A Gödel-style translation from positive calculus into strict implication logic

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In this thesis, we develop a Gödel-style translation of a positive calculus, that is, a calculus that is equivalent to the positive fragment of classical propositional logic that is sound and complete with respect to bounded distributive lattices, into a suitable expansion of classical logic. In order to accomplish this, we build on a known correspondence between bounded distributive lattices and Boolean algebras with so-called lattice subordinations. We introduce a strict implication calculus that is sound and complete with respect to the class of Boolean algebras with a lattice subordination, which, as a consequence of the duality between the category of Boolean algebras with a lattice subordination and the category of quasi-ordered Priestley spaces, will also serve to reason about quasi-ordered Priestley spaces. For the positive calculus, we chose to work with a hypersequent framework, because it allows for nontrivial extensions of the calculus that correspond to proper subclasses of the class of bounded distributive lattices. We present a translation $\mathrm{Tr}(-)$ from the positive calculus to the strict implication calculus and show that it is full and faithful. We consider extensions of both calculi and show that every extension of the positive calculus is embedded via $\mathrm{Tr}(-)$ into some extension of strict implication calculus and vice versa, that for every extension of the strict implication calculus there exists a positive calculus that is embeddable in it via $\mathrm{Tr}(-)$.

Contents

Chapter 1

Introduction

This thesis is concerned with the development of a Gödel-style translation of a positive calculus, that is, a calculus that is equivalent to the positive fragment of classical propositional logic, into a suitable expansion of classical logic.

In 1933, Gödel [24] described an embedding of the intuitionistic propositional logic IPC, as axiomatised by Heyting [26], into the modal expansion $\mathbf{S4}$ of classical logic. The embedding prefixes the modal operator $\Box$ to every subformula of a given intuitionistic propositional formula φ, with the intended meaning of $\Box$ to be 'is provable', hereby providing a formal interpretation of the (informal) intuitionistic semantics that a statement is true if it has a proof.[1] Gödel's conjecture, that a formula φ is provable in IPC if and only if its translation $T(\varphi)$ is provable in $\mathbf{S4}$, was later shown to be true by McKinsey and Tarski in the 40s [32]. They established the translation, henceforth known as the Gödel-McKinsey-Tarksi translation T, to be full and faithfull. McKinsey and Tarski also presented the Gödel embedding in algebraic form, by showing that IPC and $\mathbf{S4}$ are strongly sound and complete with respect to Heyting algebras and $\mathsf{S4}$-algebras respectively.[2] In the 50s the Gödel translation was lifted to the whole class of intermediate logics by Dummett and Lemmon [17], associating with each intermediate logic $\mathbf{L} = \mathsf{IPC} + \{\varphi_i\}_{i\in\mathcal{I}}$ the $\mathbf{S4}$ extension $\tau\mathbf{L} = \mathbf{S4} \oplus \{T(\varphi_i)\}_{i\in\mathcal{I}}$. Grzegorszyk later discovered that IPC can also be embedded into $\mathbf{S4Grz} = \mathbf{S4} \oplus \Box(\Box(p \to \Box p) \to p) \to p$, a proper extension of $\mathbf{S4}$

[1] Actually the initial translation by Gödel in [24] does not prefix $\Box$ to conjunctions and disjunctions. However, this difference does not affect the embeddings of IPC nor of its extensions.

[2] Throughout this thesis, we will use the typesetting '$\mathbf{S4}$' to denote the logic and will use '$\mathsf{S4}$' when referring to the corresponding algebra.

[25].[3] Systematic further investigations of the lattice ExtIPC of extensions of IPC and the lattice $\mathsf{NExt}\mathbf{S4}$ of normal extensions of $\mathbf{S4}$, launched by, among others Maksimova and Rybakov [30], Blok [9], and Esakia [20, 21], have shown that for each normal extension $\mathbf{M}$ of $\mathbf{S4}$ there is a unique intermediate logic $\rho\mathbf{M}$ embeddable in it, termed its *superintuitionistic fragment*, with $\mathbf{M}$ the *modal companion* of $\rho\mathbf{M}$. It was demonstrated that there are a continuum of modal companions for each intermediate logic $\mathbf{L}$ [36], with a smallest companion $\tau\mathbf{L}$ and a largest companion $\sigma\mathbf{L}$. Moreover, the mapping τ between ExtIPC and $\mathsf{NExt}\mathbf{S4}$ transpired to be a lattice isomorphism. Blok [9] and Esakia [20] further discovered the largest companion for each $\mathbf{L}$ to be $\sigma\mathbf{L} = \tau\mathbf{L} \oplus \mathbf{S4Grz}$, with their research culminating in the Blok-Esakia theorem that the map σ is a lattice isomorphism of the ExtIPC onto $\mathsf{NExt}\mathbf{Grz}$ (for a detailed survey of the development of the theory of intermediate logics and modal companions, see for instance [12]).

