SILVER MEASURABILITY AND ITS RELATION TO OTHER REGULARITY PROPERTIES

JÖRG BRENDLE, LORENZ HALBEISEN, AND BENEDIKT LÖWE

Abstract

For $a \subseteq b \subseteq \omega$ with $b \backslash a$ infinite, the set $D = \{x \in [\omega]^\omega : a \subseteq x \subseteq b\}$ is called a *doughnut*. Doughnuts are equivalent to conditions of Silver forcing, and so, a set $S \subseteq [\omega]^\omega$ is called *Silver measurable*, also known as *completely doughnut*, if for every doughnut D there is a doughnut $D' \subseteq D$ which is contained or disjoint from S. In this paper, we investigate the Silver measurability of $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ sets of reals and compare it to other regularity properties like the Baire and the Ramsey property and Miller and Sacks measurability.

0. *Introduction*

Most forcings that are used in *Set Theory of the Reals* belong to a class called **arboreal forcing notions**. A forcing notion $\mathbb{P}$ is called **arboreal** if its conditions are trees on either $2 = \{0, 1\}$ or ω ordered by inclusion and for each $T \in \mathbb{P}$, the set of all branches through T is homeomorphic to either 2^ω or ω^ω.

Each arboreal forcing notion is canonically related to a notion of measurability and an ideal:

If $\mathbb{P}$ is an arboreal forcing notion, we define

$$\mathfrak{A}_\mathbb{P} := \{A : \forall T \in \mathbb{P}\,(\exists S \leq T([S] \subseteq A \text{ or } [S] \cap A = \varnothing))\}, \text{ and}$$

$$\mathfrak{I}_\mathbb{P} := \{A : \forall T \in P\,(\exists S \leq T([S] \cap A = \varnothing))\}.$$

We call the elements of $\mathfrak{A}_\mathbb{P}$ **$\mathbb{P}$-measurable sets** and the elements of $\mathfrak{I}_\mathbb{P}$ **$\mathbb{P}$-null sets**.† Standard examples of arboreal forcing notions are **Cohen forcing** $\mathbb{C}$ (the set of basic open sets), **Sacks forcing** $\mathbb{S}$ (the set of perfect trees), **Miller forcing** $\mathbb{M}$ (the set of superperfect trees), **Silver forcing** $\mathbb{V}$ (the set of uniform perfect trees), **Mathias forcing** $\mathbb{R}$ (the set of basic Ellentuck neighbourhoods).‡ The corresponding notions of measurability and smallness have been investigated in many contexts, and some of them are known under different names: the sets in $\mathfrak{I}_\mathbb{S}$ are also called **Marczewski null**, the sets in $\mathfrak{A}_\mathbb{R}$ are also said to be **completely Ramsey**, and the sets in $\mathfrak{A}_\mathbb{V}$ are said to be **completely Doughnut** (*cf.* Section 1.2).

Note that the measurability property connected to Cohen forcing is the Baire

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†This general approach to regularity properties connected to forcing notions was considered in [**Bre95**], and continued in [**Löw98**], [**BreLöw99**], [**Bre00**] and [**Löw∞**]. Even more general are the notions of **Marczewski field** and **Marczewski ideal** from [**Bal+01/02**]. In these publications, $\mathfrak{I}_\mathbb{P}$ was denoted by (p^0), p^0 or $s^0(\mathbb{P})$.

‡*Cf.* Section 1.1 for more detailed definitions.

property (a set A has the **Baire property** if there is an open set P such that $A \triangle P$ is meagre) which is not the same as membership in $\mathfrak{A}_{\mathbb{C}}$.

Being $\mathbb{P}$-measurable is considered a regularity property of a set, and people have investigated the extent of these regularity properties: usually, all $\mathbf{\Sigma}^1_1$ sets are $\mathbb{P}$-measurable[†], there are $\mathbf{\Delta}^1_2$ sets that are not $\mathbb{P}$-measurable in the constructible universe $\mathbf{L}$, and very often the statements "Every $\mathbf{\Delta}^1_2$ set is $\mathbb{P}$-measurable" and "Every $\mathbf{\Sigma}^1_2$ set is $\mathbb{P}$-measurable" can be characterized in terms of transcendence over $\mathbf{L}$ as exemplified in Fact 0.1.

In the following, we will write $\Gamma(\mathscr{B})$, $\Gamma(\mathscr{D})$, $\Gamma(\mathscr{M})$, $\Gamma(\mathscr{R})$, $\Gamma(\mathscr{S})$ for "Every Γ set has the Baire property (is completely Doughnut, is Miller measurable, is completely Ramsey, is Sacks measurable)".

FACT 0.1.

(i) *(Solovay/Folklore)* $\mathbf{\Sigma}^1_2(\mathscr{B})$ *is equivalent to "for all* $r \in {}^\omega\omega$ *there is a comeager set of Cohen reals over* $\mathbf{L}[r]$*",*

(ii) *[***JudShe89***, Theorem 3.1]* $\mathbf{\Delta}^1_2(\mathscr{B})$ *is equivalent to "for all* $r \in {}^\omega\omega$ *there is a Cohen real over* $\mathbf{L}[r]$*",*

(iii) *[***JudShe89***, Theorem 2.10]* $\mathbf{\Delta}^1_2(\mathscr{R})$ *and* $\mathbf{\Sigma}^1_2(\mathscr{R})$ *are equivalent,*

(iv) *[***JudShe89***, Theorem 3.5(iv)]* $\mathbf{\Sigma}^1_2(\mathscr{R})$ *does not imply* $\mathbf{\Delta}^1_2(\mathscr{B})$*,*

(v) *[***BreLöw99***, Theorem 6.1]* $\mathbf{\Sigma}^1_2(\mathscr{M})$ *and* $\mathbf{\Delta}^1_2(\mathscr{M})$ *are equivalent, and equivalent to "for all* $r \in {}^\omega\omega$ $({}^\omega\omega \cap \mathbf{L}[r]$ *is not dominating)",*

(vi) *[***BreLöw99***, Theorem 7.1]* $\mathbf{\Sigma}^1_2(\mathscr{S})$ *and* $\mathbf{\Delta}^1_2(\mathscr{S})$ *are equivalent, and equivalent to "for all* $r \in {}^\omega\omega$ $({}^\omega\omega \cap \mathbf{L}[r] \neq {}^\omega\omega)$*".*

Abstractly, you could describe Fact 0.1 (1) as "Measurability of $\mathbf{\Sigma}^1_2$ sets corresponds to the existence of a large set of generics over $\mathbf{L}[r]$," while you could describe Fact 0.1 (2) as "Measurability of $\mathbf{\Delta}^1_2$ sets corresponds to the existence of generics over $\mathbf{L}[r]$." We follow [**BreLöw99**] and call theorems of type (1) **"Solovay-type characterization"** and theorems of type (2) **"Judah-Shelah-type characterizations"**.

