A HIERARCHY OF NORMS DEFINED VIA BLACKWELL GAMES

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Abstract. We define a hierarchy of norms using strongly optimal strategies in Blackwell games and prove that the resulting hierarchy is a prewellordering.

1. Introduction

The Axiom of Blackwell Determinacy was introduced by Vervoort [Ve95] as an imperfect information analogue of the Axiom of Determinacy AD. Tony Martin [Mar98] proved that perfect information determinacy implies imperfect information determinacy classwise (*i.e.*, for a boldface pointclass $\mathbf{\Gamma}$, if all sets in $\mathbf{\Gamma}$ are determined as perfect information games then all sets in $\mathbf{\Gamma}$ are determined as imperfect information games). Martin conjectured that perfect information and imperfect information determinacy are equivalent.

This conjecture is still unproven, but a lot of progress has been made: Martin, Neeman and Vervoort showed in [MarNeeVer∞] that the Axiom of Blackwell Determinacy is equiconsistent to AD, and moreover, if the Axiom of Blackwell Determinacy holds in $\mathbf{L}(\mathbb{R})$ (the least model of set theory containing all reals), then AD holds there as well.

The present author proved some of the very characteristic combinatorial consequences of AD from Blackwell Determinacy in [Löw∞]. (For a survey, *cf.* [Löw02b].) But so far, most consequences of Blackwell Determinacy have been local or bounded in character: they talk

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about projective ordinals or set theory below $\Theta^{\mathbf{L}(\mathbb{R})}$, but not about the set theoretic universe up to Θ itself.

One striking global consequence of AD that we would like to mimic under $\mathsf{Bl}\text{-}\mathsf{AD}$ is the existence of a delicate structure theory of the Lipschitz and Wadge degrees. Using mixed strategies instead of pure strategies, we are able to define a **Blackwell Lipschitz hierarchy** and a **Blackwell Wadge hierarchy** (*cf.* [Löw02b, Theorem 4.6]), but we are unable to prove that these hierarchies are wellfounded (the Martin-Monk method founders).

In this paper, we look at a different hierarchy with a global structure theory under AD: the **Hierarchy of Norms** going back to Moschovakis' First Periodicity Theorem, and further investigated by Chalons [Cha00] and Duparc [Dup03]. Under AD, the Hierarchy of Norms is a prewellordering and serves as a measure of complexity on norms.

We define the **Blackwell Hierarchy of Norms** (which by the mentioned result of Martin, Neeman and Vervoort coincides with the Hierarchy of Norms in $\mathbf{L}(\mathbb{R})$) and prove that it is a prewellordering under the assumption of $\mathsf{Bl}\text{-}\mathsf{AD}$.

2. Prerequisites

2.1. **Blackwell Determinacy.** Blackwell determinacy goes back to imperfect information games of finite length due to von Neumann and was introduced for infinite games by Blackwell [Bla69]. Since the full axiom of Blackwell determinacy contradicts the full Axiom of Choice AC, we shall work throughout this paper in the theory $\mathsf{ZF} + \mathsf{DC}$.

We will be working on Baire space $\mathbb{N}^{\mathbb{N}}$, endowed with the product topology of the discrete topology on $\mathbb{N}$, $\mathbb{N}^{<\mathbb{N}}$ is the set of finite sequences of natural numbers. Let us write $\mathbb{N}^{\text{even}}$ and $\mathbb{N}^{\text{odd}}$ for finite sequences of even and odd length, respectively, and $\text{Prob}(\mathbb{N})$ for the set of probability measures on $\mathbb{N}$.

We shall be using the standard notation for infinite games: If $x \in \mathbb{N}^{\mathbb{N}}$ is the sequences of moves for player I and $y \in \mathbb{N}^{\mathbb{N}}$ is the sequence of moves for player II, we let $x * y$ be the sequence constructed by playing x against y, *i.e.*,

$$(x * y)(n) := \begin{cases} x(k) & \text{if } n = 2k, \\ y(k) & \text{if } n = 2k+1. \end{cases}$$

Conversely, if $x \in \mathbb{N}^{\mathbb{N}}$ is a run of a game, then we let x_{I} be the part played by player I and x_{II} be the part played by player II, *i.e.*, $x_{\mathrm{I}}(n) = x(2n)$ and $x_{\mathrm{II}}(n) = x(2n+1)$.

We call a function $\sigma : \mathbb{N}^{\mathrm{Even}} \to \mathrm{Prob}(\mathbb{N})$ a **mixed strategy for player I** and a function $\sigma : \mathbb{N}^{\mathrm{Odd}} \to \mathrm{Prob}(\mathbb{N})$ a **mixed strategy for player II**. A mixed strategy σ is called **pure** if for all $s \in \mathrm{dom}(\sigma)$ the measure $\sigma(s)$ is a Dirac measure, *i.e.*, there is a natural number n such that $\sigma(s)(\{n\}) = 1$. This is of course equivalent to being a strategy in the usual (perfect information) sense.

