On Modal μ-Calculus with Explicit Interpolants

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October 23, 2002

Abstract This paper deals with the extension $\tilde{\mu}$ of Kozen μ-calculus with the so-called " bisimulation quantifiers". These quantifiers allow to look for a subset P satisfying $\phi(P)$ not only within the model, but also in any other model which is bisimilar to the given one. By using these quantifiers one can describe explicitly the uniform interpolant of any formula of the μ-calculus. In this work we provide a complete calculus for $\tilde{\mu}$.

1 Introduction

The uniform interpolant of a formula ϕ with respect to a sublanguage L' of the language of ϕ is a formula ψ in the language L' which *behaves* like ϕ when the language L' is concerned, in the sense that ψ has the same L'-logical consequences as ϕ. A logic that contains all uniform interpolants of its formulas is said to enjoy uniform interpolation. Since uniform interpolation implies Craig interpolation, a logic having this property has a good interplay between syntactical and semantical behaviour. Moreover uniform interpolation allows *modularization* and this is in general not possible if only Craig interpolation is present: if we are interested only in L'-consequences of ϕ then we may consider the (hopefully simpler) uniform interpolant ψ instead of ϕ, and this would be like a module for this subtask. Notice however that in order to use this property in the applications it is important to construct a uniform interpolant ψ explicitly from the starting formula ϕ.

In this paper we consider the Kozen μ-calculus (see [4],[5]), a temporal logic with the same expressive power than Monadic Second Order Logic, when only bisimulation invariant properties are considered. In [[1]] it is shown that the

μ-calculus enjoys uniform interpolation but no explicit form for uniform interpolants in the original logic is given, although a semantical characterization in terms of bisimulation is proved: if L is the language of ϕ minus P, for a propositional constant P, then the uniform interpolant of ϕ with respect to L is true in a model M iff we can find an interpretation of P under which ϕ is true, not necessarily in M but in a model N which is L'-bisimilar to N. This property suggests to enrich the original μ-language with an existential bisimulation quantifier $\widetilde{\exists}P$, and to denote the uniform interpolant of ϕ with respect to the language of ϕ minus P by $\widetilde{\exists}P\phi$. This logic, denoted by $\tilde{\mu}$, is not more powerful then the original one, but in the extended language we gain the possibility of expressing uniform interpolants explicitly. However, to *use* this new logic we need at least to find a complete calculus for it.

In this paper we give an axiomatization of this μ-calculus with explicit interpolants. The strategy is to first find explicit uniform interpolants in the original μ-language for disjunctive formulas, a well known class which is representative of the whole μ-calculus, and then find a system of axioms and rules that are sufficient to *prove* within the system that these are indeed uniform interpolants.

We first start with the simple case of basic modal logic K in Section 3 and then consider the μ-calculus in Section 4.

2 Preliminaries and Notation

In the following, we only consider structures for the language containing a unary predicate P for each symbol in $Prop$, a binary predicate R, and a constant r, i.e. tuples of the form

$$M = (D^M, r^M, R^M, P_1^M, \ldots\}.$$

We will describe properties of these structures using propositional temporal logics. We are mainly interested in *Basic Modal Logic* K and in the *μ-calculus*, and in their extensions by means of bisimulation quantifiers.

2.1 Logics

We consider *modal formulas* as constructed from $\bot, \top$ and propositional constants (one for each unary predicate for each symbol P in $Prop$) using $\neg, \vee$ and the n-ary *cover operators*: if σ is a conjunctions of literals from $Prop$ and $\phi_1, \ldots, \phi_n$ are formulas, then

$$\sigma \wedge Cover(\phi_1, \ldots, \phi_n)$$

is a formula.
A cover operator of type $\sigma \wedge Cover(\phi_1, \ldots, \phi_n)$ is true in a point w of a model M if and only if w satisfies σ and the successors of z are *covered* by $\phi_1, \ldots, \phi_n$. More formally, we define:

$$M \models \sigma \wedge Cover(\phi_1, \ldots, \phi_n) \Leftrightarrow \qquad r^M \models \sigma \qquad \text{and:}$$

1. for all $i = 1, \dots, n$ there exists w with $(r^M, w) \in R^M$ and $(M, w) \models \phi_i$;
2. for all w with $(r^M, w) \in R^M$ there exists $i = 1, .., n$ with $(M, w) \models \phi_i$;

Since cover operators are definable in terms of $\Box, \Diamond$ and

$$\Box\phi \Leftrightarrow Cover(\emptyset) \vee Cover(\phi), \qquad \Diamond\phi \Leftrightarrow Cover(\phi, \top),$$

this logic is equivalent to the familiar modal logic constructed from $\Box$ and $\Diamond$. Next, we consider an important class of modal formulas:

