Strong limits and Inaccessibility with non-wellorderable powersets.

MSc Thesis (*Afstudeerscriptie*)

written by

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0 Introduction.

This thesis is about set theory without the axiom of choice. The theory of ordinals and their powersets without the axiom of choice is not a popular subject in set theoretic practise; in this thesis, we will shed a little light on some basic questions in this area.

Our basic theory is ZF, unless otherwise stated. In this thesis we will be interested in the concept of a cardinal being a strong limit. This concept is one of the basic properties of the ordinary theory of cardinals and their powersets. It is well studied in the ZFC context and is typically defined as follows:

$$\kappa \text{ is a strong limit } \overset{\text{def}}{\iff} \forall \lambda < \kappa (2^\lambda < \kappa)$$

where we read "$2^\lambda < \kappa$" as "some (any) ordinal in bijection with the powerset of λ is smaller than κ". In the ZFC context, this ordinal always exists, if 2^λ is not wellorderable, it may not. As it turns out, this definition is equivalent in ZFC to four other definitions (where $<$ is replaced by relations $<_{\mathbf{s}}$, $<_{\mathbf{i}}$, $<_{\bar{\mathbf{s}}}$, $<_{\bar{\mathbf{i}}}$) that are more appropriate for an investigation without the axiom of choice.

We look at this subject from two different points of view, thus this thesis includes two parts. The first part is looking at the problem from an axiomatic point of view, i.e., we see what different answers we can have when we assume different axioms. It starts with the axiom of choice, the axiom of determinacy, weaker forms of them which are involved in this study and some generalisations of statements incompatible with the more famous axioms above. We end this part with a discussion on several notions of being an inaccessible cardinal, i.e., a regular strong limit cardinal. These are defined using the alternative definitions of strong-limitedness we mentioned above. We also define the notion of being a β-inaccessible cardinal that uses the set of ultrafilters on a cardinal and it is connected with the axiom of determinacy.

The notion of inaccessibility is connected with a metamathematical point of view in set theory. It is known that the existence of inaccessible cardinals is equivalent to ZFC having a set model (see [1, Ch.IV, Lemma 6.3]). By Gödel's Incompleteness and Completeness Theorems this is actually a metamathematical proof that these cardinals' existence cannot be proven in ZFC. From this metamathematical point of view, theorems that talk about the consistency of a theory motivate us to define a consistency strength hierarchy between theories that contain ZF. This is because we conventionally accept ZF as consistent.

We will not go into the details of this hierarchy but we will just state that a theory[1] T has stronger consistency strength than a theory T' if T can prove the consistency of the same or more theories than T' can. Therefore the theory ZFC+"there is an inaccessible cardinal" is stronger than ZFC. All this creates the natural question of what happens in non-AC environments.

This leads us at the second part of the thesis, where we look at the problem by constructing generic models by forcing. First we take a brief look at a model

[1] By theory we mean a recursively axiomatisable theory that contains ZF.

by Blass where all ultrafilters are principal and where the notion of being a β-strong limit becomes trivial. Afterwards we will describe the method of taking symmetric submodels of generic extensions and then we will study in depth the Feferman-Lévy model, a symmetric submodel. This model will answer most of our questions in this part and this will lead us to attempt a generalisation of it, hoping this will solve our last question. This attempt will fail but this failure will make the problem clearer and might help to lead us in a new way to approach this in the future.

1 Notation and preliminaries.

We assume that the reader is familiar with the basic ZF theory, which at large is covered by Jech's "Set Theory" ([2]). Nevertheless, below we define notions essential for this thesis and we state some more or less standard notation.

Definition 1 (Relations). A relation R is defined to be any set of ordered pairs. The domain of the relation is the set of all elements that occur as the first coordinate of a pair in the relation and it's denoted by $\mathsf{dom}(R)$. The range is the set of all elements that occur as the second coordinate of a pair in the relation and it's denoted by $\mathsf{rng}(R)$.

Definition 2 (Functions). Let A and B be sets. If there is a surjection from A onto B, we write $A \twoheadrightarrow B$. If there is an injection from A into B we write $A \rightarrowtail B$ and if there is a bijection from A onto B then we write $A \rightarrowtail\!\!\!\!\!\twoheadrightarrow B$. For A, B sets, the set ${}^{A}B$ is the set of all functions from A to B.

For two functions f and g, if $\mathsf{rng}(f) \subseteq \mathsf{dom}(g)$ then we can define their composition to be the following set.

$$f \circ g \stackrel{\text{def}}{=} \{(x, y) \,;\, \exists z \in \mathsf{rng}(f) \cap \mathsf{dom}(g)[(x, z) \in f \text{ and } (z, y) \in g]\}.$$

For two sets A, B with relations R_A and R_B respectively, we write $\langle A, R_A\rangle \cong \langle B, R_B\rangle$ if there is an isomorphism between the structures $\langle A, R_A\rangle$ and $\langle B, R_B\rangle$.

Definition 3 (Sequences). A finite sequence of elements of a set X is an element of the set ${}^{<\omega}X$. For such a sequence s we denote by $\mathsf{lh}(s)$ the length of s, i.e., the smallest $n \in \omega$ such that $s \in {}^{n}X$. We denote the last element of s as $\mathsf{last}(s)$, i.e., $\mathsf{last}(s) \stackrel{\text{def}}{=} s(\mathsf{lh}(s) - 1)$. A countable sequence of elements of a set X is an element of the set ${}^{\omega}X$.

The ordinals are defined as usual, we write Ord for the class of all ordinals. The relation $<$ between ordinals is defined to be equal to $\in\restriction \mathrm{Ord} \times \mathrm{Ord}$.

For a set x, its transitive closure is defined as $\mathsf{trcl}(x) \stackrel{\text{def}}{=} \bigcup_{n\in\omega} x_n$, where $x_0 \stackrel{\text{def}}{=} x$ and $x_{n+1} \stackrel{\text{def}}{=} \bigcup x_n$.

We will now describe the constructible universe, $\mathbb{L}$ and the constructible universe relative to a set a, $\mathbb{L}[a]$. For a detailed introduction in these constructions, see [9, Ch.II].

Definition 4 ($\mathbb{L}$ and $\mathbb{L}[a]$). For a set X, $\mathsf{Def}[X]$ denotes the set of all subsets of X which are definable in the structure $\langle X, \in\rangle$ from a formula φ of set theory that has only one free variable. $\mathbb{L}$ is defined recursively as follows.

$$\mathbb{L}_0 \stackrel{\text{def}}{=} \varnothing \qquad\qquad \mathbb{L}_{\alpha+1} \stackrel{\text{def}}{=} \mathsf{Def}(\mathbb{L}_\alpha)$$

$$\mathbb{L}_\lambda \stackrel{\text{def}}{=} \bigcup_{\beta\in\lambda} \mathbb{L}_\beta \quad \text{for } \lambda \text{ limit, and} \quad \mathbb{L} \stackrel{\text{def}}{=} \bigcup_{\alpha\in\mathrm{Ord}} \mathbb{L}_\alpha$$

It is well known that this is the smallest inner model of ZFC and that it satisfies the generalised continuum hypothesis (GCH).

For a, X sets, $\mathsf{Def}^a[X]$ denotes the set of all subsets of X which are definable in the structure $\langle X, \in, a \cap X\rangle$ from a formula $\varphi \in \mathsf{Fmas}(\mathcal{L}_X(\mathring{a}))$ that has only one free variable. The unary predicate symbol $\mathring{a}(x)$ is intended to be interpreted as $x \in a$. Similarly to $\mathbb{L}$, $\mathbb{L}[a]$ is defined recursively as follows

$$\mathbb{L}_0[a] \stackrel{\text{def}}{=} \varnothing \qquad\qquad \mathbb{L}_{\alpha+1}[a] \stackrel{\text{def}}{=} \mathsf{Def}^a(\mathbb{L}_\alpha[a])$$

$$\mathbb{L}_\lambda[a] \stackrel{\text{def}}{=} \bigcup_{\beta\in\lambda} \mathbb{L}_\beta[a] \quad \text{for } \lambda \text{ limit, and} \quad \mathbb{L} \stackrel{\text{def}}{=} \bigcup_{\alpha\in\mathrm{Ord}} \mathbb{L}_\alpha[a]$$

It is known that $\mathbb{L}[a]$ is a model of ZFC and it's clear that $\mathbb{L} \subseteq \mathbb{L}[a]$, for every set a.

While working without the axiom of choice, we have to be very careful with the notion of cardinality. This notion is meant to describe the size of a set. When all sets are wellorderable, then we can use some kind of numbering, namely the ordinals, to measure the size of any set. But when choice fails there are non wellorderable sets and thus ordinals cannot be used to describe the size of any set. This is because the original definition says that the cardinality of a set X is the first ordinal that is in bijection with X. Since ordinals are wellordered by $\in$, they cannot be used for this definition in a non-choice environment. We define cardinality as follows.

Definition 5 (Cardinality). For every set X, the cardinality of X is defined to be

$$||X|| \stackrel{\text{def}}{=} \{Y \,;\, Y \rightarrowtail\!\!\!\!\twoheadrightarrow X\}.$$

In ZF this is a proper class if X is non-empty, which is strange since cardinality defined under AC is always a set, i.e., a cardinal. This will not be a problem for us but we mention that there is a set-definition of cardinality without choice due to Scott. He defined cardinality as $||X|| \cap \mathbb{V}_\alpha$ where α is minimal such that the defined set is non-empty and in some papers this set is called a cardinal. In this thesis we define cardinals as follows.

Definition 6 (Cardinals). An ordinal κ is said to be a cardinal if it is an initial ordinal, i.e., if

$$\bigcap(\mathrm{Ord} \cap ||\kappa||) = \kappa.$$

Denote the class of all cardinals by Card. Note that this is equivalent to the traditional definition;

$$\kappa \in \mathrm{Card} \iff \kappa = \mathsf{min}\{\alpha \in \mathrm{Ord} \,;\, \alpha \in ||\kappa||\}.$$

If A is a wellorderable set then there is a unique cardinal κ that is the minimal ordinal in $||A||$. We write $|A|$ for κ. For wellorderable sets we often say "the cardinality of A" when we mean "the cardinal of A", i.e., $|A|$. Conversely, when for a set A we say "the cardinality of A is κ" then we imply that A is wellorderable.

Since cardinals are ordinals, we order them by $<_c$ which is defined to be equal to $<\restriction \mathrm{Card} \times \mathrm{Card}$. For ease of notation we will denote this relation also as $<$. By the context it will be clear which one of the two relations we are using.

Definition 7 (Successor and limit cardinal). For a cardinal κ, we write κ^+ for the first ordinal α for which there is no surjection from κ onto α. Clearly, as an initial ordinal, κ^+ is a cardinal. The cardinal κ^+ is called a successor cardinal. A cardinal κ is said to be a limit cardinal if it is not a successor.

For every wellorderable set A, we denote by $2^{|A|}$ the cardinality of $\wp(A)$ and for a non-wellorderable set B we denote the cardinality of its powerset by $2^{||B||}$.

Since we defined only wellorderable cardinals, we can still make use of cardinal arithmetic up to the point that it does not involve any exponentiation of cardinals. Because then we are not guaranteed that the result of the exponentiation is wellorderable. But we can still do addition and multiplication at this point.

If we had allowed non wellorderable cardinals to exist, then nor addition nor multiplication would be definable. This is because these definitions use transfinite recursion and so apply to wellorderable sets only.

1.1 The reals

In this thesis, we define the reals as the powerset of the natural numbers, i.e.,

$$\mathbb{R} \stackrel{\text{def}}{=} \wp(\omega),$$

which has cardinality the continuum. By "the continuum" we mean the cardinality of the reals when they are defined the standard way, i.e., as the unique ordered field in which every non-empty bounded set has a supremum.

Naturally, the continuum is not always a cardinal. In fact, the statement "the continuum is a cardinal" is equivalent to the axiom of choice for subsets of the reals (see [3, Forms 79, 79A]). The statement "for every set X, X is wellorderable" (and therefore X has a cardinal) is equivalent to the axiom of choice itself (see [3, Forms 1, 1E]). We will discuss the axiom of choice and axiomatic fragments of it in §2.1.

We are interested mostly in cardinalities that include only non-wellorderable sets and in particular we are interested mostly in the cardinality of the reals. Therefore any set with cardinality 2^ω could be used instead of the actual reals. Thus we are justified in our definition of the reals for the purposes of this thesis. Only in one proof, the proof of Theorem 3.21, we will study a topological property of the reals. There we are going to use the Cantor set, i.e., ${}^{\omega}2$ which is homeomorphic to a compact subset of $\mathbb{R} \setminus \mathbb{Q}$, where by $\mathbb{Q}$ we mean here the rationals. Note that the Cantor set consists of countable sequences of zeroes and ones, which actually are characteristic functions for subsets of ω.

According to our definition for $\mathbb{R}$, we will refer to a subset of ω as a real number.

In our topological discussion we are going to need some basics from the projective hierarchy. We refer the reader to [7, 1E] for more details on this hierarchy and to [7, 1B] for more details on the Borel hierarchy. We will define only the sets we will need for our proof of Theorem 3.21.

Definition 8 (Topology of the reals and the set $\underset{\sim}{\mathbf{\Pi}}^1_1$). If we assume the reals to be the Cantor set, then we can define a metric on 2^ω as follows. For $x, y \in 2^\omega$, define

$$d(x,y) \overset{\text{def}}{=} \begin{cases} 0 & \text{if } x = y \\ 2^{-n} & \text{if } n \text{ is least such that } x(n+1) \neq y(n+1) \end{cases}$$

The set $\underset{\sim}{\mathbf{\Sigma}}^0_1$ is defined to be the set of all open subsets of the reals, according to the metric d. The set $\underset{\sim}{\mathbf{\Pi}}^0_1$ is the set of complements of sets in $\underset{\sim}{\mathbf{\Sigma}}^0_1$, i.e., as the set of all closed subsets of $\mathbb{R}$ according to the metric d. For A a subset of $2^\omega \times 2^\omega$ we define its projection $\exists^{\mathbb{R}} A$ as follows:

$$\exists^{\mathbb{R}} A \overset{\text{def}}{=} \{y \in 2^\omega \; ; \exists x \in 2^\omega (x,y) \in A\}.$$

The set $\underset{\sim}{\mathbf{\Sigma}}^1_1$ is defined as follows:

$$\underset{\sim}{\mathbf{\Sigma}}^1_1 \overset{\text{def}}{=} \{\exists^{\mathbb{R}} A \; ; A \in \underset{\sim}{\mathbf{\Pi}}^0_1\},$$

i.e., as the set of all projections of closed sets of reals according to the metric d. Finally, the set $\underset{\sim}{\mathbf{\Pi}}^1_1$ is the set of all complements of sets in $\underset{\sim}{\mathbf{\Sigma}}^1_1$.

1.2 New cardinal notions

If we assume AC, then a cardinal is called a strong limit if for all $\lambda < \kappa$ we have $2^\lambda < \kappa$. Without the axiom of choice, 2^λ is not necessarily an ordinal, so this definition of strong limit does not always make sense. If we are to work in non-choice environments we need another notion of strong limit. The purpose of the next definition is to state possible other ways of talking about strong limits and later about different notions of inaccessibility, without involving the axiom of choice.

Definition 9 ($\mathbf{s}, \mathbf{i}, \bar{\mathbf{s}}, \bar{\mathbf{\imath}}$-strong, strong limit). For sets X,Y and for ordinals α, β, we define the following relations.

$X <_{\mathbf{s}} \alpha \overset{\text{def}}{\iff}$ there is some $\beta < \alpha$ and a surjection from β onto X.

$X <_{\mathbf{i}} \alpha \overset{\text{def}}{\iff}$ there is some $\beta < \alpha$ and an injection from X into β.

$X <_{\bar{\mathbf{s}}} Y \overset{\text{def}}{\iff}$ there is no surjection from X onto Y.

$X <_{\bar{\mathbf{\imath}}} Y \overset{\text{def}}{\iff}$ there is no injection from Y into X.

Note that for the first two cases we need the right hand side to be an ordinal, for the others we don't. Now, for $\mathbf{x} \in \{\mathbf{s}, \mathbf{i}, \bar{\mathbf{s}}, \bar{\mathbf{i}}\}$, a cardinal κ is called $\mathbf{x}$-strong if for all cardinals λ with $\lambda <_{\mathbf{x}} \kappa$ we have $2^\lambda <_{\mathbf{x}} \kappa$. If the cardinal is also a limit cardinal, then we call it an $\mathbf{x}$-strong limit.

First we'll prove the following easy lemma.

Lemma 1.1.
For every X, Y sets and for every $\alpha \in \mathrm{Card}$, the following hold:

(a) $X \rightarrowtail Y$ implies that $Y \twoheadrightarrow X$.

(b) $X <_{\mathbf{s}} \alpha$ is equivalent to $X <_{\mathbf{i}} \alpha$.

(c) $X <_{\mathbf{i}} \alpha$ implies that $X <_{\bar{\mathbf{s}}} \alpha$.

(d) $X <_{\bar{\mathbf{s}}} Y$ implies that $X <_{\bar{\mathbf{i}}} Y$.

(e) If $\lambda < \alpha$, then $2^\lambda <_{\mathbf{s}} \alpha$ implies that 2^λ is well orderable.

Proof. Let X, Y be arbitrary sets and α an arbitrary cardinal.

(a) Without loss of generality assume that X is not empty and let x_0 be an element of X. Let f be the injection from X into Y. Define $g\colon Y \to X$ as follows.

$$g(y) \overset{\mathrm{def}}{=} \begin{cases} f^{-1}(y) & \text{if } y \in f[X] \\ x_0 & \text{otherwise.} \end{cases}$$

(b) To show from left to right, assume that there is a β in α and a function $f\colon \beta \twoheadrightarrow X$. Fix β, f and define $g\colon X \to \beta$ for every x in X to be

$$g(x) \overset{\mathrm{def}}{=} \mathsf{min} f^{-1}(x).$$

Since $f^{-1}(x)$ is a subset of β, g is well defined and clearly an injection.

To show now the direction from right to left, assume that there is a β in α such that $X \rightarrowtail \beta$. By (a) this implies that $\beta \twoheadrightarrow X$ and thus $X <_{\mathbf{s}} \alpha$ holds.

