Non-well founded semantics for belief revision

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Chapter 1

Introduction

Among the literature in belief revision, we can roughly classify it between two main approaches[6]. The classical approach represented in the work done by [2] and called the AGM theory, based on a first order logic, suitable for static revision of factual information. The DEL approach, originated on the work of [22, 19, 18], and appropriate for multi-agent learning actions, and revision of higher order beliefs.

A fusion of the above mentioned theories, can be found in the Baltag and Smets approach. The advantages of the previous approaches, is taken into account here, systematizing several fine-grained distinctions into a unified framework, and the changes induced by the learning actions are emphasized.

One of the advantages of having a unified systematic framework is that it sheds light over the specific needs of the logic we want to work with, both at the syntactic and the semantic level. Two of them are the need of taking into account infinitary logics, and the consideration of non-well founded orders in the models used.

Infinitary examples of belief revision are not unusual. The Consecutive numbers puzzle[26] and others are a sample of this. Belief revision seen as a learning method requires non- well founded orders[20]. However, non of these features have received enough attention.

A non-well founded set semantics seems to us a suitable mean to create a framework which take care of these aspects. Moreover, the links between modal logics and non-well founded sets have received few attention, notwithstanding, the research in this field ([10],[11],[4],[15],[18]) has shown it is an area worthy to keep studying.

In this thesis we give a non-well founded set semantics for the logic $\mathcal{L}_{K\Box}$ and $\mathcal{L}^{\infty}_{K\Box}$ developed by Baltag and Smets[6]. It is structured as follows. In Chapter 2 we present the needed background in set theory as well as important results that will be used in the following chapters, and we settle the conventions in the notation used, most of it standard.

In Chapter 3 we give all the details for the static logic $\mathcal{L}_{K\Box}$ and $\mathcal{L}^{\infty}_{K\Box}$. We remark some of the differences in expressiveness among these languages, provide the standard results for the logic. We show also how in this setting we can have tighter results than just identifying a class of models for the sentences that true on it, but we can go here also the other way around, which is not possible in the usual semantics.

Chapter 4 deals with the definition of the three base operations used in belief revision: update, radical upgrade and conservative upgrade. We explained in detail the requirements for the definition of such operations in this setting in order to preserve the desired range of them. For the content of the definitions of this chapter, we preserve properties of the usual semantics, as well as the same difficulties.

Chapter 5 presents the generalization of the above operations, into the concepts of *Questions and Answers*. We preserve the advantages of functional composition while we have a more intricate behavior of the iteration of the operations defined when dealing with infinitary formulas. Notwithstanding, we have a suitable framework to get a further generalization of these model transformers seen as a learning methods, as is illustrated. Finally, we present some conclusions and future work.

Chapter 2

Background in Set-Theory

Non-well founded Set Theory forms part of the contemporary development of mathematics (or, if you want, of its recent history). It arose as an alternative system in which the Axiom of Foundation (AF) was given up from the Zermelo-Fraenkel axiomatization with the axiom of choice, and the now called Anti-foundation Axiom was added. The set of axioms resulting from giving up AF is usually denoted by ZFC^{-}. The reader should recall that it consists of the following axioms:

Axiom 2.1. Extensionality $(\forall x)(\forall y)[\forall z z \in x \leftrightarrow z \in y] \rightarrow x = y$

Axiom 2.2. Pair $(\forall x)(\forall y)[\exists z x \in z \wedge y \in z]$

Axiom 2.3. Union $(\forall x)(\exists y)(\forall z \in x)(\forall w \in z) w \in y$

Axiom 2.4. Power Set $(\forall x)(\exists y)(\forall z)[z \subseteq x \leftrightarrow z \in y]$

Axiom 2.5. Infinity $(\exists x)(\varnothing \in x \wedge (\forall y)[y \in x \rightarrow (\exists z \in x) \wedge z = y \cup \{y\}])$

Axiom 2.6. Replacement

$$(\forall x)(\forall y \in x)(\exists z)\varphi(y,z) \rightarrow (\exists w)(\forall y \in x)(\exists z \in w)\varphi(y,z)$$

Axiom 2.7. Subset Selection $(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow z \in x \wedge \varphi(z))$

Axiom 2.8. Choice $(\forall x)(\exists y)[y \textit{ is a well order of } x]$

where φ is a FOL formula with $\in$ as a relation symbol.

It is usually stated [21] that the Anti-foundation axiom was introduced by Marco Forti and Furio Honsell in their 1983 article [17]. However, some variants were already introduced since 1926. As a matter of fact, the axiom became widespread known until the publication of the by now classical Aczel's book [1]. Since then, the axiom has been called AFA, and thus the name of the different variants mentioned was coined: SAFA, (Scott's AFA) due to Dana Scott [24]; FAFA due to Paul Finsler [16]; and BAFA, due to Maurice Boffa[13]. Later, one of the firsts text-books on Set Theory in including information about non-well founded sets was [14] (just in its second edition).

To achieve an extension of the kind of sets we will deal with, it is need a different representation of them. Aczel's representation[1] is given by means of graphs. Barwise and Moss' representation[12] is given by means of equations. We can transit between both representations.

Here, we will work within the framework favored by Barwise and Moss. This means not only to work with equations but also to include entities that are not sets within our objects. The class of those entities is denoted by $\mathcal{U}$, and are called urelements. This leads to the need of adding an special axiom for them:

Axiom 2.9. Urelements $(\forall x)(\forall y)[\mathcal{U}(x) \rightarrow \neg(y \in x)]$

Here $\mathcal{U}$ is used as a predicate. The way these elements intervene in the construction of sets can be seen in the following definitions

Definition 2.1. *Given a set s, its transitive closure is denoted by TC, and can be defined by:*

$$TC(s) = \bigcup\{\bigcup^* s\}$$

The urelements involved in the formation of a set are given by its support.

Definition 2.2. *Given any set s, its support is the set $TC(s) \cap \mathcal{U}$. We will denote it by spp(s).*

To have an idea of how much of such elements we can have, it is needed an axiom. A version assuring that for any cardinality, we can find that much (fresh) urelements:

Axiom of Plenitude For every set s ,there is an injective function $f : s \to \mathcal{U}$so that $f[s] \cap s = \varnothing$.
is not enough. We may need a proper class as well. This is guaranteed by the following stronger version:

Axiom 2.10. Strong Axiom of Plenitude

There is an operation `new`*(s, A) so that*

- *For all sets s, and for all $A \subseteq \mathcal{U}$,* `new`*(s,A) $\in \mathcal{U} \setminus s$.*
- *For all $s \neq s'$, and all $A \subseteq \mathcal{U}$,* `new`*(s,A)≠* `new`*(s',A)*

and this is the axiom added in the frame of work of [12] due to the need of it in defining class operators.

An special subclass of $\mathcal{U}$ is the class of indeterminates. Sets of indeterminates will be denoted by $Z, Y, X, \ldots$, and members of them by the low case letters $z, y, x \ldots$. Indeterminates will allow us to define systems of equations. Sets of urelements will be denoted by $A, B, C, \ldots$. Elements of these classes will be denoted by the low case letters $p, q, r, \ldots$. Whenever we refer to the universe of pure sets it will be denoted by V_{afa}, and the universe of all sets with urelements by $V_{afa}[\mathcal{U}]$. In general, whenever we have any $A \subseteq \mathcal{U}$, it is denoted by $V_{afa}[A]$ the class of all sets constructed over A.

Also, as customary, for sets s, t such that $s \subseteq t$ we denote by $\overline{s}$ the set $t \setminus s$. By s^t the set of all functions from t to s, i.e. $s^t = \{f | \ f : t \to s\}$ The cardinality of a set will be denoted by $|s|$, and $dom(f)$, $rng(s)$ will denote the domain and the range of a function respectively.

Before we can give the first version of AFA we will work with, we need to introduce some concepts. The first one is the concept of a system of equations.

Definition 2.3. *A flat system of equations is a tuple* $\mathcal{E} = \langle X, A, e\rangle$ *where* X *and* A *are sets so that* $A \subseteq \mathcal{U}$, *and* $X \cap A = \varnothing$; *and* e *is a function* $e : X \to \wp(X \cup A)$

Given a system of equations we want to have a solution to it.

Definition 2.4. *A* solution *to a flat system of equations* $\mathcal{E} = \langle X, A, e\rangle$ *is a function* $\mathscr{s} : X \to V_{afa}[A]$ *such that*

$$\mathscr{s}(x) = \{ \ \mathscr{s}(y) \mid y \in e(x)\} \cup (e(x) \cap A) \text{ for all } x \in X$$

Flat systems of equations may be a bit unpractical, but are very useful in theory. A general, more handy (and equivalent) version of the above definitions will be given latter. However, this version allows to present a first formulation of AFA

Axiom 2.11. The Solution Lemma formulation of AFA *Every flat system of equations* $\mathcal{E}$ *has a unique solution*

The solution set of any system $\mathcal{E} = \langle X, A, e \rangle$ will be denoted by $sol - set$ and can be defined over the justification of this form of the axiom as:

$$sol - set(\mathcal{E}) = \{ \, \mathscr{s}(\mathrm{y}) \, | y \in X \}$$

To realize that every set has a representation of this kind, we need a slight generalization in the above version of this lemma. A generalized flat system is a tuple $\mathcal{E} = \langle X, A, e \rangle$ where X and A are any sets so that $X \cap A \neq \varnothing$ and e is as before.

With this change we can have a canonical representation for any set s as a system of equations, taking $X = TC(s)$, $A = spp(s)$ and $e = Id_X$. The solution lemma assure us that whenever we have a system of the kind defined, we know that it represents a set, and it is unique.

So far, all sets in V are represented. However, we have included as well the solution to the system $\mathcal{E} = \langle X, A, e \rangle$ with $X = \{x\}$, $A = \varnothing$, and $e(x) = \{x\}$. This solution is denoted Ω.

The addition of the axiom AFA, and the lack of FA leads also to some changes into customary tools we work with. The extensionality axiom needs to be strengthened. The way this stronger form is required to work needs of the following definitions.

Definition 2.5. *Let $A \subseteq \mathcal{U}$, and $\mathcal{E} = \langle X, A, e \rangle$, $\mathcal{E}' = \langle X', A, e' \rangle$ two generalized flat systems.*

- *A relation $R \subseteq X \times X'$ so that*
  - *If xRx', then $\forall y \in x \cap e(x)$, $\exists y' \in x' \cap e(x')$ such that yRy'*
  - *If xRx', then $\forall y' \in x' \cap e(x')$, $\exists y \in x \cap e(x)$, so that yRy'*
  - *If xRx', then $A \cap e(x) = A \cap e(x')$*

is called an A-bisimulation between $\mathcal{E}$ and $\mathcal{E}'$

- *Two systems $\mathcal{E}$, $\mathcal{E}'$ are A-bisimilar, denoted by $\mathcal{E} \equiv \mathcal{E}$ iff there exists an A-bisimulation between them so that*
  - $\forall x \in X\ \exists x' \in X'$ *so that* xRx'
  - $\forall x' \in X'\ \exists x \in X$ *so that* xRx'

The following theorem

Theorem 2.1. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two generalized flat systems over A, then $\mathcal{E}$ and $\mathcal{E}'$ have the same solution set iff $\mathcal{E} \equiv \mathcal{E}'$*

allows us to directly apply the relation of bisimulation to sets:

Definition 2.6. *Given two sets s, t and a binary relation R so that if sRt, then*

- $\forall s' \in s\ \exists\ t' \in t$ *so that* $s'Rt'$
- $\forall t' \in t\ \exists\ s' \in s$ *so that* $s'Rt'$
- $s \cap \mathcal{U} = t \cap \mathcal{U}$

we say that s and s are bisimilar $s \equiv t$ iff there exists a bisimulation R so that sRt.

and then have a stronger form of the extensionality axiom:

Theorem 2.2. STRONG EXTENSIONALITY *For all $s, t \in V_{afa}[\mathcal{U}]$, we have that $s = t$ iff $s \equiv t$*

of which the axiom of extensionality is a particular case.

Now, as we do not have FA in the axiomatization we will work with, we need an alternative to the induction principle (which still works restricted to well-founded sets).

A classical construction by recursion is that of naturals, which are the finite ordinals. In general, we will denote ordinals by $\alpha, \beta \ldots$ The principle of induction should be used in a weaker form:

Principle 2.1. $\in$ –Induction on Well-founded Sets *Let φ be a property such that for all $s \in V$, if $\varphi(t)$ for all $t \in s$, then also $\varphi(s)$, then you can conclude that for all $s \in V$ $\varphi(s)$*

The alternative we are looking for is given by a principle of co-induction. Also, to define functions or operations on proper classes we need a principle of co-recursion. To define these principles we need to recall some concepts.

We will understand by an operator $\Gamma : V_{afa}[\mathcal{U}] \to V_{afa}[\mathcal{U}]$ a transformation form sets into sets. We say that Γ is monotone if $\forall s, t \in V_{afa}[\mathcal{U}]$ so that $s \subseteq t$ we have $\Gamma(s) \subseteq \Gamma(t)$ We say that a class C is a fixed point of Γ if $C = \Gamma(C)$. We say that a class C is the greatest (respectively least) fixed point of Γ iff for every fixed point G of Γ we have that $G \subseteq C$ (respectively $C \subseteq G$)

Example, for a fixed $A \subseteq \mathcal{U}$ the operator

$\Gamma(s) = \{t \mid t \subseteq s \cap A, tisfinite\}$

has as a fixed point the class of all hereditarily finite sets over A. We can characterize that class a such a fixed point, Also we can have at hand the following definition:

Definition 2.7. *Let $A \subseteq \mathcal{U}$. A set is hereditarily finite over A if every set $t \in TC(\{b\})$ is finite and spp$(t) \subseteq A$*

this class is denoted by $HF^1[A]$. In general, whenever we talk of a set whose elements and elements of elements have the property X, we say that they conform the class of hereditarily X sets, and will be denoted by HX

Every monotone operator Γ has a greatest (respectively least) fixed point denoted Γ^* (Γ_*), and it can be characterized as the greatest (least) class such that $\Gamma(C) \subseteq C$ ($C \subseteq \Gamma(C)$).