Definition 2.1 *The class of* disjunctive modal formulas *is the least class containing $\top, \bot$, and non-contradictory conjunctions of literals which is closed under disjunctions and non contradictory cover operator: if $\phi_1, \dots, \phi_n$ are disjunctive formulas and σ is a conjunction of a set of literals not containing both P and $\neg P$, for any $P \in Prop$, then $\sigma \wedge Cover(\phi_1, \dots, \phi_n)$ is a disjunctive formula.*

Using the notion of bounded bisimulation, one can see that any modal formula is equivalent to a disjunctive one (this is because the r-bisimulation type of a model M is in fact a disjunctive formula).

The *μ-formulas* are constructed from literals $Prop \cup \{\neg P : P \in Prop\}$ and variables in a set Var using $\vee, \wedge$, cover formulas of type $\sigma \wedge Cover(\phi_1, \dots, \phi_n)$ (where σ is a conjunctions of elements from $Prop \cup \{\neg P : P \in Prop\} \cup Var$), and the fixed point operators $\mu X.\phi, \nu X.\phi$.

The semantics of a μ-calculus formula ϕ in a structure M depends on a valuation V of the free variables of ϕ and it is defined as usual for the modal operators, while $\mu X.\phi$ and $\nu X.\phi$ are interpreted respectively as the least and the greatest fixed point of the operator induced by the formula $\phi(X)$ (for a precise definition see [4]).

An important class of μ-formulas are the disjunctive ones:

Definition 2.2 *The class of* disjunctive μ formulas *is the least class containing $\top, \bot$, and non-contradictory conjunctions of literals which is closed under disjunctions and non contradictory cover operators (if $\phi_1, \dots, \phi_n$ are in the class and σ is a conjunction of a set of literals not containing both P and $\neg P$, then $\sigma \wedge Cover(\phi_1, \dots, \phi_n)$ is in the class), and such that if ϕ is disjunctive and does not contain $X \wedge \gamma$ as a subformula, for any formula γ, then $\mu X.\phi, \nu X.\phi$ are disjunctive.*

As in the modal case (but the proof is more difficult!) one has :

Theorem 2.3 *([2]) Any μ-calculus formula is equivalent to a disjunctive one.*

2.2 Bisimulation Quantifiers

To introduce bisimulation quantifiers we first need the notion of bisimulation.

Definition 2.4 *Let M, N be structures with D^M, D^N as respective domains. Let $Prop' \subseteq Prop$. A relation $Z \subseteq D^M \times D^N$ is a $(Prop')$-*bisimulation* between M and N if $r^M Z r^N$, bisimilar points agree on propositional constants in $Prop'$ (i.e. if wZv then $w \in P^M$ iff $v \in P^M$, for all $P \in Prop'$) and whenever wZv the following conditions hold:*
if $wR^M w'$, then there exists a v' such that $vR^N v'$ and $w'Zv'$;
if $vR^N v'$, then there exists a w' such that $wR^M w'$ and $w'Zv'$.
*Two structures M, N are $Prop'$-*bisimilar* (notation: $M \sim_{Prop'} N$) if there exists a $Prop'$-bisimulation between them (but we write $M \sim N$ if $Prop' = Prop$).*
If $Var' \subseteq Var$ we may also need the notion of a bisimulation between two structures (M, V_1) and (N, V_2) were V_1, V_2 are valuations of the variables in Var': in this case we additionally require: if wZv then $w \in V_1(X)$ iff $v \in V_2(X)$, for all $X \in Var'$.

An existential bisimulation quantifier is then defined as a classical Monadic Second Order quantifier, except that we look for a subset satisfying a certain property not only in the model, but in any other model which is bisimilar to the given one:

Definition 2.5 *(Existential Bisimulation Quantifier) We enrich the modal and μ-grammar with a propositional quantifier $\widetilde{\exists} P\phi$, for $P \in Prop$, whose semantics is defined as follows. If M is a structure and V is a valuation of the free variables of ϕ, then*

$$M, V \models \widetilde{\exists} P\phi$$

$$\Updownarrow$$

$$\exists N, \exists V' \text{ and } \exists P \subseteq D^N \text{ with } (N, V') \sim_{Prop\setminus\{P\}} (M, V) \text{ and } (N, V', P) \models \phi,$$

where V' is a valuation of the free variables of ϕ in N, and (N, V', P) stands for the structure which is like (N, V') except for the propositional constant P that receives now a new interpretation (denoted again by P).