(c) Assume there is a β in α such that $X \rightarrowtail \beta$. By (a) we have that there is a function $g\colon \beta \twoheadrightarrow X$. Assume for a contradiction that there is a surjection f from X onto α. Then $g \circ f$ would be a surjection from β onto α. But $\beta < \alpha$, contradiction.

(d) By contraposition of (a), $X \not\twoheadrightarrow \alpha$ implies $\alpha \not\rightarrowtail X$.

(e) Let $\lambda < \alpha$ such that $2^\lambda <_{\mathbf{i}} \alpha$, i.e., such that there exists a γ in α and a function $g\colon 2^\beta \rightarrowtail \alpha$. Define a relation $<^*$ on 2^β:

$$x <^* y \overset{\mathrm{def}}{\iff} g(x) < g(y).$$

Clearly, this is a wellorder.

qed

Schematically, the above lemma says the following.

$$<_{\mathbf{s}} \Longleftrightarrow <_{\mathbf{i}} \Longrightarrow <_{\bar{\mathbf{s}}} \Longrightarrow <_{\bar{\mathbf{\imath}}}$$

The next lemma with the next corollary show that these newly defined relations are all equivalent under the axiom of choice.

Lemma 1.2.
Assume AC*. Then for two sets A and B it holds that if $A \twoheadrightarrow B$ then $B \rightarrowtail A$.*

Proof. Let A, B be arbitrary non-empty sets. If one of them was empty, then the lemma would trivially hold. Let there be a function $f\colon A \twoheadrightarrow B$. Look at the set $\{f(a)\ ;\ a \in A\}$. This is a set of non-empty sets because f is a total function. By AC there is a function g such that for every a in A, $g(f(a)) \in f(a)$. Define a function $h\colon B \to A$ for every b in B to be $h(b) \stackrel{\text{def}}{=} g(f(a))$. Since for every b in B there is an a in A such that $f(a) = b$, this is a total function. And since f is a function, h is an injection. qed

Corollary 1.3.
Assume AC*. Then $< = <_{\mathbf{s}} = <_{\mathbf{i}} = <_{\bar{\mathbf{s}}} = <_{\bar{\mathbf{\imath}}}$.*

Proof. Work with the notation in Lemma 1.1. By [3, Forms 1, 1E], there are ordinals α, β such that $\alpha \rightarrowtail\!\!\!\!\twoheadrightarrow X$ and $Y \rightarrowtail\!\!\!\!\twoheadrightarrow \beta$. Therefore we work with the ordinals α, β instead of the sets. By Lemma 1.1 it suffices to show that $\alpha < \beta$ implies that $\alpha <_{\bar{\mathbf{\imath}}} \beta$, that $\alpha <_{\bar{\mathbf{\imath}}} \beta$ implies that $\alpha <_{\bar{\mathbf{s}}} \beta$ and that the latter implies that $\alpha <_{\mathbf{i}} \beta$. But these relations are obviously equal for ordinals. qed

The result of Lemma 1.1 is a general result about these relations. We are interested in the notions of **x**-strong cardinals and that splits in two studies, one of **x**-strong successor cardinals and one for **x**-strong limit cardinals. From ZF we have a basic lemma that involves powersets and surjections and since it's a useful one we prove it below.

Lemma 1.4.
For every infinite cardinal λ, $\wp(\lambda) \twoheadrightarrow \lambda^+$.

Proof. Fix a cardinal λ. If $\alpha < \lambda^+$ then by definition there is a cardinal $\kappa \leq \lambda$ and a bijection $f\colon \kappa \rightarrowtail\!\!\!\!\twoheadrightarrow \alpha$. For every bijection f from a cardinal κ below λ^+ to an ordinal α below λ^+, define a relation $R_f \subseteq \lambda \times \lambda$ by

$$\gamma R_f \beta \stackrel{\text{def}}{\Longleftrightarrow} f(\gamma) \in f(\beta).$$

Clearly, $\langle \lambda, R_f \rangle \cong \langle \alpha, \in \rangle$ for every such f. Now we can define $F\colon \wp(\lambda \times \lambda) \to \lambda^+$ as follows.

$$F(R) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \langle \lambda, R \rangle \text{ is not isomorphic to an ordinal, and} \\ \alpha & \text{if } \alpha \text{ is the unique ordinal such that } \langle \lambda, R \rangle \cong \langle \alpha, \in \rangle \text{ holds.} \end{cases}$$

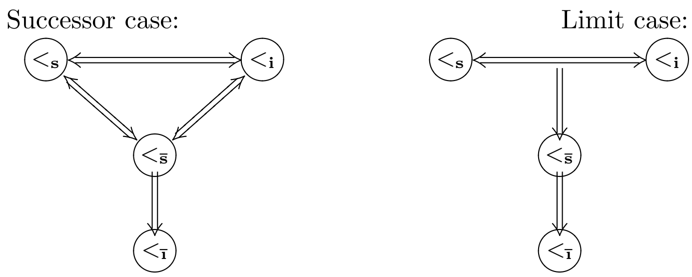
This is a surjection from $\wp(\lambda \times \lambda)$ onto λ^+ and since $\lambda \times \lambda \rightarrowtail\!\!\!\!\twoheadrightarrow \lambda$ holds, we have that $\wp(\lambda)$ surjects onto λ^+. qed

This lemma is essential, as we'll see in the next lemma, since it makes $\bar{\mathbf{s}}$-strongness impossible for successor cardinals.

Lemma 1.5 (Successor case).
Successor cardinals can never be $\mathbf{s}$, $\mathbf{i}$, $\bar{\mathbf{s}}$*-strong.*

Proof. By Lemma 1.1 we know that it's enough to show that no successor cardinal can be $\bar{\mathbf{s}}$-strong. Assume for a contradiction that for $\kappa \in \text{Card}$, κ^+ is $\bar{\mathbf{s}}$-strong, i.e., that for every $\lambda < \kappa^+$, $2^\lambda \not\twoheadrightarrow \kappa^+$. But $\kappa < \kappa^+$ and thus we have that there is no surjection from 2^κ onto κ^+, which is a contradiction to Lemma 1.4. qed

What about $\bar{\mathbf{i}}$-strongness? In §3.3 we'll see that the existence of $\bar{\mathbf{i}}$-strong successor cardinals is consistent with ZF. For the limit case we will see that in the diagram below, the arrow from $<_{\mathbf{s}}$ and $<_{\mathbf{i}}$ to $<_{\bar{\mathbf{s}}}$ cannot be reversed because it is consistent with ZF that there is an $\bar{\mathbf{s}}$-strong limit cardinal which is not an $\mathbf{s}$-strong limit cardinal. The question whether $\bar{\mathbf{i}}$ implies or not $\bar{\mathbf{s}}$ is an open question for this thesis, but one would expect that it does not. At the moment we have the following picture:



Note that according to Corollary 1.3, if we assume AC then the diagrams above collapse.

2 Part I. Axiomatic approach

In this part we are going to examine different axiomatic systems that might give answers or insights on our questions. We are going to see ZFC which is ZF + AC and ZF + AD. Also we are going to investigate weaker systems, some systems in between ZF and ZFC and some systems in between ZF and ZF + AD. We will see that even though AC is incompatible with AD, there are systems which are weaker than both and stronger than ZF and with them we will look at different notions of inaccessibility.

2.1 Fragments of AC

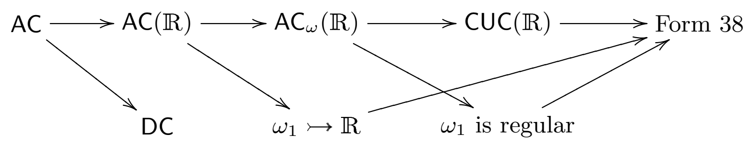
We are going to present some axiomatic fragments of AC. We do that because as we will see later, they are helpful to our questions. These axiomatic fragments are listed in the lemma below. After the lemma we give a diagram that shows precisely how the implications between these fragments are. The role of the lemma is actually more of a definition since the proofs are not given.

Next to each statement in the lemma there is a statement like [Form n] where n is a natural number. This refers to [3] where these statements have these form numbers. If there is no other abbreviation and one is needed, we use the form number of a statement. The book [3] is part of the project "Consequences of AC" by P. Howard and J.E. Rubin. This is one of the most useful books for this kind of research since it includes a large amount of known implications between statements that follow from the axiom of choice and of course the references.

Lemma 2.1.
The following are implied by AC*, the axiom of choice [Form 1].*

- DC*; the axiom of dependent choices [Form 43].*
- $\mathsf{AC}(\mathbb{R})$*; the axiom of choice for families of sets of reals [Form 79A].*
- $\mathsf{AC}_\omega(\mathbb{R})$*; the axiom of choice for countable families of sets of reals [Form 94].*
- $\mathsf{CUC}(\mathbb{R})$*; the statement "Countable unions of countable sets of reals are countable" [Form 6].*
- *"ω_1 is regular" [Form 34].*
- *"$\omega_1 \rightarrowtail \mathbb{R}$" [Form 170].*
- *"$\mathbb{R}$ is not a countable union of countable sets" [Form 38].*

In particular, the implications are as follows.



In this diagram, an arrow means proper implication. Most of the proofs of the implications above are rather simple and well known. In this thesis we have included the proof of "$\omega_1 \rightarrowtail \mathbb{R} \Rightarrow$ Form 38" and of "ω_1 is regular $\Rightarrow$ Form 38", see Theorem 2.7(b) for $\kappa = \omega$.

Next we are going to see the axiom of determinacy, an axiom inconsistent with $\mathsf{AC}(\mathbb{R})$ and therefore AC itself.

2.2 Determinacy

Here we assume that the reader is a bit familiar with the theory of determinacy. For details and information on this theory we refer the reader to Kanamori's "The Higher Infinite" ([8, Chapter 6]). The axiom of determinacy or AD states that for every subset $A \subseteq {}^\omega\omega$, the ω-game on A is determined. The axiom of determinacy is broadly used in descriptive set theory and has some very nice consequences. As seen in [8, Corollary 27.4] AD is incompatible with $\mathsf{AC}(\mathbb{R})$. Under the axiom of determinacy ω_1 is not injectable into the reals and the reals are not injectable in ω_1 either. This is also a motivation for this study, since this is a axiomatic system that has answers for our questions. The following theorem states important consequences of AD.

Theorem 2.2.
The following hold:

(a) AD *implies the perfect set property of* $\mathbb{R}$ *(*PSP*), i.e., that every subset of* $\mathbb{R}$ *is either countable or contains a subset that is homeomorphic to* ${}^{\omega}2$ *(a perfect set).*

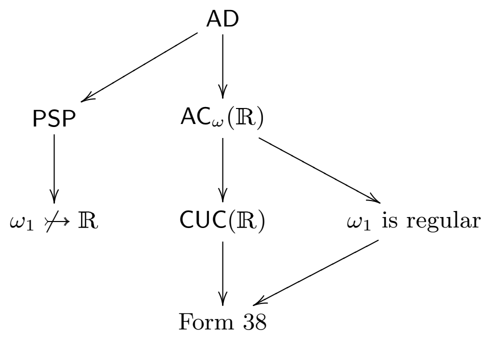
(b) PSP *implies that* $\omega_1 \not\leq 2^{\omega}$*.*

(c) AD *implies* $\mathsf{AC}_{\omega}(\mathbb{R})$*, i.e., that every countable set of subsets of* $\mathbb{R}$ *has a choice function.*

Proof. For the proof of (a) see [8, Proposition 27.5] and for (c) see [8, Proposition 27.10]. We will prove (b).

Let PSP hold. Assume for a contradiction that $\omega_1 \leq \mathbb{R}$ holds, and call f the injection from ω_1 into $\mathbb{R}$. Then $X \stackrel{\text{def}}{=} f[\omega_1]$ is an uncountable subset of $\mathbb{R}$ and by PSP this means that X is homeomorphic to the reals. But X is in bijection with ω_1 via f. This makes $\mathbb{R}$ wellordered and thus makes $\mathsf{AC}(\mathbb{R})$ hold. Contradiction. qed

Schematically, this is as follows, where an arrow means proper implication.



In the next lemma we see that the axiom of determinacy does give an answer to our questions. In particular, it separates the notion of $\bar{\mathsf{s}}$ from $\bar{\mathsf{\imath}}$-strong cardinal.

Lemma 2.3.
Assume AD*. Then* ω_1 *is an* $\bar{\imath}$*-strong cardinal and not an* $\bar{\mathsf{s}}$*-strong cardinal.*

Proof. This is clear since ω_1 being a $\bar{\mathsf{\imath}}$-strong cardinal is equivalent to ω_1 not being injectable into $\wp(\mathbb{R})$, i.e., the reals. The latter is true under AD by Theorem 2.2(b). Also, by Lemma 1.5 we have that ω_1 as a successor can never be $\bar{\mathsf{s}}$-strong. qed

This is a nice result, since now we know that $\bar{\mathsf{\imath}}$-strongness for successor cardinals can be a meaningful notion.

So we see that AD does indeed have answers for our questions and these answers are well known. One might think here that there is no reason for researching further our questions, but the reason comes from a theorem of Woodin which showed that AD has the same consistency strength with a strong large cardinal axiom[2](see [8, Theorem 32.16]). These strong interconnections of AD

[2] In particular AD is equiconsistent with the statement "There exist infinitely many Woodin cardinals".

with large cardinal axioms lead us to consider our questions still open and to continue our reasearch without involving AD.

As seen in the proof of Lemma 2.3, AD gave us a result only because it implies that $\omega_1 \not\leq \mathbb{R}$. Next we are going to look at some interesting statements that imply $\omega_1 \not\hookrightarrow 2^\omega$ and generalisations of this that do not require strong statements like AD.

2.3 Figura's generalisations

The statement "countable unions of countable sets of reals are not always countable" or $\neg\mathsf{CUC}$, is inconsistent with both the axiom of determinacy and the axiom of choice. We will see that it can be a fruitful statement.

Lemma 2.4.
If X is a countable union of countable sets, then there is no surjection from X onto ${\omega_1}^\omega$.

Proof. Let $X = \bigcup_{n\in\omega} X_n$ for some countable sets X_n. Let f be any function from X to ${\omega_1}^\omega$. Since f is a function, for every $n \in \omega$ the set $f[X_n]$ is countable and therefore $\omega_1 \setminus \{f(x)(n) \; ; \; x \in X_n\}$ is not empty. Let α_n be the minimal element of this set. Define the function $Z \colon \omega \to \omega_1$ as $Z(n) \stackrel{\text{def}}{=} \alpha_n$.

We claim that Z is not in the range of f. Assume for a contradiction that it is, then there must be some $z \in X$ such that $f(z) = Z$. Let n be such that $z \in X_n$. By construction $Z(n) = \alpha_n \neq f(z)(n)$, so $Z \neq f(z)$. Therefore, f cannot be such a surjection. qed

Theorem 2.5.
If $\mathbb{R}$ is a countable union of countable sets, every wellorderable subset of the reals has cardinality $\leq \omega$.

Proof. We prove this by contraposition. Assume that there is an injection from ω_1 into 2^ω. Then there is an injection from ${\omega_1}^\omega$ into $(2^\omega)^\omega = (2^{\omega\times\omega}) = 2^\omega$. But clearly there is an injection from 2^ω into ${\omega_1}^\omega$. By the Schröder-Bernstein theorem, we get that $2^\omega = {\omega_1}^\omega$. If $\mathbb{R}$ was a countable union of countable sets, we would have a contradiction to Lemma 2.4. qed

Therefore, if we manage to have a model where $\mathbb{R}$ is a countable union of countable sets (a statement that contradicts both AC and AD), then that model will be a witness of the difference between $\bar{\mathsf{s}}$ and $\bar{\mathsf{\imath}}$ -strong cardinals. Such a model was constructed by Feferman and Lévy in 1963. For the abstract see [5] and for a more detailed version see [2, Example 15.57]. We will study this model in Part II, §3.3.

The statements that are studied above are going to be useful in our study for successor cardinals, so we take the time to look at some generalisations of them hoping to reach a separation result for limit cardinals. To do that we'll use the following generalisations that are due to Figura (see [6]).

Definition 10. Let κ be a cardinal. Then,

$$\mathsf{CP}(\kappa) \overset{\text{def}}{\iff} \text{“}\kappa^+ \text{ is a singular cardinal.”}$$
$$\mathsf{WOP}(\kappa) \overset{\text{def}}{\iff} \text{“every wellorderable subset of } \wp(\kappa) \text{ has cardinality} \leq \kappa.\text{”}$$
$$\mathsf{DP}(\kappa) \overset{\text{def}}{\iff} \text{“}\wp(\kappa) \text{ is a union of } \kappa \text{ sets of cardinality} \leq \kappa.\text{”}$$

These notions are generalisations of the ones for $\kappa = \omega$ which are more well known. In particular, $\mathsf{WOP}(\omega)$ means that "every wellorderable subset of $\mathbb{R}$ has cardinality $\leq \omega$". This is clearly equivalent to saying that $\omega_1 \not\hookrightarrow \mathbb{R}$, i.e., ¬Form 170. $\mathsf{DP}(\omega)$ is "$\mathbb{R}$ is a union of ω sets of cardinality $\leq \omega$, i.e., ¬Form 38 and $\mathsf{CP}(\omega)$ is "ω_1 is singular", i.e., ¬Form 34. It is known that Form 34 implies Form 38, so with contraposition it is true that $\mathsf{DP}(\omega)$ implies $\mathsf{CP}(\omega)$. The next theorem generalizes this result as well as the result of Theorem 2.5. But before we start we need a lemma similar to Lemma 2.4.

Lemma 2.6.
If X is a union of κ-many sets of cardinality $\leq \kappa$, then there cannot be a surjection from 2^κ onto ${\kappa^+}^\kappa$.

Proof. The proof is similar to the proof of Lemma 2.4.

Let $X = \bigcup_{\alpha\in\kappa} X_\alpha$ for some sets X_α of cardinality $\leq \kappa$. Let f be any function from X to ${\kappa^+}^\kappa$. Since f is a function, for every $\alpha \in \kappa$ the set $f[X_\alpha]$ has cardinality $\leq \kappa$ and therefore for every $\alpha \in \kappa$, the set $\kappa^+ \setminus \{f(x)(\alpha) \,;\, x \in X_\alpha\}$ is not empty. Let β_α be the minimal element of this set. Define the function $Z \colon \kappa \to \kappa^+$ as $Z(\alpha) \overset{\text{def}}{=} \beta_\alpha$.