Thus, at the syntactic level, the Gödel translation results in an embedding of every intermediate logic $\mathbf{L}$ into an interval $[\tau\mathbf{L}, \sigma\mathbf{L}]$ of normal extensions of $\mathbf{S4}$. At the algebraic level, where we interpret intuitionistic formulas in Heyting algebras and modal formulas in $\mathsf{S4}$-algebras, this embedding manifests itself by the fact the algebras of open elements (that is, the elements a of an $\mathsf{S4}$-algebra for which $a = \Box a$) of $\mathsf{S4}$-algebras are precisely Heyting algebras and that every Heyting algebra is isomorphic to the algebra of open elements of a befitting $\mathsf{S4}$-algebra.

The algebraical side of the Gödel embedding has been generalised by [3] to an embedding of bounded distributive lattices into analogues of $\mathsf{S4}$-algebras. In particular, [3] introduces binary relations $\prec$ on Boolean algebras called lattice subordinations, which resemble de Vries proximities from [16] and naturally generalise to so-called subordinations by dropping some of the specified conditions (the exact definitions will be presented in the following chapter). The theory of subordinations is closely related to several other lines of research. They are in fact the dual concept of the pre-contact relations (also known as proximity relations) introduced by Düntsch and Vakarelov in [18] and are in a 1-1 correspondence with the quasi-modal operators introduced by Celani [11]. These three notions have been shown to be equivalent, and hence in this work we could take any one of them as primitive. In this thesis we choose to work with the signatures $\prec$ and $\rightsquigarrow$ in line with our main references, in order to directly apply their

[3] In fact, the axiom above, though known as Grzegorczyk's axiom, is due to Sobocinński [39]. Grzegorczyk [25] used the axiom $\Box(\Box(p \to \Box q) \to \Box q) \wedge \Box(\Box(\neg p \to \Box q) \to \Box q) \to \Box q)$ which turned out to be equivalent in the system $\mathbf{S4}$.

key concepts and results. In [3] it has been shown that, for every Boolean algebra with a lattice subordination, the lattice of reflexive elements (that is, elements $a \in B$ for which $a \prec a$) is a bounded distributive lattice and conversely, each bounded distributive lattice is isomorphic to the bounded distributive lattice of reflexive elements of a suitable Boolean algebra with a lattice subordination.

The main objective of this thesis is to develop a syntactic counterpart for the aforementioned generalised Gödel embedding, that is, to define a positive calculus that is sound and complete with respect to bounded distributive lattices and one likewise for Boolean algebras with a lattice subordination and furthermore, to develop analogues for the concepts of modal companions and superintuitionistic fragments. In this thesis, we define such a translation $\mathrm{Tr}(-)$ and our main result establishes it to be full and faithful. In our choice for a positive calculus whe are faced with the problem that there can be no nontrivial extension of a positive sequent calculus (and thus no nontrivial extensions of a positive logic). However, we will see that for hypersequent calculi, such extensions do exist. As we are interested in developing analogues for nontrivial superintuitionistic logics, that is, extensions of a positive calculus that correspond to subclasses of the class of bounded distributive lattices, the hypersequent framework is a *comme il faut* choice. As for Boolean algebra with a lattice subordinations, there are two observations to make, concerning semantics and the nature of the calculus respectively. First, due to the presence of the binary relation $\prec$, a Boolean algebra with a lattice subordination is formally not an algebra. We follow the solution of [5], that every (lattice) subordination can be described equivalently by a characteristic function $\rightsquigarrow$ on a Boolean algebra.[4] For our choice of a calculus, the class of Boolean algebras with a lattice subordination is not a universal class, but an inductive one. We build on the work of [5] that introduces a calculus $\mathbf{RC}_{\rightsquigarrow}$ that is sound and complete with respect to the class of Boolean algebras with a reflexive subordination. They introduce a specific sort of nonstandard rules, so-called Π_2-rules and show that, $\mathbf{RC}_{\rightsquigarrow}$ extended by Π_2-rules correspond to inductive subclasses.

1.1 Outline

The structure of this thesis is as follows. In Chapter 2, we define the concepts and structures that are used throughout this thesis, among which subordinations,

[4]The work in [5] is based on the Master's thesis [38]. In this thesis, we will be drawing extensively on both sources though we mostly refer to [5].

lattice subordinations, and Heyting lattice subordinations on Boolean algebras, as well as their corresponding strict implication algebras. We present Stone-like dualities for Boolean algebras with a subordinations and restrictions hereof, and corresponding categories of topological spaces. In order to do so, we first recall relevant concepts and definitions and known results among which Stone and Priestley duality.

In Chapter 3, we present a positive propositional language and an algebraic semantics based on bounded distributive lattices. We present a positive hypersequent calculus $\mathbf{PC}_+$ and show that it is sound and complete with respect to the class of bounded distributive lattices. We motivate our choice of a hypersequent setting, by showing that there can be no nontrivial extension of a positive sequent calculus (and thus no nontrivial extensions of a positive logic), whereas we show that, for hypersequent calculi, such extensions do exist. Although soundness and completeness of the hypersequent calculus with respect to the class of bounded distributive lattices has been known, to the knowledge of the author there has been no spelled-out proof in the literature, hence we present the proof in full detail.

