The Johansson/Heyting letters and the birth of minimal logic

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Abstract

In 1935 and 1936 Johansson and Heyting exchanged a series of letters. This exchange inspired Johansson to develop his minimal logic which he ultimately published in an article in 1937. This report summarises Johansson's article and letters, and discusses a number of interesting details of the letters.

Introduction

In 1937 the article '*Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus*' by the Norwegian mathematician Ingebrigt Johansson (1904–1987) appeared in the *Compositio Mathematica*. In this article Johansson introduced his minimal logic, a weakened version of the intuitionistic formal system developed by Arend Heyting (1898–1980) in [1930]. Johansson's article was the result of a series of letters exchanged between Johansson and Heyting in 1935 and 1936. In this report I will summarise Johansson's article and letters, and discuss a number of interesting details of the letters.

Johansson wrote Heyting six letters and one postcard. The first letter was dated 29 August 1935; the postcard, completing the series, was dated 28 January 1936. While Johansson's writings to Heyting have been preserved, those of Heyting's, unfortunately, have not. The content of Heyting's letters can be inferred only from a carbon copy of one of Heyting's letters, from the few notes Heyting scribbled on Johansson's letters, and from citations and other references in Johansson's letters.

In this report a formula is called *negative* if the logical constant $\neg$ occurs in it, and *positive* if it is not negative.

Summary of Johansson's article

In his article Johansson claims that among Heyting's axioms there are two dubious ones. These are

$$b \to (a \to b) \tag{2.14}$$

$$\neg a \to (a \to b). \tag{4.1}$$

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Axiom 2.14 is dubious because it is not evident that $a \to b$ is implied by the truth of b. Similarly, in the case of Axiom 4.1 it is not evident that $a \to b$ is implied by the absurdity of a. Axiom 4.1 is especially problematic because, according to Johansson, it constitutes a considerable stretch of the implication relation. As Johansson writes in his second letter to Heyting, Axiom 4.1 'says that once $\neg a$ has been proved, b follows from a, even if this had not been the case before'. Deeming Axiom 4.1 unacceptable, Johansson proposes to remove it from Heyting's system. The resulting system Johansson calls 'minimal logic' (*Minimalkalkül*).

In § 2, Johansson points out that every positive formula provable in Heyting's system may be proved in the exact same way in minimal logic. In the case of negative formulas, however, there are important differences. Here minimal logic, because of its rejection of Axiom 4.1, has at its disposal only Heyting's second axiom for negation:

$$((a \to b) \wedge (a \to \neg b)) \to \neg a. \tag{4.11}$$

As a result, proofs of several of Heyting's negative formulas are not valid in minimal logic. Some of these formulas can be proved in a different way. Nine formulas turn out to be unprovable. However, for eight of these, weaker variants can be proved. Of course, Axiom 4.1 is among the unprovable formulas.

The avoidance of Axiom 4.1, however, comes at a considerable price. In § 3, Johansson shows that the formula

$$((a \wedge \neg a) \vee b) \to b \tag{4.41}$$

is no longer provable. The two weaker variants that are,

$$\begin{aligned} &((a \wedge \neg a) \vee b) \to \neg\neg b \\ &((a \wedge \neg a) \vee \neg b) \to \neg b, \end{aligned}$$

are not sufficient. The provability of Formula 4.41 in minimal logic is a desideratum because it stems from the disjunction property. The disjunction property is a property shared by all the usual intuitionistic formal systems. It states that if we can produce a proof of $a \vee b$, then we can also produce a proof of a or a proof of b. So, if $(a \wedge \neg a) \vee b$ (the antecedent of 4.41) has been proved, then, by the disjunction property, $a \wedge \neg a$ is provable or b is. In a consistent system like minimal logic $a \wedge \neg a$ is not provable. Therefore, b (the consequent of 4.41) must be provable. This indicates that Formula 4.41 should hold in minimal logic.

As Formula 4.41 is not derivable from the axioms, it may seem plausible to add it as a new axiom. But this does not help either, because doing so would open the door to a proof of Axiom 4.1 – the very formula Johansson set out to avoid.

