VARIETIES OF TWO-DIMENSIONAL CYLINDRIC ALGEBRAS. PART II

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Abstract

In [2] we investigated the lattice $\Lambda(\mathbf{Df}_2)$ of all subvarieties of the variety $\mathbf{Df}_2$ of two-dimensional diagonal free cylindric algebras. In the present paper we investigate the lattice $\Lambda(\mathbf{CA}_2)$ of all subvarieties of the variety $\mathbf{CA}_2$ of two-dimensional cylindric algebras. We give a dual characterization of representable two-dimensional cylindric algebras, prove that the cardinality of $\Lambda(\mathbf{CA}_2)$ is that of continuum, give a criterion for a subvariety of $\mathbf{CA}_2$ to be locally finite, and describe the only pre locally finite subvariety of $\mathbf{CA}_2$. We also characterize finitely generated subvarieties of $\mathbf{CA}_2$ by describing all fifteen pre finitely generated subvarieties of $\mathbf{CA}_2$. Finally, we give a rough picture of $\Lambda(\mathbf{CA}_2)$, and investigate algebraic properties preserved and reflected by the reduct functors $\mathbb{F} : \mathbf{CA}_2 \to \mathbf{Df}_2$ and $\Phi : \Lambda(\mathbf{CA}_2) \to \Lambda(\mathbf{Df}_2)$.

1 Introduction

This paper is a sequel to [2] and in it we investigate the lattice $\Lambda(\mathbf{CA}_2)$ of all subvarieties of the variety $\mathbf{CA}_2$ of two-dimensional cylindric algebras. The variety $\mathbf{CA}_2$ is widely studied in the literature. One of the main references is the fundamental work by Henkin, Monk, and Tarski [5]. Among many other things it is well known that unlike the variety $\mathbf{Df}_2$ of two-dimensional diagonal free cylindric algebras, not every member of $\mathbf{CA}_2$ is representable, that the representable members of $\mathbf{CA}_2$ form a proper subvariety of $\mathbf{CA}_2$, usually denoted by $\mathbf{RCA}_2$, that the equational theories of both $\mathbf{CA}_2$ and $\mathbf{RCA}_2$ are decidable, and that both $\mathbf{CA}_2$ and $\mathbf{RCA}_2$ are finitely approximable, but that neither of them is locally finite. To these results we add a criterion for a variety of two-dimensional cylindric algebras to be locally finite, a characterization of finitely generated and pre finitely generated varieties of two-dimensional cylindric algebras, and a rough description of the lattice $\Lambda(\mathbf{CA}_2)$.

The paper is organized as follows. Section 2 has a preliminary purpose and it contains all the information on $\mathbf{Df}_2$ and $\mathbf{CA}_2$ needed in subsequent sections. In Section 3 we characterize representable two-dimensional cylindric algebras. In Section 4 we show that there exists a continuum of subvarieties of $\mathbf{RCA}_2$, and that there exists a continuum of varieties in between $\mathbf{RCA}_2$ and $\mathbf{CA}_2$. In Section 5 we describe the only pre locally finite subvariety of $\mathbf{CA}_2$, and characterize locally finite varieties of two-dimensional cylindric algebras. In Section 6 we characterize finitely generated subvarieties of $\mathbf{CA}_2$ by describing all fifteen pre finitely generated subvarieties of $\mathbf{CA}_2$. Finally, in Section 7 we give a rough picture of the lattice structure of $\Lambda(\mathbf{CA}_2)$, define the reduct functors $\mathbb{F} : \mathbf{CA}_2 \to \mathbf{Df}_2$ and $\Phi : \Lambda(\mathbf{CA}_2) \to \Lambda(\mathbf{Df}_2)$, and investigate algebraic properties preserved and reflected by $\mathbb{F}$ and Φ.

2 Preliminaries

2.1 $\mathbf{Df}_2$

In this subsection we recall all the needed information about two-dimensional diagonal free cylindric algebras which will be used subsequently. For references consult [5] and [2].

Definition 2.1. *Suppose* $(B, \wedge, \vee, -, 0, 1)$ *is a Boolean algebra. A unary operation* $\exists : B \to B$ *is called a* monadic operator *on* B *if the following three conditions are satisfied for all* $a, b \in B$*:*

$\exists 0 = 0$;

$a \leq \exists a$;

$\exists(\exists a \wedge b) = \exists a \wedge \exists b$.

Definition 2.2. *A triple* $(B, \exists_1, \exists_2)$ *is called a* two-dimensional diagonal-free cylindric algebra*, or a* $\mathbf{Df}_2$-algebra *for short, if* B *is a Boolean algebra, and* $\exists_1, \exists_2$ *are monadic operators on* B *satisfying the following condition for all* $a \in B$*:*

$$\exists_1 \exists_2 a = \exists_2 \exists_1 a.$$

The variety of two-dimensional diagonal-free cylindric algebras is denoted by $\mathbf{Df}_2$.

Suppose X is a nonempty set, R is a binary relation on X, $x \in X$ and $A \subseteq X$. Let

$R(x) = \{y \in X : xRy\}$,

$R^{-1}(x) = \{y \in X : yRx\}$,

$R(A) = \bigcup_{x \in A} R(x)$,

$R^{-1}(A) = \bigcup_{x \in A} R^{-1}(x)$.

We call $R(x)$ the *R-saturation* of x, and $R(A)$ - the *R-saturation* of A. Note that if R is an equivalence relation, then $R(x) = R^{-1}(x)$ and $R(A) = R^{-1}(A)$.

Recall that a topological space X is called a *Stone space* if X is 0-dimensional, compact, and Hausdorff. A subset A of X is called a *clopen* subset of X if it is simultaneously a closed and an open subset of X. Denote by $CP(X)$ the Boolean algebra of all clopen subsets of a Stone space X. A relation R on a Stone space X is said to be a *clopen* relation if $A \in CP(X)$ implies $R^{-1}(A) \in CP(X)$. We call R *point-closed* if $R(x)$ is a closed subset of X for every $x \in X$.

Definition 2.3. *A triple* (X, E_1, E_2) *is said to be a* $\mathbf{Df}_2$-space *if* X *is a Stone space and* E_1 *and* E_2 *are point-closed and clopen equivalence relations on* X *with* $E_1E_2(x) = E_2E_1(x)$ *for every* $x \in X$.

Given two $\mathbf{Df}_2$-spaces (X, E_1, E_2) and (X', E_1', E_2'), a function $f : X \to X'$ is said to be a $\mathbf{Df}_2$*-morphism* if f is continuous and $fE_i(x) = E_i'f(x)$ for every $x \in X$, $i = 1, 2$. We denote the category of $\mathbf{Df}_2$-spaces and $\mathbf{Df}_2$-morphisms by $\mathbf{DS}$. Then we have the following representation of $\mathbf{Df}_2$-algebras:

Theorem 2.4. $\mathbf{Df}_2$ *is dual to* $\mathbf{DS}$. *In particular, every* $\mathbf{Df}_2$*-algebra can be represented as* $(CP(X), E_1, E_2)$ *for the corresponding* $\mathbf{Df}_2$*-space* (X, E_1, E_2).

For a $\mathbf{Df}_2$-space (X, E_1, E_2), let $E_0 = E_1 \cap E_2$. It is routine to check that E_0 is an equivalence relation on X. Call the sets of the form $E_i(x)$ *E_i-clusters* $(i = 0, 1, 2)$. A subset A of X is called *saturated* if $E_1E_2(A) = A$. A $\mathbf{Df}_2$-space (X, E_1, E_2) is called a *component* if $E_1E_2(x) = X$ for each $x \in X$. A partition R of X is called *correct* if

(1) From $x \not R y$ it follows that there exists a R-saturated clopen A such that $x \in A$ and $y \notin A$,

(2) $RE_i(x) \subseteq E_i R(x)$ for every $x \in X$ and $i = 1, 2$.

Then we have the following dual characterization of congruences and subalgebras of $\mathbf{Df}_2$-algebras, as well as subdirectly irreducible and simple $\mathbf{Df}_2$-algebras.

Theorem 2.5. *1. The lattice of congruences of a $\mathbf{Df}_2$-algebra $(B, \exists_1, \exists_2)$ is isomorphic to the lattice of open saturated subsets of its dual (X, E_1, E_2).*

2. The lattice of subalgebras of $(B, \exists_1, \exists_2) \in \mathbf{Df}_2$ is dually isomorphic to the lattice of correct partitions of its dual (X, E_1, E_2).

3. $(B, \exists_1, \exists_2) \in \mathbf{Df}_2$ is subdirectly irreducible iff $(B, \exists_1, \exists_2)$ is simple iff its dual (X, E_1, E_2) is a component.

2.2 $\mathbf{CA}_2$

Definition 2.6. *[5] A quadruple $(B, \exists_1, \exists_2, d)$ is said to be a* two-dimensional cylindric algebra*, or a* $\mathbf{CA}_2$-algebra *for short, if $(B, \exists_1, \exists_2)$ is a $\mathbf{Df}_2$-algebra and $d \in B$ is a constant satisfying the following conditions for all $a \in B$ and $i = 1, 2$.*

$$\exists_i(d) = 1;$$

$$\exists_i(d \wedge a) = -\exists_i(d \wedge -a).$$

Denote the variety of all two-dimensional cylindric algebras by $\mathbf{CA}_2$.

Since in this paper we only deal with two-dimensional cylindric algebras, we will simply call them cylindric algebras. Below we will generalize the duality for $\mathbf{Df}_2$-algebras to $\mathbf{CA}_2$-algebras.

Definition 2.7. *A quadruple (X, E_1, E_2, D) is said to be a* cylindric space *if the triple (X, E_1, E_2) is a $\mathbf{Df}_2$-space and D is a clopen subset of X such that every E_i-cluster of X contains a unique point from D for $i = 1, 2$.*

A routine consequence of this definition is the following proposition.

Proposition 2.8. *Suppose $\mathcal{X}$ is a cylindric space. Then the cardinality of the set of all E_1-clusters of $\mathcal{X}$ is equal to the cardinality of the set of all E_2-clusters of $\mathcal{X}$.*

Proof. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the sets of all E_1- and E_2-clusters of $\mathcal{X}$, respectively. Define $f : \mathcal{E}_1 \to \mathcal{E}_2$ by putting $f(C) = E_2(C \cap D)$. Suppose $C_1, C_2 \in \mathcal{E}_1$, $C_1 \neq C_2$, $C_1 \cap D = \{x\}$, and $C_2 \cap D = \{y\}$. Since every E_i-cluster of $\mathcal{X}$ contains a unique point from D, it follows that $f(C_1) = E_2(x) \neq E_2(y) = f(C_2)$. Therefore, f is an injection. Now suppose $C' \in \mathcal{E}_2$ and $C' \cap D = \{x\}$. If we let $C = E_1(x)$, then $f(C) = E_2(x) = C'$. Thus, f is a surjection. Hence, we obtain that f is a bijection. $\square$

Given two cylindric spaces (X, E_1, E_2, D) and (X', E_1', E_2', D'), a function $f : X \to X'$ is said to be a *cylindric morphism* if f is a $\mathbf{Df}_2$-morphism and $f^{-1}(D') = D$. We denote the category of cylindric spaces and cylindric morphisms by $\mathbf{CS}$. Then we have the following representation of cylindric algebras:

Theorem 2.9. $\mathbf{CA}_2$ *is dual to* $\mathbf{CS}$. *In particular, every cylindric algebra $\mathcal{B} = (B, \exists_1, \exists_2, d)$ can be represented as $(CP(X), E_1, E_2, D)$ for the corresponding cylindric space $\mathcal{X} = (X, E_1, E_2, D)$.*

Proof. Everything goes similarly to the $\mathbf{Df}_2$-case. The only additional fact that has to be verified is the following. In the dual space $\mathcal{X}$ of $\mathcal{B}$, every E_i-cluster contains a unique point from D for $i = 1, 2$; and conversely, in every cylindric space we have $E_i(D) = X$ and $E_i(D \cap A) = -E_i(D \cap -A)$ for $i = 1, 2$ and $A \in CP(X)$.

To show the latter claim, suppose $\mathcal{X}$ is a cylindric space. Since D intersects each E_i-cluster of $\mathcal{X}$, it is obvious that $E_i(D) = X$ for $i = 1, 2$. Moreover, if $x \in E_i(D \cap A)$, then there exists $y \in X$ such that xE_iy and $y \in D \cap A$. Since $E_i(x)$ contains a unique point from D, for any $z \in E_i(x)$ we have either $z \notin D$ or $z = y$, and so $z \in A$. In either case, $E_i(x) \cap (D \cap -A) = \varnothing$. Therefore, $x \in -E_i(D \cap -A)$, and so $E_i(D \cap A) \subseteq -E_i(D \cap -A)$. The converse inclusion is proved similarly.

