'$. So yRy', yrx, and $y'rx'$, which by the forth condition gives xRx'. Therefore, $x' \in R[\varphi(a)]$, yielding $x' \in \varphi(b)$. Thus, $y' \in \Box_r \varphi(b)$. Consequently, $R[r^{-1}[\varphi(a)]] \subseteq \Box_r \varphi(b)$, and hence h satisfies (M3).

(3) We recall that if $S \subseteq A$, then $\varphi(\bigvee S) = \overline{\bigcup\{\varphi(s) : s \in S\}}$. Therefore, for each $a \in A$, we have $\varphi(h(a)) = \Box_r \varphi(a)$ and

$$\begin{aligned} \varphi(\bigvee\{h(b) : b \prec a\}) &= \overline{\bigcup\{\Box_r \varphi(b) : R[\varphi(b)] \subseteq \varphi(a)\}} \\ &= \overline{\bigcup\{\Box_r \varphi(b) : \varphi(b) \subseteq \Box_R \varphi(a)\}} \\ &= \overline{\Box_r \Box_R \varphi(a)}. \end{aligned}$$

Thus, h satisfies (M4) iff $\Box_r \varphi(a) = \overline{\Box_r \Box_R \varphi(a)}$ for each $a \in A$. This is equivalent to $Y - r^{-1}[U] = Y - \text{int}(r^{-1}R^{-1}[U])$ for each $U \in \mathsf{Clop}(U)$. This in turn is equivalent to $r^{-1}[U] = \text{int}(r^{-1}R^{-1}[U])$ for each $U \in \mathsf{Clop}(U)$, yielding that h satisfies (M4) iff r satisfies the de Vries condition. □

Definition 8.12. Let (Y, R) and (X, R) be Gleason spaces. We call a relation $r \subseteq Y \times X$ a *de Vries relation* provided r is a cofinal Boolean relation satisfying the forth and de Vries conditions.

As follows from Lemma 8.11, de Vries relations dually correspond to de Vries morphisms. As with de Vries morphisms, because of the de Vries condition, the composition of two de Vries relations may not be a de Vries relation. Thus, for two de Vries relations $r_1 \subseteq X_1 \times X_2$ and $r_2 \times X_2 \times X_3$, we define $r_2 * r_1 \subseteq X_1 \times X_3$ as follows. Let $h_1 : \mathsf{Clop}(X_2) \to \mathsf{Clop}(X_1)$ be the dual of r_1 and $h_2 : \mathsf{Clop}(X_3) \to \mathsf{Clop}(X_2)$ be the dual of r_2. Let $h_3 = h_1 * h_2$ be the composition of h_1 and h_2 in the category DeV of de Vries algebras. Then $h_3 : \mathsf{Clop}(X_3) \to \mathsf{Clop}(X_1)$ is a de Vries morphism. Let $r_3 \subseteq X_1 \times X_3$ be the dual of h_3, and set $r_3 = r_2 * r_1$. With this composition, Gleason spaces and de Vries relations form a category we denote by Gle. We also let KHaus denote the category of compact Hausdorff spaces and continuous maps. The following is an immediate consequence of the above observations.

Theorem 8.13. Gle is dually equivalent to DeV, hence Gle is equivalent to KHaus.

Thus, Gle is another dual category to DeV. This provides an alternative more "modal-like" duality to de Vries duality.

Remark 8.14. The functor $\Phi : \mathsf{Gle} \to \mathsf{KHaus}$ establishing an equivalence of Gle and KHaus can be constructed directly, without first passing to DeV. For $(X, R) \in \mathsf{Gle}$, let $\Phi(X, R) = X/R$. Clearly $X/R \in \mathsf{KHaus}$. For $r : Y \times X$ a morphism in Gle, let $\Phi(r) = f$, where $f : Y/R \to X/R$ is defined as follows. Let $\pi : X \to X/R$ be the quotient map. Since r is cofinal, for each $y \in Y$ there is $x \in X$ with yrx. We set $f(\pi(y)) = \pi(x)$, where yrx. Since r satisfies the forth condition, f is well defined, and as r is a Boolean relation, f is continuous. Thus, f is a morphism in KHaus. From this it is easy to see that Φ is a functor. We already saw that there is a 1-1 correspondence between Gleason spaces and compact Hausdorff spaces. The functor Φ is full because for each continuous function $f : Y \to X$ between compact Hausdorff spaces, $f = \Phi(r)$, where r is the de Vries relation corresponding to the de Vries dual of f. Finally, the functor is faithful because among the cofinal Boolean relations r that satisfy the forth condition and yield the same continuous function $f : Y \to X$ in KHaus, there is the largest one, which satisfies the de Vries condition. Consequently, by [22, Thm. IV.4.1], $\Phi : \mathsf{Gle} \to \mathsf{KHaus}$ is an equivalence.