One thing the reader should be aware of is that under this axiomatization, the least and greatest fixed points do not have to coincide. Consider the following examples[12]:

$$\Gamma(s) = \{t \mid t \subseteq s, t \in t\}$$

we have that in ZFC $\Gamma_* = \varnothing = \Gamma^*$. While, under this axiomatization we have that $\Gamma_* = \varnothing$ while $\Gamma^* = \Omega$.

A clearer illustration is the operator

$$\Gamma(s) = \wp(s)$$

Observe that this Γ under ZFC has as its least and greatest fixed point the class of all well-founded sets WF, while under this axiomatization, $\Gamma_* = WF$ and $\Gamma^* = V_{afa}$

We can define now the principles we have mentioned

Principle 2.2. Coinduction Principle *For a given* Γ *monotone operator, to prove that* $s \in \Gamma^*$*, prove that* $s \subseteq \Gamma(s)$

Principle 2.3. Corecursion Principle *For a given* Γ*, monotone, given a function* $\pi : C \to \Gamma(C)$ *there exists a unique function* $F : C \to \Gamma^*$ *such that*

$$F(C) = F[\pi(C)] \cup (C \cap A)$$

An example of the kind of functions we can construct with this principle are the following. Consider an operation skl which takes a set $s \in V_{afa}[\mathcal{U}]$ and takes away its urelements, returning just its structure. Such an operation can be defined as follows.

Let $\Gamma(s) = \{TC(\{s\}) \smallsetminus \mathcal{U}\}$. This operator is monotone. Its fixed point is the class V_{afa} Now let $\pi : V_{afa}[\mathcal{U}] \to V_{afa}$ defined as follows: $\pi(s) = TC(\{s\}) \cap V_{afa}[\mathcal{U}]$. Then, there exists a unique function $sk : V_{afa}[\mathcal{U}] \to V_{afa}$ such that

$$sk(s) = sk[\pi(s)] \tag{2.1}$$

returning the skeleton of a set.

Also, we can present now a general version of the Flat Solution Lemma. First, the generalization of the notion of a system of equations is as follows:

Definition 2.8. *A general system of equations is a tuple $\mathcal{E} = \langle X, A, e\rangle$ where $X, A \subseteq \mathcal{U}$, and $X \cap A = \varnothing$, and $e : X \to V_{afa}[X \cup A]$*

To generalize the notion of solution we need to introduce the notion of a substitution. A substitution is a function $\mathfrak{s}$ which has as a domain a set of urelements. For $l \in \mathcal{U} \cup V_{afa}[\mathcal{U}]$, it will be denoted by $l[\mathfrak{s}]$ the result of substituting the urelements in l by $\mathfrak{s}(l)$

The formal definition is given as follows

Let $\wp$ be the class of all substitutions, define

$$\pi : \wp \times (\mathcal{U} \cup V_{afa}[\mathcal{U}]) \to \mathcal{U} \cup V_{afa}[\mathcal{U}]$$

as follows:

$$\pi(\langle \mathfrak{s}, l\rangle) = \begin{cases} l[\mathfrak{s}] & \text{if } l \in dom(\mathfrak{s}) \\ l & \text{if } l \in \mathcal{U} \setminus dom(\mathfrak{s}) \\ \mathfrak{s}'(l) & \text{if } l \in V_{afa}[\mathcal{U}] \end{cases}$$

where $\mathfrak{s}'$ is the solution to the system $\mathcal{E} = \langle X, A, e\rangle$ with

$$X = (TC(\{l\}) \cup rng(\mathfrak{s})) \setminus \mathcal{U}$$
$$A = (TC(\{l\}) \cup rng(\mathfrak{s})) \cap \mathcal{U}$$
$$e(z) = \{x[\mathfrak{s}] \mid x \in z \cap dom(\mathfrak{s})\} \cup \{x \mid x \in z \cap (A \setminus dom(\mathfrak{s}))\} \cup (z \cap X)$$

Then, $sub(\langle \mathfrak{s}, b\rangle) = sub[\pi(\langle \mathfrak{s}, b\rangle)]$. To avoid cumbersome notation we will keep using $\mathfrak{s}$ for a substitution, which will be clearly specified when used with the squared brackets.

Definition 2.9. *A solution to a general system of equations $\mathcal{E} = \langle X, A, e\rangle$ is a function $\mathfrak{s}$ with domain X and such that*

$$\mathfrak{s}(x) = e(x)[\mathfrak{s}]$$

the general version of the axiom is given by this theorem:

Theorem 2.3. *Every general system of equations $\mathcal{E}$ has a unique solution $\mathfrak{s}$*

A further useful step may allow us to use operators in the definition of the equations of our systems. This should be done under the some restrictions given by the following definitions.

Definition 2.10. *A set $X \subseteq \mathcal{U}$ is new for an operator Γ if for all substitutions $\mathfrak{s}: X \to V_{afa}[\mathcal{U}]$ and for all sets s we have that*

$$\Gamma(s[\mathfrak{s}]) = \Gamma(s)[\mathfrak{s}]$$

Definition 2.11. *An operator* Γ *has the property that almost all urelements are new for* Γ *if there is a set* $X_\Gamma \subseteq \mathcal{U}$ *so that for all* $Y \subseteq \mathcal{U}$ *with* $Y \cap X = \varnothing$, Y *is new for* Γ*. Such a set* X_Γ *is called the avoidance set for* Γ

Definition 2.12. *An operator* Γ *is proper if for all* s *we have that* $\Gamma(s) \subseteq V_{afa}[\mathcal{U}]$

Then we can use an operator Γ in defining our equations if Γ

- Is proper
- Is monotone
- almost all urelements are new for Γ
- Its use is guarded by set brackets (i.e. its application comes within a set in the form $\{\Gamma(s)\}$),

We will call any such operator Γ an *appropriate* operator

The last restriction comes from the fact that certain equations may not have a solution. Consider for instance:

$$e(x) = \wp(x)$$

which cannot have a solution due to Cantor's theorem. The third restriction comes form the fact that omitting it may lead to lack of uniqueness in the solution of the systems so defined.

The quid to define what a solution for such a system of equations is lies in the interaction that a substitution function will have with the operator. We will talk then about a $\Gamma - term$, to be an expression of the kind $\Gamma(x)$, with x either a set or an urelement. Then the required interaction with these expressions is given by the following functions.

Proposition 2.1. *Let $\mathcal{s}$ be a substitution, and Γ an appropriate operator. Then, there exists a unique operation $[\mathcal{s}]_\Gamma$ so that for all sets s*

$$s[\mathcal{s}]_\Gamma = \{\Gamma(s'[\mathcal{s}]_\Gamma) \mid \forall s' \in s,\ s'\Gamma - term\} \cup \{s'[\mathcal{s}]_\Gamma \mid \forall s' \in s, s' not\ a\ \Gamma - term\}$$

Given a system of equations $\mathcal{E}$ where e is defined by means of an appropriate operator Γ, a solution $\mathcal{s}$ to $\mathcal{E}$ is a function $\mathcal{s}\colon X \to V_{afa}[\mathcal{U}]$ so that

$$\mathcal{s}(x) = e(x)[\mathcal{s}]_\Gamma$$

We can have as well then a version of the solution lemma that we can apply to work with the above definitions:

Theorem 2.4. *Let Γ be an appropriate operator. Then every equation system using Γ to define its equations has a unique $\mathcal{s}_\Gamma$ solution*

As in this setting we can have properties that are not present in the common Kripke semantics, to have an illustration of it we also will refer to some results from [3] that involve the use of modal logic.

Recall its syntax for a fixed set $A \subseteq \mathcal{U}$:

$$p \mid \neg\varphi \mid \bigwedge \Phi \mid \Box\varphi$$

where $p \in A$, Φ is a set of formulas, and we define the special constant $\bigwedge \varnothing = \top$ and $\neg\Box\neg = \Diamond$ the dual of the box. When we do restrict Φ to finite sets we talk about the finitary fragment denoted by $\mathcal{L}_\Box(A)$, and the infinitary fragment by $\mathcal{L}_\Box^\infty(A)$. In this setting we will always refer to a language depending on a fixed set A. To keep the simplest possible the notation we will avoid that reference, and we will specify it whenever it may lead to confusion.

The following results show to us the classes of sets that we are able to characterize in agreement with the language we use

Theorem 2.5. *Let $s, t \in V_{afa}[A]$. If s, t satisfy the same sentences of $\mathcal{L}^\infty(A)$, then $s \equiv t$*

Theorem 2.6. *Let $s, t \in HF^1[A]$. If s, t satisfy the same sentences of $\mathcal{L}(A)$, then $s \equiv t$*

This alert us of the differences in expressiveness that we will have, and also of the nice features of modal logic as a tool for doing set theory. Even more, we can have more precise results:

Definition 2.13. *Let $s \in V_{affa}[A]$, and ϑ a sentence in $\mathcal{L}(A)$ or $\mathcal{L}^\infty(A)$. We say that ϑ characterizes s in $V_{afa}[A]$ if for any $t \in V_{afa}[A]$ we have that $t \in [\![\vartheta]\!]_t$ iff $t = s$*

and in fact we have such a property for any set:

Theorem 2.7. *Every set $s \in V_{afa}[A]$ is characterizable in $V_{afa}[A]$ by some sentence ϑ^s of $\mathcal{L}^\infty(A)$*

Of course, the technical problem is to specify such a sentence. In order to do that, the following sentence will play a special role:

Definition 2.14. *For any set Φ of sentences, $\triangle\Phi$ is the following sentence:*

$$\bigwedge_{\varphi\in\Phi} \Diamond\varphi \wedge \Box \bigvee_{\varphi\in\Phi} \varphi$$

and here there is the first step in the definition of the characteristic sentence of a set:

Definition 2.15. *Let $A \subseteq \mathcal{U}$, fixed. For each $s \in V_{afa}[A]$ it is defined by recursion a transfinite sequence φ_α^s of formulas in $\mathcal{L}^\infty(A)$ as follows:*

$$\begin{aligned} \varphi_0^s &= \bigwedge_{p \in A \cap s} p \wedge \bigwedge_{p \in A \setminus s} \neg p \\ \varphi_{\alpha+1}^s &= \varphi_0^s \wedge \triangle\{\varphi_\alpha^{s_0} \mid s_0 \in s\} \\ \varphi_\lambda^s &= \bigwedge_{\alpha<\lambda} \varphi_\alpha^s \ \lambda \textit{limit ordinal} \end{aligned}$$

the only ingredient missing to complete its definition is a specific ordinal which depends on each set:

Theorem 2.8. *For every set $s \in V_{afa}[A]$, there exists an ordinal α so that φ_α^s characterizes s*

Definition 2.16. *For any $s \in V_{afa}[A]$ the degree of s is the least ordinal α such that φ_α^s characterizes s. It is denoted by $deg(s)$*

Finally, we will also make use or the following proposition:

Proposition 2.2. *Let W be transitive on sets. Then, the set of all infinitary sentences φ such that $models_W \varphi$ is closed under necessitation.*

Let T be some set or class of infinitary sentences closed under the rule of necessitation, and let $W = \{a \in V_{afa}[A] | a \vDash T\}$. Then, W is transitive on sets.

Let's proceed.

Chapter 3

Static Logic

In this chapter we present the static logic that will be at the base of the dynamic modalities for belief revision. First we present the class of sets that will constitute the semantics for the logics presented. We proceed then to give its syntax and semantics. Standard validity of the relevant axioms, soundness and completeness, as well an application of the characterization results mentioned in the previous chapter are included.

3.1 Plausibility-set Models

In this section we will construct the class of sets that will constitute the semantics for the logics we will work with. We want to express it as the greatest fixed point of a monotone operator to be able to use the principles presented in the previous chapter. Thus, the aim of this section is to show that the class of sets we will work with can be obtained in such a way. Recall that we denote by R,T and C the class of reflexive, transitive and connected sets respectively.

Let $A \subseteq \mathcal{U}$, fixed, and $s \in V_{afa}$. Consider the operator:

$$\Phi(s) = \{y \subseteq s \cup A \mid y \in C, \wedge y \subseteq (R \cap T) \cup A\}$$

It is easy to see that Φ is monotone, thus it has a greatest fix point. Denote it by Φ^*.

Proposition 3.1.

$$\Phi^* = \{y \mid spp(y) \subseteq A,\ \ y \in HC, \wedge \forall z \in y,\ \ z \in HRT\}$$

Let‘s verify our affirmation. First, we need to show that $\Phi^* \subseteq \Phi(\Phi^*)$.

Let $c \in \phi^*$. Then, $spp(c) \subseteq A$. Let $x \in c$, in particular $x \in HC$ and $\forall y \in x$, $y \in HRT$. Thus $x \in \Phi^*$. Then $c \subseteq \Phi^* \cup A$.

Second, as $c \in \Phi^*$, then $c \in C$, and $\forall x \in c$, $x \in RT$. Then $c \in \Phi(\Phi^*)$.

Now, let W be such that $W \subseteq \Phi(W)$. Let $w \in W$. Then $w \in \Phi(W)$ implies that $x \subseteq W \cup A$, $w \in C$, and $\forall y \in w\ y \in RT$.

Let $x \in w$, and let $x_1, x_2 \in x$. As $x \in T$, $x_1, x_2 \in w$, then either $x_1 \in x_2$ or $x_2 \in x_1$ by connectedness of w. Then x is connected. As those elements were arbitrary we can see that $w \in HC$.

Now, take $x \in w$ and $x_1 \in x$. As $x \in T$, then $x_1 \in w$ thus $x_1 \in R$. As those elements were arbitrary we can conclude that $x \in HR$.

Finally, by an analogous reasoning as before, we can see that $x \in HT$. Therefore $w \in \Phi^*$

□

We can now use the principle of coinduction and corecursion for this operator.

Definition 3.1. PLAUSIBILITY-SET MODELS

For any $A \subseteq \mathcal{U}$, fixed, let $\mathcal{M} = \Phi^* \cap T \cap V_{afa}[A]$. We define $\mathcal{M}$ as the class of sets that will constitute our models.

Observe that

$$\mathcal{M} = HTC \cap \wp(HR) \cap V_{afa}[A]$$

Claim 3.1. *For any A fixed, $\mathcal{M} \subseteq \Phi^*$*

Let $x \in \Phi^* \cap T \cap V_{afa}[A]$. We want to show $x \subseteq \Phi(x)$. Let $x' \in x$. Observe that $x' \subseteq x \cup A$ because $x \in \Phi^*$ implies that $spp(x) \subseteq A$, and as x is transitive, $x' \subseteq x$. Also, $x \in \Phi^*$ implies that $x' \in HRTC$, thus in particular, x' is connected and $\forall x'' \in x'$, $x'' \in HTR$. Hence $x' \in \Phi(x)$.