In this paper, we shall investigate Silver measurability, continuing research from the paper [**Hal03**], in order to give a complete diagram of the implications between the three properties $\mathscr{B}$, $\mathscr{D}$ and $\mathscr{R}$ for $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ sets. Further we will compare Silver measurability with Miller measurability $\mathscr{M}$ and Sacks measurability $\mathscr{S}$.

In particular, it will be shown that $\mathbf{\Delta}^1_2(\mathscr{B})$ implies that all projective sets are Silver measurable, that $\mathbf{\Delta}^1_2(\mathscr{D})$ implies that there are splitting reals over each $\mathbf{L}[r]$, and that $\mathbf{\Sigma}^1_2(\mathscr{D})$ implies that there are unbounded reals over each $\mathbf{L}[r]$.

We will introduce some notation in Section 1 and list what was known about Silver measurability before our work. In Section 2 and Section 3 we show how to get models for $\mathbf{\Delta}^1_2(\mathscr{D})$ and $\mathbf{\Sigma}^1_2(\mathscr{D})$, respectively, and how $\mathbf{\Delta}^1_2(\mathscr{D})$ and $\mathbf{\Sigma}^1_2(\mathscr{D})$ are related to certain ideals and to splitting and unbounded reals, respectively. In Section 4, we will summarize our results and list some open questions.

[†]There is a uniform approach via game proofs of analytic measurability for these regularity properties in [**Löw98**].

1. *Definitions*

Throughout this paper we will use standard set theoretic terminology which the reader can find, *e.g.*, in textbooks like [**BarJud95**]. We will introduce some of the notions which are of particular interest for our paper in this section.

1.1. *Trees*

As usual, a **tree on** X is a subset of $X^{<\omega}$ closed under initial segments where $X^{<\omega}$ is the set of all finite sequences of elements of X. If $x \in {}^{\omega}X$ is a function from ω to X and $n \in \omega$ is a natural number, we denote the finite sequence $\langle x(0), x(1), ..., x(n-1)\rangle$ by $x{\restriction}n$ and call it **the restriction of** x **to** n. If $s \in X^{<\omega}$ and $t \in X^{<\omega}$ or $x \in {}^{\omega}X$, we can define the **concatenation of** s **and** t **(of** s **and** x**)**, denoted by $s^\frown t$ ($s^\frown x$) in the obvious way.

A tree on $2 = \{0, 1\}$ is called **uniform**, if for all $s, t \in T$ of the same length we have

$$s^\frown 0 \in T \iff t^\frown 0 \in T \quad \text{and} \quad s^\frown 1 \in T \iff t^\frown 1 \in T\,.$$

If T is a tree, then a function $x \in {}^{\omega}X$ is called a **branch** through T, if for all $n \in \omega$, we have that $x{\restriction}n \in T$. The set of all branches through T is denoted by $[T]$. A tree T on 2 is called **perfect**, if for every $s \in T$ there is a $t \in T$ with $s \subseteq t$ such that both $t^\frown 0$ and $t^\frown 1$ belong to T; such a sequence t is called a **splitting node** of T.

A perfect T tree is canonically (order) isomorphic to the full binary tree $2^{<\omega}$, and the order isomorphism induces a homeomorphism $\Theta_T : [T] \to 2^{\omega}$. Note that if $B \subseteq [T]$ is a Borel set with a Borel code in $\mathbf{L}[r]$, then $\Theta_T[B]$ is a Borel set with a Borel code in $\mathbf{L}[r, T]$ since the homeomorphism can be read off in a recursive way from the tree T. This will be used later.

Similarly, if T is a tree on ω, we can call $s \in T$ an ω**-splitting node** if s has infinitely many immediate successors. A tree T is called **superperfect** if for each $s \in T$ there is an ω-splitting node $t \supseteq s$ with $t \in T$.

We can now use the special kinds of trees just defined to define the forcing notions mentioned in the introduction:

Silver forcing $\mathbb{V}$ is the set of all uniform perfect trees ordered by inclusion,† **Sacks forcing** $\mathbb{S}$ is the set of all perfect trees ordered by inclusion, and **Miller forcing** $\mathbb{M}$ is the set of all superperfect trees ordered by inclusion.

1.2. *Doughnuts*

Investigating arrow partition properties, Carlos DiPrisco and James Henle introduced in [**DiPHen00**] the so-called doughnut property: Let $[\omega]^{\omega} := \{x \subseteq \omega : |x| = \omega\}$. Then, for $a \subseteq b \subseteq \omega$ with $b \setminus a \in [\omega]^{\omega}$, the set $D = \{x \in [\omega]^{\omega} : a \subseteq x \subseteq b\}$ is called a **doughnut**, or more precisely, the (a, b)-doughnut, denoted by $[a, b]^{\omega}$.

Doughnuts are equivalent to uniform perfect trees in the following sense (*cf.* [**Hal03**]):

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†Uniform perfect trees have been used in recursion theory, and are called **Lachlan 1-trees** there. *Cf.* [**Lac71**].

FACT 1.1. *Each uniform perfect tree $T \subseteq \{0,1\}^{<\omega}$ corresponds in a unique way to a doughnut, and vice versa.*

Di Prisco and Henle said that a set A has the **Doughnut property** if it either contains or is disjoint from a doughnut, and that it is **completely Doughnut** if for every doughnut D there is a doughnut $D^* \subseteq D$ such that either $D^* \subseteq A$ or $D^* \cap A = \varnothing$.

By virtue of Fact 1.1, being completely Doughnut is just equivalent to being Silver measurable in the sense of the introduction.

The Ramsey property, originally defined in terms of the Baire property in the Ellentuck topology or in terms of partitions†, can be equivalently defined in terms of doughnuts: a set $S \subseteq [\omega]^\omega$ is **completely Ramsey**, denoted by $\mathscr{R}$, if it for each doughnut $[\varnothing, a]^\omega$ there is a doughnut $[\varnothing, b]^\omega$ such that $[\varnothing, b]^\omega \subseteq S$ or $[\varnothing, b]^\omega \cap S = \varnothing$.

Silver measurability or the doughnut property was investigated by the first author in [**Bre95**], for analytic sets in terms of games by the third author in [**Löw98**], and for $\mathbf{\Sigma}^1_2$-sets by the second author in [**Hal03**].