Thereafter, in Chapter 4, we set out a classical propositional language enriched by a binary connective $\rightsquigarrow$ called a strict implication. We present an algebraic semantics for this language based on Boolean algebras with a strict implication. We recall the results from [5] that presents a calculus $\mathbf{RC}_{\rightsquigarrow}$ (by [5] termed SIC) which is sound and complete with respect to the class of Boolean algebras with a reflexive subordination. Following [5], we define so-called Π_2-rules and recall their result that $\mathbf{RC}_{\rightsquigarrow}$ extended with Π_2-rules is sound and complete with respect to inductive subclasses of the aforementioned class. We introduce a specific Π_2-rule, ρ_{qp} and show that the system $\mathbf{BC}_{\rightsquigarrow}$, the extension of $\mathbf{RC}_{\rightsquigarrow}$ with ρ_{qp}, is sound and complete with respect to the class of Boolean algebras with a lattice subordination.

In Chapter 5 we present the main result of our thesis, namely, a translation $\mathrm{Tr}(-)$ from positive hypersequent rules into sequent rules in the strict implication language. We show that every such strict implication sequent rule is equivalent to a strict implication formula. We proceed to show that a positive hypersequent rule is derivable in $\mathbf{PC}_+$ if and only if the strict implication formula that corresponds to its translation is derivable in the calculus $\mathbf{BC}_{\rightsquigarrow}$ and hence, that $\mathrm{Tr}(-)$ defines an embedding of $\mathbf{PC}_+$ into $\mathbf{BC}_{\rightsquigarrow}$. We explain why this is, in fact, a generalisation of the Gödel translation and lift it to extensions of $\mathbf{PC}_+$ and corresponding extensions of $\mathbf{BC}_{\rightsquigarrow}$. We define the notions of superpositive fragment and strict implication companions and provide examples thereof.

Chapter 2

Preliminaries

The study of categorical dualities between classes of algebras and classes of topological spaces finds its early beginnings in Marshall's Stone's 1936 seminal paper on the representation of Boolean algebras [40]. Stone showed that Boolean algebras can be represented as algebras of clopen sets of associated compact zero-dimensional Hausdorff spaces (now commonly known as Stone spaces) and, that such Stone spaces can be represented via associated Boolean algebras. This extends to a dual equivalence between the category of Boolean algebras together with Boolean algebra homomorphisms and the category of Stone spaces and continuous maps. Such (dual) equivalences between categories allow us to translate mathematical structures, theorems, problems, and concepts from one category into structures, theorems, problems, and concepts of its associated category - and to alternate between them. Hence, following Stone, in the course of the second half of the 20$^{\text{th}}$ century, other categories of algebras and of topological spaces have been investigated and related to one another by categorical dual equivalences. Generalisations and extensions of Stone duality that are specifically relevant to this thesis, are Priestley duality for bounded distributive lattices (presented below), de Vries duality for compact Hausdorff spaces, Esakia duality for Heyting algebras, and Jónnson-Tarski duality for modal algebras.

In this chapter, we recall the formal results of Stone and Priestley dualities, followed by a specific Stone-type duality from [3] between Boolean algebras with a so-called subordination and Stone spaces ordered by a closed relation. Following [3] we then present a restriction of this duality between Boolean algebras with a so-called lattice subordination and quasi-ordered Priestley spaces. As shown in [3], from a lattice subordination, we can identify a bounded distributive sublattice of a Boolean algebra. We proceed by connecting back the Priestley spaces obtained from bounded sublattices to the dual quasi-ordered Priestley space of the Boolean algebra with a lattice subordination in question. We conclude this section by restricting the above correspondences to the special case of Heyting lattice subordinations.

2.1 Stone Duality and Priestley Duality

In this section, we review the celebrated Stone and Priestley dualities. The first establishes a dual equivalence between the category of Boolean algebras together with Boolean algebra homomorphisms and the category of Stone spaces (topological spaces that are compact, Hausdorff, and zero-dimensional) together with continuous maps. By the latter duality, the category of bounded distributive lattices with bounded lattice homomorphisms is dually equivalent to the category of Priestley spaces (which are Stone spaces partially ordered by a relation R that satisfies the Priestley axiom) and order-preserving continuous maps. Though Stone duality chronologically precedes Priestley duality, we will first set out the latter since, as we will see, Priestley duality generalizes Stone duality. First, we give a brief overview of the basic concepts and properties needed to set out these dualities. The material of this section is based on [10, 15], and, unless stated otherwise, the full details and proofs can be found therein.