We claim that Z is not in the range of f. Assume for a contradiction that it is, then there must be some $z \in X$ such that $f(z) = Z$. Let α be such that $z \in X_\alpha$. By construction $Z(\alpha) = \beta_\alpha \neq f(z)(\alpha)$, so $Z \neq f(z)$. Therefore, f cannot be such a surjection. qed

Theorem 2.7 (Figura).
Let $\kappa \in$ Card.

(a) $\mathsf{DP}(\kappa)$ *implies* $\mathsf{WOP}(\kappa)$.

(b) $\mathsf{DP}(\kappa)$ *implies* $\mathsf{CP}(\kappa)$.

Proof. Let κ be a cardinal.

(a) The proof is similar to the one of Theorem 2.5. It is a proof by contraposition, i.e., we'll prove that if there is an injection from κ^+ into 2^κ, then 2^κ cannot be a κ-union of sets of cardinality $\leq \kappa$. So by our assumption, there is an injection from κ^+ into 2^κ. Then there is an injection from ${\kappa^+}^\kappa$ into $(2^\kappa)^\kappa = (2^{\kappa\times\kappa}) = 2^\kappa$. But clearly there is an injection from 2^κ into ${\kappa^+}^\kappa$. By the Schröder-Bernstein theorem, we get that $2^\kappa = {\kappa^+}^\kappa$. If 2^κ was a κ-union of sets with cardinality $\leq \kappa$, we would have a contradiction to Lemma 2.6.

(b) Assume that $\mathsf{DP}(\kappa)$ holds, i.e., that $\wp(\kappa)$ is a union of κ sets of cardinality $\leq \kappa$. Then, look at

$$\mathsf{cof}(\kappa^+) = \mathsf{inf}\{\alpha \in \text{Ord} \,;\, \exists\langle\beta_i \,;\, i \in \alpha\rangle[\forall i \in \alpha, \beta_i < \kappa^+ \text{ and } \kappa^+ = \bigcup_{i\in\alpha} \beta_i]\}$$

Assume towards contradiction that κ^+ is regular, i.e., for every sequence $\langle \beta_i \,;\, i \in \alpha \rangle$ such that for every i in α, $\beta_i < \kappa^+$, if $\alpha < \kappa^+$, then $\bigcup_{i\in\alpha} \beta_i < \kappa^+$. But now $\wp(\kappa)$ is such a union, therefore $\wp(\kappa) < \kappa^+$, which makes $\wp(\kappa)$ wellordered and so of cardinality κ which is a contradiction.

qed

2.4 Inaccessibilities

Since strong limit cardinals are defined only where choice holds, the same happens with the definition of inaccessible cardinals. A cardinal κ is called an inaccessible cardinal if it is a strong limit and it is a regular cardinal. If we define **x**-inaccessibility using our notion of **x**-strong limit, then under the axiom of choice this notion is equivalent with the the notion of inaccessibility, for every $\mathbf{x} \in \{\mathbf{s}, \mathbf{i}, \bar{\mathbf{s}}, \bar{\mathbf{\imath}}\}$ (see Corollary 1.3).

There is also a notion of weak inaccessibility that does not require choice. A cardinal κ is defined to be weakly inaccessible if it is a regular limit cardinal. Of course inaccessible cardinals are also weakly inaccessible and the existence of the latter is not provable in ZFC (see [1, Ch.VI, Corollary 4.13]).

Let's take a look at what fragments of AC have to say on this notion of inaccessibility and strong limitedness. We remind the reader that $\mathsf{AC}_\omega(\mathbb{R})$ implies that ω_1 is regular. This fact combined with the following is the usual proof that $\mathsf{AC}_\omega(\mathbb{R}) + \omega_1 \not\rightarrowtail \mathbb{R}$ has the consistency strength of an inaccessible cardinal. Therefore $\mathsf{AC}_\omega(\mathbb{R})$ is not fully needed for such a consistency strength.

Theorem 2.8.
If the cardinal ω_1 does not inject into 2^ω, then for every M inner model of ZFC *it holds that $\omega_1^M < \omega_1^{\mathbb{V}}$.*

Proof. Assume that in $\omega_1 \not\rightarrowtail \mathbb{R}$ and fix M an inner model of ZFC. Since $M \subseteq \mathbb{V}$ we have that $\omega_1^M \leq \omega_1^{\mathbb{V}}$. Because $M \models \mathsf{ZFC}$ we have that $\omega_1^M \rightarrowtail \mathbb{R}^M$. Since $\mathbb{R}^M \subseteq \mathbb{R}^{\mathbb{V}}$ we have that $\omega_1^M \rightarrowtail \mathbb{R}$. This means that $\omega_1^M \neq \omega_1^{\mathbb{V}}$ and therefore $\omega_1^M < \omega_1^{\mathbb{V}}$. qed

Theorem 2.9.
Assume $\omega_1^M < \omega_1^{\mathbb{V}}$. Then in any M inner model of ZFC*, $\omega_1^{\mathbb{V}}$ is a strong limit cardinal.*

Proof. Assume that for every inner model M of ZFC, it holds that $\omega_1^M < \omega_1^{\mathbb{V}}$. Fix M and assume for a contradiction that in M, $\omega_1^{\mathbb{V}}$ is not a strong limit, i.e., that in M there is a $\gamma < \omega_1^{\mathbb{V}}$ such that $\wp(\gamma) \not< \omega_1$. Fix γ. Since $M \models \mathsf{ZFC}$ we get that in M it holds that $\omega_1^V \rightarrowtail \wp^M(\gamma)$ and by $M \subseteq \mathbb{V}$ we get that in $\mathbb{V}$ it holds that for the countable γ there is an injection from ω_1 into $\wp^M(\gamma) \subseteq \wp(\gamma)$. Since γ is countable, its powerset injects into the reals and therefore we have an injection from ω_1 into the reals. Contradiction to our assumption. qed

In §3.3 we will see that if the Feferman-Lévy model is our universe, then ω_1 is a strong limit cardinal in the constructible universe $\mathbb{L}$ but since it's not regular, it is not inaccessible in $\mathbb{L}$.

Now we will introduce a new notion of inaccessibility which is due to Kechris, that is called β-inaccessibility.

Definition 11 (β-strong limit, β-inaccessible). Let λ be a cardinal and let $\beta\lambda$ be the set of ultrafilters on λ (the Stone-Čech compactification of λ). For a cardinal κ we say that κ a β-strong limit if for all $\lambda < \kappa$, we have that $\beta\lambda < \kappa$. The cardinal κ is called β-inaccessible if it is a β-strong limit and it is regular.

We will see that in ZFC this is equivalent to being a strong limit. For that we'll use the following theorem from topology.

Theorem 2.10.
Assume AC. *Then for every cardinal $\kappa \geq \omega$, the Stone-Čech compactification $\beta\kappa$ of κ has cardinality 2^{2^κ}.*

For a proof of Theorem 2.10 see [11, Theorem 3.6.11].

Theorem 2.11.
Assume AC. *Then a cardinal κ is a strong limit cardinal if and only if it is a β-strong limit cardinal.*

Proof. Let κ be a strong limit cardinal and let λ be a cardinal in κ. Then by Theorem 2.10, $\beta\lambda \rightarrowtail\!\!\!\!\twoheadrightarrow 2^{2^\lambda}$. Since κ is a strong limit, we have that $2^\lambda < \kappa$ and again for the same reason we have that $2^{2^\lambda} < \kappa$. Therefore, $\beta\lambda \rightarrowtail \kappa$ and therefore κ is a β-strong limit cardinal.

Now assume that κ is a β-strong limit cardinal and let λ be a cardinal in κ. Then $\beta\lambda < \kappa$ and by Theorem 2.10, $2^{2^\lambda} < \kappa$. But it clearly holds that $2^\lambda \rightarrowtail 2^{2^\lambda}$ and therefore $2^\lambda < \kappa$. Therefore κ is a strong limit cardinal. qed

With this theorem we showed that this is a good definition, i.e., compatible with the standard AC-definition of being a strong limit. Without AC this has the same problem as the original notion, as $\beta\lambda$ might not be wellorderable. Kunen proved that if we assume $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$, then for each $\lambda < \Theta$, the set of ultrafilters on λ is wellorderable. So under $\mathsf{AD} + \mathsf{DC}$ the question whether Θ is a strong limit does make sense.

With the following theorem and using the aforementioned theorem of Kunen, Kechris proved that being a β-strong limit is meaningful.

Theorem 2.12.
Assume AD. *Then in $\mathbb{L}(\mathbb{R})$ there is a cardinal that is β-inaccessible and there is a cardinal that is not.*

For a proof of this see [4, §3.2, (c)Theorem and (d)Theorem].

This is a very interesting notion but it can trivialise. In §3.1 we present a model from Blass, in which every cardinal is a β-strong limit cardinal. Therefore we are not going to concentrate much on this notion and we'll continue to focus on the notions of $\mathbf{x}$-strongness for $\mathbf{x} \in \{\mathbf{s}, \mathbf{i}, \bar{\mathbf{s}}, \bar{\mathbf{\imath}}\}$, defined in §1.2.

3 Part II. Approach by construction

The axiomatic approach has many results to offer but cardinal notions are basic in set theory and therefore we feel they should be studied in ZF alone. This part is dedicated in studying ZF models and seeing what they have to say about our questions.

Forcing is an extremely helpful tool for studying questions like ours. We assume that the reader is familiar with the basics of the forcing method which can be found in Kunen's "Set Theory: An Introduction to Independence Proofs" ([1]).

As we'll need to work in non-AC environments, a simple generic extension will not do because that is always an AC model. For constructing models of $\mathsf{ZF} + \neg\mathsf{AC}$ we need the method of symmetric submodels. Before we go into that method, we take a brief look at a model by Blass which is constructed by taking sets that are hereditarily definable from ordinals and members of a certain set S.

3.1 The Blass model

In 1977, Andreas Blass published a paper [10] with the model we will present below, which is based on a model by Feferman and has as main property that in this model, all ultrafilters are principal. One result this model is good for is that all cardinals in this model are β-inaccessible cardinals and therefore β-inaccessibility trivialises. But this is all that we use it for and therefore we will not describe it in depth and by this we mean we will not give most proofs. This model was constructed with the method of taking sets that are hereditarily ordinal definable over a set S. We briefly describe this method in the next paragraph. For details on this method, see [1, Ch.V and in particular Exercise 9].

Hereditarily Ordinal Definable over a set S

Similarly to Definition 4, for a set X and an $n \in \omega$, define $\mathsf{Def}(X, n)$ to be the set of all sets that are definable in the structure $\langle X, \in \rangle$ from a formula φ of set theory with n free variables. The class $\mathbb{OD}$ is defined as the class of all sets x for which there is a set X of n-many ordinals such that $x \in \mathsf{Def}(X, n)$. The class $\mathbb{HOD}$ of hereditarily ordinal definable sets is defined to be $\mathbb{HOD} \stackrel{\text{def}}{=} \{x \,;\, \mathsf{trcl}(x) \subseteq \mathbb{OD}\}$ and it is a model of ZFC.

As seen in [1, Ch.V, Exercise 9], given a transitive set S we can define similarly the class of all sets that are hereditarily ordinal definable over S as the class of those sets that are definable from a finite number of elements of $\mathrm{Ord} \cup S \cup \{S\}$. This class is a model of ZF.

Construction

Let M be the ground model that satisfies $\mathsf{ZF} + \mathbb{V} = \mathbb{L}$, use the partial order $\mathsf{Fn}(\omega \times \omega, 2)^M$ and take G to be an M-generic filter on $\mathsf{Fn}(\omega \times \omega, 2)^M$. We know that $\bigcup G$ is a function and so we can define for every $n \in \omega$, $a_n(m) \stackrel{\text{def}}{=} (\bigcup G)(n, m)$. As we said in §1.1, functions from ω to 2 are in fact characteristic functions for subsets of ω and therefore can be identified with the subset of ω that they correspond to. Such a_n's are called generic reals.

For every $x \subseteq \omega$, define $\overline{x} \stackrel{\text{def}}{=} \{y \subseteq \omega \ ; \ x \triangle y < \omega\}$, where for two sets x, y, $x \triangle y$ denotes their symmetric difference, i.e., $x \triangle y \stackrel{\text{def}}{=} (x \setminus y) \cup (y \setminus x)$. Let $f \in M[G]$ be the function defined as $f(n) \stackrel{\text{def}}{=} \left\{\overline{a_n}, \overline{\omega \setminus a_n}\right\}$. Finally, let

$$S \stackrel{\text{def}}{=} \bigcup_{n \in \omega} (\overline{a_n} \cup \overline{\omega \setminus a_n}) \cup \{f\} \, .$$

Take the class $\mathbb{HOD}(S)$ that consists of all sets that are hereditarily ordinal definable by ordinals and S. By [1, Ch.V, Exercise 9] this is a model of ZF and we will call this model $\mathcal{M}15$, a name taken by [3].

We define a class W as the smallest class to contain singletons and to be closed under wellordered unions. By the latter we mean that if $\alpha \in \mathrm{Ord}$ and for every $\xi < \alpha$, $x_\xi \in W$ for some sets x_ξ, then $(\bigcup_{\xi < \alpha} x_\xi) \in W$. This class is definable by a ZF formula and it's called the class of almost wellordered sets.

Proposition 3.1.
The class W of almost wellordered sets is closed under W-indexed unions, subsets, images, finite products and sets of finite sequences.

For this proof see [10, Lemma 3]. The following lemma gives some of the important properties of $\mathcal{M}15$. The following lemma can be found in [10], (a) as Lemma 1, (b) as Lemma 2 and (c) as Lemma 4.

Lemma 3.2 (Properties of $\mathcal{M}15$).
In $\mathcal{M}15$ the following hold.

(a) All ultrafilters on ω are principal.

(b) All ultrafilters on a wellorderable set are principal.

(c) All ultrafilters on sets in W are principal.

β-inaccessibility

The main result of this model is the following.

Theorem 3.3.
In $\mathcal{M}15$, all ultrafilters are principal.

Sketch proof. By Lemma 3.2 it suffices to show that in $\mathcal{M}15$, all sets are in W.

From the definition of $\mathcal{M}15$ we see that there is a map $g \colon \mathrm{Ord}^M \times {}^{<\omega}S \to \mathcal{M}15$ that is onto and definable in $M[G]$ from S. Therefore every set $x \in \mathcal{M}15$ is the image via a function in $\mathcal{M}15$ of an $\mathcal{M}15$-subset of $\alpha \times {}^{<\omega}S$ for some $\alpha \in \mathrm{Ord}$. By Proposition 3.1 we see that it's enough to show that $S \in W$; then we'll have that $x \in W$ as an image of a function and therefore the theorem will be proved.

In $\mathcal{M}15$ each $\overline{a_n} \cup \overline{\omega \setminus a_n}$ is in W because they are countable. As a singleton, $\{f\}$ is in W. Since $f \in \mathcal{M}15$, the family $A \overset{\text{def}}{=} \{\{f\}, \overline{a_n} \cup \overline{\omega \setminus a_n}\ ; n \in \omega\}$ is in W because it is countable. Finally, since W is closed under wellordered unions, $S = \bigcup A$ is in W. qed

At this point we remind the reader that a cardinal κ is a β-strong limit cardinal if for all $\lambda < \kappa$, the set $\beta\lambda$ of all ultrafilters on λ is wellorderable and injectable into κ. The model $\mathcal{M}15$ is a witness that this definition can trivialise.

Theorem 3.4.
In $\mathcal{M}15$, every cardinal κ is a β-strong limit cardinal.

Proof. Fix a cardinal κ and let $\lambda < \kappa$ be arbitrary. Since every ultrafilter is principal, it is generated by a singleton $\{\alpha\} \subset \lambda$. Call u_α the corresponding ultrafilter and define $h\colon \beta\lambda \to \lambda$ for every $u_\alpha \in \beta\lambda$ to be $h(u_\alpha) \overset{\text{def}}{=} \alpha$. This is clearly a bijection and therefore $\beta\lambda < \kappa$. qed

By this result we see that at least so far we only have interesting results if we assume strong axioms (see Theorem 2.12) and in particular AD which has large cardinal strength.

For the modal logician

It is worth noting that this construction, $\mathcal{M}15$, models a very strange situation from the point of view of the modal logician. One of the reasons is because ultrafilters play an important role in modal logic. To construct the ultrafilter extension of a model we do not use any choice. Therefore in $\mathcal{M}15$ we can still make ultrafilter extensions of frames, but then this construction is trivial. If one takes a look at the proof of [12, Equation 2.1 in page 95] then one would easily see that when all ultrafilters are principal, then the resulting ultrafilter extension is isomorphic to the original model.

Another very important modal theoretic construction is the one of canonical models. This construction gives a model for every system of modal logic and thus is used for proving completeness for normal modal logics. If we do not restrict ourselves to countable languages but we talk about *any*[3] modal language then what we need for this construction is precisely the axiom of choice which does not hold in $\mathcal{M}15$ (see [3]).

But since normal[4] modal logics usually have only countably many formulas, then we usually would not need the axiom of choice. This is because the construction of a canonical model is based on [12, Lemma 4.17] which is known as Lindenbaum's Lemma. It states that "If Σ is a consistent set of formulas then there is a maximal consistent set Σ^+ such that $\Sigma \subseteq \Sigma^+$". Since wellorderable languages have an enumeration of their formulas, this construction does not involve the axiom of choice and therefore in $\mathcal{M}15$, completeness does hold for

[3]In particular, if we talk about a non well orderable modal language.
[4]For the definition of a normal modal logic see [12, Definition 4.1 and 4.3].

wellorderable modal languages.

Finally we take a look at the Stone Representation Theorem (SRT), which is of great importance in modal logic. In ZF, the Stone Representation Theorem is equivalent to the Boolean Prime Ideal (BPI), a known weaker former of choice which states that "Every Boolean algebra has a maximal (or prime) ideal". For this equivalence we refer the reader to [3] where SRT is Form 14B and BPI is Form 14. Note that in [3] equivalent forms have the same form number.

Form 14 or BPI doesn't hold in the model $\mathcal{M}15$ because it implies the statement "There exist non principal ultrafilters" (see [3, Form 206]). This can also be seen in the "Models" section of [3]. Thus, the SRT does not hold in $\mathcal{M}15$ which could be a strong motivation for a modal logician to prefer a formalisation where at least BPI holds.

3.2 Symmetric submodels

From now on we will concentrate in our other definitions for strong limits but before that we will introduce this very useful technique of symmetric submodels.