One simple consequence of this analysis that we shall use later is

OBSERVATION 1.2. *Every Borel set either contains the branches through a uniform perfect tree or is disjoint from the set of branches through a uniform perfect tree.*

1.3. *Weak Measurability*

The notion of $\mathbb{P}$-measurability is a Π_2 notion. By dropping the first universal quantifier you arrive at a weaker Σ_1 notion that is called weak $\mathbb{P}$-measurability: A set A is said to be **weakly $\mathbb{P}$-measurable** if there is a $T \in \mathbb{P}$ such that either $[T] \subseteq A$ or $[T] \cap A = \varnothing$. In general, the notion of weak measurability is not a statement about the regularity of a set: a set can contain a $\mathbb{P}$-condition T and be completely irregular outside of T. Compare this to the Doughnut property from Section 1.2: as Silver measurability is equivalent to being completely Doughnut, weak Silver measurability is equivalent to the Doughnut property.

Although weak measurability of a single set doesn't imply its regularity, classwise statements of weak measurability suffice to prove full measurability as the following general lemma from [**BreLöw99**] shows:

LEMMA 1.3 Brendle-Löwe (1999). *Let $\mathbf{\Gamma}$ be a boldface pointclass closed under intersections with closed sets (in this paper, $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ are the only examples). Then the following are equivalent:*

(i) *Every set in $\mathbf{\Gamma}$ is Silver measurable, and*
(ii) *every set in $\mathbf{\Gamma}$ is weakly Silver measurable.*

Lemma 1.3 was proved in an abstract setting in [**BreLöw99**, Lemma 2.1].

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† *Cf.* [**Kec95**, 19.D].

1.4. *Quasigenericity*

Let $\mathfrak{I}$ be an ideal and M be a model of set theory. We write $\mathrm{N}(\mathfrak{I}, M)$ for the set of all Borel sets B such that

– $B \in \mathfrak{I}$, and
– there is a Borel code for the set B in M.

It is well-known that there are characterizations of the generics of random and Cohen forcing via the ideals $\mathfrak{N}$ of Lebesgue null and $\mathfrak{M}$ of meagre sets, respectively:†

Fact 1.4 Solovay.
– *A real r is random over M if and only if $r \notin \bigcup \mathrm{N}(\mathfrak{N}, M)$, and*
– *a real c is Cohen over M if and only if $c \notin \bigcup \mathrm{N}(\mathfrak{M}, M)$.*

For arbitrary arboreal forcings $\mathbb{P}$, the set $\omega^\omega \setminus \bigcup \mathrm{N}(\mathfrak{I}_{\mathbb{P}}, M)$ is not in general the set of generics. But we can use Fact 1.4 to define a notion of quasi-genericity:

Let $\mathfrak{I}$ be an ideal and M be a model of set theory. We set

$$\mathrm{QG}(\mathfrak{I}, M) := \omega^\omega \setminus \bigcup \mathrm{N}(\mathfrak{I}, M),$$

and call the elements of $\mathrm{QG}(\mathfrak{I}, M)$ **$\mathfrak{I}$-M-quasigeneric**.

Our notation for Borel codes will be standard: if c is a Borel code, we denote the decoded set by B_c.

1.5. *The least non-smooth equivalence relation E_0 and Silver Homogeneity*

The equivalence relation E_0, defined by $xE_0y \iff \forall^\infty n(x(n) = y(n))$, is well-known from Descriptive Set Theory. It is the least non-smooth countable Borel equivalence relation and as such the object of the famous Generalized Glimm-Effros Dichotomy of Harrington, Kechris and Louveau.‡ We call a Borel set $A \subseteq {}^\omega 2$ an **E_0-selector** if for any distinct $x, y \in A$ there are infinitely many $n \in \omega$ such that $x(n) \neq y(n)$. This makes sure that A selects at most one element from each equivalence class of E_0 (see [**Zap**∞, Section 2.3.10]). Denote the set of E_0-selectors with Sel_{E_0}.

Now, let $\mathfrak{I}_{E_0}$ be the σ-ideal of sets σ-generated by Borel E_0-selectors.

An ideal $\mathfrak{I}$ is called **Silver homogenous** if for each $T \in \mathbb{V}$, the canonical homeomorphism $\Theta_T : [T] \to 2^\omega$ preserves membership in $\mathfrak{I}$, *i.e.*, if $A \in \mathfrak{I}$, then $\Theta_T[A] \in \mathfrak{I}$.§

Observation 1.5. *Both $\mathfrak{I}_{\mathbb{V}}$ and $\mathfrak{I}_{E_0}$ are Silver homogeneous.*

Lemma 1.6 First Homogeneity Lemma. *Let $\mathfrak{I}$ be Silver homogeneous and $T \in \mathbb{V}$. Suppose that there is an $\mathfrak{I}$-$\mathbf{L}[r, T]$-quasigeneric real x, then $\Theta_T^{-1}(x)$ is also $\mathfrak{I}$-$\mathbf{L}[r, T]$-quasigeneric.*

Proof. Let $x \in \mathrm{QG}(\mathfrak{I}, \mathbf{L}[r, T])$. We claim that $y := \Theta_T^{-1}(x)$ is also $\mathfrak{I}$-$\mathbf{L}[r, T]$-quasigeneric.

But this is a direct consequence of Silver homogeneity: take any Borel set $B \in \mathfrak{I}$

† *Cf.* [**Kan94**, Theorem 11.10].
‡ *Cf.* [**HarKecLou90**] and the survey paper [**Kec99**, p. 166-167].
§ This is a slight generalization of Zapletal's notion of homogeneity [**Zap**∞].

coded in $\mathbf{L}[r,T]$, then $B \cap [T]$ is still a Borel set from $\mathfrak{I}$ coded in $\mathbf{L}[r,T]$. We shift it from $[T]$ to 2^ω via Θ_T. By Silver homogeneity, it is still in $\mathfrak{I}$. But since Θ_T is recursively defined from T, $\Theta_T[B \cap [T]]$ is in $\mathrm{N}(\mathfrak{I}, \mathbf{L}[r,T])$. If $y \in B$, then $x \in \Theta_T[B \cap [T]]$, contradicting x's quasigenericity; thus, y can't lie in B. □

Note that Θ_T and Θ_T^{-1} preserve the property of being a uniform perfect tree: If S is a uniform perfect tree, then $\Theta_T^{-1}[S]$ is the set of branches through a uniform perfect subtree of T.