□

3.2 The logics $\mathcal{L}_{K\Box}$ and $\mathcal{L}^{\infty}_{K\Box}$

Ww now proceed to present the static logics we will work with. The logic $\mathcal{L}_{K\Box}$ (and $\mathcal{L}^{\infty}_{K\Box}$) were created by Baltag and Smets in their article [6]. There, K is the normal operator of knowledge, and $\Box$ is the operator called "safe belief". Conceptually, it was introduced by Stalnaker [25]. The idea encoded by this operator is that any formula guarded by it is a formula that will not be revised under truthful information. This is why is a belief safe to hold.

It is called also an operator of knowledge as $\Box$ is the box for the $\in$ relation. In the order given by the $\ni$ relation, we want to specify the following special cases. Its *strict* version is present when $s \ni t$ but $t \not\ni s$ then we will denote it by $>$. Also, we say that two sets are equiplausible $s \simeq t$ iff $s \ni t$ and $t \ni s$.

Given $c \in \mathcal{M}$, we define $bests(c) = Min_{\ni}\{t \mid t \ \in c\}$ A relation $\ni$ is called "quasi-well-founded"when the *strict* $>$ relation is converse well-founded.

3.2.1 Syntax

Now we will present the syntax for these logics.

$$\varphi := p \mid \neg\varphi \mid \ \bigwedge \Phi \mid \ K\varphi \mid \ \Box\,\varphi$$

where Φ is any set of formulas. Whenever we refer to the finitary fragment of the logic, we restrict $\bigwedge \Phi$ to any finite set Φ. It will be denoted by $\mathcal{L}_{K\Box}$. The full infinitary language will be denoted by $\mathcal{L}^{\infty}_{K\Box}$.

As usual the defined connectives and constant are:

$$\top := \bigwedge \varnothing$$

$$\bigvee \Phi := \neg \bigwedge_{\varphi \in \Phi} \neg\varphi$$

$$\hat{\mathrm{K}}\varphi := \neg K \neg\varphi$$

$$\Diamond\varphi := \neg \Box \neg\varphi$$

3.2.2 Semantics

For $c \in \mathcal{M}$, we have the following semantics:

$$[\![p]\!]_c = \{t \in c \mid p \in t\}$$

$$[\![\neg\varphi]\!]_c = c \setminus [\![\varphi]\!]_c$$

$$[\![\varphi \wedge \psi]\!]_c = [\![\varphi]\!]_c \cap [\![\psi]\!]_c$$

$$[\![\bigwedge_{\varphi \in \Phi} \varphi]\!]_c = \bigcap_{\varphi \in \Phi} [\![\varphi]\!]_c$$

$$[\![\Box\varphi]\!]_c = c \cap \wp([\![\varphi]\!]_c)$$

$$[\![K\varphi]\!]_c = \begin{cases} c & \text{if } [\![\varphi]\!]_c = c \\ \varnothing & \text{else} \end{cases}$$

Proposition 3.2. .

$$[\![\top]\!]_c = c$$
$$[\![\varphi \to \psi]\!]_c = \overline{[\![\varphi]\!]_c} \cup [\![\psi]\!]_c$$
$$[\![\bigvee \Phi]\!]_c = \overline{\bigcap_{\varphi \in \Phi} \overline{[\![\varphi]\!]_c}}$$

$$[\![\hat{K}\varphi]\!]_c = \begin{cases} c & \textit{if } [\![\varphi]\!]_c \neq \varnothing \\ \varnothing & \textit{else} \end{cases}$$

$$[\![\Diamond\varphi]\!]_c = \{t \in c |\ t \cap [\![\varphi]\!]_c \neq \varnothing\}$$

One of the nice features of this logic is that we can have as derived operators the belief operators as follows:

$$B^{\psi}\varphi = \hat{\mathrm{K}}\psi \to \hat{\mathrm{K}}(\psi \wedge \Box(\psi \to \varphi))$$

$$B\varphi = B^{\top}\varphi$$

Proposition 3.3.

$$[\![B\varphi]\!]_c = \begin{cases} c & \textit{if } \exists t \in c \textit{ so that } t \subseteq [\![\varphi]\!]_c \\ \varnothing & \textit{else} \end{cases}$$

$$[\![B^{\psi}\varphi]\!]_c = \begin{cases} c & if\ \exists t \in c, with\ t \cap [\![\psi]\!]_c \neq \varnothing,\ \ and\ \ t \cap [\![\psi]\!]_c \subseteq [\![\varphi]\!]_c \\ \varnothing & else \end{cases}$$

For a sentence $\varphi \in \mathcal{L}_{K\Box}$ or in $\mathcal{L}^{\infty}_{K\Box}$ we say that

- φ is satisfied for a set $t \in c$ with $c \in \mathcal{M}$ iff $t \in [\![\varphi]\!]_c$. We can also denote such relation by $t \vDash^{\mathcal{M}} \varphi$
- φ is valid with respect to the class $\mathcal{M}$ iff $\forall c \in \mathcal{M}$ we have that $[\![\varphi]\!]_c \neq \varnothing$. We denote this notion by $\vDash^{\mathcal{M}} \varphi$
- φ is a consequence of the set of sentence Φ iff $[\![\psi]\!]_c \subseteq [\![\varphi]\!]_c, \forall c \in \mathcal{M}$. This is also denoted $\Phi \vDash^{\mathcal{M}} \varphi$

whenever no superscript is present, we refer to the relation as holding for any class of sets. As well whenever we have any other class than $\mathcal{M}$ as superscript we refer to this notions as holding for that particular class.

Standard models are those sets $c \in \mathcal{M}$ in which the $>$ relation is well founded. This is, there are no infinite descending sequences:

$$c_0 > c_1 > c_2 > \dots$$

Any $c \in \mathcal{M}$ where $>$ is not well founded will be called *non-standard*. The finitary language is blind to this difference.

Consider the following example, which is a modification of an example offered in [12, 3].

Example 3.1. *Consider the set ω. By the axiom of plenitude, we can have a set X of indeterminates such that $X \sim \omega$. Now, considering the set X, by the strong axiom of plenitude we can have a set $A \subseteq \mathcal{U}$ so that $X \cap A = \varnothing$. Let $e : X \to V_{afa}[A]$ given as follows:*

$$e(x_i) = \{x_j\}_{j\leq i} \cup \{p_j\}_{j\leq i}$$

Let $\mathcal{E}$ denote such system of equations, and s its solution, and denote by $c_\omega = sol = set(\mathcal{E})$. Observe that $c_m \in \mathcal{M}$ and that $>$ are well founded. This is, it is an example of an standard model.

Now, consider the system of equations $\mathcal{E}^+$ with $X^+ = X \cup \{x_\omega\}$, $A^+ = A$ and

$$e^+(x_i) = \begin{cases} X^+ \cup A+ & \text{if } x_i = x_\omega \\ \{x_j\}_{j\geq i} \cup \{p_j\}_{j\geq i} & else \end{cases}$$

As before, let $\mathcal{s}$ be its solution, and denote by $c^+_\omega = sol - set(\mathcal{E}^+)$ Observe that $c^+_\omega \in \mathcal{M}$ and that in this case we do have infinite descending $>$ sequence of elements in c^+_ω.

Proposition 3.4. *c_ω and c^+_ω satisfy the same sentences in $\mathcal{L}_{K\Box}$*

Now, we will give a sentence $\vartheta \in \mathcal{L}^{\infty}_{K\Box}$ in which they differ. Consider the following sequence of formulas.

$$\tau_0 = p_0 \wedge \textstyle\bigwedge_{0<i} \neg p_i$$
$$\tau_{n+1} = \diamond\tau_n \wedge \textstyle\bigwedge_{i\leq n+1} p_{n+1} \wedge \textstyle\bigwedge_{n+1<j} \neg p_j$$

then, define the following sentence:

$$\vartheta := \diamond \bigwedge_{n\in\mathbb{N}} \tau_n$$

we have that $c^+_\omega \vDash \vartheta$ while $c_\omega \nvDash \vartheta$

3.3 Soundness and Completeness issues

As was noted in the previous chapter, we can characterize in infinitary modal languages infinite sets. We will return to this in the last section of this chapter. In particular, for the language $\mathcal{L}^{\infty}_{K\Box}$ means that correspondence of classes of sets and modal logics can be established at the level of models.

This correspondence for axioms in $\mathcal{L}_{K\Box}$ can only hold at the level of "frames". Let's illustrate the known results for these logics.

3.3.1 Known results on frame correspondance for $\mathcal{L}_{K\Box}$

Given any set, the operation *sk* defined in 2.1 returns the structure of such a set. Having the skeleton of a set, we can have the class of all sets that we can form out of that skeleton by adding the desired urelements from a fixed $A \subseteq \mathcal{U}$. An instance of such a class will be given by an operation *val* taking an skeleton or frame and returning a set in $V_{afa}[A]$.

In a general form, we can refer to a skeleton as a system $\mathcal{E} = \langle X, e \rangle$ Then, a valuation *val* depending on a set of urelements $A \subseteq \mathcal{U}$ will return a set $s_v = sol - set(\mathcal{E}_{val})$ where $\mathcal{E}_{val} = \langle X_v, A, e_v \rangle$ with $X_v = X$, and

$$e_v(x) = e(x) \cup val(x)$$

and $val(x) \subseteq A$

We can abstract away the conditions that the skeleton of a set must have in order to ensure that an axiom scheme can be valid on the class of sets that we can obtain by adding different urelements to it. These conditions are given as follows:

Definition 3.2. $\in^e$ *−Memebership*

Given a system $\mathcal{E} = \langle X, A, e\rangle$ *we define the relation* $\in^e$*, with* e *the equation function of* $\mathcal{E}$ *as follows*

$$x \in^e y \Leftrightarrow x \in e(y)$$

Then, the conditions to ensure validity of the scheme axioms are the following

Class	Defining Property	Axiom
Re	$x \in^e x$	$\Box\varphi \to \varphi$
Tra	$\forall x_1, x_2, x_3 \in X, \quad x_1 \in^e x_2 \in^e x_3 \Rightarrow x_1 \in^e x_3$	$\Box\varphi \to \Box\Box\varphi$
Sym	$\forall x_1, x_2 \in X$ if $x_1 \in^e x_2$, then $x_2 \in^e x_1$	$\varphi \to \Box\Diamond\varphi$
Con	$\forall x_1, x_2 \in X$ $x_1 \in^e x_2$ or $x_2 \in^e x_1$	$\Box(\Box\varphi \wedge \neg\psi) \vee \Box(\Box\psi \wedge \neg\varphi)$

Denote by Re, Tra, Sym and Con the class of sentences that are instances of the axiom $\Box\varphi \to \varphi$, $\Box\varphi \to \Box\Box\varphi$, $\varphi \to \Box\Diamond\varphi$ and $\Box(\Box\varphi \wedge \neg\psi) \vee \Box(\Box\psi \wedge \neg\varphi)$ respectively. We can proof then standard validity of the axioms we want. We show this just for transitivity as an example.

Proposition 3.5. *A skeleton* $\mathcal{E} = \langle X, e\rangle$ *is transitive iff* $\Box\varphi \to \Box\Box\varphi$ *is valid on it.*

$\Rightarrow$

Let $\mathcal{E} = \langle X, e\rangle$ be a transitive frame. Let val be any valuation such that for s_v with $s_v = sol - set(\mathcal{E}_{val} = \langle X, A, e\rangle)$ we have that $s_v = [\![\Box\varphi]\!]_{s_v}$. This means that $s_v = [\![\varphi]\!]_{s_v}$ Let x, y such that $y \in x \in s_v$. By assumption of transitivity, we have that $y \in s_v$. Thus, $y \in [\![\varphi]\!]_{s_v}$. We have then that $y \in [\![\Box\Box\varphi]\!]_{s_v}$. But x, y were arbitrary, then, $s_v \in [\![\Box\Box\varphi]\!]_{s_v}$

$\Leftarrow$

Let $\mathcal{E} = \langle X, e \rangle$ be a frame such that $\Box\varphi \to \Box\Box\varphi$ is valid on it. Suppose that $\mathcal{E}$ is not transitive. Then there exists $x_1, x_2 x_3 \in X$ so that $x_1 \in^e x_2 \in^e x_3$ but $x_1 \notin^e x_3$. The consider a valuation over the set $A = \{p\}$ returning the system $\mathcal{E}_{val} = \langle X_v, A, e_V \rangle$ where:

$$e_v(x_i) = \begin{cases} e(x_i) = e(x) \cup p & \text{if } x_i \in x_3 \\ e(x) & \text{else} \end{cases}$$

let s be its solution. Then

$s(x_3) \in [\![\Box p]\!]_{s_v}$ but $s(x_3) \notin [\![\Box\Box p]\!]_{s_v}$ contradicting our assumption. Hence, $\mathcal{E}$ should be transitive.

