Atomless Varieties

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Abstract

We define a nontrivial variety of boolean algebras with operators such that every member of the variety is atomless. This shows that not every variety of boolean algebras with operators is generated by its atomic members, and thus establishes a strong incompleteness result in (multi-)modal logic.

Keywords Boolean algebras with operators, atoms, modal logic, incompleteness.

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1 Introduction

One of the basic negative results in the theory of modal logic is that not every modal logic is complete with respect to its class of Kripke frames, cf. THOMASON [6]. Algebraically, this means that not every variety of boolean algebras with operators (BAOs) is *complete*, that is, generated by those of its members that are complete, atomic and have completely additive operators, cf. GOLDBLATT [3]. For general context on modal logic and its algebraic side we refer to BLACKBURN, DE RIJKE & VENEMA [1].

One might wonder whether some weaker form of general completeness might hold. For instance, suppose that we drop the (second order) conditions of completeness and complete additivity from the requirements on the generating class of algebras. Will we find that every variety of BAOs is generated by its atomic members? The aim of this note is to show that the answer to this question is negative; in fact, we can prove something stronger:

Theorem 1 *There exists a nonempty variety* V *of boolean algebras with operators such that all members of* V *are atomless.*

In terms of modal logic, this implies there is a (multi-)modal logic which is incomplete with respect its class of *discrete* general frames (a general frame is called discrete, or atomic, if all singletons are admissible). Theorem 1 can thus be read as a strong incompleteness result in modal logic.

The key step in the *proof* of Theorem 1 is the construction of a particular BAO $\mathbb{A} = (A, \vee, -, \perp, (f_i)_{i\in I})$ and the definition of a unary term $\pi(x)$ such that the formula

$$\forall x\,(\perp < x \to \perp < \pi(x) < x) \qquad (\alpha)$$

holds in $\mathbb{A}$. The proof of Theorem 1 then proceeds through a fairly standard argument using the theory of discriminator varieties.

The ideas underlying our construction are similar to, yet obtained independently of, those in KRACHT & KOWALSKI [5]. The authors of this paper intend to define a variety of modal algebras containing no atomic members, but unfortunately, a number of errors makes it difficult to judge whether their approach might succeed or not. Observe that our Theorem 1, stating the existence of a variety in which every member is atomless rather than non-atomic, is a strengthening of the result announced in [5].

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2 The construction

In this section we give a step by step definition of the algebra $\mathbb{A}$. We start with an informal explanation of the basic idea.

The basic idea The starting point of our construction is a countable, atomless BA. Recall that all such algebras are isomorphic; we wrote 'a' countable, atomless BA since we make good use of a rather particular representation of this algebra.

Consider the sets $2^{<\omega}$ and 2^{ω} of finite (infinite, respectively) strings over the alphabet $2 = \{0, 1\}$. Given a finite string x, let $x\Uparrow$ be the collection of infinite strings that have x as an initial segment; sets of the form $x\Uparrow$ will be called *i-cones*. It is well-known (and not very difficult to show) that the collection Ω of finite unions of i-cones forms an atomless boolean algebra $\mathbb{O}$ under the standard set-theoretical operation.

Basically, our aim would be to define a map $H : \Omega \to \Omega$ such that for all non-empty $U \in \Omega$ we have that $\varnothing \subset H(U) \subset U$. The crucial idea underlying our construction is that if U is an i-cone, say $U = x\Uparrow$, we could define $H(x\Uparrow) = x0\Uparrow$. For a finite union $U = \bigcup_{x \in X} x\Uparrow$ we would like to define $H(U) = \bigcup_{x \in X} x0\Uparrow$ but the problem is that there may be *various* ways to represent U as a finite union of i-cones. This problem will be avoided by canonically choosing, for an element $U \in \Omega$, a set Z_U such that $U = \bigcup_{z \in Z_U} z\Uparrow$; this set Z_U will be called the *origin* of U. Given this, it is easy to show that the following map H, given by

$$H(U) = \bigcup_{z \in Z_U} z0\Uparrow,$$

indeed satisfies $\varnothing \subset H(U) \subset U$ for all non-empty $U \in \Omega$.