Let

$$\nu(\sigma,\tau)(s) := \begin{cases} \sigma(s) & \text{if } \mathrm{lh}(s) \text{ is even, and} \\ \tau(s) & \text{if } \mathrm{lh}(s) \text{ is odd.} \end{cases}$$

Then for any $s \in \mathbb{N}^{<\mathbb{N}}$, we can define

$$\mu_{\sigma,\tau}([s]) := \prod_{i=0}^{\mathrm{lh}(s)-1} \nu(\sigma,\tau)(s{\restriction}i)(\{s_i\}).$$

This generates a Borel probability measure on $\mathbb{N}^{\mathbb{N}}$. If B is a Borel set, $\mu_{\sigma,\tau}(B)$ is interpreted as the probability that the result of the game ends up in the set B when player I randomizes according to σ and player II according to τ. If σ and τ are both pure, then $\mu_{\sigma,\tau}$ is a Dirac measure concentrated on the unique real that is the outcome of this game, denoted by $\sigma * \tau$. As usual, we call a pure strategy σ for player I (τ for player II) a **winning strategy** if for all pure counterstrategies τ (σ), we have that $\sigma * \tau \in A$ ($\sigma * \tau \notin A$).

For mixed strategies σ (for player I) or τ (for player II) we define a measure of quality (the **mixed value of the strategy**) by

$$\mathrm{mval}_{\mathrm{I}}^{A}(\sigma) := \inf\{\mu^{-}_{\sigma,\tau}(A)\,;\, \tau \text{ is a mixed strategy for player II}\}, \text{ and}$$

$$\mathrm{mval}_{\mathrm{II}}^{A}(\tau) := \sup\{\mu^{+}_{\sigma,\tau}(A)\,;\, \sigma \text{ is a mixed strategy for player I}\}.^{1}$$

A mixed strategy for player I is now called **strongly optimal for** A if $\mathrm{mval}_{\mathrm{I}}^{A}(\sigma) = 1$, and and a mixed strategy τ for player II is called **strongly optimal for** A if $\mathrm{mval}_{\mathrm{II}}^{A}(\tau) = 0$.

We call a set A **perfect information Blackwell determined** if either player I or player II has a strongly optimal strategy and we call a pointclass Γ perfect information Blackwell determined (in symbols:

[1]Here, μ^{+} denotes outer measure and μ^{-} denotes inner measure with respect to μ in the usual sense of measure theory. If A is Borel, then $\mu^{+}(A) = \mu^{-}(A) = \mu(A)$ for Borel measures μ.

$\mathsf{pBl\text{-}Det}(\Gamma)$) if for all $A \in \Gamma$, the set A is perfect information Blackwell determined.[2]

2.2. **The Hierarchy of Norms.** We investigate norms $\varphi : \mathbb{R} \to \alpha$ (for some ordinal α).

For two norms φ and ψ, we say that φ is **FPT-reducible to** ψ (for "**F**irst **P**eriodicity **T**heorem"; in symbols: $\varphi \leq_{\mathrm{FPT}} \psi$) if there is a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have

$$\varphi(F(x)) \leq \psi(x).$$

FPT-reducibility can be expressed in game terms: Look at the two-player perfect information game where player I plays x, player II plays y and is allowed to pass provided he plays infinitely often, and player II wins if and only if $\varphi(x) \leq \psi(y)$. We call this game $\mathbb{G}_{\leq}(\varphi, \psi)$; $\varphi \leq_{\mathrm{FPT}} \psi$ if and only if player II has a winning strategy in $\mathbb{G}_{\leq}(\varphi, \psi)$. We call this relation FPT-reducibility because the game (and the well-ordering proof) goes back to the proof of the First Periodicity Theorem of Moschovakis [AddMos68]. That proof essentially shows that in $\mathsf{ZF} + \mathsf{DC}(\mathbb{R}) + \mathsf{AD}$, $<_{\mathrm{FPT}}$ is a prewellordering. For more on the proof, *cf.* [Mos80, 6B].

FPT-reducibility is called "Steel reducibility" in [Cha00] and [Dup03], and the Hierarchy of Norms is called the "Steel hierarchy". The reader can find many details about this hierarchy restricted to Borel sets in [Dup03].

3. The Blackwell Hierarchy of Norms

We will now use the game-theoretic definition of the Hierarchy of Norms and define a Blackwell version of $<_{\mathrm{FPT}}$ as follows:

For two FPT-functions φ and ψ, we say that φ is **Blackwell FPT-reducible** to ψ and write

$$\varphi \leq^{\mathrm{Bl}}_{\mathrm{FPT}} \psi$$

if player II has a strongly optimal strategy in the game $\mathbb{G}_{\leq}(\varphi, \psi)$.