LEMMA 1.7 Second Homogeneity Lemma. *Let $A = 2^\omega \setminus \bigcup \mathrm{QG}(\mathfrak{I}, \mathbf{L}[r])$. Suppose that the following conditions are met:*

(i) *A is weakly Silver measurable,*
(ii) *$\mathfrak{I}$ is Silver homogeneous,*
(iii) *for each s there is an $\mathfrak{I}$-$\mathbf{L}[r,s]$-quasigeneric.*

Then there is a uniform perfect tree of $\mathfrak{I}$-$\mathbf{L}[r]$-quasigenerics.

Proof. Since A is weakly Silver measurable, there is either a uniform perfect tree whose branches are disjoint from A or one whose branches are all in A.

In the former case, all of the branches of that tree are quasigeneric by definition of A and we're done immediately.

In the latter case, all of the branches of T are non-quasigeneric. By the assumption, we can pick some $\mathfrak{I}$-$\mathbf{L}[r,T]$-quasigeneric real. Now the assumptions of the First Homogeneity Lemma 1.6 are satisfied, so we get a $\mathfrak{I}$-$\mathbf{L}[r,T]$-quasigeneric inside $[T]$. But since $\mathrm{QG}(\mathfrak{I}, \mathbf{L}[r,T]) \subseteq \mathrm{QG}(\mathfrak{I}, \mathbf{L}[r])$, this is absurd. □

2. $\mathbf{\Delta}^1_2$ *sets*

Let us start by forcing a model in which all $\mathbf{\Delta}^1_2$-sets are Silver measurable.

THEOREM 2.1. *An ω_1-iteration with countable support of Silver forcing, starting from $\mathbf{L}$, yields a model in which every $\mathbf{\Delta}^1_2$-set is Silver measurable.*

Proof. Let $\mathbb{V}$ be Silver forcing, $\mathbb{V}_{\omega_1}$ be the ω_1-iteration with countable support, starting from $\mathbf{L}$, and let $\mathbf{W}$ be the $\mathbb{V}_{\omega_1}$-extension. Let $A \subseteq [\omega]^\omega$ be a $\mathbf{\Delta}^1_2$-set in $\mathbf{W}$. Thus, there are $\mathbf{\Sigma}^1_2$-formulas φ and ψ such that $\mathbf{W} \models A = \{y \in [\omega]^\omega : \varphi(y)\} = \{y \in [\omega]^\omega : \neg\psi(y)\}$. So, $\mathbf{W} \models \forall y\big(\varphi(y) \leftrightarrow \neg\psi(y)\big)$, which is a $\mathbf{\Pi}^1_3$-sentence and therefore downward absolute. Let $[a,b]^\omega$ be any doughnut in $\mathbf{W}$. Without loss of generality we may assume that the parameters of φ and ψ, as well as a and b belong to the ground model $\mathbf{L}$.

We claim that there is a doughnut $[a',b']^\omega \subseteq [a,b]^\omega$ such that either $[a',b']^\omega \subseteq A$ or $[a',b']^\omega \cap A = \varnothing$.

Let $\dot{z}$ be the canonical $\mathbb{V}$-name for the $\mathbb{V}$-generic real. Let $M \preccurlyeq H(\chi)$ be a countable elementary submodel of $H(\chi)$ (for some χ) such that $a, b, \dot{z}$ belong to M. Let p be a $\mathbb{V}$-generic condition over M, and without loss of generality let us assume that $p = [a'', b'']^\omega \subseteq [a,b]^\omega$ and that $p \Vdash_{\mathbb{V}} \varphi(\dot{z})$ (the case when $p \Vdash_{\mathbb{V}} \psi(\dot{z})$ is similar).

We claim that there is a doughnut $[a',b']^\omega \subseteq [a'',b'']^\omega$ so that whenever $z \in [a',b']^\omega$, then z is $\mathbb{V}$-generic over M. To see this, let $\{\mathcal{A}_n : n \in \omega\}$ be an enumeration of all antichains of $\mathbb{V}$ in M. Then, by fusion, we can construct a doughnut $[a',b']^\omega \subseteq$

$[a'', b'']^\omega$ such that for each $\forall z \in [a', b']^\omega\, \forall n \in \omega\, \exists F \in \mathcal{A}_n(z \in F)$, and hence, z is $\mathbb{V}$-generic over M.

Since we assumed $p \Vdash_{\mathbb{V}} \varphi(\dot{z})$, and since each $z \in [a', b']^\omega \subseteq p$ is $\mathbb{V}$-generic over M, for all $z \in [a', b']^\omega$ we have $M[z] \models \varphi(z)$. Now, because $\mathbf{\Sigma}^1_2$ formulae are upwards absolute for countable models, we also have $\mathbf{W} \models \varphi(z)$ for all $z \in [a', b']^\omega$, which implies $\mathbf{W} \models [a', b']^\omega \subseteq \{y : \varphi(y)\} = A$ and completes the proof. □

PROPOSITION 2.2. *If for all $r \in \omega^\omega$ there is $\mathfrak{I}_{\mathbb{V}}$-$\mathbf{L}[r]$-quasigeneric real, then every $\mathbf{\Delta}^1_2$ set is Silver measurable.*

Proof. By Lemma 1.3 we only have to show that for every $\mathbf{\Delta}^1_2$ set X there is a T in S such that either $[T] \subseteq X$ or $[T] \cap X = \varnothing$.

Given a $\mathbf{\Delta}^1_2$ set X with parameter r, we define Y, X_α, Y_α as follows:

Let $\varphi(v_0, v_1)$ and $\psi(v_0, v_1)$ be $\mathbf{\Sigma}^1_2$-formulae such that for $X = \{x : \varphi(x, r)\}$ and $Y = \{y : \psi(y, r)\}$ we have $X \cup Y = [\omega]^\omega$ and $X \cap Y = \varnothing$. Now we use the representation of $\mathbf{\Sigma}^1_2$ sets as unions of ω_1 Borel sets and let $X = \bigcup_{\alpha<\omega_1} X_\alpha$ and $Y = \bigcup_{\alpha<\omega_1} Y_\alpha$, where the X_α's and Y_α's are Borel sets with Borel code in $\mathbf{L}[r]$.

Case 1: There is an α such that $X_\alpha \notin \mathfrak{I}_{\mathbb{V}}$. Since X_α is Borel, this means by Observation 1.2 that there is T in S such that $[T] \subseteq X_\alpha \subseteq X$.

Case 2: There is an α such that $Y_\alpha \notin \mathfrak{I}_{\mathbb{V}}$. Since Y_α is Borel, this means that there is T in S such that $[T] \subseteq Y_\alpha \subseteq Y$.