Hence, if we could expand $\mathbb{O}$ with a number of operators in such a way that the map H is term definable, we would have established our goal. Unfortunately, this problem seems to be very hard and may in fact be impossible to solve directly; the reason for this is that our definition of H crucially involves *finite* strings, and these are not directly 'available' in $\mathbb{O}$. A solution to this problem is to somehow *add* finite strings to the algebra.

In an earlier incarnation of this paper, we constructed a 'hybrid' boolean algebra $\mathbb{H}$ as a particular field of sets over the *disjoint union* of the sets $2^{<\omega}$ and 2^{ω}. More precisely, $\mathbb{H}$ itself was the direct product of the algebra $\mathbb{O}$ with the boolean algebra of the finite and cofinite sets of finite strings. Note however, that such an algebra cannot satisfy (α) since it has atoms; as a consequence, the algebra $\mathbb{H}$ could only be used to show the existence of varieties that do not contain atomic members. In order to prove that there are *atomless* varieties, we need to work with a boolean algebra that has no atoms at all.

The construction in the present paper is based on the *product* $W = 2^{<\omega} \times 2^{\omega}$ of the sets of finite and infinite strings, respectively. The domain of our algebra $\mathbb{A}$ will be the collection A of those subsets a of W of which each *slice* $a_s = \{\sigma \in 2^{\omega} \mid (s, \sigma) \in a\}$ is a finite union of i-cones. The boolean part of our algebra will thus be a countable, atomless subalgebra of $\mathcal{P}(W)$; the additional operations will be induced by some relations on the set W. As we will see in a moment, this particular representation of the countable, atomless boolean algebra will allow us to associate an admissible subset of W with each finite set of strings, and conversely.

Finally, in our product perspective, an arbitrary admissible set corresponds to a $2^{<\omega}$-indexed *family* of finite unions of i-cones, rather than to a *single* element of Ω. Can we still associate a *single* origin with an admissible set? The idea here is to find for a non-empty admissible set, the *first* non-empty slice — 'first', that is, in some fixed enumeration of $2^{<\omega}$.

Preliminary definitions Let $\alpha \leq \omega$ be an ordinal, and recall that 2 denotes the set $\{0,1\}$. Elements of the set 2^α are called *strings of length* α; we define $2^{<\alpha} = \bigcup_{\beta<\alpha} 2^\beta$, so $2^{<\omega}$ is the set of finite strings. In the sequel we use lower case roman letters (x, y, z, s, t, ...) to denote finite strings, and lower case greek letters (σ, τ, ...) for infinite strings. The *proper initial segment* relation $\prec$ on $2^{<\omega} \cup 2^\omega$ is a strict partial order on the set of all strings.

Given a finite string x we let, for $i \in \{0,1\}$, xi denote the string x extended with the symbol i; the length of x is denoted as $|x|$. We also assume that we have an enumeration $(s_n)_{n\in\omega}$ of the set of all finite strings. (To be concrete, we could define s_0 as the empty string ϵ, and let s_{n+1} be the string denoting n in binary notation.) This means that any set of finite strings has a *first* element.

An *i-cone* is a set of the form $x\Uparrow = \{y \in 2^\omega \mid x \prec y\}$; Ω denotes the collection of finite unions of i-cones. In order to find a canonical representation of an arbitrary element of Ω, first observe that (i) for any two finite strings x and y we have $x \prec y$ iff $y\Uparrow \subset x\Uparrow$, and (ii) for any finite string x, it holds that $x0\Uparrow \cup x1\Uparrow = x\Uparrow$. Using these two observations we can, given a representation of an element $a \in \Omega$ as $a = \bigcup_{x\in X} x\Uparrow$, step by step simplify the index set X, until we obtain a representation of a in the form $a = \bigcup_{z\in Z} z\Uparrow$ where Z is minimal in the sense that $y\Uparrow \subseteq a$ for *no* proper initial segment y of any string $z \in Z$. It can be shown that such a set Z is uniquely determined by a; we call it the *origin* of a, notation: Z_U. The canonical representation of an element $U \in \Omega$ is defined through this origin of U: simply write $U = \bigcup_{z\in Z_U} z\Uparrow$. For future reference we mention the following fact:

$$\text{for all nonempty } U \in \Omega\text{:} \bigcup_{z\in Z_U} z0\Uparrow \text{ is a \textit{proper}, nonempty subset of } U. \tag{1}$$

The boolean part We define W as the set $2^{<\omega} \times 2^\omega$. Given a subset $a \subseteq W$ and a finite string s, we define the *s-slice* of a as the set $a_s = \{\sigma \in 2^\omega \mid (s,\sigma) \in a\}$, and the *support set* of a as the set $W_a = \{s \in 2^{<\omega} \mid a_s \neq \varnothing\}$.