The logic obtained by enriching the μ-calculus (or modal logic K) by the existential bisimulation quantifier $\widetilde{\exists}$ is denoted by $\tilde{\mu}$ ($\tilde{K}$, respectively). In these logics, the operators $\wedge, \rightarrow$ are defined as usual and the universal bisimulation quantifier $\widetilde{\forall}$ is defined as the dual of the existential one: $\widetilde{\forall} P\phi = \neg\widetilde{\exists} P\neg\phi$. In the following, we denote by $L(\phi)$ the set of propositional contants which occur in ϕ and by $\models$ the relation of logical consequence between $\tilde{\mu}$-sentences.

2.3 Uniform interpolation

Bisimulation quantifiers are related to the property of uniform interpolation:

Definition 2.6 *Given a μ-sentence ϕ and a language $L' \subseteq L(\phi)$, the* uniform interpolant *of ϕ with respect to L' is a formula θ such that:*

1. *$\phi \models \theta$;*

2. *Whenever* $\phi \models \psi$ *and* $L(\phi) \cap L(\psi) \subseteq L'$ *then* $\theta \models \psi$.

3. $L(\theta) \subseteq L'$.

When we say that the μ-calculus has uniform interpolation we mean that we can always find a uniform interpolant when the appropriate conditions are satisfied. Clearly, if the μ-calculus has uniform interpolation, it also have Craig interpolation: if $\phi \models \psi$, simply choose $L' = L(\phi) \cap L(\psi)$. The interpolant is then the uniform interpolant of ϕ relative to L'. This explains why we call this formula a *uniform* interpolant: no information is needed about the formula ψ except which non-logical symbols it has in common with ϕ.

In [1] it is proved that the μ-calculus can express all uniform interpolants of all its formulas and that the uniform interpolant of a formula ϕ with respect to the language $L(\phi) \setminus \{P\}$ is semantically equivalent to the formula $\widetilde{\exists} P \phi$:

Theorem 2.7 *If* ϕ *is a* μ*-formula then there exists a* μ*-formula* ψ *which is a uniform interpolant of* ϕ *w.r.t. the language* $L(\phi) \setminus \{P\}$ *and moreover, considering* ψ *as a* $\tilde{\mu}$*-formula one has*

$$\models \psi \leftrightarrow \widetilde{\exists} P \phi.$$

3 A Complete Calculus for Modal Logic with Explicit Uniform Interpolants

Which are the principles that the existential bisimulation quantifier satisfies? By a simple inspection on the meaning of the existential quantifier we recognize that the following properties hold:

Existential Elimination Rule (EER):
if $\phi \rightarrow \psi$ is valid, then $\widetilde{\exists} P \phi \rightarrow \psi$ is valid, provided P is not free in ψ (i.e. if it does occur, it is in the scope of a bisimulation quantifier).

Existential Introduction Axiom (EIA):
the formula $\phi[P/\psi] \rightarrow \widetilde{\exists} P \phi$ is valid, where the substitution of ψ for P in ϕ is admissible, in the sense that no free variables of ψ must become bounded after the substitution for P in ϕ.

To prove for example that the Existential Introduction Axiom is valid we may proceed as follows. Given a model M, consider the model $M[P/\psi]$ which is like M except for the valuation of P that is now interpreted as the set $\{w : (M, w) \models \psi\}$, where (M, w) denotes the model M with the new root w. One can easily check that

$$M \models \phi[P/\psi] \Leftrightarrow M[P/\psi] \models \phi.$$

Hence, if $M \models \phi[P/\psi]$ then $M \models \widetilde{\exists} P \phi$, because the model $M[P/\psi]$ is bisimilar w.r.t. $L(\phi) \setminus \{P\}$ to M.

Our aim is to describe completely the principles satisfied by $\widetilde{\exists}P$ over the modal logic K. We do this by means of a complete calculus for the logic $\tilde{K}$. One could think that adding the Existential Introduction Axiom and the Existential Elimination Rule to a system K which is complete for modal logic would suffice, like in first order logic, but it is easy to see that this is not the case. Consider for example a valid principle as

$$\phi = \Diamond\top \to \widetilde{\exists}P(\Diamond(P) \wedge \Diamond(\neg P)).$$

It is easy to see that the system $K + EER + EIA$ cannot prove ϕ: this is because all axioms and rules of $K + EER + EIA$ are valid when we interpret the bisimulation quantifier $\widetilde{\exists}$ as a standard second order quantifier, while ϕ is not valid under this interpretation. Hence we need to add some more principles to $K + EER + EIA$ in order to obtain a complete system. In this section we show that it is enough to add some simple *commutativity axioms*, relating the bisimulation quantifier $\widetilde{\exists}$ to disjunction and cover operators. First of all we prove that the existential bisimulation quantifier $\widetilde{\exists}P$ commutes with disjunctions and cover operators.
In the following lemma, we denote by $\sigma[P/\top, \neg P/\top]$ the formula obtained from a conjunction σ of literals by deleting P and $\neg P$.