Johansson presents an interesting solution to this problem. He proposes to replace Formula 4.41 with the proof schema (*Schlußschema*)

$$\frac{b \vee (a \wedge \neg a)}{b.} \tag{7}$$

This schema means that if we have a proof of $b \vee (a \wedge \neg a)$, then we can transform that into a proof of b. Every formula that is provable with the schema also is provable without it. Johansson keenly notes that the schema is not part of the system; rather, it expresses a property of the system. At the end of the article Johansson, using the sequent calculus, proves the correctness of the schema through what he notes are considerations *about* the system. Schema 7 thus in fact is an admissible rule *avant la lettre*. It is quite remarkable that Johansson had developed this insight twenty years before admissible rules were introduced as a proper object of study by Paul Lorenzen [1955].

Having Schema 7 at his disposal, Johansson is able to derive proof schemata for other formulas that are unprovable yet desired. For instance, the schema

$$\frac{(a \vee b) \wedge \neg a}{b} \tag{8}$$

serves as a subsitute for the formula

$$((a \vee b) \wedge \neg a) \rightarrow b. \tag{4.42}$$

In § 4, Johansson describes an alternative way to construct minimal logic. To this end, he introduces $\curlywedge$ as a primitive symbol, representing an arbitrary contradiction, and uses it to define $\neg$ by

$$\neg a \equiv a \rightarrow \curlywedge. \tag{22}$$

Johansson notes that $\curlywedge$, when occurring in proofs, may be used just like any other ordinary formula variable – that is, a formula without any special properties. This way, the formula

$$((a \rightarrow b) \wedge (a \rightarrow (b \rightarrow \curlywedge))) \rightarrow (a \rightarrow \curlywedge)$$

can be derived from the positive axioms. By Definition 22 this formula is equivalent to Axiom 4.11. It thus follows that all formulas valid in minimal logic can be derived from the positive axioms and Definition 22.

More generally, the following can be proved. Let φ be a formula consisting only of $\rightarrow$, $\wedge$, $\vee$, $\neg$ and proposition variables, and let φ' be the result of substituting $\curlywedge$ for $\neg$ in φ in the way prescribed by Definition 22. Then φ is valid in minimal logic iff φ' is derivable from the positive axioms only. As a special case, from this it follows that every logically valid positive formula can be derived from the positive axioms alone.

Johansson remarks that $\curlywedge$ bears a resemblance with Kolmogorov's [1932] 'problem-solving' interpretation of intuitionistic logic. There, $\neg a$ is interpreted as the task to derive a contradiction given a solution of a. This parallels Definition 22. Johansson also shows that a problem-solving interpretation of minimal logic can be obtained by making two minor modifications in Kolmogorov's interpretation.

In § 5, following Gentzen [1935], Johansson constructs minimal versions of natural deduction and sequent calculus and shows they are equivalent to minimal logic as presented in the article. He goes on to prove cut elimination and the disjunction property for the minimal sequent calculus. Finally, Johansson shows how the disjunction property leads to a proof of Schema 7. It is notable that Johansson was one of the first to make essential use of Gentzen's results in order to prove properties about a logical system [Gentzen, 2016: 30].

Summary of Johansson's letters

In his first letter to Heyting, Johansson writes about his rejection of Axiom 4.1 and the equivalent formula

$$(a \wedge \neg a) \rightarrow b. \tag{4.4}$$

He recognises that the rejection of Axiom 4.1 implies the loss of Formula 4.42. It is interesting to note that in the article Johansson is concerned with the loss of Formula 4.41 instead. I shall return to this point later. Johansson describes how he is able to reconstruct a considerable part of Heyting's system without Axiom 4.1. Additionally, he notes that the weaker variants

$$((a \vee b) \wedge \neg a) \rightarrow \neg\neg b$$
$$((a \vee \neg b) \wedge \neg a) \rightarrow \neg b$$

of Formula 4.42, containing negative occurrences of b, can be proved. He ends his letter asking if Heyting considers it absolutely necessary to preserve Formula 4.42 for positive values of b.

