Subintuitionistic Logics with Kripke Semantics

Dick de Jongh[1], Fatemeh Shirmohammadzadeh Maleki[2]

[1] Institute for Logic, Language and Computation, University van Amsterdam.
[2] Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran.

Abstract. The subintuitionistic logics introduced by Corsi and Restall are developed in a uniform manner. Proof systems are given for derivations without and with assumptions. The results are applied to give conservation theorems for intuitionistic logic IPC over Corsi's system F. For Visser's basic logic strong completeness is proved and additional conservation results are obtained.

Keywords: subintuitionistic logic, intuitionistic logic, basic logic, conservativity, completeness, modus ponens.

1 Introduction

G. Corsi, in [1], introduced sublogics of intuitionistic propositional logic IPC which are characterized by classes of Kripke models in which no assumption of truth preservation is made, and proved strong completeness for those systems. Her basic system was called F. G. Restall [4] made a similar study, also considering truth preservation, with somewhat different methods. His basic system was called SJ and can be considered to be equivalent to F. The proofs in his paper are somewhat sketchy.

In 1981, A. Visser [7] had already introduced Basic logic (BPC), an extension of F with truth preservation, in the natural deduction form, and proved completeness of BPC for finite, transitive, irreflexive Kripke models. Then in 1997, Suzuki and Ono [6] introduced a Hilbert style proof system for BPC as an extension of Corsi's system [1]. They proved a weak completeness theorem.

The structure of this paper is as follows. In Section 2 we introduce the logics, provide proof systems without and with assumptions and prove weak and strong completeness theorems. In Section 3 we will show that any prime theory Π satisfying some specific properties can be treated in much the same way as F with the same proofs, and a form of strong completeness for F due to Restall [4] is shown. We then apply the results to logics stronger than F. Thus we will prove a strong completeness theorem for BPC with a Hilbert style proof system. In all of this we clarify the role of the rules of modus ponens, conjunction and a fortiori. In Section 4 we will introduce two special classes of formulas and show that IPC is conservative over F with respect to these classes. This makes very clear what proof of IPC can be proved in F. We will prove that IPC is in addition conservative over BPC with respect to the NNIL formulas of [8]. This clarifies what more BPC can prove than F.

2 Subintuitionistic Logic

The Kripke models of subintuitionistic logics have a relation R that lacks the properties of reflexivity, transitivity and preservation of intuitionistic Kripke models.

Definition 1. *A **rooted subintuitionistic Kripke frame** is a triple $\langle W, g, R\rangle$. R is a binary relation on W; $g \in W$, the **root** is **omniscient**, i.e. gRw for each $w \in W$. A **root subintuitionistic Kripke model** is a quadruple $\langle W, g, R, V\rangle$ with $V : P \to 2^W$ a valuation function on the set of propositional variables P. The binary relation $\Vdash$ is defined on $w \in W$ as follows.*

1. $w \Vdash p \quad \Leftrightarrow \quad w \in V(p)$, *for any* $p \in P$,
2. $w \Vdash A \wedge B \quad \Leftrightarrow \quad w \Vdash A$ *and* $w \Vdash B$,
3. $w \Vdash A \vee B \quad \Leftrightarrow \quad w \Vdash A$ *or* $w \Vdash B$,
4. $w \Vdash A \to B \quad \Leftrightarrow \quad$ *for each v with wRv, if $v \Vdash A$ then $v \Vdash B$.*

*The constant f representing the **contradiction** is treated as a propositional variable. $M \Vdash A$ if, for all $w \in W$, $M, w \Vdash A$, and if all models force A, we write $\Vdash A$ and call A **valid**.*

This validity notion is Corsi's. We will discuss Restall's notion in Section 3.2.

Definition 2. F *is the logic given by the following axioms and rules,*

1. $A \to A \vee B$	7. $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
2. $B \to A \vee B$	8. $(A \to B) \wedge (B \to C) \to (A \to C)$
3. $A \wedge B \to A$	9. $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
4. $A \wedge B \to B$	10. $A \to A$
5. $\frac{A \quad B}{A \wedge B}$	11. $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
6. $\frac{A \quad A \to B}{B}$	12. $\frac{A}{B \to A}$

The rules are to be applied in such a way that, if the formulas above the line are theorems of F, then the formula below the line is a theorem as well. We may write $\vdash$ for $\vdash_{\mathsf{F}}$. We will call rule 5 *the conjunction rule* and, after Corsi, rule 12 the *a fortiori rule*. We return to the rules when we discuss deduction from hypotheses. In [4] SJ has different rules and axioms but is essentially the same system as F though it misses the a fortiori rule 12. For clarity's sake we prefer to prove weak completeness first using only direct deduction without hypotheses.

Proposition 1. (Soundness of F) In any root subintuitionistic Kripke model $\langle W, g, R, V\rangle$, for each $w \in W$ and each formula A, if $\vdash_{\mathsf{F}} A$ then $w \Vdash A$.

Proof. Easy. □

From the above proposition it does not follow that $\vdash A \to (B \to A)$. The following example shows that $\nvdash p \to (q \to p)$.

Example 1. Let $W = \{g, w_0, w_1\}$ and define $M = \langle W, g, R, V\rangle$ as follows:
$R = \{(g, g), (g, w_0), (g, w_1), (w_0, w_1)\}$.
$V(p) = \{w_0\}$, $V(q) = \{g, w_1\}$.
In this model $M, g \nVdash p \to (q \to p)$.

Next we will show F to be complete. First we will show that F has the disjunction property.

Definition 3. [2] We define $|A$ by induction on A, as follows

1. $|p$ iff $\vdash p$,
2. $|A \wedge B$ iff $|A$ and $|B$,
3. $|A \vee B$ iff $|A$ or $|B$,
4. $|A \to B$ iff $\vdash A \to B$ and (if $|A$ then $|B$).

Theorem 1. $|A \;\Leftrightarrow\; \vdash A$

Proof. The proof is a trivial modifciation of the standard one for IPC. □

Theorem 2. *If* $\vdash A \vee B$ *then* $\vdash A$ *or* $\vdash B$.

Proof. Assume $\vdash A \vee B$, by Theorem 1($\Leftarrow$), $|A \vee B$. So $|A$ or $|B$. By Theorem 1, ($\Rightarrow$), $\vdash A$ or $\vdash B$. □

Remark 1. Now that we have the disjunction property the following rules adopted by Restall follow from the corresponding rules without $\vee$.

$$\frac{A \vee C \quad (A \to B) \vee C}{B \vee C} \qquad \textit{and} \qquad \frac{(A \to B) \vee E \quad (C \to D) \vee E}{((B \to C) \to (A \to D)) \vee E}$$

Because let $\vdash A \vee C$ and $\vdash (A \to B) \vee C$. By Theorem 2, $\vdash A$ or $\vdash C$, and $\vdash A \to B$ or $\vdash C$. If $\vdash C$ then $\vdash B \vee C$. So, let $\nvdash C$. Then $\vdash A$ and $\vdash A \to B$. By rule 6 of F we conclude that $\vdash B$ and hence $\vdash B \vee C$. The proof of the other rule is similar to this.

We show that we do not need Restall's rule $\frac{(A \to B) \quad (C \to D)}{(B \to C) \to (A \to D)}$, because it follows from the a fortiori rule.

Proposition 2. *Let* $\vdash A \to B$ *and* $\vdash C \to D$ *then* $\vdash (B \to C) \to (A \to D)$.