Let us briefly mention that the same has been done for the Lipschitz games $\mathbb{G}_{\mathrm{L}}(A, B)$ and the Wadge games $\mathbb{G}_{\mathrm{W}}(A, B)$ by the present author (unpublished, but discussed in [Löw02b]). We can recover some of the AD theory of Lipschitz and Wadge reducibility for the new relations

[2] It is not at all obvious that this definition yields useful axioms of Blackwell determinacy. In order to show that $\mathsf{pBl\text{-}Det}(\boldsymbol{\Delta}^1_1)$ holds, it is necessary to employ techniques from [MarNeeVer∞] (in particular the **Martin-Vervoort Zero-One Law** and the **Vervoort Strong Zero-One Law**) for games on finite sets, and from [Löw02a] for games on infinite sets.

$\leq_{\mathrm{Bl}\ell}$ and $\leq_{\mathrm{Bl}W}$, but we were not able to prove the two key theorems: the analogue of the Martin-Monk theorem ("$\leq_{\mathrm{Bl}\ell}$ is wellfounded") and the analogue of the Steel-van Wesep theorem ("$A \leq_{\mathrm{Bl}\ell} \mathbb{N}^{\mathbb{N}}\backslash A$ if and only if $A \leq_{\mathrm{Bl}W} \mathbb{N}^{\mathbb{N}}\backslash A$"). Both of these theorems employ the Martin-Monk technique (constructing a set which is not Lebesgue measurable from a family of infinite game diagrams) which doesn't seem to work with mixed strategies.

In order to see that we didn't define a trivial relation, note that Blackwell FPT-reducibility and usual FPT-reducibility coincide if the game $\mathbb{G}_{\leq}(\varphi,\psi)$ is determined: suppose for a contradiction that player I wins $\mathbb{G}_{\leq}(\varphi,\psi)$ via a winning strategy σ, but player II has a strongly optimal strategy τ. If you play σ against τ, then $\mu_{\sigma,\tau}$ concentrates on winning plays for player I, so player I wins with probability 1, contradicting strong optimality of τ.

In particular, for all Borel sets A and B,

$$A \leq_{\mathrm{FPT}} B \iff A \leq^{\mathrm{Bl}}_{\mathrm{FPT}} B,$$

and similarly for projective sets under the assumption of PD and for all sets under the assumption of AD.

The following could be called "Wadge's Lemma" for the Blackwell Hierarchy of Norms:

Proposition 3.1. *Assume* $\mathsf{pBl\text{-}AD}$. *Then* $\leq^{\mathrm{Bl}}_{\mathrm{FPT}}$ *is a linear preordering.*

Proof. If player II doesn't have a strongly optimal strategy in the game $\mathbb{G}_{\leq}(\varphi,\psi)$, then player I has one. This can be used as a strategy for player II in the game $\mathbb{G}_{\leq}(\psi,\varphi)$. □

The proof that the Hierarchy of Norms is wellfounded is slightly less involved than the Martin-Monk theorem (in terms of our notation used later, it needs only a single diagram instead of an $\mathbb{R}$-indexed family of diagrams). This allows us to prove its analogue for Blackwell determinacy.

We now provide some notation for the proof:

Let's call an element of $\mathbb{N}^{\mathbb{N}\times\mathbb{N}}$ a **diagram**. We interpret diagrams as $\mathbb{N}\times\mathbb{N}$ matrices and speak of **rows** and **columns** of a diagram. The ith column of the diagram d will be denoted by d_i. We imagine a diagram d as an infinite sequence of games where column i and $i+1$ are interpreted as a run of the game in which player I plays d_i and player II plays d_{i+1}.

A sequence $\langle s_0,\ldots,s_n\rangle$ is called a **triangle** (*cf.* Figure 1 for a picture) if $s_i \in \mathbb{N}^{<\mathbb{N}}$ and $\mathrm{lh}(s_i) = n-i$. If $\langle s_0,\ldots,s_n\rangle$ is a triangle, we can imagine columns i and $i+1$ to be the moves of players I and II in a game and get a sequence $s_i * s_{i+1}$ of length $(n-i)+(n-(i+1)) = 2(n-i)-1$.

A (basic open) subset B of $\mathbb{N}^{\mathbb{N}\times\mathbb{N}}$ is called a **triangle** if there is a triangle $\langle s_0, \ldots, s_n\rangle$ such that

$$B = \{\, d\,;\, \forall i \le n\, \forall j < n-1\, (d_i(j) = s_i(j))\,\}.$$

Note that in order to determine a Borel measure on the space of all diagrams it is enough to fix the measure on triangles.