We recall that a binary relation R on a set S is called a *quasi-order* or a *pre-order* if it is reflexive and transitive, and a *partial order* if, additionally, it is anti-symmetric. We will often denote R by $\leq$ if it is a pre-order. Then, for a pre-order R on a set S, a subset P of S is called an *upset* if, for all $a, b \in S$, whenever $a \in P$ and aRb we also have $b \in P$. A nonempty set L with a partial order $\leq$ is called a *lattice* if, for all $a, b \in L$, the least upper bound and greatest lower bound of a and b exist. The least upper bound of a and b is denoted by $a \vee b$ and the greatest lower bound of a and b by $a \wedge b$, which are called *meet* and *join* respectively. A lattice is *bounded* if it has both a least and greatest element, that we denote by 0 and 1. We call a subset F of a lattice L a *filter* if it satisfies for all $a, b \in L$, (i) $F \neq \emptyset$; (ii) if $a \in F$ and $a \leq b$ then $b \in F$; (iii) if $a, b \in F$ then $a \wedge b \in F$. Furthermore, a proper filter F on L is called a *prime filter* if it satisfies (iv) $a \vee b \in F$ implies $a \in F$ or $b \in F$. And, a proper filter F on L is called an *maximal filter* if it is maximal among all proper filters, that is, if G is a proper filter on L such that $F \subseteq G$ then $F = G$.

Proposition 2.1.1. A structure $(L, \vee, \wedge, 0, 1)$ with L a nonempty set and binary functions $\wedge : L^2 \to L$ and $\vee : L^2 \to L$ on L is a bounded lattice if and only if, for all $a, b, c \in L$ it satisfies the following identities,

(L1)	(i)	$a \vee b = b \vee a$	
	(ii)	$a \wedge b = b \wedge a$	(Commutative laws)
(L2)	(i)	$a \vee (b \vee c) = (a \vee b) \vee c$	
	(ii)	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$	(Associative laws)
(L3)	(i)	$a \vee a = a$	
	(ii)	$a \wedge a = a$	(Idempotent laws)
(L4)	(i)	$a = a \vee (a \wedge b)$	
	(ii)	$a = a \wedge (a \vee b)$	(Absorption laws)
(L5)	(i)	$a \vee 0 = a$	
	(ii)	$a \wedge 1 = a$	

Proof. The right-to-left direction boils down to verifying that each bounded lattice satisfies (L1)-(L5). For the converse, we define a partial order $\leq$ by

$$a \leq b \text{ if and only if } a \wedge b = a \text{ or, equivalently, } a \vee b = b.$$

The details of the proof are spelled out in [15, Thm. 2.9,2.10]. $\square$

We say that a lattice $(L, \leq)$ is *distributive* if, for all $a, b, c \in L$ it satisfies,

(D1): $a \wedge (b \vee c) = (a \vee b) \wedge (a \vee c)$;
(D2): $a \vee (b \wedge c) = (a \wedge b) \vee (a \wedge c)$.

A structure $(B, \vee, \wedge, \neg, 0, 1)$ is called a *Boolean algebra* if $(B, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $\neg : B \to B$ a unary operator that satisfies for all $a \in B$, $a \vee \neg a = 1$ and $a \wedge \neg a = 0$.

2.1.1 Priestley duality

Let BDL denote the category whose objects are bounded distributive lattices and whose morphisms are bounded lattice homomorphisms. Given a topological space X, a pre-order R on X is said to satisfy the *Priestley separation axiom* if, for all $x, y \in X$ with $\neg(xRy)$, there exists a clopen upset U of X with $x \in U$ and $y \notin U$. The relation R is called a *Priestley order* if it is a partial order that satisfies the aforementioned Priestley axiom. We recall that a collection C of subsets of X is said to *cover* X, and is called a *cover of* X, if the union of its elements is equal to X. Then, a covering C of X is called an *open covering* if it is a collection of open subsets of X. We say that X is *compact* if every open covering C contains a finite subset $C^{\mathsf{Fin}} \subseteq C$ that also covers X. Furthermore, X

is called *zero-dimensional* if it has a basis of clopen sets and is called *Hausdorff* if, for distinct points $x, y \in X$, there exist an open set U containing x and an open set V containing y so that $U \cap V = \emptyset$. We call X a *Stone space* provided it is Hausdorff, compact, and zero-dimensional [33]. We call a pair (X, R) a *Priestley space* if X is a Stone space and R a Priestley order on X. For topological spaces X and Y, a map $f : X \to Y$ is said to be *continuous* if, for every open $U \subseteq Y$, its inverse image $f^{-1}(U)$ is open in X. Then, by Priest we refer to the category whose objects are Priestley spaces and whose morphisms are continuous order-preserving maps.

Let X_D be the set of all prime filters in a bounded distributive lattice D. We define the map $\phi : D \to \mathcal{P}(X_D)$ by $\phi(a) := \{x \in X_D \mid a \in x\}$.