The second letter contains a first, rough version of minimal logic. Johansson introduces a distinction between 'false' and 'absurd' such that in a consistent system the former follows from the latter. The expression $\sim a$ denotes that a is false. Several definitions follow, including

$$\curlywedge \equiv \exists x(x \wedge \sim x)$$
$$\neg a \equiv a \rightarrow \curlywedge.$$

In this system Johansson is able to derive the formulas

$$(a \rightarrow (b \wedge \neg b)) \rightarrow \neg a$$
$$(a \wedge \neg a) \rightarrow (b \vee \curlywedge).$$

The first of these two formulas derives the same formulas as Axiom 4.11 does and, in conjunction with the second formula, derives weaker variants of all of Heyting's negative formulas. In order to derive a from $a \vee \curlywedge$ and $\sim b$ from $\neg b$, Johansson introduces two proof schemata:

$$\frac{\begin{array}{l} a \vee \curlywedge \text{ proved} \\ \curlywedge \text{ unprovable} \end{array}}{a \text{ proved}} \qquad \frac{\begin{array}{l} a \rightarrow \curlywedge \text{ proved} \\ \curlywedge \text{ unprovable} \end{array}}{\sim a \text{ proved.}}$$

Note that the first schema is very similar to Schema 7. Thus, as Johansson notes, where Heyting proves $\varphi \rightarrow a$, Johansson merely proves $\varphi \rightarrow (a \vee \curlywedge)$, but in a consistent system Johansson, using his proof schema, is able to prove a in the end.

In the third letter, we learn that Heyting wrote that he interprets $a \rightarrow b$ as 'b follows from a, or a is absurd'. Johansson presents an objection to this interpretation. To this end, he introduces a stricter form of implication: $a \succ b$ denotes that b follows from a (but it does *not* denote that a is impossible). He then defines

$$a \rightarrow b \equiv (a \succ b) \vee \neg a$$

which he believes corresponds to Heyting's interpretation. (This belief turns out to be false. I shall return to this point later.) With this definition, as Johansson shows, modus ponens is no longer valid. Instead, at most $b \vee (a \wedge \neg a)$ can be derived from $a \to b$ and a, using the schema

$$\frac{\begin{array}{l} a \succ b \\ a \end{array}}{b.}$$

At the end of the letter a second version of his system appears. Now $\curlywedge$ is a primitive symbol rather than a defined one. The system is able to prove all formulas in Heyting's system that are provable without Axiom 4.1. Schema 7 also makes its first appearance. Johansson notes that with the help of this schema other schemata an be derived that serve as replacements for the unprovable formulas from Heyting's system. Furthermore, Johansson notes that when his system is used as the foundation for a certain discipline, $\curlywedge$ can no longer remain an undefined primitive symbol, but should denote the disjunction of all false formulas. Finally, Johansson describes how versions of Gentzen's sequence calculus and natural deduction for minimal logic can be obtained.

The fourth letter is concerned mostly with clarifications in response to questions from Heyting. Near the end, Johansson remarks that he does not mean to criticise Heyting's system. For given Heyting's interpretation of implication, the system is perfectly well in order. It is merely that Johansson prefers a stricter interpretation, and minimal logic shows that such an interpretation is tenable.

In the fifth letter, however, Johansson writes that he has found a problem with Heyting's interpretation of implication after all. Consider minimal logic as constructed using the $\succ$ implication introduced in the third letter. Then with the definition

$$a \to b \equiv (a \succ b) \vee (a \succ \curlywedge)$$

(which Johansson believes corresponds to Heyting's interpretation of implication) it is not possible to obtain Heyting's intuitionistic formal system from minimal logic, because the formulas

$$(a \wedge (a \to b)) \to b \tag{2.15}$$
$$((a \to c) \wedge (b \to c)) \to ((a \vee b) \to c) \tag{3.12}$$

cannot be derived. However, if the definition

$$a \to b \equiv a \succ (b \vee \curlywedge),$$

is used instead, then Heyting's system can be obtained in full. Furthermore, modus ponens is recoverable using the second definition of $\to$ together with the two schemata

$$\frac{\begin{array}{l} a \vee \curlywedge \text{ proved} \\ \curlywedge \text{ unprovable} \end{array}}{a \text{ proved}} \qquad \frac{\begin{array}{l} a \succ b \\ a \end{array}}{b}$$

introduced in the second and third letters, respectively.

