A BiOT Account of Gricean Reasoning

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1 Introduction

- Tea or Coffee?

- Yes

- Huh? No, I mean, would you like tea or coffee?

- Yes

- Christ! Do you want tea?

- Yes

- So... no coffee?

- I'd prefer coffee.

- But... alright, so coffee it is. Milk or sugar?

- Yes

- ..., milk?

- no

- One lump or two?

- Yes

- No, I mean..., how many do you want?

- Three

- Would you like some biscuits or choc... never mind...

One could wonder if it's such a good idea to use logic to study natural language. Actually one could wonder if it is such a good idea to study logic at all. But, let's suppose it is.

Many people with a more practical disposition in life consider it a lost cause to say anything logical about language because it's too intuitive. They're probably right. But, let's suppose they aren't.

The art would be to capture the intuitive part of language in logic. Clearly language doesn't quite behave as strictly logical as, well, logic does. If I want coffee, logically the sentence "I want tea or coffee" is true. Somehow

"Yes" doesn't quite do it for us as a response to the question: "Would you like tea or coffee?".

There is a difference between the semantical content of an utterance and the conversational intentions of the speaker of an utterance. When I ask you if you want tea or coffee, I give you three options to choose from namely tea, coffee or nothing. I expect you to choose precisely one of those options. This is the conversational intention but it's not part of the semantical content. In the field of linguistics these conversational intentions are commonly referred to as conversational implicatures or just *implicatures*.

There are many different approaches to give a formal account for implicatures. Paul Grice has been most influential in this area (if not only for coining the term implicatures).[1] Much contemporary research is based on the findings of Grice.

Grice suggested that a speaker obeys certain conversational conventions. For instance, someone wants to say something he believes is true and relevant. The speaker has an intended meaning and by means of these conversational conventions this intended meaning is mapped to an utterance. The hearer of the utterance can infer the intended meaning by taking the speakers perspective into account.

Grice proposed a list of four conversational conventions (maxims). In [1] Maria Aloni proposes to formalize these maxims in bidirectional optimality theory (BiOT). With this paper I try to investigate how well Aloni's approach works and how to improve Aloni's findings.

This paper contains nine sections. The first three set out the field. Discussing some common ways of dealing with implicatures. Section four and five build up to Aloni's approach. In section six the problems with Aloni's approach are discussed. This leads to some suggestions to improve this approach (worked out in section seven). The last two chapter are reserved for the conclusion and discussion of the suggested improvements.

2 What are implicatures?

First things first. What is the problem we're facing?

It is not hard to come up with a sentence where the meaning of the sentence and the conversational intentions are not the same. Consider the utterance (1).

(1) John is married or in love.

[1] In [12] for example

This sentence is true if John is married but not in love and vice versa. Similarly, if John is both married and in love the sentence is logically true. This is, however, not how we tend to interpret such a sentence. There are a number of conclusions we draw when we hear such a sentence that are not explained by the meaning. For instance, the speaker holds it possible John is married and he also holds it possible that John is in love. He doesn't know which one of the two is the case. If the speaker would had that John is married, he would simply have stated:

(2) John is married

If the speaker had known that John is in love, he would have said:

(3) John is in love

Another conclusion we draw is that the speaker doesn't know that John is *both* married *and* in love. Even if this were a realistic situation, the speaker would have said:

(4) John is married and in love

So, after hearing a sentence like "John is married or in love" we draw a number of conclusions on what the speaker believes. These intentions are typically not logically implied by the meaning of the sentence. However, they are *implicated* by the speaker[2].

Grice One could dream up many kind of implicatures. And of course, finally we would like to have an analysis that can account for all imaginable implicatures, as we would like to live in a world of peace and prosperity. This paper will be mainly concerned with disjunctive sentences (sentences containing the word 'or') and its implicatures.

As said before Grice offered a general way of reasoning about such sentences that accounts for implicatures. Before studying Grice's approach further, let's have a look at the kind of sentences we're considering.

[2]The implicatures of an utterance are generally not part of the meaning. But, as Kent Bach (in [**?**]) notes, there are some exceptions. Consider example (5).

(5) Q Nobody voted Berlusconi I hope?

A Well, Alessia voted for Berlusconi.

⇒ Someone voted for Berlusconi.

The implicature that someone voted for Berlusconi is actually implied by the sentence "Allessia voted for Berlusconi".

Disjunctive sentences have been thoroughly studied. Most research seems to have reached some consensus on what implicatures of a simple disjunction should be. The examples of inferences here are taken from [4]. Though much related research uses very similar examples and inferences ([**?**], [**?**], ...). Consider the sentence "John is married or in love".

(6) The implicatures for "John is married or in love":

(a) Basic meaning:
Either John is married or in love or both.

(b) Ignorance inferences:
The speaker does not know that John is married.
The speaker does not know that John is in love.

(c) Scalar implicatures:
John is not both married and in love.

(d) Further consequences:
The speaker thinks it's possible that John is married
The speaker thinks it's possible that John is in love

I adopted the terminology (*ignorance inferences* and *scalar implicatures*) from Fox. Not all research uses the same terminology. Sauerland refers to ignorance inferences as *primary implicatures* and scalar implicatures as *secondary implicatures*. Gazdar and Schulz & van Rooij use the term *clausal implicatures* rather than ignorance inferences.

2.1 Classic Gricean reasoning

These implicatures arise from the inference of the 'direct' meaning of a sentence and general communication principles. These principles as formulated by Grice confine the way we communicate information.

Quantity Be as informative as needed, but not more than that

Quality Be correct

Manner Be brief and orderly

Relation Be relevant

The implicatures as described in (6) are generally taken to be the consequence of the quantity principle[3]. Suppose the speaker utters a disjunctive

[3]Sauerland, Fox, van Rooij

sentence:

$$A \vee B$$

Does the speaker know that A? No, A is stronger then $A \vee B$. By the quantity principle the speaker should have said A. Similarly the speaker doesn't know B. So we derive:

$$\neg\Box A \wedge \neg\Box B$$

The same goes for $A \wedge B$. If the speaker would have known $A \wedge B$ he would have said so. He didn't, so:

$$\neg\Box(A \wedge B)$$

Let's assume the speaker to be competent on the issue. When he doesn't know something it is because it isn't true. This requires some consideration. By saying $A \vee B$ the speaker already indicates that there are things he doesn't know. Assuming the speaker has full knowledge of the world would lead to contradictions. If we assume that $\neg\Box A$ implies $\neg A$ and $\neg\Box B$ implies $\neg B$, we would derive $\neg(A \vee B)$. This is inconsistent with the initial sentence $A \vee B$. We can however consistently derive

$$\neg(A \wedge B)$$

With this last assumption (competence) we have to be careful. We suppose that a speaker is well informed, but only if this doesn't give rise to contradictions. Not all research agrees on how to deal with this.

2.2 Alternatives

Until now we only considered the alternatives $A \wedge B$, A and B to $A \vee B$. But we can not just take any ϕ and compare it to $A \vee B$. Consider $(A \vee B) \wedge \neg(A \wedge B)$. This is stronger then $A \vee B$. So by quantity we get $\neg\Box[(A \vee B) \wedge \neg(A \wedge B)]$. Together with $\Box\neg(A \wedge B)$ this yields $\neg\Box(A \vee B)$. Freely applying the quantity principle on any sentence leads to contradictions. Sauerland refers to this as the symmetry problem.

A common way to deal with this is by defining a set of alternatives for a sentence. The alternatives for ϕ (generally written as $\mathrm{Alt}(\phi)$) are the sentences that could have been uttered in stead of ϕ. The example above will work well if we take $\mathrm{Alt}(A \vee B) = \{A, B, A \wedge B\}$. To define this set of alternatives is not straightforward, many different approaches can be found in the literature.

2.3 Applications

Most theoretical research based on Grice's maxim's goes along these lines. They all use the maxim of quantity, quality and relevance. (With the example above we implicitly assumed the speaker to say something correct and relevant.) Manner is generally ignored. In [1] Aloni proposes a system that incorporates this Manner principle as well. This proposal forms the foundation of the research presented in this paper. But before that we take a look at some alternative approaches.

3 Some Gricean Approaches

There are many solutions to the problem of implicatures. To set out the field, I'll briefly discuss two solutions. They're far enough apart to at least give an impression of the field that makes up the context of my research. In the following I'll discuss Fox's syntactic approach to the problem. Then I'll discuss a more 'classical' solution by Schulz & van Rooij.

3.1 Fox

Normally to find scalar implicatures one first derives ignorance inferences. Those ignorance inferences together with the assumption that the speaker is competent leads to the scalar implicatures.

But, this kind of analysis causes problems with the free-choice inferences. This is the main reason for Fox to dismiss any such analysis. To show how such an analysis can go wrong consider Sauerlands definition of alternatives[4]. He suggests a system that generates the alternatives

$$\{A, B, A \wedge B\}$$

for $A \vee B$. So for $A \vee B$ this generates the right implicatures. But it doesn't work for free choice. I'll take $\Diamond_D$ and $\Box_D$ as the deontic modals. For $\Diamond_D(A \vee B)$ it gives the alternatives

$$\Diamond_D A, \Diamond_D B, \Diamond_D(A \wedge B)$$

Quantity yields the ignorance inferences $\neg\Box\Diamond_D A$ and $\neg\Box\Diamond_D B$. This is unwanted. It contradicts the standard free choice implicature $\Box(\Diamond_D A \wedge \Diamond_D B)$.

Fox observes that scalar impicatures can be derived in a syntactical way. The implicatures arise when we reinterpret the sentence as if it contained

[4] In [**?**], Sauerland (2004)

the word ‘only’. When we say “John send a love letter to Mary or Sue” we’d interpret the sentence as “John send a love letter to *only* Mary or Sue”.

It should be clear that for a simple disjunct this gives the right reading. This turns the whole reasoning around. The neo-gricean approach first determines the ignorance inference and derives the scalar implicatures from there. Fox turns this around. The ‘only’ operator gives the scalar implicatures, and from this strong reading (basic meaning + scalar implicatures) we derive the ignorance inferences. $A \vee B$ will be read as only$(A \vee B)$. This gives the exclusive reading $A \triangledown B$. When we apply the quantity maxim to the strong reading $A \triangledown B$ we find the right ignorance inferences.

3.1.1 The syntactic solution

Fox assumes that there is a covert exhaust operator. Intuitively, when we apply this operator to a sentence S (Exh(S)) it gives something in the likes of “S and nothing else(/stronger)”

Trying to avoid some obstacles on the way, the final definition that Fox comes up with is:

$$[[Exh]](A_{st,t})(p_{st})(w) \Leftrightarrow p(w) \ \& \ \forall q \in \text{I-E}(p, A) \rightarrow \neg q(w)$$

That is, the exhaustive interpretation of a sentence p given a set of alternatives A is the set of worlds w where:

- (1) p is true
- And (2) if q is innocently excludable given A (can be excluded without deriving any contradictions) then q doesn’t hold in w.

Innocently excludable (I-E) is defined as:

$$\text{I-E}(p, A) = \bigcap\{A' \subseteq A | A' \text{ is a maximal set in } A \text{ such that } A'^{\neg} \cup \{p\} \text{ is consistent}\}$$

$$(\text{with } S^{\neg} = \{\neg x | x \in S\})$$

This is a rather convoluted definition. I’ll use an example to illustrate the idea. Consider the sentence $X \vee Y$ then $A = \{X, Y, X \wedge Y\}$. Then $\{X, X \wedge Y\}$ is a maximal set such that $\{X \vee Y, \neg X, \neg(X \wedge Y)\}$ is consistent. The same holds for $\{Y, X \wedge Y\}$. The set $\{X \vee Y, \neg Y, \neg X, \neg(X \wedge Y)\}$ is inconsistent. So the maximal sets that are consistent are $\{X, X \wedge Y\}$ and $\{Y, X \wedge Y\}$. The intersection gives I-E$(X \vee Y, A) = \{X \wedge Y\}$.

Applying this exh operator gives a stronger reading of the sentence. Sometimes after applying the exh operator we don’t get a strong enough reading. In these cases the exh should be recursively reapplied till it gives a strong enough reading. We'll return to this issue below.

The set of alternatives is defined syntactically in terms of horn clauses. Horn clauses are sets of similar operators. In this case: $\{\Diamond, \Box\}$ and $\{\wedge, \vee, L, R\}$. The alternatives of ϕ ($Alt(\phi)$) can be generated by replacing one or more operators in ϕ with an operator from the same horn clause. Note that L and R are not actual operators. They're just a means to get rid of one of the disjuncts/conjuncts in a disjunction/conjunction ($\phi L \psi = \phi$ and $\phi R \psi = \psi$).

Simple disjunction One application of exh operator directly gives the right results for a simple disjunction. Take $A \vee B$. The set of alternatives is $\{A, B, A \wedge B\}$. As shown before, the only innocently excludable alternative is $A \wedge B$. So we get the strong reading $(A \vee B) \wedge \neg(A \wedge B)$.

After establishing the strong reading application of the quantity maxim should give the right ignorance inferences. In this case A and B are both stronger then the strong reading $(A \vee B) \wedge \neg(A \wedge B)$. The speaker didn't say them, so he doesn't know that they are true. This gives the ignorance inferences $\neg\Box A$ and $\neg\Box B$.

Free choice For the free choice effect a single application of exh doesn't suffice. Take $\Diamond_D(A \vee B)$. The alternatives for this sentence are $\{\Diamond_D B, \Diamond_D A, \Diamond_D(A \wedge B)\}$. The only innocently excludable alternative is $\Diamond_D(A \wedge B)$. So, the strong reading after one pass of exh is $\Diamond_D(A \vee B) \wedge \neg\Diamond_D(A \wedge B)$.

For a second application of exh we have to determine the strong reading for all the alternatives in Alt (the alternatives of $\Diamond_D(A \vee B)$):

1. $\text{Exh}(\text{Alt})(\Diamond_D(A \vee B)) = \Diamond_D(A \vee B) \wedge \neg\Diamond_D(A \wedge B)$
2. $\text{Exh}(\text{Alt})(\Diamond_D A) = \Diamond_D A \wedge \neg\Diamond_D B$
3. $\text{Exh}(\text{Alt})(\Diamond_D B) = \Diamond_D B \wedge \neg\Diamond_D A$
4. $\text{Exh}(\text{Alt})(\Diamond_D(A \wedge B)) = \Diamond_D(A \wedge B)$

The innocently excludable (strong) alternatives are 2 and 3. So the second pass of exh gives the strong reading $\Diamond_D(A \vee B) \wedge \neg\Diamond_D(A \wedge B) \wedge \neg(\Diamond_D A \wedge \neg\Diamond_D B) \wedge \neg(\Diamond_D B \wedge \neg\Diamond_D A)$. This, together with the original sentence, gives the free choice reading: $\Diamond_D A \wedge \Diamond_D B$ $(\wedge\neg\Diamond_D(A \wedge B))$.

3.2 Problems with Fox

So, according to Fox, scalar implicatures are the result of exhaustification and primary implicatures are derived from the strong meaning. This solutions works to explain free choice implicatures. However, this syntactic solution has its vices.

Ad hoc First of all I'd like to point out a minor weakness in Fox' argument for why his approach is better then many other approaches. Fox suggests that his approach is less ad-hoc then many others. Fox starts out by pointing out that neo-gricean solutions rely on a rather "arbitrary" set alternatives to get ignorance inferences. These sets of alternatives are indeed ad-hoc and a solution that can avoid them would be very welcome. And, indeed, given the scalar implicatures of a sentence, Fox doesn't need such a set of alternatives to derive ignorance inferences. However, to compute the scalar implicatures he does need alternatives. So indirectly they're also used to derive the ignorance inferences.

This is not to say that it is bad thing to use alternatives. But I don't see how this solution can be considered less ad-hoc the any other solution. Above that Fox defines this rather obscure 'innocently excludable', to make sure we don't get any contradictory implicatures. If anything that makes it even more ad-hoc then the classical gricean approach.

Pragmatic A more fundamental problem is the way the exh operator is applied. To get the right implicatures, the exh operator should sometimes applied once ($A \vee B$), sometimes twice ($\Diamond(A \vee B)$, and sometimes not at all.

To see that you don't always have an exhaustive interpretation, consider example (8).

(7) Who voted (of Allessia, Maria and Jean-Louis?)
Not Allessia and Maria

It is not directly clear what the implicatures of such a sentence should be, but it doesn't necessarily get an exhaustive interpretation. The sentence $\neg(A \wedge B)$ (or equivalently $\neg A \vee \neg B$) generates the alternatives:

$$\text{Alt}(\neg A \vee \neg B) = \{\neg A, \neg B, \neg A \wedge \neg B\}$$

The only innocently excludable alternative is $\neg A \wedge \neg B$. So we derive $\neg(\neg A \wedge \neg B)$ or

$$A \vee B$$

But intuitively the utterance "Not Allessia and Maria" does not give rise to the exhaustive interpretation that "Allessia or Maria" voted. Often Gricean reasoning gives an exhaustive interpretation of a sentence. But it doesn't always need to.