$\Box$

Definition 3.3. *Given any plausibility model $M = \langle W, \geq, V \rangle$, consider the following flat system $\mathcal{E}_M = \langle X, A, e \rangle$ with $X = W$, $A = dom(V)$ and $e(x) = \{y \mid x \geq y\} \cup V(x)$ Let $c_M = sol = set(\mathcal{E})$, we call $\mathcal{E}_M$ the canonical equational representation of M, and to c_M the canonical set associated with M*

Let PC be the class of plausibility models then we have that for any $M \in PC$, $c_m \models Tra \wedge Con$, and for any $c'_m \in c_m$ $c'_m \models Re \wedge Tra \wedge Con$

Let $\boldsymbol{K}\Box$ be the system with axioms and rules 1-6, and $\boldsymbol{K}\Box^\infty$ the whole set of axioms and rules from 1-11. See[6, 3]

1. Necessitation Rules for K and $\Box$
2. Modus Ponens

3. The $S5$–axioms for K
4. The $S4$–axioms for $\Box$
5. $K\varphi \to \Box\varphi$
6. $\Box(\Box\varphi \wedge \neg\psi) \vee \Box(\Box\psi \wedge \neg\varphi)$
7. From any countable set Φ infer $\bigwedge \Phi$
8. $\bigwedge_{\alpha<\beta} \varphi_\alpha \to \varphi_\alpha$
9. $\bigvee_{i\in I} \bigwedge_{j\in J} \varphi_{i,j} \to \bigwedge_{f\in J^I} \bigvee_{i\in I} \varphi_{i,f(i)}$
10. $K \bigwedge_{\alpha<\beta} \varphi_\alpha \to \bigwedge_{\alpha<\beta} K\varphi_\alpha$
11. $\Box \bigwedge_{\alpha<\beta} \varphi_\alpha \to \bigwedge_{\alpha<\beta} \Box\varphi_\alpha$

Definition 3.4. *A sentence φ is provable in $\boldsymbol{K\Box}$ (or in $\boldsymbol{K\Box^\infty}$) if it is in the smallest set of sentences containing all the instances of the axioms in $\boldsymbol{K\Box}$ (or $\boldsymbol{K\Box^\infty}$) and closed under MP and necessitation of K and $\Box$. It is denoted by $\vdash_{K\Box} \varphi$ if φ is provable in $\boldsymbol{K\Box}$, respectively this relation will be denoted with $\vdash_{\boldsymbol{K\Box^\infty}}$ for the system $\boldsymbol{K\Box^\infty}$ In general, φ is provable from a set T if there are $\psi_i \in T$ so that $\bigwedge \psi_i \to \varphi$. Then, this is denoted by $T \vdash_{K\Box} \varphi$ (respectively with the subscript $\boldsymbol{K\Box^\infty}$ for that system)*

Proposition 3.6. *Soundness.- For $\varphi \in \mathcal{L}^\infty_{K\Box}$, we have that if $T \vdash_{K\Box^\infty} \varphi$, then $\vDash_{\mathcal{M}} \varphi$*

It is easy to see that the axioms are valid with respect to $\mathcal{M}$. Thus, if φ is an instance of them we have directly that $\vDash^{\mathcal{M}} \varphi$. If φ was obtained by MP from a set of sentences $\{\psi \to \varphi, \psi\}$ assumed valid. Suppose that $\nvDash^{\mathcal{M}} \varphi$, then $\exists c \in \mathcal{M}$ so that $\nvDash^{\mathcal{M}} \varphi$. As $c \in \mathcal{M}$ then $c \vDash^{\mathcal{M}} \psi \to \varphi$ and $c \vDash^{\mathcal{M}} \psi$, but then $c \nvDash^{\mathcal{M}} \psi \to \varphi$, which is a contradiction. Hence we should have $\vDash^{\mathcal{M}} \varphi$ If φ was obtained by necessitation of either K or $\Box$ by proposition 2.2, we have that

$\vDash^{\mathcal{M}} \varphi$ Finally, if φ was obtained by the rule of infinitary conjunction, form a set Φ assumed to be valid, then we directly have has well that $\vDash^{\mathcal{M}} \varphi$

Proposition 3.7. *The class $\mathcal{M}$ is closed under equational representation*

Then we can apply the following theorem from Barwise and Moss[12]

Theorem 3.1. *Let C be a complete semantics for a theory T, and closed under equational representation, then for all $\varphi \in \mathcal{L}_{K\Box}$ we have that*

$$If\ T \vDash^{HCP} \varphi\ then\ T \vdash_{K\Box} \varphi$$

It has been shown [6, 3] that the class of connected preorders CP is a complete semantics for $\boldsymbol{K\Box}$. Thus for $\varphi \in \mathcal{L}_{K\Box}$ we have that

$$If\ T \vDash^{HPC} \varphi\ then\ T \vdash_{\boldsymbol{K\Box}} \varphi$$

Then, $\mathcal{M}$ is a complete semantics for $\mathcal{L}_{K\Box}$.

Conjecture 3.1. *The logic $\mathcal{L}^{\infty}_{K\Box}$ is complete with respect to the calculus $\boldsymbol{K\Box}^{\infty}$*

3.4 Characterizing Formulas

In this section we want to clarify how we can apply the results due to Baltag [3] mentioned in the previous chapter about the construction of a characteristic formula in our setting.

Thus, the first thing we need to do is tho have an analogous base sentence:

Definition 3.5. *Let $A \subseteq \mathcal{U}$, fixed. For each $s \in m \in \mathcal{M}$ it is defined by recursion on On a transfinite sequence φ^s_α of formulas in $\mathcal{L}^{\infty}_{K\Box}(A)$ as follows:*

$$\begin{aligned}
\varphi_0^{s,m} &= \bigwedge_{p \in A \cap s} p \wedge \bigwedge_{p \in A \smallsetminus s} \neg p \\
\varphi_{\alpha+1}^{s,m} &= \varphi_0^{s,m} \wedge \triangle_K \{\varphi_\alpha^{t,m} \mid t \in m\} \\
. & \qquad \wedge \triangle_\Box \{\varphi_\alpha^{t,m} \mid t \in s\} \\
\varphi_\lambda^{s,m} &= \bigwedge_{\alpha < \lambda} \varphi_\alpha^{s,m} \ \lambda \mathit{limit\ ordinal}
\end{aligned}$$

where $\triangle_\odot \Phi$ is the following sentence:

$$\bigwedge_{\varphi \in \Phi} \neg \odot \neg \varphi \wedge \odot \bigvee_{\varphi \in \Phi} \varphi$$

for $\odot$, in $\{K, \Box\}$

We need the following analogous lemma.

Lemma 3.1. *For $A \subseteq \mathcal{U}$, $s \in m \in \mathcal{M}$ and $s' \in m' \in \mathcal{M}$ with $\alpha, \beta \in On$ we have:*

1. $s' \in [\![\varphi_0^{s,m}]\!]_{m'}$ *iff* $s' \cap A = s \cap A$
2. $s \in [\![\varphi_\alpha^{s,m}]\!]_m$
3. $\alpha \geq \beta \Rightarrow \vDash \varphi_\alpha^{s,m} \to \varphi_\beta^{s,m}$
4. $\varphi_\alpha^{s,m}$, $\varphi_\alpha^{s',m'}$ *jointly satisfiable* $\Rightarrow$ $\varphi_\alpha^{s,m} = \varphi_\alpha^{s',m'}$
5. $\varphi_\alpha^{s,m} = \varphi_\alpha^{s',m'} \Rightarrow \varphi_\beta^{s,m} = \varphi_\beta^{s,m}$ $\forall \beta < \alpha$

We will see the first two cases. Part 1. is direct as $s' \in [\![\varphi_0^{s,m}]\!]_{m'}$ iff $\forall p, p \in s \cap A \Rightarrow p \in s'$ and $\forall p, p \in A \smallsetminus s \cap A \Rightarrow \neg p \in s'$ iff $s' \cap A = s \cap A$.

Part 2. Is proved by induction on $\alpha \in On$. $\alpha = 0$ is direct
Assume that $s \vDash \varphi_\beta^{s,m}$ for some $\beta \in On$. We want to prove that $s \vDash \varphi_\alpha^{s,m}$ for $\alpha = \beta + 1$.

It is clear that s satisfies the first conjunct. Then, let's see that

$$s \vDash \triangle_K \{\varphi_\beta^{t,m} | t \in m\}$$

as $s \vDash \varphi_\beta^{s,m}$, for $\gamma \leq \beta$ we have $s \vDash \triangle_K\{\varphi_\gamma^{t,m}|\ t \in m\}$, This means that for $t \in m$ there exists $s_t \in s$ so that $s_t \vDash \triangle\{\varphi_\beta^{t,m}|t \in m\}$. This means that $s \vDash \triangle_K\{\varphi_\beta^{t,m} \mid t \in m\}$

an analogous reasoning applies for $\Box$

Now let $\alpha := \lambda$ for λ limit ordinal.

Suppose $s \nvDash \bigwedge_{\beta<\lambda} \varphi_\beta^{s,m}$ then, there exists $\gamma \in On$ so that $s \nvDash \varphi_\gamma^{s,m}$ but this means that $s \nvDash \varphi_\xi^{s,m}$ for $\xi \leq \gamma$. Then $s \nvDash \varphi_0^{s,m}$, but this is a contradiction. Hence $s \vDash \bigwedge_{\beta<\gamma} \varphi_\beta^{s,m}$

□

We want the have the analogue of theorem 2.8 for our setting.

Theorem 3.2. *For any set $s \in m \in \mathcal{M}$ we will always have an ordinal α such that $\varphi_\alpha^{s,m}$ characterizes m.*

This will follow by the analogous of Baltag's lemma:

Let κ be a regular cardinal, then we define the relation R on $V_{afa}[A]$

$$xRy \quad \text{iff} \quad \exists z \in H_\kappa[A] \quad \text{so that} \quad \varphi_\kappa^{s,x} = \varphi_\kappa^{s,z} = \varphi_\kappa^{s,y}$$

The proof of these statements is also analogous as the original proof.

3.5 Correspondence

The aim of this section is to show that in this setting, we can have the traditional correspondence frame results not only at the level of frames as in the finitary case, but at the level of models due to the results mentioned in the previous section.

Definition 3.6. *A set S is closed under $\Box$ if whenever $\varphi \in S$, we have $\Box\varphi \in S$*

Let the following sets:

- $\boldsymbol{R}$ and $\boldsymbol{R}_\infty$
- $\boldsymbol{T}$ and $\boldsymbol{T}_\infty$
- $\boldsymbol{C}$ and $\boldsymbol{T}_\infty$

be the closure under necessitation of all the instances of the axioms $\Box\varphi \to \varphi$ $\Box\varphi \to \Box\Box\varphi$ and $\Box(\Box\varphi \wedge \neg\psi) \vee \Box(\Box\psi \wedge \neg\varphi)$, respectively, where $\varphi \in \mathcal{L}_{K\Box}$ when no subscript is present, and $\varphi \in \mathcal{L}^\infty_{K\Box}$ when we have the ∞ subscript. Then, we have the following theorem

Theorem 3.3. *For any set $s \in V_{afa}[A]$ in each case, affirmations i and ii are equivalent*

$i. - s \in HR \quad i. - s \in HT \quad i. - s \in HC$

$ii. - s \vDash \boldsymbol{R}_\infty \quad ii. - s \vDash \boldsymbol{T}_\infty \quad ii. - s \vDash \boldsymbol{C}_\infty$

as well as the finitary cases.

We will prove as an example the second case. $i \Rightarrow ii$ Let $s \in HT$, by proposition 2.2 it is enough to prove that $s \vDash \Box\varphi \to \Box\Box\varphi \ \forall\varphi \in \mathcal{L}^\infty_{K\Box}$. Thus, assume $s \vDash \Box\varphi$ and let $s_0, s_1 \in s$ so that $s_0 \in s_1 \in s$. By transitivity of s, we have that $s_0 \in s$, hence $s_0 \vDash \varphi$. As s_0, s_1 were arbitrary, thus $s \vDash \Box\varphi \to \Box\Box\varphi$

$ii. \Rightarrow i.$

Let $W = \{s | s \vDash \boldsymbol{T}_\infty\}$ Let $x \in W$, and $z \in y \in x$. Then, we have that $\vDash \vartheta^x \to \Diamond\vartheta^y$, and $\vDash \vartheta^y \to \Diamond\vartheta^z$. But then, by transitivity of $\to$ we have that $\vDash \vartheta^x \to \Diamond\vartheta^z$, thus $z \in x$, i.e. x is transitive. The same reasoning applies for any element $y \in x$. Thus as HT is the largest class with this property, then $x \in HT$.

Corollary 3.1. *For a fixed A, $c \in \mathcal{M}$ iff $c \models \boldsymbol{T}_\infty \wedge \boldsymbol{C}_\infty$ and for all $c' \in c$ we have $c' \models \boldsymbol{R}_\infty$*

Chapter 4

Doxastic Upgrades

First introduced by [23], update, radical upgrade and conservative upgrade are operations on pointed models which make us revise our information, with different degrees of trust in the source of information.

An infallible source corresponds to an update, in which we erase the worlds that do not satisfy the formula announced. A radical upgrade corresponds to a highly trusted source, but not know to be infallible, so we make the worlds satisfying the announced formula more plausible than the ones not satisfying it.

Finally, a conservative upgrade corresponds to a source in which we barely trust. There, we make only the most plausible world satisfying the sentence announced the most plausible overall. The general term to refer to any of these model transformers is *doxastic upgrades*[7, 9, 6].

In this chapter we will define those transformations over $\mathcal{M}$. We will proceed by giving first some extra conditions that we need in the functions we will work with, so we can remain within the range we want. Later we will propose the π functions needed to define a base function for doxastic upgrades. Finally we will give some relevant examples of their properties

4.1 General Requirements

Recall operator Φ defined in the previous section and the class $\mathcal{M}$ of models we are working on. Recall also the principle of corecursion for Φ

Principle 4.1. *Let $c \in \Phi^*$, and $\pi : c \to \Phi(c)$, then, there exists a unique function $F_\pi : c \to \Phi^*$ such that*

$$F_\pi(x) = F_\pi[\pi(x)] \cup (x \cap A) \ \forall x \in c$$

In this section, we want to establish some properties that will ensure that the range of the functions we will define,is the one we need. These properties or restrictions will be proposed for the pump functions we will use for the functions we define by corecursion. For these functions, we want them to have as range $\bigcup \mathcal{M}$.

Observe that for a fixed $A \subseteq \mathcal{U}$

$$\bigcup \mathcal{M} = HTCR \cap V_{afa}[A]$$

That will be the range of our base functions. Recall that we remain in Φ^* as $(\mathcal{M} \cap R) \subseteq \Phi^*$.

The restrictions we need to impose then, are encoded in the following properties.

Definition 4.1. $\pi - Membership$. *Given any $\pi : D \to V_{afa}$ we define*

$$x \in^\pi y \text{ iff } x \in \pi(y)$$

Definition 4.2. *Let D be any class and $\pi : D \to V_{afa}[\mathcal{U}]$ so that*

$$\forall x \in D \cap R \text{ we have that } x \in^\pi x$$

then, we say that π has the R_π property

Definition 4.3. *Let D be any class and $\pi : D \to V_{afa}[\mathcal{U}]$ so that*

$$\forall x, y, z \in D \cap T, \text{ if } x \in^\pi y \in^\pi z, \text{ then } x \in^\pi z$$

then, we say that π has the T_π property.