The method of taking symmetric submodels of a generic extension is used to create models where AC does not hold everywhere. This method is inspired by the older method of permutation models. In that method, the base model is a model of ZFCA, that is ZFC with the axiom for the existence of atoms. Atoms have the same defining property of the empty set but are not equal to it. Therefore atoms are excluded from the axiom of extensionality. These atoms can be as many as we want (by adding an axiom as "the set of atoms is κ big") and they are indistinguishable from each other.

By permuting these atoms we can construct models of $\mathsf{ZFA} + \neg\mathsf{AC}$. In this paragraph we'll see we'll see that the way to do that is very similar to the way we construct symmetric submodels. That is by having a permutation group of some sort; then a filter on it and allowing in our new model to exist only sets that remain intact from the permutations of a set in the filter. These sets are called symmetric, thus the name "symmetric submodels".

The definitions, lemmata and theorems below will have references in [2, Ch.15], where they are stated for Boolean-valued models. One can translate them for separative[5] partial orders by using [2, 14.38 and Lemma 14.37], where a separative partial order is a partial order $\langle \mathbb{P}, \leq \rangle$ such that for every $p, q \in \mathbb{P}$ it holds that

$$\text{if } p \not\leq q \text{ then there exists an } r \leq p \text{ that is incompatible with } q.$$

All the partial orders we are going to use in this thesis are separative and below in the definition, lemmata and theorems when we say "partial order" we mean "separative partial order".

[5] In fact the restriction that the partial order is separative is not necessary. The construction works for non-separative partial orders as well.

Definition 12. Let $\mathbb{P}$, $\mathbb{Q}$ be partial orders. If $i \colon \mathbb{P} \to \mathbb{Q}$, then define by recursion on $\tau \in M^{\mathbb{P}}$,

$$i_*(\tau) = \{(i_*(\sigma), i(p)) \; ; \; (\sigma, p) \in \tau\}.$$

It's easy to see that $i_*(\tau) \in M^{\mathbb{Q}}$ and that the definition of i_* is absolute for M thus if $\mathbb{P}, \mathbb{Q} \in M$ then $i_* \colon M^{\mathbb{P}} \to M^{\mathbb{Q}}$.

Definition 13. If $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$ is a partial order, an automorphism of $\mathbb{P}$ is a bijection of $\mathbb{P}$ to itself, which preserves $\leq$ and satisfies $i(\mathbb{1}) = \mathbb{1}$.

First let M be a countable transitive model and let $\mathbb{P}$ a partial order in M. Take $\mathcal{G}$ to be a group of automorphisms of $\mathbb{P}$ and $\mathcal{F}$ a filter on $\mathcal{G}$. According to Definition 12, for every $\pi \in G$, we can recursively define an automorphism π_* of the class of names $M^{\mathbb{P}}$.

Lemma 3.5 (Symmetry Lemma).
Let $\mathbb{P}$ be any separative partial order in M, let π be an automorphism of $\mathbb{P}$ and let φ be a formula. Then for the extended automorphism π_* of $M^{\mathbb{P}}$ and for all names $\tau_1, \ldots, \tau_n$, it holds that

$$p \Vdash \varphi(\tau_1, \ldots, \tau_n) \Longleftrightarrow \pi p \Vdash \varphi(\pi_* \tau_1, \ldots, \pi_* \tau_n)$$

For the proof of this see [2, 14.37].

Definition 14 (Symmetry group). Let $\mathcal{G}$ be a group of automorphisms of $\mathbb{P}$ and let $\mathcal{F}$ be a filter over $\mathcal{G}$. For each $\tau \in M^{\mathbb{P}}$, we define its symmetry group:

$$\mathsf{sym}(\tau) \stackrel{\text{def}}{=} \{\pi \in \mathcal{G} \; ; \; \pi_* \tau = \tau\}$$

Given a group $\mathcal{G}$ and a filter $\mathcal{F}$ over $\mathcal{G}$, we say that τ is symmetric if $\mathsf{sym}(\tau) \in \mathcal{F}$. The class of all hereditarily symmetric names is denoted by HS and contains all names in $M^{\mathbb{P}}$ that are symmetric and all elements in their transitive closure are also symmetric.

Lemma 3.6.
If π is an automorphism of $\mathbb{P}$, then it holds that for every $\check{x} \in M^{\mathbb{P}}$, it holds that $\pi_* \check{x} = \check{x}$ and therefore all $\check{x}$ are in HS.

The proof of this lemma is a simple induction using the definition of $\check{x}$.

Definition 15 (Symmetric submodel). Let M be the ground model, $\mathbb{P}$ a separative partial order in M, $\mathcal{G}$ a group of automorphisms of $\mathbb{P}$ in M and let $\mathcal{F}$ be a filter over $\mathcal{G}$ that is also in M. If G is a filter on $\mathbb{P}$ that is M-generic, then we define the symmetric submodel of $M[G]$ with respect to $\mathcal{G}$, $\mathcal{F}$ as follows,

$$N[\mathcal{G}, \mathcal{F}] \stackrel{\text{def}}{==} \{(\tau)^G \; ; \; \tau \in M^{\mathbb{P}}\}$$

Theorem 3.7.
With the notation of Definition 15, the symmetric submodel $N[\mathcal{G}, \mathcal{F}]$ is a transitive model of ZF and $M \subseteq N[\mathcal{G}, \mathcal{F}] \subseteq M[G]$.

This theorem is proved in [2, Lemma 15.51].

3.3 The Feferman-Lévy model

This model was constructed by Feferman and Lévy in 1963 (for the abstract see [5]). In this model, the reals are a countable union of countable sets and therefore both AC and AD fail. By looking at Theorem 2.5 we see that in this model, ω_1 is a $\bar{\mathfrak{l}}$-strong cardinal and by Lemma 1.4 that it's not $\bar{\mathfrak{s}}$-strong. This is an interesting model and we will prove some nice properties of it and some results from it.

Construction

Let M be a countable transitive model of $\mathsf{ZFC}+\mathbb{V}=\mathbb{L}$. Our goal is to construct a model $N[\mathcal{G},\mathcal{F}]$ such that $\omega_1^{N[\mathcal{G},\mathcal{F}]}$ is singular. We know that ω_ω^M is singular so we'll need to collapse all cardinals smaller than ω_ω^M into ω, and on the same time make sure ω_ω^M is not collapsing. Note that if all ω_n^M were collapsed onto ω, then $\mathsf{CUC}(\mathbb{R})$ suffices to make ω_ω^M countable as well. This is why we are going to use the technique of symmetric submodels.

Firstly, we are going to construct a generic extension $M[G]$ by adjoining collapsing maps for every ordinal smaller than ω_ω^M, i.e., for every $n\in\omega$, we are going to construct a function $f_n\colon\omega\twoheadrightarrow\omega_n^M$. Consider the set

$$\mathbb{P}\overset{\text{def}}{=}\{p\in{}^{\omega\times\omega}\omega_\omega\ ;\ \mathsf{dom}(p)<\omega\text{ and for every }(n,i)\in\mathsf{dom}(p),\ p(n,i)<\omega_n\}$$

with the partial order of extension ($p\leq q\iff p\supseteq q$). Take G to be an M-generic filter on $\mathbb{P}$.

Lemma 3.8.
In $M[G]$, for every $n\in\omega$, there is a function $f_n\colon\omega\twoheadrightarrow\omega_n^M$.

Proof. We know that $f=\bigcup G$ is a function on $\omega\times\omega$. Note that for $(n,i)\in\omega\times\omega$ and $\alpha\in\omega_n^M$, we have that

$$f(n,i)=\alpha\iff\text{ there is a }p\in G\text{ such that }(n,i)\in\mathsf{dom}(p)\text{ and }p(n,i)=\alpha.$$

Moreover it holds that

$$\text{for every }p\in G\text{ if }(n,i)\in\mathsf{dom}(p)\text{, then }p(n,i)=f(n,i).\qquad(1)$$

For every $n\in\omega$, define $f_n(i)=f(n,i)$. We will show that for every $n\in\omega$, f_n is a surjection of ω onto ω_n^M. Fix $n\in\omega$ and note that f_n is always a function because f is. Take arbitrary $\alpha\in\omega_n^M$ and define

$$E_\alpha\overset{\text{def}}{=}\{p\in\mathbb{P}\ ;\ \alpha\in\mathsf{rng}(p)\text{ and }\exists i\in\omega(n,i)\in\mathsf{dom}(p)\}$$

To show that these are dense sets, fix α and take $p\in\mathbb{P}\setminus E_\alpha$. We have that $\alpha\notin E_\alpha$ but since $\mathsf{dom}(p)<\omega$, there is an $i\in\omega$, such that $(n,i)\notin\mathsf{dom}(p)$ and for this i we can see that:

$$p\subseteq p\cup\{(n,i,\alpha)\}\in E_\alpha$$

Therefore E_α is a dense set for every α, therefore for every $\alpha\in\omega_n^M$, $G\cap E_\alpha\neq\varnothing$ and so for every $\alpha\in\omega_n^M$ there is a $p\in G$ such that for some $i\in\omega$, $p(n,i)=\alpha$, i.e., $f_n(i)=\alpha$ which means that for every $n\in\omega$, $\mathsf{rng}(f_n)=\omega_n^M$. qed

So we constructed surjections from ω to ω_n^M for every $n \in \omega$ but since the generic models satisfies AC we have that in $M[G]$ all ω_n^M are now countable. This means that since ω_ω^M is now the (countable) union of these countable sets and AC holds, ω_ω^M has also collapsed to a countable ordinal. To stop ω_ω^M from collapsing, we will construct a symmetric submodel $N[\mathcal{G}, \mathcal{F}]$ of $M[G]$ where all f_n will remain, but countable unions of countable sets will not be necessarily countable (in particular, ω_ω^M will be the first uncountable ordinal).

Let $\mathsf{Perm}_{\omega\times\omega}$ be the group of all permutations of $\omega \times \omega$ and let

$$\mathcal{G} \stackrel{\text{def}}{=} \{\pi \in \mathsf{Perm}_{\omega\times\omega} \; ; \forall n, i \in \omega \exists j \in \omega(\pi(n,i) = (n,j))\}$$

Definition 16. Every $\pi \in \mathcal{G}$ induces an automorphism of $\mathbb{P}$ as follows:
for $p \in \mathbb{P}$,

$$\mathsf{dom}(\hat{\pi}p) = \{\pi(n,i) \; ; (n,i) \in \mathsf{dom}(p)\}$$
$$\hat{\pi}(p)(\pi(n,i)) = p(n,i)$$

We will denote the induced automorphism also by π. Note that this is indeed an automorphism of $\mathbb{P}$ because for every $p \in \mathbb{P}$ and every $\pi \in \mathcal{G}$, since $\mathcal{G}$ contains only those permutations of $\omega \times \omega$ that keep the first coordinate stable, we have that $\pi(p) \in \mathbb{P}$. This means that $\pi p(n,i)$ will still map to an ordinal $< \omega_n$ and therefore πp is still in $\mathbb{P}$.

Now, for every $n \in \omega$, define the set

$$K_n \stackrel{\text{def}}{=} \{\pi \in \mathcal{G} \; ; \forall k \leq n \forall i \in \omega(\pi(k,i) = (k,i))\}$$

and let $\mathcal{F}$ be the filter on $\mathcal{G}$ that is generated by $\{K_n \; ; \; n \in \omega\}$. As we saw in the previous section in Theorem 3.7 this permutation group and this filter create a symmetric submodel $N[\mathcal{G}, \mathcal{F}]$ of $M[G]$ by taking the interpretation of all hereditarily symmetric names.

This model $N[\mathcal{G}, \mathcal{F}]$ we just described, we are going to call $\mathcal{M}9$ which is a name taken from [3].

Some properties

We will show that all the collapsing functions are in $\mathcal{M}9$. The following definition will be useful.

Definition 17 (Restriction). For any $p \in \mathbb{P}$ and any $m \in \omega$, we define $[q]^m$ to be the restriction of q to the set $\{(k, \beta, \beta) \; ; k \leq n \text{ and } \beta \in \omega_\omega^M\}$.

Lemma 3.9.
For every $n \in \omega$, $f_n \in \mathcal{M}9$.

Proof. For every $n \in \omega$, look at the set

$$F_n \stackrel{\text{def}}{=} \{(\widecheck{(i,\alpha)}, [p]^n) \; ; p \in \mathbb{P} \text{ and } (n,i) \in \mathsf{dom}(p) \text{ and } p(n,i) = \alpha\}$$

This is a name of f_n, for every $n \in \omega$, because:

$$\begin{aligned}(F_n)_G &= \{(\check{\widehat{(i,\alpha)}})_G \; ; \; [p]^n \in G \text{ and } (n,i) \in \mathsf{dom}(p) \text{ and } p(n,i) = \alpha\} \\ &= \{(i,\alpha) \; ; \; p \in G \text{ and } (n,i) \in \mathsf{dom}(p) \text{ and } p(n,i) = \alpha\} \\ &= f_n\end{aligned}$$

Of course, the last equality is because of (1) in Lemma 3.8. Note that by Lemma 3.6, for every $(i,\alpha) \in \omega \times \omega_\omega$, $\check{\widehat{(i,\alpha)}} \in \mathsf{HS}$. So it suffices to show that f_n itself has a symmetric name as well. Let $\pi \in K_n$. We have that:

$$\begin{aligned}\pi(F_n) &= \{(\pi_* \check{\widehat{(i,\alpha)}}, \pi([p]^n)) \; ; \; p \in \mathbb{P} \text{ and } (n,i) \in \mathsf{dom}(p) \text{ and } p(n,i) = \alpha\} \\ &= \{(\check{\widehat{(i,\alpha)}}, [p]^n) \; ; \; p \in \mathbb{P} \text{ and } (n,i) \in \mathsf{dom}(p) \text{ and } p(n,i) = \alpha\} \\ &= F_n\end{aligned}$$

Therefore $F_n \in \mathsf{HS}$, i.e., $f_n \in \mathcal{M}9$. qed

To show that ω_ω^M is a cardinal in $\mathcal{M}9$, we use the following proposition:

Proposition 3.10.
For every name τ, if $\mathsf{sym}(\tau) \supseteq K_n$ and $p \Vdash \varphi(\tau)$, then $[p]^n \Vdash \varphi(\tau)$.

Proof. Assume that $[p]^n$ does not force $\varphi(\tau)$. By [2, Theorem 14.7(ii)(a)], there is a $q \supset [p]^n$ in $\mathbb{P}$, such that $q \Vdash \neg\varphi(\tau)$. Let $m \overset{\text{def}}{=} \mathsf{max}\{k \; ; \; \exists j (k,j) \in \mathsf{dom}(p)\}$ and $\ell \overset{\text{def}}{=} \mathsf{max}\{k \; ; \; \exists j (k,j) \in \mathsf{dom}(q)\}$.

- If $m \leq n$ then $[p]^n = p$ and so $[p]^n \Vdash \varphi(\tau)$, contradiction.
- If $m > n$ then define a permutation π of $\omega \times \omega$ as follows:

  $$\pi(k,j) = \begin{cases} (k,j) & k \leq n \\ (\ell + i, j) & k \in (n, m] \\ (n + i, j) & k \in (m, m + \ell - n) \\ (k,j) & k \geq m + \ell - n \end{cases}$$

  We want to show that for this π, it holds that $\pi p \parallel q$. We know that $[p]^n \subseteq q$ so also $\pi[p]^n \subseteq q$. Look at $\mathsf{dom}(\pi p) = \{\pi(k,j) \; ; \; (k,j) \in \mathsf{dom}(p)\}$. It's clear that

  $$\mathsf{dom}(\pi p) = \{(k,j) \; ; \; k \leq n\} \cup \{(\ell + k, j) \; ; \; k \in (n, m]\}$$

  So, $\mathsf{dom}(\pi p) \cap \mathsf{dom}(q) \subseteq \{(k,j) \; ; \; k \leq n\}$ and $\pi[p]^n = [q]^n$. So for every $(k,j) \in \omega \times \omega$, let

  $$r(k,j) = \begin{cases} \pi(k,j) = q(k,j) & \text{if } k \leq n \\ q(k,j) & \text{if } k \in (n,m] \\ \pi p(k,j) & \text{if } k \in (m, m + \ell - n) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

  Clearly, $r \supseteq \pi p$ and $r \supseteq q$, therefore $\pi p \parallel q$

Now by Lemma 3.5 we have that $\pi p \Vdash \varphi(\pi\tau)$. But $\pi \in K_n \subseteq \mathsf{sym}(\tau)$, therefore $\pi p \Vdash \varphi(\tau)$. Since $\pi p, q$ are compatible by r, by [2, Theorem 14.7(i)(a)] and by what we have so far:

$$r \Vdash \varphi(\tau) \text{ and } r \leq q \text{ and } q \Vdash \neg\varphi$$

which is a contradiction to [2, Theorem 14.7(ii)(a)].

qed

Theorem 3.11.
The ordinal ω_ω^M is a cardinal in $\mathcal{M}9$.

Proof. Assume towards contradiction that g is a surjection from ω onto ω_ω^M that is in $\mathcal{M}9$ and let $\dot{g}$ be a symmetric name for g. By the forcing theorem (see [2, Theorem 14.6]), there is a $p_0 \in G$ such that p_0 forces "$\dot{g}$ is a function from $\check{\omega}$ onto $\check{\omega}_\omega^M$". Since $g \in \mathcal{M}9$, $\mathsf{sym}(\dot{g}) \in \mathcal{F}$, i.e., there is an $n \in \omega$ such that $K_n \subseteq \mathsf{sym}(\dot{g})$. Fix n.

For every $k \in \omega$, define

$$A_k \overset{\text{def}}{=} \{\alpha \in \omega_\omega^M \ ; \exists p \in \mathbb{P}(p \supseteq p_0 \text{ and } p \Vdash \dot{g}(k) = \alpha)\}.$$

Note that the requirement $p_0 \subseteq p$ is just to make sure that g is still a surjection from ω to ω_ω^M (look at [2, Theorem 14.7(i)(a)]). If for every $k \in \omega$, it was true that $|A_\kappa| \leq \omega_n^M$ then $\omega_\omega^M \leq \bigcup_{k\in\omega} \omega_n^M$ which is a contradiction in M. Therefore, for at least one $k \in \omega$, $|A_k| \geq \omega_{n+1}^M$. Fix this k.

For every $\alpha \in \omega_\omega^M$, define $B_\alpha = \{p \in \mathbb{P} \ ; p \supset p_0 \text{ and } p \Vdash \dot{g}(k) = \alpha\}$.