Analogously, the $\forall$ quantifier must be replaced by a weaker variant $\forall'$, defined by

$$\forall' x\varphi(x) \equiv \forall x(\varphi(x) \vee \curlywedge).$$

This variant is necessary, because the proof schema

$$\frac{\begin{array}{l}\varphi \to \psi(x)\\ x \text{ does not occur free in } \varphi\end{array}}{\varphi \to \forall x\psi(x)} \qquad \text{i.e.} \qquad \frac{\begin{array}{l}\varphi \succ (\psi(x) \vee \curlywedge)\\ x \text{ does not occur free in } \varphi\end{array}}{\varphi \succ (\forall x\psi(x) \vee \curlywedge)}$$

is not valid. This is proved by the counterexample $\varphi = \forall x(\psi(x) \vee \curlywedge)$.

Johansson has no objections to this second definition of $\to$, although he regrets that this definition does not allow $\succ$ to be defined in terms of $\to$ – unlike $\neg$ and $\curlywedge$, which can be defined in terms of one another.

The sixth letter and the postcard, finally, are devoted primarily to practical details of the publication of the article.

The rejection of Axiom 4.1

Johansson was not the first to criticise the axiom

$$\neg a \to (a \to b) \tag{4.1}$$

which expresses that *ex falso quodlibet*. Already in 1925, Kolmogorov, when discussing Hilbert's axioms, criticises the slightly different but equivalent axiom $a \to (\neg a \to b)$ thus:

> Hilbert's first axiom of negation, 'Anything follows from the false', [. . .] does not and cannot have any intuitive foundation since it asserts something about the consequences of something impossible: we have to accept b if the true judgment a is regarded as false. Thus, Hilbert's first axiom of negation cannot be an axiom of the intuitionistic logic of judgments [. . .]. [Kolmogorov, 1925: 421]

However, Kolmogorov also notes that the axiom is

> used only in a symbolic presentation of the logic of judgments; therefore it is not affected by Brouwer's critique, especially since it has no intuitionistic foundation either. [Kolmogorov, 1925: 419]

G. F. C. Griss [1955], too, as part of his negationless mathematics, proposed to reject Axiom 4.1. Responding to Griss and other proponents of negationless mathematics, L. E. J. Brouwer [1948] presented an example of a negative property that cannot be tranformed into a constructive one. This example showed that negations are essential in intuitionistic mathematics.

A discrepancy between Johansson's letters and his article

As mentioned earlier, there is a slight inconsistency of sorts between Johansson's first letter to Heyting and his article. In the letter, Johansson is concerned about the loss of

$$((a \vee b) \wedge \neg a) \to b, \tag{4.42}$$

but in the article his concern is about the loss of

$$((a \wedge \neg a) \vee b) \rightarrow b) \tag{4.41}$$

instead. One can only speculate as to the reason for this change. Possibly the most straightforward explanation is that Johansson simply had a change of heart about which of the two formulas he considered more fundamental.

Another explanation for the change is more involved and has to do with the two proof schemata that replace Formulas 4.41 and 4.42, respectively. As we have seen, Formula 4.41 is substituted by the proof schema

$$\frac{b \vee (a \wedge \neg a)}{b} \tag{7}$$

and Formula 4.42 by

$$\frac{(a \vee b) \wedge \neg a}{b.} \tag{8}$$

As Johansson shows in the article, Schema 8 can be derived from Schema 7. It is not clear, however, whether the other direction is possible as well – that is to say, whether 7 is derivable from 8. Although it is decidable whether an admissible rule (or proof schema) is intuitionisticaly derivable, it is not decidable whether, given one admissible rule, another is intuitionistically derivable.