Proof. Let $\vdash A \to B$ and $\vdash C \to D$ then,

1. $\vdash (B \to C) \to (A \to B)$ rule 12
2. $\vdash (B \to C) \to (B \to C)$
3. $\vdash ((B \to C) \to (A \to B)) \wedge ((B \to C) \to (B \to C))$
4. $\vdash (B \to C) \to (A \to B) \wedge (B \to C)$ From 2,3 using axiom 9
5. $\vdash (A \to B) \wedge (B \to C) \to (A \to C)$
6. $\vdash (B \to C) \to (A \to C)$ From 4,5 using axiom 8
7. $\vdash (B \to C) \to (C \to D)$ From assumption and rule 12
8. $\vdash (B \to C) \to (A \to C) \wedge (C \to D)$ From 6,7 using axiom 9

9. $\vdash (B \to C) \to (A \to D)$ Frome 8 using axiom 8.

□

To show weak completeness of F we need some definitions.

Definition 4. *1. A set of sentences Δ is a **theory** if and only if*
(a) $A, B \in \Delta \;\Rightarrow\; A \wedge B \in \Delta$,
(b) $\vdash A \to B \;\Rightarrow$ (if $A \in \Delta$, then $B \in \Delta$),
(c) F is contained in Δ.
2. For theories Γ, Δ, $\Gamma R \Delta$ iff, for all $A \to B \in \Gamma$, $A \in \Delta \Rightarrow B \in \Delta$.
*3. A set of sentences Δ is **prime** if and only if*

$$\text{if } A \vee B \in \Delta, \text{ then } A \in \Delta \text{ or } B \in \Delta.$$

Theorem 3. *Let Γ be a prime theory and $C \to D \notin \Gamma$. Then there is a prime theory Δ such that $\Gamma R \Delta$, $C \in \Delta$ and $D \notin \Delta$.*

Proof. Enumerate all formulas, with infinitely many repetitions: $B_0, B_1, ...$ and define

$\Delta_0 = \{E \mid C \to E \in \Gamma\}$,
$\Delta_{n+1} = \Delta_n \cup \{B_n\}$ if for no $\bar{B}_1, ..., \bar{B}_m \in \Delta_n$, $\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B_n \to D \in \Gamma$,
$\Delta_{n+1} = \Delta_n$ otherwise.

Take Δ to be the union of all Δ_n. We will show that Δ is a theory. Assume that $F \in \Delta$, $G \in \Delta$ and $F \wedge G \notin \Delta$. Let $F = B_i$, $G = B_j$ and $F \wedge G = B_n$ such that, $i \geq n$ and $j \geq n$. So there exist $\bar{B}_1, ..., \bar{B}_m \in \Delta_n$ such that

$$\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge F \wedge G \to D \in \Gamma \tag{1}$$

W.l.o.g. let $i \geq j$, then $\bar{B}_1, ..., \bar{B}_m, G \in \Delta_i$. By (1) we conclude that $F \notin \Delta$ and this is in contradiction with our assumption.

Now let $\vdash A \to B$ and $A \in \Delta$. We must show that $B \in \Delta$. Let $B = B_n$ and $B \notin \Delta$. So there exist $\bar{B}_1, ..., \bar{B}_m \in \Delta_n$, such that

$$\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B \to D \in \Gamma$$

We know $\vdash A \to B$. We conclude by axiom 9 and Modus Ponens that

$$\vdash \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B$$

and so $\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B \in \Gamma$. Now we have

$$(\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B) \wedge (\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B \to D) \in \Gamma \tag{2}$$

Γ is a theory so by (2) and axiom 8 we have $\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to D \in \Gamma$ and this is a contradiction, because $A \in \Delta$.

Assume that $F \vee G \in \Delta$, and $F \notin \Delta$, $G \notin \Delta$. Let $F = B_n$ and $G = B_k$. Then there exist $\bar{B}_1, ..., \bar{B}_m \in \Delta_n$ such that $\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge F \to D \in \Gamma$ and also there exist $B'_1, ..., B'_{m'} \in \Delta_k$ such that $B'_1 \wedge ... \wedge B'_{m'} \wedge G \to D \in \Gamma$. W.l.o.g. take

$n \geq k$, then $\bar{B}_1, ..., \bar{B}_m, B_1^{'}, ..., B_{m'}^{'} \in \Delta_n$. Thus by axiom 11 and some steps we will have

$$\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B_1^{'} \wedge ... \wedge B_{m'}^{'} \wedge (F \vee G) \to D \in \Gamma$$

But that cannot be true since $F \vee G \in \Delta$.

We know that $C \to C \in \Gamma$, so by definition $C \in \Delta_0$ and hence $C \in \Delta$. Also we have $D \to D \in \Gamma$, so $D \notin \Delta$.

Now assume that $\vdash F$, we want to show that $F \in \Delta$. Assume $F = B_n$ and $F \notin \Delta$, then for some $\bar{B}_1, ..., \bar{B}_m \in \Delta_n$, we have

$$\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge F \to D \in \Gamma \tag{3}$$

We have $\vdash \bar{B}_1 \wedge ... \wedge \bar{B}_m \to F$, so

$$\bar{B}_1 \wedge ... \wedge \bar{B}_m \to \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge F \in \Gamma \tag{4}$$

Γ is a theory therefore by (3) and (4) and axiom 8 we conclude that

$$\bar{B}_1 \wedge ... \wedge \bar{B}_m \to D \in \Gamma$$

and this is a contradiction. Hence $F \in \Delta$.

Finally we will show that $\Gamma R \Delta$. let $A \to B \in \Gamma$ and $A \in \Delta$. Let $B \notin \Delta$ and $B = B_n$. Then there exist $\bar{B}_1, ..., \bar{B}_m \in \Delta_n$ such that

$$\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B \to D \in \Gamma \tag{5}$$

We call $\bar{B}_1 \wedge ... \wedge \bar{B}_m = C$. We have

$$\vdash (C \wedge A \to A) \wedge (A \to B) \to (C \wedge A \to B) \tag{6}$$

We know that $(C \wedge A \to A) \wedge (A \to B) \in \Gamma$ and Γ is a theory. So by 6, $C \wedge A \to B \in \Gamma$. On the other hand we have

$$\vdash (C \wedge A \to B) \wedge (C \wedge A \to C) \to (C \wedge A \to B \wedge C) \tag{7}$$

We know that $(C \wedge A \to B) \wedge (C \wedge A \to C) \in \Gamma$ and Γ is a theory, so by 7, $C \wedge A \to B \wedge C \in \Gamma$. That is

$$\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B \in \Gamma \tag{8}$$

Γ is a theory so by (5), (8) and axiom 8 we have $\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to D \in \Gamma$ and this is a contradiction, because $A \in \Delta$. So $\Gamma R \Delta$. □

Definition 5. *We call* $\{A \,|\, \vdash A\}$ *the* ***empty theory****.*

Proposition 3. *The empty theory* Δ *is a prime theory.*

Proof. (1) Let $A, B \in \Delta$, then $\vdash A$ and $\vdash B$, so $\vdash A \wedge B$. By definition, $A \wedge B \in \Delta$. (2) Let $\vdash A \to B$ and $A \in \Delta$. Then $\vdash A$, so $\vdash B$. By definition $B \in \Delta$. (3) Trivial.

To prove that Δ is prime, assume $A \vee B \in \Delta$. Then $\vdash A \vee B$. By Theorem 2, $\vdash A$ or $\vdash B$. That is $A \in \Delta$ or $B \in \Delta$. □

Definition 6. *The* ***Canonical Model*** $M_{\mathsf{F}} = \langle W_{\mathsf{F}}, \Delta, R, \Vdash \rangle$ *of* F *is defined by:*

1. *Δ is the empty theory,*
2. *W_{F} is the set of all prime theories,*
3. *The canonical valuation is defined by $\Gamma \Vdash p$ iff $p \in \Gamma$.*

In the canonical model $M_{\mathsf{F}} = \langle W_{\mathsf{F}}, \Delta, R, \Vdash \rangle$, Δ is omniscient. Because let $\Gamma \in W_{\mathsf{F}}$. If $A \to B \in \Delta$, then $\vdash A \to B$. So, if $A \in \Gamma$, then $B \in \Gamma$.