Lemma 3.1 *If σ is a conjunction of a set of literals not containing both P and $\neg P$, and $\phi_1, \ldots, \phi_n$ are μ-formulas, then the following are valid formulas.*

$$\widetilde{\exists}P\sigma \leftrightarrow \sigma[P/\top, \neg P/\top], \qquad \widetilde{\exists}P(\phi_1 \vee \phi_2) \leftrightarrow \widetilde{\exists}P\phi_1 \vee \widetilde{\exists}P\phi_2$$

$$\widetilde{\exists}P(\sigma \wedge Cover(\phi_1, \ldots, \phi_n) \leftrightarrow \widetilde{\exists}P\sigma \wedge Cover(\widetilde{\exists}P\phi_1, \ldots, \widetilde{\exists}P\phi_n),$$

Proof.
We only prove the implication

$$\widetilde{\exists}P\sigma \wedge Cover(\widetilde{\exists}P\phi_1, \ldots, \widetilde{\exists}P\phi_n) \to \widetilde{\exists}P\ (\sigma \wedge Cover(\phi_1, \ldots, \phi_n)).$$

Suppose $\widetilde{\exists}P\sigma \wedge Cover(\widetilde{\exists}P\phi_1, \ldots, \widetilde{\exists}P\phi_n)$ holds in a model (M, V), where V is a valuation of the free variables in $\phi_1, \ldots, \phi_n$. Fix a successor v of the root r^M of M and consider all formulas of type $\widetilde{\exists}P\phi_i$ it satisfies: for any such formula there exists a model $N_{v,i}$ and a valuation $V_{v,i}$ of the free variables of ϕ_i such that $(N_{v,i}, V_{v,i})$ is $Prop \setminus \{P\}$-bisimilar to (M, V) and $(N_{v,i}, V_{v,i}) \models \phi_i$. Consider a new model M' with a new root satisfying the same propositional constants as r^M and connected to all these $N_{v,i}$, when v varies in the successors of r^M. Then the model M' and the valuation V' defined on a variable X as $V'(X) = \cup_{v,i} V_{v,i}(X)$ is $Prop \setminus \{P\}$-bisimilar to (M, V) and verifies the formula $\sigma \wedge Cover(\phi_1, \ldots, \phi_n)$.

Using this Lemma we can easily find explicit interpolants (in the original modal language) for the class of disjunctive modal formulas:

Corollary 3.2 *If ϕ is a disjunctive modal formula, then*

$$\models \widetilde{\exists}P\phi \leftrightarrow \phi[P/\top, \neg P/\top],$$

where $\phi[P/\top, \neg P/\top]$ is defined from ϕ by simultaneously substituting the literals P and $\neg P$ with $\top$ (i.e. $(P \vee \neg P)[P/\top, \neg P/\top] = \top \vee \top$).

The disjunctive class is an important one because any μ-formula is equivalent to a disjunctive one. We use this fact and Corollary 3.2 to find a complete system for $\tilde{K}$.

Theorem 3.3 *The logic $\tilde{K}$ is complete with respect to the following axioms and rule:*

1. *a complete system of axioms and rules for modal logic K;*
2. *the existential introduction axiom EIA and the existential elimination rule EER;*
3. *$\tilde{\exists} P(P) \leftrightarrow \tilde{\exists} P(\neg P) \leftrightarrow \top$;*
4. *if σ is a non contradictory conjunction of literals and $\phi_1, \ldots, \phi_n$ are formulas, then*

$$\tilde{\exists} P(\sigma \wedge Cover(\phi_1, \ldots, \phi_n)) \leftrightarrow \sigma[P/\top, \neg P/\top] \wedge Cover(\tilde{\exists} P\phi_1, \ldots, \tilde{\exists} P\phi_n).$$

Proof.

To prove that the above system is complete, it is enough to show that any $\tilde{K}$-formula ϕ is provably equivalent to a modal formula. This can be achieved by induction on the structural complexity of the formula, the only interesting case being $\phi = \tilde{\exists} P\psi$. By induction, we suppose that ψ is provably equivalent to a modal formula ψ^-, which is in turn provably equivalent to a disjunctive formula ψ_d^- since our system contains the complete system K. By the rule of existential elimination and introduction we have

$$\tilde{\exists} P\psi \leftrightarrow \tilde{\exists} P\psi_d^-$$

and we only have to prove that $\tilde{\exists} P\psi_d^-$ is provably equivalent to a modal formula. But this can be easily seen because Corollary 3.2 can be proved in our system by using the cover axioms, EER, and EIA.