Theorem 4.1. *Let $\pi : c \to \Phi(c)$ for $c \in \mathcal{M}$ so that π satisfies the R_π and T_π properties. Then, there exists a unique function $F_\pi : c \to (\Phi^* \cap T) \cap R$ so that*

$$F_\pi(x) = F_\pi[\pi(x)] \cup (x \cap A) \; \forall x \in c$$

Existence and uniqueness follow from the principle of corecursion for Φ. This also means that by construction $F_\pi(x) \in \Phi^*$. Let $x \in c$ for $c \in \mathcal{M}$. This means that $x \in R$ then, by R_π, we have that $x \in \pi(x)$. Thus $F_\pi(x) \in F[\pi(x)] = F_\pi(x)$, i.e. $F_\pi(x) \in R$ Finally let $x \in c$, and y, z so that $y \in F_\pi(x)$ and $z \in y$. Observe that $F_\pi(x) \in \Phi^*$ implies that $y, z \in T$. Now, $y \in F_\pi(x)$ implies $\exists x_y \in \pi(x)$ so that $y = F_\pi(x_y)$. Analogously, $z \in y$ implies $\exists x_z \in \pi(x_y)$ so that $z = F_\pi(x_z)$ By T_π we have that $x_z \in \pi(x)$. Hence $z \in F_\pi(x)$

□

We can now characterize the kind of functions we will work with.

Corollary 4.1. *Assume that $\forall c \in \mathcal{M}$, there exists $\pi_c : c \to \Phi(c)$ satisfying the R_π and T_π properties. Then, there exists a unique $G : \mathcal{M} \to \mathcal{M}$ such that*

$$G(c) = F_{\pi_c}[c] \;\; \forall c \in \mathcal{M}$$

Existence and uniqueness of these functions follows by the existence and uniqueness of the base F_φ. Now, we have to prove that we have the desired range for the functions of the kind of G.

Claim 4.1. *For $c \in \mathcal{M}$, $G(c) \in \mathcal{M}$*

First, we need to show that for a fixed $A \subseteq \mathcal{U}$, we have $G(c) \in \Phi^*$.

By coinduction it is enough to prove that $G(c) \subseteq \Phi(G(c))$.

- We need to show that any $x \in G(c)$ is so that $x \subseteq G(c) \cap A$.
  Let $x \in G(c)$, then there exists c_x so that $x = F_{\pi_c}(c_x)$ then by definition of F_{π_c} we have $spp(x) \subseteq A$
  Let $y \in x$. This means that there exists $c_y \in c_x$ so that $y = F_{\pi_c}(c_y)$. By transitivity of c we have that $c_y \in c$. Thus $y \in G(c)$. Therefore $x \subseteq G(c) \cup A$
- We will show that x is connected. Let $y, z \in x = F_{\pi_c}[c_x]$, i.e. there exists $c_y, c_z \in c_x$ such that $y = F_{\pi_c}(c_y)$, $z = F_{\pi_c}(c_z)$. Observe that $c_x \in c$ implies that c_x is connected. As $c_y, c_z \in c_x$, then either $c_y \in c_z$, or $c_z \in c_y$. Assume $c_y \in c_z$, then $y = F_{\pi_c}(c_y) \in F_{\pi_c}(c_z) = z$. the other case is analogous.
- We need to show that any $y \in x$ is HRT Observe that $\forall y \in x$, $y = F_{\pi_c}(c_y)$. Then, by 4.1, we know that $y \in R$ and $y \in T$. Moreover, as $F_{\pi_c}[c] \subseteq \Phi^*$ then, any $z \in y$ is reflexive and transitive. Thus, we know that $y \in HRT$. Therefore $x \in \Phi(G(c))$

Second, let's see that$G(c)$ is transitive.

Let y, w be such that $y \in w \in G(c) = F_{\pi_c}[c]$, i.e. there exists $c_w \in c$ such that $w = F_{\pi_c}(c_w)$, and there exists $c_y \in c_w$ such that $y = F_{\pi_c}(c_y)$. By transitivity of c, $c_y \in c$, thus $y \in G(c)$

Therefore $G(c) \in \mathcal{M}$

□

4.2 Upgrades

We will define basic doxastic upgrades over $\mathcal{M}$. In all cases we will proceed as follows. First, we give the π functions for each case and check they fulfill the required properties. Second, we define by corecursion base functions that will be used to define the final functions that will constitute our doxastic upgrades.

4.2.1 Pump functions

Definition 4.4. *Given any $\varphi \in \mathcal{L}_{K\Box}$, $c \in \mathcal{M}$, for a fixed $A \subseteq \mathcal{U}$, consider the following functions $\pi : c \to \Phi(c)$*

$$\pi_{!\varphi}(t) = \begin{cases} (\llbracket\varphi\rrbracket_c \cap t) \cup (t \cap A) & \textit{if} \quad \llbracket\varphi\rrbracket_c \neq \varnothing \textit{and } t \in \llbracket\varphi\rrbracket_c \\ \uparrow & \textit{else} \end{cases} \tag{4.1}$$

$$\pi_{\Uparrow\varphi}(t) = \begin{cases} (\llbracket\varphi\rrbracket_c \cap t) \cup (t \cap A) & \textit{if} \quad t \in \llbracket\varphi\rrbracket_c \\ (\llbracket\varphi\rrbracket_c \cup t) \cup (t \cap A) & \textit{if} \quad t \in c \setminus \llbracket\varphi\rrbracket_c \end{cases} \tag{4.2}$$

$$\pi_{\uparrow\varphi}(t) = \begin{cases} (\bigcap[\![\varphi]\!]_c \cap t) \cup (t \cap A) & \textit{if } t \in \bigcap[\![\varphi]\!]_c \\ (\bigcap[\![\varphi]\!]_c \cup t) \cup (t \cap A) & \textit{if } t \in c \setminus \bigcap[\![\varphi]\!]_c \end{cases} \tag{4.3}$$

Proposition 4.1. *For π defined in equations (4.1),(4.2) and (4.3) above,* $\pi(c) \in \Phi(c) \forall c \in \mathcal{M}$

In order to prove the above proposition we need to observe that in all cases $[\![\varphi]\!]_c \subseteq c$ and $\forall x \in c$ implies that $spp([\![\varphi]\!]_c)$, $spp(x) \subseteq A$. Then, in all cases $spp(\pi(x)) \subseteq A$.

1. We will prove this for $\pi_{!\varphi}$

   First let's prove that $\pi_{!\varphi}(c) \subseteq c \cup A$ Let $x \in \pi_{!\varphi}(c)$, whenever $\pi_{!\varphi}$ is defined we have that $x \in [\![\varphi]\!]_c$ by definition of $\pi_{!\varphi}$. As $[\![\varphi]\!]_c \subseteq c$, then $x \in c$ and we already noted that it has the right support. Thus $\pi_{!\varphi}(c) \subseteq c \cup A$

   Second $\forall x \in \pi_{!\varphi}(c)$ we have that $x \in c$ (as we note in the previous part) This means that $x \in R$ and $x \in T$

   Finally, we need to show that $\pi_{!\varphi}(c)$ is connected. Let $y, z \in \pi_{!\varphi}(c)$. Then as before, we have that $y, z \in [\![\varphi]\!]_c$ and then $y, z \in c$. By connectedness of c we have that either $y \in z$ or $z \in y$. Thus $\pi_{!\varphi}(c)$ is connected.

   Therefore $\pi_{!\varphi} \in \Phi(c)$

   □

2. We will prove now $\pi_{\Uparrow\varphi}(c) \in \Phi(c)$

   As before let's see first that $\pi_{\Uparrow\varphi} \subseteq \Phi(c)$ Let $x \in \pi_{\Uparrow\varphi}$, then either $x \in [\![\varphi]\!]_c \cap t$ for some $t \in c$ or $x \in [\![\varphi]\!]_c \cup t$ for some $t \in c$. In both cases we can see that $x \in c$. Thus $\pi_{\Uparrow\varphi}(c) \subseteq \Phi(c)$

Second, as in the previous case it follows directly that for $x \in \pi_{\Uparrow\varphi}(x)$, in any of the two cases for this function, $x \in c$. Thus, $x \in R$ and $x \in T$ Finally, we need to check that $\pi_{\Uparrow\varphi}(c)$ is connected. Let $y, z \in \pi_{\Uparrow\varphi}(c)$. Then we have two cases.

- $y, z \in [\![\varphi]\!]_c \cap x$ for some $x \in c$ with $x \in [\![\varphi]\!]_c$.

  Observe that $x \in c$ implies that x is connected. Also, $\forall y, z \in [\![\varphi]\!]_c$, $y, z \in c$. Thus $y \in z$ or $z \in y$. Hence $[\![\varphi]\!]_c$ is connected, thus $[\![\varphi]\!]_c \cap x$ is connected. Thus, for $y, z \in [\![\varphi]\!]_c \cap x$ either $y \in z$ or $z \in y$.
- $y, z \in [\![\varphi]\!]_c \cup x$ for some $x \in c \setminus [\![\varphi]\!]_c$.

  Note that if $y, z \in [\![\varphi]\!]_c$, or $y, z \in x$, then by connectedness of those sets we are done. If $y \in [\![\varphi]\!]_c$ and $z \in x \setminus [\![\varphi]\!]_c$, as $[\![\varphi]\!]_c \subseteq c$, then $y \in c$. Also, as $x \in c \setminus [\![\varphi]\!]_c$, then $z \in c$ by transitivity of c Thus, either $z \in y$ or $y \in z$ by connectedness of c. The other case is analogous. Therefore $\pi_{\Uparrow\varphi}(c)$ is connected

Therefore $\pi_{\Uparrow\varphi}(c) \in \Phi(c)$

3. The case for $\pi_{\uparrow\varphi}$ is analogous to the case for $\pi_{\Uparrow\varphi}$ considering $\bigcap[\![\varphi]\!]_c$ instead of $[\![\varphi]\!]_c$.

□

Proposition 4.2. *The functions π defined in 4.1, 4.2 and 4.3 above satisfy the properties R_π and T_π*

As before, we make a a general remark valid for all cases, namely, that $x \in c$ implies that $x \in R$ and $x \in T$.

1. The case for $\pi_{!\varphi}$

   - We will show that $\pi_{!\varphi}$ has the R_π property Let $x \in c$. Note that whenever $\pi_{!\varphi}$ is defined $x \in [\![\varphi]\!]_c$. Also, $x \in R$ implies that $x \in x$ Thus $x \in [\![\varphi]\!]_c \cap x$. Then, $x \in^{\pi_{!\varphi}} x$ by definition of $\pi_{!\varphi}$.

- Let's see that $\pi_{!\varphi}$ has the T_π property Let y, z be such that $z \in^{\pi_{!\varphi}} y$ and $y \in^{\pi_{!\varphi}} x$ for some $x \in [\![\varphi]\!]_c$. Then, $y \in [\![\varphi]\!]_c \cap x$ and $z \in [\![\varphi]\!]_c \cap y$ by definition of $\pi_{!\varphi}$. In particular, we have $z \in y \in x$. By transitivity of x, we have $z \in x$. Thus $z \in [\![\varphi]\!]_c \cap x$ this means that $z \in^{\pi_{!\varphi}} x$ again by definition of $\pi_{!\varphi}$

2. Now, we will prove those properties for $\pi_{\Uparrow\varphi}$

   - $\pi_{\Uparrow\varphi}$ has the R_π property Let $x \in c$. If $x \in [\![\varphi]\!]_c$, then, $x \in [\![\varphi]\!]_c \cap x$, thus by definition of $\pi_{\Uparrow\varphi}$ $x \in^{\pi_{\Uparrow\varphi}} x$. If $x \in c \setminus [\![\varphi]\!]_c$, then $\pi_{\Uparrow\varphi}(x) = ([\![\varphi]\!]_c \cup x) \cup (x \cap A)$, then it is direct that $x \in^{\pi_{\Uparrow\varphi}} x$
   - We will prove now that $\pi_{\Uparrow\varphi}$ has the T_π property.

     Let x, y, z be so that $z \in^{\pi_{\Uparrow\varphi}} y \in^{\pi_{\Uparrow\varphi}} x$ We have the following cases:

     – $x \in [\![\varphi]\!]_c$. Then, by definition of $\pi_{\Uparrow\varphi}$ $y \in [\![\varphi]\!]_c \cap x_s$, and this implies that $z \in [\![\varphi]\!]_c \cap y$ because $y \in [\![\varphi]\!]_c$ and by definition of $\pi_{\Uparrow\varphi}$. Observe that in this case we have that $z \in y \in x$. Hence by transitivity of x, we have $z \in x$, and also $z \in [\![\varphi]\!]_c \cap x$. By definition of $\pi_{\Uparrow\varphi}$ this means that $z \in^{\pi_{\Uparrow\varphi}} x$.

     – $x \in c \setminus [\![\varphi]\!]_c$. Then $y \in^{\pi_{\Uparrow\varphi}} x$ implies that $y \in [\![\varphi]\!]_c \cup x$ by definition of $\pi_{\Uparrow\varphi}$. We have the following sub-cases:

       * $y \in [\![\varphi]\!]_c$. Then $z \in^{\pi_{\Uparrow\varphi}} y$ implies that $z \in [\![\varphi]\!]_c \cap y$. This means that $z \in cap$ implies that $z \in \cap \cup x$ and then $z \in^{\pi_{\Uparrow\varphi}} x$ by definition of $\pi_{\Uparrow\varphi}$.
       * $y \in x \setminus [\![\varphi]\!]_c$ Then, $z \in^{\pi_{\Uparrow\varphi}} y$ means that $z \in [\![\varphi]\!]_c \cup y$ If $z \in [\![\varphi]\!]_c$, then as in the previous case we have that $z \in^{\pi_{\Uparrow\varphi}} x$.

If $z \in y \setminus [\![\varphi]\!]_c$, then by transitivity of x, we have that $z \in x$. Then, $z \in [\![\varphi]\!]_c \cap x$, i.e. $z \in^{\pi_{\Uparrow\varphi}} x$ by definition of $\pi_{\Uparrow\varphi}$

3. The case for $\pi_{\uparrow\varphi}$ is totally analogous to the case in (2), by considering $\bigcap[\![\varphi]\!]_c$ instead of $[\![\varphi]\!]_c$

4.2.2 Upgrades

We have already verified that all the proposed pump functions satisfy the requirements we need Now, we will define some base functions by corecursion.