Lemma 1. (Truth lemma) For each $\Gamma \in W_{\mathsf{F}}$ and for every formula C,

$$\Gamma \Vdash C \text{ iff } C \in \Gamma.$$

Proof. By induction on C. The atomic case holds by definition.

($C := A \wedge B$) Let $\Gamma \Vdash A \wedge B$ then $\Gamma \Vdash A$ and $\Gamma \Vdash B$. By the induction hypothesis, $A \in \Gamma$ and $B \in \Gamma$. Γ is a theory so $A \wedge B \in \Gamma$.

Now let $A \wedge B \in \Gamma$. We have $\vdash A \wedge B \to A$ and $\vdash A \wedge B \to B$, hence by definition of theory we conclude that $A \in \Gamma$ and $B \in \Gamma$. By induction hypothesis, $\Gamma \Vdash A$ and $\Gamma \Vdash B$ so $\Gamma \Vdash A \wedge B$.

($C := A \vee B$) $\Gamma \Vdash A \vee B$ then $\Gamma \Vdash A$ or $\Gamma \Vdash B$. By the induction hypothesis, $A \in \Gamma$ or $B \in \Gamma$. We have $\vdash A \to A \vee B$ and $\vdash B \to A \vee B$ so by definition of theory we conclude that $A \vee B \in \Gamma$.

Now let $A \vee B \in \Gamma$. Γ is a prime, so $A \in \Gamma$ or $B \in \Gamma$. By induction hypothesis, we conclude that $\Gamma \Vdash A$ or $\Gamma \Vdash B$. That is $\Gamma \Vdash A \vee B$.

($C := A \to B$) Let $A \to B \in \Gamma$, $\Sigma \in W_{\mathsf{F}}$ and $\Gamma R \Sigma, \Sigma \Vdash A$. By induction hypothesis $A \in \Sigma$. Then by definition of R, $B \in \Sigma$. Again by induction hypothesis, $\Sigma \Vdash B$. that is $\Gamma \Vdash A \to B$.

Now let $A \to B \notin \Gamma$. Then by Theorem 3, there is a prime theory Δ such that $A \in \Delta$, $B \notin \Delta$ and $\Gamma R \Delta$. So $\Gamma \nVdash A \to B$. □

Theorem 4. (Weak Completeness) For any formula A if $\Vdash A$, then $\vdash A$.

Proof. Let $\nvdash A$ and let Δ be the empty theory. By the definition of empty theory $A \notin \Delta$. So, we have $M_{\mathsf{F}}, \Delta \nVdash A$. That is, $\nVdash A$. □

Next we prove strong completeness with the semantics as in Corsi [1]. But we introduce a notion of derivation from hypotheses.

Definition 7. *(a) We define $\Gamma \vdash A$ if there is a derivation of A from Γ and theorems of F using the rules $\frac{A \quad B}{A \wedge B}$, and $\frac{A \quad A \to B}{B}$ (only if $\vdash_{\mathsf{F}} A \to B$).*
(b) We define $\Gamma \Vdash A$ iff for all $M, w \in M$, if $M, w \Vdash \Gamma$ then $M, w \Vdash A$.

Remark 2. Note that if $\Gamma \vdash A$ then it does not follow that $\Gamma \vdash B \to A$. For example if we assume that $\Gamma = \mathsf{F} \cup \{p\}$, then $\Gamma \vdash p$ and $\Gamma \nvdash q \to p$.

Surprisingly, the weak Deduction Theorem holds for F and $\vdash$.

Theorem 5. (Weak Deduction Theorem) $A \vdash B$ if and only if $\vdash A \to B$.

Proof. $\Rightarrow$: By induction on the length of the proof.
If B is a theorem of F. Then $\vdash B$, so by rule 12, $\vdash A \to B$.
$A \vdash A$ is covered by $\vdash A \to A$.
If $A \vdash B$ and $A \vdash C$. By induction hypothesis $\vdash A \to B$ and $\vdash A \to C$, so $\vdash A \to B \wedge C$.
If $A \vdash B$ and $\vdash B \to C$. Then by induction hypothesis $\vdash A \to B$, so $\vdash A \to C$.
$\Leftarrow$: By definition this direction is straightforward. □

Corollary 1. *1. $A_1, ..., A_n \vdash B$ iff $\vdash A_1 \wedge ... \wedge A_n \to B$.*
2. $\Delta \vdash B$ iff $A_1 \wedge ... \wedge A_n \vdash B$ for some $A_1, ..., A_n \in \Delta$.

Proof. The proof is easy. □

Proposition 4. *Δ is a theory $\Longleftrightarrow$ $\Delta \vdash A$ if and only if $A \in \Delta$.*

Proof. $\Rightarrow$: The proof from right to left is immediate. The other direction is by induction on the length of the derivation. If $A \in \Delta$ there is nothing to prove. If A is a theorem of F, then by definition of theory $A \in \Delta$.

If $\Delta \vdash A$ and $\Delta \vdash B$, by induction hypothesis $A \in \Delta$ and $B \in \Delta$. So, by the definition of theory $A \wedge B \in \Delta$.

If $\vdash A \to B$ and $\Delta \vdash A$, by induction hypothesis $A \in \Delta$, and by definition of theory $B \in \Delta$.
$\Leftarrow$: This is straightforward. □

Theorem 6. *If $\Sigma \nvdash D$ then there is a prime theory Δ such that $\Delta \supseteq \Sigma,\ D \notin \Delta$.*

Proof. By assumption and by definition of provability we conclude that $D \notin \Sigma$. Enumerate all formulas, with infinitely many repetitions: $B_0, B_1, ...$ and define

$\Delta_0 = \Sigma \cup \mathsf{F}$,
$\Delta_{n+1} = \Delta_n \cup \{B_n\}$ if $\Delta_n, B_n \nvdash D$,
$\Delta_{n+1} = \Delta_n$ otherwise.

Take Δ to be the union of all Δ_n. The proof now runs as for Theorem 3. □

Theorem 7. (Strong Completeness) For any formula A, $\Sigma \vdash A$ if and only if $\Sigma \Vdash A$.

Proof. Left to right is easy. For the other direction, Let $\Sigma \nvdash A$. Then by Theorem 6, there is a prime theory $\Gamma \supseteq \Sigma$ such that $A \notin \Gamma$. So, we will have $M_{\mathsf{F}}, \Gamma \Vdash \Sigma$ and $M_{\mathsf{F}}, \Gamma \nVdash A$. That is $\Sigma \nVdash A$. □

We will not discuss the finite model property in this paper, or translations into modal logic. We have no new results in that area and refer the reader to Corsi [1] and Sano and Ma [5].

3 Π-Provability and Restall's strong completeness

In the first subsection we introduce provabality in a theory Π with good properties. In the second subsection we will apply this notion to discuss Restall's validity notion. He considers validity in a model as truth in the root, and validity of a consequence accordingly. This gives rise to another consequence relation. We define a suitable proof relation $\vdash_r$ to prove the connected form of strong completeness. In the third subsection we further develop the notion of Π-provability and connect it with provability in logics stronger than F, in particular BPC.

3.1 Π-Provability

Definition 8. *Δ is a Π-theory if and only if:*

1. *If $A, B \in \Delta$, then $A \wedge B \in \Delta$,*
2. *If $A \to B \in \Pi$ and $A \in \Delta$, then $B \in \Delta$,*
3. *The set $\Pi_{\to}$ of members of Π of the form $A \to B$ is contained in Δ,*
4. *F is contained in Δ.*

We will assume Π to be a theory with good properties as coded in the following definition.