A subset $a \subseteq W$ is called *small* if W_a is finite; *atom-based* if W_a is a singleton; *digital* if each of its slices is either empty or equal to 2^ω. A subset a of W is *admissible* if $a_s \in \Omega$ for each finite string s; the collection of admissible sets is denoted as A. Observe that every digital set is admissible. The proof of the following proposition is straightforward.

Proposition 1 *The collection A of admissible subsets of W is closed under the standard set-theoretic operations: $(A, \cup, -_W, \varnothing)$ is a boolean algebra.*

Relations and operations We will expand the boolean algebra $(A, \cup, -_W, \varnothing)$ with some additional operations that are defined via the following binary relations on W:

$$\begin{array}{lll}
(s,\sigma) \sqsubset (t,\tau) & \text{if} & s \text{ occurs before } t \text{ in the enumeration of finite strings,} \\
(s,\sigma) < (t,\tau) & \text{if} & s \prec t, \\
(s,\sigma) \lhd (t,\tau) & \text{if} & s \prec \tau, \\
(s,\sigma) L (t,\tau) & \text{if} & s = t0.
\end{array}$$

The converses of the relations $\sqsubset$, $<$ and $\lhd$ will be denoted by $\sqsupset$, $>$ and $\rhd$, respectively. Observe that the relations $\sqsubset$, $<$ and L only depend on the first coordinates of points in W; these relations can be seen as the induced liftings of natural relations on finite strings.

Given a relation R on W, we define the following operations on the power set $\mathcal{P}(W)$ of W:

$$\begin{array}{rcl}
\langle R\rangle Y &=& \{w \in W \mid wRu \text{ for some } u \in Y\},\\
{[R]}Y &=& -_W\langle R\rangle -_W Y\\
\overline{\langle R\rangle} Y &=& \begin{cases} \langle R\rangle Y & \text{if } Y \text{ is small,}\\ W & \text{otherwise.}\end{cases}
\end{array}$$

Clearly, any operation of the form $\langle R\rangle$ is an operator (that is, it preserves all finite unions). That the same holds for $\overline{\langle R\rangle}$ immediately follows from the fact that the collection of small sets is closed under taking finite unions.

The following Proposition is the key towards the definition of our algebra $\mathbb{A}$.

Proposition 2 *The collection A of admissible subsets of W is closed under the operations $\langle\sqsupset\rangle$, $\langle>\rangle$, $\overline{\langle L\rangle}$, $\langle\lhd\rangle$ and $\overline{\langle\rhd\rangle}$. As a corollary, the structure*

$$(A, \cup, -_W, \varnothing, \langle\sqsupset\rangle, \langle>\rangle, \overline{\langle L\rangle}, \langle\lhd\rangle, \overline{\langle\rhd\rangle})$$

is a boolean algebra with operators.

Proof. We leave it for the reader to verify that A is closed under the operations $\langle\sqsupset\rangle$, $\langle>\rangle$, $\overline{\langle L\rangle}$ and $\langle\lhd\rangle$; the key fact needed is that each of these operations map admissible sets to digital ones. In order to check that A is closed under $\overline{\langle\rhd\rangle}$ as well, we may confine our attention to admissible sets that are *small*. Now every small set is the finite union of sets of the form $\{s\} \times x\Uparrow$, so by additivity of $\overline{\langle\rhd\rangle}$ it suffices to show that for each $x, s \in 2^{<\omega}$, the set $\overline{\langle\rhd\rangle}(\{s\} \times x\Uparrow)$ is admissible. This is in fact easy to check, since $\overline{\langle\rhd\rangle}(\{s\} \times x\Uparrow) = \langle\rhd\rangle(\{s\} \times x\Uparrow) = \{(t,\tau) \mid s \prec \tau\}$; that is, *every* slice of $\overline{\langle\rhd\rangle}(\{s\} \times x\Uparrow)$ is of the form $\{s\} \times x\Uparrow$.