Fox accounts for this by saying that we pragmatically decide whether or not to apply the exh operator. We first interpret the sentence without any exh operator. If that gives a pragmatically implausible reading we reevaluate the sentence with the application of exh. If that still gives an implausible reading we apply exh again, etc...

But what does this intuitively mean? The exh operator is based on the Gricean maxims. One application of exh is saying as much as 'interpret the utterance as if the speaker obeys Grice's maxims'.

But then, as a consequence by saying that exh should by pragmatically applied as needed, means that the speaker sometime does obey Grice's maxims and sometimes doesn't *depending on the particular sentence*.

Why would a speaker decide to obey Grice when he says $A \vee B$, but ignore Grice when he says $\neg(A \wedge B)$?

And, even worse, what does the double application of exh mean for the sentence $\Diamond_D(A \vee B)$? The speaker obeys the Gricean maxims doubly?

Grice's maxims are intended as general rules of conduct in communication. If they're used to explain certain implicatures they can not be discarded of when they're in the way.

Indeed, we don't *always* get an exhaustive interpretation. In Fox' approach this is explained by supposing that sometimes Gricean reasoning is used and sometimes it isn't. I think that defies the point of Grice's maxim's. It would be more systematic to suppose the Gricean maxims and, explain how they sometimes give rise to an exhaustive interpretation and sometimes don't.

3.3 Schulz & van Rooij

Schulz & van Rooij offer an alternative approach in [**?**]. They propose to unify the competence and quantity principle in one operator they call *eps*. Their *eps* operator can be seen as an exhaustivity operator as well, but differs much in dynamic from Fox' exh.

Alternatives according to Schulz & van Rooij are alternative answers to a background question. This avoids the need for the syntactic hustle with alternatives. Intuitively, if a sentence ϕ is the response to a question Q any sentence that could have been a response to the same question is an alternative for ϕ (relative to question Q).

The *eps* operator selects the worlds that best explain a utterance ϕ. First it selects those worlds for which sentence ϕ respects the quantity maxim. From the worlds left those are selected that maximize the knowledge of the speaker. Formally:

$$eps^C(A,P) =_{def} \{w \in grice^C(A,P) | \forall w' \in grice^C(A,P) : w \not\sqsubset_{A,P} w'\}$$

Where $eps^C(A,P)$ is the set of worlds that best explain sentence A. A is the sentence of interest, $C = \langle W, R \rangle$ is the context in which this sentence is uttered and P is the background question. The quantity part is taken care of by the $grice^C$ operator. $w \in grice^C(A,P)$ means that w is a world where A is true and A obeys quantity in that world. And $\sqsubset$ is the competence ordering. If $w \sqsubset_{A,P} w'$ then the speaker knows more about A relative to P in w then in w'. First let's look at the definition of *grice*.

$$grice^C(A,P) =_{def} \{w \in [\mathbf{K}A]^C | \forall w' \in [\mathbf{K}A]^C : w \preceq_{P,A} w'\}$$

$$[\mathbf{K}A]^C =_{def} \{w | \forall v \in R(w) : v \models A\}$$

$$w_1 \preceq_{P,A} w_2 \text{ iff}_{def} \ \forall v_2 \in R(w_2) \exists v_1 \in R(w_1) : v_1 \leq_{P,A} v_2$$

where

$$v_1 \leq_{P,A} v_2 \text{ iff}_{def} \ [P](v_1) \subseteq [P](v_2) \text{ and } [\Phi](v_1) = [\Phi](v_2)$$

with Φ is any non-logic vocabulary [sic][5](except for P of course). To break it down: $v_1 \leq_{P,A} v_2$ holds if less of P is true, for instance if $[P](v_1) = \{a,b\}$ and $[P](v_2) = \{a,b,c\}$ Then $w_1 \preceq_{P,A} w_2$ if as little as possible knowledge about P is assumed. For instance if $R(w_1) = \{v_1, v_2\}$ and $R(w_2) = \{v_1\}$. The grice operator selects those worlds where the speaker know A, and doesn't know anything else that's relevant.

The second part of the *eps* operator is to maximize competence. We assume the speaker to be as well informed as consistent with what he said. This is done with the $\sqsubset$ ordering

$$w_1 \sqsubset_{A,P} w_2 \text{ iff}_{def} \ \forall v_2 \in R(w_2) \exists v_1 \in R(w_1) : v_1 \equiv_{P,A} v_2$$

With $v_1 \equiv_{P,A} v_2$ iff $v_1 \leq_{P,A} v_2$ & $v_2 \leq_{P,A} v_1$. This basically prefers worlds which have less security. If we take $R(w_1) = \{v_1\}$ and $R(w_2) = \{v_1, v_2\}$ then $w_1 \sqsubset_{A,P} w_2$.

[5]Schulz & van Rooij do not specify this non-logic vocabulary. Is suppose it means only predicates (nothing containing any logical operators like $\neg, \wedge, \ldots$)

As we can see in the formal definition of *eps*, it first selects those models that account for the ignorance inferences with *grice*. From the models left the most competent one is selected. Fox used the innocently excludable principle to ensure consistency of the implicatures. Schulz & van Rooij don't have to worry about consistency, models are consistent by definition.

3.4 Problems Schulz & van Rooij

With this approach the problem of using exhaustification becomes even more apparent then with Fox. Consider the sentence 8.

(8) I know that John voted

The exhaustive interpretation doesn't give the right predictions. There is some exhaustification going on. But not in the way Schulz & van Rooij intended it. To put it in terms of 'only', intuitively, this sentence should read:

(9) I *only* know that John voted

However, the way Schulz & van Rooij define their exhaustification it reads:

(10) I know that *only* John voted

It is not the factual information but the speakers knowledge that is exhausted in 8. This is more formally worked out in appendix *A*.

Schulz & van Rooij run into to trouble explaining modal sentences. Now, for them it doesn't matter. Schulz & van Rooij were only out to show how exhaustification could explain scalar implicatures. But precisely this is the problem. They devised a formalism that is specifically tailored to derive scalar implicatures, they don't give a general formalism of Grice's Maxims.

In [2] Benjamin Spector argues that exhaustification (at least as defined by Schulz & van Rooij) can not be used as the basis of scalar implicatures. Consider the example given by Spector:

(11) Q Among the chemists and the philosophers, who came?
A Less than two of the chemists

Here exhaustivity would predict the wrong implicature:

(12) Exactly one chemist and all the philosophers came.

Grice made four assumptions about conversational behavior in general. The objective is to formalize these four assumptions and show how they explain implicatures. By using the exhaustivity operator both Fox and Schulz & van Rooij covertly assert two other assumptions:

1. Grice's maxims always give rise to an exhaustive interpretation
2. All of the further implicatures can be explained in terms of this exhaustive interpretation

It seems to me that we should limit our assumptions to Grice's maxims and the way they're formalized, and show how a proper formalization of these maxims account for both scalar implicatures and exhaustivity.

3.5 Comparison Schulz & van Rooij and Fox

Before studying the BiOT approach in the next sections it might be useful to take a look at how the above mentioned approaches relate. I started out be showing how the classical gricean reasoning worked. Here I'll show how the two approaches of Fox and Schulz & van Rooij diverge from this classical approach.

There are three main points where Fox and Schulz & van Rooij differ from the classical approach. First of all there is the way competence is dealt with. The classical approach assumes competence of the speaker if it doesn't contradict any of the earlier made ignorance inferences. Fox and Schulz & van Rooij, assume competence where it doesn't contradict any other competence assumption that could be made.

A second point of disagreement are the way alternatives are defined. Fox uses the syntactical definition of alternatives (using horn clauses) from the classical approach. Schulz & van Rooij consider an utterance to be the answer to a (covert) question. The alternatives are defined as alternative answers to these questions.

Thirdly the three approaches compute the implicatures differently. Fox first determines the scalar implicatures of a sentence. The basic meaning plus the scalar implicatures make up the strong meaning of the sentence. The ignorance inferences are derived from this stronger meaning. Schulz & van Rooij use the more classical approach. They first determine ignorance inferences and derive scalar implicatures from there. Figure 1 shows the main differences.

Both Fox and Schulz & van Rooij solve the problems they set out for themselves. But both solution are very specifically tailored and tweaked

	Classical approach	Fox	Schulz & Van Rooij
Competence	Only if consistent with ignorance inferences	Only if consistent with other competence assumptions	
Alternatives	Syntactical definition. Using Horn clauses		Depending on issue
Computation order	Ignorance inferences → scalar implicatures	Scalar implicatures → ignorance inferences	Ignorance inferences → scalar implicatures

Figure 1: ...

around those problems. They need quite some ad-hoc moves to get the right implicatures. Both approaches take exhaustivity as the basis. Which, as I argued before, is an additional assumption. And neither definition of the exhaustivity operator is very intuitive. This paper aims to develop a system that does not assume exhaustivity. The aim is to give an intuitive as possible formalization of Grice's maxims and show how exhaustivity and scalar implicatures follow from these maxim's.

Granted, we'll still have to make some assumptions on how to formalize the maxims, but they're isolated and relatively easy to study independently (this will become clear later on).

4 Grice and reasoning with constraints

In the previous sections I introduced the problem of implicatures and we've seen some ways of dealing with them. We're heading for a very different approach. Building upon the work of Aloni in [1] I'll be using bidirectional optimality theory (BiOT) to formalize the gricean constraints. For this we'll have to sidestep a bit and study BiOT itself first. The next chapter we'll be mainly concerned with a more formal studies of BiOT. In the chapter to follow (chapter 6) we'll see how BiOT can be used to implement Grice's maxims. Before going into the technical details of BiOT, I'll first try to explain the kind of reasoning we'd like to formalize at an intuitive level.

Aloni's approach doesn't rely on a formal definition of exhaustification. Aloni proposed to take Grice's maxims at face value. No ad-hoc moves.

No rummaging about with alternatives, but rather a direct formalization of reasoning with Grice's maxims.

To see how this could work consider the following example. Suppose the hearer (mr. H) asked the question:

(13) Who voted for Berlusconi (of Alessia and Maria)?

And the speaker (mr. S) is well informed on this matter. He knows that *only* Alessia voted and he wants to communicate this. He'd have to look for a good sentence to communicate this. Obviously he would't say:

(14) Alessia and Maria.

It's just not true. So he could choose between

(15) Alessia and not Maria.

and simply

(16) Alessia.

Clearly 15 is more precise then 16. So by the Quantity maxim of Grice mr. S should opt for 15. On the other hand there is the Manner maxim that dictates that mr. S should be brief and orderly. According to the Manner maxim mr. S should go for option 16.

So the Manner and Quantity principles do not agree on which utterance is best. We have to choose which one is more important. Normally most people would take option 16 so apparently Manner wins this battle. So, finally, mr. S chooses to say

(17) Alessia.

Now it is up to mr. H to determine what mr. S could have meant by saying "Alessia". Clearly it is the case that Alessia voted for Berlusconi. But what about Maria? Mr. H has two options to consider.

(18)
1. Alessia and Maria voted Berlusconi.
2. Alessia voted Berlusconi and Maria didn't.

So now mr. H takes the speakers perspective. What would I have said in both cases?

Suppose that Alessia and Maria both voted for Berlusconi then he wouldn't have said Alessia voted. Rather he would have said "Alessia and Maria".

Mr. H realizes this so he considers it unlikely that Alessia and Maria both voted.

Suppose that Alessia voted Berlusconi but Maria didn't. Then using the same reasoning as mr. S did, mr. H comes to the conclusion that he would have said "Alessia". Since mr. S indeed said "Alessia", it is probably correct to conclude that Alessia voted but Maria didn't.

There are two elements in this scenario that led to successful communication.

- Reasoning with Grice maxims.
- Considering the other's perspective.

To successfully formalize Grice we need to consider both aspects. Bidirectional optimality (BiOT) does precisely this. The rest of this section will cover BiOT. The next section explains Aloni's approach.

4.1 Definition BiOT

Given a utterance f we'd like to have a way to find a meaning m that best explains the utterance (respecting Grice's maxims). Bidirectional optimality offers a way to do this.

Bidirectional optimality is a way to match an input with an optimal output (in our case the input is a sentence and the output is it's meaning). Generally speaking BiOT is a formal system to reason about constraints. There are constraints on the input and output. The more of those constraints are violated, the less good an output matches an input. In this paper we use logical sentences as input and models as output. For the constraints we take the gricean maxims. One (quite trivial) constraint for example is that a model should be correct for an output (the QUALITY constraint).

Not all constraints are equally important. They are ranked. For instance in Aloni's proposal QUALITY is more important than MANNER. It is more important to be correct than to be brief. So QUALITY will be a higher ranked constraint than MANNER (notation: QUALITY $\prec$ MANNER. In BiOT, higher ranked constraints take absolute precedence over lower ranked constraints. Take A,B and C constraints with ranking $A \prec B \prec C$. A violation of A is worse than one of B or C even if the violation of B or C is much stronger than the one of A.

For the remainder of the paper I'll stick to Jäger's ([5]) definition of bidirectional optimality. We use **GEN** to denote the possible input/output pairs. In our case, if f is a sentence in modal logic and m is a model then $\langle f, m\rangle \in \textbf{GEN}$.

Along the lines of Jäger we suppose that the constraints yield an ordering over **GEN**. Where $\langle f, m\rangle < \langle f', m'\rangle$ means that $\langle f, m\rangle$ is preferred over $\langle f', m'\rangle$. How this ordering is obtained precisely will be explained below. Given such an ordering the optimal pairs can be calculated. There are two different definitions of which pairs are optimal, strong optimality and weak optimality. In this paper we'll only be using weak optimality, but for completeness sake, I'll give both definitions. Strong optimality:

Definition $\langle f, m\rangle$ is strongly optimal *iff*

- there is no $\langle f', m\rangle$ such that $\langle f', m\rangle < \langle f, m\rangle$
- there is no $\langle f, m'\rangle$ such that $\langle f, m'\rangle < \langle f, m\rangle$

Weak optimality:

Definition $\langle f, m\rangle$ is weakly optimal *iff*

- there is no weakly optimal $\langle f', m\rangle$ such that $\langle f', m\rangle < \langle f, m\rangle$
- there is no weakly optimal $\langle f, m'\rangle$ such that $\langle f, m'\rangle < \langle f, m\rangle$

Weak optimality has an additional recursive part. In [5] Jäger shows that if $\langle \mathbf{GEN}, <\rangle$ is well founded this recursive definition is well defined (i.e. for every pair $\langle f, m\rangle \in \mathbf{GEN}$ it is defined if it is weakly optimal or not).

To see the difference between weak optimality and strong optimality consider figure 2. In this figure $\mathbf{GEN} = \{\langle 0,0\rangle, \langle 1,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle\}$. The

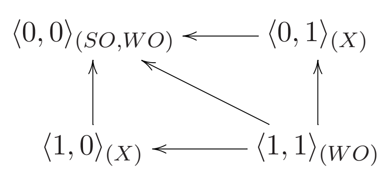


Figure 2: Example, optimality. SO: Strongly optimal, WO: Weakly optimal, X: Neither

arrows show the ordering over **GEN**. The pair $\langle 0,0\rangle$ is both weakly optimal and strongly optimal, simply because there isn't a smaller pair. The pairs $\langle 1,0\rangle$ and $\langle 0,1\rangle$ are neither weakly nor strongly optimal. They're blocked by $\langle 0,0\rangle$. Pair $\langle 1,1\rangle$ *isn't strongly optimal*. It is blocked by both $\langle 1,0\rangle$ and $\langle 0,1\rangle$. But since $\langle 1,0\rangle$ and $\langle 0,1\rangle$ aren't weakly optimal, $\langle 1,1\rangle$ *is weakly optimal*. (Note that $\langle 0,0\rangle$ doesn't block $\langle 1,1\rangle$ even though $\langle 0,0\rangle < \langle 1,1\rangle$; they differ in both arguments.)

One of the main challenges with optimality theory is to define the ordering on **GEN**. There are two kind of constraints, markedness constraints and faithfulness constraints. Markedness constraints are constraints on either only the input or the output. Faithfulness constraints are constraints on the relation between the input and output. For instance MANNER is a markedness constraint (a brief and orderly sentence is preferred over a more verbose sentence). This doesn't say anything about the relation between input and output (sentence and model). QUALITY, on the other hand is a faithfulness constraint (a sentence-model pair that is consistent is preferred over an inconsistent pair).

Take F to be the set of all possible inputs (sentences) and M the set of all possible outputs (models)[6]. Markedness constraints are orderings on either F or M. Faithfulness constraints are (in it's most general sense) orderings on $F \times M$.