Definition 9. *Π is an **adequate theory** if Π a prime Π-theory closed under the **restricted a fortiori rule**, if $A \in \Pi_{\to}$, then for all B, $B \to A \in \Pi$.*

Lemma 2. *Π is an adequate theory iff Π a prime theory closed under modus ponens and the restricted a fortiori rule.*

Proof. Obvious. ☐

In all of Section 3, Π will be assumed to be an adequate theory. This will turn out to make Π suitable to be the set of formulas true in the root of a model.

Definition 10. *We define $\Gamma \vdash_\Pi A$ as: there is a derivation of A from $\Gamma \cup \Pi_{\to} \cup \mathsf{F}$ using the rules $\frac{A \quad B}{A \wedge B}$, $\frac{A \quad A \to B}{B}$ with $A \to B \in \Pi$ in the latter case.*

Proposition 5. *Δ is a Π-theory $\iff$ $\Delta \vdash_\Pi A$ if and only if $A \in \Delta$.*

Proof. $\Rightarrow$: From right to left is trivial. The other direction is by induction on the length of the proof. If $A \in \Delta \cup \mathsf{F} \cup \Pi_{\to}$ there is nothing to prove.

Let $\Delta \vdash_\Pi A$ and $\Delta \vdash_\Pi B$. By induction hypothesis $A \in \Delta$ and $B \in \Delta$. So, by definition of Π-theory $A \wedge B \in \Delta$.

If $A \to B \in \Pi$ and $\Delta \vdash_\Pi A$. By induction hypothesis $A \in \Delta$ and by definition of Π-theory $B \in \Delta$.

$\Leftarrow$: Straightforward. ☐

Theorem 8. *If Γ is a prime Π-theory with $C \to D \notin \Gamma$, then there is a prime Π-theory Δ with $\Gamma R \Delta$, $C \in \Delta$ and $D \notin \Delta$.*

Proof. Enumerate all formulas, with infinitely many repetitions: $B_0, B_1, ...$ and define

$\Delta_0 = \{E \mid C \to E \in \Gamma\} \cup \mathsf{F}$,
$\Delta_{n+1} = \Delta_n \cup \{B_n\}$ if for all $\bar{B}_1, ..., \bar{B}_m \in \Delta_n,\ \Gamma \nvdash_{\Pi} \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B_n \to D$,
$\Delta_{n+1} = \Delta_n$ otherwise.

Take Δ to be the union of all Δ_n. We show that

1. $\Gamma R \Delta$,
2. Δ is a prime Π-theory,
3. $C \in \Delta$,
4. $D \notin \Delta$.

1. Let $A \to B \in \Gamma$ and $A \in \Delta$. We must show that $B \in \Delta$. Let $B = B_n$ and $B \notin \Delta$. So there exist $\bar{B}_1, ..., \bar{B}_m \in \Delta_n$ such that

$$\Gamma \vdash_{\Pi} \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B \to D \tag{9}$$

$A \to B \in \Gamma$ so, $\Gamma \vdash_{\Pi} A \to B$ and $\Gamma \vdash_{\Pi} \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to A$ then,

$$\Gamma \vdash_{\Pi} \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to B$$

Also we have $\Gamma \vdash_{\Pi} \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to \bar{B}_1 \wedge ... \wedge \bar{B}_m$, therefore

$$\Gamma \vdash_{\Pi} (\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to \bar{B}_1 \wedge ... \wedge \bar{B}_m) \wedge (\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to B). \tag{10}$$

By (10) and axiom 9 we conclude that

$$\Gamma \vdash_{\Pi} \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B. \tag{11}$$

By (9) and (11), we have

$$\Gamma \vdash_{\Pi} (\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B) \wedge (\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B \to D). \tag{12}$$

Again by (12) and axiom 8 we have $\Gamma \vdash_{\Pi} \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge A \to D$ and this is a contradiction, because $A \in \Delta$. So $B \in \Delta$ and hence $\Gamma R \Delta$.
2. Let $A \in \Delta$, $A \to B \in \Pi$. So, also $A \to B \in \Gamma$. As in (1.) $B \in \Delta$ follows.

Let $F \in \Pi_{\to}$. We want to show that $F \in \Delta$. We know that Π is closed under the restricted a fortiori rule, so $C \to F \in \Pi_{\to}$ and therefore $C \to F \in \Gamma$. So by definition of Δ_0, $F \in \Delta_o$ and thus $F \in \Delta$.

As in the proof of Theorem 3, we can conclude that Δ is prime and closed under conjunction.
3. We know that $C \to C \in \Gamma$, so $C \in \Delta_0$ and then $C \in \Delta$.
4. $D \notin \Delta$, since $\Gamma \vdash_{\Pi} D \to D$. □

Definition 11. *In the* ***Π-canonical model*** $M^{\Pi} = \langle W_{\Pi}, \Pi, R, \Vdash \rangle$ *of* F

1. *W_{Π} is the set of all prime Π-theories,*
2. *The canonical valuation is defined by $\Gamma \Vdash p$ if and only if $p \in \Gamma$.*

Lemma 3. (Truth lemma) For each $\Gamma \in W_{\Pi}$ and for every formula A,

$$\Gamma \Vdash C \text{ iff } \mathrm{C} \in \Gamma.$$

Proof. By induction on C exactly as in the proof of Lemma 1. □

This truth lemma will lead as usual to completeness of Π-provability, but we leave that for the last subsection in which we strengthen the conditions on Π. But we can already state the following proposition which makes the situation clearer.

Proposition 6. $\Pi \vdash_{\Pi} A \;\Leftrightarrow\; M^{\Pi}, \Pi \Vdash A.$

Proof. Immediate by Proposition 5 and Lemma 3. □

3.2 Restall's form of strong completeness

We first introduce a stronger notion of proof $\Gamma \vdash_r$ from a set of assumptions Γ. It will include full modus ponens as well as the restricted a fortiori rule. We do not assume that the set of assumptions is a prime theory or even a theory. The fact that no disjunction property is assumed means that the proof rules will have to be more complex. The idea is that it will be ultimately be possible to extend the set of assumptions to an adequate theory.

Definition 12. (*a*) *We define $\Gamma \vdash_r A$ if there is a derivation of A from Γ and theorems of* F *using the rules*

$$\frac{A \quad B}{A \wedge B}\,,\quad \frac{B \quad B \to A}{A}\,,\quad \frac{B \vee C \quad (B \to A) \vee C}{A \vee C}$$

and

$$\frac{A}{B \to A}\quad,\quad \frac{A \vee C}{(B \to A) \vee C}$$

with in the latter two cases the restriction that A has to be an implication.
(*b*) *We define, $\Gamma \vDash_r A$ iff for all $M = \langle W, g, R, V \rangle$, if $M, g \Vdash \Gamma$ then $M, g \Vdash A$.*

Proposition 7. $\Pi \vdash_r A \;\Leftrightarrow\; A \in \Pi.$

Proof. $\Rightarrow$: By induction on the length of the proof.

If $A \in \Pi$ there is nothing to prove.

If $\Pi \vdash_r A$ and $\Pi \vdash_r B$, then, by induction hypothesis, $A \in \Pi$ and $B \in \Pi$. So, by the definition of Π-theory $A \wedge B \in \Pi$.

If $\Pi \vdash_r A \to B$ and $\Pi \vdash_r A$, then, by induction hypothesis, $A \in \Pi$ and $A \to B \in \Pi$. So, by the definition of Π-theory $B \in \Pi$.