Case 3: For all α, both X_α and Y_α are Silver null. Then $\bigcup_{\alpha<\omega_1}(X_\alpha \cup Y_\alpha) \subseteq \bigcup \mathrm{N}(\mathfrak{I}_{\mathbb{V}}, \mathbf{L}[r])$, hence it can't contain a quasigeneric. But

$$\bigcup_{\alpha<\omega_1} (X_\alpha \cup Y_\alpha) = X \cup Y = 2^\omega,$$

contradicting the existence of quasigenerics. □

PROPOSITION 2.3. *$\mathbf{\Delta}^1_2(\mathscr{D})$ implies that for all $r \in {}^\omega 2$ there is a $\mathfrak{I}_{E_0}$-$\mathbf{L}[r]$-quasigeneric.*

Proof. Assume towards a contradiction that there is an r such that $\mathrm{QG}(\mathfrak{I}_{E_0}, \mathbf{L}[r]) = \varnothing$. Now, for each $x \in {}^\omega 2$ define the set

$$C_x := \{c \in \mathbf{L}[r] \,:\, B_c \in \mathrm{Sel}_{E_0} \,\&\, \exists y \in B_c (y \triangle x) \text{ is finite}\}.$$

We fix some $x \in {}^\omega 2$. By our assumption, x is not $\mathfrak{I}_{E_0}$-$\mathbf{L}[r]$-quasigeneric, so is in some set in $\mathfrak{I}_{E_0}$, hence in some E_0-selector, so C_x is a non-empty $\Sigma^1_2(r, x)$ set. Pick the $<_{\mathbf{L}[r]}$-least element of C_x and call it c_x. Note that B_{c_x} contains exactly one y such that $y \triangle x$ is finite: since B_{c_x} is an E_0-selector, any distinct y_0 and y_1 in B_{c_x} must have infinite symmetric difference, so only one of them can have finite symmetric difference with x. Thus we can define n_x to be the number of $y \triangle x$ for this uniquely defined y.

Define $C_0 := \{x : n_x \text{ even}\}$ and $C_1 := \{x : n_x \text{ odd}\}$. Both of these sets are $\mathbf{\Sigma}^1_2$ sets (with parameter r), and hence $\mathbf{\Delta}^1_2$ sets (by our assumption, we have $C_0 \cup C_1 = 2^\omega$).

But neither C_0 nor C_1 contains a uniform perfect tree: If $z \in C_0$ and T is a uniform perfect tree with $z \in [T]$, then $[T]$ contains infinitely many elements $\{z_n : n \in \omega\}$ that differ in exactly one place from z (say, $z(k_n) \neq z_n(k_n)$).

Note that $c_z = c_{z_m}$ and

$$n_z = n_{z_m} \iff k_m \in z \triangle z_m,$$

hence some of the z_n don't lie in C_0.

The same argument works for C_1. Consequently, neither C_0 nor C_1 contain a uniform perfect tree, and thus they can't be Silver measurable. $\square$

With a similar technique, we can show:

Proposition 2.4. *$\mathbf{\Delta}^1_2(\mathscr{D})$ implies that for all reals r there is a splitting real over $\mathbf{L}[r]$.*

Proof. For $x \in [\omega]^\omega$ let $\tau_x \in {}^\omega\omega$ be an increasing one-to-one mapping from ω onto $\{k : k \in x\}$ and let $\widehat{x} \in [\omega]^\omega$ be defined as follows:

$$k \in \widehat{x} \iff \exists n \in \omega\big(\tau_x(2n) < k \leq \tau_x(2n+1)\big)\,.$$

Assume towards a contradiction that there is $r \in [\omega]^\omega$ such that there is no splitting real over $\mathbf{L}[r]$, which is equivalent to

$$\exists r \in [\omega]^\omega\, \forall x \in [\omega]^\omega\, \exists y \in [\omega]^\omega \cap \mathbf{L}[r]\, \big(y \cap x \text{ or } y \setminus x \text{ is finite}\big)\,.$$

Now, for each $x \in [\omega]^\omega$ pick the $<_{\mathbf{L}[r]}$-least $y_x \in [\omega]^\omega \cap \mathbf{L}[r]$ such that $y_x \cap \widehat{x}$ or $y_x \setminus \widehat{x}$ is finite, and let $A \subseteq [\omega]^\omega$ be the set of all $\widehat{x}$ for which the former case holds. It is easy to see that A is a $\mathbf{\Delta}^1_2$-set (with parameter r) and that A does neither contain nor is it disjoint from any uniform perfect tree, which completes the proof. $\square$

3. $\mathbf{\Sigma}^1_2$ *sets*

First of all, let us remark that by work of the second author on Cohen reals and doughnuts in [**Hal03**], we know that the existence of Cohen reals implies Silver measurability at the second level of the projective hierarchy.

Lemma 3.1. *Suppose that A is a $\Sigma^1_2(r)$ set for some real number r and c is a Cohen real over $\mathbf{L}[r]$. Then there is a uniform perfect tree $T \in \mathbf{L}[r,c]$ such that either $[T] \subseteq A$ or $T \cap A = \varnothing$.*

Proof. See (the proof of) [**Hal03**, Lemma 2.1]. $\square$

Corollary 3.2. *$\mathbf{\Delta}^1_2(\mathscr{B})$ implies $\mathbf{\Sigma}^1_2(\mathscr{D})$.*

Proof. Immediate from Lemma 3.1 and Fact 0.1 (ii). $\square$

We can use the Second Homogeneity Lemma 1.7 to derive a result about $\mathbf{\Sigma}^1_2(\mathscr{D})$ and the existence of quasigenerics:

Lemma 3.3. *The following are equivalent:*
(i) *For all r, we have $\mathrm{QG}(\mathfrak{I}_{\mathbb{V}}, \mathbf{L}[r]) \neq \varnothing$ and $\mathbf{\Sigma}^1_2(\mathscr{D})$ holds, and*
(ii) *for all r, the set $\mathrm{QG}(\mathfrak{I}_{\mathbb{V}}, \mathbf{L}[r])$ is co-Silver null (i.e., its complement is in $\mathfrak{I}_{\mathbb{V}}$.*

Proof. "⇒": Consider the $\mathbf{\Sigma}^1_2$ set $X = \bigcup \mathrm{N}(\mathfrak{I}_{\mathbb{V}}, \mathbf{L}[r])$. $\mathbf{\Sigma}^1_2(\mathscr{D})$ implies that A is weakly Silver measurable. Let T be an arbitrary uniform perfect tree. We have to show that there is a uniform perfect subtree $S \subseteq T$ that consists of quasigenerics.