4 A Complete Calculus for the μ-Calculus with Explicit Uniform Interpolants

The strategy is the same as for $\tilde{K}$, that is, to find a class of μ-formulas which is representative of the whole logic and for which we can give explicit forms to the uniform interpolants without enlarging the language with $\tilde{\exists} P$. This is the class of disjunctive μ-formulas.

4.1 Explicit Uniform Interpolant for Disjunctive Formulas

In this section we prove that if $\phi(P, \neg P)$ is a disjunctive μ-calculus formula, then its uniform interpolant $\widetilde{\exists} P\phi$ is equivalent to $\phi[P/\top, \neg P/\top]$. This will allow us to find a complete calculus for $\tilde{\mu}$ in Section 4.2.

Lemma 4.1 *If ϕ is a disjunctive μ-calculus formula, then its uniform interpolant $\widetilde{\exists} P\phi$ is equivalent to $\phi[P/\top, \neg P/\top]$.*

Lemma 4.1 is proved by using tableaux for disjunctive formulas (see [2]). We first recall some definitions (which are semplified compared with [2] since we are only dealing with the disjunctive case) and then prove a preliminary result on consistent markings of disjunctive formulas' tableaux.

A μ-formula γ is *guarded* if every occurrence of the bound variable X in every subformula $\tau X.\alpha$ (for $\tau \in \{\mu, \nu\}$) is under the scope of a cover, and it is *well-named* if, for every bound variable X, there exists only one subformula of type $\tau X.\alpha$ in γ. It is possible to prove that any μ-formula is equivalent to a well-named, guarded formulas, and in this section we will only deal with this kind of formulas (henceforth called simply *formulas*). A bound variable X of such a formula γ is a μ-variable (ν-variable) if $\mu X.\alpha$ is a subformula of γ ($\nu X.\alpha$, respectively). We define a partial order $\preceq_\gamma$ on the set of the bound variables of a formula γ as the least partial order such that if $\tau X.\alpha$ is a γ-subformula, $\tau' Y.\beta$ is an α-subformula, and X occurs in β then $X \preceq_\gamma Y$. In other words, the bound variable Y is an immediate successor of the bound variable X if the scope of X contains Y and X occurs free in the scope of Y.

Definition 4.2 *A* tableau $\mathcal{T} = (T, L)$ *for a disjunctive formula γ is a tree T with a labeling function L such that the root is labeled by γ and the sons of every node are created and labeled with formulas according to the following rules:*

$$\frac{\alpha \quad \beta}{\alpha \vee \beta}$$

$$\frac{\alpha}{\tau X.\alpha}$$

$$\frac{\alpha}{X} \text{ where } \tau X.\alpha \text{ is a subformula of } \gamma$$

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\sigma \wedge Cover(\alpha_1 \quad \dots \quad \alpha_n)}$$

In a tableau, nodes labelled by cover formulas are called *modal nodes*, while sons of modal nodes are called *choice nodes*.

A variable X is *regenerated* in a node m of a tableau $\mathcal{T} = (T, L)$ if the regeneration rule is applied on m, that is, if $L(m) = X$ and the only son n of m is labeled α where $\tau X.\alpha$ is a γ-subformula.

An infinite path is called a *μ-path* if the least variable (w.r.t. the $\preceq_\gamma$-order) which is regenerated infinitely often is a μ-variable.

Let M be a structure, V a valuation of the free variables of the formula γ, and $\mathcal{T} = (T, L)$ a tableau for γ. A *consistent marking* of $\mathcal{T}$ with respect to (M, V) is a function $C : T \to Pow(M)$ such that the following conditions hold.

1. $C(r^T) = \{r^M\}$.

2. If $L(m) = \tau X.\alpha$ and n is the only son of m then $C(n) = C(m)$.

3. If $L(m) = \alpha \vee \beta$ and n_1, n_2 are the two sons of m then $C(n_1), C(n_2) \subseteq C(m)$ and $C(n_1) = C(m) \setminus C(n_2)$.

4. If $L(m) = \sigma \wedge Cover(\alpha_1, \ldots, \alpha_n)$ and $t \in C(m)$ then:

   (a) for every son n of the node m there exists a successor t' of t with $t' \in C(n)$;

   (b) for every successor t' of t in M there exists a son n of m in T with $t' \in C(n)$.

5. If m is a modal node with $L(m) = \sigma \wedge Cover(\alpha_1, \ldots, \alpha_n)$ and $t \in C(m)$, then $(M, t, V) \models \sigma$, while if m is a leaf and $t \in C(m)$ then (M, t, V) satisfies (the conjunction of literals) $L(m)$.