1. Considering $\pi_{!\varphi}$ defined in 4.1, by the corecursion principle for Φ, we have unique function $!_\varphi : c \to \mathcal{M}$, such that

$$!_\varphi(x) = !_\varphi[\pi_{!\varphi}(x)] \cup (x \cap A) \tag{4.4}$$

2. Analogously, with $\pi_{\Uparrow\varphi}$ defined in 4.2, we have unique function$\Uparrow_\varphi : c \to \mathcal{M}$ such that

$$\Uparrow_\varphi (t) = \Uparrow_\varphi [\pi_{\Uparrow\varphi}(t)] \cup (t \cap A) \tag{4.5}$$

3. Finally, by means of $\pi_{\uparrow\varphi}$ defined in 4.3, we have unique function $\uparrow_\varphi$: $c \to \mathcal{M}$ such that

$$\uparrow_\varphi (t) = \uparrow_\varphi [\pi_{\uparrow\varphi}(t)] \cup (t \cap A) \tag{4.6}$$

Observe that in each case of the above functions, its definition is independent of the c taken. Then we may assume we can have any such function $\forall c \in \mathcal{M}$. Now, considering in each case, the functions defined above, we can have our final functions as follows

1. Consider $\forall c \in \mathcal{M}$ functions $!^c_\varphi$ as defined in 4.4. Then, we have a unique function $!\varphi : \mathcal{M} \to \mathcal{M}$ such that:

$$!\varphi(c) = \Uparrow^c_\varphi [c] \quad \forall c \in \mathcal{M} \tag{4.7}$$

2. Now, take $\forall c \in \mathcal{M}$ functions $\Uparrow^c_\varphi$ as defined in 4.2. Then, we have a unique function $\Uparrow \varphi : \mathcal{M} \to \mathcal{M}$ such that

$$\Uparrow \varphi(c) = \Uparrow^c_\varphi [c] \ \forall c \in \mathcal{M} \tag{4.8}$$

3. Finally $\forall c \in \mathcal{M}$ take functions $\uparrow^c_\varphi$ as defined in 4.3. Then, we have a unique function $\uparrow \varphi : \mathcal{M} \to \mathcal{M}$ such that

$$\uparrow \varphi(c) = \uparrow^c_\varphi [c] \ \forall c \in \mathcal{M} \tag{4.9}$$

Proposition 4.3. *Existence and uniqueness of functions defined in 4.7, 4.8 and 4.9 can be derived from corollary 4.1*

We can add now the dynamic modality $\langle \dagger\varphi \rangle \psi$ for $\dagger \in \{!, \Uparrow, \uparrow\}$.

$$[\![\langle \dagger\varphi \rangle \psi]\!]_c = \dagger\varphi(c)^{-1}[[\![\psi]\!]]_c$$

where $\dagger\varphi(c)^{-1}$ means the pre-image of the transformation applied to the set in question.

4.3 Preserving Properties

During the proofs in the previous section, we have been suggesting a known property of the conservative upgrade, namely that it is a particular case of the radical upgrade. We can verify that it is the case for the functions we just defined. Take $bests$ to be a function such that $bests(c) = \bigcap c$, then:

Proposition 4.4. *For all* $c \in \mathcal{M}, and A \subseteq \mathcal{U}$ *we have that* $\uparrow \varphi(c) = \Uparrow bests([\![\varphi]\!]_c)(c)$

Let $c \in \mathcal{M} and A \subseteq \mathcal{U}$. We want to show that $\uparrow \varphi(c) = \Uparrow bests([\![\varphi]\!]_c)(c)$. Consider the following relation:

$$xRy \text{ iff } \exists c' \in c \text{ so that } x = \uparrow^c_\varphi (c') \text{ and } y = \Uparrow^c_{bests([\![\varphi]\!]_c)} (c')$$

we claim that R is a bisimulation for the sets in question.

Let $x \in \uparrow \varphi(c)$ and $y \in \Uparrow bests([\![\varphi]\!]_c)(c)$ so that xRy. This means that there exists $c' \in c$ so that $x = !^c_\varphi(c')$ and $y = \Uparrow^c_{bests([\![\varphi]\!]_c)} (c')$.

Then we have that $x \cap A = c' \cap A = y \cap A$ by definition of $\uparrow^c_\varphi$ and $\Uparrow^c_{bests([\![\varphi]\!]_c)}$ (c') respectively.

Now, let $x' \in x$. We have two cases:

- Case 1. $c' \in bests([\![\varphi]\!]_c)$

  Then, we have that $\exists c'_x \in c' \cap \bigcap [\![\varphi]\!]_c$ so that $x' = \uparrow^c_\varphi (c'_x)$ by definition of $\pi_{\uparrow\varphi}$ and $\uparrow^c_\varphi$. Then, this means that $c'_x \in c' \cap bests([\![\varphi]\!]_c)$ by definition of $bests$. Let $y' = \Uparrow^c_{bests([\![\varphi]\!]_c)} (c'_x)$. Thus we have that $y' \in y$ by definition of $\Uparrow^c_{bests([\![\varphi]\!]_c)}$, also that $x' \cap A = c'_x \cap A = y' \cap A$ by definition of $\pi_{\uparrow\varphi}$ and rup and $x'Ry'$ by our definition of R. The back condition is proved in an analogous way.

- Case 2. $c' \in c \setminus \bigcap [\![\varphi]\!]_c$ Then, we have that $x = \pi_{\uparrow\varphi}[c' \cup \bigcap [\![\varphi]\!]_c]$ and $y = \Uparrow^c_{bests([\![\varphi]\!]_c)} (c')[c' \cup bests([\![\varphi]\!]_c)]$ Then let $x' \in x$ This means that $\exists c'_x \in c' \cup \bigcap [\![\varphi]\!]_c$ so that $x' = \pi_{\uparrow\varphi}(c'_x)$. If $c'_x \in c'$ then we have that $c'_x \in c' \cup bests([\![\varphi]\!]_c)$ and by reasoning analogous to the previous case we have that $y' = \Uparrow^{c}_{bests(\llbracket\varphi\rrbracket_c)} (c')(c'_x)$ satisfies that $y' \in y$ and $x'Ry'$. The same follows if $c'_x \in \bigcap\llbracket\varphi\rrbracket_c$. The back condition holds also by an analogous reasoning.

Then, we have $\uparrow \varphi(c) = \Uparrow bests(\llbracket\varphi\rrbracket_c)(c)$. As c was arbitrary, we have proved our proposition.

□

It should be noted that here, we used in a direct way a set in the place where we had been taken a formula as a parameter of our transformation. This "sloppy"use will not hurt. We should think of a formula as a function from a model to its satisfaction set. We should formalize this notion in the next chapter.

Closure under composition of doxastic upgrades is not always possible. Doxastic updates have this closure property, and we can verify that indeed we preserve that property as well.

Proposition 4.5. *For all $c \in \mathcal{M}$ we have that $\llbracket[!\varphi][!\psi]\rrbracket_c = \llbracket[!(\varphi \wedge [!\varphi]\psi)]$*

Let $c \in \mathcal{M}$. We want to prove that $\llbracket[!\varphi][!\psi]\rrbracket_c = \llbracket[!(\varphi \wedge [!\varphi]\psi)]$

Consider the following relation:

$$xRy \text{ iff } x, y \in \llbracket\varphi \wedge \psi\rrbracket_c$$

We will show that this is a bisimulation for $\llbracket[!\varphi][!\psi]\rrbracket_c$ and $\llbracket[!(\varphi \wedge [!\varphi]\psi)]$

First observe that $x \in \llbracket[!\varphi][!\psi]\rrbracket_c$ implies that

i $x \in \llbracket[!\psi]\rrbracket_{!\varphi(c)}$ and also that this means that

ii $x \in !\psi(\llbracket\psi\rrbracket_{!\varphi(c)})$

On the other hand, $y \in \llbracket[!(\varphi \wedge [!\varphi]\psi)]$ implies that i' $y \in !(\varphi \wedge [!\varphi]\psi)(c)$, this means

ii' $y \in !(\varphi \wedge [!\varphi]\psi)(\llbracket\varphi\rrbracket_c \cap \llbracket[!\varphi]\rrbracket\psi)$

Thus, let $x \in \llbracket[!\varphi][!\psi]\rrbracket_c$ and $y \in \llbracket[!(\varphi \wedge [!\varphi]\psi)]$ so that xRy.

Now, let $x' \in x$. As $x \in \llbracket\varphi\rrbracket_c$, and $x \in !\varphi(c)$, this means that $x' \in \llbracket\varphi\rrbracket_c$ by definition of $!\varphi$. Also, as $x \in \llbracket\psi\rrbracket_c$ this means that $x' \in \llbracket\psi\rrbracket_{!\varphi(c)}$ by ii and definition of $!\psi$. This means that $x' \in \llbracket\varphi\rrbracket_c \cap \llbracket[!\varphi]\psi\rrbracket_c$ Then, let $y' = !(\varphi \wedge [!\varphi]\psi)(x')$. By ii' we have that $y' \in y$, and by definition of the updates involved we have that $y \in \llbracket\varphi \wedge \psi\rrbracket_c$ thus, $x'Ry'$

Now, let $y' \in y$. By ii' this means that there exist $c'_y \in \llbracket\varphi\rrbracket_c \cap \llbracket[!\varphi]\psi\llbracket_c$ so that $y' = !(\varphi \wedge [!\varphi]\psi)(c'_y)$ Then, as $c'_y \in \llbracket\varphi\rrbracket_c$, we know that $c'_y \in \llbracket\psi\rrbracket_{!\varphi(c)}$ for otherwise it would not belong to $\llbracket[!\varphi]\psi\rrbracket_c$ Thus, $x' = !\psi(c'_y)$ is so that $x' \in x$ and $x'Ry'$ by ii and definition of $!\psi$

Hence, $\llbracket[!\varphi][!\psi]\rrbracket_c = \llbracket[!(\varphi \wedge [!\varphi]\psi)]$ as we wanted. As c was arbitrary, we have prove our proposition

□

4.4 Preserving Difficulties

However neither for radical nor conservative upgrades we have this closure, and we inherited that feature. Consider the following variation of a scenario from [9, 26] A researcher is wondering whether it is raining or whether it is sunny in Amsterdam. He believes that it is more likely that it is raining and that it is not sunny, and that if it is sunny it is more likely that it is not raining than both things at the time. Take as symbols for *raining*; = r, and *sunny* := s

The model for this scenario is given by the following set $c = sol - set(\mathcal{E})$ where $\mathcal{E} = \langle X, A, e\rangle$ and $X = \{x, y, z\}$, $A = \{s, r\}$ and

$$e(x) = \{x, r\}$$
$$e(y) = \{y, x, s\}$$
$$e(z) = \{z, y, x, r, s\}$$

A radical upgrade with the least plausible scenario is given by $\Uparrow (r \wedge s)(c)$ Observe that $[\![r \wedge s]\!]_c = \mathcal{s}(z)$, Then,

$$\begin{aligned} \Uparrow (r \wedge s)(c) = & \qquad \Uparrow_{r\wedge s} [\pi_{\Uparrow_{r\wedge s}}(c)] \\ = & \quad \{\Uparrow^c_{r\wedge s} (\mathcal{s}(x)) = \Uparrow^c_{r\wedge s} [\mathcal{s}(x) \cup \mathcal{s}(z)] \cup \{r\} \\ & \quad \Uparrow^c_{r\wedge s} (\mathcal{s}(y)) = \Uparrow^c_{r\wedge s} [\mathcal{s}(y) \cup \mathcal{s}(z)] \cup \{s\} \\ & \quad \Uparrow^c_{r\wedge s} (\mathcal{s}(z)) = \Uparrow^c_{r\wedge s} [\mathcal{s}(z) \cap \mathcal{s}(z)] \cup \{s, r\}\} \end{aligned}$$

note that we have the following order:

$$\Uparrow^c_{r\wedge s} [\mathcal{s}(y)] \ni \Uparrow^c_{r\wedge s} [\mathcal{s}(x)] \ni \Uparrow^c_{r\wedge s} [\mathcal{s}(z)]$$

not showing the reflexive and transitive relations. A second radical upgrade with the formula s, is obtained by $\Uparrow s(\Uparrow (s \wedge r)(c))$: Observe that $[\![s]\!]_{\Uparrow s \wedge r(c)} = \{\Uparrow^c_{s\wedge r} (\mathcal{s}(y)) \Uparrow^c_{s\wedge r} (\mathcal{s}(z))\}$. To simplify the exposition of this example denote by $s_1 = \Uparrow^c_{r\wedge s} [\mathcal{s}(x)]$, $s_2 = \Uparrow^c_{r\wedge s} [\mathcal{s}(y)]$ and $s_3 = \Uparrow^c_{r\wedge s} [\mathcal{s}(z)]$

Then we have that

$$\begin{aligned}
\Uparrow s(\Uparrow (s \wedge r)(c)) = & \Uparrow_s^{\Uparrow(s\wedge r)(c)} [\pi_{\neg s}(\Uparrow (s \wedge r)(c))] \\
= & \{\Uparrow_s^{\Uparrow(s\wedge r)(c)} (s_1) = \Uparrow_s^{\Uparrow(s\wedge r)(c)} [s_1 \cup [\![s]\!]_{\Uparrow s\wedge r(c)}] \cup \{r\}, \\
& \Uparrow_s^{\Uparrow(s\wedge r)(c)} (s_2) = \Uparrow_s^{\Uparrow(s\wedge r)(c)} [s_2 \cap [\![s]\!]_{\Uparrow s\wedge r(c)}] \cup \{s\}, \\
& \Uparrow_s^{\Uparrow(s\wedge r)(c)} (s_3) = \Uparrow_s^{\Uparrow(s\wedge r)(c)} [s_3 \cap [\![s]\!]_{\Uparrow s\wedge r(c)}] \cup \{s, r\}\}
\end{aligned}$$

now we have the following order:

$$\Uparrow_s^{\Uparrow(s\wedge r)(c)} [s_1] \ni \Uparrow_s^{\Uparrow(s\wedge r)(c)} [s_2] \ni \Uparrow_s^{\Uparrow(s\wedge r)(c)} [s_3]$$

without denoting the reflexive and transitive part. This was an application of two radical upgrades that, notwithstanding, cannot be expressed as a single radical upgrade (i.e. an upgrade that lead us from c to $\Uparrow s(\Uparrow (s \wedge r)(c))$ directly)

Finally, we want to emphasize the above remarks. It can be seen that in order to have a study of the iteration of these operations, it should be found a way to have a composition of them. Also that in so doing, it will be also necessary to have a new way to conceptualize formulas in $\mathcal{L}_{K\Box}$ when used as parameters of these transformations. These remarks are incorporated into the general framework of *Questions* given in the next chapter.