This shows that the structure, as defined in the statement of the Proposition, is indeed an algebra, and hence, a boolean algebra with operators (as we saw already that all extra-boolean operations are operators). QED

Hence, we are ready to define the key algebra of the paper:

$$\mathbb{A} := (A, \cup, -_W, \varnothing, \langle\sqsupset\rangle, \langle>\rangle, \overline{\langle L\rangle}, \langle\lhd\rangle, \overline{\langle\rhd\rangle}).$$

3 Results

Now that we have constructed our BAO $\mathbb{A}$, all that is left is to come up with a term $\pi(x)$ for which the formula (α) holds in $\mathbb{A}$. For the definition of π, we extend the language of boolean algebras with five unary operation symbols, $\Diamond_\sqsupset$, $\Diamond_>$, $\Diamond_L$, $\Diamond_\lhd$, and $\Diamond_\rhd$.

Consider the following terms, the meaning of which will become clear in the proof of Proposition 3 below:

$$\begin{array}{rcl}
\varphi(x) &=& x \wedge \neg\Diamond_\sqsupset x,\\
\zeta(x) &=& \Box_\lhd x \wedge \neg\Diamond_> \Box_\lhd x,\\
\delta(x) &=& \Diamond_\rhd \Diamond_L x,\\
\pi(x) &=& \varphi(x) \wedge \delta(\zeta(\varphi(x))).
\end{array}$$

We now have all the necessary material to state and prove the key technical result of this paper. Note that the algebra $\mathbb{A}$ is an appropriate structure for the language just described.

Proposition 3 *The formula (α) holds in $\mathbb{A}$:*

$$\mathbb{A} \models \forall x\,(\bot < x \to \bot < \pi(x) < x).$$

Proof. We first prove three claims that will clarify the meaning of the terms φ, ζ and δ, respectively. Recall that each term $\beta(x_1, \ldots, x_n)$ is interpreted in $\mathbb{A}$ as a map $\beta^{\mathbb{A}} : A^n \to A$.

The first claim states that $\varphi^{\mathbb{A}}$ is a function mapping each nonempty subset a of W to an atom-based subset of a. We leave the straightforward proof as an exercise to the reader.

Claim 1 Let a be a nonempty subset of W, and let $f \in 2^{<\omega}$ be the first finite string (in our fixed enumeration of $2^{<\omega}$) in W_a. Then

$$\varphi^{\mathbb{A}}(a) \;=\; a_f.$$

Now let $a = \{s\} \times U$ be an atom-based admissible subset of W; recall from our earlier discussion that Z_U, the *origin* of U, is a finite set of finite strings which forms a unique representation of U. The next claim states that for such a, the set $\zeta^{\mathbb{A}}(a)$ is the manifestation of Z_U in its $\mathbb{A}$-disguise.

Claim 2 Let $a = \{s\} \times U$ be an atom-based subset of W. Then $\zeta^{\mathbb{A}}(a) = Z_U \times 2^{\omega}$.

Proof of Claim If a is small, then by definition of $\mathbb{A}$ we have that

$$\zeta^{\mathbb{A}}(a) \;=\; [\lhd]a \cap -_W \langle > \rangle [\lhd] a. \tag{2}$$

Using this, the proof of the Claim is routine. ◀

Conversely, our last Claim shows that $\delta^{\mathbb{A}}$ transfers information from finite strings to infinite strings.

Claim 3 Let a be a small subset of W. Then

$$\delta^{\mathbb{A}}(a) \;=\; 2^{<\omega} \times \bigcup_{x \in W_a} x0{\Uparrow}.$$

Proof of Claim Let a be an arbitrary small subset of W. Then $\overline{\langle L \rangle} a = \langle L \rangle a$, and since

$$\langle L \rangle(a) \;=\; \{(x0, \tau) \mid x \in W_a\}, \tag{3}$$

the set $\overline{\langle L \rangle}(a)$ is small as well. Thus

$$\delta^{\mathbb{A}}(a) \;=\; \langle \rhd \rangle \langle L \rangle (a).$$

Now, by the definition of $\langle \rhd \rangle$ and $\langle L \rangle$, we have, for an arbitrary element (s, σ) of W:

$$(s, \sigma) \in \langle \rhd \rangle \langle L \rangle (a) \text{ iff } x0 \prec \sigma \text{ for some } x \in W_a.$$

This proves the claim. ◀

Now we are ready to prove the Proposition itself. Let a be an arbitrary, nonempty, admissible subset of W. We have to show that $\pi^{\mathbb{A}}(a)$ is a nonempty, proper subset of a.