6. There exists no infinite μ-path $n_0, \ldots, n_i, \ldots$ of T with $C(n_i) \neq \emptyset$ for every i.

From [2] we get:

Theorem 4.3 *A disjunctive formula γ is satisfied in a finitely branching structure M under a valuation V of the free variables of γ if and only its tableau has a consistent marking w.r.t. (M, V).*

Notice that without loss of generality we may suppose that the consistent marking in Theorem 4.3 has the following properties:

(6) if $C(m) = \emptyset$ then for every descendant n of m we have $C(n) = \emptyset$;

(7) if m is modal and n is one of its sons then for every $t \in C(n)$ there exists $s \in C(m)$ with sRt.

To find explicit interpolants for disjunctive formulas we first prove that we can always *separate* consistent markings at sons of modal nodes, modulo bisimulation. We remember that the degree of a tree is the maximal number of arrows starting from a node, and that a k-expanded structure is a tree in which any successor of a given node w has at least k-bisimilar brothers.

Lemma 4.4 *Suppose the tableau $\mathcal{T} = (T, L)$ for γ has degree k. Then a k-expanded, finitely branching structure (M, V) satisfies γ if and only if there exists a consistent marking C of (M, V) such that, for every modal node m of T with sons $n_1, \ldots, n_k$ and $i \neq j$, it holds $C(n_i) \cap C(n_j) = \emptyset$.*

Proof.
Consider the structure $\overline{M}$ defined by identifying two points t, t' if they are sons of the same node and (M, t, V), (M, t', V) are bisimilar. More precisely we define the equivalence relation $\sim$ on M as

$$t \sim t' \quad \Leftrightarrow \quad \exists s \in M \text{ with } sRt, sRt' \text{ and } (M, t, V) \text{ is bisimilar to } (M, t', V).$$

The domain of $\overline{M}$ is the set of equivalence classes of $\sim$, its root $[r^M]$, its accessibility relation $\overline{R}$ is defined as

$$[s]\overline{R}[t] \Leftrightarrow sRt,$$

and for every propositional constant P we have $[s] \in P^{\overline{M}}$ if and only if $s \in P^M$. We also define a valuation $\overline{V}$ of the γ-free variables by $[s] \in \overline{\mathcal{V}}(X)$ if and only if $s \in \mathcal{V}(X)$.

One can easily see that $(\overline{M}, \overline{V})$ is bisimilar to (M, V); hence if (M, V) satisfies γ, then so does $(\overline{M}, \overline{V})$. By Theorem 4.3 we know that the tableau $\mathcal{T}$ has a consistent marking $\overline{C}$ w.r.t. $(\overline{M}, \overline{V})$. From $\overline{C}$ we define a consistent marking C of $\mathcal{T}$ w.r.t. (M, V) as follows. If m is a node of T which is not a choice node, then $C(m) = \bigcup\{[s] : [s] \in \overline{C}(m)\}$. For choice nodes we proceed differently. Suppose m is the father modal node with choice sons $n_1, \ldots, n_h$. We cannot define $C(n_i)$ as before, because a class $[t]$ could belong to $\overline{C}(n_i)$ as well as to $\overline{C}(n_j)$ and we want our marking to be disjoint at choice nodes. Hence, for each class $[t]$, we consider all choice nodes n between $n_1, \ldots, n_h$ with $[t] \in \overline{C}(n)$, say there are l such nodes; since $l \leq h \leq k$ and (M, V) is k-expanded, we can split the set $[t]$ into l non empty, disjoint subsets $[t]_n$, one for each n with $[t] \in \overline{C}(n)$, putting all points of $[t]_n$ in $C(n)$.

We now prove that C satisfies all conditions of a consistent marking. Conditions $(1), (2)$ are easily checked. Consider condition $4.a, 4.b$, regarding a modal node m and $t \in C(m)$. By construction $[t] \in \overline{C}(m)$, and if n is a son of m, there exists a $[t']$ such that $[t]R[t']$ and $[t'] \in \overline{C}(n)$. By definition, $[t']_n$ is non empty and contained in $C(n)$: if $u \in [t']_n$ we have tRu and $u \in C(n)$. This proves condition $4.a$.

As for condition $4.b$, suppose tRu. Then $[t]R[u]$, and there exists a son n of m with $[u] \in \overline{C}(n)$. By the definition of C, the set $[u]$ to which u belongs has been split into subsets $[u]_{n'}$, with $[u]_{n'} \subseteq C(n')$. For one of them it must hold $u \in [u]_{n'} \subseteq C(n')$. Hence n' is a son of m for which $u \in C(n')$, and this proves $4.b$.