We can apply the Second Homogeneity Lemma 1.7, and get a uniform perfect tree of quasigenerics. Now we can use the First Homogeneity Lemma 1.6 to copy that tree into T.

"⇐": Let $X = \{x : \varphi(x, r)\}$ be a $\mathbf{\Sigma}^1_2$-set with parameter r, so, $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where the X_α's are Borel sets with Borel code in $\mathbf{L}[r]$. Further, let S be a uniform perfect tree with code in $\mathbf{L}[r]$.

If for all $\alpha < \omega_1$, $X_\alpha \cap [S] \in \mathrm{N}(\mathfrak{I}_{\mathbb{V}}, \mathbf{L}[r])$, then, by assumption, there is a uniform perfect tree $T \subseteq S$ of quasigenerics. For this tree T, we have $[T] \cap X_\alpha = \varnothing$ for all α, which implies that $[T] \subseteq {}^\omega 2 \setminus X$.

On the other hand, if there is an $\alpha < \omega$ such that $X_\alpha \cap [S] \notin \mathrm{N}(\mathfrak{I}_{\mathbb{V}}, \mathbf{L}[r])$, then we find a uniform perfect tree $T \subseteq S$ such that $[T] \subseteq X_\alpha \subseteq X$ using Observation 1.2. □

The next Propositions 3.4 is not exactly a characterization of $\mathbf{\Sigma}^1_2(\mathscr{D})$, but very close, since the ideals $\mathfrak{I}_{\mathbb{V}}$ and $\mathfrak{I}_{E_0}$ are very similar.

PROPOSITION 3.4.

(i) *If for each r the set of $\mathfrak{I}_{\mathbb{V}}$-$\mathbf{L}[r]$-quasigenerics is co-Silver null, then $\mathbf{\Sigma}^1_2(\mathscr{D})$ holds.*

(ii) *If $\mathbf{\Sigma}^1_2(\mathscr{D})$ holds, then for each r the set of $\mathfrak{I}_{E_0}$-$\mathbf{L}[r]$-quasigenerics is co-Silver null.*

Proof. "(1)": This is an immediate consequence of Lemma 3.3.

"(2)": For the second implication, we apply the Homogeneity Lemmas again as in Lemma 3.3:

Consider the $\mathbf{\Sigma}^1_2$ set $X = \bigcup \mathrm{N}(\mathfrak{I}_{E_0}, \mathbf{L}[r])$. $\mathbf{\Sigma}^1_2(\mathscr{D})$ implies that A is weakly Silver measurable. This time, we use the Silver homogeneity of $\mathfrak{I}_{E_0}$ (Observation 1.5). After we fixed a uniform perfect tree T. we can apply the Second Homogeneity Lemma 1.7, and again get a uniform perfect tree of quasigenerics which we paste into T by use of the First Homogeneity Lemma 1.6. □

We can also connect $\mathbf{\Sigma}^1_2(\mathscr{D})$ to splitting reals, and almost get a converse to Proposition 2.4. We will later see (Corollary 3.9) that in Proposition 3.6, the conclusion cannot be strengthened to "weakly Silver measurable".

LEMMA 3.5. *If $s \in [\omega]^\omega$ splits the set A (i.e., for all $a \in A$, both $a \cap s$ and $a \setminus s$ are infinite), then there is a uniform perfect tree T such that $[T] \cap A = \varnothing$.*

Proof. Define

$$U_s := \{t \in 2^{<\omega} : (n \notin s \ \& \ n \in \mathrm{dom}(t)) \rightarrow t(n) = 0\}.$$

Since s is an infinite set, U_s is a uniform perfect tree. If now $a \in A$, then by the assumption there is an n such that $n \in a \setminus s$, so the real associated to a cannot belong to $[U_s]$. □

Proposition 3.6. *If for each r there is a splitting real over $\mathbf{L}[r]$, then every $\mathbf{\Sigma}^1_2$ set either contains the branches through a perfect tree or its complement contains the branches through a uniform perfect tree.*

Proof. By Mansfield-Solovay [**Kan94**, Corollary 14.9], every $\mathbf{\Sigma}^1_2$ set A either contains a perfect subset or is contained in $\mathbf{L}[r]$. But if it's contained in $\mathbf{L}[r]$, we can take the splitting real and construct a uniform perfect tree in the complement of A by Lemma 3.5. □

As mentioned, Proposition 3.6 can not be improved to "If for each r there is a splitting real over $\mathbf{L}[r]$, then every $\mathbf{\Sigma}^1_2$ set is weakly Silver measurable". It is still interesting to ask whether some other form of a Mansfield-Solovay dichotomy for $\mathbf{\Sigma}^1_2$ sets holds.[†]

Proposition 3.7. *$\mathbf{\Sigma}^1_2(\mathscr{D})$ implies that for each $r \in {}^\omega\omega$ there is an unbounded real over $\mathbf{L}[r]$.*

Proof. For every strictly increasing function $f \in {}^\omega\omega$ we will construct a tree $P_f \subseteq \{0,1\}^{<\omega}$ which belongs to $\mathfrak{I}_{E_0}$; and for every uniform perfect tree T we will construct a function $g_T \in {}^\omega\omega$, such that $f > g_T$ implies $[P_f] \cap [T] \neq \varnothing$. The conclusion follows then easily by construction.

For $T \in \mathbb{V}$, g_T is just the increasing enumeration of the split levels of $[T]$.

For $f \in {}^\omega\omega$, let $k_0 = 0$ and $k_{n+1} = f(k_n + 1)$. We construct the tree P_f by induction. For $n = 0$, let $P^n_f = \{0,1\}^{<\omega}$ be the full binary tree. Assume we have already constructed P^n_f for some $n \in \omega$. Let $P^n_f|_{k_{n+1}} = \big\{t \in P^n_f : |t| \leq k_{n+1}\big\}$. Further, for every $t \in \{0,1\}^{<\omega}$ with $|t| = k_{n+1}$ let $\xi^t_n \in \{0,1\}$ be defined as follows:

$$\xi^t_n = \begin{cases} 0 & \text{if } \ t(n) \equiv |\{m : n < m < k_{n+1} \text{ and } \ t(m) = 0\}| \mod 2, \\ 1 & \text{otherwise.} \end{cases}$$

Now, define

$$(P^n_f)^* := \big\{s \in P^n_f : \exists t, t' \in 2^{<\omega}\,(\,|t| = k_{n+1} \ \&\ s = t^\frown \xi^t_n{}^\frown t'\,)\,\big\}, \text{ and}$$

$$P^{n+1}_f = P^n_f|_{k_{n+1}} \cup (P^n_f)^*.$$

Finally, let $P_f = \bigcap_{n\in\omega} P^n_f$, then, by construction, $[P_f]$ is a closed set in $\mathfrak{I}_{E_0}$ with parameter f. To see that $[P_f] \in \mathfrak{I}_{E_0}$, assume towards a contradiction that there are two distinct $x, y \in [P_f]$ and an $m \in \omega$ such that $x(m) \neq y(m)$ and for all $m' > m$, $x(m') = y(m')$. Then, by construction, we get $x(k_{m+1}) \neq y(k_{m+1})$, and since $k_{m+1} > m$, this is a contradiction.