The overall ordering on **GEN** is the lexical composition of all those orderings. So, if the ordering is determined by $A \succ B \succ \ldots \succ N$ then:

$$\begin{array}{lll} t < t' & \text{iff} & t <_A t' \\ & \vee & t \not<_A t' \wedge t \not>_A t' \wedge t <_B t \\ & \vee & \ldots \\ & \vee & t \not<_A t' \wedge t \not>_A t' \wedge t \not<_B t' \wedge t \not>_B t' \wedge \ldots \wedge t <_N t \end{array}$$

With the note that if two constraints A and B are equally ranked we get that:

$$\begin{array}{lll} t <_{A,B} t' & \text{iff} & t <_A t' \wedge t \leq_B t' \\ & \vee & t \leq_A t' \wedge t <_B t' \end{array}$$

4.2 Weak optimality: Markedness implies Markedness

For the rest of the paper we'll be using weak optimality. We'll see later on that weak optimality captures the dynamics of the implicatures of interest better. In weak optimality, a marked input is not entirely blocked for any output but will generally be optimal for a marked output.

Consider figure 3. In this example there are two markedness constraint, one for input f and one for output m. I use $\odot$ to indicate that a pair is

[6]The definition of both strong and weak optimality is symmetric. Considering a sentence input is taking the hearers perspective. But, due to it's symmetry we can just as easily turn it around and say that the model is the input (and thus taking the speakers perspective)

optimal and $\otimes$ to indicate non optimality. $f*$ indicates marked input and $m*$ indicates marked output. With strong optimality (figure 3(a)), only the pair $\langle f, m \rangle$ is optimal. With weak optimality (figure 3(b)) both $\langle f, m \rangle$ and $\langle f*, m* \rangle$ are optimal.

(a) Strong optimality (b) Weak optimality

Figure 3: Markedness implies markedness

Here we see that the marked input takes a marked output. In appendix B a more complete and formal study can be found of this.

4.3 Lexical indifference

The ranking of constraints can in general make a big difference for any BiOT analysis. In some cases it doesn't however. If we have two constraints $<_M$ and $<_F$ on M and F (I'll drop the subscripts if it's clear about which order we're dealing with), the respective ranking of the two constraints doesn't matter. This follows almost directly from the definitions.

$\langle f, m \rangle$ is w-opt iff

- there is no $\langle f', m \rangle <_t \langle f, m \rangle$ where $\langle f', m \rangle$ is w-opt.
- there is no $\langle f, m' \rangle <_t \langle f, m \rangle$ where $\langle f, m' \rangle$ is w-opt.

Depending on which constraint is higher ranked $<_t$ is defined as follows:

- $<_{t,lex1}$: $\langle f, m \rangle <_t \langle f', m' \rangle$ iff $f <_f f' \vee (f \not<_f f' \wedge f \not>_f f' \wedge m <_m m')$
- $<_{t,lex2}$: $\langle f, m \rangle <_t \langle f', m' \rangle$ iff $m <_m m' \vee (m \not<_m m' \wedge m \not>_m m' \wedge f <_f f')$

We see that:

Lemma 1. *$\langle f, m, \rangle$ is weakly optimal with respect to $<_{t,lex1}$ iff it is weakly optimal with respect to $<_{t,lex2}$*

Proof. Lexical indifference

$\langle f', m \rangle <_{t,lex1} \langle f, m \rangle$ iff (since $m = m$) $f' <_f f$ iff $\langle f', m \rangle <_{t,lex2} \langle f, m \rangle$.

$\langle f, m' \rangle <_{t,lex1} \langle f, m \rangle$ iff (since $f = f$) $m' <_m m$ iff $\langle f, m' \rangle <_{t,lex2} \langle f, m \rangle$

It directly follows that $\langle f, m, \rangle$ is weakly optimal with respect to $<_{t,lex1}$ iff it is weakly optimal with respect to $<_{t,lex2}$

In a weakly optimal system without faithfulness constraints we can rewrite the definition of weak optimality as: $\langle f, m \rangle$ is w-opt iff

- there is no $f' < f$ where $\langle f', m \rangle$ is w-opt.
- there is no $m' < m$ where $\langle f, m' \rangle$ is w-opt.

The order of these constraints doesn't make any difference. The relevance of this will become clearer later on.

5 Aloni's BiOT approach

In [1], Aloni uses BiOT to account for the implicatures we've seen. Let's take another look at example 16, repeated here:

Q Who voted for Berlusconi (of Alessia and Maria)?

A Alessia voted.

⇒ Maria didn't vote.

In the third section of this paper we've seen two approaches that use exhaustification to account for these kind of implicatures. Exhaustification essentially means that the answer "Alessia voted" should be interpreted as if it says "only Alessia voted". The main objection I discussed is that these approaches can be a bit ad hoc. Besides one can think of sentences that can not be explained in terms of exhaustification (like: "Not both Alessia and Maria voted.").

In this section I'll discuss Aloni's approach. As with the other approaches we assume a logical representation of the sentences we're taking into consideration. In this case we use a propositional modal logic. The meaning of the logical sentence is determined by a semantical and a pragmatical part. The semantics are not part of this research (nor of Aloni's) so I won't go into more detail than needed to study the pragmatical side.

5.1 Semantics

First of all we'd need to find a logic system to represent the sentences. For the examples we're considering modal proposition logic suffices. Aloni uses the update semantics as defined by F. Veltman in [10] with $\heartsuit$ as additional operator. The $\heartsuit$ is a modal operator that is used to indicate if someone cares about something. "Cares if ϕ is the case" is written as $\heartsuit?\phi$. This is needed to formalize the RELATION maxim.

The update semantics is defined over a set of worlds W and a valuation V a function from worlds and propositions to truth values. The contexts that are updated encode the knowledge and interests of the speaker. A context is a pair $\langle s, Q\rangle$. s is the state (a set of possible worlds that encodes the speakers knowledge) and Q is the issue (an equivalence relation over W). If $C = \langle W, W^2\rangle$ the speaker knows nothing and cares about nothing. If $C = \langle\{w\}, \{\langle x, x\rangle | x \in W\}\rangle$ the speaker knows everything and cares about everything.

Definition [Updates]

- $C[p] = C'$ iff $s_{C'} = \{W \in s_C | V(p)(w) = 1\}$ & $Q_{C'} = Q_C$
- $C[\neg\phi] = C'$ iff $s_{C'} = s_C \backslash s_{C[\phi]}$ & $Q_{C'} = Q_C$
- $C[\phi \wedge \psi] = C[\phi][\psi]$
- $C[\Box\phi] = \begin{cases} C \text{ if } C[\phi] = C \\ \langle\emptyset, Q_C\rangle \text{ otherwise} \end{cases}$
- $C[?\phi] = C'$ iff $s_{C'} = s_C$ & $Q_{C'} = \{(w, v) \in Q_C | \langle\{w\}, Q_C\rangle[\phi] = \langle\{w\}, Q_C\rangle \text{ iff } \langle\{v\}, Q_C\rangle[\phi] = \langle\{v\}, Q_C\rangle\}$
- $C[\heartsuit?\phi] = \begin{cases} C \text{ if } C[?\phi] = C \\ \langle\emptyset, Q_C\rangle \text{ otherwise} \end{cases}$

Other operators can be derived in the normal way. Entailment is defined in terms of updates as follows:

Definition [Entailment] $C \models \phi$ iff $C[\phi] = C$

To see how this works, consider (again) example 16. Let W be the set of worlds we take into consideration and $s \subseteq W$ the set of worlds that encode the speakers knowledge. We start out without any issue ($Q = W^2$). First the issue is determined by the question:

Q Who voted for Berlusconi (of Alessia and Maria)?

This question updates the context with $C = \langle s, W^2 \rangle[?A \wedge ?M]$ (A means "Alessia voted" and M means "Maria voted"). The first update $\langle s, W^2 \rangle[?A]$ divides the issue into two equivalence classes: those worlds where Alessia voted and those where she didn't. The second update $\langle s, W^2 \rangle[?M]$ further divides the issue into worlds where Maria voted and where she didn't. So Q_C contains four equivalence classes: The one where neither Alessia nor Maria voted, the one where only Alessia voted, the one where only Maria voted and the one where they both voted. For the issue we can derive:

$$C \models \heartsuit ?A \tag{19}$$
$$C \models \heartsuit ?M \tag{20}$$
$$C \models \heartsuit ?A \wedge M \tag{21}$$
$$C \models \heartsuit ?A \vee M \tag{22}$$

Now consider the second part of the dialogue, the answer:

A Alessia voted.

Let C' be the update of the context C with the answer A ($C' = C[A]$). The state s'_C thus (according to the definition) contains only those worlds ($w \in s$) where Alessia voted ($V(A)(w) = 1$).

We're interested in the speakers knowledge. I.e. what are the possible contexts C that can explain the utterance "Alessia voted". These are the contexts where:

$$C \models A$$

In other words (according to the definition) those contexts C for which the update with A doesn't change anything ($C[A] = C$). Thus (in this case) those contexts where s_C only contains worlds where Alessia voted.

One of the contexts that explains the utterance A could for instance be $\langle [w_{AM}, w_A], Q \rangle$, a context with a world where only Maria voted and one where both Maria and Alessia voted. For this context it doesn't hold that $\neg M$. However we want to be able to explain the implicature $\neg M$. Clearly this semantics alone isn't enough. This is where the pragmatics come in. With the help of BiOT we'll select those context that are not only correct but also explain the implicatures.

For the rest of the paper I'll adopt Aloni's shorthand $Q_{?\phi}$ to denote $W^2[?\phi]$ and $Q_{?\phi ?\psi}$ to denote $Q_{?\phi}[?\psi]$.

5.2 Pragmatics

So for the sentence A we're interested in finding those sentence-context pairs $\langle A, C\rangle$ where not only $C \models A$ but also $C \models \neg M$. The above mentioned semantics doesn't explain this exclusive reading. Using Grice's maxims, the pragmatics part is going to find an optimal context for a sentence (i.e. a model that *best* explains an utterance). To do so, every Gricean constraint needs to be formulated as an ordering.

Quantity Be as informative as needed, but not more than that

Quality Be correct

Manner Be brief and orderly

Relation Be relevant

Relation and quality are clearly to be encoded as faithfulness constraints. Both are taken to be binary constraint. Either a sentence is true (or known as true by the speaker) or not; and it is either relevant or not. Manner and Quantity are markedness constraints on the logical sentence. Aloni introduces an additional markedness constraint: The Minimal Model Principle. This last constraint specifies which contexts (models) are preferred. All the markedness constraints are gradual constraints.

With these constraints we want to find optimal models for a sentence. For the sentence-context pair $\langle \phi, C\rangle$ the faithfulness constraints are defined as binary constraints.

Definition [Faithfulness]

Quality (QUAL): $C \models \Box\phi$

Relation (REL): $C \models \heartsuit?\phi$

The markedness constraints over ϕ and C are gradient constraints.

Definition [Markedness]

Quantity (QUAN): $\phi <_Q \psi$ iff $\phi \models \psi$ and $\psi \not\models \phi$.

Manner (MAN): If $m(\phi)$ is the number of occurrences of modal operators and negations in ϕ then $\phi <_M \psi$ iff $m(\phi) <_M m(\psi)$.

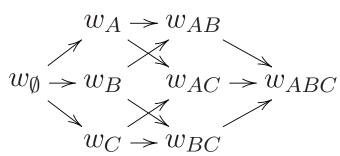
Minimal models (MM): The minimal model principle is defined as follows:

$$C \leq_{MM} C' \text{ iff}_{def}\ Q_C = Q_{C'}\ \&\ \forall v \in s_C \exists v' \in s_{C'} v \leq_C v'$$

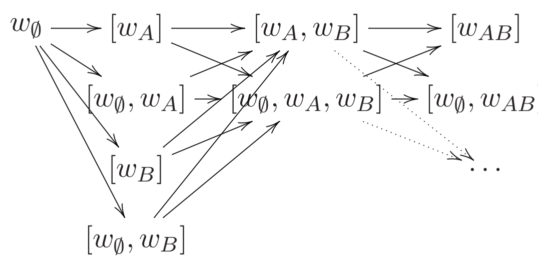
Context C is smaller then C' if they're about the same issue and all worlds in C are smaller then some world in C'. Of course we still need to define an ordering over worlds ($\leq_Q$):

$$v <_C v' \text{ iff}_{def} \ \forall p \text{ s.th. } p \text{ is atomic and } C \models \heartsuit ?p : v \models p \Rightarrow v' \models p$$

Figure 4 shows an example of what this ordering looks like. The arrows indicate the ordering.



(a) ordering worlds



(b) ordering contexts

Figure 4: Minimal model ordering for issue $Q_{?A?B}$

5.3 Ranking

The semantics and constraints are defined. The last thing that needs our attention is the ranking of the constraints. As above we'll use $\prec$ for de ordering of the constraints. Quality and Relation are taken to be the most important constraints. For practical reasons, that will become apparent later on, Aloni chooses Manner to be more important Quantity. This last part doesn't directly appeal to our intuitions and is only chosen as such for the very practical reason that otherwise we won't get the right results. As shown earlier in the section on BiOT, the ranking of the Minimal Models constraint doesn't matter as long as it doesn't precede the faithfulness constraints. So we get:

Definition [Ranking] $\text{QUAL}, \text{REL} \succ \text{MAN} \succ \text{QUAN} \succ \text{MM}$

(Which is equivalent to $\text{QUAL}, \text{REL} \succ \text{MAN} \succ \text{MM} \succ \text{QUAN}$ and to $\text{QUAL}, \text{REL} \succ \text{MM} \succ \text{MAN} \succ \text{QUAN}$).

This gives us enough information to determine if a pair $\langle \phi, C \rangle$ is weakly optimal.

Definition [Implicature]
We say that ψ is an implicature of ϕ (write $\phi \mid\approx \psi$) if for all C such that $\langle \phi, C\rangle$ is weakly optimal $C \models \psi$

For the rest of the paper I'll use the following formulations:

- Verbosity: $f <_M f'$ (manner ordering): f is less verbose then f'
- Strength : $f <_Q f'$ (quantity): f is stronger then f'
- Size : $m <_{MM} m'$ (minimal models): m is smaller then m'

To see how this works take another look at example 16. For now, suppose that we have only three sentences to consider, A, $A \wedge M$ and $A \wedge \neg M$; and two models $[w_A]$ and $[w_{AM}]$ (For this example we'll forget the issue). The sentences A and $A \wedge M$ are minimally verbose (there are no sentences that are less verbose). The sentence $A \wedge \neg M$ is more verbose (it contains a negation). The sentence A is the weakest sentence. The sentence $A \wedge M$ and $A \wedge \neg M$ are incomparable as to strength. The model $[w_A]$ is smaller than $[w_{AM}]$. According to the definition of weak optimality the pair

$$\langle A \wedge M, [w_{AM}]\rangle$$

is weakly optimal. There is no pair that is smaller ($\langle A, [w_{AM}]\rangle$ is dispreferred by strength and $\langle A \wedge M, [w_A]\rangle$ violates Quality). The pair

$$\langle A, [w_A]\rangle$$

is also weakly optimal. The pair $\langle A \wedge \neg M, [w_A]\rangle$ is dispreffered by Manner. Since $[w_A]$ is the only optimal model for A, we get that:

$$A \mid\approx \neg M$$

This example is quite trivial. In the next section I'll work out some examples to show that the principle also works if we take all possible sentences and all possible models into account.

6 Discussion of Aloni's approach

To give an idea of how the approach works I'll start out with working out some examples. This should make the definitions above a bit more insightful. The second part of this section discusses where this approach goes wrong.

6.1 Examples

For the remainder of the paper I'll drop the issue if it is not relevant. When I write $[w_0, w_1, \dots w_n]$ as a context instead of $\langle [w_0, w_1, \dots w_n], Q\rangle$, I suppose that we have some issue Q such that for all sentences ϕ we take into consideration we have $\langle [w_0, w_1, \dots w_n], Q\rangle \models \heartsuit ?\phi$. In other words if the issue is dropped, we suppose that the relation constraint is not violated.

6.1.1 Exhaustification

Consider example 23.

(23)

Q Who voted (of John and Mary)?