As for condition 5, $t \in C(n)$ implies $[t] \in \overline{C}(m)$, and $(M, t, V), (\overline{M}, [t], \overline{V})$ agree on propositional constants and free variables of γ.

Finally, for each node m if $C(m)$ is non empty then so is $\overline{C}(m)$, and this proves condition (6).

Remark We call a consistent marking C of a k-expanded model with the property stated in Lemma 4.4 a *disjoint consistent marking*. If the tableau $\mathcal{T}$ has such a marking and n_1, n_2 are nodes of T then either n_1, n_2 are comparable in the tree order or $C(n_1) \cap C(n_2) = \emptyset$. This can be seen as follows. If n_1, n_2 were not comparable one could consider their bigger common ancestor (w.r.t.

tree order), say m. This node must have more than one son, so it should be either a modal node or a node in which the *or* rule is applied. There are then two sons m_1, m_2 of m such that n_i is either m_i or an m_i-descendent, for $i = 1, 2$, and $C(m_1) \cap C(m_2) = \emptyset$. But a point t in $C(n_i)$ is either in $C(m_i)$ or it is a descendent of an element in $C(m_i)$, hence $C(n_1) \cap C(n_2) = \emptyset$.

Lemma 4.5 *The uniform interpolant $\widetilde{\exists} P\phi$ of a disjunctive μ-formula ϕ is equivalent to the μ-formula $\phi[P/\top, \neg P/\top]$, where $\phi[P/\top, \neg P/\top]$ is defined from ϕ by simultaneously substituting the literals P and $\neg P$ with $\top$.*

Proof.
Since ϕ is positive in both P and $\neg P$, the formula $\phi \to \phi[P/\top, \neg P/\top]$ is valid, and hence $\widetilde{\exists} P\phi \to \phi[P/\top, \neg P/\top]$ is valid as well.

Conversely, let $\mathcal{T} = (T, L)$ be a tableau for $\phi[P/\top, \neg P/\top]$, and let k be the degree of T. We prove that if a k-expanded model (M, V) satisfies $\phi[P/\top, \neg P/\top]$ then there is a subset $P \subseteq M$ such that (M, P, V) satisfies ϕ. This is enough to prove that $\phi[P/\top, \neg P/\top] \to \widetilde{\exists} P\phi$ is valid.

First notice that a tableau $\mathcal{T}' = (T', L')$ for ϕ has $T' = T$, while for each node m one has $L'(m)[P/\top, \neg P/\top)] = L(m)$. By Theorem 4.3 if the k-expanded model (M, V) satisfies $\phi[P/\top, \neg P/\top]$ then $\mathcal{T}$ has a consistent marking C w.r.t. (M, V). We will define a subset P of M in such a way that C will be a consistent marking for the tableau $\mathcal{T}'$ w.r.t. (M, P, V) as well.

The subset P is defined as follows: $t \in P$ iff there exists a leaf or modal node m such that $t \in C(m)$ and the formula $L'(m) \to P$ is valid.

We now check that C is a consistent marking for the tableau $\mathcal{T}'$ w.r.t. (M, P, V). The only non trivial condition to verify is condition 4. Suppose m is a modal node with $L'(m) = \sigma \wedge Cover(\alpha_1, \ldots, \alpha_n)$ and $t \in C(m)$, then since C is a consistent $\mathcal{T}$ marking w.r.t. M, t, V we know that $(M, t, V) \models \sigma[P/\top, \neg P/\top]$. We then only have to check that $t \in P$ iff σ contains P. If $t \in P$ then by definition there exists a leaf or modal node m' with $t \in C(m')$ such that $L'(m') \to P$ is valid. Since $t \in C(m) \cap C(m') \neq \emptyset$, by the Remark after Lemma 4.4 we know that m, m' must be comparable, and since m is modal it follows that $m = m'$. Hence, $L'(m') = \sigma \wedge Cover(\alpha_1, \ldots, \alpha_n)$, and from the validity of $L'(m') \to P$ we obtain that σ contains P.
Conversely, if σ contains P then $L'(m) \to P$ is valid, and $t \in P$ by definition.

Suppose now that m is a leaf and $t \in C(m)$. As before we know that $(M, t, V) \models \sigma[P/\top, \neg P/\top]$, and we only have to check that $t \in P$ iff σ contains P. If $t \in P$ then by definition there exists a leaf or modal node m' with $t \in C(m')$ such that $L'(m') \to P$ is valid. Since $t \in C(m) \cap C(m') \neq \emptyset$, by the Remark after Lemma 4.4 we know that m, m' must be comparable. But then $m = m'$, because otherwise m' cannot be a leaf or a modal node, and we may proceed as before.