If $\Pi \vdash_r (A \to B) \vee C$ and $\Pi \vdash_r A \vee C$, then, by induction hypothesis, $(A \to B) \vee C \in \Pi$ and $A \vee C \in \Pi$. So, $A \to B \in \Pi$ or $C \in \Pi$, and $A \in \Pi$ or $C \in \Pi$, since Π is prime. Therefore $C \in \Pi$, or $A \to B \in \Pi$ and $A \in \Pi$. In the latter case, by definition of Π-theory, $B \in \Pi$. So, in both cases $B \vee C \in \Pi$.

If $\Pi \vdash_r A$ and A is an implication, then, by the induction hypothesis and the closure of Π under the a restricted a fortiori rule, for all B, $B \to A \in \Pi$.

The other cases are similar.

$\Leftarrow$: If $A \in \Pi$, then by definition, $\Pi \vdash_r A$. □

Corollary 2. *$\Pi \vdash_r A \;\Leftrightarrow\; \Pi \vdash_\Pi A$.*

Proof. Immediate from Propositions 5 and 7. □

Next we show how to reason with $\vee$ in case we do not have the disjunction property.

Lemma 4. *If $A \vdash_r B$ then $C \vee A \vdash_r C \vee B$.*

Proof. By induction on the length of the proof.

If B is a theorem of F, then $C \vee A \vdash_r B$ and $C \vee A \vdash_r B \to C \vee B$, so $C \vee A \vdash_r C \vee B$.

If $A \vdash_r A$, also $C \vee A \vdash_r C \vee A$.

If $A \vdash_r B$ and $A \vdash_r D$, then by induction hypothesis $C \vee A \vdash_r C \vee B$ and $C \vee A \vdash_r C \vee D$, therefore $C \vee A \vdash_r (C \vee B) \wedge (C \vee D)$.
After some steps we will have $C \vee A \vdash_r C \vee (B \wedge D)$.

If $A \vdash_r F$ and F is implication, we want to prove for all B, $C \vee A \vdash_r C \vee (B \to F)$. By induction hypothesis $C \vee A \vdash_r C \vee F$. Then $C \vee A \vdash_r C \vee (B \to F)$.

If $A \vdash_r F$ and $A \vdash_r F \to B$, then by induction hypothesis $C \vee A \vdash_r C \vee F$ and $C \vee A \vdash_r C \vee (F \to B)$, so $C \vee A \vdash_r C \vee B$.

The remaining cases are easy. □

Proposition 8. *If $A \vdash_r C$ and $B \vdash_r C$, then $A \vee B \vdash_r C$.*

Proof. By Lemma 4, $A \vee B \vdash_r C \vee B$, and also, $C \vee B \vdash_r C \vee C$. It is simple to show that $\vdash C \vee C \to C$, so $A \vee B \vdash_r C$. □

Now we have shown that reasoning from disjunctions can be executed properly we have reached the point at which we can show that an arbitrary set of formulas can be extended to an adequate theory.

Lemma 5. *If $\Sigma \nvdash_r A$, then there is a $\Pi \supseteq \Sigma$ such that Π is an adequate theory and $\Pi \nvdash_r A$.*

Proof. Enumerate all formulas, with infinitely many repetitions: $B_0, B_1, ...$ and define

$\Pi_0 = \{B \mid \Sigma \vdash_r B\}$,
$\Pi_{n+1} = \Pi_n \cup \{B_n\}$ if for no $\bar{B}_1, ..., \bar{B}_m \in \Pi_n,\ \bar{B}_1, ..., \bar{B}_m, B_n \vdash_r A$,
$\Pi_{n+1} = \Pi_n$ otherwise.

Take Π to be the union of all Π_n. By definition of Π, it is clear that $\Pi \nvdash_r A$.

We must show that Π is a Π-theory. Assume that $E \in \Pi$, $F \in \Pi$ and $E \wedge F \notin \Pi$. Let $E = B_i$, $F = B_j$ and $E \wedge F = B_n$ such that, $i \geq n$ and $j \geq n$. So there exist $\bar{B}_1, ..., \bar{B}_m \in \Pi_n$, such that

$$\bar{B}_1, ..., \bar{B}_m, E \wedge F \vdash_r A$$

and so,

$$\bar{B}_1, ..., \bar{B}_m, E, F \vdash_r A$$

But $E, F, \bar{B}_1, ..., \bar{B}_m \in \Pi_j$, so $\bar{B}_1, ..., \bar{B}_m, E, F \nvdash_r A$, a contradiction.

Now let $C \to D \in \Pi$ and $C \in \Pi$ we must show that $D \in \Pi$. Let $D = B_n$ and $D \notin \Delta$. So there exist $\bar{B}_1, ..., \bar{B}_m \in \Delta_n$, such that

$$\bar{B}_1, ..., \bar{B}_m, D \vdash_r A$$

and so

$$\bar{B}_1, ..., \bar{B}_m, C \to D, C \vdash_r A$$

This is a contradiction.

Assume that $E \vee F \in \Pi$, and $E \notin \Pi$, $F \notin \Pi$. Let $E = B_n$ and $F = B_k$. Then there exist $\bar{B}_1, ..., \bar{B}_m \in \Delta_n$, such that $\bar{B}_1, ..., \bar{B}_m, E \vdash_r A$ and therefore

$$\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge E \vdash_r A \tag{13}$$

and also there exist $B_1^{'}, ..., B_{m'}^{'} \in \Delta_k$, such that $B_1^{'}, ..., B_{m'}^{'}, F \vdash_r A$ and therefore

$$B_1^{'} \wedge ... \wedge B_{m'}^{'} \wedge F \vdash_r A \tag{14}$$

By (13), (14), the distributive law and Proposition 8 we can conclude $\bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge B_1^{'} \wedge ... \wedge B_{m'}^{'} \wedge (E \vee F) \vdash_r A$ and hence

$$\bar{B}_1, ..., \bar{B}_m, B_1^{'}, ..., B_{m'}^{'}, E \vee F \vdash_r A \tag{15}$$

But this is a contradiction, since $E \vee F \in \Pi$. So Π is a prime Π-theory.

Finally let $E \in \Pi$ be an implication. We need to show that for all B, $B \to E \in \Pi$. Let $B \to E \notin \Pi$, then there exist $\bar{B}_1, ..., \bar{B}_m \in \Delta_n$, such that

$$\bar{B}_1, ..., \bar{B}_m, B \to E \vdash_r A \tag{16}$$

So, by (16) we have $\bar{B}_1, ..., \bar{B}_m, E \vdash_r A$, since from E we can derive $B \to E$. But this is a contradiction, hence $B \to E \in \Pi$. □

Theorem 9. (Completeness Theorem) $\Sigma \vdash_r A$ if and only if $\Sigma \vDash_r A$.

Proof. $\Rightarrow$: Suppose $\Sigma \vdash_r A$. We use induction on the length of the derivation of A from Σ to prove that $\Sigma \vDash_r A$. We only check one case.

Let A be an implication and $\Sigma \vDash_r A$. We want to show that for all formulas B, $\Sigma \vDash_r B \to A$. Let $M = \langle W, g, R, \Vdash \rangle$ and $M, g \Vdash \Sigma$. By assumption $M, g \Vdash A$, the root g is omniscient and A is an implication formula. Therefore for all $v \in W$, we have $M, v \Vdash A$, so $M, g \Vdash B \to A$.

$\Leftarrow$: Let $\Sigma \nvdash_r A$. By Lemma 5 there is a prime Π-theory, $\Pi \supseteq \Sigma$ such that $A \notin \Pi$. So, in the canonical model $M^{\Pi} = \langle W_{\Pi}, \Pi, R, \Vdash \rangle$, $M^{\Pi}, \Pi \Vdash \Sigma$ and $M^{\Pi}, \Pi \nVdash A$, since $A \notin \Pi$. So $\Sigma \nvDash_r A$. □

3.3 Π-provability and stronger logics

In this subsection we strengthen the conditions on Π to ensure that it satisfies the full a fortiori rule.