To show the former claim, suppose $\mathcal{X}$ is the dual space of $\mathcal{B}$. Then $E_i(D) = X$ and $E_i(D \cap A) = -E_i(D \cap -A)$ for each $i = 1, 2$ and $A \in CP(X)$. Since $E_i(D) = X$, every E_i-cluster of $\mathcal{X}$ contains at least one point from D for $i = 1, 2$. Suppose there exists an E_i-cluster C such that $C \cap D$ contains at least two different points, say x and y. Since X is a Stone space, there exists a clopen A such that $x \in A$ and $y \notin A$. Therefore, $x \in D \cap A$, and so $C \subseteq E_i(D \cap A)$. On the other hand, since $y \in D \cap -A$, we have $C \subseteq E_i(D \cap -A)$. But then $C \cap -E_i(D \cap -A) = \emptyset$, which contradicts the fact that $E_i(D \cap A) = -E_i(D \cap -A)$ for each $i = 1, 2$ and $A \in CP(X)$. Thus, every E_i-cluster of $\mathcal{X}$ contains a unique point from D for $i = 1, 2$. $\square$

As an easy corollary of Theorem 2.9 we obtain that the category Fin$\mathbf{CA}_2$ of finite cylindric algebras is dual to the category Fin$\mathbf{CS}$ of finite cylindric spaces with the discrete topology. In particular, every finite cylindric algebra is represented as the algebra $(P(X), E_1, E_2, D)$ for the corresponding finite cylindric space (X, E_1, E_2, D), where $P(X)$ denotes the power set of X.

To obtain the dual description of homomorphic images and subalgebras of cylindric algebras, as well as subdirectly irreducible and simple cylindric algebras, we need the following two definitions. Suppose $\mathcal{X}$ is a cylindric space. A correct partition R of X is called a *cylindric partition* if $R(D) = D$. A cylindric space $\mathcal{X}$ is called a *quasi-square* if $E_1E_2(x) = X$ for every $x \in X$.

Theorem 2.10.

1. *The lattice of congruences of a cylindric algebra $\mathcal{B}$ is isomorphic to the lattice of open saturated subsets of its dual $\mathcal{X}$.*
2. *The lattice of subalgebras of a cylindric algebra $\mathcal{B}$ is dually isomorphic to the lattice of cylindric partitions of its dual $\mathcal{X}$.*
3. *A cylindric algebra $\mathcal{B}$ is subdirectly irreducible iff it is simple iff its dual $\mathcal{X}$ is a quasi-square.*

Proof. A routine adaptation of Theorem 2.5 to cylindric algebras. $\square$

Corollary 2.11. *([5, Theorem 2.4.43])*

1. $\mathbf{CA}_2$ *is semi-simple.*
2. $\mathbf{CA}_2$ *is congruence-distributive.*
3. $\mathbf{CA}_2$ *has the congruence extension property.*

Now define the reduct functor $\mathbb{F} : \mathbf{CA}_2 \to \mathbf{Df}_2$ by putting

$$\mathbb{F}(B, \exists_1, \exists_2, d) = (B, \exists_1, \exists_2).$$

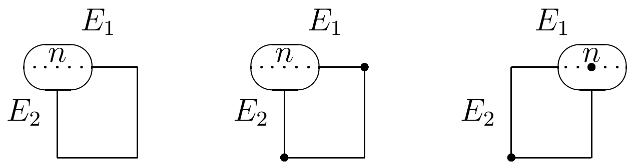
In other words, $\mathbb{F}$ forgets the diagonal element d from the signature of cylindric algebras. We are in a position to show that $\mathbb{F}$ is not onto. In fact, the set $\mathbf{Df}_2 - \mathbb{F}(\mathbf{CA}_2)$ is infinite.

For this, define the reduct functor $\mathbb{R} : \mathbf{CS} \to \mathbf{DS}$ by putting

$$\mathbb{R}(X, E_1, E_2, D) = (X, E_1, E_2).$$

Suppose $(Y, E_1, E_2) \in \mathbf{DS}$ is a component. Call (Y, E_1, E_2) a *quasi-square* if the cardinality of the sets of all E_1- and E_2-clusters coincide with each other. Note that not every component from $\mathbf{DS}$ is a quasi-square. Examples of components which are not quasi-squares are rectangle $\mathbf{Df}_2$-spaces (consult [2] for the definition of a rectangle). Since there are infinitely many rectangle $\mathbf{Df}_2$-spaces, the set $\mathbf{DS} - \mathbb{R}(\mathbf{CS})$ is infinite.

Now call a $\mathbf{Df}_2$-algebra a *quasi-square algebra* if its dual space is a quasi-square. As follows from Proposition 2.8 and Theorem 2.10, for every simple cylindric algebra $\mathcal{B}$, its $\mathbf{Df}_2$-reduct is a quasi-square algebra. Therefore, the set $\mathbf{Df}_2 - \mathbb{F}(\mathbf{CA}_2)$ is infinite. Moreover, one $\mathbf{Df}_2$-algebra can be the reduct of many non-isomorphic cylindric algebras. For instance, a $\mathbf{Df}_2$-algebra whose dual space is shown in Fig.1(a) is the reduct of the cylindric algebras whose dual cylindric spaces are shown in Fig.1(b),(c) where bold points represent the diagonal points.



(a) (b) (c)

Fig.1

3 Representable cylindric algebras

For any cardinal κ, define on the cartesian square $\kappa \times \kappa$ two equivalence relations E_1 and E_2 by putting

$$(i_1, i_2)E_1(j_1, j_2) \text{ iff } i_2 = j_2,$$

$$(i_1, i_2)E_2(j_1, j_2) \text{ iff } i_1 = j_1,$$

for $i_1, i_2, j_1, j_2 \in \kappa$. Let also $D = \{(i, i) : i \in \kappa\}$ and call $(\kappa \times \kappa, E_1, E_2, D)$ a *square*. Obviously $(P(\kappa \times \kappa), E_1, E_2, D)$ is a cylindric algebra, which we call a *square algebra*. Denote the class of all square algebras by $\mathbf{Sq}$.

Definition 3.1. *[5] A cylindric algebra $\mathcal{B}$ is called* representable *if $\mathcal{B} \in$ $\mathbf{SP}(\mathbf{Sq})$, where $\mathbf{S}$ and $\mathbf{P}$ denote the operations of taking subalgebras and direct products, respectively.*

It is known that the class of representable cylindric algebras is also closed under homomorphic images, and so forms a variety which is usually denoted by $\mathbf{RCA}_2$. It is known that $\mathbf{RCA}_2$ is a proper subvariety of $\mathbf{CA}_2$, that $\mathbf{RCA}_2$ is generated by finite square algebras, and that $\mathbf{RCA}_2$ can be axiomatized by adding the following *Henkin axioms* to the axiom system of $\mathbf{CA}_2$ (see [5]):

$$(H) \qquad \exists_i(a \wedge -b \wedge \exists_j(a \wedge b)) \leq \exists_j(-d \wedge \exists_i a), \quad i \neq j, \quad i,j = 1,2.$$

In [10] Venema has simplified these equations to the following ones:

$$(V) \qquad d \wedge \exists_i(-a \wedge \exists_j a) \leq \exists_j(-d \wedge \exists_i a), \quad i \neq j, \quad i,j = 1,2.$$

Below we will give the dual characterization of representable cylindric algebras, which will allow us to construct rather simple finite non-representable cylindric algebras.

Suppose (X, E_1, E_2, D) is a cylindric space. Call $x \in D$ a *diagonal point*, and $x \in X - D$ a *non-diagonal point*. Also call an E_0-cluster C a *diagonal E_0-cluster* if it contains a diagonal point. Otherwise call C a *non-diagonal E_0-cluster*.

Lemma 3.2. *Let $\mathcal{X}$ be a cylindric space. If a diagonal point $x \in D$ is not an isolated point, then $E_0(x) \neq \{x\}$.*

Proof. Suppose $x \in D$ is not an isolated point. Then x is a limit point, and so there exists a sequence $\{x_i\}_{i\in\omega}$ converging to x. Since D is a clopen, we can assume that each x_i belongs to D. Moreover, since $\{x_i\}_{i\in\omega}$ converges to x, we can assume without loss of generality that $E_1(x_i) \cap E_2(x) \neq \varnothing$ for every x_i. Let $y_i \in E_1(x_i) \cap E_2(x)$. Since X is compact, $\{y_i\}_{i\in\omega}$ converges to some point $y \in X$. Moreover, $y \in E_2(x)$ because $\{y_i\}_{i\in\omega} \subseteq E_2(x)$ and $E_2(x)$ is closed. Since D contains a unique point from every E_i-cluster, we have that $\{y_i\}_{i\in\omega} \subseteq -D$. But then $y \in -D$ because $-D$ is a clopen. Therefore, $y \neq x$. Let $E_1(y) \cap D = \{z\}$. If $z \neq x$, then there exists a clopen $A \subseteq D$ such that $z \in A$ and all but finitely many members of $\{x_i\}_{i\in\omega}$ do not belong to A. But then all but finitely many members of $\{y_i\}_{i\in\omega}$ do not belong to a clopen $E_1(A)$, which is impossible since $y \in E_1(A)$ and y is the limit of $\{y_i\}_{i\in\omega}$. Thus, $z = x$, implying that yE_1x. Therefore, $y \in E_0(x)$, and so $E_0(x) \neq \{x\}$. $\square$

Definition 3.3. *A cylindric space $\mathcal{X}$ is said to satisfy $(*)$ if there exists a diagonal point $x_0 \in D$ such that $E_0(x_0) = \{x_0\}$ and there exists a non-singleton E_0-cluster C which is either E_1- or E_2-related to x_0.*

Theorem 3.4. *A cylindric algebra $\mathcal{B}$ is representable iff its dual cylindric space $\mathcal{X}$ does not satusfy $(*)$.*

Proof. Suppose $\mathcal{X}$ satisfies $(*)$. We show that (V) does not hold in $\mathcal{B}$, implying that $\mathcal{B}$ is not representable. Let x_0 be a diagonal point with $E_0(x_0) = \{x_0\}$ and C be a non-singleton E_0-cluster say E_1-related to x_0 (the case when C is E_2-related to x_0 is proved similarly). It follows from Lemma 3.2 that x_0 is an isolated point. Therefore, $E_1(x_0)$ is a clopen. Choose two different points y and z from C, and consider an open set $E_1(x_0) - \{x_0, y\}$. Let $A \subseteq E_1(x_0) - \{x_0, y\}$ be a clopen containing z. Then $y \in -A \cap E_2(A)$, and so $x_0 \in D \cap E_1(-A \cap E_2(A))$. On the other hand, $E_1(A) = E_1(x_0)$. Therefore, $x_0 \notin E_2(-D \cap E_1(A))$, implying that (V) does not hold in $\mathcal{B}$. Thus, $\mathcal{B}$ is not representable.

Conversely, suppose $\mathcal{B}$ is not representable. We show that $(*)$ holds in $\mathcal{X}$. We know that (V) does not hold in $\mathcal{B}$. Therefore, there exist a point $x \in X$ and a clopen $A \subseteq X$ such that $x \in D \cap E_i(-A \cap E_j(A))$ but $x \notin E_j(-D \cap E_i(A))$, for $i \neq j$ and $i, j = 1, 2$. Since $x \in D \cap E_i(-A \cap E_j(A))$, then $x \in D$ and there exist points $y, z \in X$ such that xE_iy, yE_jz, $y \notin A$ and $z \in A$. From $y \notin A$ and $z \in A$ it follows that y and z are different. Also xE_iy and yE_jz imply that there exists a point $u \in X$ such that xE_ju and uE_iz. If $u \neq x$, then u is a non-diagonal point, and so $u \in -D \cap E_i(A)$. But then $x \in E_j(-D \cap E_i(A))$, which contradicts our assumption. Thus, $u = x$ and xE_iz. Therefore, yE_0z and both y and z are E_i-related to x. Moreover, if $E_0(x) \neq \{x\}$, then by choosing a point $u \in E_0(x)$ different from x we obtain again that $u \in -D \cap E_i(A)$, and so $x \in E_j(-D \cap E_1(A))$, which is impossible. Therefore, $E_0(x) = \{x\}$ and $E_0(y)$ is a non-singleton E_0-cluster E_i-related to x_0. Thus, $(*)$ holds in $\mathcal{X}$. □

Using this criterion it is easy to see that the cylindric algebras corresponding to the cylindric spaces shown in Fig.1(c) (bold points denote the diagonal points) are representable, while the cylindric algebras corresponding to the cylindric spaces shown in Fig.1(b) are not. Moreover, the smallest non-representable cylindric algebra is the algebra corresponding to the cylindric space shown in Fig.1(b), where the non-singleton E_0-cluster contains only two points.