Claim 1. Let $p, q \in \mathbb{P}$ such that $p, q \supseteq p_0$ and let α, $\beta \in \omega_\omega^M$. If $p \in B_\alpha$ and $q \in B_\beta$ and $\alpha \neq \beta$ then $q \perp p$.

Proof of Claim. Assume towards contradiction that $p \parallel q$, i.e., that there is an $r \supset p_0$ such that $r \leq p$ and $r \leq q$. For this r, since $p \in B_\alpha$ and by [2, Theorem 14.7(i)(a)], $r \Vdash \dot{g}(k) = \alpha$. Similarly because $q \in B_\beta$, $r \Vdash \dot{g}(k) = \beta \neq \alpha$. Contradiction. ⊣

Therefore there must be at least ω_{n+1}^M incompatible conditions that force g to take different values with each condition. So define W to be this set of pairwise incompatible conditions such that for every $p \in W$, $p \supset p_0$. Also, there must be more than ω_{n+1}^M distinct ordinals α_p (one for every $p \in W$), such that for every $p \in W$, it holds that $p \Vdash \dot{g}(k) = \alpha_p$.

For every $p \in W$, by Lemma 3.10 we have that

$$[p]^n \Vdash \dot{g}(k) = \alpha_p.$$

But note that the set $\{[p]^n \ ; p \in \mathbb{P}\}$ has cardinality only ω_n^M, by definition of $\mathbb{P}$ and therefore this result from Lemma 3.10 shows that there must be at least two distinct $p, q \in W$ such that $[p]^n \Vdash \dot{g}(k) = \alpha_q$. Since $p \supseteq [p]^n$, this means that $p \Vdash \dot{g}(k) = \alpha_q$. But by definition, $p \Vdash \dot{g}(k) = \alpha_p$ and we assumed $p \neq q$ to mean also that $\alpha_p \neq \alpha_q$. Then $\dot{g}$ cannot be a function so g cannot be a function. Contradiction. qed

The next lemma is an observation about the ordinals in $\mathcal{M}9$ compared to the ordinals in $M[G]$.

Lemma 3.12.
The ordinal $\omega_1^{\mathcal{M}9}$ is singular in $\mathcal{M}9$ and for every $n \in \omega$, $\omega_{n+2}^{\mathcal{M}9} = \omega_{n+1}^{M[G]}$.

Proof. Since ω is absolute, $\omega^{\mathcal{M}9} = \omega^{M[G]}$. By Lemma 3.9, we have that in $\mathcal{M}9$, there is no function that makes ω_ω^M countable but there are functions f_n that make all infinite ordinals smaller than ω_ω^M countable. By Theorem 3.11 we have that ω_ω^M is not countable in $\mathcal{M}9$. This means firstly that ω_ω^M is a cardinal and in particular $\omega_\omega^M = \omega_1^{\mathcal{M}9}$. Secondly, we have that $(\mathsf{cof}(\omega_\omega^M))^M = \omega$. Since for models $M \subseteq N$, the cofinality can only decrease from M to N, we have that $\mathsf{cof}(\omega_1^{\mathcal{M}9}) \leq \mathsf{cof}(\omega_\omega^M) = \omega$ and therefore in $\mathcal{M}9$, ω_1 is singular.

Since $M[G] \models \mathsf{ZFC}$ (see [2, Theorem 14.5]), it holds that $\mathsf{CUC}^{M[G]}$ and since $\omega_\omega^M = \bigcup_{n\in\omega} \omega_n^M$ holds, by Lemma 3.8 we get that ω_ω^M is countable in $M[G]$ and therefore cannot be $\omega_1^{M[G]}$. Note that the partial order $\mathbb{P}$ is a subset of the partial order $\mathsf{Fn}(\omega \times \omega, \omega_\omega)$ and by [1, Lemma 6.10], $\mathsf{Fn}(\omega \times \omega, \omega_\omega)$ has the $(\omega_\omega^{<\omega})^+$-c.c. in M. We have that

$$(\omega_\omega^{<\omega})^+ = (\omega_\omega)^+ = \omega_{\omega+1}$$

and so in M the partial order $\mathsf{Fn}(\omega \times \omega)$ has the $\omega_{\omega+1}$-c.c.. By [1, Lemma 6.9] and since ω is regular, this means that $\mathsf{Fn}(\omega \times \omega)$ preserves cardinals and cofinalities $\geq \omega_{\omega+1}^M$. So none of these ordinals above and with $\omega_{\omega+1}^M$ has collapsed in $M[G]$ and therefore neither in $\mathcal{M}9$. So, $\omega_1^{M[G]} = \omega_{\omega+1}^M = \omega_2^{\mathcal{M}9}$ and for the same reasons, for every $n \in \omega$, $\omega_{n+2}^{\mathcal{M}9} = \omega_{n+1}^{M[G]}$ holds. qed

So indeed in the model $\mathcal{M}9$, $\omega_1^{\mathcal{M}9}$ is singular and moreover countable unions of countable sets of reals are not necessarily countable ($\neg\mathsf{CUC}(\mathbb{R})$). Now we are going to see that $\mathbb{R}$ is such a countable union of countable sets.

Theorem 3.13.
The set of all reals in the symmetric model $\mathcal{M}9$ is a countable union of countable sets.

Proof. Using AC in $\mathbb{V}$, we get a function $\dot{}\ : x \mapsto \dot{x}$ from $\mathcal{M}9$ to HS, such that $(\dot{x})_G = x$. If $x \in \mathcal{M}9$, then we know that $\dot{x} \in \mathsf{HS}$ therefore there is n such that $K_n \subseteq \mathsf{sym}(\dot{x})$. Define $C_n \stackrel{\text{def}}{=} \{x \in \mathbb{R}^{\mathcal{M}9} \,;\, K_n \subseteq \mathsf{sym}(\dot{x})\}$ and note that

$$\mathbb{R}^{\mathcal{M}9} = \bigcup_{n\in\omega} C_n.$$

Therefore if we prove that for every $n \in \omega$, C_n is countable, then we proved the theorem.

For $x \in \mathbb{R}^{\mathcal{M}9}$ we define a name $\ddot{x}$ such that

$$\ddot{x} \stackrel{\text{def}}{=} \{(\check{k}, [p]^n) \,;\, p \Vdash \check{k} \in \dot{x}\}.$$

It's clear that $(\dot{x})_G = (\ddot{x})_G = x$. Define

$$C_n' \stackrel{\text{def}}{=} \{\ddot{x} \,;\, x \in C_n\} \subseteq M^{\mathbb{P}}$$

and note that $\ddot{}$ is an injection from C_n into C'_n. So if $\mathcal{M}9 \models$ "C'_n is countable", then C_n will be countable and we will have proved the theorem.

Clearly, $C'_n \subseteq \wp^{(k)}\omega_n^M$ for some finite[6] k. In $M \subseteq \mathcal{M}9$ there is a bijection $\wp^{(k)}\omega_n^M \rightarrowtail\!\!\!\!\rightarrow \omega_{n+k}^M$ because GCH holds in M. Thus we can get an injection

$$C_n \rightarrowtail C'_n \rightarrowtail \wp^{(k)}\omega_n^M \rightarrowtail\!\!\!\!\rightarrow \omega_{n+k}^M.$$

But in $\mathcal{M}9$, ω_{n+k} is countable, therefore C_n is countable in $\mathcal{M}9$. qed

Consequences

Here we will see two separation results for our diagrams of §1.2. In particular we will see that the model $\mathcal{M}9$ separates $<_{\bar{\mathbf{s}}}$ from $<_{\mathbf{s}}$ in the limit case and it also separates $<_{\bar{\mathbf{s}}}$ from $<_{\bar{\mathbf{i}}}$ in the successor stage. Moreover we will see two more interesting results for this model, one over the inaccessibility by reals discussed in §2.1 and one topological result for co-analytic sets of reals.

For our first separation result we will first prove a more general proposition.

Proposition 3.14.
Whenever a partial order has cardinality κ and $\lambda > \kappa$, then $\left|(2^\lambda)^{M[G]}\right| = \left|(2^\lambda)^M\right|$.

Proof. As in the proof of Theorem 3.13, we use AC in $\mathbb{V}$ to get a function $\dot{} : (2^\lambda)^{M[G]} \to M^{\mathbb{P}}$ such that $\dot{x}_G = x$ for every $x \in (2^\lambda)^{M[G]}$. Then we can define

$$\ddot{x} \stackrel{\text{def}}{=} \{(\check{\beta}, p) \,;\, p \Vdash \check{\beta} \in \dot{x} \text{ and } \beta \in \lambda\}.$$

For every $x \in (2^\lambda)^{M[G]}$, the name $\ddot{x}$ is an M-subset of $\lambda \times \mathbb{P}$.

As $\mathbb{P}$ has M-cardinality $\kappa < \lambda$, the set $\lambda \times \kappa$ can be coded as a subset of λ, and thus there are at most as many $M[G]$-subsets of λ as there are M-subsets of λ, and therefore we have that $\left|(2^\lambda)^{M[G]}\right| = \left|(2^\lambda)^M\right|$. qed

Theorem 3.15 (Separation of limit case for $<_{\bar{\mathbf{s}}}$ and $<_{\mathbf{i}}$).
In the model $\mathcal{M}9$, ω_ω is an $\bar{\mathbf{s}}$-strong limit, but not an $\mathbf{i}$-strong limit.

Proof. We are working in $\mathcal{M}9$. The cardinal ω_ω is not an $\mathbf{i}$-strong limit because $\omega < \omega_\omega$ holds but 2^ω cannot be injected into any ordinal and therefore not into any ordinal below ω_ω.

By Lemma 3.12 we have that for every $n > 0$, $\omega_n^{M[G]} = \omega_{\omega+n}^M$, so by Proposition 3.14, we have that for every $n > 0$, $\left|(2^{\omega_n^{M[G]}})^{M[G]}\right| = \left|(2^{\omega_{\omega+n}^M})^M\right|$. Since GCH^M holds, we get that

$$\left|(2^{\omega_n^{M[G]}})^{M[G]}\right| = \omega_{\omega+n+1}^M = \omega_{n+1}^{M[G]}. \tag{2}$$

So in particular, in $M[G]$, there cannot be a surjection from $(2^{\omega_n^{M[G]}})^{M[G]}$ onto $\omega_{n+2}^{M[G]}$, for $n > 0$.

[6]Probably $k = 6$.

***Claim* 1.** In $\mathcal{M}9$, for every $n \in \omega$, there is no surjection from $(2^{\omega_n^{\mathcal{M}9}})^{\mathcal{M}9}$ onto $\omega_{n+2}^{\mathcal{M}9}$.

Proof of Claim. If $n > 0$ then assume for a contradiction that

$$\mathcal{M}9 \models (2^{\omega_n}) \twoheadrightarrow \omega_{n+2}.$$

Then since $M[G] \supset \mathcal{M}9$ and because of Lemma 3.12, we have that

$$M[G] \models (2^{\omega_n^{\mathcal{M}9}})^{\mathcal{M}9} \twoheadrightarrow \omega_{n+1}.$$

Call this surjection $g \colon (2^{\omega_n^{\mathcal{M}9}})^{\mathcal{M}9} \to \omega_{n+1}^{M[G]}$.

We work now in $M[G]$. Note that $(2^{\omega_n^{\mathcal{M}9}})^{\mathcal{M}9} = (2^{\omega_{n-1}^{M[G]}})^{\mathcal{M}9} \subseteq 2^{\omega_{n-1}}$ holds and therefore we can use g to construct a surjection $g' \colon 2^{\omega_{n-1}} \twoheadrightarrow \omega_{n+1}$, contradiction to 2 which states that in $M[G]$ we have that $|2^{\omega_{n-1}}| = \omega_n$.

If $n = 0$ then in $M[G]$, the set $(2^{\omega})^{\mathcal{M}9}$ is countable and therefore it cannot surject onto ω_1 which by Lemma 3.12 is $\omega_2^{\mathcal{M}9}$. Therefore in $\mathcal{M}9 \subseteq M[G]$, there cannot exist a surjection from 2^{ω} onto ω_2 either. $\dashv$

A fortiori, in $\mathcal{M}9$ none of the 2^{ω_n} can surject onto ω_ω. Therefore in $\mathcal{M}9$ the cardinal ω_ω is an $\bar{\mathbf{s}}$-strong limit cardinal. qed

By the proof of Theorem 3.15 it follows that no uncountable limit cardinal is an $\mathbf{i}$-strong limit cardinal. Note that all cardinals above $\omega_\omega^{\mathcal{M}9}$ remain the same as in the ground model and that $\mathsf{GCH}^{\mathcal{M}9}$ holds above $\omega_\omega^{\mathcal{M}9}$. Now, since our partial order has too small cardinality to add surjections from $2^\alpha \twoheadrightarrow \lambda$ for $\alpha < \lambda$ an uncountable limit cardinal above $\omega_\omega^{\mathcal{M}9}$, we have that all uncountable limit cardinals in $\mathcal{M}9$ are $\bar{\mathbf{s}}$-strong and not $\mathbf{i}$-strong.

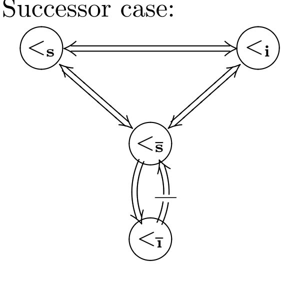
Theorem 3.16 (Separation of successor case for $<_{\bar{\mathbf{s}}}$ and $<_{\bar{\mathbf{i}}}$).
In $\mathcal{M}9$, ω_1 is a $\bar{\mathbf{i}}$-strong cardinal but not a $\bar{\mathbf{s}}$-strong cardinal.

Proof. We already saw in Lemma 1.5 that as a successor, ω_1 can never be $\bar{\mathbf{s}}$-strong. To show that ω_1 is $\bar{\mathbf{i}}$-strong, we must show that ω_1 is not injectable into the reals; but that is true in this model because the reals are a countable union of countable sets (see Theorem 3.13) and by Theorem 2.5 this means that every wellorderable subset of the reals is countable. Therefore, ω_1 cannot be injected into $\mathbb{R}$ and thus is an $\bar{\mathbf{i}}$-strong cardinal. qed

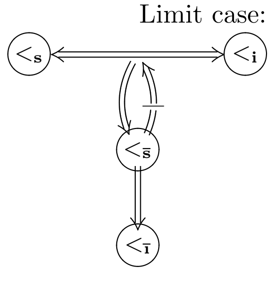
Corollary 3.17.
It is consistent with ZF *that there exist a $\bar{\mathbf{i}}$-strong successor cardinal.*

Therefore we can now update our diagrams from § 1.2 as follows.

Successor case:



Limit case:



There is still one separation result left to study, that is the separation of $<_{\bar{s}}$ from $<_{\bar{i}}$ for limit cardinals. Later we will see that this cannot be answered easily. But now let's focus at two more interesting properties of $\mathcal{M}9$. These properties are in Theorem 3.18 and Theorem 3.21 below. These theorems are motivated by notes based on e-mail discussions of Dr. Benedikt Löwe with Prof. Ralf Schindler. The proof of Theorem 3.21 comes from those notes.

Firstly we remind the reader that in §2.4 we promised to show that $\omega_1^{\mathcal{M}9}$ is a strong limit cardinal in $\mathbb{L}$ and now we're going to do just that.

Theorem 3.18 (Inaccessibility by reals).
In $\mathcal{M}9$, $\omega_1^{\mathbb{L}[a]} < \omega_1$, for any $a \in \mathcal{M}9$

Proof. As we said in Definition 4, for every $a \in \mathcal{M}9$, $\mathbb{L}[a]$ is an inner model of ZFC. By Theorem 2.8 it suffices to show that in $\mathcal{M}9$, $\omega_1 \not\hookrightarrow \mathbb{R}$. By Theorem 3.13 we know that $\mathbb{R}^{\mathcal{M}9}$ is a countable union of countable sets and by Theorem 2.5 this implies that every wellorderable subset of $\mathbb{R}$ has cardinality $\leq \omega$. Therefore in $\mathcal{M}9$, ω_1 cannot inject into $\mathbb{R}$ and by Theorem 2.8 this means that for every $a \in \mathcal{M}9$, $\omega_1^{\mathbb{L}[a]} < \omega_1^{\mathcal{M}9}$. qed

By Theorem 2.9, $\omega_1^{\mathcal{M}9}$ is a strong limit cardinal in every $\mathbb{L}[a]$. If ω_1 was regular, then it would be inaccessible. In that case we would say that ω_1 is inaccessible by reals. Just for the record, we state that Figura showed that it is possible to have $\omega_1 \not\hookrightarrow \mathbb{R}$ and ω_1 is regular (see [6, Theorem 1.8]).

At this point we turn our attention to the topological properties. As we already mentioned in §1.1, for this part we consider $\mathbb{R}$ to be the Cantor space ${}^{\omega}2$, i.e., all binary sequences of length ω. A perfect set is a subset of a topological space that is homeomorphic to the Cantor space; in our case this is the set of all reals.

There is a theorem attributed to Gödel but never published by him (see [8, Theorem 13.12] and the discussion before the theorem) that states that if all $\utilde{\mathbf{\Pi}}^1_1$ sets of reals have the perfect set property (i.e., all their subsets are either countable or contain a perfect set), then $\omega_1^{\mathbb{L}} < \omega_1$. Theorem 2.9 gives that under AC which makes ω_1 regular, the hypothesis "all $\utilde{\mathbf{\Pi}}^1_1$ sets of reals have the perfect set property" implies the existence of an inaccessible cardinal in $\mathbb{L}$.

We'll see that the Feferman-Lévy model $\mathcal{M}9$ satisfies the hypothesis above and therefore it's the combination with the statement "ω_1 is regular" the one that has high consistency strength.

Before we go to the last theorem, we'll prove two useful lemmas. The first lemma talks about what makes a filter on a partial order generic. For this, note that a subset $D \subseteq \mathbb{P}$ is called predense (in $\mathbb{P}$) if for every $p \in \mathbb{P}$, there is a $d \in D$ such that $d \parallel p$, i.e., such that there is an $e \in \mathbb{P}$ with $e \leq d$ and $e \leq p$.

Lemma 3.19 (Equivalent ways of saying "generic").
Let G be a filter on $\mathbb{P}$. The following are equivalent:

(a) The filter G on $\mathbb{P}$ is M-generic.