Lemma 2.1.2 ([15, Prop. 11.2, Prop. 11.3, Sec. 11.17])**.** *Let D be a bounded distributive lattice and X_D the set of prime filters in D. Then we have that the set $S := \{\phi(a) \mid a \in D\} \cup \{\phi(a)^{\mathsf{c}} \mid a \in D\}$ forms a subbasis for a topology $\mathcal{T}$ on X_D that is compact, Hausdorff, and zero-dimensional. In other words, $(X_D, \mathcal{T})$ is a Stone space.*

We define a map $(-)_* : \mathsf{BDL} \to \mathsf{Priest}$ as follows.

(i) For an object $D \in \mathsf{BDL}$, let $D_* := (X_D, R)$, where X_D denotes the set of prime filters in D with a topology $\mathcal{T}$ generated by the subbasis S as described in Lemma 2.1.2, and R the subset relation on X_D.

(ii) For a bounded lattice homomorphism $f : D \to C$, define $f_* : C_* \to D_*$ by $f_* := f^{-1}$.

Now define $(-)^* : \mathsf{Priest} \to \mathsf{BDL}$ as follows.

(i) For a Priestley space (X, R), let $(X, R)^* := \mathsf{ClopUp}(X)$, the bounded distributive lattice of clopen upsets under $\subseteq$.

(ii) For a continuous order-preserving map $f : Y \to X$, define $f^* : X^* \to Y^*$ by $f^* := f^{-1}$.

From [15, Sec. 11.30] we know that $(-)_* : \mathsf{BDL} \to \mathsf{Priest}$ and $(-)^* : \mathsf{Priest} \to \mathsf{BDL}$ are well-defined contravariant functors. Now, for a category C, let Id_{C} denote the identity functor on C. Following [15, Sec. 11.30], we define a function $\eta : \mathsf{Id}_{\mathsf{BDL}} \to (-)^* \circ (-)_*$ that assigns to each object $D \in \mathsf{BDL}$ and arrow

$\eta_D : D \to (D_*)^*$ of BDL by mapping each $a \in D$ to $\phi(a)$. Secondly, we define a function $\varepsilon : \mathsf{Id}_{\mathsf{Priest}} \to (-)_* \circ (-)^*$ assigning to each object $X \in \mathsf{Priest}$ an arrow $\varepsilon_X : X \to (X^*)_*$ of Priest by letting $\varepsilon_X(x) := \{V \in \mathsf{ClopUp}(X) \mid x \in V\}$. In [15, Sec. 11.30, Thm. 11.31] it is shown that the maps η and ε are natural transformation such that, for each object $D \in \mathsf{BDL}$ the arrow η_D defines isomorphism between D and $(D_*)^*$ and, for each $X \in \mathsf{Priest}$, ε_X an isomorphism between X and $(X^*)_*$. Hence, the functions $\eta : \mathsf{Id}_{\mathsf{BDL}} \to (-)^* \circ (-)_*$ and $\varepsilon : \mathsf{Id}_{\mathsf{Priest}} \to (-)_* \circ (-)^*$ are natural isomorphisms which, together with the functors $(-)_* : \mathsf{BDL} \to \mathsf{Priest}$ and $(-)^* : \mathsf{Priest} \to \mathsf{BDL}$, define a dual equivalence between the categories BDL and Priest. Thus, we have the following theorem.

Theorem 2.1.3 (Priestley Duality [35])**.** BDL *is dually equivalent to* Priest*.*

2.1.2 Stone duality

Let Stone denote the category whose objects are Stone spaces and whose morphisms are continuous maps. By Bool, we refer to the category whose objects are Boolean algebras and whose morphisms are Boolean algebra homomorphisms. Observe that every Stone space can be identified with Priestley space of the form $(X, =)$, where $=$ denotes the discrete order, and every Boolean algebra is a distributive lattice. Moreover, the morphisms of Stone are exactly those of Priest between Stone spaces with a discrete order and, bounded lattice homomorphisms between Boolean algebras are in fact Boolean algebra-homomorphisms. Thus, Stone and Bool may be identified with full subcategories of Priest and BDL respectively. Furthermore, note that the functor $(-)_*$ takes Boolean algebras to Stone spaces. We know that, for every Boolean algebra B, its prime filters coincide with its ultrafilters. Hence, for all prime filters F and G on a Boolean algebra B, we have that $F \subseteq G$ implies $F = G$ and thus the order on the dual space X_B is discrete. When working with Boolean algebras, we can simplify the set S defined in Lemma 2.1.2 that generates the topology associated with the functor $(-)_*$ to the set $\{\phi(a) \mid a \in B\}$ since, for all $a \in B$, we have $\phi(a)^{\mathsf{c}} = \phi(\neg a)$. Also, observe that the second functor $(-)^*$ takes Stone spaces to Boolean algebras. Since, for a Stone space X with the discrete order, the family of clopen upsets in X coincide with the family of clopen sets of X and, for any topological space X, the clopen subsets of X form a Boolean algebra under $\subseteq$. Thus, the maps $(-)_*$ and $(-)^*$ are two well-defined contravariant functors between the categories Bool and Stone (for a proof hereof, see for instance [15, Sec. 11.30]). As for natural isomorphisms, we can reuse the maps η and ε and restrict them to $\mathsf{Id}_{\mathsf{Bool}}$ and

$\mathsf{Id}_{\mathsf{Stone}}$ respectively. Of course, in this case we may define $\varepsilon_X : X \to (X^*)_*$ by $\varepsilon_X(x) := \{V \in \mathsf{Clop}(X) \mid x \in V\}$ since the set of clopen upsets under a discrete order equals the set op clopens. From [10, Thm. 4.6] we know that the maps η and ε in the Bool and Stone context also define natural isomorphisms. Hence, we obtain the following theorem.