Definition 13. *Π is a **fully adequate theory** if Π is an adequate theory containing no formulas without implications.*

A fully adequate theory can be said to make no purely local statements.

Lemma 6. *A fully adequate theory Π is closed under the (unrestricted) a fortiori rule.*

Proof. We have to prove $A \in \Pi \Rightarrow D \to A \in \Pi$. We prove it by induction on the complexity of A. Note that the base case is that A is an implication. Statements without implication are not in Π.

If $A \in \Pi$ is an implication, then $D \to A \in \Pi$ by assumption.

If $A \in \Pi$ is $B \wedge C$, then, by axioms 3 and 4 and modus ponens $B \in \Pi$ and $C \in \Pi$. By induction hypothesis, $D \to B \in \Pi$ and $D \to C \in \Pi$. Then, by the conjunction rule, axiom 9 and modus ponens, $D \to B \wedge C \in \Pi$.

If $A \in \Pi$ is $B \vee C$, then, since Π is prime, $B \in \Pi$ or $C \in \Pi$. By induction hypothesis $D \to B \in \Pi$ or $D \to C \in \Pi$, and by axiom 1 or 2, axiom 8 and modus ponens, $D \to B \vee C \in \Pi$. □

Theorem 10. *If Π is a fully adequate theory and $\Sigma \nvdash_\Pi D$, then there is a prime Π-theory $\Delta \supseteq \Sigma$ such that $D \notin \Delta$.*

Proof. Enumerate all formulas, with infinitely many repetitions: $B_0, B_1, ...$ and define

$\Delta_0 = \Sigma \cup \Pi_\to \cup \mathsf{F}$,
$\Delta_{n+1} = \Delta_n \cup \{B_n\}$ if for no $\bar{B}_1, ..., \bar{B}_m \in \Delta_n$, $\vdash_\Pi \bar{B}_1 \wedge \bar{B}_m \wedge ... \wedge B_n \to D$,
$\Delta_{n+1} = \Delta_n$ otherwise.

Take Δ to be the union of all Δ_n. By assumption $D \notin \Delta_0$ and also we have $\vdash_\Pi D \to D$, so $D \notin \Delta$.

We show that Δ is a prime Π-theory. This simply goes exactly as in the proof of Theorem 6 and 3, Π has all the relevant properties of F that were used in these proofs. □

Theorem 11. *If Π is a fully adequate theory then $\Sigma \vdash_\Pi A$ if and only if for all Γ in the Π-Canonical model M^Π, if $\Gamma \Vdash \Sigma$ then $\Gamma \Vdash A$.*

Proof. Left to right is easy by induction on the length of the proof. The other direction follows by Theorem 10. □

Definition 14. *We define $\Delta \vDash_\Pi A$ iff for all $M = \langle W, g, R, V \rangle$ such that $M, g \Vdash \Pi$, and all $w \in W$, if $M, w \Vdash \Delta$, then $M, w \Vdash A$.*

This now allows us to state a very general completeness theorem.

Theorem 12. **(Π-completeness theorem)** If Π is a fully adequate theory, then $\Delta \vdash_{\Pi} A \;\Leftrightarrow\; \Delta \vDash_{\Pi} A$.

Proof. Left to right is easy, the other direction follows by Theorem 11. □

This theorem can be applied to any logic extending F as long as it has the rules of modus ponens, conjunction and the a fortiori rule. Of course, a logic will usually be closed under substitution but there is no need for this. To get useful completeness theorems we of course will have to prove that the canonical model of the logic has the desired properties. We just consider the case of Basic Propositional Calculus (BPC).

BPC is interpreted in Kripke models similarly to intuitionistic propositional logic except that the accessibility relation is not necessarily reflexive. Suzuki and Ono [6] introdused a Hilbert style proof system for BPC. Their axiomatization is an extension of the logic F with the axioms, $A \to (B \to A)$, $f \to A$ and $A \to (B \to A \wedge B)$.

We show that we do not need $A \to (B \to A \wedge B)$ in BPC, because it follows from $\mathsf{F} + (A \to (B \to A))$ as follows.

1. $\vdash A \to (B \to A)$
2. $\vdash B \to B$
3. $\vdash A \to (B \to B)$ Follows from 2 using rule 12
4. $\vdash (A \to (B \to A)) \wedge (A \to (B \to B))$ Follows from 1,3 using rule 5
5. $\vdash A \to (B \to A) \wedge (B \to B)$ Follows from 4 using axiom 9
6. $\vdash (B \to A) \wedge (B \to B) \to (B \to A \wedge B)$
7. $\vdash A \to (B \to A \wedge B)$ Follows from 5,6 using axiom 8

The a fortiori rule follows from the axiom $A \to (B \to A)$ and modus ponens. In reasoning without assumptions the conjunction rules superfluous because $A \to (B \to A \wedge B)$ is provable in BPC. But in reasoning with assumptions we do not seem to be able to do without it.

The next lemma is due Restall [4] and Corsi [1].

Lemma 7. *Let $M^{\Pi} = \langle W^{\Pi}, \Pi, R, \Vdash \rangle$ be the Π-canonical model for some Π containing $A \to (B \to A)$ for all A, B. Then the relation R is transitive and satisfies preservation of truth.*

Proof. First we will show that R is transitive. Let Γ, Δ and Σ are in W^{Π} and let $\Gamma R \Delta$ and $\Delta R \Sigma$, we want to prove that $\Gamma R \Sigma$. So let $A \to B \in \Gamma$ and $A \in \Sigma$. We have $(A \to B) \to (\top \to (A \to B)) \in \Pi$. So by definition of Π-theory $\top \to (A \to B) \in \Gamma$. However $\top \in \Delta$ and $\Gamma R \Delta$, so by definition of R, $A \to B \in \Delta$. Again by definition of R, $B \in \Sigma$, since $A \in \Sigma$ and $\Delta R \Sigma$. That is, R is transitive.

Now we will show that $\Vdash$ preserves truth in the Π-canonical model. Assume that $A \in \Gamma$ and $\Gamma R \Delta$. As Δ is nonempty, there is a $B \in \Delta$. The assumption gives $B \to A \in \Gamma$ (since $A \to (B \to A) \in \Pi$), and so $A \in \Delta$. □

Theorem 13. **(Completeness Theorem for BPC)** $\Sigma \vdash_{\mathsf{BPC}} A \Leftrightarrow \Sigma \vDash_{\mathsf{BPC}} A$.

Proof. Immediate by Lemma 7 and Theorem 12. □

4 Relation of F to Intuitionistic Propositional Logic

In this section, we prove conservativity results for IPC over F and over BPC. This clarifies what part of IPC these systems can prove.

4.1 Conservativity results for IPC over F

We will provide two classes of formulas with respect to which IPC is conservative over F, the class of simple implications and the class of basic implications.