4 Cardinality of $\Lambda(\mathbf{CA}_2)$

Denote the lattice of subvarieties of $\mathbf{CA}_2$ by $\Lambda(\mathbf{CA}_2)$, and the lattice of subvarieties of $\mathbf{RCA}_2$ by $\Lambda(\mathbf{RCA}_2)$. We want to show that the cardinality of $\Lambda(\mathbf{RCA}_2)$ as well as the cardinality of $\Lambda(\mathbf{CA}_2) - \Lambda(\mathbf{RCA}_2)$ is that of continuum. For this define a partial order on the class of all non-isomorphic finite simple cylindric algebras by putting

$$\mathcal{A} \leq \mathcal{B} \text{ iff } \mathcal{A} \in \mathbf{S}(\mathcal{B}).$$

Lemma 4.1. *Every two non-isomorphic finite square algebras are $\leq$-incomparable.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two non-isomorphic finite square algebras and let $\mathcal{X}_{\mathcal{A}}$ and $\mathcal{X}_{\mathcal{B}}$ be their dual spaces. Then $\mathcal{X}_{\mathcal{A}}$ is isomorphic to $(n \times n, E_1, E_2, D)$ and $\mathcal{X}_{\mathcal{B}}$ is isomorphic to $(m \times m, E_1', E_2', D')$ where $n \neq m$. Without loss of generality we can assume that $n > m$. Then obviously $\mathcal{A}$ can not be a subalgebra of $\mathcal{B}$. Suppose $\mathcal{B}$ is a proper subalgebra of $\mathcal{A}$. Then there exists a cylindric partition R of $\mathcal{X}_{\mathcal{A}}$ such that $\mathcal{X}_{\mathcal{A}}/R$ is isomorphic to $\mathcal{X}_{\mathcal{B}}$. Therefore, R must identify points from different either E_1- or E_2-clusters of $\mathcal{X}_{\mathcal{A}}$. Without loss of generality we can assume that R identifies points from different E_1-clusters C_1 and C_2. Let $x_1 \in C_1$ be the diagonal point of C_1 and $x_2 \in C_2$ be the diagonal point of C_2. Since $R(D) = D$, we have that $x_1 R x_2$. Let $E_1(x_1) \cap E_2(x_2) = \{y_1\}$. Since $x_2 R x_1$ and $x_1 E_1 y_1$, there exists $y_2 \in \mathcal{X}_{\mathcal{A}}$ such that $y_1 R y_2$ and $y_2 E_1 x_2$. Consider $R(x_1)$ and $R(y_1)$. It is obvious that $R(x_1) E_0 R(y_1)$. Also $R(x_1) \neq R(y_1)$ since $R(D) = D$. Therefore, there exists a non-singleton E_0-cluster of $\mathcal{X}_{\mathcal{B}}$, which is impossible since $\mathcal{X}_{\mathcal{B}}$ is a square. Thus, $\mathcal{B}$ is not a proper subalgebra of $\mathcal{A}$, and so every two non-isomorphic finite square algebras are $\leq$-incomparable. $\square$

As an immediate consequence of Lemma 4.1 we obtain the following theorem.

Theorem 4.2. *The cardinality of* $\Lambda(\mathbf{RCA}_2)$ *is that of continuum.*

Proof. Let $\mathcal{X}_n$ be the square $(n \times n, E_1, E_2, D)$ and $\mathcal{B}_n$ be the square algebra $(P(n \times n), E_1, E_2, D)$. Consider the family $\Delta = \{\mathcal{B}_n\}_{n \in \omega}$. From Lemma 4.1 it follows that Δ forms a $\leq$-anti-chain. For any subset Γ of Δ, let $\mathbf{V}_\Gamma$ denote the variety generated by Γ.[1] Using the standard splitting technique (see, e.g.,

[1] That is $\mathbf{V}_\Gamma = \mathbf{HSP}(\Gamma)$.

Kracht [6] for details), we can easily show that $\mathbf{V}_\Gamma \neq \mathbf{V}_{\Gamma'}$ whenever $\Gamma \neq \Gamma'$. Therefore, there exist $2^{\aleph_0}$-many subvarieties of $\mathbf{RCA}_2$. $\Box$

For $n > 1$ let $\mathcal{Y}_n$ denote the finite cylindric space obtained from the $n \times n$ square by substituting a singleton non-diagonal E_0-cluster by a two-element E_0-cluster. For example, $\mathcal{Y}_2$ is shown in Fig.1(b) where $n = 2$. Denote by $\mathcal{A}_n$ the cylindric algebra corresponding to $\mathcal{Y}_n$. Obviously $\mathcal{Y}_n$ satisfies $(*)$, and so $\mathcal{A}_n$ is not representable. Similarly to Lemma 4.1, we can prove the following lemma.

Lemma 4.3. *The family $\{\mathcal{A}_n\}_{n\in\omega}$ forms a $\leq$-anti-chain.*

As an immediate consequence of Lemma 4.3 and the fact that $\{\mathcal{A}_n\}_{n\in\omega} \subseteq \mathbf{CA}_2 - \mathbf{RCA}_2$ we obtain the following theorem.

Theorem 4.4. *The cardinality of $\Lambda(\mathbf{CA}_2) - \Lambda(\mathbf{RCA}_2)$ is that of continuum.*

Finally, for $\Gamma, \Gamma' \subseteq \{\mathcal{A}_n\}_{n\in\omega}$ it is obvious that $\Gamma \neq \Gamma'$ implies $\mathbf{RCA} \vee \mathbf{V}_\Gamma \neq \mathbf{RCA} \vee \mathbf{V}_{\Gamma'}$. Therefore, we have the following corollary.

Corollary 4.5. *There exist continuum many varieties in between $\mathbf{RCA}_2$ and $\mathbf{CA}_2$.*

5 Locally finite subvarieties of $\mathbf{CA}_2$

Recall that a variety $\mathbf{V}$ of universal algebras is said to be *locally finite* if every finitely generated $\mathbf{V}$-algebra is finite. It is called *pre locally finite* if it is not locally finite but all its proper subvarieties are. It was first noticed by Tarski (see e.g. [5]) that $\mathbf{RCA}_2$, and hence any variety in the interval $[\mathbf{RCA}_2, \mathbf{CA}_2]$, is not locally finite. In this section, we present a criterion for a variety of cylindric algebras to be locally finite, and show that there exists exactly one pre locally finite subvariety of $\mathbf{CA}_2$.

Let $\mathcal{B}$ be a cylindric algebra and $\mathcal{X}$ be its corresponding dual cylindric space. We have that $\mathcal{B}$ is simple iff $\mathcal{X}$ is a quasi-square. We also have that the cardinalities of the sets of E_1- and E_2-clusters of $\mathcal{X}$ coincide.

Definition 5.1. *1. A quasi-square $\mathcal{X}$ is said to be of* depth n $(0 < n < \omega)$ *if the cardinality of the set of E_1-clusters (E_2-clusters) of $\mathcal{X}$ is equal to n.*

2. *A quasi-square $\mathcal{X}$ is said to be of an* infinite depth *if the cardinality of the set of E_1-clusters (E_2-clusters) of $\mathcal{X}$ is infinite.*

3. *A simple cylindric algebra $\mathcal{B}$ is said to be of* depth n *if its dual quasi-component $\mathcal{X}$ is of depth n.*

4. *A simple cylindric algebra $\mathcal{B}$ is said to be of an* infinite depth *if its dual quasi-component $\mathcal{X}$ is of an infinite depth.*

5. *A variety $\mathbf{V}$ of cylindric algebras is said to be of* depth n *if there is a simple $\mathbf{V}$-algebra of depth n and the depth of every other simple $\mathbf{V}$-algebra is less than or equal to n.*

6. *A variety $\mathbf{V}$ is said to be of* depth ω *if the depth of simple members of $\mathbf{V}$ is not bounded by any natural number.*

We note that there exists a formula measuring the depth of a variety of cylindric algebras (see [2, Theorem 4.2]). Let $d(\mathbf{V})$ denote the depth of the variety $\mathbf{V}$. Our goal is to show that a variety $\mathbf{V}$ of cylindric algebras is locally finite iff $d(\mathbf{V}) < \omega$. For this we need the following definition.

Definition 5.2. 1. *Call a quasi-square $\mathcal{X}$* uniform *if every non-diagonal E_0-cluster of $\mathcal{X}$ is a singleton set, and every diagonal E_0-cluster of $\mathcal{X}$ contains only two points.*

2. *Call a simple cylindric algebra $\mathcal{B}$* uniform *if its dual quasi-square $\mathcal{X}$ is uniform.*

Finite uniform quasi-squares are shown in Fig.2 below, where the bold points denote the diagonal points.

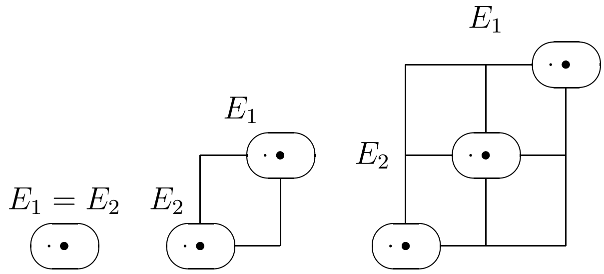


Fig.2

Denote by $\mathcal{X}_n$ the uniform quasi-square of depth n. Also let $\mathcal{B}_n$ denote the uniform cylindric algebra of depth n. It is obvious that $\mathcal{X}_n$ is (isomorphic to) the dual cylindric space of $\mathcal{B}_n$. Let $\mathbf{U}$ denote the variety generated by all finite uniform cylindric algebras, that is $\mathbf{U} = \mathbf{HSP}(\{\mathcal{B}_n\}_{n\in\omega})$.

Proposition 5.3. $\mathbf{U} \subseteq \mathbf{RCA}_2$.

Proof. Since none of the diagonal E_0-clusters of $\mathcal{X}_n$ is a singleton set, $\mathcal{X}_n$ does not satisfy $(*)$. Therefore, each $\mathcal{B}_n$ is representable by Theorem 3.4. Thus, $\{\mathcal{B}_n\}_{n\in\omega} \subseteq \mathbf{RCA}_2$, implying that $\mathbf{U} \subseteq \mathbf{RCA}_2$. □

Lemma 5.4. *1. If $\mathcal{B}$ is a simple cylindric algebra of an infinite depth, then each $\mathcal{B}_n$ is a subalgebra of $\mathcal{B}$.*

2. If $\mathcal{B}$ is a simple cylindric algebra of depth $2n$, then $\mathcal{B}_n$ is a subalgebra of $\mathcal{B}$.

Proof. (1) Suppose $\mathcal{B}$ is a simple cylindric algebra of an infinite depth and $\mathcal{X}$ is its dual cylindric space. Then $\mathcal{X}$ is a quasi-square with infinitely many E_1- and E_2-clusters. In the same way as in the proof of Claim 4.7 of [2], for every n we can divide $\mathcal{X}$ into n-many E_1-saturated disjoint clopen sets $G_1, \ldots, G_n$. We let $D_i = D \cap G_i$ and $F_i = E_2(D_i)$ for $i = 1, \ldots, n$. Obviously each of D_i's and F_i's is clopen. Define a partition R of $\mathcal{X}$ by putting:

xRy iff $x, y \in D$ and there exists $i = 1, \ldots, n$ such that $x, y \in D_i$;

xRy iff $x, y \in X - D$ and there exist $1 \leq j, k \leq n$ such that $x, y \in G_j \cap F_k$.

It is easy to check, or transform the proof of Claim 4.7 of [2], that R is a cylindric partition of $\mathcal{X}$, and that $\mathcal{X}/R$ is isomorphic to $\mathcal{X}_n$. Therefore, by Theorem 2.10(2), each $\mathcal{B}_n$ is a subalgebra of $\mathcal{B}$.

(2) Suppose $\mathcal{B}$ is a simple cylindric algebra of depth $2n$ and $\mathcal{X}$ is its dual cylindric space. Then $\mathcal{X}$ is a quasi-square. Moreover, there are exactly $2n$-many E_1- and E_2-clusters of $\mathcal{X}$. Obviously all of them are clopens. Let $C_1, \ldots, C_{2n}$ be E_1-clusters of $\mathcal{X}$ and let $G_i = C_{2i-1} \cup C_{2i}$ for $i = 1, \ldots, n$. Obviously every G_i is E_1-saturated clopen. Moreover, the same arguments as in (1) show that $\mathcal{B}_n$ is a subalgebra of $\mathcal{B}$. □

Theorem 5.5. *For a variety* $\mathbf{V}$ *of cylindric algebras,* $d(\mathbf{V}) = \omega$ *iff* $\mathbf{U} \subseteq \mathbf{V}$.

Proof. It is obvious that $d(\mathbf{U}) = \omega$. So, if $\mathbf{U} \subseteq \mathbf{V}$, then obviously $d(\mathbf{V}) = \omega$. Conversely, suppose $d(\mathbf{V}) = \omega$. We want to show that every finite uniform cylindric algebra belongs to $\mathbf{V}$. Since $d(\mathbf{V}) = \omega$, the depth of simple members of $\mathbf{V}$ is not restricted to any natural number. So, either there exists a family of simple $\mathbf{V}$-algebras of increasing finite depth, or there exists a simple $\mathbf{V}$-algebra of an infinite depth. In either case, it follows from Lemma 5.4 that $\{\mathcal{B}_n\}_{n\in\omega} \subseteq \mathbf{V}$. Therefore, $\mathbf{U} \subseteq \mathbf{V}$ since $\{\mathcal{B}_n\}_{n\in\omega}$ generates $\mathbf{U}$. □

Our next task is to show that $\mathbf{U}$ is not a locally finite variety. For this we recall the coloring technique first introduced in Esakia and Grigolia [4]. Suppose $\mathcal{B}$ is a cylindric algebra, $\mathcal{X}$ is its dual cylindric space, and $g_1, \ldots, g_n$ are elements of $\mathcal{B}$. Let $\mathbf{n} = \{0, \ldots, n-1\}$. For $i \in \mathbf{n}$ let $G_i \in CP(X)$ be the clopen corresponding to $g_i \in \mathcal{B}$. Also let $F_p = G_1^{\varepsilon_1} \cap \ldots \cap G_n^{\varepsilon_n}$, where $\varepsilon_i \in \{0, 1\}$, $p = \{i : \varepsilon_i = 1\}$ and

$$G_i^{\varepsilon_i} = \begin{cases} G_i, & \text{if } \varepsilon_i = 1 \\ -G_i, & \text{otherwise.} \end{cases}$$

It is obvious that $\{F_p\}_{p\subseteq\mathbf{n}}$ is a partition of X into 2^n-many equivalence classes, which we call the *coloring* of X. A given $x \in F_p$ is said to have the *color* p, written as $Col(x) = p$.