A John

$|\approx$ not Mary

The issue is $Q_{J?M?}$. The sentence of interest J. I'll use the common BiOT practice of tables to show which contexts are optimal for J. $\Rightarrow$ marks the optimal pairs. $*$ marks the violation of a constraint (relative to the other pairs in the table). To proof that $J, \langle Q_{J?M?}, [w_J]\rangle$ is optimal requires to

			QUAL	REL	MAN	QUAN	MM
$\Rightarrow$	J	$\langle Q_{J?M?}, [w_J]\rangle$				*	
	M	$\langle Q_{J?M?}, [w_J]\rangle$	*			*	
	$J \wedge M$	$\langle Q_{J?M?}, [w_J]\rangle$	*				
	$J \wedge \neg M$	$\langle Q_{J?M?}, [w_J]\rangle$			*		
	$J \vee M$	$\langle Q_{J?M?}, [w_J]\rangle$				**	
	Z	$\langle Q_{J?M?}, [w_J]\rangle$	*	*		*	
	J	$\langle Q_{J?M?}, [w_M]\rangle$	*			*	
$\Rightarrow$	M	$\langle Q_{J?M?}, [w_M]\rangle$				*	
	J	$\langle Q_{J?M?}, [w_J, w_\emptyset]\rangle$	*			*	
	J	$\langle Q_{J?M?}, [w_{JM}]\rangle$				*	*
$\Rightarrow$	$J \wedge M$	$\langle Q_{J?M?}, [w_{JM}]\rangle$					*
	J	$\langle Q_{J?M?}, [w_{JM}, w_J]\rangle$				*	*

Figure 5: Table for J. If row X contains more * then row Y for a certain constrain, it means that the corresponding pair X violates that constraint more then pair Y.

list all possible models and sentences. Table 5 shows how other candidate models for J violate more constraints. This obviously does not constitute a proper proof. It does, however, illustrate the principle.

We can see that $\langle J, \langle Q_{J?M?}, [w_J] \rangle \rangle$ is weakly optimal. The context $\langle Q_{J?M?}, [w_{JM}] \rangle$ is dispreferred on grounds of Minimal Models, and above that it is blocked by the stronger sentence $J \wedge M$. The context $[w_{JM}, w_J]$ ins't optimal for J either (by Minimal Models). So, the model $[w_J]$ is preferred.

But, this is not all we need to check. We're dealing with *bidirectional* optimality. So we also have to check that there is no sentence ϕ that is preferred over J by the model $[w_J]$. J is already minimal as to Manner, so ϕ can not contain any negations and modals. So even though $J \wedge \neg M$ is stronger then J, it isn't optimal for this model since it violates Manner. (Here we see why it is important that Manner is a higher ranked constraint then Quantity). $J \wedge M$ simply violates Quality. $\Box J$ also violates Manner. So we cannot find a sentence ϕ that is stronger then J, that doesn't violate Manner. So we get an exhaustive reading for J (namely $J \mid\approx \neg M$).

6.1.2 Exclusive reading of disjuncts

For the reading of disjuncts look at the example 24

(24)

Q Who voted (of John and Mary)?

A John or Mary

$\mid\approx$ not both John and Mary

Besides the exclusive reading we'd also like to conclude that the speaker holds it possible that John voted and that Mary voted. This example is worked out in table 6

The sentence $J \vee M$ is optimal for the context $\langle Q_{J?M?}, [w_J, w_M] \rangle$. This gives the implicatures we aimed for. Namely the exclusive reading: $J \vee M \mid\approx \neg(J \wedge M)$ and also $J \vee M \mid\approx \Diamond J \wedge \Diamond M$

6.1.3 Modals

We saw that exhaustivity can run into problems with modal sentences. Aloni's account does not explicitly apply exhaustivity. It does follow when required (as in the above examples), but the effect is not always there. Unfortunately we do run into problems with modals with Aloni's system.

			QUAL	REL	MAN	QUAN	MM
$\Rightarrow$	$J \vee M$	$\langle Q_{J?M?}, [w_J, w_M]\rangle$				**	
	J	$\langle Q_{J?M?}, [w_J, w_M]\rangle$	*			*	
	$J \vee M$	$\langle Q_{J?M?}, [w_J]\rangle$				**	
$\Rightarrow$	J	$\langle Q_{J?M?}, [w_J]\rangle$				*	
	$J \vee M$	$\langle Q_{J?M?}, [w_{JM}]\rangle$				**	*
$\Rightarrow$	$J \wedge M$	$\langle Q_{J?M?}, [w_{JM}]\rangle$					*
	$J \vee M$	$\langle Q_{J?M?}, [w_J, w_M, w_{JM}]\rangle$				**	*

Figure 6: Table for J

This, however is not due to the exhaustivity effect but rather a contingent problem.

Consider the example 25

(25)

Q Who voted (of John and Mary)?

A I know that John voted

$|\approx$ Mary might or might not have voted

We'd like to find that $\Box J$ is optimal for the model $\langle Q_{J?M?}, [w_{JM}, w_J]\rangle$. In this case $\Box J \;|\approx\; \Diamond M \wedge \Diamond\neg M$. We don't have to fear anything from J, this sentence already takes the smaller model $[w_J]$. But, there is another sentence that blocks $\langle Q_{J?M?}, [w_{JM}, w_J]\rangle$, the stronger sentence $J \wedge \Diamond M$ (with the current update semantics $\phi \Leftrightarrow \Box\phi$ holds). According to the definition of Manner $\Box J$ and $J \wedge \Diamond M$ are equally verbose. However if we'd say that $J \wedge \Diamond M$ is dispreferred by manner, we would get the right implicatures for $\Box J$. So the fact that $\Box J$ doesn't get the right implicatures is just due to the definition of Manner. This is not a fundamental problem with the BiOT solution. It can be solved by defining Manner differently.

6.2 What doesn't work

As a butterfly in China could theoretically cause a tornado in Europe, one small change somewhere in the BiOT system can have big overall consequences (though, I fear, the BiOT-effect will never become as popular an expression). In order to explain the implicatures of sentences we're interested in, we have to consider the optimal pairs for sentences we're not at all interested in, sentences we might not have any intuitions about to begin with. This makes any analysis in BiOT rather involved. For example, to know the implicatures of $\Box A$ we need to know what happens with $A \wedge \Diamond B$.

With Aloni's approach we run into some problems due to this effect. In this section I'll discuss these problems. In the sections to follow, I'll finally come up with a proposal that aims to fix these problems.

6.2.1 Problems with the definition of Manner

Since there is no straightforward definition of manner, and the definition that is given by Aloni, lacks an independent motivation, it is not surprising that this definition yields some problems. It is not my intention to come up with a better founded definition, but I'll propose one that I think should work slightly better. Here, I'll point out some problems we run into with the current definition. Manner is defined as "be brief and orderly". This intuitive notion of manner is implemented as the constraint "Modals and negations should be avoided". So $A <_m \Box A$ and $\neg C <_m \neg C \wedge \Box B$. Clearly this does not give a complete account for the intuitive interpretation of manner.

This of course needn't be a problem. We're only studying certain (relatively simple) implicatures. It might be that for those examples a definition purely in terms of modals suffices. The problem is that even for the implicatures we're considering here this definition runs in to trouble. Consider the following two examples:

(26) Q Who voted (Paul and Maria)?
A I know that Paul voted
$|\approx$ Paul and possibly Maria voted

(27) Q Who voted (Paul and Maria)?
A Paul and possibly Maria
$|\approx$ Paul and possibly Maria voted

As we have seen before, $\Box P$ should take $[w_{MP}, w_P]$ as optimal model but it doesn't. The phrase $P \wedge \Diamond M$ is preferred by quantity over $\Box P$ (they're equal as to manner). Since $P \wedge \Diamond M$ is w-opt for $[w_{MP}, w_P]$ it blocks this model for $\Box P$.

We can find many similar problems. For instance $\Box(A \vee B)$ should take $[w_A, w_B, w_{AB}]$, but this one is blocked by $(A \vee B) \wedge \Diamond(A \wedge B)$

6.2.2 Manner always precedes quantity

Not only the definition of Manner itself yields problems but also the ranking of the constraint. As it is defined by Aloni it always precedes Quantity. This is done to get the right results (for instance to ensure that the sentence $A \wedge \neg B$ is not preferred over A). This ranking, however, causes come problems as well.

Example 1 Let's take another look at example 26 and 27. Both expressions should yield the same optimal model, namely $[w_{PM}, w_P]$. Above we saw that $[w_{PM}, w_P]$ is blocked for $\Box P$ by $P \wedge \Diamond M$. This could trivially be solved by defining the manner constraint such that $\Box P$ is preferred by manner over $P \wedge \Diamond M$. But in this case $P \wedge \Diamond M$ will not get $[w_{PM}, w_P]$ as optimal model.

As said before, we're not particulary interested in the implicatures of the utterance "Paul and possibly Maria voted". So why not just accept that this utterance doesn't get the right model?

As also said before, the disadvantage of BiOT is that a small change in some part of the system can have great overall effects. If $P \wedge \Diamond M$ doesn't take the model $[w_{PM}, w_P]$ it will take a more marked model, like $[w_{PMJ}, w_{PM}, w_P]$. This could lead to undesired effects.

So we've got two contesting sentences ($\Box P$ and $P \wedge \Diamond P$) for the some model. However we'll define manner, either $\Box P$ will block $P \wedge \Diamond P$ or the other way around. Not all can be solved by finding the right definition for manner.

Let's look at another example that makes the same point:

Q Who voted (Paul and Maria)?

A Paul voted

$|\approx$ Paul and not Maria

Q Who voted (Paul and Maria)?

A Paul voted, Maria didn't

$|\approx$ Paul and not Maria

The model we would like both sentences (P and $P \wedge \neg M$) to take the model $[w_P]$ as optimal. Unfortunately P can only take $[w_P]$ if P is preferred by manner over $P \wedge \neg M$. But in this case $P \wedge \neg M$ cannot take $[w_P]$ as optimal model.

So, there are some problems that arise with the current definition of Manner that can not be solved by changing the definition. The ranking of Manner causes problems as well.

Example 2 Sometimes we don't mind being more verbose, if that allows us to be clearer to the speaker. With the previous example we saw that the sentences "I know that Paul voted" and "Paul and maybe Maria voted" get the same model. For this example both sentences mean the same. The second one leaves less to the interpretation of the hearer. At the cost of being more verbose we can be more precise (leaving less up to interpretation). This tendency is not accounted for by Aloni. Similar problems arise with $\neg C$ (representing, for instance, the sentence "Carl didn't vote"). Only here our intuitions on what the proper implicatures should be, are less clear. The example might be useful to illustrate the point nonetheless.

Figure 7 gives us a list of possible models for $\neg C$[7]. Which ones of these are optimal? For every model we will have to consider if there exists a sentence that is smaller than $\neg C$ but doesn't have a smaller optimal model itself.

1	$[w_{AB}\quad]$	$A \wedge B$
2	$[w_{AB}, w_0]$	...
3	$[w_{AB}, w_B]$	$B \wedge \Diamond A \wedge \neg C$
4	$[w_{AB}, w_B, w_0]$	...
5	$[w_{AB}, w_A]$	...
6	$[w_{AB}, w_A, w_0]$	...
7	$[w_{AB}, w_A, w_B]$	...
8	$[w_{AB}, w_A, w_B, w_0]$	...

Figure 7: Some possible models for $\neg C$

Some of the models in this example are not optimal for $\neg C$. For instance let's take model 1. $A \wedge B$ holds in this model. There is clearly not a smaller model where $A \wedge B$ holds and $A \wedge B$ is preferred by manner over $\neg C$. Hence, Model 1 is not weakly optimal for $\neg C$.

For the other proposed models it isn't as easy to find a proper formula. Let's take model 3. $B \wedge \Diamond A \wedge \neg C$ holds in model 3. And, is preferred by quantity over $\neg C$. But, on the other hand, depending on your definition of manner, it could very well be dispreferred by manner. Let's say that all

[7] I left out all models that don't contain w_{AB}. Those models are all smaller and it's easy to show that they're not optimal for $\neg C$

formula's for model 3 are dispreferred either by manner or quantity. Then model 3 is an optimal model for $\neg C$.

Which models are optimal for $\neg C$ heavily depends on the definition of Manner. To make the point, let's suppose we have such a definition of Manner that the models 2 to 8 are all optimal for $\neg C$.

When the speaker says $\neg C$ the hearer doesn't know which one of the models 2 to 8 represents the knowledge of the speaker. It might not matter. It might be that the speaker did not intend to communicate all his knowledge. The speaker might for instance know that A and B always vote together, if they vote. In this case the model that represents the knowledge of the speaker is model 2 ($[w_{AB}, W_{\emptyset}]$).

If the speaker would want to be more precise and communicate this, he'd have to say something like "Carl didn't vote and I'm not sure about Alessia and Bert but they always go together". This would be represented by something hideously verbose like $\neg C \wedge ((A \wedge B) \vee (\neg A \wedge \neg B))$. Unfortunately, this sentence cannot be optimal for model 2. The sentence $\neg C$ already blocks $\neg C \wedge ((A \wedge B) \vee (\neg A \wedge \neg B))$, since it's preferred by Manner. The only way for the speaker communicate model 2, is by saying $\neg C$.

So, having multiple optimal models for one sentence represents the uncertainty of the hearer (the hearer doesn't know which model it precisely is that the speaker wants to communicate). This in itself is not a problem. Often we don't care to be so precise. But, with the current setup it is not possible for the speaker to be more precise at the cost of being more verbose.

Initially this is not something Aloni wanted to model, so it doesn't need to be a problem. But, this problem is closely related to the formentioned problems. It is another consequence of having Manner taking priority over Quantity. As we will see in the coming sections, we can solve all these problems at once by changing the way Manner is dealt with.

General problem If the intuitions for the above examples (for $\neg C$ and $\Box P$) are correct we should conclude that Manner is less strict a constraint than it is taken for. We have two orderings. The one for manner and the one for quantity ($<_M$ and $<_Q$). With the current setup if some sentence S is dispreferred by manner over S', it is dispreferred in general $S <_M S' \Rightarrow S < S'$. Though the above examples lead us to conclude that if S' is dispreferred by manner over S but *preferred* by quantity, they might very well be equally good candidates for a certain model. This is currently not accounted for with Aloni's approach. I'll try to offer a solution for this in the sections to follow.

6.2.3 Ordering of the models

There's one last problem I'd like to point out. The current ordering on de models (the minimal model principle) works and is needed to get some of the implicatures right. But, we don't have any independent argument for using minimal models. Why would it be a good constraint?

The whole point of using BiOT is that we can avoid ad-hoc moves. To preserve this spirit, I think it to be needed to find a definition that is more closely tied to our intuitions.

7 New proposal: Separate form and meaning & use the competence principle

Most problems mentioned above are not really fundamental problems. That is to say, the concept seems to work. A little tweaking could give us the proper results. Tweaking, however, is not a very highly regarded waste of time in the world of academics.

A more highly regarded waste of time is to find independent arguments for the assumptions we make and then show that those assumptions give the expected results.

If the observations in the previous section are correct, I think the problems we're facing can be summarized as follows:

1. Manner does not always precede Quantity. (discussed in section 6.2.2)
2. The current definition of Manner does not give the right results (discussed in section 6.2.1)
3. There is no independent argument for using the minimal model principle

In this section I'll give solution for these problems. I aim to get the interaction between Manner and Quantity right, and, secondly I'd like to replace the markedness constraint on the models by a definition that is closer to our intuition.

As for the definition of Manner, I'll make some assumptions on how it should be defined; but, I'll conveniently ban a proper treatment of this constraint from the scope of this paper.

7.1 Separate form and meaning

As seen in the examples above, a lot hinges on the definition of manner. The definition of manner in its turn depends on the specific kind of logic we choose to use. In this case we're working with propositional modal logic. Manner is defined in terms of modals and negations. But what if we had opted for some other logic; say, predicate logic?

Take a sentence like 28

(28) Some student

This can be very easily coded into predicate logic:

$$\exists x : S(x)$$

In propositional logic this will however give a very long disjunct, listing all students:

$$s_1 \vee s_2 \vee s_3 \vee \ldots \vee s_n$$

If we define the manner constraint in terms of the chosen logic we would have to account for the fact that both $\exists x : S(x)$ and $s_1 \vee s_2 \vee s_3 \vee \ldots \vee s_n$ are equally verbose.

Another point of concern is the fact that one phrase with the same semantical content can be uttered in different ways; not all being equally verbose. Consider example 29.

(29)
- Balkenende is a wuss.
- It is true that Balkenende is a wuss.

Both sentences have the same semantical content, but the second utterance is clearly more verbose then the first. This should lift the meaning. "It is true that Balkenende is a wuss" seems to implicate that something else which is to be expected isn't true. This could implicate something like: "But, that doesn't make him a bad politician."

The precise implicatures don't really matter for the point I'm trying to make. The point is that by linking manner directly to the logical sentence, this can not be accounted for. The same semantical content can come from different phrases.

The final point of concern involves the interaction between quantity and manner. In the discussion of Aloni's approach we saw that we don't want to punish being overly precise. For issue $Q_{A?B?}$ the sentence A takes $[w_A]$ as optimal model. But, $A \wedge \neg B$ should also take $[w_A]$ as optimal model. With the current analysis this doesn't work out. The sentence $A \wedge \neg B$ is more verbose, so A is preferred even though $A \wedge \neg B$ is stronger.

If the above two points are valid and relevant, I think we should introduce 'verbosity' as a separate dimension in our analysis. This way we can have a less strict relation between the logical sentence and the verbosity. One logical sentence can be expressed in more and less verbose ways.