Chapter 5

Generalization and Applications

In this chapter we will present a natural generalization of the above formalized operations in the notion of a *Question*. Also related applications like iterated upgrades and its relation with Learning Theory. By means of the last illustration we want to show the pertinence of this kind of semantics.

5.1 Questions

The first step in generalizing the operations defined in the previous chapter, is to take any formula $\varphi \in \mathcal{L}_{K\Box}$ (or $\mathcal{L}^{\infty}_{K\Box}$) as a function $\boldsymbol{\varphi} : c \to \wp(c)$ in the following way. For $c \in \mathcal{M}$ $\boldsymbol{p}(c) = [\![p]\!]_c$, $(\neg\boldsymbol{\varphi})(c) = c \setminus [\![\varphi]\!]_c$ $(\bigwedge_{\alpha<\beta} \boldsymbol{\varphi_\alpha})(c) = \bigwedge_{\alpha<\beta}(\boldsymbol{\varphi_\alpha})(c)$ $(K\boldsymbol{\varphi})(c) = K\boldsymbol{\varphi}(c)$ $(\Box\boldsymbol{\varphi})(c) = \Box\boldsymbol{\varphi}(c)$

Given in this way we will refer to them as *doxastic propositions* Whenever we refer to the image of such function applied to a given $c \in \mathcal{M}$ we will denote it by φ_c.

Definition 5.1. *A question Q is finite non-empty family of doxastic propositions*

$$Q = \{\varphi^1, \dots, \varphi^n\}$$

such that

$$\bigvee_{i=1}^{n} \varphi^i = \top$$

and $\varphi^i \wedge \varphi^j = \bot$ *for* $i \neq j$

In this setting, for any $c \in \mathcal{M}$ we want to refer to a distinguished element $c^* \in c$ which represents the real world. The correct answer to a question Q in $c \in \mathcal{M}$ is the unique φ^i such that $c^* \in \varphi^i_c$.

Learning with certainty the answer of a question Q in a model c is modeled by an action $!\varphi(c)$, which corresponds to the function $!\varphi(c)$.

Learning uncertain information corresponds to a *belief upgrade*. Given a question $Q = \{\varphi^1, \dots, \varphi^n\}$ a belief upgrade is a subset $\mathcal{A} \subseteq \{\varphi^1, \dots, \varphi^n\}$ together with an order $\geq$ we want to create on a given c. Such upgrade will be denoted by $\alpha = \langle \mathcal{A}, \geq \rangle$ Intuitively, this action will correspond to learn that the answer to a Question $Q = \{\varphi^1, \dots, \varphi^n\}$ is in a subset $\mathcal{A} \subseteq \{\varphi^1, \dots, \varphi^n\}$ and that there are some answers more plausible than others.

Proposition 5.1. *For every $\alpha = \langle \mathcal{A}, \leq \rangle$, there exists an equivalent upgrade $\alpha' = \langle \mathcal{A}, < \rangle$*

Such upgrade will be called *standard* and the special notation for it will be $(\varphi^1 \dots \varphi^n)$ for $\varphi^i \in \mathcal{A}$

A sequence of doxastic propositions $[\varphi^1, \ldots, \varphi^n]$ induces a standard upgrade given by:

$$(\varphi^1, \neg\varphi^1 \wedge \varphi^2, \ldots \bigwedge_{1 \leq i \leq n-1} \neg\varphi^i \wedge \varphi^n)$$

Let's give the function for this action step by step. Given a question $Q = \{\varphi^1, \ldots, \varphi^n\}$, and an answer $\alpha = \langle \mathcal{A}, > \rangle$ consider the following pump $\pi_\alpha : c \to \Phi(c)$, with $A = spp(c)$ and defined in each case when the condition $EX := [\![\bigvee_{i=1}^n \mathcal{A}]\!]_c \neq \varnothing$ is satisfied for $\varphi^i \in \mathcal{A}$:

$$\pi_\alpha(x) = \begin{cases} (x \cap \varphi^1_c) \cup (x \cap A) & \text{if } x \in \varphi^1_c, \text{ and } EX \\ (x \cap \varphi^i_c) \cup \bigcup\{\varphi^j_c\}_{j<i} \cup (x \cap A) & \text{if } x \in \varphi^i_c \setminus \bigcup\{\varphi^j_c\}_{j<i} \text{and } EX \\ \uparrow & \text{else} \end{cases} \tag{5.1}$$

Claim 5.1. *For $c \in \mathcal{M}$ and $A \subseteq \mathcal{U}$ we have that $\pi_\alpha(c) \in \Phi(c)$*

Observe that in the first case, when $\pi_\alpha(x) = (x \cap \varphi^1_c) \cup (x \cap A)$ we are in the particular case of the function $\pi_{!\varphi}$ for $\varphi := \varphi^1$, and we have already seen in the previous chapter that for this function we have the desired property. Now, for the remaining cases we have seen that for each i with $1 \leq i \leq n$ $x \cap \varphi^i_c \in \Phi(c)$. Also that for any j with $j < i$, $\varphi^j_c \in \Phi(c)$, then $(x \cap \varphi^i_c) \cup \bigcup\{\varphi^j_c\}_{j<i} \in \Phi(c)$

Moreover, we have as well that for any $c \in M$ π_α has the R_π and the T_π property given the fact that each defined case is a special case of the pump functions involved in the previous actions and that we had already checked.

Then, by our corecursion theorem, for any $Q = \{\varphi^1, \ldots, \varphi^n\}$, $\alpha = \langle \mathcal{A}, > \rangle$ and $c \in \mathcal{M}$ we have a function $\gamma_{\pi_\alpha} : c \to \mathcal{M}$ so that

$$\gamma_{\pi_\alpha}(x) = \gamma_{\pi_\alpha}[\pi_\alpha(x)] \cup (x \cap A)$$

Corollary 5.1. *Assume that for any $c \in \mathcal{M}$ a given question $Q = \{\boldsymbol{\varphi}^1, \dots, \boldsymbol{\varphi}^n\}$ and answer $\alpha = \langle \mathcal{A}, > \rangle$ we have functions $\gamma^c_{\pi_\alpha}$ defined as above. Then, we have a unique function $\alpha : \mathcal{M} \to \mathcal{M}$ such that:*

$$\alpha(c) = \gamma^c_{\pi_\alpha}[c] \ \forall \ c \in \mathcal{M}$$

existence and uniqueness follow by corollary 4.1

The dynamic modality associated with this operation has the following semantics:

$$[\![[\alpha]\boldsymbol{\varphi}]\!]_c = \boldsymbol{\varphi}_{\alpha(c)}$$

We preserve as well the properties for the previous defined operations $\dagger\varphi$ with $\dagger \in \{!, \Uparrow, \uparrow\}$ For instance, being a particular case of an answer.

Proposition 5.2. .

- $!\varphi = [\boldsymbol{\varphi}]$
- $\Uparrow \varphi = [\boldsymbol{\varphi}, \neg\boldsymbol{\varphi}]$
- $\uparrow \varphi = [\bigcap\boldsymbol{\varphi}, \neg\bigcap\boldsymbol{\varphi}]$

Let's prove the second case. Denote by $\alpha = (\boldsymbol{\varphi}, \neg\boldsymbol{\varphi})$ Let $A \subseteq \mathcal{U}$ fixed and take $c \in \mathcal{M}$ We want to show that $\Uparrow \varphi(c) = \alpha(c)$

Consider the following relation:

$$xRy \text{ iff } \exists c' \in c \text{ so that } x = \Uparrow^c_\varphi (c') \text{ and } y = \gamma^c_{\pi_\alpha}(c')$$

We will prove that R is a bisimulation. Consider $x \in \Uparrow \varphi(c)$ and $y \in \alpha(c)$ so that xRy This means that there exists $c' \in c$ so that $x = \Uparrow^c_\varphi (c')$ and $y = \gamma^c_{\pi_\alpha}(c')$. Then we have two cases:

- Case 1. $c' \in \boldsymbol{\varphi}_c$

  Then, $x = \Uparrow^c_\varphi [c' \cap \boldsymbol{\varphi}_c]$ and $y = \gamma^c_{\pi_\alpha}[c' \cap \boldsymbol{\varphi}_c]$ by definition of $\Uparrow^c_\varphi$ and $\gamma^c_{\pi_\alpha}$. Let $x' \in x$, then $\exists c'_x \in c' \cap \boldsymbol{\varphi}_c$ so that $x' = \Uparrow^c_\varphi (c'_x)$. Let $y' = \gamma^c_{\pi_\alpha}(c'_x)$, then $y' \in y$, $x' \cap A = c'_x \cap A = y' \cap A$ and $x'Ry'$. The back condition follows by an analogous reasoning.

- Case 2. $c' \in c \smallsetminus \boldsymbol{\varphi}_c$

  In this case we have that

  $x = \Uparrow^c_\varphi [c' \cup \boldsymbol{\varphi}_c]$ and $y = \gamma^c_{\pi_\alpha}[c' \cap (c' \smallsetminus \boldsymbol{\varphi}_c) \cup \boldsymbol{\varphi}_c]$ by definition of $\Uparrow^c_\varphi$ and ans. Let $x' \in x$, then $\exists c'_x \in c' \cup \boldsymbol{\varphi}_c$ so that $x' = \Uparrow^c_\varphi (c'_x)$. If $c'_x \in c'$ then $c'_x \in c' \cap (c' \smallsetminus \boldsymbol{\varphi}_c)$ thus let $y' = \gamma^c_{\pi_\alpha}(c'_x)$. We have that $x' \cap A = c'_x \cap A = y' \cap A$ and $x'Ry'$. If $c'_x \in \boldsymbol{\varphi}_c$, we also have that $y' = \gamma^c_{\pi_\alpha}(c'_x) \in y$ $x' \cap A = c'_x \cap A = y' \cap A$ and $x'Ry'$.

  The back condition follows by an analogous reasoning.

□

This general definition was created to have functional composition. We preserve this property in our setting as well. Composition of two belief upgrades is given as follows. Let $\alpha = [\boldsymbol{\varphi}^1, \ldots \boldsymbol{\varphi}^n]$ and $\beta = [\boldsymbol{\psi}^1, \ldots, \boldsymbol{\psi}^m]$ then

$$\alpha;\beta = [\boldsymbol{\varphi}^1 \wedge [\alpha]\boldsymbol{\psi}^1, \ldots \boldsymbol{\varphi}^n \wedge [\alpha]\boldsymbol{\psi}^1, \ldots, \ldots \boldsymbol{\varphi}^1 \wedge [\alpha]\boldsymbol{\psi}^m, \ldots \boldsymbol{\varphi}^n \wedge [\alpha]\boldsymbol{\psi}^m]$$

The immediate question that arises is what happens when we iterate these operations. The interest has been focussed on investigate the properties of their fixed points. Thus the first thing we need to check is whether the operations already mentioned are monotone.

Proposition 5.3. *$!\varphi$ is a monotone operation.*

Let $c, c' \in \mathcal{M}$ so that $c \subseteq c'$. We want to show that $!\boldsymbol{\varphi}(c) \subseteq !\boldsymbol{\varphi}(c')$. We will work with its equivalent update form. Let $x \in !\varphi(c)$. Then, $\exists c_x \in c$ so that $x = !^c_\varphi(c_x)$. As $c \subseteq c'$ then $c_x \in c'$. Let $x' = !^{c'}_\varphi(c_x)$.

Claim $x \equiv x'$

We propose a bisimulation as before, considering

$$xRy \text{ iff } \exists c^* \in c \text{ so that } x = !^c_\varphi(c^*) \text{ and } y = !^{c'}_\varphi(c^*)$$

Observe that $x \cap A = c_x \cap A = x' \cap A$, and also that xRx'. Let $y \in x$, then $\exists c_y \in c_x$ so that $y = !^c_\varphi(c_y)$. By transitivity of c' we know that $c_y \in c'$ Let $y' = !^{c'}_\varphi(c_y)$. Then we can see that $y' \in x'$ $y \cap A = c_y \cap A = y' \cap A$ and yRy'.

Now, for the back condition, let $y' \in x'$, thus $\exists c'_y \in c_x$ so that $!^{c'}_\varphi(c'_y)$ Again by transitivity of c we have $c'_y \in c$ Let $y = !^c_\varphi(c'_y)$. Then it is direct that $y \in x$ $y \cap A = c'_y \cap A = y' \cap A$, and yRy'. Thus $x \equiv x'$, therefore $x \in !^c_\varphi(c')$

□

However, this does not happen with the rest of the transformations formalized. Consider the following example. Let $c = sol - set(\mathcal{E})$ and $c' = sol - set(\mathcal{E}')$ with $\mathcal{E} = \langle X, A, e\rangle$ and $\mathcal{E}' = \langle X', A', e'\rangle$ as follows: $X = \{x, y, z\}$, $X' = \{x', y', z', w'\}$, $A = \{p\} = A'$ and e, e' given by the tables:

$e(x) = \{x\}$	$e'(x) = \{x\}$
$e(y) = \{y, x, p\}$	$e'(y) = \{y, x, p\}$
$e(z) = \{z, y, x, \}$	$e'(z) = \{z, y, x, \}$
	$e(w) = \{w, z, y, x, p\}$

let $\mathfrak{s}$ be the solution to $\mathcal{E}$, and $\mathfrak{s}$' the solution to $\mathcal{E}'$ Then, $c \subseteq c'$. However consider $\Uparrow \boldsymbol{p}(c)$, and $\Uparrow \boldsymbol{p}(c')$. $\gamma^c_{\pi_{\boldsymbol{p}}}(\mathfrak{s}(x)) \in \Uparrow \boldsymbol{p}(c)$ but $\gamma^c_{\pi_{\boldsymbol{p}}}(\mathfrak{s}(x)) \notin \Uparrow \boldsymbol{p}(c)$

This means that we do not have warranty about the existence of a fixed point for this operator, and neither for the conservative one. Thus, it has come into the agenda of the logics of belief revision to study the conditions under which those operators reach a fixed point.