(b) For every $D \subseteq \mathbb{P}$ such that $D \in M$, if D is predense then $G \cap D \neq \varnothing$.

Proof. It's easy to see that (b) implies (a) because every dense set is predense. For the implication from (a) to (b), let D be a predense subset of $\mathbb{P}$, i.e., be such that for every $p \in \mathbb{P}$, there is a $d \in D$, with $d \parallel p$. Consider the set

$$E \stackrel{\text{def}}{=} \{q \in \mathbb{P} \,;\, \exists d \in D(q \leq d)\}.$$

This E is dense in $\mathbb{P}$ because for every $r \in \mathbb{P}$ there is a $d \in D$ such that $r \parallel d$, i.e., for every $r \in \mathbb{P}$ there is a $d \in D$ and an $e \in \mathbb{P}$ such that $e \leq r$ and $e \leq d$. This e is in E, by definition of E, so we have that for evert $r \in \mathbb{P}$ there is an $e \in E$ such that $e \leq r$, i.e., we have that E is dense in $\mathbb{P}$. Note that $\leq\, \in M$ and $A \in M$, so the defining formula of E is also in M, since Δ_0 formulas are absolute for transitive models.

Since $E \in M$ and G is a M-generic filter, $G \cap E \neq \varnothing$. Take $g \in G \cap E$. Since $g \in E$ we have that there is a $d \in D$ with $g \leq d$. But G is a filter and $g \in G$ so this $d \in D$ is also in G. So $G \cap D \neq \varnothing$. qed

For the second lemma define the following partial orders.

$$\mathbb{P}_n \stackrel{\text{def}}{=} \{p \in \mathbb{P} \,;\, \forall (k, m) \in \mathsf{dom}(p)[k \leq n]\},$$
$$\mathbb{P}'_n \stackrel{\text{def}}{=} \{p \in \mathbb{P} \,;\, \forall (k, m) \in \mathsf{dom}(p)[k > n]\},$$

In other words, the partial order $\mathbb{P}_n$ is all $[p]^n$ for $p \in \mathbb{P}$ and the partial order $\mathbb{P}'_n$ is the set of all $p \setminus [p]^n$ for $p \in \mathbb{P}$.

Lemma 3.20.
For every $a \in \mathcal{M}9$, there is an $m \in \omega$ and a g that is an M-generic filter on $\mathbb{P}_m$ and such that $a \in M[g]$.

Proof. Fix $a \in \mathcal{M}9$. As a symmetric model, i.e., by Theorem 3.7, $\mathcal{M}9$ is transitive and therefore $\mathsf{trcl}(a) \in \mathcal{M}9$. Take a $\mathbb{P}$-name $\tau \in \mathsf{HS}$ for $\mathsf{trcl}(a)$. Clearly, for every $\sigma \in \mathsf{trcl}(\tau) \cap M^{\mathbb{P}}$, σ is a hereditarily symmetric $\mathbb{P}$-name for a set in $\mathsf{trcl}(a)$. This is because the names occurring in $\mathsf{trcl}(\tau) \cap M^{\mathbb{P}}$ are the relevant ones for the elements of $\mathsf{trcl}(a)$ and therefore interpreted as such.

Since τ is in HS, there is an $m \in \omega$ such that $K_m \subseteq \mathsf{sym}(\tau)$. So for every $x \in \mathsf{trcl}(a)$, look at the set $S_x \stackrel{\text{def}}{=} \{\sigma \,;\, \sigma \in \mathsf{trcl}(\tau) \cap M^{\mathbb{P}} \text{ and } (\sigma)_G = x\}$ and by AC take a function $\dot{} : S_x \mapsto \dot{x} \in S_x$. By definition, for every $x \in \mathsf{trcl}(a)$ it holds that $\dot{x} \in \mathsf{trcl}(\tau) \cap M^{\mathbb{P}}$ and therefore $\dot{x}$ is a hereditarily symmetric $\mathbb{P}$-name. Moreover, $(\dot{x})_G = x$ and so $\dot{x}$ is a hereditarily symmetric $\mathbb{P}$-name for x.

Define $g \stackrel{\text{def}}{=} \{[p]^m \,;\, p \in G\}$.

***Claim* 1.** g is an M-generic filter on $\mathbb{P}_m$.

Proof of Claim. First we show that g satisfies the two defining properties of being a filter on $\mathbb{P}_m$. First, take arbitrary $p, q \in g$. This means that there are $p', q' \in G$ such that $[p']^m = p$ and $[q']^m = q$. Since G is a filter on $\mathbb{P}$, there is an $r' \in G$ such that $r' \leq p'$ and $r' \leq q'$, i.e., $r' \supseteq p'$ and $r' \supseteq q'$. But then we have that

$$[r']^m \supseteq [p']^m \text{ and } [r']^m \supseteq [q']^m.$$

So we found an $r = [r']^m \in g$ such that $r \leq p$ and $r \leq q$.

Second, let $p \in G$ and take $q \in \mathbb{P}_m$ such that $p \supseteq q$. Let $p' \in G$ be such that $[p']^m = p$. We have that $p' \supseteq [p']^m = p \supseteq q$. Since G is a filter we get $q \in G$ and since $q \in \mathbb{P}_m$, $q \in g$.

So we proved that g is a filter on $\mathbb{P}_m$. Now if we take a dense set $D \subseteq \mathbb{P}_m$, we see that it is predense in $\mathbb{P}$. For this, let $p \in \mathbb{P}$. Then $[p]^m \in \mathbb{P}_m$ and since D is dense in $\mathbb{P}_m$, there is a $d \in D$ such that $d \supseteq [p]^m$. Since $\mathsf{dom}(d) \subseteq (n+1) \times \omega$, we have that $d \cup [p]^{-m} \supseteq d$ and of course $d \cup [p]^{-m} \supseteq p$. Therefore $d \parallel p$.

So we have that any D dense in $\mathbb{P}_m$ is predense in $\mathbb{P}$ and thus by Lemma 3.19 it holds $G \cap D \neq \varnothing$, let $d \in G \cap D$. But $d \in D$ so $d \in \mathbb{P}_m$, i.e., $[d]^m = d \in G$. So $d \in g$, i.e., $g \cap D \neq \varnothing$. So g is an M-generic filter on $\mathbb{P}_m$. ⊣

To show that $a \in M[g]$. For every $x \in \mathsf{trcl}(a)$, define recursively the following symmetric $\mathbb{P}_m$-names

$$\ddot{x} \stackrel{\text{def}}{=} \{(\ddot{y}, [p]^m) \; ; \; p \Vdash \dot{y} \in \dot{x}\}.$$

We will show by induction on $\rho^M(\ddot{x})$, i.e., the M-rank of $\ddot{x}$, that for every $x \in \mathsf{trcl}(a)$, $K_m \subseteq \mathsf{sym}(\ddot{x})$.

For $x = \varnothing$ it's trivial. Assume that for every $\ddot{y} \in \mathsf{trcl}(\ddot{x}) \cap M^{\mathbb{P}}$, $K_m \subseteq \mathsf{sym}(\ddot{x})$ and take any $\pi \in K_m$. We have that

$$\begin{aligned} \pi\ddot{x} &= \{(\pi\ddot{y}, \pi[p]^m) \; ; \; p \Vdash \dot{y} \in \dot{x}\} \\ &= \{(\ddot{y}, [p]^m) \; ; \; p \Vdash \dot{y} \in \dot{x}\} \\ &= \ddot{x} \end{aligned} \tag{3}$$

Therefore for every $x \in \mathsf{trcl}(a)$, $\ddot{x}$ is a hereditarily symmetric $\mathbb{P}_m$-name. But is it for x?

We want to show that for every $x \in \mathsf{trcl}(a)$, $(\ddot{x})_g = x$. Note that for a $p \in \mathbb{P}$ it holds that

$$[p]^m \Vdash \ddot{y} \in \ddot{x} \iff p \Vdash \dot{y} \in \dot{x}$$

From right to left because of Lemma 3.10 and from left to right because of [2, Theorem 14.7(i)(a)]. We remind the reader that $(\dot{x})_G = x$.

We will show by induction on $\rho^M(\ddot{x})$, i.e., the M-rank of $\ddot{x}$, that for every $x \in \mathsf{trcl}(a)$, $(\ddot{x})_g = (\dot{x})_G$.

For $x = \varnothing$ it's trivial. Assume that for every $\ddot{y} \in \mathsf{trcl}(\ddot{x}) \cap M^{\mathbb{P}}$, $(\ddot{y})_g = (\dot{y})_G$. We have that

$$\begin{aligned} (\ddot{x})_g &= \{(\ddot{y})_g \; ; \; \exists [p]^m \in g[[p]^m \Vdash \ddot{y} \in \ddot{x}]\} \\ &= \{(\dot{y})_G \; ; \; \exists p \in G[p \Vdash \dot{y} \in \dot{x}]\} \\ &= (\dot{x})_G. \end{aligned}$$

Therefore also for $a \in \mathsf{trcl}(a)$ it holds that $(\ddot{a})_g = (\dot{a})_G = a$ and therefore $a \in M[g]$. qed

Now as promised we are going to show that in the model $\mathcal{M}9$, all coanalytic (i.e., $\boldsymbol{\Pi}^1_1$) sets of reals contain a perfect set.

Theorem 3.21.
Every uncountable coanalytic set of reals contains a perfect set.

Proof. In $\mathcal{M}9$, let $A \subseteq {}^{\omega}2$ be an uncountable $\boldsymbol{\Pi}^1_1$ set of reals. We want to show that A contains a perfect tree. First of all we'll see that there is an $a \in A \setminus M$. Assume towards contradiction that $A \subseteq M$. Since $M \subseteq \mathcal{M}9$, the set A is an uncountable subset of the reals of M. Let R be the relation that wellorders A in M. Since $M \subseteq \mathcal{M}9$, the relation R is in $\mathcal{M}9$ as well. Therefore A is wellordered by R in $\mathcal{M}9$. This is not possible in $\mathcal{M}9$ because of Theorem 3.13 and Theorem 2.5. So indeed there is an $a \in A \setminus M$. Fix a.

By Lemma 3.20 there is an $n \in \omega$ such that $\mathbb{R}^{\mathcal{M}9} \in M[g]$ for some g that is an M-generic filter on $\mathbb{P}_n$, so fix these n, g and note that by the proof of Lemma 3.20 we have that $K_n \subseteq \mathsf{sym}(\sigma)$ for every $\sigma \in \mathsf{HS}$ such that $(\sigma)_g \in \mathbb{R}^{\mathcal{M}9}$ (see 3). $\mathbb{P}_n$ has cardinality at most ω_n^M so it has at most ω_{n+1}^M many subsets in M, since GCH^M holds. So in $\mathcal{M}9$, $\mathbb{P}_n$ has at most countably many subsets and thus in $\mathcal{M}9$ we can enumerate the dense subsets of $\mathbb{P}_n$ that are in M. Let $\{D_i \, ; \, i \in \omega\}$ be such an enumeration. Now let φ be the Π^1_1 formula describing A and let $\dot{a}$ be a $\mathbb{P}_n$-name for a.

In $\mathcal{M}9$ we do the following. Since $\varphi(a)$ holds and since $a \notin M$, then according to the forcing lemma, there is a $p'_\varnothing \in g$ such that $p'_\varnothing \Vdash \varphi(\dot{a}) \wedge \dot{a} \notin \check{M}$, where $\check{M}$ is the canonical name for the ground model. The set D_0 is dense in $\mathbb{P}_n$ so there is a $p_\varnothing \in D_0$ such that $p_\varnothing \leq p'_\varnothing$, fix $p_\varnothing$. By the properties of the forcing relation, $p_\varnothing \Vdash \varphi(\dot{a}) \wedge \dot{a} \notin \check{M}$. Note that when we want to take a condition in g that forces a certain formula, then we can take that condition in one of the dense sets.

Since $a \notin M$, there must be an $i_\varnothing \in \omega$ and conditions $p_{\langle 0\rangle}, \ p_{\langle 1\rangle} \in D_1$ such that $p_{\langle 0\rangle}, \ p_{\langle 1\rangle} \leq p_\varnothing$, $p_{\langle 0\rangle} \Vdash \dot{a}(\check{\imath}_\varnothing) = \check{0}$ and $p_{\langle 1\rangle} \Vdash \dot{a}(\check{\imath}_\varnothing) = \check{1}$.

We iterate this construction by induction. Note that since ω and $\mathbb{P}_n$ are wellorderable in $\mathcal{M}9$, we are not using AC for this construction. Let $\ell \in \omega$ and assume that for every finite sequence s' with $\mathsf{lh}(s') < \ell$, there is $i'_s \in \omega$ and there are $p_t \in D_{\mathsf{lh}(t)}$ for all sequences t with $\mathsf{lh}(t) \leq \ell$, such that for all $t, t' \in {}^{<\ell}2$, if $t \subseteq t'$ then $p'_t \leq p_t$ and such that for every p_t we have that $p_t \Vdash \dot{a}(\check{\imath}_{t \restriction (\mathsf{lh}(t)-1)}) = \widecheck{\mathsf{last(t)}}$.

Fix $s \in {}^{\ell}2$ and let $C \stackrel{\text{def}}{=} \{i \in \omega \, ; \, \exists k < \ell [i = i_{s\restriction k}]\}$ which is a finite set. Any $y \in \mathbb{R}^{\mathcal{M}9}$ such that for every $m \in \omega \setminus C$, $y(m) = a(m)$, cannot be in M because otherwise we'd have that

$$a = \langle a(k) \, ; \, k \in C\rangle \cup \langle a(m) \, ; \, m \in \omega \setminus C\rangle = \langle a(k) \, ; \, k \in C\rangle \cup \langle y(m) \, ; \, m \in \omega \setminus C\rangle$$

and since $\langle a(k) \,;\, k \in C\rangle \in M$, C is finite and $\langle y(m) \,;\, m \in \omega \setminus C\rangle \subseteq y \in M$, we would get that $a \in M$ which is a contradiction.

So there is an $i_s \in \omega$ and there are $p_{s^\frown\langle 0\rangle},\ p_{s^\frown\langle 1\rangle} \in D_\ell$ with $p_{s^\frown\langle 0\rangle},\ p_{s^\frown\langle 1\rangle} \leq p_s$ such that

$$p_{s^\frown\langle 0\rangle} \Vdash \dot{a}(\check{\imath}_s) = \check{0} \text{ and } p_{s^\frown\langle 1\rangle} \Vdash \dot{a}(\check{\imath}_s) = \check{1}.$$

Therefore, for every finite ℓ and for every finite binary sequence t with $\mathsf{lh}(t) = \ell + 1$, there are $p_t \in D_\ell$ as described above.

Now in $M[g]$. For every $z \in \mathbb{R}^{M[g]}$, define

$$G_z \stackrel{\text{def}}{=} \{p \in \mathbb{P}_n \,;\, \exists k \in \omega(p_{z\restriction k} \leq p)\}.$$

Claim 2. For every $z \in \mathbb{R}^{M[g]}$, G_z is a M-generic filter on $\mathbb{P}_n$.

Proof of Claim. Fix a $z \in \mathbb{R}^{M[g]}$. For arbitrary $p, q \in G_z$, there are $m_p, m_q \in \omega$ such that

$$p_{z\restriction m_p} \leq p \text{ and } p_{z\restriction m_q} \leq q.$$

Take $m \stackrel{\text{def}}{=} \mathsf{min}\,\{m_p, m_q\}$ and for this m we have $p_{z\restriction m} \leq p, q$ and $p_{z\restriction m} \in G_z$. Now take $p \in G_z$ and $q \in \mathbb{P}_n$ such that $p \leq q$. Since $p \in G_z$, there is an $m \in \omega$ such that $p_{z\restriction m} \leq p \leq q$ and thus $q \in G_z$. So G_z is a filter. Since for every $m \in \omega$, $p_{z\restriction m}$ is in G_z we have that for every $m \in \omega$, $G_z \cap D_m \neq \varnothing$. So G_z is an M-generic filter on $\mathbb{P}_n$. $\dashv$

It's easy to see that for every $z \in \mathbb{R}^{M[G]}$ we have that $G_z \subseteq g$, so define $f\colon \mathbb{R}^{M[g]} \to M[g]$ such that $f(z) = (\dot{a})_{G_z}$.

Claim 3. f is an injective function from $\mathbb{R}^{M[g]}$ into $\{y \,;\, M[g] \models \varphi(y)\}$.

Proof of Claim. Let $x, y \in \mathbb{R}^{M[g]}$ such that $x \neq y$. Then there must be an $m \in \omega$ for which it holds that

$$\forall k < m(x(k) = y(k)) \text{ and } x(m) \neq y(m).$$

Fix m. This means that $x\restriction(m-1) = y\restriction(m-1)$ where if $m = 0$ we define $x\restriction(m-1) = y\restriction(m-1) = \varnothing$. We also have that $\mathsf{last}(x\restriction m) \neq \mathsf{last}(y\restriction m)$. Define $s \stackrel{\text{def}}{=} x\restriction(m-1) = y\restriction(m-1)$ and without loss of generality assume that $\mathsf{last}(x\restriction m) = 0$ and $\mathsf{last}(y\restriction m) = 1$.

By the way p_s is defined we have that:

$$p_{x\restriction m} = p_{s^\frown\langle 0\rangle} \Vdash \dot{a}(\check{\imath}_s) = \check{0}$$
$$p_{y\restriction m} = p_{s^\frown\langle 1\rangle} \Vdash \dot{a}(\check{\imath}_s) = \check{1}$$

Now assume for a contradiction that $(\dot{a})_{G_x} = (\dot{a})_{G_y}$ and note that for every $z \in \mathbb{R}^{M[g]}$,

$$(\dot{a})_{G_z} = \{(\check{\ell})_{G_z} \,;\, \exists p \in G_z(p \Vdash \dot{a}(\check{\ell}) = \check{1})\}.$$

But then our assumption would imply that for every $\ell \in \omega$ and every $j \in 2$ it holds that

$$\exists p \in G_x(p \Vdash \dot{a}(\check{\ell}) = \check{\jmath}) \iff \exists q \in G_y(q \Vdash \dot{a}(\check{\ell}) = \check{1}).$$

But for $\ell = i_s$ this cannot be the case. Contradiction. Therefore f is injective.