Further, if $f > g_T$, then $g_T(k_n) < k_{n+1}$, which implies that for any $n \in \omega$, there is a split level of T between k_n and k_{n+1}, and thus, by construction, we have $[P_f] \cap [T] \neq \varnothing$. □

Proposition 3.8. *Let $\mathbb{C}_{\omega_1}$ be the ω_1-product with finite support of Cohen forc-*

[†] E.g., compare Spinas' theorem that every analytic set is either not dominating or contains the branches through a weakly uniform tree. [**Spi94**, Theorem 1] (Spinas calls the trees "uniform trees"; we changed the notation to avoid confusion).

ing. Then

$$\mathbf{V}^{\mathbb{C}_{\omega_1}} \models \text{“all projective sets are Silver measurable”.}$$

Proof. Let $A = \{y : \varphi(y)\}$, where φ is a $\boldsymbol{\Sigma^1_n}$-formula with some parameter r. Given $[a,b]^\omega \in \mathbf{V}^{\mathbb{C}_{\omega_1}}$, we want to find $[a',b']^\omega \subseteq [a,b]^\omega$ such that either $[a',b']^\omega \subseteq A$ or $[a',b']^\omega \cap A = \varnothing$. Without loss of generality, let us assume that $a, b, r \ldots$ belong to $\mathbf{V}$. Recall that $\mathbb{C}_{\omega_1}$ is homogeneous, and therefore, for every sentence σ of the forcing language with parameters in $\mathbf{V}$ we have either $[\![\sigma]\!]_{\mathbb{C}_{\omega_1}} = \mathbf{1}$ or $[\![\sigma]\!]_{\mathbb{C}_{\omega_1}} = \mathbf{0}$. Notice also that if $c \in \mathbf{V}^{\mathbb{C}_{\omega_1}}$ is Cohen-generic over $\mathbf{V}$, then $\mathbb{C}_{\omega_1} = \mathbb{C} * \dot{\mathbb{A}}$, where $\Vdash_{\mathbb{C}_{\omega_1}} \dot{\mathbb{A}} \cong \dot{\mathbb{C}}_{\omega_1}$ steps into $\mathbf{V}[c]$.

Let us consider $\varphi(c)$: By homogeneity, in $\mathbf{V}[c]$ we have either $[\![\varphi(c)]\!]_{\mathbb{C}_{\omega_1}} = \mathbf{1}$ or $[\![\varphi(c)]\!]_{\mathbb{C}_{\omega_1}} = \mathbf{0}$. Hence, in $\mathbf{V}$, we have either $\big[\!\big[[\![\varphi(\dot{c})]\!]_{\dot{\mathbb{C}}_{\omega_1}} = \dot{\mathbf{1}}\big]\!\big]_{\mathbb{C}} = p(\mathbf{1})$ or $\big[\!\big[[\![\varphi(\dot{c})]\!]_{\dot{\mathbb{C}}_{\omega_1}} = \dot{\mathbf{0}}\big]\!\big]_{\mathbb{C}} = p(\mathbf{0})$, where $p(\mathbf{1}) \vee p(\mathbf{0}) = \mathbf{1}$ and $p(\mathbf{1}) \wedge p(\mathbf{0}) = \mathbf{0}$. Now, in $\mathbf{V}[c]$ we find a doughnut $[a',b']^\omega \subseteq [a,b]^\omega$ such that for all $x \in [a',b']^\omega$, x is Cohen-generic over $\mathbf{V}$. By shrinking $[a',b']^\omega$ if necessary, we may assume that $[a',b']^\omega \subseteq p(\mathbf{1})$ or $[a',b']^\omega \subseteq p(\mathbf{0})$. Let us consider just the former case, since the latter case is similar.

We claim that $[a',b']^\omega \subseteq A$: If $x \in [a',b']^\omega \subseteq p(\mathbf{1})$, then x is Cohen-generic over $\mathbf{V}$ (no matter where x is). Thus, $\mathbf{V}[x] \models [\![\varphi(x)]\!]_{\mathbb{C}_{\omega_1}} = \mathbf{1}$. But the extension leading to $\mathbf{V}^{\mathbb{C}_{\omega_1}}$ is a $\mathbb{C}_{\omega_1}$-extension, hence, $\mathbf{V}^{\mathbb{C}_{\omega_1}} \models \varphi(x)$. □

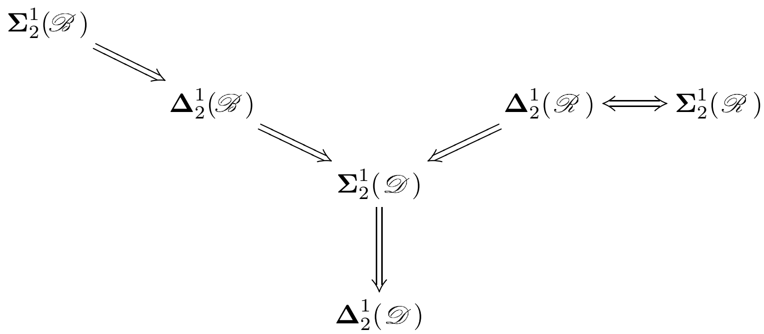
As a consequence we get:

COROLLARY 3.9. *An ω_1-iteration with countable support of Silver forcing, starting from $\mathbf{L}$, yields a model $\mathbf{W}$ in which we have $\boldsymbol{\Delta}^1_2(\mathscr{D}) + \neg\boldsymbol{\Sigma}^1_2(\mathscr{D}) + \neg\boldsymbol{\Delta}^1_2(\mathscr{B}) + \neg\boldsymbol{\Delta}^1_2(\mathscr{R})$.*