Theorem 5.6. *(Coloring Theorem)* $\mathcal{B}$ *is generated by* $g_1, \ldots, g_n$ *iff for every non-trivial cylindric partition* R *of* $\mathcal{X}$*, there exists an equivalence class of* R *containing the points of different color.*

Proof. A straightforward adaptation of the Generation Theorem of [4]. Suppose $\mathcal{B}$ is generated by $g_1, \ldots, g_n$ and R is a non-trivial cylindric partition of $\mathcal{X}$. Consider the set $\mathcal{P}_R$ of the R-saturated elements of $CP(X)$. Since $g_1, \ldots, g_n$ generate $\mathcal{B}$ and R is a non-trivial cylindric partition, there exists $i \in \mathbf{n}$ such that $G_i \notin \mathcal{P}_R$. Therefore, there exists $p \subseteq \mathbf{n}$ such that F_p is not R-saturated. But then there exists $x \in F_p$ such that $R(x) \cap F_p \neq \emptyset$ and $R(x) \cap -F_p \neq \emptyset$. Hence, $R(x)$ contains points of different color.

Conversely, suppose $\mathcal{B}$ is not generated by $g_1, \ldots, g_n$. Denote by $\mathcal{B}_0$ the subalgebra of $\mathcal{B}$ generated by $g_1, \ldots, g_n$. Obviously $\mathcal{B}_0$ is a proper subalgebra of $\mathcal{B}$ and the cylindric partition R of $\mathcal{X}$ corresponding to $\mathcal{B}_0$ is non-trivial.[2] Moreover, since $g_1, \ldots, g_n \in \mathcal{B}_0$, we have $R(G_i) = G_i$ $(i \in \mathbf{n})$, and hence $R(F_p) = F_p$ $(p \subseteq \mathbf{n})$. But then every equivalence class of R contains points of the same color. $\square$

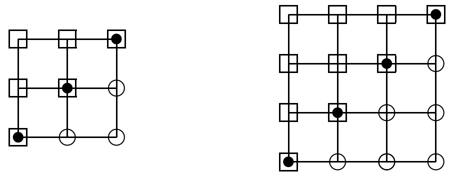
Lemma 5.7. *1. Every finite square algebra is cyclic.*

2. Every finite uniform algebra is cyclic.

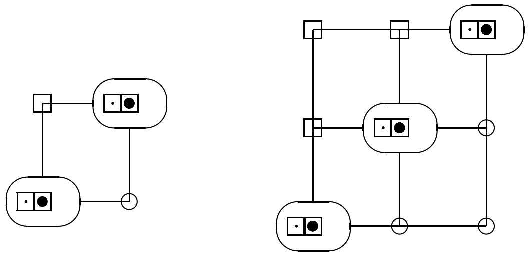
Proof. (1) For a finite square $\mathcal{X} = (n \times n, E_1, E_2, D)$, consider the set $G = \{(k, m) : k < m\}$. We show that the square algebra $(P(n \times n), E_1, E_2, D)$ is generated by G. For this observe that G gives rise to two colors, the 0-color corresponding to $-G$ and the 1-color corresponding to G. This coloring is shown in Fig.3(a), where the points of the empty color are in squares and the points of the 1-color are in circles. Let R be a non-trivial cylindric partition of $\mathcal{X}$. Since R is non-trivial, R must identify at least two different diagonal points, say x and y. This immediately implies that the points $z \in E_1(x) \cap E_2(y)$ and $u \in E_2(x) \cap E_1(y)$ should also be identified by R. However, one of z and u belongs to G and another to $-G$, and points from G and $-G$ have different colors. So, every non-trivial cylindric partition identifies the points of different color, which by the Coloring Theorem implies that G generates $(P(n \times n), E_1, E_2, D)$.

(2) is proved analogously to (1). If $\mathcal{B}$ is a finite uniform algebra and $\mathcal{X}$ is its dual cylindric space, then $\mathcal{X}$ is obtained from a finite square by replacing the diagonal by the two point E_0-cluster containing one diagonal point, and the same arguments as above imply that the lower triangle G generates $\mathcal{B}$ (see Fig.3(b)). $\square$

[2] R is defined on X by putting xRy iff $a \in x \Leftrightarrow a \in y$ for every $a \in \mathcal{B}_0$.



(a)



(b)

Fig.3

Remark 5.8. Note that the $\mathbf{Df}_2$-reducts of finite uniform algebras are not generated by G. Indeed, since the equivalence relation R identifying the points of one color in the non-singleton E_0-clusters is a correct $\mathbf{Df}_2$-partition, there exists a non-trivial correct $\mathbf{Df}_2$-partition identifying the points of the same color, and by the $\mathbf{Df}_2$-version of the Coloring Theorem we obtain that G does not generate $\mathcal{B}$. Of course, R is not a cylindric partition because $R(D) \neq D$. Actually, no finite uniform algebra is a cyclic $\mathbf{Df}_2$-algebra since we can show that the following theorem holds true: A quasi-square $\mathbf{Df}_2$-algebra is cyclic iff either it is a square algebra, or every E_0-cluster of its dual space is a singleton set except one E_0-cluster that contains exactly two points.

Now in order to conclude that $\mathbf{U}$ is not locally finite all we need is to remember the following characterization of locally finite varieties from G. Bezhanishvili [1].

Theorem 5.9. *A variety* $\mathbf{V}$ *of a finite signature is locally finite iff for every natural number* n *there exists a natural number* $M(n)$ *such that the cardinality of every* n*-generated subdirectly irreducible* $\mathbf{V}$*-algebra is less than or equal to* $M(n)$*.*

Corollary 5.10. $\mathbf{U}$ *is not locally finite.*

Proof. Follows from Lemma 5.7 and Theorem 5.9. $\square$

It is left to be shown that if $d(\mathbf{V}) < \omega$, then $\mathbf{V}$ is locally finite. For this we need the following lemma.

Lemma 5.11. *If $d(\mathbf{V}) < \omega$, then for every n-generated simple $\mathbf{V}$-algebra $\mathcal{B}$ there exists a natural number $M(n)$ such that the cardinality of $\mathcal{B}$ is bounded by $M(n)$.*

Proof. Let $d(\mathbf{V}) = k < \omega$ and consider a simple n-generated $\mathbf{V}$-algebra $\mathcal{B}$. Let $g_1, \ldots, g_n$ be the generators of $\mathcal{B}$ and $\mathcal{X}$ be the dual cylindric space of $\mathcal{B}$. Then $\mathcal{X}$ is a quasi-square and there exists a partition of $\mathcal{X}$ into 2^n different colors. If there is a non-diagonal E_0-cluster C of $\mathcal{X}$ containing more than 2^n points, then consider the smallest equivalence relation R on $\mathcal{X}$ identifying two points of the same color in C. Obviously R is a cylindric partition and each R-equivalence class has the same color. Therefore, by Coloring Theorem, $\mathcal{B}$ is not generated by $g_1, \ldots, g_n$. Thus, every non-diagonal E_0-cluster of $\mathcal{X}$ contains no more than 2^n points. Similar arguments show that every diagonal E_0-cluster contains at most $2^n + 1$ points. Now since there exist at most k^2 different E_0-clusters of $\mathcal{X}$, the cardinality of $\mathcal{X}$ is bounded by $b(n) = k^2(2^n+1)$. Thus, the cardinality of $\mathcal{B}$ is bounded by $M(n) = 2^{b(n)}$. $\square$

Corollary 5.12. *If $d(\mathbf{V}) < \omega$, then $\mathbf{V}$ is locally finite.*

Proof. Follows from Lemma 5.11 and Theorem 5.9. $\square$

Finally, combining Corollary 5.10 with Corollary 5.12 we obtain the following characterization of locally finite varieties of cylindric algebras.

Theorem 5.13. *1. For $\mathbf{V} \subseteq \mathbf{CA}_2$ the following conditions are equivalent:*

(a) $\mathbf{V}$ is locally finite;

(b) $d(\mathbf{V}) < \omega$;

(c) $\mathbf{U} \not\subseteq \mathbf{V}$.

2. $\mathbf{U}$ is the only pre locally finite subvariety of $\mathbf{CA}_2$.

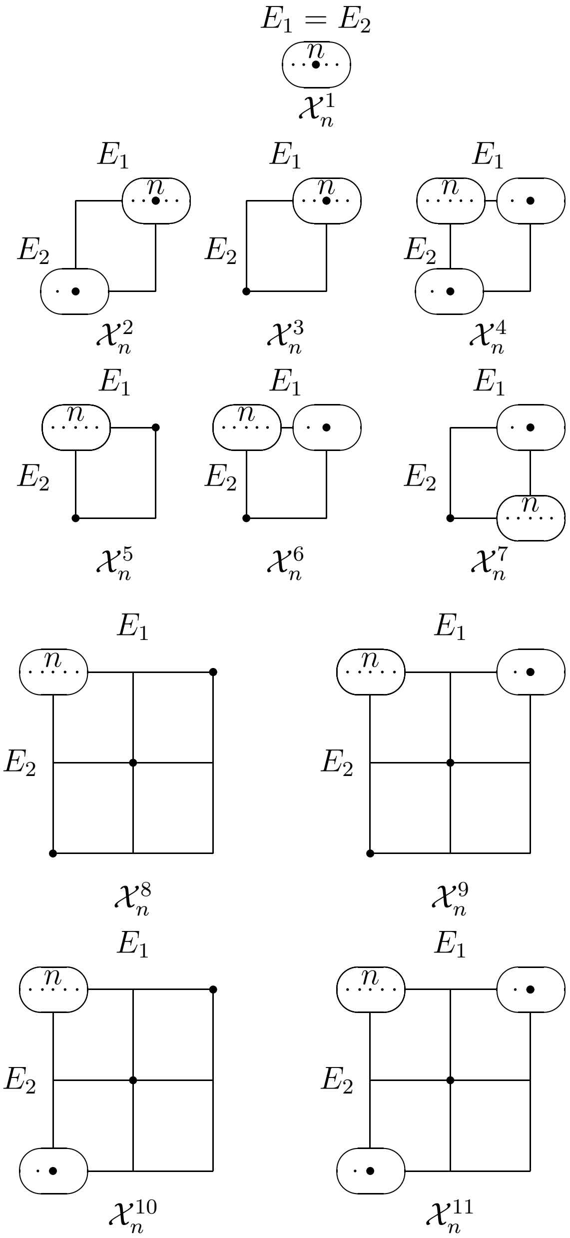
Therefore, in contrast to the diagonal-free case, there exist uncountably many subvarieties of $\mathbf{CA}_2$ ($\mathbf{RCA}_2$) which are not locally finite. Since every locally finite variety is obviously generated by its finite members, we obtain from Theorem 5.13 that every subvariety of $\mathbf{CA}_2$ of a finite depth is generated by its finite members. We conjecture that every subvariety of $\mathbf{CA}_2$ is in fact generated by its finite members.

6 Finitely generated and pre finitely generated subvarieties of $\mathbf{CA}_2$

Recall that a variety of universal algebras is said to be *finitely generated* if it is generated by a finite universal algebra. We call a variety *pre finitely generated* if it is not finitely generated but all its proper subvarieties are. It was shown in [2, Theorem 5.4] that there exist exactly six pre finitely generated varieties in $\Lambda(\mathbf{Df}_2)$. The situation is more complex in $\Lambda(\mathbf{CA}_2)$. In this section, we show that there exist exactly fifteen pre finitely generated varieties in $\Lambda(\mathbf{CA}_2)$, and that six of them belong to $\Lambda(\mathbf{RCA}_2)$. It trivially implies a characterization of finitely generated subvarieties of $\Lambda(\mathbf{CA}_2)$.

Consider the finite quasi-squares $\mathcal{X}_n^i$ shown in Fig.4, where $i = 1, \dots, 15$ and $n \in \omega$. It is easy to see that each E_0-cluster of $\mathcal{X}_n^i$ consists of either one, two or n points[3]. Let $\mathcal{B}_n^i$ denote the cylindric algebra corresponding to $\mathcal{X}_n^i$. For fixed $i = 1, \dots, 15$ let $\mathbf{V}_i$ denote the variety generated by the family $\{\mathcal{B}_n^i : n \in \omega\}$. From Theorem 3.4 it follows that only $\mathcal{B}_n^1, \dots, \mathcal{B}_n^4$, $\mathcal{B}_n^{11}$ and $\mathcal{B}_n^{12}$ are representable algebras, and so only $\mathbf{V}_1 - \mathbf{V}_4$, $\mathbf{V}_{11}$ and $\mathbf{V}_{12}$ belong to $\Lambda(\mathbf{RCA}_2)$.

[3] In $\mathcal{X}_n^1$, $\mathcal{X}_n^2$ and $\mathcal{X}_n^3$ we assume that $n > 1$.