Definition 15. *Let us call a formula $A \to B$ with A and B containing only $\wedge$ and $\vee$ a **simple implication**, and a formula that is obtained by applying only $\wedge$ and $\vee$ to simple implications a **basic formula**. Finally a formula $A \to B$ with A and B basic formulas is a **basic implication**.*

Theorem 14. *If $\vdash_{\mathsf{F}} A \leftrightarrow B$, then $\vdash_{\mathsf{F}} E\,[A/p] \leftrightarrow E\,[B/p]$, where p is an atom.*

Proof. The proof is easy by induction on E. We only check the implication case. Let $E = C \to D$. Then, by induction hypothesis, we have:

$$\vdash_{\mathsf{F}} C\,[A/p] \to C\,[B/p]$$

$$\vdash_{\mathsf{F}} D\,[B/p] \to D\,[A/p]$$

By Proposition 2, we can conclude that

$$\vdash_{\mathsf{F}} (C\,[B/p] \to D\,[B/p]) \to (C\,[A/p] \to D\,[A/p]).$$

The other direction is the same. So we can conclude that

$$\vdash_{\mathsf{F}} (C\,[A/p] \to D\,[A/p]) \leftrightarrow (C\,[B/p] \to D\,[B/p]).$$

That is,

$$\vdash_{\mathsf{F}} E\,[A/p] \leftrightarrow E\,[B/p]\,.$$

□

Theorem 15. *Let A be a formula such that A is constructed by applying only $\wedge$ and $\vee$ to formulas from a class Θ. Then there are formulas $A^{'}, A^{''}$ such that*

1. *$\vdash A \leftrightarrow A^{'}$ and $A^{'}$ is a disjunction of conjunctions of formulas in Θ.*
2. *$\vdash A \leftrightarrow A^{''}$ and $A^{''}$ is a conjunction of disjunctions of formulas in Θ.*

Proof. The proof is straightforward. □

We will apply Theorem 15 to Θ as the class of atoms, and as the class of very simple implications. Now by the previous theorems, a simple implication $A \to B$ can be replaced by an F- and IPC-equivalent $A^{'} \to B^{'}$ such that $A^{'}$ is a disjunction of conjunctions and $B^{'}$ is a conjunction of disjunctions.

Lemma 8. *For all formulas $A_i, 1 \leq i \leq k$ and $B_i, 1 \leq j \leq m$ we have,*

$$\vdash A_1 \vee ... \vee A_K \to B_1 \wedge ... \wedge B_m$$

$$\text{iff}$$

$$\vdash A_i \to B_j \ \text{ for all i, j,}$$

where $\vdash$ can be read as $\vdash_{\mathsf{F}}$ as well as $\vdash_{\mathsf{IPC}}$.

Proof. Easy. □

Definition 16. *A formula $A \to B$ called a **very simple implication** if A is conjunction of atoms and B is disjunction of atoms. A formula $A \to B$ is called **very basic implication** if A is conjunction of very simple implications and B is disjunction of very simple implications.*

By the previous lemma we can conclude that to show that IPC is conservative over F with respect to simple implications and basic implications it is sufficient to do so for very simple implications and very basic implications. We can do so now for very simple implications, and in fact even for CPC instead of IPC.

Theorem 16. *If CPC proves a very simple implication (and a fortiori if IPC does), then F proves it as well.*

Proof. Let $A \to B$ is a very simple implication, so $A = \bigwedge_i (P_i)$ and $B = \bigvee_j (q_j)$. Assume $\nvdash_{\mathsf{F}} A \to B$. Then by the completeness theorem there exists a root subintuitionistic model M and $w \in M$, such that $M, w \nVdash A \to B$. So there exists $v \in M$, such that $M, v \Vdash A$ and $M, v \nVdash B$. Now we select this point v from M and then we make the one point CPC model $M_{\mathsf{CPC}} = \langle v, (v, v), \vDash \rangle$ such that for all propositional variables p, $M_{\mathsf{CPC}}, v \vDash p$ if and only if $M, v \Vdash p$. Clearly

$$M_{\mathsf{CPC}}, v \vDash p_i, \ \textit{for all } i$$

$$M_{\mathsf{CPC}}, v \nvDash q_j, \ \textit{for all } j$$

That is $M_{\mathsf{CPC}}, v \nvDash A \to B$, so $\mathsf{CPC} \nvDash A \to B$. □

Up to now CPC (classical logic) did just as well as IPC, but to restrict the class of very basic implications further we need disjunction properties only available in (sub)intuitionistic logics. We need the slash | for this purpose also under assumptions. The following both applies if $\vdash$ is read as $\vdash_{\mathsf{F}}$ and as $\vdash_{\mathsf{IPC}}$. Similarly for the | defined in terms of $\vdash$.

Definition 17. *Let Γ be a set of formulas. We define the slash $\Gamma | A$ inductively on the structure of A as follows*

1. $\Gamma \mid p$ *iff* $\Gamma \vdash p$,
2. $\Gamma | A \wedge B$ *iff* $\Gamma | A$ *and* $\Gamma | B$,
3. $\Gamma | A \vee B$ *iff* $\Gamma | A$ *or* $\Gamma | B$,
4. $\Gamma | A \to B$ *iff* $\Gamma \vdash A \to B$ *and (if* $\Gamma | A$ *then* $\Gamma | B$*).*

Theorem 17. [2] If $\Gamma|A$ for all $A \in \Gamma$, then $(\Gamma|B \Leftrightarrow \Gamma \vdash B)$.

Proof. As in [2]. □

Theorem 18. [2] If $\Gamma|C$ for all $C \in \Gamma$ and $\Gamma \vdash A \vee B$, then $\Gamma \vdash A$ or $\Gamma \vdash B$.

Proof. Let $\Gamma \vdash A \vee B$. By Theorem 17, $\Gamma|A \vee B$. So $\Gamma|A$ or $\Gamma|B$. Again by Theorem 17, $\Gamma \vdash A$ or $\Gamma \vdash B$. □

Lemma 9. *If $A = A_1 \wedge ... \wedge A_k$, such that for all $1 \leq i \leq k$, A_i is a very simple implication, then $A|A$. Similarly if, for all $1 \leq i \leq k$, A_i is an atom.*

Proof. By assumption for all $1 \leq i \leq n$, $A_i = B_i \to C_i$. Clearly, $A_1 \wedge ... \wedge A_k \vdash B_i \to C_i$. We have $A_1 \wedge ... \wedge A_k \nvdash B_i$, because we can make a model M such that $M \Vdash A_1 \wedge ... \wedge A_k$ and $M \nVdash B_i$ (we make all atoms false). So $A_1 \wedge ... \wedge A_k \nmid B_i$ and therefore $A_1 \wedge ... \wedge A_k|B_i \to C_i$. So, $A_1 \wedge ... \wedge A_k|A_1 \wedge ... \wedge A_k$. □

Theorem 19. *For arbitrary A, D, if $A|A$ and $\vdash A \to D$, then $A|D$.*

Proof. Let $\Gamma = \{A\}$. Then by Theorem 17, $\Gamma \vdash A$, and by assumption $\vdash A \to D$. So, $\Gamma \vdash D$. Again by Theorem 17, $\Gamma|D$. That is $A|D$. □

Lemma 10. *If $A|A$, then $\vdash A \to E \vee C \ \Leftrightarrow\ \vdash A \to E$ or $\vdash A \to C$, for both* F *and* IPC.

Proof. By Theorem 19, we have $A|E \vee C$, so $A|E$ or $A|C$. By Theorem 17, $A \vdash E$ or $A \vdash C$. Therefore by the weak Deduction Theorem we conclude that $\vdash A \to E$ or $\vdash A \to C$. By using F rules and axioms the other direction is easy. □

Definition 18. *A very basic implication $A \to B$ is an* ***extremely basic implication*** *if B is a sole very simple implication.*

By Lemma 8 and 10, we can conclude that to show that IPC is conservative over F with respect to basic implications it is sufficient to do so for extremely basic implications.

Theorem 20. IPC *is conservative over* F *with respect to basic implications.*

Proof. By the above, we can assume $\nvdash_{\mathsf{F}} A$ with A an extremely basic implication:

$$A = (A_1 \to B_1) \wedge ... \wedge (A_n \to B_n) \to (C \to D).$$

Then there exist $M, w \in M$ such that

$$M, w \nVdash (A_1 \to B_1) \wedge ... \wedge (A_n \to B_n) \to (C \to D)$$

So, there exists $v \in M$ with wRv and $M, v \Vdash A_i \to B_i$ for each $1 \leq i \leq n$ and

$$M, v \nVdash C \to D$$

So, there exists vRu with $M, u \Vdash C$ and $M, u \nVdash D$, and, if $M, u \Vdash A_i$ then $M, u \Vdash B_i$ for each i.