Theorem 2.1.4 (Stone Duality [40])**.** Bool *is dually equivalent to* Stone*.*

Thus, we can view Stone duality as a special case of Priestley duality. Stone duality follows from Priestley duality when we restrict BDL to the full subcategory Bool and Priest to the full subcategory that has as objects all and only the Stone spaces with a discrete order.

2.2 Subordinations on Boolean algebras

In this section, we present the formal definition of a binary relation $\prec$ on a Boolean algebra B, called a subordination, and distinguish so-called reflexive, transitive, and compingent subordinations amongst them. We note that subordinations on a Boolean algebra correspond to the precontact relations (also known as proximity relations) introduced by Düntsch and Vakarelov in [18] and the quasi-modal operators of [11]. We further observe that subordinations can be characterised in terms of a characteristic function $\rightsquigarrow$, called a strict implication. These notions have been shown to be equivalent, so we could take any of them as primitive. In this thesis, we will be working with subordinations primarily and make use of their corresponding strict implications. The choice of signatures $\prec$ and $\rightsquigarrow$ is that of our main references, in order to directly apply their key concepts and results. Following [3], we then show that the category of Boolean algebras with a subordination is dually equivalent to Stone spaces ordered by a closed relation R; that the category of Boolean algebras with a reflexive subordination corresponds to the category of Stone spaces ordered by a closed relation R that is reflexive; and Boolean algebras with a transitive subordination to Stone spaces ordered by a closed relation R that is a transitive.

Definition 2.2.1 (Subordination [6, Def. 2.3])**.** A binary relation $\prec$ on a Boolean algebra B is called a *subordination* if it satisfies for all $a, b, c, d \in B$,

(B1) $0 \prec 0$ and $1 \prec 1$;
(B2) $a \prec b, c$ implies $a \prec b \wedge c$;
(B3) $a, b \prec c$ implies $a \vee b \prec c$;

(B4) $a \leq b \prec c \leq d$ implies $a \prec d$. ⊣

We will often use $a \prec b \prec c$ to abbreviate "$a \prec b$ and $b \prec c$", for $a, b, c \in B$. Observe that subordinations on Boolean algebras are the dual concept of precontact relations, defined as follows.

Definition 2.2.2 (Precontact relation [18])**.** A binary relation δ on a Boolean algebra B is called a *precontact relation* or a *proximity* on B if it satisfies the following conditions,

(P1) $a\delta b$ implies $a, b \neq 0$;
(P2) $a\delta(b \vee c)$ if and only if $a\delta b$ or $a\delta c$;
(P3) $(a \vee b)\delta c$ if and only if $a\delta c$ or $b\delta c$. ⊣

Given a subordination $\prec$ on a Boolean algebra B, we can define a precontact relation $\delta_\prec$ on B by $a\delta_\prec b$ iff $a \not\prec \neg b$, where $\not\prec$ denotes the complement of the relation $\prec$. Conversely, if δ is a precontact relation on B, the relation $\prec_\delta$ defined by $a \prec_\delta b$ iff $a\not\delta\neg b$, is a precontact relation, where $\not\delta$ denotes the complement of the relation δ. Moreover, we have $\delta = \delta_{\prec_\delta}$ and $\prec = \prec_{\delta_\prec}$. Hence, subordinations are in 1-1 correspondence with precontact relations (see [6, Sec. 2]). As mentioned above, another closely related concept to both subordinations and precontact relations is that of a quasi-modal operator.

Definition 2.2.3 (Quasi-modal operator [11])**.** Let B be a Boolean algebra and $\mathcal{I}(B)$ denote the lattice of ideals of B. A *quasi-modal operator* on B is a function $\Delta : B \to \mathcal{I}(B)$ such that, for all $a, b \in B$,

Q1. $\Delta(a \wedge b) = \Delta a \cap \Delta b$;
Q2. $\Delta 1 = B$. ⊣

Precontact relations and subordinations are also in 1-1 correspondence with quasi-modal operators on Boolean algebras. Let $\prec$ be a subordination on a Boolean algebra B. For every $a \in B$, define $\Delta_\prec(a) := \{b \in B \mid b \prec a\}$. Then $\Delta_\prec(a)$ is an ideal of B and thus $\Delta_\prec : B \to \mathcal{I}(B)$ is well-defined. Moreover, it is easily verified that $\Delta_\prec$ is a quasi-modal operator. Conversely, if Δ is a quasi-modal operator on B, define $\prec$ by $a \prec_\Delta b$ iff $b \in \Delta(a)$. Again, it is easy to check that $\prec_\Delta$ is a subordination on B. Moreover, $\Delta = \Delta_{\prec_\Delta}$ and $\prec = \prec_{\Delta_\prec}$ (see, e.g., [4, Remark 2.6]). Thus, we see that precontact relations, subordinations, and quasi-modal operators are interdefinable and hence equivalent notions. In this thesis, we chose to work with subordinations primarily. In Chapter 4, we will present a language that we interpret in Boolean algebras with a subordination.