To do so we have two options. We could consider take the input to be tuples $\langle \phi, f \rangle$ (with ϕ as the semantical content and f the form/verbosity.

Another option is to consider form to be a third dimension in the OT analysis and redefine OT such that it can deal with three dimensions. As we will see the two approaches amount to more or less the same. But, for reasons that will become clear, I'll be using the later approach.

7.1.1 Redefine OT to deal with three dimensions

It is fairly straightforward to redefine OT for three dimensions:

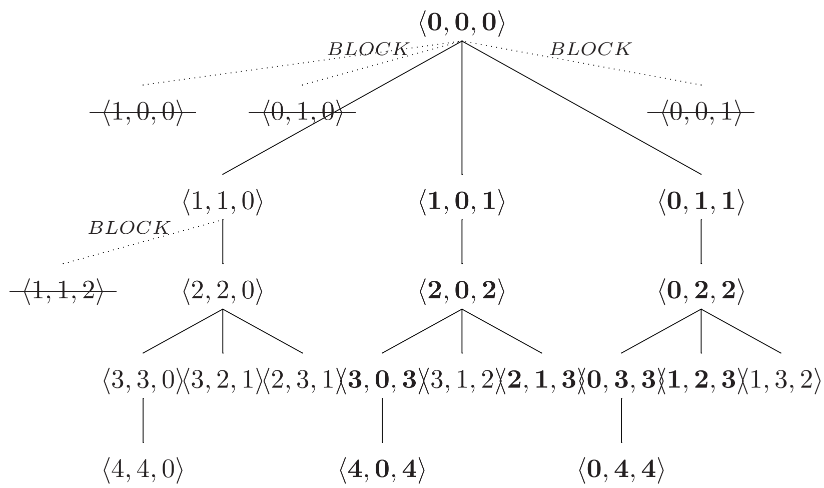
Symmetric $\langle a, b, c \rangle$ is weakly optimal iff there is no:

- $\langle a', b, c \rangle < \langle a, b, c \rangle$ s.th. $\langle a', b, c \rangle$ is w-opt
- $\langle a, b', c \rangle < \langle a, b, c \rangle$ s.th. $\langle a, b', c \rangle$ is w-opt
- $\langle a, b, c' \rangle < \langle a, b, c \rangle$ s.th. $\langle a, b, c' \rangle$ is w-opt

Unfortunately with this definition we run into problems right away. This system is symmetrically defined. If we'd apply this definition to natural numbers, $\langle 1, 2, 3 \rangle$ is super optimal. But since the definition is entirely symmetric $\langle 3, 2, 1 \rangle$ is also super optimal. For our purpose this is unwanted. We'd like to model the behavior that a weak and verbose sentence takes a relatively big model. With this symmetric definition this is not always the case.

It might be useful to have a look at how this definition behaves with natural numbers. Here below we see a part of the resulting tree for this definition. All the blotted out pairs are not weakly optimal. The other pairs are. The bold w-opt pairs are the ones we actually want. The unwanted triples can block wanted triples. For example $\langle 1, 1, 2 \rangle$ is blocked by $\langle 1, 1, 0 \rangle$.

While we would like the first one and not the later to be w-opt.

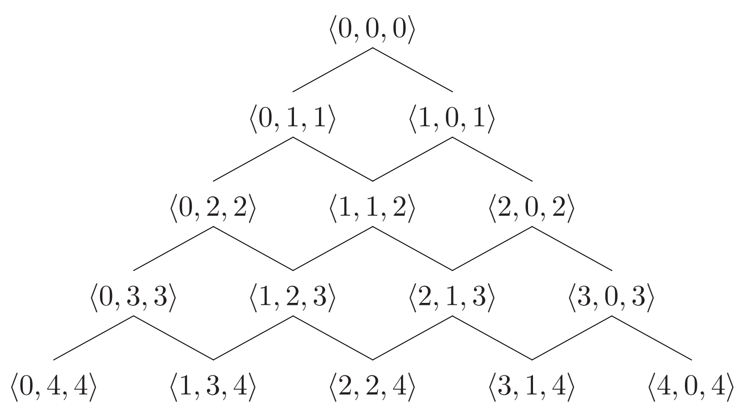


Though we introduced an extra dimension (the one that encodes the verbosity of a sentence), we don't want all dimension to be treated symmetrically. The first two are both input, the last one is the output. We need an alternative definition.

Asymmetric $\langle a, b, c\rangle$ is w-opt iff there is no:

- $\langle a', b', c\rangle < \langle a, b, c\rangle$ s.th. $\langle a', b', c\rangle$ is w-opt
- $\langle a, b, c'\rangle < \langle a, b, c\rangle$ s.th. $\langle a, b, c'\rangle$ is w-opt

The resulting tree of optimal pairs does look much more like what we'd want for later definition:



(30)

The dynamics we're after is that a weaker sentence enforces a bigger model, a more verbose sentence should also give a bigger model, but higher strength and higher verbosity should cancel each other out. We see this pattern emerge in the tree.

If we stick to natural numbers we get that $\langle a, b, c\rangle$ is w-opt iff $a + b = c$

Proof: $\langle a, b, n\rangle$ is w-opt iff $a + b = n$. By induction on c:

Initial step. $\langle 0, 0, 0\rangle$: 0+0=0

Induction hypothesis $\forall c < n : \langle a, b, c\rangle$ is w-opt iff $a + b = c$

$\Rightarrow$ Suppose $\langle x, y, n\rangle$ is w-opt.

To show $x + y = n$. Suppose not. $x + y \neq n$. Then either:

$x+y < n$. Take $z = x+y$. We know $z < n$, so by i.h $\langle x, y, z\rangle$ is w-opt. So $\langle x, y, n\rangle$ isn't. Contradiction.

$x + y > n$. Take $x' + y' = n$ s.th. neither $x' > x$ nor $y' > y$ (evidently this is always possible). Then $\langle x', y', n\rangle$ is *not* w-opt (otherwise it would block $\langle x, y, n\rangle$). But the either:

- there is $n' < n$ s.th. $\langle x', y', n'\rangle$ is w-opt. But $x' + y' < n$, so according to i.h. not w-opt. Contradiction
- or $x'' < x'$ or $y'' < y'$ (or both) s.th. $\langle x'', y'', n\rangle$ is w-opt. But $x'' < x$ and $y'' < y$ so $\langle x, y, n\rangle$ isn't w-opt. Contradiction.

Contradiction. So $x + y = n$

So $\langle x, y, n\rangle$ is w-opt $\Rightarrow x + y = n$.

$\Leftarrow$ Suppose $x + y = n$.

To show $\langle x, y, n\rangle$ is w-opt. Take $\langle x', y', n'\rangle$ w-opt s.th. $n' < n$. Then $x' + y' = n'$ either $x' < x$ or $y' < y$ (or both) assume w.l.o.g. that $x' < x$ then both $x' < x$ *and* $n' < n$. So $\langle x', y', n'\rangle$ cannot block $\langle x, y, n\rangle$. Our only assumption was that $\langle x', y', n'\rangle$ is w-opt. No w-opt triple can block $\langle x, y, n\rangle$. So $\langle x, y, n\rangle$ is w-opt.

In principle one could obtain the same results by using two dimensional BiOT and take tuples as input. So rather the determining the optimality of something $\langle a, b, c \rangle$ we could define the constraints and determine the optimality of something like $\langle\langle a, b\rangle, c\rangle$. The advantage of separating the three dimensions in BiOT is that we can define the structure of the first two arguments in terms of faithfulness constraints between a and b.

7.2 Minimal model principle vs. competence

Aloni uses the minimal model constraint to ensure we get 'reasonable' models for a certain sentence. When we say:

(31) Alessia voted for Berlusconi

We'd like to conclude that Maria didn't do so. Quantity takes partly care of this. All the models that make $A \wedge M$ true are blocked by the sentence $A \wedge M$. Because of quantity:

$$[w_{AM}] \text{ is not optimal for } A$$

Unfortunately quantity itself is not enough to conclude that Maria didn't vote. The model

$$[w_A, w_{AM}]$$

can still be optimal for A. There is no phrase ϕ that is preferred over A that holds in $[w_A, w_{AM}]$.

This is where Minimal Models come in. This constraint says that $[w_A]$ is a better (smaller) model then $[w_A, w_{AM}]$. Without Minimal Models we can derive the ignorance inferences but not the scalar implicatures. But why would we use Minimal Models and not some other ordering over the models?

The minimal model principle basically states that as little as possible should be true. But is this an assumption that corresponds to our intuitions?

Indeed exhaustiveness suggest that we tend to assume that all that is not affirmed is not true. But this dynamic is already explained by Quantity. Just as with the other Gricean solutions, Quantity derives the ignorance inferences.

Standard gricean accounts use the competence assumption to get scalar implicatures. As discussed earlier, gricean reasoning and Aloni's system work very similarly. Figure 8 suggests the relation with gricean based reasoning and OT. It is the competence principle that yields the scalar implicatures. Without any constraint on the models, it's precisely those scalar implicatures that can't be derived. Therefore I propose to use the competence principle as a constraint on the models rather then the minimal model principle.

	Neo Gricean account	OT account
1	Determine all $\psi \in \text{Alt}(\phi)$	Manner/Relation
2	Get ignorance inferences $\neg\Box\psi$	Quantity
3	Competence, if possible: $\neg\psi$	?

Figure 8: Relation between Neo Grice account and OT account

How can we formalize the competence principle? A model that assumes more knowledge of the speaker is preferred. So one obvious definition would be:

Bad definition $C <_Q C'$ iff $Q_C = Q'_C \wedge \forall\phi : C \models \heartsuit?\phi : C' \models \Box\phi \Rightarrow C \models \Box\phi$

Unfortunately this definition does not work for this particular semantics. With thee current semantics we have that $C \models \Box\phi \Leftrightarrow C \models \phi$. Thus this definition reduces to:

Bad definition' $C <_Q C'$ iff $Q_C = Q'_C \wedge \forall\phi : C \models \heartsuit?\phi : C' \models \phi \Rightarrow C \models \phi$

This is surely not what we want. The definition should for instance yield that $[w_A] < [w_A, w_\emptyset]$ (In $[w_A]$ there's no doubt about the truth value of A). The current definition does not achieve this: $[w_A, w_\emptyset] \models \Diamond\neg A$ and $[w_A] \not\models \Diamond\neg A$.

This is clearly not what we want. This problem could be solved by considering only non-modal sentences, but this is a rather ad-hoc solution. Fortunately such an ad-hoc solution isn't needed.

A better formulation of competence would be: "If the speaker thinks something is possible, he knows it is the case."

Better definition $C <_Q C'$ iff $Q_C = Q'_C \wedge \forall\phi : C \models \heartsuit?\phi : C' \models (\Diamond\phi \rightarrow \Box\phi) \Rightarrow C \models (\Diamond\phi \rightarrow \Box\phi)$

Note that this definition is equivalent to $C <_Q C'$ iff $Q_C = Q'_C \wedge \forall\phi : C \models \heartsuit?\phi : C' \models (\Box\phi \vee \Box\neg\phi) \Rightarrow C \models (\Box\phi \vee \Box\neg\phi)$. (This might be a more intuitive definition to some.)

What are the consequences of this definition? In general if a context contains less worlds there is less doubt about the truth values of the propositions. This intuition is reflected in the fact that the competence ordering is more or less the same as the inclusion relation.

If we take the issue to be a conjunction of atomic questions (i.e $Q_{?A?B...?N}$) and we only consider contexts containing worlds that make a subset of those atoms true, then we can derive:

$$C' <_Q C \text{ iff } C' \subset C \vee |C'| = 1$$

So, if C contains only one world[8] it is automatically smaller than any other context and otherwise $<_C$ behaves like the subset relation.

This is a very limited case but suffices since, for the examples to come, the issue is always such a conjunction of atoms. A complete account of the behavior of competence can be found in appendix C.

This gives a different ordering than minimal models. Though, as I'll show later on, with little practical difference. This ordering has two advantages. (1) It is easy to analyze (as subset ordering). And, (2) it is based on a well established assumption: Competence.

7.3 How not to define manner

With Aloni's approach, manner was considered a property of the logical sentence. By separating the two, a logical sentence can have any verbosity. That is to say, a natural sentence decomposes into a logical representation and a level of verbosity.

Though verbosity is not a property of a logical sentence anymore, for further analysis we'd still need to know the minimal level of verbosity a logical sentence can have. A sentence ϕ can be the logical representation of a number of natural language utterances. We need to know what verbosity the least verbose utterance of ϕ has. This will be a constraint in the system called Minimal Verbosity

To give an independent argument for some definition of Manner would require much more research. First of all one would have to establish what manner precisely is.

In this paper I'm referring to the verbosity of a sentence in relation to manner. Suggesting that the Manner constraint has something to do with the information density of the sentence. This is hardly, if at all, accurate.

One could go many ways in defining manner. Could it be expressed in terms of the length of a sentence? The cognitive effort it takes to analyze the sentence? The relative frequency of the expression in natural language? The number of occurrences of the word 'supercalifragilisticexpialidocious' in the sentence?

[8] Contexts with only one world form an exception since $\Diamond$ and $\Box$ coincide. So, for such contexts $m \models \Diamond\phi \rightarrow \Box\phi$ is trivially true.

In any case, it would require quite some empirical research to establish a proper definition. This is far beyond the scope of this paper (not in the last place because I defined the scope myself).

Finding a good definition of manner hardly falls within the realm of logic anymore. But, to be able to study some examples we'll have to give some kind of definition.

The easiest way to look at it might be in terms of alternatives. As we saw in the discussion of Aloni's approach, $A \wedge B$ can block the model $[w_{AB}]$ for A. In other words, $A \wedge B$ is an alternative to A. This can only be achieved if $A \wedge B$ is not worse in terms of Manner ($A \wedge B \leq_M A$).

It would be very counter intuitive if $A \wedge B <_M A$, so it seems reasonable to assume that conjunction contributes nothing to the verbosity of the sentence (i.e. $A \wedge B \equiv_M A$).

On the other hand we saw that $A \wedge \Diamond B$ should be more verbose then $\Box A$. Otherwise $\Box A$ doesn't get the model $[w_{AB}, w_B]$. So it should be the case that $\Box A <_M A \wedge \Diamond B$. So in this case using the conjunction *does* increase the verbosity of a sentence.

So the contribution of conjunction to the overall verbosity of the sentences $A \wedge \Diamond B$ and $A \wedge B$ is not the same. Perhaps the easiest way to explain this difference is that the first is a list of different types of sentences (A and $\Diamond B$ don't have the same modality) and the second is a list of sentences of a similar type (A and B do have the same type of modality).

I'll first formulate some intuitions which will be used to give a definition of Manner:

- Atomic sentences are the least verbose kind of sentences.
- If a conjunction lists sentences of the same modality it doesn't contribute to the overall verbosity. (i.e. A and $A \wedge B \wedge C$ have the same verbosity)
- If a conjunction lists sentence of different modalities it does contribute to the overall verbosity. (i.e. $A \wedge \Diamond B$ is more verbose then $\Diamond A$).
- Adding modals and negations increases the verbosity. $A <_M \neg A <_M \neg A \wedge \Box B$
- Using disjuncts increases the verbosity ("A or B" doesn't really seem a better way of saying then "Maybe A").

If one is to come up with a better (properly investigated) version of Manner, it is relatively easy to plug it into the current system.

For this purpose it suffices to just assign a natural number to a sentence as a score for it's verbosity. I can imagine that in a more complete analysis this isn't sufficient, but for now it will do. We need a formal definition of the minimal verbosity of any logical sentence.

We need a formal definition to work with, so for now $M(\phi) = n$ iff:

- $n = 1$ iff ϕ atomic
- $n = max(M(\psi), M(\xi))$ iff $\phi = \psi \wedge \xi$ and ψ and ξ are of the same modality.
- $n = max(M(\psi), M(\xi)) + 1$ iff $\phi = \psi \wedge \xi$ and ψ and ξ are of a different modality.
- $n = max(M(\psi), M(\xi)) + 1$ iff $\phi = \psi \vee \xi$
- $n = M(\psi) + 1$ iff $\phi = C\psi$ (with $C \in \{\neg, \Box, \Diamond\}$)

The function M stands for Minimal Verbosity. So a logical sentence ϕ can have any verbosity n such that $n \geq M(\phi)$

7.4 Summary model

To summarize the above, the new system differs from the one of Aloni in three points. The sentence in natural language has a semantical content and a form (level of verbosity). The two are separate dimensions in the OT system. The Minimal Model principle is replaced by the Competence principle. And for practical reasons Manner is redefined.

For the rest the system is the same. So we still use the update semantics as defined on (page ??.)