$E_1 = E_2$
n
$\mathcal{X}_n^1$
E_1 E_2 n $\mathcal{X}_n^2$
E_1 E_2 n $\mathcal{X}_n^3$
E_1 E_2 n $\mathcal{X}_n^4$
E_1 E_2 n $\mathcal{X}_n^5$
E_1 E_2 n $\mathcal{X}_n^6$
E_1 E_2 n $\mathcal{X}_n^7$
E_1 E_2 n $\mathcal{X}_n^8$
E_1 E_2 n $\mathcal{X}_n^9$
E_1 E_2 n $\mathcal{X}_n^{10}$
E_1 E_2 n $\mathcal{X}_n^{11}$

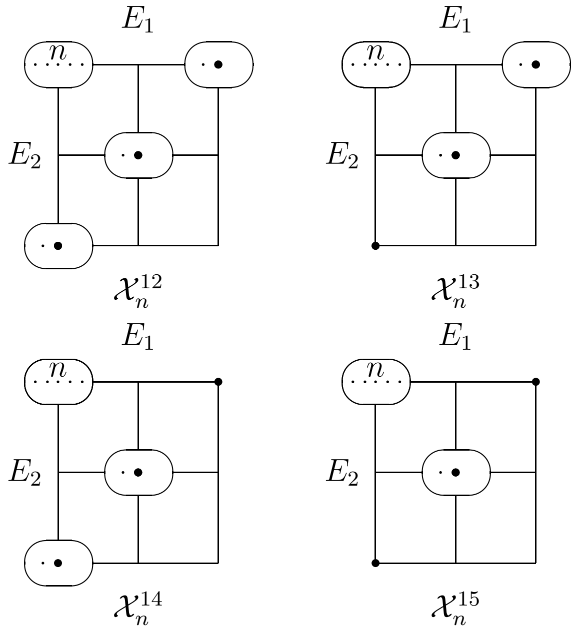


Fig.4

Now we are in a position to prove that $\mathbf{V}_1 - \mathbf{V}_{15}$ are the only pre finitely generated subvarieties of $\mathbf{CA}_2$. For this we need to show that $\mathbf{V}_1 - \mathbf{V}_{15}$ are not finitely generated, which follows from their definition, and that every variety of cylindric algebras which is not finitely generated contains exactly one of $\mathbf{V}_1 - \mathbf{V}_{15}$.

Lemma 6.1. $\mathbf{V}_2 \subset \mathbf{U}$.

Proof. Suppose $\mathcal{B}_n$ is the finite uniform algebra of depth n. We show that $\mathcal{B}_n^2$ is a subalgebra of $\mathcal{B}_n$. Consider the uniform square $\mathcal{X}_n$ of depth n, fix a diagonal E_0-cluster, say C, and let $D \cap C = \{x_0\}$. Define an equivalence relation R on X by putting:

xRy iff $x = y$ for all $x, y \in C$;

xRy for all $x, y \in E_1(C) - C$;

xRy for all $x, y \in E_2(C) - C$;

xRy for all $x, y \in D - \{x_0\}$;

Let any other R-equivalence class consist of exactly $n-1$ points chosen so that the class contains just one point from each E_i-cluster of $X - (E_1(C) \cup E_2(C) \cup D)$ for $i = 1, 2$.

It is a matter of routine verification that R is a cylindric partition, and that $\mathcal{X}_n/R$ is isomorphic to $\mathcal{X}_n^2$. Therefore, $\mathcal{B}_n^2$ is a subalgebra of $\mathcal{B}_n$ for every n, implying that $\mathbf{V}_2 \subset \mathbf{U}$. □

Therefore, we obtain that if $d(\mathbf{V}) = \omega$, then $\mathbf{V}_2 \subseteq \mathbf{V}$. Suppose $d(\mathbf{V}) < \omega$. Then $\mathbf{V}$ is locally finite by Theorem 5.13. Let $\mathrm{Fin}\mathbf{V}_S$ denote the class of all finite simple $\mathbf{V}$-algebras. Since $\mathbf{V}$ is locally finite, $\mathbf{V}$ is generated by $\mathrm{Fin}\mathbf{V}_S$. Suppose $\mathcal{B} \in \mathrm{Fin}\mathbf{V}_S$ and $\mathcal{X}$ is its dual cylindric space. Then $\mathcal{X}$ is a finite quasi-square. Fix $x \in X$.

Definition 6.2. *1. Call the number of elements of $E_0(x)$ the* girth *of x.*

2. *The maximum of the girths of all $x \in E_0(D)$ is called the* diagonal girth *of $\mathcal{X}$.*

3. *The maximum of the girths of all $x \in X - E_0(D)$ is called the* non-diagonal girth *of $\mathcal{X}$.*

4. *The* diagonal girth *of $\mathcal{B}$ is the diagonal girth of $\mathcal{X}$.*

5. *The* non-diagonal girth *of $\mathcal{B}$ is the non-diagonal girth of $\mathcal{X}$.*

6. *The diagonal girth of $\mathbf{V}$ is said to be n if there is $\mathcal{B} \in Fin\mathbf{V}_S$ whose diagonal girth is n, and the diagonal girths of all the other members of $Fin\mathbf{V}_S$ are less than or equal to n.*

7. *The diagonal girth of $\mathbf{V}$ is said to be ω if the diagonal girths of the members of $Fin\mathbf{V}_S$ are not bounded by any integer.*

8. *The non-diagonal girth of $\mathbf{V}$ is said to be n if there is $\mathcal{B} \in Fin\mathbf{V}_S$ whose non-diagonal girth is n, and the non-diagonal girths of all the other members of $Fin\mathbf{V}_S$ are less than or equal to n.*

9. *The non-diagonal girth of $\mathbf{V}$ is said to be ω if the non-diagonal girths of the members of $Fin\mathbf{V}_S$ are not bounded by any integer.*

Lemma 6.3. *If* $\mathbf{V}$ *is a variety of cylindric algebras of finite depth whose diagonal and non-diagonal girths are bounded by some integer, then* $\mathbf{V}$ *is a finitely generated variety.*

Proof. There exist only finitely many non-isomorphic finite simple cylindric algebras whose depth, the diagonal girth and the non-diagonal girth are bounded by some integer. Therefore, there are only finitely many non-isomorphic finite simple $\mathbf{V}$-algebras, implying that $\mathbf{V}$ is finitely generated. □

It follows that if a variety $\mathbf{V}$ of a finite depth is not finitely generated, then either the diagonal girth or the non-diagonal girth of $\mathbf{V}$ must be ω.

Lemma 6.4. *If* $\mathbf{V}$ *is a variety of finite depth whose diagonal girth is* ω*, then one of* $\mathbf{V}_1 - \mathbf{V}_3$ *is contained in* $\mathbf{V}$.

Proof. Since the diagonal girth of $\mathbf{V}$ is ω, for each n there is $\mathcal{B} \in \mathrm{Fin}\mathbf{V}_S$ whose diagonal girth is n. Let $\mathcal{X}$ be the dual cylindric space of $\mathcal{B}$. Then $\mathcal{X}$ is a quasi-square. Denote by C the diagonal E_0-cluster of $\mathcal{X}$ containing n points. Then two cases are possible. Either $d(\mathcal{X}) = 1$ or $d(\mathcal{X}) \geq 2$ for infinitely many n. In the former case, it is obvious that $\mathcal{X}$ is isomorphic to $\mathcal{X}_n^1$, and so $\mathbf{V}_1 \subseteq \mathbf{V}$. In the latter case, define an equivalence relation R on $\mathcal{X}$ by putting:

xRy iff $x = y$ for any $x, y \in C \cup D$;

xRy iff xE_0y for any $x, y \in X - (C \cup D)$.

Clearly R is a cylindric partition. Denote $\mathcal{X}/R$ by $\mathcal{Y}$. Then every non-diagonal E_0-cluster of $\mathcal{Y}$ is a singleton set and every diagonal E_0-cluster different from C contains either one or two points.

Again there are two cases possible. Either $d(\mathcal{Y}) = 2$ or $d(\mathcal{Y}) > 2$ for infinitely many n. In the former case, $\mathcal{Y}$ is isomorphic to either $\mathcal{X}_n^2$ or $\mathcal{X}_n^3$ for infinitely many n. Therefore, either $\mathbf{V}_2 \subset \mathbf{V}$ or $\mathbf{V}_3 \subset \mathbf{V}$. And in the latter case, define an equivalence relation Q on Y by putting:

xQy iff $x = y$ for any $x, y \in C$;

xQy for any $x, y \in E_1(C) - C$;

xQy for any $x, y \in E_2(C) - C$;

xQy for any $x, y \in D - C$;

xQy for any $x, y \in Y - (E_1(C) \cup E_2(C) \cup D)$.

It is a matter of routine verification that Q is a cylindric partition, and that $\mathcal{Y}/Q$ is isomorphic to $\mathcal{X}_n^2$. Thus, $\mathbf{V}_2 \subset \mathbf{V}$. □

Lemma 6.5. *If* $\mathbf{V}$ *is a variety of finite depth whose non-diagonal girth is* ω*, then one of* $\mathbf{V}_4 - \mathbf{V}_{15}$ *is contained in* $\mathbf{V}$*.*

Proof. Since the non-diagonal girth of $\mathbf{V}$ is ω, for each n there is $\mathcal{B} \in \mathrm{Fin}\mathbf{V}_S$ whose non-diagonal girth is n. Let $\mathcal{X}$ be the dual cylindric space of $\mathcal{B}$. Then $\mathcal{X}$ is a quasi-square. Denote by C the non-diagonal E_0-cluster of $\mathcal{X}$ containing n points. Since the non-diagonal E_0-clusters exist only in cylindric spaces of depth > 1, we have $d(\mathcal{X}) > 1$. Define an equivalence relation R on $\mathcal{X}$ by putting:

xRy iff $x = y$ for any $x, y \in C \cup D$;

xRy iff xE_0y for any $x, y \in X - (C \cup D)$.

Clearly R is a cylindric partition. Since $d(\mathcal{X}) > 1$, there are three cases possible. Either $d(\mathcal{X}) = 2$, $d(\mathcal{X}) = 3$, or $d(\mathcal{X}) > 3$ for infinitely many n.

If $d(\mathcal{X}) = 2$ for infinitely many n, then $\mathcal{X}/R$ is isomorphic to one of $\mathcal{X}_n^4 - \mathcal{X}_n^7$ for infinitely many n, implying that one of $\mathbf{V}_4 - \mathbf{V}_7$ is contained in $\mathbf{V}$.

If $d(\mathcal{X}) = 3$ for infinitely many n, then $\mathcal{X}/R$ is isomorphic to one of $\mathcal{X}_n^8 - \mathcal{X}_n^{15}$ for infinitely many n, implying that one of $\mathbf{V}_8 - \mathbf{V}_{15}$ is contained in $\mathbf{V}$.

Finally, let $3 < d(\mathcal{X}) < \omega$ for infinitely many n. Denote by C' the diagonal E_0-cluster E_1-related to C, and by C'' - the diagonal E_0-cluster E_2-related to C. Define an equivalence relation R on $\mathcal{X}$ by putting:

xRy iff $x = y$ for any $x, y \in C \cup ((C' \cup C'') \cap D)$;

xRy for any $x, y \in D - (C' \cup C'')$;

xRy for any $x, y \in X - (D \cup E_1(C') \cup E_2(C') \cup E_1(C'') \cup E_2(C'')$;

xRy iff xE_0y for any $x, y \in (E_2(C') \cap E_1(C'')) \cup ((C' \cup C'') - D)$;

xRy for any $x, y \in E_2(C) - (C \cup C'')$;

xRy for any $x, y \in E_1(C) - (C \cup C')$;

xRy for any $x, y \in E_2(C') - (E_1(C'') \cup C')$;

xRy for any $x, y \in E_1(C'') - (E_2(C') \cup C')$.

It is a matter of routine verification that R is a cylindric partition. Moreover, there are four cases possible. Either both C' and C'' are singleton sets, C' is a singleton set and C'' is not, C'' is a singleton set and C' is not, or neither C' nor C'' are singleton sets, for infinitely many n. In the first case $\mathcal{X}/R$ is isomorphic to $\mathcal{X}_n^{15}$, in the second case $\mathcal{X}/R$ is isomorphic to $\mathcal{X}_n^{14}$, in the third case $\mathcal{X}/R$ is isomorphic to $\mathcal{X}_n^{13}$, and finally in the fourth case $\mathcal{X}/R$ is isomorphic to $\mathcal{X}_n^{12}$. Therefore, one of $\mathbf{V}_{12} - \mathbf{V}_{15}$ is contained in $\mathbf{V}$.

Thus, going through all the cases we obtained that one of $\mathbf{V}_4 - \mathbf{V}_{15}$ is contained in $\mathbf{V}$. □

Corollary 6.6. *1.* $\mathbf{V}_1 - \mathbf{V}_{15}$ *are the only pre finitely generated varieties in* $\Lambda(\mathbf{CA}_2)$.