From the previous result we can now easily prove that the existential bisimulation quantifier $\widetilde{\exists} P$ commutes with the fixed point operators $\mu X, \nu X$, provided the *contexts* $\mu X.\phi$, $\nu X.\phi$ are disjunctive.

Corollary 4.6 *If $\mu X.\phi$ and $\nu X.\phi$ are disjunctive formulas then*

$$\models \tilde{\exists} P\mu X.\phi \leftrightarrow \mu X.\tilde{\exists} P\phi \qquad\qquad \models \tilde{\exists} P\nu X.\phi \leftrightarrow \nu X.\tilde{\exists} P\phi.$$

Proof.
This is simply because by the previous result one has e.g. that $\tilde{\exists}P\mu X.\phi$ is equivalent to $(\mu X.\phi)[P/\top, \neg P/\top]$, which is the same as $\mu X.(\phi[P/\top, \neg P/\top])$, which is in turn equivalent to $\mu\tilde{\exists}P\phi$.
Notice that the second equivalence is not true in general. Consider e.g. the formula $\phi = P \wedge \Diamond(\neg P) \wedge \Box X$. We have $\tilde{\exists}P\phi = \Diamond\top \wedge \Box X$, hence $\nu X.\tilde{\exists}P\phi = \nu X.\Diamond\top \wedge \Box X$, which is true in a model M iff all nodes accessible from the root satisfy $\Diamond\top$. On the other hand, the formula $\nu X.\phi$ is equivalent to $\bot$, and so is $\tilde{\exists}P\nu X.\phi$.

4.2 A Complete System for $\tilde{\mu}$

In the next theorem we show that adding commutativity between $\tilde{\exists}$ and fixed point operators in a disjunctive context to the principles of logic $\tilde{K}$ is enough to obtain a complete system for $\tilde{\mu}$.

Theorem 4.7 *$\tilde{\mu}$ is complete with respect to the following axioms and rules:*

1. *a complete system of axioms and rules for the μ-calculus, e.g. the Kozen system (see [5]);*
2. *the Existential Elimination Rule and the Existential Introduction axiom (see section 3);*
3. $\tilde{\exists}P(P) \leftrightarrow \tilde{\exists}P(\neg P) \leftrightarrow \top$;
4. *if σ is a non contradictory conjunction of literals and $\phi_1, \ldots, \phi_n$ are formulas, then*

$$\tilde{\exists}P(\sigma \wedge Cover(\phi_1, \ldots, \phi_n)) \leftrightarrow \tilde{\exists}P\sigma \wedge Cover(\tilde{\exists}P\phi_1, \ldots, \tilde{\exists}P\phi_n));$$

5. *if $\mu X.\phi$ is disjunctive, then $\tilde{\exists}P\mu x.\phi \leftrightarrow \mu x.\tilde{\exists}P\phi$;*
6. *if $\nu X.\phi$ is disjunctive, then $\tilde{\exists}P\nu x.\phi \leftrightarrow \nu x.\tilde{\exists}P\phi$.*

To prove that the above system is complete, it is enough to show that any $\tilde{\mu}$-formula ϕ is provably equivalent under this system to a μ-formula. This can be achieved by induction on the structural complexity of the formula, the only interesting case being $\phi = \tilde{\exists}P\psi$.

By induction, we suppose that ψ is provably equivalent to a μ-formula ψ^-, which is in turn provably equivalent to a disjuntive formula ψ_d^-. By induction on the structure of the disjunctive formula ψ_d^- we prove using the axioms and the rules above that $\tilde{\exists}P\psi_d^- \leftrightarrow \psi_d^-(P/\top, \neg P/\top)$. Hence $\tilde{\exists}P\psi$ is provably equivalent to the μ-formula $\psi_d^-(P/\top, \neg P/\top)$.

5 Conclusions and Further Work

In this paper we gave an axiomatixation of the μ-calculus with explicit interpolants, allowing in this way the possibility of computing with a logic as powerful as the μ-calculus in which we have a way to denote uniform interpolants. A related question is the axiomatization of BQL ([3]) which is defined as the bisimulation quantifier closure of Propositional Dynamic Logic PDL. The logic BQL is semantically equivalent to the μ-calculus, but has the advantage of replacing all fixed point operators (which are difficult to read, expecially when nesting of two or more operators occur) by the Kleene star and the existential bisimulation quantifier. To obtain a complete axiom system for BQL one could simply translate the complete system $\tilde{\mu}$ into BQL, but the resulting system is not a natural one for BQL. The question of finding a set of natural principles governing the behaviour of the bisimulation quantifier w.r.t. PDL-operators is left open.

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