For the pragmatical part we use a weak optimality system. The elements that make up the OT system are: *Sem* the set of all modal logic sentences, *Form* the set of "ways of saying" (which we, for now, assume to be the set of natural numbers) and *Model* is the set of all possible models. The markedness constraints are simply orderings over *Sem*, *Form* and *Model*. The faithfulness constraint are binary relations.

The markedness constraints are:

- Quantity, $<_q$ over *Sem*. $\phi, \psi \in Sem$: $\phi <_q \psi$ iff $\phi \models \psi$ and $\psi \not\models \phi$
- Manner, $<_m$ over *Form*. $<_m$ is defined as the usual order over $\mathbb{N}$

- Competence, $<_c$ over $Model$. $C <_c C'$ iff for all relevant ϕ ($C' \models \Diamond\phi \rightarrow \Box\phi) \Rightarrow (C \models \Diamond\phi \rightarrow \Box\phi)$ (I.e. knowledge in C' implies knowledge in C)

Faithfulness constraints:

- Quality, $m \models \Box\phi$
- Relation, $m \models \heartsuit?\phi$
- Minimal Verbosity. ϕ can be expressed with verbosity f. (And, if ϕ and f don't violate MV then ϕ and f' such that $f' > f$ doesn't violate MV either.)

The ordering of the constraints:

$$\text{Relation}, \text{Quality}, \text{Minimal Verbosity} \succ \text{Quantity, Manner} \succ \text{Competence}$$

Not that the ranking of Competence is irrelevant with respect to Quantity and Manner. In the next section I'll go through some examples to show how the system as defined here behaves. For most examples the faithfulness constraints are never violated. If such is the case they'll be left out. So, for most examples we'll only be concerned with the constraints:

$$\text{Quantity, Manner} \succ \text{Competence}$$

Which is equivalent to:

$$\text{Competence} \succ \text{Quantity, Manner}$$

8 Examples

We studied the system of Aloni. This system gives quite accurate predictions. Though there were implicatures that did not come out right in this system. This is caused by several problems: The nature of the minimal model constraint was one, the definition of Manner another. Even if we would redefine this constraint, we would run into trouble. This is mainly caused by the fact that Manner takes absolute precedence over Quantity (i.e. Manner is a higher ranked constraint then Quantity).

The system we defined in this paper solves these problems. Rather than using BiOT on tuples we use the same principles but then on triplets. The main reason to do this is the intuitive notion that the logical form of a sentence has very little to do with its verbosity. That is to say to express $A \wedge B$ one could say "A and B" but also "I know that A and B are both true". Both sentences have the same semantics, but the later is much more verbose then the first. Of course, there is some relation between the two. Logical sentences do have a minimal verbosity. That is, the easiest way to express A is "A". The easiest way to express $\Box A$ is "I know that A". We find this relation back in the minimal verbosity constraint.

Now we will look at some examples to see if these changes indeed solved the problems. Since the determination of optimality in BiOT is quite involved, I'll use a couple of shorthand to keep everything manageable. We're interested in the optimality of triplets like $\langle \phi, f, C \rangle$ (where ϕ is the logical sentence, f the verbosity and C the context).

I'll use tables to compare the triples with all other possible triples. So $\langle A, 0, \langle Q_{A?B?}[w_A] \rangle \rangle$ is represented in a table like:

Sent	Verb	Model	Qual Rel	Min verb	Mann Quan	Comp
A	0	$\langle Q_{A?B?}[w_A] \rangle$				

The tabular doesn't only mention the values of the triple but also the constraints (equally ranked constraints are not separated by \|). I'll use a star to denote a violation of the corresponding constraint:

Sent	Verb	Model	Qual Rel	Min verb	Mann Quan	Comp
A	0	$\langle Q_{A?B?}[w_A] \rangle$				
A	0	$\langle Q_{A?B?}[w_B] \rangle$	*			

In this table the triple $\langle A, 0, \langle Q_{A?B?}[w_B] \rangle \rangle$ violates quality.

When the tuples that violate any of the faithfulness constraints (Quality, Relation and Minimal Verbosity) are not relevant for the example, the respective columns are left out:

Sent	Verb	Model	Mann Quan	Comp
A	0	$\langle Q_{A?B?}[w_A] \rangle$		
B	0	$\langle Q_{A?B?}[w_B] \rangle$		

The number of $*$'s indicate the amount of violation relative to other constraints:

Sent	Verb	Model	Mann Quan	Comp
A	0	$\langle Q_{A?B?}[w_{AB}] \rangle$	*	
$A \wedge B$	0	$\langle Q_{A?B?}[w_{AB}] \rangle$		
$A \wedge B$	1	$\langle Q_{A?B?}[w_{AB}] \rangle$	*	
$A \wedge B$	2	$\langle Q_{A?B?}[w_{AB}] \rangle$	**	

In this table, the triple $\langle A, 0, \langle Q_{A?B?}[w_{AB}] \rangle \rangle$ violates quantity more then $\langle A \wedge B, 0, \langle Q_{A?B?}[w_{AB}] \rangle \rangle$. The triples $\langle A, 0, \langle Q_{A?B?}[w_{AB}] \rangle \rangle$,$\langle A \wedge B, 0, \langle Q_{A?B?}[w_{AB}] \rangle \rangle$ violate manner less then $\langle A \wedge B, 1, \langle Q_{A?B?}[w_{AB}] \rangle \rangle$. The triple $\langle A \wedge B, 2, \langle Q_{A?B?}[w_{AB}] \rangle \rangle$

violates manner even more.

If all examples have the same issue, it will be left out:

Sent	Verb	Model	Mann	Quan	Comp
A	0	$[w_{AB}]$		*	
$A \wedge B$	0	$[w_{AB}]$			
$A \wedge B$	1	$[w_{AB}]$	*		
$A \wedge B$	2	$[w_{AB}]$	**		

And, rather then reserving a column for the verbosity it will be denoted by the number of $*$'s in front of the sentence. This results in the table:

Sent	Model	Mann	Quan	Comp
A	$[w_{AB}]$		*	
$A \wedge B$	$[w_{AB}]$			
$*(A \wedge B)$	$[w_{AB}]$	*		
$**(A \wedge B)$	$[w_{AB}]$	**		

The $\Rightarrow$ marks the optimal tuples:

Sent	Model	Mann	Quan	Comp
A	$[w_{AB}]$		*	
$\Rightarrow$ $A \wedge B$	$[w_{AB}]$			
$*(A \wedge B)$	$[w_{AB}]$	*		
$**(A \wedge B)$	$[w_{AB}]$	**		

Now let's have a look at some examples. For the examples below the issue will be $Q_{A?B?}$ unless explicitly mentioned otherwise.

8.1 Basic examples: A, $A \wedge B$, $A \vee B$

First of all consider the very basic cases A, $A \wedge B$ and $A \vee B$. Those already worked in Aloni's system. Let's see if they still work. Table 9 shows the dynamic of these examples:

Though these examples are simple enough to be more or less self-evident. They can be explained as follows:

- A takes the model $[w_A]$. It cannot take $[w_{AB}]$ since it's already blocked by $A \wedge B$. (This requires that the minimal verbosity of $A \wedge B$ is the same as the one for A)

- $A \vee B$ takes $[w_A, w_B]$. The alternative candidates $[w_A]$,$[w_B]$,$[w_{AB}, w_B]$ and $[w_{AB}, w_A]$ are blocked by A,B,$*A$,$*B$. Note that here the value of disconnecting verbosity and meaning of a sentence becomes important. $A \vee B$ doesn't take $[w_{AB}, w_A]$ since it is blocked by the stronger $*A$.

	Sentence	Model	Manner	Quantity	Competence
$\Rightarrow$	A	$[w_A]$		*	
	A	$[w_{AB}]$		*	
	A	$[w_{AB}, w_A]$		*	*
$\Rightarrow$	$*A$	$[w_{AB}, w_A]$	*	*	*
$\Rightarrow$	$A \wedge B$	$[w_{AB}]$			
	$A \vee B$	$[w_A]$	*	**	
	$A \vee B$	$[w_{AB}]$	*	**	
$\Rightarrow$	$A \vee B$	$[w_A, w_B]$	*	**	*
	$A \vee B$	$[w_{AB}, w_A]$	*	**	*

Figure 9: Basic examples

If verbosity would be a property of the logical sentence, we wouldn't have $*A$ available.

- $A \wedge B$ takes $[w_{AB}]$ since it's the smallest (and only model for this issue) that doesn't violate Quality.

So we get the implicatures: $A \mid\approx \neg B$, $A \vee B \mid\approx \neg\Box A, \neg\Box B, \neg(A \wedge B)$. In other words for these examples we get the right exhaustivity, ignorance inferences and scalar implicatures.

8.2 Epistemic examples: $\Box A$, $\Diamond A$

Next we get the simple modal cases. These didn't work quite correctly in Aloni's system. Consider tabel 10.

Observe that $*A$ and $\Box A$ take the same model. This shouldn't be surprising. With the update semantics we're using A and $\Box A$ are semantically equivalent. The only difference is that the minimal verbosity of $\Box A$ is higher then the one of A. Basically $\Box A$ is just a marked version of A. This examples show the following predictions:

- $\Box A$ takes $[w_{AB}, w_A]$. So, $\Box A \mid\approx \Diamond B \wedge \Diamond\neg B$,
- $\Diamond A$ takes $[w_A, w_\emptyset]$. The model $[w_B, w_A]$ is blocked by the stronger $\Diamond A \wedge \Diamond B$

So for $\Box A$ we automatically get the right implicatures. For $\Diamond A$ the implicatures highly depend on manner. This uncertainty might reflect the lack of intuition we have for this example.

	Sentence	Model	Manner	Quantity	Competence
$\Rightarrow$	A	$[w_A]$			
	$\Box A$	$[w_A]$	*		
	$\Diamond A$	$[w_A]$	*	**	
$\Rightarrow$	$*A$	$[w_{AB}, w_A]$	*		*
$\Rightarrow$	$\Box A$	$[w_{AB}, w_A]$	*		*
	$\Diamond A$	$[w_{AB}, w_A]$	*	**	*
$\Rightarrow$	$\Diamond A$	$[w_A, w_\emptyset]$	*	*	*
$\Rightarrow$	$A \vee B$	$[w_A, w_B]$	*	*	*
	$\Diamond A$	$[w_A, w_B]$	*	**	*
$\Rightarrow$	$\Diamond A \wedge \Diamond B$	$[w_A, w_B]$	*	*	*

Figure 10: Simple modals

8.3 Being overly verbose: $A \wedge \Diamond B$, $A \wedge \neg B$

Finally we get to some examples where the difference between Aloni's system and the one proposed here very clearly come to bear. As a matter of personal preference some people like to be overly precise. In spite of being tedious and boring, those people aren't wrong. The system should allow for this. Table 11 and 12 show what happens with $A \wedge \Diamond B$ and $A \wedge \neg B$. The sentence $A \wedge \Diamond B$

	Sentence	Model	Manner	Quantity	Competence
$\Rightarrow$	A	$[w_A]$		**	
$\Rightarrow$	$\Box A$	$[w_{AB}, w_A]$	*	**	*
$\Rightarrow$	$*A$	$[w_{AB}, w_A]$	*	**	*
$\Rightarrow$	$A \wedge \Diamond B$	$[w_{AB}, w_A]$	**	*	*
	$A \wedge \Diamond B$	$[w_{AB}]$	**	*	
$\Rightarrow$	$A \wedge B$	$[w_{AB}]$	*		

Figure 11: Being to verbose

	Sentence	Model	Manner	Quantity	Competence
$\Rightarrow$	A	$[w_A]$		*	
$\Rightarrow$	$A \wedge \neg B$	$[w_A]$	*		

Figure 12: Being to verbose

is stronger then $\Box A$. On the other hand it's more verbose. Better in terms of Quantity, worse in terms of Manner. As we can see. These two cancel each other out. Both sentences take the same model. Fortunately. . . this was one of the main reasons to change the system to begin with. It would have been a rather ghastly waist of time, if this wouldn't come out right.

Similarly $A \wedge \neg B$ and A take the same model.

8.4 Free choice, indifference, ignorance $A \vee B$, $\Diamond(A \vee B)$, $\Box(A \vee B)$

Now back to the difference between indifference and ignorance. Table 13 shows the ignorance analysis. Here we assume (as with all the previous examples) the issue $Q_{A?B?}$ As we can see, $\Box(A \vee B)$ takes $[w_{AB}, w_A, w_B]$

	Sentence	Model	Manner	Quantity	Competence
$\Rightarrow$	$A \vee B$	$[w_A, w_B]$	*	*	
	$\Box(A \vee B)$	$[w_A, w_B]$	**	*	
	$\Diamond(A \vee B)$	$[w_A, w_B]$	**	**	
$\Rightarrow$	$\Diamond(A \wedge B)$	$[w_{AB}, w_\emptyset]$	*		
	$\Diamond(A \vee B)$	$[w_{AB}, w_\emptyset]$	**	**	
$\Rightarrow$	$*A$	$[w_{AB}, w_A]$	*		
	$\Box(A \vee B)$	$[w_{AB}, w_A]$	**	*	
	$\Diamond(A \vee B)$	$[w_{AB}, w_A]$	**	**	
$\Rightarrow$	$\Diamond(A)$	$[w_A, w_\emptyset]$	*	*	
	$\Diamond(A \vee B)$	$[w_A, w_\emptyset]$	**	**	
$\Rightarrow$	$\Box(A \vee B)$	$[w_{AB}, w_A, w_B]$	**	*	*
$\Rightarrow$	$\Diamond(A \vee B)$	$[w_A, w_B, w_\emptyset]$	**	**	*
$\Rightarrow$	$*\Diamond(A \wedge B)$	$[w_{AB}, w_A, w_\emptyset]$	**	*	*
$\Rightarrow$	$*\Diamond(A \wedge B)$	$[w_{AB}, w_B, w_\emptyset]$	**	*	*

Figure 13: Ignorance

and $\Diamond(A \vee B)$ takes $[w_A, w_B, w_\emptyset]$. All possible smaller models are already blocked by better sentences. These predictions are exactly the same as the ones for Aloni's proposal.

Indifference is much easier to analyze. We take the issue $Q_{?(A \vee B)}$. In this case anything that is stronger then $A \vee B$ violates the relation constraint. Due to the fact that anything stronger then $A \vee B$ violates relation, $A \vee B$ takes all the smallest models that don't violate Quality. Now $\Box(A \vee B)$ and $\Diamond(A \vee B)$ are very easy to establish. $\Box(A \vee B)$ is semantically

	Sentence	Model	Relation	Manner	Quantity	Competence
$\Rightarrow$	$A \vee B$	$[w_A]$		*	*	
	A	$[w_A]$	*			
$\Rightarrow$	$A \vee B$	$[w_B]$		*	*	
	B	$[w_B]$	*			
$\Rightarrow$	$A \vee B$	$[w_{AB}]$		*	*	
	$A \wedge B$	$[w_{AB}]$	*			

Figure 14: Indifference

equivalent to $A \vee B$. It takes the smallest models for which $\Box(A \vee B)$ doesn't violate quantity and which are not already blocked by $A \vee B$ (that is: $[w_A, w_B]$,$[w_{AB}, w_B]$,$[w_{AB}, w_A]$).

Similar reasoning shows that $\Diamond(A \vee B)$ takes $[w_A, w_\emptyset]$, $[w_B, w_\emptyset]$, $[w_{AB}, w_\emptyset]$.

8.5 Problems for larger contexts: $\neg A$, $\Box A$

As the context grows the number of sentences we have to take into consideration explodes. This has the nasty downside that the definition of the Manner constraint becomes more important. Since this definition is not part of the research and we'd like to depend on it as little as possible, I'll show that the implicatures we get are relative insensitive to the definition of Manner. This requires a lot of 'what if?"s and a very lengthy explanation. For this, my apologies.

As pointed out before we do run into problems with sentences that cancel the competence principle. The competence principle states that the speaker has as much knowledge on the issue as we can consistently assume. Why else would we ask someone a question to begin with?

The sentence "I know that A", kind of implies that I don't know anything else that is relevant. This can be explained in two ways:

- By quantity: Suppose that the speaker says $\Box A$. If the speaker would have known $\Box A \wedge \Box B$ then the quantity maxim would have forced the speaker to say $\Box A \wedge \Box B$. He didn't, so we can derive $\neg(\Box A \wedge \Box B$.
- By explicit canceling: As soon as someone says "I know that …" it cancels the competence constraint.

If the later would be the proper explanation, this system cannot account for it. Competence is one of the presupposed constraints. Having to change the constraints depending on the kind of sentence we have, would be very problematic.