Now we select the point u from M and then we make the one point model $M_{\mathsf{IPC}} = \langle u, (u, u), \Vdash^I \rangle$, such that for all propositional variables p, $M_{\mathsf{IPC}}, u \Vdash^I p$ if and only if $M, u \Vdash p$. We will show that

$$M_{\mathsf{IPC}}, u \nVdash^I (A_1 \to B_1) \wedge ... \wedge (A_n \to B_n) \to (C \to D).$$

It is easy to see that for any conjunction or disjunction E of atoms $M_{\mathsf{IPC}}, u \Vdash^I E$ iff $M, u \Vdash E$. This applies to each of the A_i, B_i, C and D. So, $M_{\mathsf{IPC}}, u \Vdash^I C$, $M_{\mathsf{IPC}}, u \nVdash^I D$, and if $M_{\mathsf{IPC}}, u \Vdash^I A_i$, then $M_{\mathsf{IPC}}, u \Vdash^I B_i$. So, $M_{\mathsf{IPC}}, u \nVdash^I (A_1 \to B_1) \wedge ... \wedge (A_n \to B_n) \to (C \to D)$. Therefore $\mathsf{IPC} \nvdash A$. □

To see that the conservativity result does not apply to CPC just note that the formula $(p \to q \vee r) \to (p \to q) \vee (p \to r)$ is a very basic implication which is not provable in IPC or F, but is provable in CPC.

The conservativity result for IPC over F can be extended to conjunctions and disjunctions of simple implications and basic implications. In the case of simple implications CPC can no longer take the role of IPC however as the following example shows: $\vdash_{\mathsf{CPC}} (p \to q) \vee (q \to p)$, but $\nvdash_{\mathsf{IPC}} (p \to q) \vee (q \to p)$

Finally it is important to note that the result cannot be extended by mixing propositional variables and implications. In F even such a simple IPC-theorem as $p \wedge (p \to q) \to q$ cannot be proved.

4.2 A conservativity result for IPC over BPC

Of course, the above conservativity results apply to stronger logics than F, but for BPC we can prove an additional theorem. In this subsection we give a formal definition of NNIL formulas [8] and we will prove that IPC is conservative over BPC with respect to NNIL formulas.

Definition 19. *The smallest class satisfying the following clauses is called* NNIL.

1. *All propositional variables are in* NNIL,
2. *if* $A, B \in \mathsf{NNIL}$ *then* $A \wedge B \in \mathsf{NNIL}$,
3. *if* $A, B \in \mathsf{NNIL}$ *then* $A \vee B \in \mathsf{NNIL}$,
4. *if* $A \in \mathsf{NNIL}$ *and* B *does not contain implications, then* $B \to A \in \mathsf{NNIL}$.

Definition 20. *The smallest class satisfying the following is the class of normal* NNIL *formulas,*

1. *All propositional variables are in normal* NNIL,
2. *if* A, B *is in normal* NNIL, *then* $A \wedge B$ *is in normal* NNIL,
3. *if* A, B *is in normal* NNIL, *then* $A \vee B$ *is in normal* NNIL,
4. *if* A *is in normal* NNIL *and* B *is conjunction of atoms then* $B \to A$ *is in normal* NNIL.

In F we have that $A \vee B \to C$ is equivalent to $(A \to C) \wedge (B \to C)$, so any NNIL formula is provably equivalent to a normal NNIL formula.

Theorem 21. *If* $A \in \mathsf{NNIL}$ *and* $\vdash_{\mathsf{IPC}} A$, *then* $\vdash_{\mathsf{BPC}} A$.

Proof. Let $M = \langle W, R, V\rangle$ be a model for Basic Logic BPC. Then we define the intuitionistic model $\bar{M} = \langle W, \bar{R}, V\rangle$, by $\bar{R} = R \cup \{(w, w) \mid w \in W\}$. By induction on the complexity of $A \in$ NNIL, we will show that for all $w \in W$, if $M, w \nVdash_{\mathsf{BPC}} A$, then $\bar{M}, w \nVdash_{\mathsf{IPC}} A$. We only check implication cases, the other cases are easy. Let $A = \wedge p_i \to C$ and $M, w \nVdash_B \wedge p_i \to C$, then there exist $v \in W$ such that, wRv and $M, v \Vdash_B \wedge p_i$, $M, v \nVdash_B C$. Then $\bar{M}, v \Vdash_{\mathsf{IPC}} \wedge p_i$ and by induction hypothesis $\bar{M}, v \nVdash_{\mathsf{IPC}} C$. We know $v\bar{R}v$, so $\bar{M}, v \nVdash_{\mathsf{IPC}} \wedge p_i \to C$. We can conclude that $\bar{M}, w \nVdash_{\mathsf{IPC}} \wedge p_i \to C$, since we have preservation and wRv.

Now, assume $\nvdash_{\mathsf{BPC}} A$. Then by the completeness theorem for BPC there exists a BPC-model M and $w \in M$, such that $M, w \nVdash_B A$, so $\bar{M}, w \nVdash_{\mathsf{IPC}} A$. Then, by soundness $\nvdash_{\mathsf{IPC}} A$. □

Finally, returning to the example $\nvdash_{\mathsf{F}} p \wedge (p \to q) \to q$, BPC is still not able to prove this formula: even in the case of BPC we cannot mix implications and atoms in the conservativity result.

5 Conclusion

We developed the subintuitionistic logics introduced by Corsi and Restall in a uniform manner. Proof systems for Corsi's basic system F are given for derivations without and with assumptions, and completeness theorems are proved, clarifying the role of the rules of modus ponens, conjunction and a fortiori. Similarly for Restall's notion of proof from a theory Π. This is then applied to obtain completeness for extensions of F. Two classes of formulas are then introduced, simple implications and basic implications, and our results are used to give a conservation theorem for intuitionistic logic IPC over Corsi's system F with respect to these two classes of formulas. For Visser's basic logic strong completeness is proved and an additional conservation results is given with respect to the class of NNIL formulas.

Our methods have been fruitfully used in investigations of a weaker logic WF characterized by neighborhood models [3]. The conservativity result applies in that case only to the simple implications, clearly showing the difference in strength of the two logics.

References

1. G. Corsi, Weak Logics with strict implication, *Zeitschrift fur Mathematische Logik und Grundlagen der Mathematic*, 33:389-406, 1987.
2. S. C. Kleene, Disjunction and existence under implication in elementary intuitionistic formalisms. *Journal of Symbolic Logic*, 27:11-18, 1962. addndum ibidem 28,154-156.
3. D. de Jongh and F. Shirmohammadzadeh Maleki, Weak Subintuitionistic Logics, in preparation.
4. G. Restall, Subintuitionistic Logics, *Notre Dame Journal of Formal Logic*, Volume 35, Number 1, Winter 1994.

5. K. Sano, M. Ma, Alternative semantics for Visser's propositional logics. M. Aher et al (Eds.): *10th International Tbilisi Symposium on Logic, Language, and Computation, TbiLLC 2013,* pp. 257-275, 2015, LNCS 8984, Springer.
6. Y. Suzuki, H. Ono, Hilbert-style proof system for BPL. *Technical report, IS-RR-97-0040F* Japan Advanced Institute of Science and Technology (1997)
7. A. Visser, A propositional logic with explicit fixed points, *Studia Logica*, 40, 2 (1981): 155-175.
8. A. Visser, J. Van Benthem, D. de Jongh and G. Renardel de Lavalette. NNIL, a study in intuitionistic propositional Logic. In A. Ponse, M. de Rijke, and Y. Venema, editors, *Modal Logics and process Algebra, a bisimulation perspective*, page 289-326, 1995