To study the properties of their iteration, we need then avoid some trivial conditions that will deviate the operator from reaching a fixed a point or to reach it in a trivial way like by telling lies.

5.2 Iterated Upgrades

The minimum conditions to avoid the cases above mentioned are those of *correctness* and *truthfulness*.

Definition 5.2. *A standard upgrade* $\alpha = (\varphi^1, \ldots, \varphi^n)$ *is said to be correct with respect to a model* $c \in \mathcal{M}$ *if its most plausible answer* (φ^1) *is correct, i.e. if for* $c^* \in c$ *we have* $c^* \in \varphi^1_c$

Definition 5.3. *We say that* $\dagger\varphi$ *are* truthful *with respect to a* $c \in \mathcal{M}$ *if we have that* $\dagger\varphi(c) \neq \varnothing$

Whenever they are met we have assured an interesting study of the iteration of the transformations defined. This study is done by means of iterated upgrades. There, we can characterize intermediate stages between reaching a fixed point and not reaching it, like reaching a stable set of knowledge, beliefs or conditional beliefs.

Definition 5.4. *An* upgrade stream $\boldsymbol{\alpha}$ *is an infinite sequence of upgrades* $(\alpha_n)_{n\in\mathbb{N}}$

A particular case is the one of a *repeated upgrade* which is nothing but an upgrade steam $(\alpha_n)_{n\in\mathbb{N}}$ where $\alpha_n = \alpha \ \forall n \in \mathbb{N}$. Any upgrade stream $\boldsymbol{\alpha}$ induces a function mapping every $c \in \mathcal{M}$ into a sequence :

$$c_0 = c \text{ and } c_{n+1} = \alpha_n(c_n) \text{ if } \alpha_n \text{ is executable on } c_n$$

An upgrade steam $\boldsymbol{\alpha}$ is executable (correct) on a model c if every α_n is executable (correct) on c_n. We will be concerned now with upgrade streams consisting only of one of the † operations defined, thus we will call those streams truthful in an analogous way to the previous concepts.

The behavior of these streams is characterized by the following cases: an upgrade stream $\boldsymbol{\alpha}$

- Stabilizes a model c if $\exists n \in \mathbb{N}$ so that $c_n = c_m \ \forall m > n$.
- Stabilizes a knowledge set on c if $\exists n \in \mathbb{N}$ so that $c_n \vDash K\varphi$ iff $c_m \vDash K\varphi$ $\forall m > n$ and $\forall \boldsymbol{\varphi} \in \mathcal{L}_{K\Box}$
- Stabilizes a belief set on c if $\exists n \in \mathbb{N}$ so that $c_n \vDash B\varphi$ iff $c_m \vDash B\varphi \ \forall m > n$ and $\forall \boldsymbol{\varphi} \in \mathcal{L}_{K\Box}$
- Stabilizes a conditional belief set on c if $\exists n \in \mathbb{N}$ so that $c_n \vDash B^{\boldsymbol{\psi}}\varphi$ iff $c_m \vDash B^{\boldsymbol{\psi}}\varphi \ \forall m > n$ and $\forall \boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{L}_{K\Box}$

A natural question in this setting is whether we preserve the results obtained by [8]. We can be sure that we preserve the results on stabilization of any model by any update stream due to its monotonicity.

However, it seems that for $\boldsymbol{\alpha}$ with $\boldsymbol{\varphi} \in \mathcal{L}^{\infty}_{K\Box}$ we do not preserve theorem 7 of that article, namely:

Theorem 5.1. Baltag and Smets *Every correct upgrade stream stabilizes the agent's beliefs.*

First, observe that countably many iterations will not be enough for this theorem to hold with $\varphi \in \mathcal{L}^{\infty}_{K\Box}$ consider the first model of example 3.1, the model c_ω. Consider the upgrade stream $(\alpha_n)_{n\in\mathbb{N}}$, with

$$\alpha_n = \bigwedge_{i\leq n} p_i$$

this upgrade stream will stabilize c_ω after countably many iterations. However, observe that $c_1 \vDash B\alpha_1$ but $c_0 \nvDash B\alpha_1$ (recall that $c_0 = c_\omega$), and so on, for each $n \in \mathbb{N}$ we have that $c_n \nvDash B\alpha_{n+1}$ and $c_{n+1} \vDash B\alpha_{n+1}$

Now, observe that if we allow transfinite-many iterations, the theorem will hold for this example. However, we can construct the same example for any $\beta \in On$. This is, for any β we can construct an analogous model with a set of indeterminates X so that $|X| = \beta$ and its correspondent set of urelements A of the same cardinality, and do the analogous stream upgrade of size β.

In general, we can expect that most results concerning the behavior of iterated upgrades for $\varphi \in \mathcal{L}^{\infty}_{K\Box}$ may not hold.

5.3 Belief Revision Methods

Belief revision methods, can be seen properly speaking as a learning method. Following the course of methods studying language acquisition, belief revision methods had been studied as a learning method in [20, 5].

Roughly speaking, a learning method is a function that given any set and some data sequence, outputs in finite time an index of an hypothesis. In this setting it can be seen as a function L that for any set $s \in V_{afa}[\mathcal{U}]$ and any positive data sequence $\sigma = (\sigma_0, \ldots, \sigma_n)$ associates some "belief set"so that $L(s; \sigma_0. \ldots, \sigma_n) \subseteq s$.

In this case the hypothesis is a subset which depends on things known in s and the stream σ received.

In what follows we want to illustrate some of the results obtained by [20, 5]. The aim here is to learn which is the real world, to come to know for $c^* \in c$ the set $c^* \cap A$ for $A \subseteq \mathcal{U}$.

In this particular shape, one is concerned in learning positive data. This reflects in the type of stream upgrades that one work with. For a fixed $A \subset \mathcal{U}$, upgrades streams consist only of positive ontic facts i.e. are the particular case in which the upgrade stream $\dagger\varphi_n$ is so that $\varphi_n \in A$.

These streams are a particular kind of "texts"or presentations of a set, and will thus be denoted in a particular way by ε. They can contain repetitions, thus $set(\varepsilon) = \{p_n | \ p_n \in \varepsilon \ \forall n \in \mathbb{N}\}$

Given a set $c^* \in c$ a positive data stream ε is *sound* with respect to c^* iff $set(\varepsilon) \subseteq c^* \cap A$. It is *complete* with respect to c^* iff $c^* \cap A \subseteq set(\varepsilon)$ These conditions can be seen as the analogous for the correctness and truthfulness conditions above defined.

In general, for a fixed $A \subseteq \mathcal{U}$ a set $s \in V_{afa}[A]$ is *identified in the limit* by a learning method L if for every world $s^* \in s$ and every ε sound and complete with respect to s^*, there exists a finite stage after which L outputs only worlds s so that $s \cap A = s^* \cap A$. A set s is *identifiable* if there exists a learning method L that identifies it.

A belief revision method is a function R that given any $c \in \mathcal{M}$ and any stream σ it returns a new model $R(c, \sigma) = c_\sigma$

A belief revision method together with a prior plausibility set model (that we can take to be generated out of an e function) generates a learning method

$$L_R(s,\sigma) = min_{\leq_{e_s}}(c,\sigma)$$

where $c = sol - set(\mathcal{E}_s)$ with $X_s = \{x_{s_n} |\ s_n \in s\}$ $A = \bigcup_n s_n$, and e_s a function fulfilling ϵ^e – reflexivity, transitivity and connectedness; $\leq_{e_s}$ the order induced by e_s with $x \leq y$ iff $x \epsilon^{e_s} y$, and $min_{\leq_{e_s}}(c,\sigma)$ is the minimum of the model generated by e_s if it exists or c otherwise.

In the particular case of a belief revision method R, a set s is identified in the limit if there exists a *prior* plausibility assignment $\geq_s$ so that the induced learning method identifies s in the limit.

A set s is standardly identified if $\geq_S$ creates a prior standard model. It is non-standard if it creates a prior non-standard model.

A learning method is *universal* if it can identify in the limit every set that is identifiable. Also, is called *standardly universal* if it can identify standardly in the limit every set that is identifiable. Then the most interesting feature for us here is that belief revision methods are no-standard.

Consider the following base of set $\mathcal{E}$ with $X \sim \omega$, and $A \cap X = \varnothing$ so that $A \sim \omega$. Consider sets

$$s_n = \{p_k |\ k \geq n\}$$

for all $n \in \mathbb{N}$. Take $s = \{s_n |\ n \in \mathbb{N}\}$. and consider the prior plausibility model given by the following e:

$$e(x_n) = \{x_n\} \cup \{x_j\}_{j\geq n} \cup s_n$$

$sol - set(\mathcal{E}) \in \mathcal{M}$ is a non-standard model, and there does not exists any prior standard model under which any belief revision method can identify such a set.

Conclusions

The study of the properties and or differences that we have within the infinitary language in this setting is far from being exhausted in the work we have done. However, we expect we have shown so far that we preserve most of the properties we may want to preserve form the finitary logics.

Notwithstanding, we win also worthy features. The most obvious is its expressiveness. But not only, we have pointed out some important theoretical features that are embraced within this semantics like the conceptualization of these models transformers as learning methods.

Also, so far, we have seen that the possible drawback will be a more intricate behavior of the iterated upgrades. However, we still need to look at the properties that we may win within this framework as so far we restricted to check what we can preserve.

This points out that we have a long agenda for future work. This includes the need to extend this work to the multi-agent setting, and an exhaustive study of the behavior of iterated upgrades. Notwithstanding, we expect to have shown some advantages of this semantics.

Bibliography

[1] Peter Aczel. *Non-Well Founded Sets*. CSLI Publications, Lecture Notes Number 14, Stanford, 1988.

[2] C.E. Alchurron, P. Gärdenfors, and D. Makinson. On the logic of theory of change: Partial meet, contraction and revision functions. *The journal fo symbolic Logic*, 50(2):510–530, 1985.

[3] Alexandru Baltag. *STS: A Structural Theory of Sets*. PhD thesis, Indiana University, 1998.

[4] Alexandru Baltag. A logic for suspicious players: epistemic action and belief-upgrades in games. *Bulletin of Economic Research*, 54(1):1–45, 2002.

[5] Alexandru Baltag, Nina Gierasimczuk, and Sonja Smets. Belief revision as a truth-tracking process. In K. Apt, editor, *Proceedings of the 13th COnference TARK*, 2011.

[6] Alexandru Baltag and Sonja Smets. *'A Qualitative Theory of Dynamic Interactive Belief Revision'*, pages 179–202. Amsterdam University Press, 2008.

[7] Alexandru Baltag and Sonja Smets. Group belief dynamics under iterated revision: Fixed points and cycles of joint upgrades. In *Proceedings of Theoretical Aspects of Rationality and Knowledge TARK 2009*, pages 41–50, 2009.

[8] Alexandru Baltag and Sonja Smets. 'Learning by Questions and Answers: From Belief-Revision Cycles to Doxastoc Fixed Points'. In *LNAI Lecture Notes in Computer Science*, pages 41–50, 2009.

[9] Alexandru Baltag, Hans van Ditsmarch, and Lawrence Moss. *Handbook of Philosophy of Information*, volume 8 of *Handbook of Philosophy of Science*, chapter Epistemic Logic and Information Update, pages 361–455. Elsevier, P. Adriaans and J.van Benthem edition, 2008. General Editors D. Gabbay, P. Thagard and J. Woods.

[10] Jon Barwise. *The situation in Logic*. CSLI, Stanford, 1989.

[11] Jon Barwise and John Etchemendy. *The Liar, An Essay on Truth and Circularity*. Oxford University Press, Oxford, 1987.

[12] Jon Barwise and Lawrence Moss. *Vicious Circles*. CSLI Publications, Stanford, 1996.

[13] Maurice Boffa. 'Forcing et négation de l' axiome de Fondement'. In *Mémoires de l'Academie royale des Sciences, des lettres et des beaux-arts de Belgique*, volume XL, 1972.

[14] Keith Devlin. *The Joy of Sets*. 2nd ed. Springer Verlag, NY, 1993.

[15] Ronald Fagin. ‘A Quantitative Analysis of Modal Logic’. *The Journal of Symbolic Logic*, 59(1):209–252, 1994.

[16] Paul Finsler. ‘‘Über die Grundlagen der Mengenlehre ’. *Mathematische Zeitschrift*, 25:683–713, 1926.

[17] Marco Forti and Furio Honsell. ‘Set Theory with Free Construction Principles’. *Annali Scuola Normale Superiore di Pisa, Classe di Scienze*, pages 493–522, 1983.

[18] Jelle Gerbrandy. *Bisimulation on Kripke Planet*. PhD thesis, University of Amsterdam, 1999.

[19] Jelle Gerbrandy and Willem Groeneveld. Reasoning about information change. *Journal of Logic, Language and Information*, (6):147–169, 1997.

[20] Nina Gierasimczuk. *Knowing One's limits*. PhD thesis, University of Amsterdam, 2010.

[21] Lawrence Moss. Non-well founded Set Theory. The Stanford Encyclopedia of Philosophy, Fall 2009 Edition. http://plato.stanford.edu/archives/fall2009/entries/nonwellfounded-set-theory.

[22] Jan Plaza. Logics of public communication. In *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems*, pages 201–216, 1989.

[23] Hans Rott. ‘Conditionals and Theory of Change: revisions, expansions and additions ’. *Synthese*, 81:91–113, 1989.

[24] Dana Scott. A Different Kind of Model of Set Theory. Unpublished paper presented at the Stanford Congress of Logic, Metodology and Philosophy of Science, 1960.

[25] Robert Stalnaker. “On Logics of Knowledge and Belief”. *Philosophical Studies*, 128(1):169–199, 2006.

[26] Hans van Ditmarsch, Wiebe van der Hoek, and Barteld Kooi. *Dynamic Epistemic Logic*, volume 337 of *Studies in Epistemology, Logic, Methodology, and Philosophy of Science*. Springer, The Netherlands, 1st edition, 2008.

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