So if we're to explain this behavior it should be on grounds of quantity. This, however, yields another problem. There is the natural tendency in the system that bigger worlds take more verbose sentences[9]. But the size of the models that go with $\Box A$ (which has a fixed verbosity) grows with the size of the issue. The examples in table 15 and 16 show how this needn't be a big problem. In table 15 we see that $\Box A$ doesn't have the same model

	Sentence	Model	Relation	Mann	Quan	Comp
	$\Box A$	$\langle Q_{A?B?C?}, [w_{AB}, w_A]\rangle$			*	
$\Rightarrow$	$A \wedge \neg C$	$\langle Q_{A?B?C?}, [w_{AB}, w_A]\rangle$				
$\Rightarrow$	$\Box A$	$\langle Q_{A?B?}, [w_{AB}, w_A]\rangle$			*	
	$A \wedge \neg C$	$\langle Q_{A?B?}, [w_{AB}, w_A]\rangle$	!*			

Figure 15: Canceling competence 1

for different context. $A \wedge \neg C$ can block $\Box A$. But in the context $Q_{A?B?}$ the sentence $A \wedge \neg C$ violates relation, so for this context it doesn't block.

Indeed we get that $\Box A$ *does* take other (bigger) models with larger context. So at least partly the canceling of competence works. Though the precise models that are optimal for $\Box A$ are very sensitive to the precise definition of manner. Table 16 lists all models that don't violate quality. As we can see, the models that are optimal for $\Box A$ depend greatly on the definition of Manner. All the logical sentences in the table are stronger then $\Box A$. If there is a way to express them in natural language that isn't more verbose then "I know ..." can block the model in question. For instance, we consider the expression "A and B" an alternative of "A". Suppose this holds under modals as well. Then 'I know that A and B" is an alternative of "I know that A". So $\Box(A \wedge B)$ blocks $[w_{ABC}, w_{AB}]$ for $\Box A$. But what about $A \wedge (B \leftrightarrow C)$? This could come from the expression "A came and B and C always come together". Maybe... It is surely not clear whether or not this sentence is an alternative to $\Box A$. What can we say though about the implicatures? We don't need to know precisely which models are optimal in order to say something about the implicatures. If we scratch all the models that are obviously blocked, we're left with the models 7,9,11,13,14 and 15 for $\Box A$. Some of these models will be optimal for $\Box A$. $\neg\Box B$ holds in all

[9] I'm purposely a bit vague. This is not always the case

	Sentence	Model	Relation	Mann	Quan	Comp
1	A	$[w_A]$			*	
2	$A \wedge C$	$[w_{AC}]$				
3	$A \wedge \neg B$	$[w_{AC}, w_A]$		*		*
4	$A \wedge B$	$[w_{AB}]$				
5	$A \wedge \neg C$	$[w_{AB}, w_A]$		*		*
6	$A \wedge (B \vee C)$	$[w_{AB}, w_{AC}]$		*		*
7	$A \wedge \neg(B \wedge C)$	$[w_{AB}, w_{AC}, w_A]$		?		**
8	$A \wedge B \wedge C$	$[w_{ABC}]$				
9	$A \wedge (B \leftrightarrow C)$	$[w_{ABC}, w_A]$		?		*
10	$\Box(A \wedge C)$	$[w_{ABC}, w_{AC}]$		*		*
11	$\Box A?$	$[w_{ABC}, w_{AC}, w_A]$		*	*	**
12	$\Box(A \wedge B)$	$[w_{ABC}, w_{AB}]$		*		*
13	$\Box A?$	$[w_{ABC}, w_{AB}, w_A]$		*	*	**
14	$*(A \wedge (B \vee C))$	$[w_{ABC}, w_{AB}, w_{AC}]$		?		**
15	$\Box A?$	$[w_{ABC}, w_{AB}, w_{AC}, w_A]$		*	*	***

Figure 16: $\Box A$ for issue $Q_{A?B?C?}$

those models. So however the precise definition of manner, $\Box A \mid\approx \neg\Box B$. Similarly for $\neg\Box C$, $\neg\Box(B \wedge C)$ and $\neg\Box(B \vee C)$. So we get the kind of ignorance inferences we would expect.

9 Conclusion

Along the lines of Aloni's research I've developed a formalization of Grice's maxims. Grice's maxims offer a general procedure to compute scalar implicatures. Some research uses grice's maxims.

I've discussed some other approaches that I think are not entirely satisfactory. They also use Grice's ideas. And, though they get at least some of the implicatures right, they need to make some ad hoc moves, they don't incorporate all maxims and the formalizations of the maxims they do incorporate are not always very intuitive. Aloni's approach formalizes all maxim's in a more intuitive way. But, this approach runs into some practical problems. The ranking of the constraints she chooses cause some unwanted effects. And, she uses the minimal model principle, which works but lacks independent motivation.

A study of the behavior of BiOT let to reconsidering the ranking of the constraints Manner and Quantity. The studies discussed earlier use the competence principle to derive scalar implicatures from ignorance inferences. Inspired by this, I formalized the competence principle as a BiOT constraint. This, together with a slightly altered definition of Manner, lead to the system as I proposed.

The examples show that this solves some problems we encountered with Aloni's approach. Some of the problems that are solved are mere technical details. Others are slightly more fundamental. For instance my proposal accounts for the fact that the sentences "John voted" and "John voted and Pete didn't" can in a certain context mean the same thing.

With these improvements I hope to give a better formalization of Grice's theory. A proper formalization will make Grice's maxims testable. This can lift his theory out of the philosophical realm. If this can help with seeing if Grice's theory works it can help us understand the dynamics of communication a bit better in general.

This proposal raises some further questions. The most urgent one is how to give a proper account of Manner. Manner stays some kind of Joker card, that can be played when one needs to confine the alternatives. The lack of a proper (independently motivated) formalization prevents any strong conclusion from being drawn.

Another interesting research would be to investigate BiOT more closely. In the paper we've stumbled upon the problem that BiOT is very sensitive to small alterations. If we make a small change in some part of the system how does this affect the rest? Can we say something about it on a formal level? What happens if the hearer and the and speaker don't use exactly the same constraints? How does that influence the communication?

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A Wrong predictions with exhaustification

Schulz & van Rooij have difficulties with getting the right reading for example 32.

(32) I know that John voted

The exhaustive interpretation doesn't give the right predictions in both cases. It shouldn't be to hard to solve. There is some exhaustification going on with example 8. The exhaustification of knowledge. To put it in terms of "only": This example could be read as

(33) I *only* know that John voted

However, the way Schulz & van Rooij define their exhaustification it reads.

(34) I know that *only* John voted

As we have seen in the BiOT system $\Box J \mid\approx \Diamond M \wedge \Diamond\neg M$. Schultz & van Rooij don't get this.

Proof For this proof we suppose the epistemic axiom $\Box\Box\phi \leftrightarrow \Box\phi$. Furthermore take the background question "Who voted, John or Mary?". The speaker said "I know that John" ($\Box J$).

First we need to determine the result of applying *grice*:

$$grice^C(A, P) =_{def} \{w \in [\mathbf{K}A]^C | \forall w' \in [\mathbf{K}A]^C : w \preceq_{P,A} w'\}$$

First we should determine the worlds where $\Box\Box J$ holds. Since we have $\Box\Box J \leftrightarrow \Box J$ we are interested in the worlds where $\Box J$ hold. This reduces the problem to the example used by Schulz & van Rooij themselves in [**?**]. The worlds of interest are graphically represented in 17.

We get that $w_i \preceq w_1$ for all i. The world w_1 is the only not minimal world so the *grice* operator selects world w_2, w_3, w_4. Now we can apply the *eps* operator:

$$eps^C(A, P) =_{def} \{w \in grice^C(A, P) | \forall w' \in grice^C(A, P) : w \not\sqsubset_{A,P} w'\}$$

The only maximal world is w_3. So we get that $eps^C(\Box J, P) = \{w_3\}$. This gives a too strong reading: $w_3 \models \neg M$

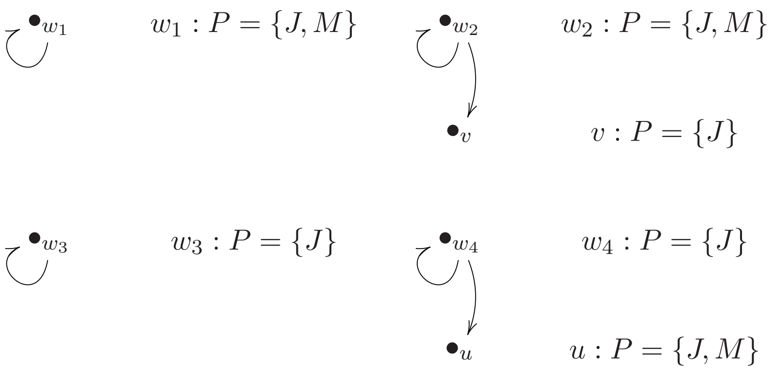


Figure 17: Possible worlds for $\Box J$

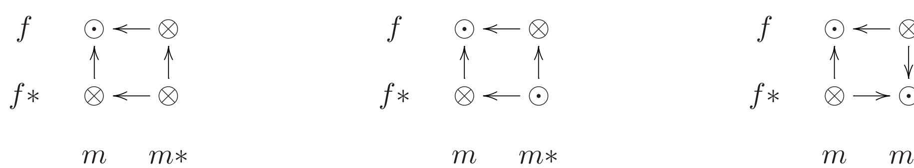
B Some properties of weak optimality

B.1 Division of Pragmatic Labor

In this paper weak optimality is chosen over strong optimality for the fact that markedness implies markedness. That is, if the input is marked then so will the output be. This tendency (Division of Pragmatic Labor, or DoPL) is pointed out by Blutner in [6].

However, he never really formalizes it. With binary constraints its self evident of how DoPl could be formally defined, but gradual constraints take a bit more of consideration. In this section Ill try to give a formal definition of DoPL.

To get a general idea consider figure 18. In this example there are two



(a) Strong optimality

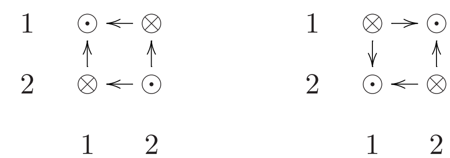
(b) Weak optimality

(c) Strong optimality with $F* \leftrightarrow M*$

Figure 18: Blutner: Strong optimality with 'markedness implies markedness'

markedness constraint, one for input f and one for output m. I use $\odot$ to indicate that a pair is optimal and $\otimes$ to indicate non optimality. $f*$ indicates marked input and $m*$ indicates marked output. With strong optimality (figure 18(a)), only the pair $\langle f, m \rangle$ is optimal. With weak optimality (figure 18(b)) both $\langle f, m \rangle$ and $\langle f*, m* \rangle$ are optimal. If we strongly optimal system and add the constraint that input and output should be equally marked (the DoPL constraint) then it seems to simulate the behavior of weak optimality (figure 18(c)).

Does DoPL always work? First of all it should be noted that DoPL does not always work. We can easily define a system where markedness does not imply markedness. Consider figure 19. In this example the input and



(a) Only markedness constraints

(b) Added faithfulness constraint: $f \neq m$

Figure 19: No DoPL

output are either 1 or 2 with the habitual ordering as markedness constraints. Normally this would give a picture like in figure 19(a). But we can add a new faithfulness constraint that explicitly counters the DoPL effect like the constraint $f \neq m$.

It should intuitively be clear (though it should be proven) that we can create any kind of relation between input and output with the help of faithfulness constraints. These relations can completely overrule the dynamics of a BiOT system. The dynamics of BiOT can be much more easily studied if we ignore faithfulness constraints and focus on the markedness constraints. For the rest of this section we'll only consider systems with nothing but markedness constraints.

If F is the set if inputs and M is the set of outputs, we'll suppose that all constraints are encoded in two orders $<_F$ and $<_M$ over respectively the input an the output. We're free to define any number of constraints with any ranking in OT. In the next two sections I'll argue that we can reduce any BiOT system with markedness constraints to a system with only one constraint for the input and one for the output. After that I'll give a formal definition of DoPL.

Well founded orders It is sufficient to only consider BiOT systems with one ordering on the input and one on the output. Any system without faithfulness constraints can be reduced to this. First let's establish the kind of orderings we're dealing with.

To proof that weak optimality is well defined Jäger assumed the ordering $<_t$ over $F \times M$ to be well founded. In this case $<_t$ is only determined by $<_F$ and $<_M$. It is defined as $\langle f', m' \rangle <_t \langle f, m \rangle$ iff $f < f' \vee (f \not< f' \wedge f \not> f' \wedge m < m')$. Or, (equivalently as shown above), it is defined as $\langle f', m' \rangle <_t \langle f, m \rangle$ iff $m < m' \vee (m \not< m' \wedge m \not> m' \wedge f < f')$. In both case it holds that[10]:

$$\langle F \times M, <_t \rangle \text{ is well founded } \textit{iff } \langle F, <_f \rangle \textit{ and } \langle M, <_m \rangle \text{ are well founded}$$

For the rest of this section we take $<_F$ and $<_M$ to be well founded orders.

Well orders We're out to define fossilization in BiOT. In the previous sections I argued that only considering a system with two markedness constraints suffice (one on the input and one on the output) and that those constraints are well founded. To simplify matters, I'll first show how fossilization works for well orders (i.e. total well founded orders).

The general idea of fossilization is that markedness implies markedness. For well orders this comes down to an isomorphism between the two orders as represented in figure 20 (the double lines connect the optimal pairs and the arrows symbolize the orderings).

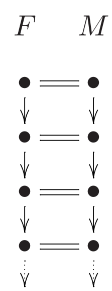


Figure 20: Optimal pairs, well orders

Let's first consider natural numbers only. Let F and M be initial segments of $\mathbb{N}$ with their usual orders.

Lemma 1. $\langle f, m \rangle$ *is weakly optimal* iff $f = m$

[10]Note that this does not need to be the case when we have faithfulness constraints.

Proof. By induction on $f + m$.

Basis: $\langle 0, 0\rangle$ is weakly optimal.

Induction hypothesis: Suppose for all $f + m < n : \langle f, m\rangle$ is weakly optimal *iff* $f = m$

Take $f + m = n$

$\Rightarrow$: Let $\langle f, m\rangle$ be w-opt.

Suppose $m < f$. Then $m + m < n$. So i.h. gives $\langle m, m\rangle$ is weakly optimal. This blocks $\langle f, m\rangle$. Contradiction

Suppose $m > f$. Similar.

$\Leftarrow$: Suppose $f = m$. Then for all $f' < f$: $f' \neq m$ so (by i.h.) $\langle f', m\rangle$ isn't w-opt. The same for all $m' < m$. Hence $\langle f, m\rangle$ is w-opt.

This proof can be trivially extended to any ordinal. If λ is a limit ordinal the proof still works.

Lemma 1. *Let W_0 and W_1 be well orders. Take $w_0 \in W_0$ and $w_1 \in W_1$ then trivially: $\langle w_0, w_1\rangle$ is w-opt* iff *$o.t.(w_0) = o.t.(w_1)$. (where o.t. denotes the order transformation of the well order)*

(Odd note: the set OPT of weakly optimal pairs of two well orders is an order isomorphism between the two well orders.)

Collary 1. *Let F and M be well orders. Take the system WO with only the constraints $<_F$ and $<_M$. Take the system SO with only the constraint $o.t.(f) = o.t.(m)$. The pair $\langle f, m\rangle$ is* weakly *optimal in WO* iff *it is* strongly *optimal in SO.*

I use the concept of order transformation since this also hold for the transfinite case. Order tranformations map a well order to the unique order isomporphic ordinal. Ordinals are a set theoretic abstraction of natural numbers. If we ignore the transfinite case we can consider o.t. the mapping of an element in a well order to natural number (where the smallest element is 0, the next smallest is 1, etc. . .). So basically lemma 1 states that the smallest element in F is optimal for the smallest element in M, the second smallest in F for the second smallest in M, etc. . . . With the final lemma this is used to define the DoPL constraint.

Well founded non total order For well orders (total) the proof is straightforward. The problem for well founded orders (not total) can be reduced to that of well orders by defining a linear extension. Not any kind of linear extension from a well founded order to a well order works. In this section the proper linear extension will be defined and I'll show it's correctness. Let $\langle F, < \rangle$ be a well founded order. Let's define a linear extension λ_F. If $S \subseteq F$, $\min(S)$ denotes the set of minimal elements of S. Since F, is well founded, every subset contains at least one minimal element so $\min(S)$ is well defined.

$$F_0 = F \tag{35}$$
$$F_{n+1} = F_n - \min(F_n) \tag{36}$$

$$x \in \min(F_n) \quad \text{iff} \quad \lambda_F(x) = n \tag{37}$$

Clearly $x < y \Rightarrow \lambda_F(x) < \lambda_F(y)$. But this is not the only property of interest. This specific extension has the (for this case) important property $\forall y \forall n < \lambda_F(y) \exists x < y \lambda_F(x) = n$. Which is to say for any y_n there is an descending chain $y_0 < y_1 < y_2 < \ldots < y_n$ such that $y_i \in \min(F_i)$.