2. $\mathbf{V}_1 - \mathbf{V}_4$, $\mathbf{V}_{11}$ *and* $\mathbf{V}_{12}$ *are the only pre finitely generated varieties in* $\Lambda(\mathbf{RCA}_2)$

Proof. It is an immediate consequence of Lemmas 6.1, 6.3 – 6.5, and the fact that all the fifteen varieties are non-comparable. □

7 Lattice structure of $\Lambda(\mathbf{CA}_2)$

In order to obtain a rough picture of the lattice structure of subvarieties of $\mathbf{CA}_2$, we need the following notation:

$\mathbf{FG} = \{\mathbf{V} \in \Lambda(\mathbf{CA}_2) : \mathbf{V} \text{ is finitely generated}\}$;

$\mathbf{D}_F = \{\mathbf{V} \in \Lambda(\mathbf{CA}_2) : d(\mathbf{V}) < \omega \text{ and } \mathbf{V} \notin \mathbf{FG}\}$;

$\mathbf{D}_\omega = \{\mathbf{V} \in \Lambda(\mathbf{CA}_2) : d(\mathbf{V}) = \omega\}$.

Let also $\mathbf{V}_\perp$ denote the trivial variety.

Theorem 7.1. *1.* $\{\mathbf{FG}, \mathbf{D}_F, \mathbf{D}_\omega\}$ *is a partition of* $\Lambda(\mathbf{CA}_2)$.

2. $\mathbf{V}_\perp$ *is a least element of* $\mathbf{FG}$.

3. $\mathbf{FG}$ *does not have maximal elements.*

4. $\mathbf{D}_F$ *has precisely fifteen minimal elements.*

5. $\mathbf{D}_F$ *does not have maximal elements.*

6. $\mathbf{U}$ *and* $\mathbf{CA}_2$ *are a least and a greatest elements of* $\mathbf{D}_\omega$*, respectively.*

Proof. Follows immediately from Theorem 5.5 and Corollary 6.6. □

The lattice $\Lambda(\mathbf{CA}_2)$ can be roughly depicted as shown in Fig.5 below.

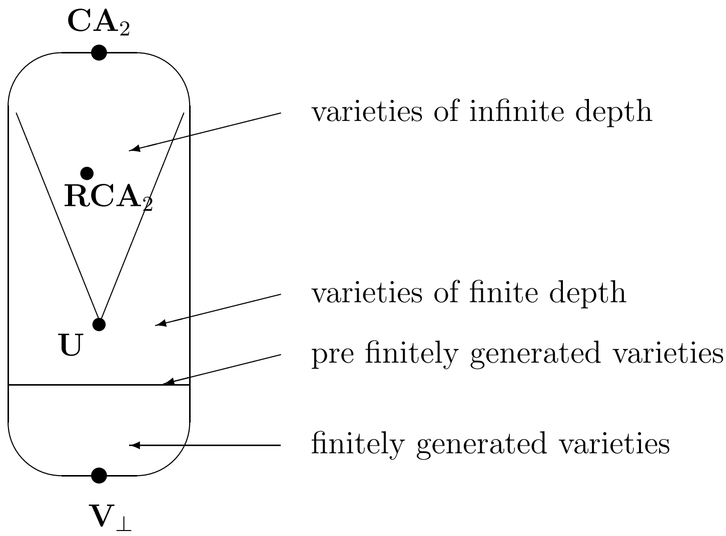


Fig.5

Now we will investigate the lower part of $\Lambda(\mathbf{CA}_2)$ in a greater detail. It follows from Corollary 6.6 that a variety $\mathbf{V} \subseteq \mathbf{CA}_2$ ($\mathbf{RCA}_2$) is finitely generated iff $\mathbf{V}$ does not contain one of the fifteen (six) pre finitely generated varieties. Another criterion is given by the following theorem.

Theorem 7.2. *For a variety* $\mathbf{V} \subseteq \mathbf{CA}_2$ *the following conditions are equivalent:*

1. $\mathbf{V}$ *is finitely generated.*

2. $\mathbf{V}$ *has only finitely many subvarieties.*

3. $\mathbf{V}$ *contains only finitely many non-isomorphic simple algebras (and all of them are finite).*

Proof. (1) $\Rightarrow$ (2) is straightforward since $\mathbf{CA}_2$ is congruence-distributive.

(2) $\Rightarrow$ (3). If $\mathbf{V}$ contains an infinite family $\{\mathcal{B}_i\}_{i\in I}$ of non-isomorphic simple algebras, then $\{\mathbf{HSP}(\mathcal{B}_i)\}_{i\in I}$ is an infinite family of subvarieties of $\mathbf{V}$, which is a contradiction.

(3) $\Rightarrow$ (1). Let $\{\mathcal{B}_i\}_{i=1}^n$ be the family of all (finite) simple non-isomorphic $\mathbf{V}$-algebras. Then $\prod_{i=1}^n \mathcal{B}_i$ generates $\mathbf{V}$. □

Corollary 7.3. *1. Every cover of a finitely generated variety of cylindric algebras is finitely generated.*

2. A finitely generated variety of cylindric algebras has only finitely many covers.

Proof. (1) If $\mathbf{V}'$ is a cover of $\mathbf{V}$ and $\mathbf{V}$ is finitely generated, then there are only finitely many subvarieties of $\mathbf{V}'$, hence $\mathbf{V}'$ is finitely generated by Theorem 7.2.

(2) The proof is analogous to the standard proof that a finitely generated variety of $\mathbf{K4}$-algebras has only finitely many covers (see, e.g., Kracht [6] or Chagrov and Zakharyaschev [3]). □

7.1 Varieties of cylindric algebras of depth one

In this subsection we give a complete charcterization of the lattice structure of the varieties of cylindric algebras of depth one.

Let $\mathbf{2}^n$ denote the 2^n-element $\mathbf{Df}_2$-algebra, where $n \geq 1$ and

$$\exists_i(a) = \begin{cases} 0 & \text{if } a{=}0, \\ 1 & \text{otherwise,} \end{cases}$$

for $i = 1, 2$. Let also d be an atom of $\mathbf{2}^n$. Then $(\mathbf{2}^n, d)$ is a cylindric algebra. We recall that in the diagonal-free case the two-element $\mathbf{Df}_2$-algebra $\mathbf{2}$ is a subalgebra of every nontrivial $\mathbf{Df}_2$-algebra. For $\mathbf{CA}_2$ the situation is different.

Proposition 7.4. *Suppose $\mathcal{B}$ is a nontrivial simple cylindric algebra.*

1. $(\mathbf{2}, 1)$ *is a subalgebra of* $\mathcal{B}$ *iff* $\mathcal{B}$ *is isomorphic to* $(\mathbf{2}, 1)$.

2. *If $\mathcal{B}$ is not isomorphic to $(\mathbf{2}, 1)$, then $(\mathbf{2}^2, d)$ is a subalgebra of $\mathcal{B}$.*

Proof. (1) If $(\mathbf{2}, 1)$ is a subalgebra of $\mathcal{B} = (B, d)$, then $d = 1$, which implies that $B = \mathbf{2}$.

(2) It is known that $\exists_1\exists_2 - d = \exists_1 - d = \exists_2 - d$ (see, e.g., [5, Theorem 1.3.18]). Since $\mathcal{B}$ is not isomorphic to $(\mathbf{2}, 1)$, we have $d \neq 1$. Hence, $-d \neq 0$. So, $\exists_1\exists_2 - d = 1$ since $\mathcal{B}$ is simple. Thus, $\{1, 0, d, -d\}$ is a cylindric subalgebra of $\mathcal{B}$.

□

Corollary 7.5. *If a variety $\mathbf{V}$ of cylindric algebras contains the two-element cylindric algebra $(\mathbf{2}, 1)$, then $\mathbf{V}$ is generated by a simple algebra iff $\mathbf{V} = \mathbf{Var}(\mathbf{2}, 1)$.*

Proof. Suppose $(\mathbf{2}, 1) \in \mathbf{V}$ and $\mathbf{V}$ is generated by a simple $\mathbf{V}$-algebra $\mathcal{B}$. Then Jónsson's Lemma implies that $(\mathbf{2}, 1) \in \mathbf{S}(\mathcal{B})$, and applying Proposition 7.4 we obtain that $\mathcal{B}$ is isomorphic to $(\mathbf{2}, 1)$. □

Let $\mathbf{V}^1 \subseteq \mathbf{CA}_2$ be the variety of all cylindric algebras of depth one. It is known from [2, Theorem 4.2] that $\mathbf{V}^1 = \mathbf{CA}_2 + (\exists_2\exists_1 a \leq \exists_1 a)$. Let $(\mathcal{F}, \leq)$ denote the partially ordered set of all non-isomorphic finite cylindric algebras of depth one. We recall from Section 3 that $\leq$ is defined on $\mathcal{F}$ by putting $\mathcal{B} \leq \mathcal{B}'$ iff $\mathcal{B} \in \mathbf{S}(\mathcal{B}')$. It follows from Proposition 7.4 that $(\mathcal{F}, \leq)$ is isomorphic to the disjoint union of the set of natural numbers $(\mathbf{N}, \leq)$ with the set consisting of one reflexive point.

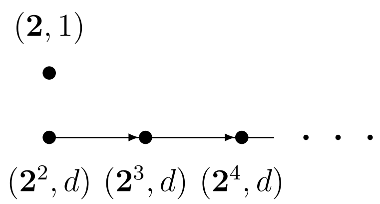


Fig.6

Recall that $\mathcal{G} \subseteq \mathcal{F}$ is called a *downset* of $\mathcal{F}$ if $\mathcal{A} \in \mathcal{G}$ and $\mathcal{B} \leq \mathcal{A}$ imply $\mathcal{B} \in \mathcal{G}$. Since every variety of cylindric algebras of finite depth is locally finite, using the standard splitting technique (see, e.g., Kracht [6]) one can easily obtain the following representation of the lattice of varieties of cylindric algebras of depth one.

Theorem 7.6. *The lattice of varieties of cylindric algebras of depth one is isomorphic to the lattice of downsets of* $(\mathcal{F}, \leq)$.

The lattice of varieties of cylindric algebras of depth one is shown in Fig.7 below. To explain the labelling, with each downset of $(\mathcal{F}, \leq)$ of the form $\downarrow(\mathbf{2}^n, d) = \{(\mathbf{2}^k, d) : 1 < k < n\}$ we associated the variety $\mathbf{V}_{(\mathbf{2}^n,d)}$ generated by $(\mathbf{2}^n, d)$; and with each downset of the form $\downarrow(\mathbf{2}^n, d) \cup \{(\mathbf{2}, 1)\}$ we associated the variety $\mathbf{V}_{(\mathbf{2},1)\times(\mathbf{2}^n,d)}$ generated by $(\mathbf{2}, 1) \times (\mathbf{2}^n, d)$; furthermore, $\mathbf{V}_1 = \mathbf{HSP}(\{(\mathbf{2}^n, d) : n > 1\})$.

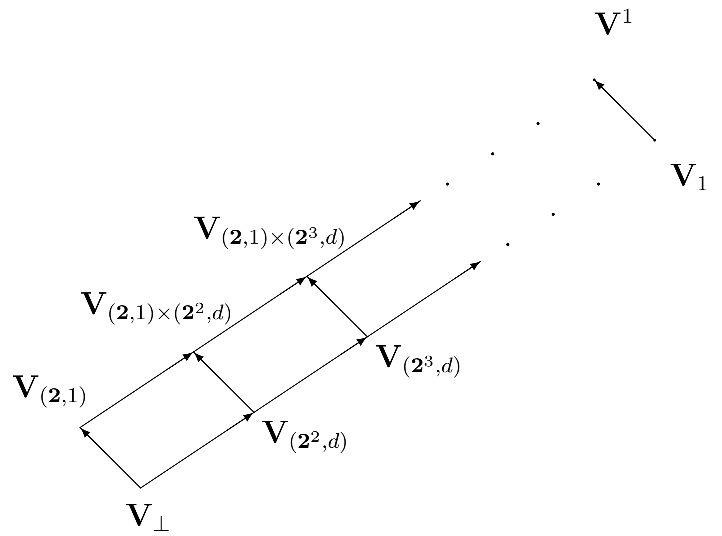


Fig. 7

Theorem 7.7. *Every subvariety of* $\mathbf{V}^1$ *is finitely axiomatizable.*

Proof. A proof similar to the proof in [8] shows that the formula

$$\bigwedge_{k=1}^{n+1} \exists_1 a_k \leq \bigwedge_{1\leq k,j\leq n+1} \exists_1(a_k \wedge a_j)$$

holds true in a simple cylindric algebra iff the corresponding quasi-square contains $\leq n$ points. Therefore, the varieties $\mathbf{V}_{(\mathbf{2},1)\times(\mathbf{2}^n,d)}$ are axiomatized by these formulas. On the other hand, the identity $\exists_1 - d = 1$ holds true in $(\mathbf{2}^n, d)$ iff $n > 1$. Therefore, the variety $\mathbf{V}_1$ is axiomatized by the identity $\exists_1 - d = 1$, while the varieties $\mathbf{V}_{(\mathbf{2}^n,d)}$ are axiomatized by adding $\exists_1 - d = 1$ to the identities of $\mathbf{V}_{(\mathbf{2},1)\times(\mathbf{2}^n,d)}$. $\square$

Remark 7.8. In fact, using the Jankov type formulas, we can show that every subvariety of $\mathbf{CA}_2$ of finite depth is finitely axiomatizable (for the Jankov formulas see, e.g., [6]).