Let f be the first finite string in W_a. It follows from Claim 1 that

$$\varphi^{\mathbb{A}}(a) \;=\; a_f. \tag{4}$$

Note that a_f is an atom-based, nonempty subset of a; say, $a_f = \{f\} \times U$. By the admissibility of a_f it follows that U is a finite union of i-cones; in particular, we may represent U via its origin Z_U as $U = \bigcup_{z \in Z_U} z{\Uparrow}$. It follows from Claim 2 and (4) that

$$\zeta^{\mathbb{A}}(\varphi^{\mathbb{A}}(a)) \;=\; Z_U \times 2^{\omega}.$$

In particular, we see that $\zeta^{\mathbb{A}}(\varphi^{\mathbb{A}}(a))$ is small and digital, with the set Z_U forming its support set. Thus by Claim 3 we obtain that

$$\delta^{\mathbb{A}}(\zeta^{\mathbb{A}}(\varphi^{\mathbb{A}}(a))) \;=\; 2^{<\omega} \times \bigcup_{z \in Z_U} z0{\Uparrow},$$

which, together with (4), yields

$$\pi^{\mathbb{A}}(a) \;=\; \{f\} \times \bigcup_{z \in Z_U} z0{\Uparrow}.$$

But then $\pi^{\mathbb{A}}$ is clearly a nonempty subset of a_f; and it follows from (1) that it is a *proper* subset of a_f, and thus, of a. This finishes the proof of the Proposition. QED

Now we are ready for the proof of our main result.

Proof of Theorem 1. Let K be the class of algebras that satisfy (α), and on which $\gamma(x)$ is a *unary discriminator term* for K, cf. Jipsen [4]. That is, assume that every algebra in K satisfies

$$\gamma(x) \;=\; \begin{cases} \bot & \text{if } x = \bot, \\ \top & \text{if } x > \bot. \end{cases} \tag{Δ}$$

If there is no such term γ, then expand the similarity type with a new unary function symbol and interpret this function symbol as the operator defined by (Δ).

Let V be the variety generated by K; since our algebra $\mathbb{A}$ belongs to K, V is not trivial. It is well known that V, being a discriminator variety, is semisimple; that is, every algebra in V is a subdirect product of simple algebras. In order to show that all algebras in V are atomless, we need a slightly stronger result.

Let $\mathbb{B}$ be an arbitrary algebra in V. It follows from Theorem 2.3 of Givant [2] that there is a family $(\mathbb{B}_i)_{i \in I}$ of simple algebras *in* K, and a subdirect embedding $e : \mathbb{B} \to \prod_{i \in I} \mathbb{B}_i$. Let $e_i : \mathbb{B} \to \mathbb{B}_i$ denote the induced surjective homomorphism.

In order to show that $\mathbb{B}$ is atomless, take an arbitrary nonzero element b of $\mathbb{B}$. First observe that since $\mathsf{K} \models \pi(x) \leq x$, the same holds for the variety V, and thus for $\mathbb{B}$. That is, we have

$$\pi^{\mathbb{B}}(b) \leq b. \tag{5}$$

Now by the subdirectness of the embedding and the fact that $b \neq \bot$, there must be some index $i \in I$ such that $e_i(b) \neq \bot_i$. Since every $\mathbb{B}_i$ belongs to K, and thus, satisfies (α), we find that

$$\bot_i < \pi^{\mathbb{B}_i}(e_i(b)) < e_i(b). \tag{6}$$

But e_i is a homomorphism, so $\pi^{\mathbb{B}_i}(e_i(b)) = e_i(\pi^{\mathbb{B}}(b))$. From this and (6) it is immediate that $\pi^{\mathbb{B}}(b) \neq \bot$ and $\pi^{\mathbb{B}}(b) \neq b$, so with (5) this gives

$$\bot < \pi^{\mathbb{B}}(b) < b, \tag{7}$$

showing that b is *not* an atom. Since b was an arbitrary nonzero element of $\mathbb{B}$, this proves that $\mathbb{B}$ is atomless. In fact, it follows from (7) that $\mathbb{B} \models \alpha$. QED

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