Proof. Firstly recall that Silver forcing does not add unbounded reals. Thus, since $\boldsymbol{\Delta}^1_2(\mathscr{R})$ implies that for all $r \in {}^\omega\omega$ there is a dominating real over $\mathbf{L}[r]$, we have $\mathbf{W} \models \neg\boldsymbol{\Delta}^1_2(\mathscr{R})$. Secondly, in Theorem 2.1 we have seen that an ω_1-iteration of Silver forcing with countable support, starting from $\mathbf{L}$, yields a model $\mathbf{W}$ in which every $\boldsymbol{\Delta}^1_2$-set is Silver measurable, and in Proposition 3.7 we have seen that $\boldsymbol{\Sigma}^1_2(\mathscr{D})$ implies that for every real r, there are unbounded reals over $\mathbf{L}[r]$. Hence, since Silver forcing does not add unbounded reals, by Corollary 3.2 we have $\mathbf{W} \models \boldsymbol{\Delta}^1_2(\mathscr{D}) + \neg\boldsymbol{\Sigma}^1_2(\mathscr{D}) + \neg\boldsymbol{\Delta}^1_2(\mathscr{B})$. □

4. *Conclusion*

THEOREM 4.1. *The only implications between the properties $\mathscr{D}$, $\mathscr{B}$ and $\mathscr{R}$ of $\boldsymbol{\Delta}^1_2$ and $\boldsymbol{\Sigma}^1_2$-sets are given in the following transitive diagram:*



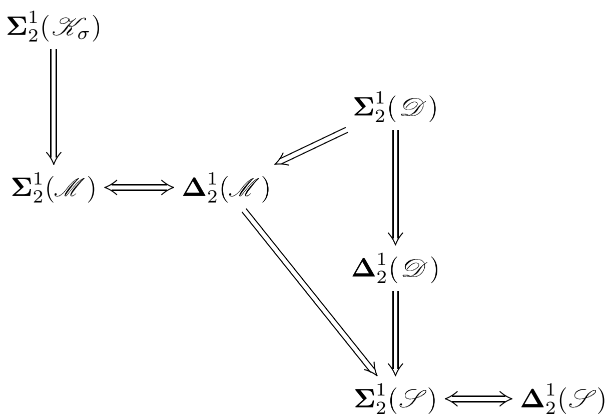
Proof. For the implications between the Baire and Ramsey property of $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ sets see [**Jud88**] and [**JudShe89**].

$\mathbf{\Sigma}^1_2(\mathscr{D}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathscr{R})$: This follows from $\mathbf{\Sigma}^1_2(\mathscr{R}) \Leftrightarrow \mathbf{\Delta}^1_2(\mathscr{R})$ (*cf.* Fact 0.1 (iii)) and $\mathbf{\Sigma}^1_2(\mathscr{D}) \not\Rightarrow \mathbf{\Sigma}^1_2(\mathscr{R})$ (*cf.* [**Hal03**]).

$\mathbf{\Sigma}^1_2(\mathscr{D}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathscr{B})$: This follows from the obvious implication $\mathbf{\Sigma}^1_2(\mathscr{R}) \Rightarrow \mathbf{\Sigma}^1_2(\mathscr{D})$ and $\mathbf{\Sigma}^1_2(\mathscr{R}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathscr{B})$ (*cf.* [**JudShe89**]).

$\mathbf{\Delta}^1_2(\mathscr{D}) \not\Rightarrow \mathbf{\Sigma}^1_2(\mathscr{D})$: This follows from Corollary 3.9. □

PROPOSITION 4.2. *Between the properties $\mathscr{D}$, $\mathscr{M}$, $\mathscr{S}$ and $\mathscr{K}_\sigma$ of $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$-sets, we have the following implications:*



Proof. In [**Jud88**] it is proved that

$$\mathbf{\Sigma}^1_2(\mathscr{K}_\sigma) \iff \forall r \in {}^\omega\omega\,({}^\omega\omega \cap \mathbf{L}[r] \text{ is bounded}).$$

The proposition follows from Fact 0.1 (v) & (vi), Proposition 3.7 and Proposition 2.3. □

We have succeeded in determining the strength of $\mathbf{\Delta}^1_2(\mathscr{D})$ and $\mathbf{\Sigma}^1_2(\mathscr{D})$ in terms of other regularity properties. What is still missing are results of Solovay- and Judah-Shelah-type: Propositions 2.2, 2.3, and 3.4 yield almost equivalences since the ideals $\mathfrak{I}_{\mathbb{V}}$ and $\mathfrak{I}_{E_0}$ are very close to each other. It is even conceivable that the existence of $\mathfrak{I}_{\mathbb{V}}$- and $\mathfrak{I}_{E_0}$-quasigenerics are equivalent. But so far, we don't know.

In terms of connections between $\mathbf{\Delta}^1_2(\mathscr{D})$ and the existence of certain reals, we have the following diagram:

$$
\begin{array}{ccc}
 & [\mathsf{v}]\ \mathfrak{I}_{\mathbb{V}}\text{-generics} & \\
 & \Downarrow & \\
 & [\mathsf{d}]\ \mathbf{\Delta}^1_2(\mathscr{D}) & \\
 \swarrow\!\!\swarrow & & \searrow\!\!\searrow \\
[\mathsf{e}]\ \mathfrak{I}_{E_0}\text{-generics} & & [\mathsf{s}]\ \text{splitting reals}
\end{array}
$$

The question is: can we get the reverse directions anywhere in this diagram?

Question 1. *Does* [d] $\Rightarrow$ [v] *hold (the converse to Proposition 2.2)?*

Note that if [d] $\Rightarrow$ [v], then we can also characterize $\mathbf{\Sigma}^1_2(\mathscr{D})$ in terms of quasigenerics by Lemma 3.3): In that case, the converse to Proposition 3.4 (1) holds as well.

Question 2. *Does the existence of a splitting real over each* $\mathbf{L}[r]$ *imply* $\mathbf{\Delta}^1_2(\mathscr{D})$ *(*[s] $\Rightarrow$ [d]*; the converse to Proposition 2.4)?*

Question 3. *Does the existence of* $\mathfrak{I}_{E_0}$*-*$\mathbf{L}[r]$*-quasigenerics imply* $\mathbf{\Delta}^1_2(\mathscr{D})$ *(*[e] $\Rightarrow$ [d]*; the converse to Proposition 2.3)?*

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Jörg Brendle
The Graduate School of Science and Technology
Kobe University
Rokko-dai 1-1, Nada
Kobe 657-8501
Japan

brendle@pascal.cs.kobe-u.ac.jp

Lorenz Halbeisen
Department of Pure Mathematics
Queen's University Belfast
Belfast BT7 1NN
Northern Ireland

halbeis@qub.ac.uk

Benedikt Löwe
Institute for Logic, Language and Computation
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam
The Netherlands

bloewe@science.uva.nl