7.2 Reduct functors

Suppose $\mathcal{B} = (B, \exists_1, \exists_2, d)$ is a cylindric algebra. In Section 2 we denoted its $\mathbf{Df}_2$-reduct by $\mathbb{F}(\mathcal{B}) = (B, \exists_1, \exists_2) \in \mathbf{Df}_2$. If K is a subclass of $\mathbf{CA}_2$, let $\mathbb{F}(K) = \{\mathbb{F}(\mathcal{B}) : \mathcal{B} \in K\}$. Also if M is a subclass of $\mathbf{Df}_2$, let $\mathbb{F}^{-1}(M) = \{\mathcal{B} \in \mathbf{CA}_2 : \mathbb{F}(\mathcal{B}) \in M\}$.

Lemma 7.9. *Suppose $K \subseteq \mathbf{CA}_2$ and $M \subseteq \mathbf{Df}_2$. Then the following hold.*

1. $\mathbf{H}\mathbb{F}(K) = \mathbb{F}\mathbf{H}(K)$.
2. $\mathbf{S}\mathbb{F}(K) \supset \mathbb{F}\mathbf{S}(K)$.
3. $\mathbf{P}\mathbb{F}(K) = \mathbb{F}\mathbf{P}(K)$.
4. $\mathbf{H}\mathbb{F}^{-1}(M) \subset \mathbb{F}^{-1}\mathbf{H}(M)$.
5. $\mathbf{S}\mathbb{F}^{-1}(M) \subset \mathbb{F}^{-1}\mathbf{S}(M)$.
6. $\mathbf{P}\mathbb{F}^{-1}(M) = \mathbb{F}^{-1}\mathbf{P}(M)$.

Proof. (1) Since every cylindric homomorphism is a also a $\mathbf{Df}_2$-homomorphism, it is obvious that $\mathbb{F}\mathbf{H}(K) \subseteq \mathbf{H}\mathbb{F}(K)$. Conversely, suppose (A, d) is a cylindric algebra and there is a $\mathbf{Df}_2$-homomorphism h from $A = \mathbb{F}(A, d)$ to a $\mathbf{Df}_2$-algebra B. Using the fact that h is onto it is easy to show that $(B, h(d))$ is a cylindric algebra, and that $h : (A, d) \to (B, h(d))$ is a cylindric homomorphism. Thus, $\mathbf{H}\mathbb{F}(K) = \mathbb{F}\mathbf{H}(K)$.

(2) It is obvious that if $\mathcal{B}$ is a cylindric subalgebra of $\mathcal{A}$, then $\mathbb{F}(\mathcal{B})$ is a $\mathbf{Df}_2$-subalgebra of $\mathbb{F}(\mathcal{A})$. Hence, $\mathbb{F}\mathbf{S}(K) \subseteq \mathbf{S}\mathbb{F}(K)$. To see that the converse inclusion does not hold, let $d(K) \geq 2$ and consider $\mathcal{B} \in K$ with $d(\mathcal{B}) \geq 2$. Denote by $\mathcal{X} = (X, E_1, E_2, D)$ the dual cylindric space of $\mathcal{B}$. Define a partition R on X by putting xRy if xE_2y. Then R is a correct $\mathbf{Df}_2$-partition and the $\mathbf{Df}_2$-algebra A corresponding to the $\mathbf{Df}_2$-space X/R belongs to $\mathbf{S}\mathbb{F}(K)$. On the other hand, the E_1-depth of X/R is 1 and the E_2-depth of X/R is ≥ 2. Therefore, X/R has different E_1 and E_2 depths, which implies that A can not be the reduct of any cylindric algebra. Thus, $\mathbf{S}\mathbb{F}(K) \not\subseteq \mathbb{F}\mathbf{S}(K)$.

(3) easily follows from the fact that for any family $\{\mathcal{B}_i\}_{i\in I}$ of cylindric algebras we have $\mathbb{F}(\prod_{i\in I}\mathcal{B}_i) = \prod_{i\in I}\mathbb{F}(\mathcal{B}_i)$.

(4) That $\mathbf{H}\mathbb{F}^{-1}(M) \subseteq \mathbb{F}^{-1}\mathbf{H}(M)$ follows from the fact that every cylindric homomorphism is a also a $\mathbf{Df}_2$-homomorphism. To show that this inclusion is proper, consider a cylindric algebra $\mathcal{B}$ and let A be a $\mathbf{Df}_2$-algebra such that $d_1(A) \neq d_2(A)$. Then $\mathbb{F}(\mathcal{B})$ is a homomorphic image of $\mathbb{F}(\mathcal{B}) \times A$, but since $d_1(A) \neq d_2(A)$, $\mathbb{F}(\mathcal{B}) \times A$ is not the reduct of any cylindric algebra. Hence, $\mathcal{B} \in \mathbb{F}^{-1}\mathbf{H}(\{\mathbb{F}(\mathcal{B}) \times A\})$, but $\mathbf{H}\mathbb{F}^{-1}(\{\mathbb{F}(\mathcal{B}) \times A\})$ is empty.

(5) That $\mathbf{S}\mathbb{F}^{-1}(M) \subseteq \mathbb{F}^{-1}\mathbf{S}(M)$ follows from the fact that if $\mathcal{B}$ is a cylindric subalgebra of $\mathcal{A}$, then $\mathbb{F}(\mathcal{B})$ is a $\mathbf{Df}_2$-subalgebra of $\mathbb{F}(\mathcal{A})$. To see that this inclusion is proper, suppose the two-element $\mathbf{Df}_2$-algebra $\mathbf{2}$ does not belong to M. Then the two-element cylindric algebra, $(\mathbf{2}, 1)$ does not belong to $\mathbb{F}^{-1}(M)$. By Proposition 7.4 $(\mathbf{2}, 1) \notin \mathbf{S}\mathbb{F}^{-1}(M)$. On the other hand, $\mathbf{2}$ is a $\mathbf{Df}_2$-subalgebra of every $\mathbf{Df}_2$-algebra. Therefore, $\mathbf{2} \in \mathbf{S}(M)$ and $(\mathbf{2}, 1) \in \mathbb{F}^{-1}(\mathbf{S}(M))$.

(6) That $\mathbf{P}\mathbb{F}^{-1}(M) \subseteq \mathbb{F}^{-1}\mathbf{P}(M)$ follows from the definition of the product of cylindric algebras. To see the converse, suppose $\mathcal{B} \in \mathbb{F}^{-1}\mathbf{P}(M)$. Then $\mathcal{B} = (B, d)$, where $B = \prod_{i\in I} B_i$ for some $\mathbf{Df}_2$-algebras $B_i \in M$. Let (B_i, d_i) be the i-th projection of $\mathcal{B}$. Since the i-th projection is an onto $\mathbf{Df}_2$-homomorphism, we can show similarly to (1) that each (B_i, d_i) is a cylindric algebra, and that $d = \langle d_i\rangle_{i\in I}$. Then $\mathcal{B}$ is isomorphic to $\prod_{i\in I}(B_i, d_i)$. Now every (B_i, d_i) belongs to $\mathbb{F}^{-1}(M)$. Hence, $\mathbb{F}^{-1}\mathbf{P}(M) \subseteq \mathbf{P}\mathbb{F}^{-1}(M)$. □

Theorem 7.10. *1. If $\mathbf{V}$ is a subvariety of $\mathbf{Df}_2$, then $\mathbb{F}^{-1}(\mathbf{V})$ is a subvariety of $\mathbf{CA}_2$.*

2. For a non-trivial subvariety $\mathbf{V}$ of $\mathbf{CA}_2$, $\mathbb{F}(\mathbf{V})$ is a subvariety of $\mathbf{Df}_2$ iff $\mathbf{V} = \mathbf{V}_{(\mathbf{2},1)\times(\mathbf{2}^n,d)}$ for some $n \in \omega$.

Proof. (1) By Lemma 7.9 we have $\mathbf{HSP}\mathbb{F}^{-1}(\mathbf{V}) \subseteq \mathbb{F}^{-1}(\mathbf{HSP}(\mathbf{V})) = \mathbb{F}^{-1}(\mathbf{V})$. Hence, $\mathbb{F}^{-1}(\mathbf{V})$ is a variety of cylindric algebras.

(2) Suppose $\mathbf{V}$ is a subvariety of $\mathbf{CA}_2$. If $d(\mathbf{V}) > 1$, then it follows from the proof of Lemma 7.9(2) that $\mathbb{F}(\mathbf{V})$ is not closed under subalgebras, hence is not a variety. If $(\mathbf{2}, 1) \notin \mathbf{V}$, then $\mathbb{F}(\mathbf{2}, 1) \notin \mathbb{F}(\mathbf{V})$ and again $\mathbb{F}(\mathbf{V})$ is not a variety since every nontrivial variety of diagonal-free cylindric algebras contains $\mathbf{2} = \mathbb{F}(\mathbf{2}, 1)$. Finally, we show that $\mathbb{F}(\mathbf{V^1})$ is not a variety. Let $\mathbb{C}$ denote the Cantor space. Then $\mathcal{X} = (\mathbb{C}, E_1, E_2)$ is a $\mathbf{Df}_2$-space, where $E_1(c) = E_2(c) = \mathbb{C}$ for any $c \in \mathbb{C}$. If $\mathcal{X}$ was the reduct of a cylindric space, then $\mathcal{X}$ would contain an isolated point. Since $\mathbb{C}$ is dense in itself,

it follows that $\mathcal{X}$ is not the reduct of any cylindric space. Let $\{y\}$ be a singleton topological space. Then $\mathcal{Y} = (\mathbb{C} \oplus \{y\}, E_1, E_2, \{y\})$ is a cylindric space, where $E_1(x) = E_2(x) = \mathbb{C} \oplus \{y\}$ for any $x \in \mathbb{C} \oplus \{y\}$. Moreover, $\mathcal{B} = (CP(\mathcal{Y}), E_1, E_2, \{y\})$ is an infinite simple cylindric algebra of depth 1, and so $\mathcal{B} \in \mathbf{V}^1$. Now consider $\mathbb{R}(\mathcal{Y}) = (\mathbb{C} \oplus \{x_0\}, E_1, E_2)$. Fix any point $c \in \mathbb{C}$ and let R be the smallest equivalence relation which identifies y and c. It is easy to check that R is a correct $\mathbf{Df}_2$-partition, and that $\mathbb{R}(\mathcal{Y})/R$ is isomorphic to $\mathcal{X}$. So, $A = (CP(\mathcal{X}), E_1, E_2)$ is isomorphic to a $\mathbf{Df}_2$-subalgebra of $\mathbb{F}(\mathcal{B})$, but it is not the reduct of any cylindric algebra. Hence, A does not belong to $\mathbb{F}(\mathbf{V}^1)$, and so $\mathbb{F}(\mathbf{V}^1)$ is not a variety. Therefore, if $\mathbf{V} \neq \mathbf{V}_{(\mathbf{2},1)\times(\mathbf{2}^n,d)}$ for any $n \in \omega$, then $\mathbb{F}(\mathbf{V})$ is not a variety. Conversely, one can easily verify that for any $n \in \omega$, $\mathbb{F}(\mathbf{V}_{(\mathbf{2},1)\times(\mathbf{2}^n,d)}) = \mathbf{V}_{\mathbf{2}^n}$, where $\mathbf{2}^n = \mathbb{F}((\mathbf{2}^n, d))$. This finishes the proof of the theorem.

□

We define a map $\Phi : \Lambda(\mathbf{CA}_2) \to \Lambda(\mathbf{Df}_2)$ from the lattice $\Lambda(\mathbf{CA}_2)$ of subvarieties of $\mathbf{CA}_2$ to the lattice $\Lambda(\mathbf{Df}_2)$ of subvarieties of $\mathbf{Df}_2$ by putting $\Phi(\mathbf{V}) = \mathbf{S}(\mathbb{F}(\mathbf{V}))$. It follows from Lemma 7.9 that Φ is well defined. The following theorem establishes basic properties of Φ.

Theorem 7.11. *1. Φ is order preserving.*

2. For $\mathbf{L} \in \Lambda(\mathbf{Df}_2)$, if $d_1(\mathbf{L}) \neq d_2(\mathbf{L})$, then $\Phi^{-1}(\mathbf{L}) = \emptyset$.

3. $\Phi^{-1}(\mathbf{Df}_2) = [\mathbf{U}, \mathbf{CA}_2]$.

4. Φ is neither surjective nor injective and does not preserve $\wedge$.

5. Φ preserves top, bottom, and $\vee$.

Proof. (1) directly follows from the definition of Φ.

(2) First we show that

$$d(\mathbf{V}) = d_1(\Phi(\mathbf{V})) = d_2(\Phi(\mathbf{V})) \qquad (D)$$

for every $\mathbf{V} \in \Lambda(\mathbf{CA}_2)$. It is obvious that $d(\mathbf{V}) \leq d_1(\Phi(\mathbf{V})), d_2(\Phi(\mathbf{V}))$. Conversely, for each finite simple algebra $A \in \Phi(\mathbf{V})$, there exists $B \in \mathbb{F}(\mathbf{V})$ such that A is a subalgebra of B. Hence, $d_i(A) \leq d_i(B) \leq d(\mathbf{V})$, and therefore, $d(\mathbf{V}) \geq d_1(\Phi(\mathbf{V})), d_2(\Phi(\mathbf{V}))$. Now suppose $\mathbf{L} \in \Lambda(\mathbf{Df}_2)$ and $d_1(\mathbf{L}) \neq d_2(\mathbf{L})$. If there exists $\mathbf{V} \in \Phi^{-1}(\mathbf{L})$, then it follows from (D) that $d(\mathbf{V}) = d_1(\mathbf{L}) = d_2(\mathbf{L})$, which is a contradiction.