However, due to the presence of the binary relation $\prec$, these are not algebras, strictly speaking. Below we present a characteristic function $\rightsquigarrow$, called a strict implication, that is also in a 1-1 correspondence with subordinations and will enable us to develop an algebraic semantics based on Boolean algebras with a subordination.

Definition 2.2.4 (Strict implication [4, Def. 3.1])**.** Let B be a Boolean algebra. A characteristic function $\rightsquigarrow: B \times B \to \{0,1\} \subseteq B$ is called a *strict implication on* B if, for all $a, b, c, d \in B$, it satisfies the following properties,

(I1) $0 \rightsquigarrow a = a \rightsquigarrow 1 = 1$;
(I2) $(a \vee b) \rightsquigarrow c = (a \rightsquigarrow c) \wedge (b \rightsquigarrow c)$;
(I3) $a \rightsquigarrow (b \wedge c) = (a \rightsquigarrow b) \wedge (a \rightsquigarrow c)$. ⊣

Given a subordination $\prec$ on a Boolean algebra B, we can define a strict implication $\rightsquigarrow_{\prec}$ by,

$$a \rightsquigarrow_{\prec} b := \begin{cases} 1 & \text{if } a \prec b, \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, given a function $\rightsquigarrow: B \times B \to \{0,1\}$, we can define a subordination $\prec_{\rightsquigarrow}$ on B by $a \prec_{\rightsquigarrow} b$ iff $a \rightsquigarrow b = 1$. Moreover, we have $\rightsquigarrow = \rightsquigarrow_{\prec_{\rightsquigarrow}}$ and $\prec = \prec_{\rightsquigarrow_{\prec}}$ (see [4, Sec. 3]). Thus, there exists a bijective correspondence between subordination relations and strict implications on Boolean algebras. Hence, we can view any pair $(B, \prec)$, where B is a Boolean algebra and $\prec$ a subordination on B, as an algebra $(B, 1, \vee, \neg, \rightsquigarrow)$ and vice versa.

Definition 2.2.5 (Reflexive subordination [6, Def. 2.3])**.** We call a subordination $\prec$ on B a *reflexive subordination* if, for all $a, b \in B$, it satisfies,

(B5) $a \prec b$ implies $a \leq b$. ⊣

Definition 2.2.6 (Transitive subordination [6, Def. 2.3])**.** We call a subordination $\prec$ on B a *transitive subordination* if, for all $a, b \in B$, it satisfies,

(B6) $a \prec b$ implies there is $c \in B$ such that $a \prec c \prec b$. ⊣

Definition 2.2.7 (Compingent relation [16] see also, [6, Def 2.8])**.** A subordination $\prec$ on a Boolean algebra B is called a *compingent relation* or a *de Vries subordination* if in addition to axioms (B5) and (B6) for all $a, b \in B$ it satisfies,

(B7) $a \prec b$ implies $\neg b \prec \neg a$;
(B8) $a \neq 0$ implies there exists $c \in B$ with $c \neq 0$ and $c \prec a$. ⊣

As noted in [6, Sec. 2], for all $a, b \in B$, both the additional axioms (B5) and (B6) have a corresponding strict implication axiom (I5) and (I6) listed below. We skip (I4) in our numbering of additional strict implication axioms to match the numbering of axioms (B5) and (B6).

(I5) $a \rightsquigarrow b \leq a \to b$;
(I6) $a \rightsquigarrow b = 1$ implies there exists $c \in B$ such that $a \rightsquigarrow c = 1$ and $c \rightsquigarrow b = 1$.

We denote by Sub the category whose objects are pairs $(B, \prec)$ that consist of a Boolean algebra B and a subordination $\prec$ on B. The morphisms of Sub are Boolean algebra homomorphisms $h : A \to B$ that satisfy, for all $a, b \in A$, if $a \prec_A b$ then $h(a) \prec_B h(b)$. We will refer to Boolean algebra homomorphisms satisfying the aforementioned condition as *subordination homomorphisms*. Furthermore, we denote by RSub the full subcategory of Sub consisting of Boolean algebras with a reflexive subordination.