This, then, might have presented a problem to Johansson when he was preparing his article. One could speculate that Johansson, concerned about the loss of Formula 4.42 as he originally was, devised Schema 8. He then found himself unable to derive Schema 7 from it. After discovering that 8 is derivable 7, he decided to switch from 4.42 to 4.41 as the formula of prime concern. Again, this is mere speculation.

Heyting's interpretation of implication

Minimal logic is the result of Johansson's rejection of Heyting's axiom

$$\neg a \rightarrow (a \rightarrow b), \tag{4.1}$$

which in turn is the result of Johansson's stricter interpretation of implication. In his article, Johansson notes that generally the assertion $a \rightarrow b$ is considered to hold if (1) b is a logical consequence of a, (2) b is true, or (3) a is false. While Johansson is only mildly opposed to case 2, it is case 3 which he deems outright unacceptable [Johansson, 1937: 120].

Heyting, clearly, has a different view on implication. On the back of Johansson's second letter, Heyting wrote a summary of his reply. This summary contains the following passage:

> I take $a \rightarrow b$ to mean: to reduce the solution of b to that of a, or to show the impossibility of the solution of a. This interpretation is efficient [*doelmatig*].

Heyting's phrasing, however, turned out to be ambiguous. Johansson presents two formalisations of Heyting's interpretation, but it does not become entirely clear if any of these are accurate.

Johansson's first formalisation is found in his third and fifth letters. As discussed before, he first introduces a stricter form of implication: $a \succ b$ denotes that b follows (logically) from a. He then formalises what he believes is Heyting's interpretation of implication as

$$a \to b \equiv (a \succ b) \vee \neg a$$

or, equivalently,

$$a \to b \equiv (a \succ b) \vee (a \succ \curlywedge).$$

In his fifth letter, Johanssons presents his second formalisation:

$$a \to b \equiv a \succ (b \vee \curlywedge).$$

He then points out that the formulas

$$(a \wedge (a \to b)) \to b \tag{2.15}$$
$$((a \to c) \wedge (b \to c)) \to ((a \vee b) \to c) \tag{3.12}$$

are provable with the second formalisation, but not with the first.

If Johansson's first formalisation is incorrect, Heyting makes no effort to correct it in his reply to Johansson's third letter. Of this reply a carbon copy has been preserved. In it Heyting merely writes:

> I think we can come to terms on the interpretation of intuitionistic logic in spite of your strong criticism. Surely you will admit that my logical system holds if, firstly, implication is interpreted like I did in my previous letter [...].

However, it seems likely that Johansson's second formalisation is the one Heyting had in mind. Johansson seems to have embraced it implicitly before he learned about Heyting's interpretation. For already in his second letter, when laying out an earlier version of his logic, Johansson observes that 'wherever you prove, say, $(a \vee b) \wedge \neg b \to a$, I merely prove $(a \vee b) \wedge \neg b \to a \vee \curlywedge$'. This is also noted by Heyting, who wrote '$a \to b$ for me $=$ $a \to b \vee \curlywedge$ for Johansson' in the margin of that letter. This clearly points at the second formalisation.

The ambiguity in Heyting's interpretation might have arisen from Heyting's formulation in terms of solving problems. Perhaps the true intent of Heyting's interpretation becomes more obvious if we remember that 'the impossibility of the solution a' means that a is absurd; that is, from a follows a contradiction. Heyting's interpretation then could be rephrased as '$a \to b$ means to derive from a either b or a contradiction'. This phrase is not naturally expressed in terms of problem solving. Hence Heyting's phrasing.

Philosophical differences between Heyting and Johansson

There appear to be several philosophical differences between Heyting and Johansson. Heyting's view, of course, is very close to the original views of Brouwer. According to Brouwer, mathematics is an edifice constructed by man who is guided by his mathematical intuition. Mathematics, therefore, is an intellectual activity. As such, mathematics precedes, and hence is independent from, language and logic. Language, however, may be used afterwards to describe the process of mathematical construction, for instance in order to communicate mathematical results. Logic, in turn, studies these linguistic expressions of mathematical activity and abstracts logical laws and principles out of regularities and recurring patterns in mathematical language [Brouwer, 1907: 125–128]. This is how Heyting's formalisation of intuitionistic logic is to be regarded. As Heyting writes in the introduction to his formalisation of intuitionistic logic:

> Intuitionistic mathematics is a thought activity. Therefore, for intuitionistic mathematics, all languages, including formal ones, merely serve as a tool for communication. It is, as a matter of principle, impossible to construct a system of formulas that is equal to mathematics, because the possibilities of thought cannot be reduced to a finite number of rules that have been set up in advance [Heyting, 1930: 43].