Lemma 1. *Let F and M be well founded orders. With $f \in F$, $m \in M$. Then $\langle f, m \rangle$ is weakly optimal* iff *$\langle \lambda_F(f), \lambda_M(m) \rangle$ is weakly optimal.*

In figure 21 we can see how this works. The function λ is represented by $\sim>$, $=$ links the optimal pairs. In figure 21(a) we can see how λ projects the well founded order on a well order. Figure 21(b) shows the optimal pairs between the well founded orders.

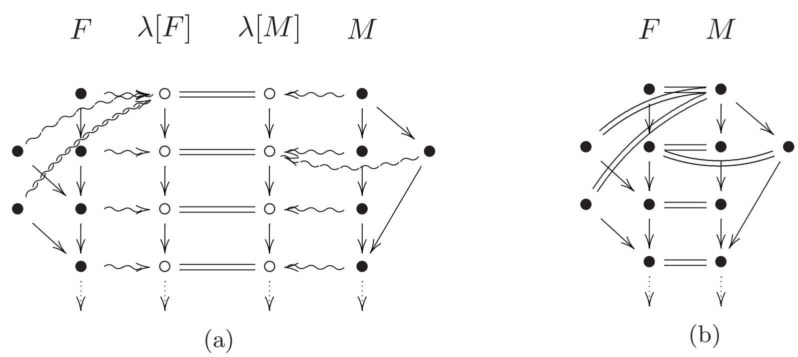


Figure 21: Partial well founded orders.

Proof. Since $\langle \lambda_F(f), \lambda_M(m) \rangle$ is w-opt iff $\lambda_F(f) = \lambda_M(m)$, we only have to proof: $\langle f, m \rangle$ is w-opt *iff* $f \in \min(F_i)$ and $m \in \min(M_i)$ for some i.

$\Rightarrow$ $\langle f, m \rangle$ is w-opt $\Rightarrow f \in \min(F_i)$ and $m \in \min(M_i)$ for some i

Basis: Clearly, $f \in \min(F_0) \Leftrightarrow m \in \min(M_0)$

Induction hypothesis: For all $j < i$: $f \in \min(F_j) \Leftrightarrow m \in \min(M_j)$

$\Rightarrow$ Take $f \in \min(F_i)$ and $m \in \min(M_k)$.

$k < i$: By i.h. $\langle f, m \rangle$ w-opt implies that $f \in \min(F_k)$. Contradiction.

$k > i$: There exists an $m' < mM \in \min(M_i)$. Now $\langle f, m' \rangle$ is not w-opt. So there is a $f' < f$ such that $\langle f', m' \rangle$ is w-opt. $f' \in \min(F_l) withl < i$. By i.h. $l = k$. So $k < i$. Contradiction.

So $k = i$.

$\Leftarrow$ Same proof. Swap all m and f.

$\Leftarrow$ $f \in \min(F_i)$ and $m \in \min(M_i)$ for some $i \Rightarrow \langle f, m \rangle$ is w-opt.
Take $f \in \min(F_i)$ and $m \in \min(M_i)$. if $\langle f, m \rangle$ isn't w-opt. Then either:

- There is $f' < f$ such that $\langle f', m \rangle$ is w-opt. This implies that $f' \in \min(F_j)$ for some $j < i$ but according to the first part of the proof $m \in \min(M_j)$. Which is a direct contradiction.
- Or, there is $m' < m$ such that $\langle f, m' \rangle$ is w-opt. Same proof yields contradiction.

So $\langle f, m \rangle$ is w-opt.

Corollary 1. *Let F and M be well founded orders. Take the system WO with only the constraints $<_F$ and $<_M$. Take the system SO with only the constraint $\lambda_F(f) = \lambda_M(m)$. The pair $\langle f, m \rangle$ is* weakly *optimal in WO* iff *it is* strongly *optimal in SO.*

This reduction can not be used to simply reduce any weakly optimal system. Most systems will make use of faithfulness constraints. It does however formalize and confirm the intuition of DoPL. The fact that markedness implies markedness is heavily used within Aloni's system and in the system proposed in this paper. The above proof should be seen as an argument for why this is reasonable.

B.2 The BLOCK constraint

There is another way to analyze weak optimality in terms of string optimality. In [3], Beaver suggests a constraint $BLOCK$ that simulates the behavior of weak optimality in a strong optimality system.

Take **CON** to be a sequence of ranked constraints that results in the order $<_{CON}$. Now define a new constraint $BLOCK$.

Definition $\langle f, m\rangle$ violates $BLOCK$ *iff*

- there is a $\langle f', m\rangle <_{CON} \langle f, m\rangle$ such that $\langle f', m\rangle$ does not violate $BLOCK$
- or there is a $\langle f, m'\rangle <_{CON} \langle f, m\rangle$ such that $\langle f, m'\rangle$ does not violate $BLOCK$

If we now take $\mathbf{CON} + BLOCK$ to be the sequence of constraints of **CON** but with $BLOCK$ as additional (and highest ranked) constraint, it is easy to see that:

Lemma 1. $\langle f, m\rangle$ *is* weakly optimal *with respect to* **CON** iff *it is* strongly optimal *with respect to* $\mathbf{CON} + BLOCK$

For completeness sake the proof for this can be found here below. But the point I wanted to make is that this reduction is correct but it doesn't help us much. We only moved the complexity of weak optimality from one place to another.

Proof Take **CON** to be a sequence of ranked constraints that results in the order $<_{CON}$. Define a new constraint $BLOCK$.

Definition $\langle f, m\rangle$ violates $BLOCK$ *iff*

- there is a $\langle f', m\rangle <_{CON} \langle f, m\rangle$ such that $\langle f', m\rangle$ does not violate $BLOCK$
- or there is a $\langle f, m'\rangle <_{CON} \langle f, m\rangle$ such that $\langle f, m'\rangle$ does not violate $BLOCK$

If we now take $\mathbf{CON} + BLOCK$ to be the sequence of constraints of **CON** but with $BLOCK$ as additional (and highest ranked) constraint, it is easy to see that:

Lemma 2. $\langle f, m\rangle$ *is* weakly optimal *with respect to* **CON** iff *it is* strongly optimal *with respect to* $\mathbf{CON} + BLOCK$

Let $<_{CON}$ be the resulting order for the constraints **CON** and $<_{CON+}$ the resulting order for the constraints $\mathbf{CON}+BLOCK$. $<_{CON}$ and $<_{CON+}$ are well-orders.

Definition Define $\langle S, <\rangle_n$ recursively as:

- $\langle S, <\rangle_0$ is the set of minimal elements of S
- $\langle S, <\rangle_{n+1}$ is $\langle S, <\rangle_n \cup$ the set of minimal elements of $S - \langle S, <\rangle_n$

This is used for the induction in the proof. $\langle S, <\rangle_0$ are the smallest elements of S,$\langle S, <\rangle_0$ are the smallest and second smallest elements of S, etc...

To show: $\langle f, m\rangle \in GEN$ is weakly optimal with respect to **CON** *iff* $\langle f, m\rangle$ is *not* marked by $BLOCK$ in $\mathbf{CON} + BLOCK$. By induction we'll show that for all n: $\langle f, m\rangle \in \langle GEN, <_{CON}\rangle_n$ is weakly optimal with respect to **CON** *iff* it is not marked by $BLOCK$.

Proof. by induction on n:

- Basis: $\langle f, m\rangle \in \langle GEN, <_{CON}\rangle_0$ is w-opt *iff* $\langle f, m\rangle$ is not marked by $BLOCK$.

  Proof. By definition all $\langle f, m\rangle \in \langle GEN, <_{CON}\rangle_0$ are w-opt. Also by definition all $\langle f, m\rangle \in \langle GEN, <_{CON}\rangle_0$ are not marked by $BLOCK$. So this trivially holds.

- Induction: Suppose all $\langle f, m\rangle \in \langle GEN, <_{CON}\rangle_n$ is w-opt *iff* $\langle f, m\rangle$ is not marked by $BLOCK$. To show: $\langle f', m'\rangle \in \langle GEN, <_{CON}\rangle_{n+1}$ is w-opt *iff* $\langle f', m'\rangle$ is not marked by $BLOCK$

  Proof. $\Rightarrow$ Suppose $\langle f', m'\rangle \in \langle GEN, <_{CON}\rangle_{n+1}$ is w-opt. So for all $\langle f'', m'\rangle < \langle f', m'\rangle$ it holds that $\langle f'', m'\rangle$ is not w-opt. Similarly for all $\langle f', m''\rangle$. By induction hypothesis we get that all $\langle f'', m'\rangle < \langle f', m'\rangle$ and $\langle f', m''\rangle < \langle f', m'\rangle$ are marked by $BLOCK$. Thus by the definition of $BLOCK$, $\langle f', m'\rangle$ is not marked by $BLOCK$.

  $\Leftarrow$ Suppose $\langle f, m\rangle$ is not marked by $BLOCK$. Then for all $\langle f'', m'\rangle < \langle f', m'\rangle$ it holds that $\langle f'', m'\rangle$ is marked by $BLOCK$. Similarly for all $\langle f', m''\rangle$. By induction hypothesis we get that all

$\langle f'', m' \rangle < \langle f', m' \rangle$ and $\langle f', m'' \rangle < \langle f', m' \rangle$ aren't w-opt. Thus by the definition of weak optimality, $\langle f', m' \rangle$ is weakly optimal.

C Behavior competence

The competence ordering behaves more or less like the subset relation. This appendix formalizes this notion of 'more or less like the subset relation'. Remember that the definition of competence:

Definition $C < C'$ iff $Q_C = Q'_C \wedge \forall\phi : C \models \heartsuit?\phi : C' \models (\Diamond\phi \to \Box\phi) \Rightarrow C \models (\Diamond\phi \to \Box\phi)$

Two contexts are only comparable if they have the same issue. To make analysis easier I'll fix the issue Q and only talk about the ordering on the models (sets of possible worlds). For a fixed issue Q, $m' < m$ denotes $\langle m', Q\rangle <_C \langle m, Q\rangle$.

We suppose that an issue is determined by a formula definable in the update semantics.

Definition For all Q it holds that there is a sentence ξ such that $Q = W^2[\xi]$

This definition yields that every equivalence class has a determining sentence (a sentence that holds for all worlds in that equivalence class and in no other world):

Lemma 1. *For issue Q and any world w there is a formula $sup_Q(w)$ such that*

$$\forall w' : w' \models sup_Q(w) \Leftrightarrow w'Qw$$

Proof. We can simply recursively construct $sup_Q(w)$. Let Q be an issue such that $Q = W^2[\xi]$ and w be any world $w \in W$.

If $\xi = ?\phi$ then define $sup_Q(w)$ as follows:

If $w \models \phi$ then $sup_Q(w) = \phi$.

If $w \models \neg\phi$ then $sup_Q(w) = \neg\phi$.

- If $\xi = \psi \wedge ?\phi$ and for we already have a $sup'_Q(w)$ for all w and $Q' = W^2[\psi]$ then define $sup_Q(w)$ as follows:

  If $w \models \phi$ the $sup_Q(w) = sub'_Q(w) \wedge \phi$.

  If $w \models \neg\phi$ the $sup_Q(w) = sub'_Q(w) \wedge \neg\phi$.

$\xi = \neg?\phi$ and $\xi = \Box?\phi$ reduce to $\xi = ?\phi$.

Another property that follows from this definition is that the truth value of every relevant sentence is determined by the $sup_Q(w)$ formulas.

Lemma 2. *For any Q, ϕ and w if $Q \models \heartsuit ?\phi$ then either $sup_Q(w) \models \phi$ or $sup_Q(w) \models \neg\phi$.*

Proof. Take ϕ and w such that $sup_Q(w) \not\models \phi$ and $sup_Q(w) \not\models \neg\phi$. Then there is a world w_1 and a world w_2 such that $w_1 \models sup_Q(w)$ and $w_2 \models sup_Q(w)$, but $w_1 \models \phi$ and $w_2 \models \neg\phi$. Since $w_1 \models sup_Q(w)$ we have that $w_1 Q w$ and since $w_2 \models sup_Q(w)$ we have that $w_1 Q w$ (as a consequence of lemma 1). Q is an equivalence relation, so $w_1 Q w_2$. Since $w_1 \models \phi$ and $w_2 \models \neg\phi$, the update $Q[?\phi] = Q'$ yields that $Q' \neq Q$. So $Q \not\models \heartsuit ?\phi$

Definition [Subset relative to Q] $m \subseteq_Q m'$ iff $\forall w \in m : \exists w' \in m' : w' Q w$

Note that if Q is taken to be the identity relation then $\subseteq_Q$ is indeed a normal subset relation.

Definition [Cardinality relative to Q] $|m|_Q = |m/Q|$ (i.e. the number of equivalence classes in m with respect to Q).

Note that if Q is taken to be the identity relation then $|m|_Q = |m|$.

With the above definition we can formalize the notion of 'a more or less subset relation:

Theorem 3. *$m <_Q m'$ iff $|m'|_Q > 1$ and ($|m|_Q = 1$ or $m \subset_Q m'$)*

The following three lemmas will that the competence ordering behaves as such.

Lemma 4. *If $|m|_Q = 1$ then $\forall m' : m \leq_C m'$*

Proof.

If $|m|_Q = 1$ then for all $w, w' \in m$ $w' Q w$.

So, for all $w \in m$ we have $\forall w' \in m : w' \models sup_Q(w)$.

Take ϕ such that $Q \models \heartsuit ?\phi$

Then $sup_Q(w) \models \phi$ or $sup_Q(w) \models \neg\phi$ (lemma 2).

Suppose $m \models \diamond\phi$ then there is a $w' \in m : w' \models \phi$

Thus, $sup_Q(w) \models \phi$

Since $\forall w' \in m : w' \models sup_Q(w)$ so $\forall w' \in m : w' \models \phi$, so $m \models \Box\phi$

So for all ϕ such that $Q \models \heartsuit?\phi$: $m \models \Diamond\phi \rightarrow \Box\phi$.

Lemma 5. *If $m \subseteq_Q m'$ then $m \leq_C m'$.*

Proof.

Take $m \subseteq_Q m'$ (i.e. $\forall w \in m : \exists w' \in m' : w'Qw$) and ϕ such that $Q \models \heartsuit?\phi$ and $m' \models \Box\phi$

To show: $m \models \Box\phi \vee \neg\Box\phi$

1. By counter position. Suppose $m \not\models \phi$ & $m \not\models \neg\phi$
2. Then there's a $w \in m$ such that $w \models \neg\phi$
3. We had that $m \subseteq_Q m'$, So there is a $w' \in m'$ such that $w'Qw$.
4. $w' \models \neg\phi$ (since $Q \models \heartsuit?\phi$).
5. So $m' \not\models \Box\phi$. Contradiction.

So $m \models \Box\phi \vee \neg\Box\phi$.

Lemma 6. *If $|m|_Q > 1$, $|m'|_Q > 1$ and $m <_C m'$ then $m \subset_Q m'$*

Proof.

Take $|m|_Q > 1$ and $|m'|_Q > 1$ and $m \not\subset_Q m'$ (i.e. $\exists w \in m : \neg\exists w' \in m' : w'Qw$)

To show: $\exists\phi$ such that $m' \models \Diamond\phi \rightarrow \Box\phi$ and $m \models \Diamond\phi$ and $m \models \neg\Box\phi$.

1. Take $w \in m$ such that $\neg\exists w' \in m' : w'Qw$.
2. $\forall w'' : w'' \models sup_Q(w) \rightarrow w''Qw$ (lemma 1)
3. (1+2) $\forall w' \in m' : w' \models \neg sup_Q(w)$
4. $m' \models \Diamond\neg sup_Q(w) \rightarrow \Box\neg sup_Q(w)$
5. Since $|m|_Q > 1$ there is a $w''' \in m$ such that $\neg w'''Qw$

6. $w''' \models \neg sup_Q(w)$ (lemma 1)
7. $m \models \Diamond \neg sup_Q(w)$
8. $w \models sup_Q(w)$ so $m \models \neg \Box \neg sup_Q(w)$

So $m \not<_C m'$.

Lemma 4, 5 and 6 together form the proof of theorem 3, the claim that:

$$m <_Q m' \text{ iff } |m'|_Q > 1 \text{ and } (|m|_Q = 1 \text{ or } m \subset_Q m')$$

This can make further analysis much easier. We're mostly interested in issues that consists of a conjunct of atoms (like $Q_{?A?B...?N}$). If we confine ourselves to the models that contain worlds with only relevant atoms, then Q becomes the identity relation.

Consquence 7. *If we consider only those worlds such that Q is the identity relation (wQw' iff $w = w'$) then $m < m'$ iff $|m'| > 1$ and ($|m| = 1$ or $m \subset m'$).*

So for the examples we're studying the competence ordering over the model can be more or less regarded as the subset relation.