It's easy to see that $\mathsf{rng}(f) \subseteq \{y \ ; \ M[g] \models \varphi(y)\}$; it follows from the fact that for every $x \in \mathbb{R}^{M[g]}$ and every $k \in \omega$, $p_{x\restriction k} \leq p_\varnothing$ and thus $p_{x\restriction k} \Vdash \varphi(\dot{a})$. $\dashv$

The last thing left to prove is that f is in $\mathcal{M}9$ because then $\mathbb{R}^{\mathcal{M}9}$ would be injected into A. For every $z \in \mathbb{R}^{\mathcal{M}9}$, define the following name

$$\dot{a}^z \stackrel{\text{def}}{=} \{(\check{n}, p_{z\restriction k}) \ ; \ \exists p \in \mathsf{rng}(\dot{a}) \exists k \in \omega (p_{z\restriction k} \leq p)\}.$$

It's clear that $(\dot{a})_{G_z} = (\dot{a}^z)_g$. By using AC take a hereditarily symmetric name $\dot{z}$ for every $z \in \mathbb{R}^{\mathcal{M}9}$ for which it holds that $K_n \subseteq \mathsf{sym}(\dot{z})$. This is possible from the proof of Lemma 3.20. Finally define the name $F \stackrel{\text{def}}{=} \{(\mathsf{op}(\dot{z}, \dot{a}^z), \mathbb{1}) \ ; \ z \in \mathbb{R}^{\mathcal{M}9}\}$. Clearly, $F_g = f$. Note that for every $z \in \mathbb{R}^{\mathcal{M}9}$, $K_n \subseteq \mathsf{sym}(\dot{z})$ and by the definition of $\dot{a}^z$ it holds that $K_n \subseteq \mathsf{sym}(\dot{a}^z)$. Therefore $K_n \subseteq \mathsf{sym}(F)$ as well and thus $f \in \mathcal{M}9$.

So A contains a copy of $\mathbb{R}^{\mathcal{M}9}$ which is a perfect set. qed

3.4 A new model

As we said in our introduction, the results that we got from $\mathcal{M}9$ motivated us to build a generalisation of it. Our ground model M is a countable transitive model of $\mathsf{ZFC} + \mathsf{GCH}$. We construct a new model via an ω-step iterated forcing construction with finite supports (for details on this method see [1, Ch.VIII, §5]). For that construction we use the canonical names for the following partial orders in M, for every $n \in \omega \setminus \{0\}$.

$\mathbb{Q}_0 \stackrel{\text{def}}{=} \mathsf{Fn}(\omega \times \omega, \omega_\omega^M, \omega)$,
$\mathbb{Q}_n \stackrel{\text{def}}{=} \mathsf{Fn}(\omega \times \omega_{\omega\cdot n+1}^M, \omega_{\omega\cdot(n+1)}^M, \omega_{\omega\cdot n+1}^M)$
$\mathbb{P}'_0 \stackrel{\text{def}}{=} \{p \in \mathbb{Q}_0 \ ; \ \forall (m,k) \in \mathsf{dom}(p)\ p(m,k) < \omega_m^M \text{ and } \mathsf{Ind}(p) < \omega\}$, and
$\mathbb{P}'_n \stackrel{\text{def}}{=} \{p \in \mathbb{Q}_n \ ; \ \forall (m,\beta) \in \mathsf{dom}(p)[p(m,\beta) < \omega_{\omega\cdot n+1+m}^M] \text{ and } \mathsf{Ind}(p) < \omega\}$,

with their standard orders and maximal elements. The ω-th step is the following partial order. We take countable sequences of names of conditions in the $\mathbb{P}'_n$ above, something like

$$\mathbb{P} \stackrel{\text{def}}{=} \{0\} \times \mathsf{dom}(\check{\mathbb{P}}_0) \times \mathsf{dom}(\check{\mathbb{P}}'_1) \times \mathsf{dom}(\check{\mathbb{P}}'_2) \times \cdots .$$

The ω-th step is the partial suborder $\mathbb{P}_\omega$ of $\mathbb{P}$ that includes only those countable sequences in $\mathbb{P}$ that have a finite support, i.e., that only finitely many names of conditions in their range are not a maximal element.

We take $G^\star$ an M-generic filter on $\mathbb{P}_\omega$ and then we take a symmetric submodel of $M[G^\star]$ as follows. Let $\mathsf{Perm}_{\omega\times\omega_{\omega^2}^M}$ be the group of all permutations of $\omega \times \omega_{\omega^2}^M$, for $n \in \omega$ let

$$\mathcal{G}_n^\star \stackrel{\text{def}}{=} \{\pi \in \mathsf{Perm}_{\omega\times\omega_{\omega^2}^M} \ ; \ \forall n \in \omega \forall \beta \in \omega_{\omega\cdot n}^M \exists \gamma \in \omega_{\omega\cdot n}^M [\pi(n,\beta) = (n,\gamma)]\}$$

and let

$$\mathcal{G}^\star \stackrel{\text{def}}{=} \bigcap_{n\in\omega} \mathcal{G}_n^\star$$

This is a group of permutations of $\omega \times \omega^M_{\omega^2}$.

For every $\pi \in \mathcal{G}^\star$, every $n \in \omega$ and every $p \in \mathbb{P}'_n$ define $\hat{\pi}\colon \mathbb{P}'_n \to \mathbb{P}'_n$ as follows:

$$\begin{aligned} \mathsf{dom}(\hat{\pi}p) &= \{\pi(n,\beta)\ ;\ (n,\beta) \in \mathsf{dom}(p)\} \\ \hat{\pi}p(\pi(n,\beta)) &= p(n,\beta), \end{aligned}$$

For every $\pi \in \mathcal{G}^\star$ and every $n \in \omega$, $\hat{\pi}$ is an automorphism of $\mathbb{P}'_n$. From now on we'll denote $\hat{\pi}$ also by π. Define $\hat{\hat{\pi}}\colon \mathbb{P}_\omega \to \mathbb{P}_\omega$ to be

$$\hat{\hat{\pi}}(\langle \check{p}_n\ ;\ n \in \omega\rangle) = \langle \widehat{\check{\pi p_n}}\ ;\ n \in \omega\rangle$$

This is an automorphism of $\mathbb{P}_\omega$ and from now on we'll also denote $\hat{\hat{\pi}}$ by π. For every $m \in \omega$, define

$$K_m := \{\pi \in \mathcal{G}\ ;\ \forall k \leq m \forall \beta \in \omega^M_{\omega^2}[\pi(k,\beta) = (k,\beta)]\}$$

and let $\mathcal{F}^\star$ be the filter over $\mathcal{G}^\star$ that is generated by $\{K_m\ ;\ m \in \omega\}$. Take the symmetric submodel $N[\mathcal{F}, \mathcal{G}]$ and call it $N^\star$.

In the definition of the partial orders $\mathbb{P}'_n$ above one can notice the asymmetry between $\mathbb{P}'_0$ and $\mathbb{P}'_n$ for $n > 0$. Conditions in $\mathbb{P}'_0$ have cardinality below ω and conditions in $\mathbb{P}'_n$ for $n > 0$ have cardinality below $\omega_{\omega \cdot n+1}$. One might have expected this to be $\omega_{\omega \cdot n}$ as a generalisation. The reason for this asymmetry is that it is problematic to use the partial order $\mathsf{Fn}(I, J, \lambda)$ when λ is not regular. This is because for regular λ we can use [1, Ch.VII, Lemma 6.13] to prove a preservation lemma similar to Lemma 3.25.

When it comes to the notation, the model $N^\star$ is a complicated construction and since we unfortunately did not get any answers to our questions from this model, we are only going to describe in detail the first two steps of the iterated forcing construction. This will be done in the next section. A full version of the model is available by request. Now we are going to state the basic facts of the model $N^\star$ and in §3.5 we will prove their restricted versions for the two-step construction.

In $N^\star$, for every $n \in \omega$ the ordinal ω^M_n is countable and for every $n, m \in \omega$ with $n > 0$, the ordinal $\omega^M_{\omega \cdot n+1+m}$ surjects onto $\omega^M_{\omega \cdot n+1}$. Also, for every $n \in \omega$ with $n > 0$, the ordinals $\omega^M_{\omega \cdot n}$ and $\omega^M_{\omega \cdot n+1}$ are cardinals in $N^\star$ and they are singular and regular respectively. Finally, the ordinal $\omega^M_{\omega^2}$ is the cardinal ω_ω in $N^\star$.

The purpose of this model was to prove that in $N^\star$, the cardinal ω_ω is not an $\bar{\mathbf{s}}$-strong limit cardinal but it is an $\bar{\mathbf{\imath}}$-strong limit cardinal. For ω_ω not being an $\bar{\mathbf{s}}$-strong limit it would suffice to find a surjection from 2^ω onto ω_ω. But we don't know whether that exists or not.

On the other hand, to show that ω_ω is an $\bar{\mathbf{\imath}}$-strong limit cardinal we would have to show that for every $n \in \omega$, ω_ω does not inject into the set 2^{ω_n}. By using Figura's theorem that if $\wp(\kappa)$ is a κ union of κ sets then κ^+ does not inject into $\wp(\kappa)$, we wanted to show that for every $n \in \omega$, it holds that ω_{2n+1} does not inject into $2^{\omega_{2n}}$ which proves that there is no injection from ω_ω into 2^{ω_m} for every m.

That would follow immediately if we knew that for every $n \in \omega$, the set $2^{\omega_{2n}}$ is an ω_{2n} union of sets of cardinality $\leq \omega_{2n}$. But we have no information on that either. We can see this if we look at 2^ω. We don't know how many reals the partial order $\mathbb{P}_\omega$ adds, so we don't have enough information to complete a proof similar to the one for the model $\mathcal{M}9$ (Theorem 3.13).

We expect that this problem is not solvable with the use of this forcing construction and symmetric submodel. Nevertheless, we will describe a two step version of this model in the next section.

3.5 The two-step version

The version we describe below is constructed with product forcing, the two-step version of iterated forcing constructions. For details on this method see [1, Ch.VIII, §1]. For easier notation define the following abbreviations.

$$\mathbb{Q}_0 \stackrel{\text{def}}{=} \mathsf{Fn}(\omega \times \omega, \omega_\omega^M, \omega),$$
$$\mathbb{Q}_1 \stackrel{\text{def}}{=} \mathsf{Fn}(\omega \times \omega_{\omega+1}^M, \omega_{\omega\cdot 2}^M, \omega_{\omega+1}^M), \text{ and}$$
$$\forall p \in \mathbb{Q}_0 \cup \mathbb{Q}_1,\ \mathsf{Ind}(p) \stackrel{\text{def}}{=} \{m \in \omega\ ;\ \exists \beta \in \omega_{\omega^2}^M((m,\beta) \in \mathsf{dom}(p))\}.$$

Let M be a countable transitive model of $\mathsf{ZFC}+\mathsf{GCH}$ and define the following partial orders in M.

$$\mathbb{P}_0 \stackrel{\text{def}}{=} \{p \in \mathbb{Q}_0\ ;\ \forall (m,k) \in \mathsf{dom}(p)\ p(m,k) < \omega_m^M \text{ and } \mathsf{Ind}(p) < \omega\}$$
$$\mathbb{P}_1 \stackrel{\text{def}}{=} \{p \in \mathbb{Q}_1\ ;\ \forall (m,\beta) \in \mathsf{dom}(p)\ p(m,\beta) < \omega_{\omega+1+m} \text{ and } \mathsf{Ind}(p) < \omega\}$$

These partial orders are ordered by extension, i.e., for $j \in 2$, $p \leq_j q \stackrel{\text{def}}{\iff} p \supseteq q$ and they have a maximal element $\mathbb{1}_j \stackrel{\text{def}}{=} 0$.

The partial order $\langle \mathbb{P}_0 \times \mathbb{P}_1, \leq, \mathbb{1}\rangle$ is the product partial order of $\langle \mathbb{P}_0, \leq_0, \mathbb{1}_0\rangle$ and $\langle \mathbb{P}_1, \leq_1, \mathbb{1}_1\rangle$ and it is defined by

$$\langle p_0, p_1\rangle \leq \langle q_0, q_1\rangle \stackrel{\text{def}}{\iff} p_0 \leq_0 q_0 \text{ and } p_1 \leq_1 q_1,$$

and $\mathbb{1} \stackrel{\text{def}}{=} \langle \mathbb{1}_0, \mathbb{1}_1\rangle$.

Let G be an M-generic filter on $\mathbb{P}_0 \times \mathbb{P}_1$. We define $i_0 \colon \mathbb{P}_0 \to \mathbb{P}_0 \times \mathbb{P}_1$ and $i_1 \colon \mathbb{P}_1 \to \mathbb{P}_0 \times \mathbb{P}_1$ by $i_0(p) = \langle p, \mathbb{1}_1\rangle$ and $i_1(p) = \langle \mathbb{1}_0, p\rangle$. By [1, Ch.VIII, Lemma 1.2], these maps are complete embeddings and by [1, Ch.VIII, Lemma 1.3] we have that $G_0 \stackrel{\text{def}}{=} i_0^{-1}(G)$ is an M-generic filter on $\mathbb{P}_0$ and $G_1 \stackrel{\text{def}}{=} i_1^{-1}(G)$ is an M-generic filter on $\mathbb{P}_1$. Moreover, $G = G_0 \times G_1$ and by [1, Ch.VIII, Theorem 1.4] we have that $M[G] = M[G_0][G_1] = M[G_1][G_0]$.

Lemma 3.22.
In $M[G]$, for every $n \in \omega$ there are functions $f_{0,n} \colon \omega \twoheadrightarrow \omega_n^M$ and $f_{1,n} \colon \omega_{\omega+1} \twoheadrightarrow \omega_{\omega+1+n}$.

Proof. Define $f_0 \stackrel{\text{def}}{=} \bigcup G_0$ and $f_1 \stackrel{\text{def}}{=} \bigcup G_1$ which are functions on $\omega \times \omega$ and $\omega \times \omega_{\omega+1}$ respectively. For every $n \in \omega$ define $f_{0,n}(\beta) \stackrel{\text{def}}{=} f_0(n, \beta)$ and $f_{1,n}(\beta) \stackrel{\text{def}}{=} f_1(n, \beta)$. These are all functions because f_0 and f_1 are. To show that for every $n \in \omega$, $f_{0,n}$ is surjective onto ω_n^M is exactly the same as in the proof of Lemma 3.8. To show that $f_{1,n}$ is surjective onto $\omega_{\omega+1+n}^M$ is exactly similar. qed

So we have collapsed all ω_n^M onto ω and all $\omega_{\omega+1+n}^M$ onto $\omega_{\omega+1}^M$. Since $M[G]$ is a ZFC model, we have automatically collapsed ω_ω^M onto ω because it has become a countable union of countable sets, and we also collapsed $\omega_{\omega\cdot 2}^M$ onto $\omega_{\omega+1}^M$ because it is now a countable union of sets of cardinality $\omega_{\omega+1}^M$. To resolve this we are going to permute in a manner similar to the one in $\mathcal{M}9$.

Let $\mathsf{Perm}_{\omega\times\omega_{\omega\cdot 2}^M}$ be the group of all permutations of $\omega \times \omega_{\omega\cdot 2}^M$ and let

$$\mathcal{G}_0 \stackrel{\text{def}}{=} \{\pi \in \mathsf{Perm}_{\omega\times\omega} \,;\, \forall n \in \omega \forall k \in \omega \exists \ell \in \omega[\pi(n, k) = (n, \ell)]\}$$
$$\mathcal{G}_1 \stackrel{\text{def}}{=} \{\pi \in \mathsf{Perm}_{\omega\times\omega} \,;\, \forall n \in \omega \forall \beta \in \omega_{\omega+1}^M \exists \gamma \in \omega_{\omega+1}^M [\pi(n, \beta) = (n, \gamma)]\}$$

and let

$$\mathcal{G} \stackrel{\text{def}}{=} \mathcal{G}_0 \cap \mathcal{G}_1$$

This is a group of permutations.

For every $\pi \in \mathcal{G}$ and every $j \in 2$ define $\hat{\pi} \colon \mathbb{P}_j \to \mathbb{P}_j$ for every $p \in \mathbb{P}_j$ as follows:

$$\mathsf{dom}(\hat{\pi}p) = \{\pi(n, \beta) \,;\, (n, \beta) \in \mathsf{dom}(p)\}$$
$$\hat{\pi}p(\pi(n, \beta)) = p(n, \beta)$$

For every $\pi \in \mathcal{G}$ and every $j \in 2$, $\hat{\pi}$ is an automorphism of $\mathbb{P}_j$. From now on we'll denote $\hat{\pi}$ also by π. Define $\hat{\hat{\pi}} \colon \mathbb{P}_0 \times \mathbb{P}_1 \to \mathbb{P}_0 \times \mathbb{P}_1$ to be

$$\hat{\hat{\pi}}(\langle p, q\rangle) = \langle \pi p, \pi q\rangle$$

This is an automorphism of $\mathbb{P}_0 \times \mathbb{P}_1$ and from now on we'll also denote $\hat{\hat{\pi}}$ by π. For every $m \in \omega$, define

$$K_m \stackrel{\text{def}}{=} \{\pi \in \mathcal{G} \,;\, \forall k \leq m \forall \beta \in \omega_{\omega\cdot 2}^M [\pi(k, \beta) = (k, \beta)]\}$$

and let $\mathcal{F}$ be the filter over $\mathcal{G}$ that is generated by $\{K_m \,;\, m \in \omega\}$. Take the symmetric submodel $N[\mathcal{G}, \mathcal{F}]$ and call it N for simpler notation.

Now we are going to look at some properties of the model N. These properties are similar to the ones of $N^\star$ we described without proof in §3.4. First we must give some useful definitions.

Definition 18 (Restrictions). Let $j \in 2$, $m \in \omega$ and $q \in \mathbb{P}_j$ be all arbitrary. We remind the reader that in Definition 17 we defined $[q]^m$ to be the restriction of q to the set $\{(k, \beta) \,;\, k \leq n\}$. Similarly, for any $r = \langle p_0, p_1\rangle \in \mathbb{P}_0 \times \mathbb{P}_1$, define the following:

$$[r]^m \stackrel{\text{def}}{=} \langle [p_0]^m, [p_1]^m\rangle$$
$$r^{\downarrow j} \stackrel{\text{def}}{=} i_j^{-1}(r) \text{ for every } j \in 2$$

Proposition 3.23.
For every $m \in \omega$ and $j \in 2$, it holds that $|\{[p]^m \; ; p \in \mathbb{P}_j\}| \leq \omega^M_{\omega \cdot j+1+m}$.