Johansson's position is less clear. On the one hand, he clearly is concerned with intuitionistic principles and appears to subscribe to the intuitionistic school of thought. This is most evident in his rejection of Axiom 4.1. In his first letter to Heyting, Johansson writes that he 'had not expected to find [... Axiom 4.1] in intuitionism', because, as he writes in his second letter, it 'contradicts my intuition in the most profound way'. On the other hand, there are various occasions where Johansson's thoughts appear to align with those of David Hilbert instead. To Johansson, it seems, mathematics *is* founded by an axiomatised logic, although the choice of axioms would be guided by intuition. This is nicely illustrated by an exchange between Heyting and Johansson. Heyting writes:

> After all, we have the mathematical intuition at our disposal which allows us to build all of mathematics without any help of logic whatsoever. This way, sentences like 'π is a transcendental number' follow from the empty axiom system.

To this, Johansson replies that he agrees only 'if by "logic" you merely mean a specific formalism' (which Heyting does not). Johansson continues:

> I do not understand [... your example about the transcendence of π] if you mean to say
>
> $\vdash \pi$ is transcendental.
>
> However, if you mean to say
>
> Number axioms $\vdash \pi$ is transcendental,
>
> then I understand you completely.

Of course, Heyting does mean the first of the two readings. The second reading, which Johansson agrees to, is examplary of Hilbert's approach. There are other examples. In the same letter, Johansson writes:

> I probably could not have understood one bit of intuitionism if I had not found the accompanying formalisation. Even if all men share the same intuition, the meaning of words and expressions still would be different between them. I can appreciate the thoughts of another man only if I know the formal rules by which he uses words.

Furthermore, in one letter and in his article, Johansson discusses how minimal logic should be used to axiomatise other branches of mathematics. This again is a rather Hilbertian idea. Finally, consistency – another typical Hilbertian notion – is a recurring theme with Johansson. In his second letter, Johansson writes:

> In short, my position presently is as follows: derivations according to the rules of usual intiutionistic logic gives us the sentences that will be valid as soon as the consistency is proved.

Remarks like these lead Heyting to observe that ‘my logic is absolute, Johansson’s is relative (to an axiom system)’. Heyting’s logic is absolute, because it formalises intuitionistic mathematical reasoning.

Still, it seems Heyting’s position is sensitive to some criticism, too. His interpretation of implication being based on purpose rather than principle does not seem to be entirely reconcilible with intuitionism. Heyting fails to present convincing arguments for the intuitionistic acceptability of his interpretation. For instance, in his *Intuitionism: an introduction*, Heyting writes:

> You remember that $p \to q$ can be asserted only if we possess a construction which, joined with the construction p, would prove q. Now suppose that $\vdash \neg p$, that is, we have deduced a contradiction from the supposition that p were carried out. Then, in a sense this can be considered as a construction, which, joined to a proof of p (which cannot exist) leads to a proof of q. I shall interpret the implication in this wider sense [Heyting, 1956: 102].

Yet it remains unclear exactly how a construction of $\neg p$ and p would *lead* to a proof of q. It is this very question that is in need of explanation and justification. Perhaps this is the reason for Hao Wang to write:

> Heyting appears rather diffident about defending the inclusion of [... Axiom 4.1]. [...] Hence it is fair to say that, as a codification of Brouwer’s ideas, [... Johansson’s minimal logic] is no less reasonable than Heyting’s propositional calculus [Wang, 1977: 414].