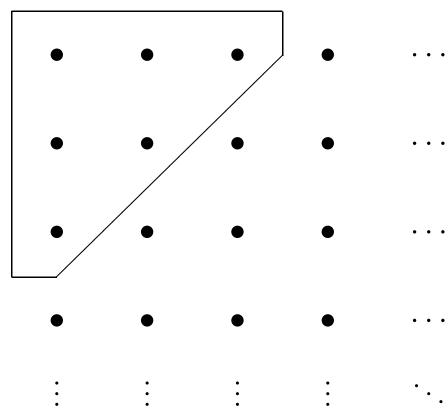

FIGURE 1. A triangle.

Theorem 3.2. pBl-AD *implies that the relation* $<^{\mathrm{Bl}}_{\mathrm{FPT}}$ *is well-founded.*

Proof. We have to show that there is no descending $<^{\mathrm{Bl}}_{\mathrm{FPT}}$-chain.

Towards a contradiction, let $\langle \varphi_n\,;\, n \in \omega\rangle$ be such that $\varphi_{n+1} <^{\mathrm{Bl}}_{\mathrm{FPT}} \varphi_n$ for all $n \in \omega$.

Using pBl-AD, we can pick strongly optimal strategies σ_n for player I in the game $\mathbb{G}_{\le}(\varphi_n, \varphi_{n+1})$ with value 1.

The sequence $\langle \sigma_n\,;\, n \in \mathbb{N}\rangle$ determines a Borel measure[3] μ on the space of all diagrams $2^{\mathbb{N}\times\mathbb{N}}$ as follows: If $s = \langle s_0, \ldots, s_n\rangle$ is a triangle and $B_s := \{\, d\,;\, \forall i \le n\, \forall j < n-1\, (d_i(j) = s_i(j))\,\}$, then let

$$\mu(B_s) := \prod_{k=0}^{n-1} \prod_{i=0}^{\mathrm{lh}(s_k)-1} \sigma_k(s_k * s_{k+1}\restriction 2i)(s_k(i)).$$

In this Borel measure, you determine the probability for the values of $d_i(0)$ first by using the strategies σ_i and then fill up the diagram by

[3]The idea of using mixed functions to define a measure on the set of diagrams in Blackwell determinacy contexts is due to Neeman, *cf.* [MarNeeVer∞].

letting player I with strategy σ_i play against the run d_{i+1} (as depicted in Figure 2).

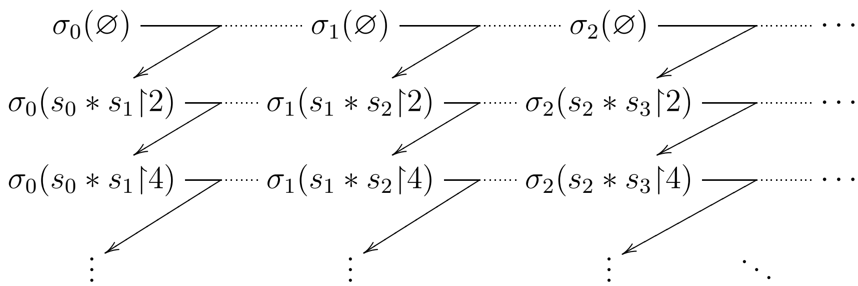

Figure 2. The game interpretation of the measure μ.

For each $i \geq 1$, we will now define a **blindfolded** strategy τ_i for player II:

$$\tau_i(s)(\{j\}) := \mu(\{d\,;\, d_i(n) = j\}) \text{ if } \mathrm{lh}(s) = 2n + 1.$$

(The passing move gets probability 0.) The strategy τ_i gives the event $d_i(k) = \ell$ the probability that it is assigned to it by the entire process of filling in the rest of the triangle to the right-hand side of $d_i(k)$ by using the strategies $\sigma_i, \ldots, \sigma_{i+k}$.

By construction, we have for all Borel sets B of real numbers that

$$(\ddagger) \qquad \mu_{\sigma_i \tau_{i+1}}(B) = \mu(\{d\,;\, d_i * d_{i+1} \in B\}).$$

Let $S_{i,i+1} := \{x * y\,;\, \varphi_i(x) > \varphi_{i+1}(y)\}$. Then $(\mu_{\sigma_i \tau_{i+1}})^{-}(S_{i,i+1}) = 1$, so by ($\ddagger$) we have $(\mu)^{-}(\{\, d\,;\, d_i * d_{i+1} \in S_{i,i+1}\}) = 1$, and thus by σ-additivity, we get

$$(\mu)^{-}(\{\, d\,;\, \forall i \in \mathbb{N}\,(d_i * d_{i+1} \in S_{i,i+1}\,)\}\,) = 1.$$

Pick one of those d in the set with measure 1. Then

$$\langle \varphi_i(d_i)\,;\, i \in \mathbb{N} \rangle$$

is an infinite strictly decreasing chain of ordinals. Contradiction. □

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