References

[1] P. Balbiani and S. Kikot. Sahlqvist theorems for precontact logics. In *Advances in Modal Logic 9, papers from the ninth conference on "Advances in Modal Logic," held in Copenhagen, Denmark, 22-25 August 2012*, pages 55–70, 2012.

[2] P. Balbiani, T. Tinchev, and D. Vakarelov. Dynamic logics of the region-based theory of discrete spaces. *J. Appl. Non-Classical Logics*, 17(1):39–61, 2007.

[3] P. Balbiani, T. Tinchev, and D. Vakarelov. Modal logics for region-based theories of space. *Fund. Inform.*, 81(1-3):29–82, 2007.

[4] G. Bezhanishvili. Stone duality and Gleason covers through de Vries duality. *Topology Appl.*, 157(6):1064–1080, 2010.

[5] G. Bezhanishvili. De Vries algebras and compact regular frames. *Appl. Categ. Structures*, 20(6):569–582, 2012.

[6] G. Bezhanishvili. Lattice subordinations and Priestley duality. *Algebra Universalis*, 70(4):359–377, 2013.

[7] G. Bezhanishvili, N. Bezhanishvili, and J. Harding. Modal compact Hausdorff spaces. *Journal of Logic and Computation*, 2012. DOI 10.1093/logcom/exs030. Available at `http://logcom.oxfordjournals.org/content/early/2012/07/07/logcom.exs030.full.pdf+html`.

[8] G. Bezhanishvili, N. Bezhanishvili, and R. Iemhoff. Stable canonical rules. Submitted. Available at `http://www.illc.uva.nl/Research/Publications/Reports/PP-2014-08.text.pdf`, 2014.

[9] G. Bezhanishvili and J. Harding. Proximity frames and regularization. *Appl. Categ. Structures*, 22(1):43–78, 2014.

[10] G. Bezhanishvili, R. Mines, and P. J. Morandi. The Priestley separation axiom for scattered spaces. *Order*, 19(1):1–10, 2002.

[11] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, Cambridge, 2001.

[12] A. Chagrov and M. Zakharyaschev. *Modal logic*, volume 35 of *Oxford Logic Guides*. The Clarendon Press, New York, 1997.

[13] H. de Vries. *Compact spaces and compactifications. An algebraic approach.* PhD thesis, University of Amsterdam, 1962.

[14] G. Dimov and D. Vakarelov. Topological representation of precontact algebras. In W. MacCaull, M. Winter, and I. Düntsch, editors, *Relational Methods in Computer Science*, volume 3929 of *Lecture Notes in Computer Science*, pages 1–16. Springer Berlin Heidelberg, 2006.

[15] I. Düntsch and D. Vakarelov. Region-based theory of discrete spaces: a proximity approach. *Ann. Math. Artif. Intell.*, 49(1-4):5–14, 2007.

[16] L. Esakia. Topological Kripke models. *Soviet Math. Dokl.*, 15:147–151, 1974.

[17] A. M. Gleason. Projective topological spaces. *Illinois J. Math.*, 2:482–489, 1958.

[18] R. Goldblatt. Varieties of complex algebras. *Annals of Pure and Applied Logic*, 44(3):173 – 242, 1989.

[19] P. R. Halmos. *Algebraic logic*. Chelsea Publishing Co., New York, 1962.

[20] B. Jónsson and A. Tarski. Boolean algebras with operators. I. *Amer. J. Math.*, 73:891–939, 1951.

[21] M. Kracht. *Tools and techniques in modal logic*. North-Holland Publishing Co., Amsterdam, 1999.

[22] S. Mac Lane. *Categories for the working mathematician*. Springer-Verlag, New York, second edition, 1998.

[23] S. A. Naimpally and B. D. Warrack. *Proximity spaces*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 59. Cambridge University Press, London, 1970.

[24] H. A. Priestley. Ordered topological spaces and the representation of distributive lattices. *Proc. London Math. Soc. (3)*, 24:507–530, 1972.

[25] V. Sofronie-Stokkermans. Duality and canonical extensions of bounded distributive lattices with operators, and applications to the semantics of non-classical logics I. *Studia Logica*, 64(1):93–132, 2000.

[26] M. H. Stone. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40(1):37–111, 1936.

[27] A. Stralka. A partially ordered space which is not a Priestley space. *Semigroup Forum*, 20(4):293–297, 1980.

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