Interpreting intuitionistic logic in minimal logic

As pointed out by Johansson [1937: 123], minimal predicate logic (MQC) is obtained by extending minimal propositional logic with the axioms

$$\forall x\varphi(x) \to \varphi(t)$$
$$\varphi(t) \to \exists x\varphi(x)$$

together with the inference rules

$$\frac{\varphi \to \psi(x) \quad x \text{ does not occur free in } \varphi}{\varphi \to \forall x\psi(x)} \qquad \frac{\varphi(x) \to \psi \quad x \text{ does not occur free in } \psi}{\exists x\varphi(x) \to \psi.}$$

Dick de Jongh observed that Johansson’s fifth letter contains the basis for an interpretation of intuitionistic predicate logic (IQC) in MQC. This translation can be made precise as follows. If p is an atom and φ and ψ are arbitrary formulas, then define the mapping * recursively by

$$
\begin{aligned}
p^* &= p \\
\bot^* &= \curlywedge \\
(\neg\varphi)^* &= \neg\varphi^* \\
(\varphi \vee \psi)^* &= \varphi^* \vee \psi^* \\
(\varphi \wedge \psi)^* &= \varphi^* \wedge \psi^* \\
(\varphi \to \psi)^* &= \varphi^* \to (\psi^* \vee \curlywedge) \\
(\exists x\varphi)^* &= \exists x\varphi^* \\
(\forall x\varphi)^* &= \forall x(\varphi^* \vee \curlywedge).
\end{aligned}
$$

Given this mapping, it holds that

$$\vdash_{\mathrm{IQC}} \varphi \Leftrightarrow\ \vdash_{\mathrm{MQC}} \varphi^*.$$

The proof of the right-to-left direction is trivial, because $\vdash_{\mathrm{IQC}} \varphi \leftrightarrow (\varphi \vee \curlywedge)$, so $\vdash_{\mathrm{IQC}} \varphi \leftrightarrow \varphi^*$. The left-to-right direction is proved by induction on the length of the proof of φ. The cases for the axioms are fairly straightforward. It will be useful first to prove

$$\vdash_{\mathrm{MQC}} (a \to b) \to (a \to (b \vee \curlywedge)). \tag{J}$$

This is proved by

1. $\vdash_{\mathrm{MQC}} (b \to c) \to ((a \to b) \to (a \to c))$ — Fml. 2.291
2. $\vdash_{\mathrm{MQC}} (b \to (b \vee \curlywedge)) \to ((a \to b) \to (a \to (b \vee \curlywedge)))$ — Subst.: 1
3. $\vdash_{\mathrm{MQC}} a \to (a \vee b)$ — Ax. 3.1
4. $\vdash_{\mathrm{MQC}} b \to (b \vee \curlywedge)$ — Subst.: 3
5. $\vdash_{\mathrm{MQC}} (a \to b) \to (a \to (b \vee \curlywedge))$ — Mod. pon.: 2, 4

Now, as a first example, a proof of the case of $b \to (a \to b)$ (Axiom 2.14):

1. $\vdash_{\mathrm{MQC}} (a \to b) \to ((b \to c) \to (a \to c))$ — Fml. 2.29
2. $\vdash_{\mathrm{MQC}} (b \to (a \to b)) \to (((a \to b) \to (a \to (b \to \curlywedge))) \to (b \to (a \to (b \vee \curlywedge))))$ — Subst.: 1
3. $\vdash_{\mathrm{MQC}} b \to (a \to b)$ — Ax. 2.14
4. $\vdash_{\mathrm{MQC}} ((a \to b) \to (a \to (b \to \curlywedge))) \to (b \to (a \to (b \vee \curlywedge)))$ — Mod. pon.: 2, 3
5. $\vdash_{\mathrm{MQC}} (a \to b) \to (a \to (b \vee \curlywedge))$ — Fml. J
6. $\vdash_{\mathrm{MQC}} b \to (a \to (b \vee \curlywedge))$ — Mod. pon.: 4, 5
7. $\vdash_{\mathrm{MQC}} (b \to (a \to (b \vee \curlywedge))) \to (b \to ((a \to (b \vee \curlywedge)) \vee \curlywedge))$ — Subst.: 5
8. $\vdash_{\mathrm{MQC}} b \to ((a \to (b \vee \curlywedge)) \vee \curlywedge)$ — Mod. pon.: 6, 7