(3) First we show that $\Phi(\mathbf{U}) = \mathbf{Df}_2$. Let $\mathcal{B}_n$ be a finite uniform cylindric algebra and $\mathcal{X}_n$ its dual uniform space. Then the quotient-space $\mathbb{R}(\mathcal{X}_n)/E_0$ is isomorphic to $n \times n$. Hence, every finite square $\mathbf{Df}_2$-algebra is a subalgebra of $\mathbb{F}(\mathcal{B}_n)$ for some $n \in \omega$. Therefore, every finite square $\mathbf{Df}_2$-algebra belongs to $\Phi(\mathbf{U})$. Since $\mathbf{Df}_2$ is generated by its finite square algebras (see, e.g., [9] or [2]), then $\Phi(\mathbf{U}) = \mathbf{Df}_2$. Now since Φ is order preserving, we get that $\Phi^{-1}(\mathbf{Df}_2) = [\mathbf{U}, \mathbf{CA}_2]$.

(4) That Φ is not a surjection follows from (2). To see that it is not an injection consider the varieties $\mathbf{V}_{\mathcal{B}_2^3}$ and $\mathbf{V}_{\mathcal{B}_2^5}$, where $\mathcal{B}_2^3$ and $\mathcal{B}_2^5$ denote the cylindric algebras of the power sets of the cylindric spaces $\mathcal{X}_2^3$ and $\mathcal{X}_2^5$ shown in Fig.4 above (see Section 6). Since $\mathcal{B}_2^3$ is representable and $\mathcal{B}_2^5$ is not, $\mathcal{B}_2^3$ is not isomorphic to $\mathcal{B}_2^5$. Therefore, $\mathbf{V}_{\mathcal{B}_2^3} \neq \mathbf{V}_{\mathcal{B}_2^5}$. However, $\mathbb{F}(\mathcal{B}_2^3)$ is isomorphic to $\mathbb{F}(\mathcal{B}_2^5)$. Hence, $\Phi(\mathbf{V}_{\mathcal{B}_2^3}) = \Phi(\mathbf{V}_{\mathcal{B}_2^5})$, and so $\Phi : \Lambda(\mathbf{CA}_2) \to \Lambda(\mathbf{Df}_2)$ is not an injection.

To show that Φ does not preserve $\wedge$ we again consider the varieties $\mathbf{V}_{\mathcal{B}_2^3}$ and $\mathbf{V}_{\mathcal{B}_2^5}$. It is easy to check that $(\mathbf{2}^2, d)$ is the only subdirectly irreducible member of the variety $\mathbf{V}_{\mathcal{B}_2^3} \cap \mathbf{V}_{\mathcal{B}_2^5}$. Therefore, $\mathbf{V}_{\mathcal{B}_2^3} \cap \mathbf{V}_{\mathcal{B}_2^5} = \mathbf{V}_{(\mathbf{2}^2,d)}$. However, since $\mathbb{F}(\mathcal{B}_2^3)$ is isomorphic to $\mathbb{F}(\mathcal{B}_2^5)$, $\mathbb{F}(\mathcal{B}_2^3)$ belongs to both $\Phi(\mathbf{V}_{\mathcal{B}_2^3})$ and $\Phi(\mathbf{V}_{\mathcal{B}_2^5})$. Hence, it also belongs to their intersection. By (D) we know that $d_1(\Phi(\mathbf{V}_{(\mathbf{2}^2,d)})) = d_2(\Phi(\mathbf{V}_{(\mathbf{2}^2,d)})) = 1$. On the other hand, $d_i(\Phi(\mathbf{V}_{\mathcal{B}_2^3}) \cap \Phi(\mathbf{V}_{\mathcal{B}_2^5})) = 2$ for $i = 1, 2$. Therefore, $\Phi(\mathbf{V}_{(\mathbf{2}^2,d)}) \neq \Phi(\mathbf{V}_{\mathcal{B}_2^3}) \cap \Phi(\mathbf{V}_{\mathcal{B}_2^5})$, and so Φ does not preserve $\wedge$.

(5) That $\Phi(\mathbf{CA}_2) = \mathbf{Df}_2$ follows from (3). Hence, Φ preserves top. Obviously the Φ-reduct of the trivial variety of cylindric algebras is the trivial variety of $\mathbf{Df}_2$-algebras. Therefore, Φ preserves bottom. Finally, we show that Φ preserves $\vee$, that is $\Phi(\mathbf{V}_1 \vee \mathbf{V}_2) = \Phi(\mathbf{V}_1) \vee \Phi(\mathbf{V}_2)$. Indeed,

$$\begin{aligned} \Phi(\mathbf{V}_1 \vee \mathbf{V}_2) &= \mathbf{HSP}(\{\mathbb{F}(\mathcal{B}) : \mathcal{B} \in \mathbf{V}_1 \vee \mathbf{V}_2\}) \\ &= \mathbf{HSP}(\{\mathbb{F}(\mathcal{B}) : \mathcal{B} \in (\mathbf{V}_1 \vee \mathbf{V}_2)_S\}). \end{aligned}$$

By Jónsson's lemma $(\mathbf{V}_1 \vee \mathbf{V}_2)_S = (\mathbf{V}_1)_S \cup (\mathbf{V}_2)_S$. Also recall that for arbitrary classes of universal algebras Γ and Δ, we have $\mathbf{HSP}(\Gamma \cup \Delta) = \mathbf{HSP}(\mathbf{HSP}(\Gamma) \cup \mathbf{HSP}(\Delta))$. Hence,

$$\begin{aligned}\Phi(\mathbf{V}_1 \vee \mathbf{V}_2) &= \mathbf{HSP}(\{\mathbb{F}(\mathcal{B}) : \mathcal{B} \in (\mathbf{V}_1)_S \cup (\mathbf{V}_2)_S\}) \\ &= \mathbf{HSP}(\{\mathbb{F}(\mathcal{B}) : \mathcal{B} \in (\mathbf{V}_1)_S\} \cup \{\mathbb{F}(\mathcal{B}) : \mathcal{B} \in (\mathbf{V}_2)_S\}) \\ &= \mathbf{HSP}(\mathbf{HSP}(\{\mathbb{F}(\mathcal{B}) : \mathcal{B} \in (\mathbf{V}_1)_S\}) \cup \\ &\quad\ \mathbf{HSP}(\{\mathbb{F}(\mathcal{B}) : \mathcal{B} \in (\mathbf{V}_2)_S\})) \\ &= \mathbf{HSP}(\Phi(\mathbf{V}_1) \cup \Phi(\mathbf{V}_2)) \\ &= \Phi(\mathbf{V}_1) \vee \Phi(\mathbf{V}_2).\end{aligned}$$

□

Note that there are subvarieties $\mathbf{L}$ of $\mathbf{Df}_2$ such that $d_1(\mathbf{L}) = d_2(\mathbf{L})$ and still $\Phi^{-1}(\mathbf{L}) = \emptyset$. For example, let $\mathbf{L}$ be a proper subvariety of $\mathbf{Df}_2$ with $d_1(\mathbf{L}) = d_2(\mathbf{L}) = \omega$. We know from [2] that such varieties exist. If $\Phi^{-1}(\mathbf{L}) \neq \emptyset$, then there exists $\mathbf{V} \in \Lambda(\mathbf{CA}_2)$ such that $\Phi(\mathbf{V}) = \mathbf{L}$. It follows from the equation (D) that $d(\mathbf{V}) = \omega$. Therefore, $\mathbf{V} \in [\mathbf{U}, \mathbf{CA}_2]$. This, together with Theorem 7.11(3), implies that $\Phi(\mathbf{V}) = \mathbf{Df}_2 \neq \mathbf{L}$, which is a contradiction. Thus, $\Phi^{-1}(\mathbf{L}) = \emptyset$, even though $d_1(\mathbf{L}) = d_2(\mathbf{L})$.

Suppose $\mathbf{V} \in \Lambda(\mathbf{CA}_2)$ and $\mathbf{L} \in \Lambda(\mathbf{Df}_2)$. For a property $\mathcal{P}$ of varieties of universal algebras, we say that Φ *preserves* $\mathcal{P}$ if $\Phi(\mathbf{V})$ has $\mathcal{P}$ whenever $\mathbf{V}$ does; and we say that Φ *reflects* $\mathcal{P}$ if every variety in $\Phi^{-1}(\mathbf{L})$ has $\mathcal{P}$ whenever $\mathbf{L}$ does.

Theorem 7.12. 1. *$\mathcal{P}$ is preserved by Φ if $\mathcal{P}$ is one of the following properties: (a) being finitely approximable; (b) being of finite depth; (c) being locally finite; (d) being pre locally finite; (e) being finitely generated.*

2. *$\mathcal{P}$ is not preserved by Φ if $\mathcal{P}$ is the property of being pre finitely generated.*

3. *$\mathcal{P}$ is reflected by Φ if $\mathcal{P}$ is one of the following properties: (a) being of finite depth; (b) being locally finite; (c) being finitely generated;*

4. *$\mathcal{P}$ is not reflected by Φ if $\mathcal{P}$ is the property of (a) being pre locally finite; (b) being pre finitely generated.*

Proof. 1. (a) is obvious since every subvariety of $\mathbf{Df}_2$ is finitely approximable (see [2, Corollary 4.9.]). (b) follows from the equation (D). (c) Suppose $\mathbf{V} \subseteq \mathbf{CA}_2$ is locally finite. Then $\mathbf{V}$ has finite depth. By (b) $\Phi(\mathbf{V})$ also has finite depth. Hence, $\Phi(\mathbf{V})$ is a proper subvariety of $\mathbf{Df}_2$. But every proper subvariety of $\mathbf{Df}_2$ is locally finite. Therefore, $\Phi(\mathbf{V})$ is locally finite. (d) The only pre locally finite subvarieties of $\mathbf{CA}_2$ and $\mathbf{Df}_2$ are $\mathbf{U}$ and $\mathbf{Df}_2$, respectively; and $\Phi(\mathbf{U}) = \mathbf{Df}_2$, by Theorem 7.11. (e) Suppose $\mathbf{V} \subseteq \mathbf{CA}_2$ is finitely generated. Then $\text{Fin}\mathbf{V}_S$ is finite by Theorem 7.2. Hence, $\mathbb{F}(\text{Fin}\mathbf{V}_S)$ is also finite. Since $\Phi(\mathbf{V})$ is generated by $\mathbb{F}(\text{Fin}\mathbf{V}_S)$ and every finite simple $\mathbf{Df}_2$-algebra has finitely many simple subalgebras, $\Phi(\mathbf{V})$ contains finitely many finite simple $\mathbf{Df}_2$-algebras. Therefore, $\Phi(\mathbf{V})$ is finitely generated (see [2]).

2. Observe that the Φ-images of pre finitely generated subvarieties of $\mathbf{CA}_2$ of depth 3 are varieties of $\mathbf{Df}_2$-algebras of both E_1- and E_2-depth 3. Also observe that subvarieties of $\mathbf{Df}_2$ of depth 3 are not pre finitely generated varieties (see [2, Theorem 5.4.]). The result follows.

3. (a) directly follows from the equation (D). (b) The only non-locally finite subvariety of $\mathbf{Df}_2$ is $\mathbf{Df}_2$ itself. By Theorem 7.11 $\Phi^{-1}(\mathbf{Df}_2) = [\mathbf{U}, \mathbf{CA}_2]$. Hence, Theorem 5.5 implies that if $\mathbf{L} \in \Lambda(\mathbf{Df}_2)$ is locally finite, then $\Phi^{-1}(\mathbf{L})$ is either empty or contains varieties of cylindric algebras of finite depth. Since every subvariety of $\mathbf{CA}_2$ of finite depth is locally finite, Φ reflects the property of being locally finite. (c) is proved similarly to 1(e).

4. (a) is obvious since $\Phi^{-1}(\mathbf{Df}_2) = [\mathbf{U}, \mathbf{CA}_2]$ and $\mathbf{Df}_2$ and $\mathbf{CA}_2$ are the only pre locally finite varieties in $\Lambda(\mathbf{Df}_2)$ and $\Lambda(\mathbf{CA}_2)$, respectively. (b) As follows from [2, Theorem 5.4.], the variety $\Phi(\mathbf{V}_1)$ is pre finitely generated. Since $\Phi^{-1}(\Phi(\mathbf{V}_1)) = \{\mathbf{V}_1, \mathbf{V}^1\}$ and $\mathbf{V}^1$ is not pre finitely generated (see Section 6), we obtain that the property of being pre finitely generated is not reflected by Φ.

$\square$

We conclude by mentioning that it is an open problem whether Φ reflects finite approximability. If every variety of cylindric algebras was finitely approximable, which we stated as an open problem at the end of Section 5, then the answer to this problem would be positive.

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