Proof. For $j = 0$ look at the set $\{[p]^m \; ; p \in \mathbb{P}_0\}$. By definition of $[p]^m$ and of $\mathbb{P}_0$, it is clear that this set can take at most ω^M_m different values. For $j = 1$ as well; the restrictions in the definition of $\mathbb{P}_1$ allow $[p]^m$ have at most $\omega^M_{\omega+1+m}$ different values. qed

Lemma 3.24.
For all $n \in \omega$ and $j \in 2$, $f_{j,n} \in N$.

Proof. Fix $j \in 2$ and $n \in \omega$ and look at the set

$$F_{j,n} \stackrel{\text{def}}{=} \left\{ \left(\check{\widehat{(\beta,\alpha)}}, [r]^n \right) ; r \in \mathbb{P}_0 \times \mathbb{P}_1 \text{ and } (n,\beta) \in \mathsf{dom}(r^{\downarrow j}) \text{ and } (r^{\downarrow j})(n,\beta) = \alpha \right\}$$

This is a name for $f_{j,n}$ because:

$$\begin{aligned}(F_{j,n})_G &= \left\{ \left(\check{\widehat{(\beta,\alpha)}}\right)_G ; r \in G \text{ and } (n,\beta) \in \mathsf{dom}(p^{\downarrow j}) \text{ and } (p^{\downarrow j})(n,\beta) = \alpha \right\} \\ &= \{(\beta,\alpha) \; ; \exists q \in \mathbb{P}_j \cap G_j[(n,\beta) \in \mathsf{dom}(q) \text{ and } q(n,\beta) = \alpha]\} \\ &= f_{j,n}\end{aligned}$$

Let $\pi \in K_n$. Note that for any $\check{x}$ and any π automorphism of $\mathbb{P}_0 \times \mathbb{P}_1$, $\pi\check{x} = \check{x}$ is true. Also, for any $\pi \in K_n$ and any $p \in \mathbb{P}_0 \times \mathbb{P}_1$ it's true that $\pi[p^{\downarrow j}]^n = [p^{\downarrow j}]^n$, therefore $K_n \subseteq \mathsf{sym}(F_{j,n})$. Thus for every $j \in 2$ and $n \in \omega$, $F_{j,n} \in \mathsf{HS}$ and so $f_{j,n} \in N$. qed

Lemma 3.25.
The partial order $\mathbb{P}_0$ preserves cardinals and cofinalities $\geq \omega^M_{\omega+1}$ and of course the ones $\leq \omega$. The partial order $\mathbb{P}_1$ preserves cardinals and cofinalities $\geq \omega^M_{\omega\cdot 2+1}$ and $\leq \omega^M_{\omega+1}$.

Proof. For $\mathbb{P}_0$, note that $\mathbb{P}_0 \subset \mathbb{Q}_0 = \mathsf{Fn}(\omega \times \omega, \omega^M_\omega, \omega)$. By [1, Ch.VII, Lemma 6.10], $\mathbb{Q}_0$ has the $\left(\left(\omega^M_\omega\right)^{<\omega}\right)^+$-c.c. and since GCH^M holds and ω^M_ω is singular, the $\omega^M_{\omega+1}$-c.c.. Therefore by [1, Ch.VII, Lemma 6.9], $\mathbb{Q}_0$ preserves cardinals and cofinalities above and with $\omega^M_{\omega+1}$, and since $\mathbb{P}_0 \subseteq \mathbb{Q}_0$, so does $\mathbb{P}_0$.

For $\mathbb{P}_1$, note that $\mathbb{P}_1 \subseteq \mathbb{Q}_1 = \mathsf{Fn}(\omega \times \omega^M_{\omega+1}, \omega^M_{\omega\cdot 2}, \omega^M_{\omega+1})$ which according to [1, Ch.VII, Lemma 6.10] has the $\left(\left(\omega^M_{\omega\cdot 2}\right)^{<\omega^M_{\omega+1}}\right)^+$-c.c. and by GCH^M and singularity of $\omega^M_{\omega\cdot 2}$, the $\omega^M_{\omega\cdot 2+1}$-c.c.. Therefore $\mathbb{Q}_1$ preserves cardinals and cofinalities above and with $\omega^M_{\omega\cdot 2+1}$, and so does $\mathbb{P}_1 \subseteq \mathbb{Q}_1$.

Also, since $\omega^M_{\omega+1}$ is regular, according to [1, Ch.VII, Lemma 6.13] the partial order $\mathbb{Q}_1$ is $\omega^M_{\omega+1}$-closed which by [1, Ch.VII, Corollary 6.15] means that $\mathbb{Q}_1$ preserves cardinals and cofinalities below and with $\omega^M_{\omega+1}$ and so does $\mathbb{P}_1 \subseteq \mathbb{Q}_1$. qed

The fact that $\omega^M_{\omega+1}$ is regular enables us to have this preservation result and this is the reason why we left this sort of gap in our forcing construction (instead of taking partial functions from $\omega \times \omega_\omega^M$). As we'll see this has as a result that the first four infinite cardinals are regular, singular, regular, singular; in this order.

At this point we want to show that ω_ω^M and $\omega_{\omega\cdot 2}^M$ are cardinals in N. To do that, we need the following proposition.

Proposition 3.26.
For every name τ, every $n \in \omega$, every $j \in 2$ and every $p \in \mathbb{P}_j$, if $p \Vdash_{\mathbb{P}_j} \varphi(\tau)$ and $\mathsf{sym}(\tau) \supseteq K_n$ then $[p]^n \Vdash_{\mathbb{P}_j} \varphi(\tau)$.

Proof. For $j = 0$ this is the same as Proposition 3.10. So let $j = 1$ and assume that $[p]^n$ does not force $\varphi(\tau)$. By [2, Theorem 14.7(ii)(a)], there is a $q \supset [p]^n$, $q \in \mathbb{P}_1$ such that $q \Vdash \mathbb{P}_1 \neg\varphi(\tau)$. Let $k \stackrel{\text{def}}{=} \mathsf{max}(\mathsf{Ind}(p))$ and $\ell \stackrel{\text{def}}{=} \mathsf{max}(\mathsf{Ind}(q))$. If $k \leq n$ then we'd have that $[p]^n = p$ and so $[p]^n \Vdash_{\mathbb{P}_1} \varphi(\tau)$, contradiction.

If $k > n$ then without loss of generality assume that $\ell > k$ and define a permutation π on $\omega \times \omega_{\omega\cdot 2}^M$ as follows:

$$\pi(a,b) = \begin{cases} (a,b) & \text{if } a \leq n \\ (\ell + a, b) & \text{if } a \in (n,k] \\ (n+a,b) & \text{if } a \in (k, \ell + k - n) \\ (a,b) & \text{otherwise.} \end{cases}$$

We want to show that for this π, it holds that $\pi p \parallel q$. We know that $[p]^n \subseteq q$ so also $\pi[p]^n \subseteq q$. Look at $\mathsf{dom}(\pi p) = \{\pi(a,b) \,;\, (a,b) \in \mathsf{dom}(p)\}$. It's clear that

$$\mathsf{dom}(\pi p) = \{(a,b) \,;\, a \leq n\} \cup \{(\ell + k, b) \,;\, k \in (n,m]\}$$

So, $\mathsf{dom}(\pi p) \cap \mathsf{dom}(q) \subseteq \{(k,b) \,;\, k \leq n\}$ and $\pi[p]^n = [q]^n$. So for every $(k,j) \in \omega \times \omega$, let

$$r(a,b) = \begin{cases} p(a,b) = q(a,b) & \text{if } a \leq n \\ q(a,b) & \text{if } a \in (n,\ell] \\ \pi p(a,b) & \text{if } a \in (\ell, \ell + k - n) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Clearly, $r \supseteq \pi p$ and $r \supseteq q$ therefore $\pi p \parallel q$. By Lemma 3.5 we have that $\pi p \Vdash_{\mathbb{P}_1} \varphi(\pi\tau)$ and we know that $\pi \in K_n \subseteq \mathsf{sym}(\tau)$. Therefore $\pi p \Vdash_{\mathbb{P}_1} \varphi(\tau)$ and since $\pi p \parallel q$, [2, Theorem 14.7] tells us that $q \Vdash_{\mathbb{P}_1} \varphi(\tau)$, contradiction. qed

Lemma 3.27.
The ordinals ω_ω^M and $\omega_{\omega\cdot 2}^M$ are cardinals in N.

Proof. The ordinal ω_ω^M is a cardinal in N for the same reasons as in Theorem 3.11 of §3.3. For $\omega_{\omega\cdot 2}^M$, assume towards contradiction that in N there is a surjection g from $\omega_{\omega+1}^M$ onto $\omega_{\omega\cdot 2}^M$ that is in N and let $\dot{g}$ be a symmetric name for g. Since $\mathbb{P}_0$ preserves $\omega_{\omega\cdot 2}^M$, there is a $p_0 \in G_1$ such that

$$p_0 \Vdash_{\mathbb{P}_1} \text{“}\dot{g} \text{ is a function from } \check{\omega}_{\omega+1}^M \text{ onto } \check{\omega}_{\omega\cdot 2}^M\text{.”} \stackrel{\text{def}}{=} \varphi(\dot{g}).$$

Let $K_\ell \subseteq \mathsf{sym}(\dot{g})$ and fix ℓ. For every $\beta \in \omega_{\omega+1}^M$ define

$$A_\beta \stackrel{\text{def}}{=} \{\alpha \in \omega_{\omega\cdot 2}^M \,;\, \exists q \in \mathbb{P}_1 [q \supseteq q_0 \text{ and } q \Vdash_{\mathbb{P}_1} \dot{g}(\beta) = \alpha]\}.$$

If for every $\beta \in \omega^M_{\omega\cdot 2}$ it was true that $A_\beta \leq \omega^M_{\omega+1+m}$, then we'd have:

$$\omega^M_{\omega\cdot 2} \leq \bigcup_{\beta\in\omega^M_{\omega+1}} \omega^M_{\omega+1+m} = \omega^M_{\omega+1+m}, \text{ contradiction.}$$

So for at least one $\beta \in \omega^M_{\omega+1}$, $A_\beta \geq \omega^M_{\omega+1+m+1}$. Fix this β. For every $\alpha \in \omega^M_{\omega\cdot 2}$ define $B_\alpha \overset{\text{def}}{=} \{q \in \mathbb{P}_1 \,;\, q \supseteq q_0 \text{ and } q \Vdash_{\mathbb{P}_1} \dot{g}(\beta) = \alpha\}$.

Claim. *Let $p, q \in \mathbb{P}_1$ such that $p, q \supseteq q_0$ and let $\alpha, \gamma \in \omega^M_{\omega\cdot 2}$. If $p \in B_\alpha$, $q \in B_\gamma$ and $\alpha \neq \gamma$ then $p \perp q$.*

Proof of Claim. Assume towards contradiction that $p \parallel q$, i.e., that there is an $r \supseteq q_0, p, q$. For this r, since $p \in B_\alpha$ and by [2, Theorem 14.7(i)(a)], $r \Vdash_{\mathbb{P}_1} \dot{g}(\beta) = \alpha$. Similarly, because $q \in B_\gamma$, $r \Vdash_{\mathbb{P}_1} \dot{g}(\beta) = \gamma \neq \alpha$. Contradiction to $r \Vdash$"$\dot{g}$ is a function". ⊣

Therefore there must be more that $\omega^M_{\omega+m+2}$-many incompatible conditions that force $\dot{g}(\beta)$ to take a different value with each condition. Let B be the set of these pairwise incompatible conditions. Also, there must be more than $\omega^M_{\omega+m+2}$-many distinct ordinals α_p (one for each $p \in B$), such that for every $p \in B$, $p \Vdash_{\mathbb{P}_1} \dot{g}(\beta) = \alpha_p$.

By Proposition 3.26 we have that for every $p \in B$,

$$[p]^\ell \Vdash_{\mathbb{P}_1} \dot{g}(\beta) = \alpha_p$$

but by Proposition 3.23, the set $\{[p]^\ell \,;\, p \in \mathbb{P}_1\}$ has cardinality less than or equal to $\omega^M_{\omega+1+\ell}$ and this is a contradiction.

Therefore, in $M[G]$ and for every $n \in \omega$, $\omega^M_{\omega\cdot 2}$ is a cardinal and so $\omega^M_{\omega\cdot 2}$ is a cardinal also in $N \subseteq M[G]$. qed

Now we have proven our claim that in N the first four infinite cardinals are regular (ω), singular ($\omega_1 = \omega^M_\omega$), regular ($\omega_2 = \omega^M_{\omega+1}$) and singular ($\omega_3 = \omega^M_{\omega\cdot 2}$).

As for the model $N^\star$, we don't know if the reals are wellorderable or not here. If one tries a similar proof as for the Feferman-Lévy model (see Theorem 3.13) then this is what happens:

Lemma 3.28.
In N, the powerset of ω is not wellorderable.

Attempt of Proof. Using AC in $\mathbb{V}$, we get a function $\dot{}\, : x \mapsto \dot{x}$ from N to HS, such that $(\dot{x})_G = x$. If $x \in N$, then we know that $\dot{x} \in$ HS therefore there is n such that $K_n \subseteq \mathsf{sym}(\dot{x})$. Define $C_n \overset{\text{def}}{=} \{x \in \mathbb{R}^N \,;\, K_n \subseteq \mathsf{sym}(\dot{x})\}$ and note that

$$\mathbb{R}^N = \bigcup_{n\in\omega} C_n.$$

Therefore if we prove that for every $n \in \omega$, C_n is countable, then we proved the theorem.

For $x \in \mathbb{R}^N$ we define a name $\ddot{x}$ such that

$$\ddot{x} \stackrel{\text{def}}{=} \{(\check{k}, [p]^n) \; ; \; p \Vdash \check{k} \in \dot{x}\}.$$

It's clear that $(\dot{x})_G = (\ddot{x})_G = x$. Define

$$C'_n \stackrel{\text{def}}{=} \{\ddot{x} \; ; \; x \in C_n\} \subseteq M^{\mathbb{P}_0 \times \mathbb{P}_1}$$

and note that $\ddot{\ }$ is an injection from C_n into C'_n. So if $N \models$ "C'_n is countable", then C_n will be countable and we will have proved the theorem.

Look at the elements of C'_n. They are sets of pairs of the form $(\check{k}, [p]^n)$. The first coordinate is a canonical name for an M-subset of ω and by GCH^M, these are ω_1^M-many. The second coordinate consists of conditions in the set $\{[p]^n \; ; \; p \in \mathbb{P}_0 \times \mathbb{P}_1 \exists x \in \mathbb{R}^N (p \Vdash \ddot{x} \in \mathbb{R}^N)\}$.

Claim. In M, for every $n \in \omega$, the set $A_n \stackrel{\text{def}}{=} \{[p]^n \; ; \; p \in \mathbb{P}_0 \times \mathbb{P}_1$ and $\exists x \in \mathbb{R}^N (p \Vdash \ddot{x} \in \mathbb{R}^N)\}$ has cardinality lower than ω_ω^M.

If this claim is true, then C'_n would be countable in N and therefore $\mathbb{R}$ would be a countable union of countable sets, as in $\mathcal{M}9$. Then it could not be injected into any ordinal and therefore not into any ordinal below $\omega_3^{\mathcal{M}9}$. Also, $\mathbb{R}$ would be contained in 2^{ω_1} and 2^{ω_2} and thus they could not be injected into any ordinal below ω_3 either. So if this claim was true, then ω_3 would be an $\bar{\mathbf{\imath}}$-strong cardinal. But then, in $M[G]$, $\mathbb{R}^N$ would be countable and therefore there could not be a surjection from $\mathbb{R}$ onto ω_3 which was our intended way to show that ω_3 is not $\bar{\mathbf{s}}$-strong. Similarly we cannot show if 2^{ω_1} or 2^{ω_2} surject onto ω_3 or not.

Now, if the claim is not true then we don't know how to show that ω_3 is $\bar{\mathbf{\imath}}$-strong or not and neither we can show that ω_3 is $\bar{\mathbf{s}}$-strong or not. We feel that this question is not answerable via this type of construction; at least not easily.

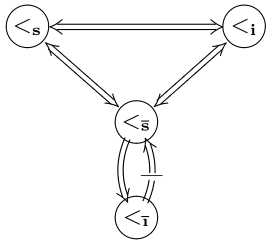
4 Conclusion and open question

Concluding, we saw that the notion of being strong limit cardinal is rather interesting when studied without the assumption of choice. We saw that the statement $\omega_1 \not\lesssim \mathbb{R}$ has high consistency strength when combined with AC, and in particular with the statement "ω_1 is regular". Without the latter statement, the consistency of $\omega_1 \not\lesssim \mathbb{R}$ is not strong; it is implied by the consistency of ZF.

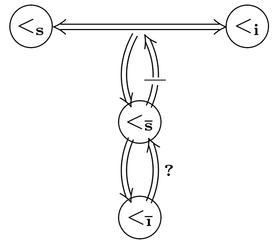
With the axiom of determinacy we saw how questions of descriptive set theory come to play; questions about topological properties of the real line. The axiom of determinacy also opened possibilities for the definition of the β-strong limit cardinal. We saw that this definition is more appropriate for descriptive set theory since when AD and AC fail, as in Blass' model $\mathcal{M}15$, β-strong limits are not always interesting.

The most interesting results in Part II of this thesis come with the construction of the Feferman-Lévy model $\mathcal{M}9$. These results can be seen in the diagrams below.

Successor case:



Limit case:



The Feferman-Lévy model $\mathcal{M}9$ is a well known construction with automorphisms of partial orders that gave all our results in the diagrams above and inspired us to create a generalisation of it to solve our final question. Even though this generalisation has failed its purpose, we understand now that having a surjection and not having its converse by AC injection is not as easy as we thought.

In a second look at $\mathcal{M}9$, we now realise that for $\mathcal{M}9$ we could simply destroy the injection $\omega_1 \rightarrowtail \mathbb{R}$ because the surjection was protected by the ZF axioms[7] and thus could not be destroyed. Therefore we feel that at least by using the kind of partial orders we used for the construction of N and N^*, we will not shed light to our separation problem; at least not easily.

After the completion of this thesis, Andreas Blass solved the open question, i.e., separated the notions of being an $\bar{\mathbf{s}}$-strong limit cardinal and an $\bar{\mathbf{i}}$-strong limit cardinal.

[7] See Lemma 1.4.

References

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