As a second example, the proof of the case of axiom $\varphi(t) \to \exists x\varphi(x)$:

1. $\vdash_{\mathrm{MQC}} \varphi(t) \to \exists x\varphi(x)$ — Axiom
2. $\vdash_{\mathrm{MQC}} \varphi^*(t) \to \exists x\varphi^*(x)$ — Subst.: 1
3. $\vdash_{\mathrm{MQC}} (a \to b) \to (a \to (b \vee \curlywedge))$ — Fml. J
4. $\vdash_{\mathrm{MQC}} (\varphi^*(t) \to \exists x\varphi^*(x)) \to (\varphi^*(t) \to (\exists x\varphi^*(x) \vee \curlywedge))$ — Subst.: 3
5. $\varphi^*(t) \to (\exists x\varphi^*(x) \vee \curlywedge))$ — Mod. pon.: 2, 4

In the inductive step, one of three inference rules may have been applied. First, modus ponens. Assume that $\vdash_{\mathrm{IQC}} \varphi$ was derived using modus ponens. Then $\vdash_{\mathrm{IQC}} \psi$ and $\vdash_{\mathrm{IQC}} \psi \to \varphi$ were derived earlier in the proof. Hence, by the induction hypothesis, $\vdash_{\mathrm{MQC}} \psi^*$ and $\vdash_{\mathrm{MQC}} \psi^* \to (\varphi^* \vee \curlywedge)$. Therefore, by modus ponens, $\vdash_{\mathrm{MQC}} \varphi^* \vee \curlywedge$. Then, by the admissible rule

$$\frac{\vdash_{\mathrm{MQC}} \chi \vee \curlywedge}{\vdash_{\mathrm{MQC}} \chi}$$

it follows that $\vdash_{\mathrm{MQC}} \varphi^*$.

Second, the inference rule

$$\frac{\begin{array}{l}\varphi \to \psi(x) \\ x \text{ does not occur free in } \varphi\end{array}}{\varphi \to \forall x \psi(x).}$$

Assume $\vdash_{\mathrm{IQC}} \varphi$ was derived using this inference rule. Then there are ψ, χ and x such that $\varphi = \psi \to \forall x \chi(x)$, $\vdash_{\mathrm{IQC}} \psi \to \chi(x)$ was derived earlier in the proof, and x does not occur free in ψ. Hence, by induction hypothesis,

$$\vdash_{\mathrm{MQC}} \psi^* \to (\chi^*(x) \vee \curlywedge).$$

Then, by the same inference rule,

$$\vdash_{\mathrm{MQC}} \psi^* \to \forall x (\chi^*(x) \vee \curlywedge).$$

Finally, suitable use of Formula J yields

$$\vdash_{\mathrm{MQC}} \psi^* \to (\forall x (\chi^*(x) \vee \curlywedge) \vee \curlywedge)$$

which is equivalent to $\vdash_{\mathrm{MQC}} \varphi^*$.

Third, the inference rule

$$\frac{\begin{array}{l}\varphi(x) \to \psi \\ x \text{ does not occur in } \psi\end{array}}{\exists x \varphi(x) \to \psi.}$$

Assume $\vdash_{\mathrm{IQC}} \varphi$ was derived using this inference rule. Then there are ψ, χ and x such that $\varphi = \exists x \psi(x) \to \chi$, $\vdash_{\mathrm{IQC}} \psi(x) \to \chi$ was derived earlier in the proof, and x does not occur free in χ. Hence, by induction hypothesis,

$$\vdash_{\mathrm{MQC}} \psi^*(x) \to (\chi^* \vee \curlywedge).$$

Then, by the same inference rule,

$$\vdash_{\mathrm{MQC}} \exists x \psi^*(x) \to (\chi^* \vee \curlywedge)$$

which is equivalent to $\vdash_{\mathrm{MQC}} \varphi^*$. This completes the proof.

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