2.2.1 Duality for Boolean algebras with a subordination

Let X be a topological space. Recall that we call a binary relation R on X a *closed relation* if R is a closed subset in the product topology on $X \times X$. If X is a Stone space and R is a closed relation on X, we call the pair (X, R) a *subordinated Stone space*. Let X_1, X_2 be sets and R_1, R_2 relations on X_1 and X_2 respectively. We call a map $f : X_1 \to X_2$ a *stable map* if, for all $x, y \in X_1$, we have that xR_1y implies $f(x)R_2f(y)$. Then, by StR we refer to the category whose objects are subordinated Stone spaces and whose morphisms are continuous stable maps.

Following [6], we present two contravariant functors $(-)_* : \mathsf{Sub} \to \mathsf{StR}$ and $(-)^* : \mathsf{StR} \to \mathsf{Sub}$ that establish a dual equivalence between the categories Sub and StR. Thereafter, we present a second functor $(-)^+ : \mathsf{StR} \to \mathsf{Sub}$ as defined in [3] and show that the subordination $\prec_R^+$ associated with the functor $(-)^+$ is a subset of the subordination $\prec_R^*$ associated with the functor $(-)^*$. We then present a counterexample to show that, in general, the converse does not hold. The relation $\prec_R^*$ is not a subset of $\prec_R^+$. However, when we restrict ourselves to subordinated Stone spaces (X, R) whereof the relation R is a pre-order that satisfies the Priestley separation axiom, the relations $\prec_R^+$ and $\prec_R^*$ do coincide.

Before we set out our duality, following [6], for $(B, \prec) \in \mathsf{Sub}$ and $S \subseteq B$ we define $\twoheaduparrow S$ to be the upset of S with respect to $\prec$, that is,

$$\twoheaduparrow S := \{b \in B \mid \exists a \in S : a \prec b\}.$$

Now, define $(-)_* : \mathsf{Sub} \to \mathsf{StR}$ as follows.

(i) For $(B, \prec) \in \mathsf{Sub}$, let $(B, \prec)_* \coloneqq (X_B, R)$, where X_B is the Stone dual of B and xRy if and only if $\twoheaduparrow x \subseteq y$.

(ii) For Sub-morphism $h : (B, \prec_B) \to (A, \prec_A)$, define $h_* : (A, \prec_A)_* \to (B, \prec_B)_*$ by $h_* \coloneqq h^{-1}$. That is, for each ultrafilter $F \in (X_A, R)$ let $h_*(F) = h^{-1}(F)$.

From Stone duality (Thm. 2.1.4) we already know that for all $(B, \prec) \in \mathsf{Sub}$, X_B defines a Stone space. In [6, Lem. 3.5] it is shown that the relation R associated with $(B, \prec)_*$ is a closed relation on X_B and thus, we have $(B, \prec)_* \in \mathsf{StR}$. Moreover, [6, Lem. 3.7] proves that h_* defines a continuous stable map. Hence, $(-)_*$ is a well-defined contravariant functor between Sub and StR.

Let X be a set and $R \subseteq X \times X$ a binary relation on X. For a subset U of X, we define $R[U]$ and $R^{-1}[U]$ by,

$$R[U] \coloneqq \{y \in X \mid \exists x \in U \text{ with } xRy\},$$
$$R^{-1}[U] \coloneqq \{x \in X \mid \exists y \in U \text{ with } xRy\}.$$

We define $(-)^* : \mathsf{StR} \to \mathsf{Sub}$ as follows.

(i) For $(X, R) \in \mathsf{StR}$ let $(X, R)^* \coloneqq (B_X, \prec_R^*)$, where B_X is the Boolean algebra of clopen subsets of X and $U \prec_R^* V$ if and only if $R[U] \subseteq V$.

(ii) For $f : (X_2, R_2) \to (X_1, R_1)$ define $f^* : (X_1, R_1)^* \to (X_2, R_2)^*$ by $f^* = f^{-1}$.

Again, by Stone duality we know that, for all $(X, R) \in \mathsf{StR}$, B_X defines a Boolean algebra. In [6, Lem. 3.10] it has been proved that $\prec_R^*$ associated with $(X, R)^*$ defines a subordination on B_X, thus, we have $(B_X, \prec_R^*) \in \mathsf{Sub}$. Moreover, from [6, Lem. 3.12] we know that for a continuous stable map f, the map f^* defines a subordination homomorphism. Thus, $(-)^*$ is a well-defined contravariant functor from Sub to StR.

Recall that the map $\eta : \mathsf{Id}_{\mathsf{Bool}} \to (-)^* \circ (-)_*$ associates with each Boolean algebra B a Boolean isomorphism $\eta_B : B \to (B_*)^*$. In [6, Lem. 3.14] it has been shown that for all Boolean algebras B with a subordination $\prec$ on B, for all $a, b \in B$ we have $a \prec b$ iff $\phi(a) \prec \phi(b)$. Thus, each $(B, \prec)$ is isomorphic to $((B, \prec)_*)^*$ via η_B and hence $\eta : \mathsf{Id}_{\mathsf{Sub}} \to (-)^* \circ (-)_*$ is a well-defined natural isomorphism. Moreover, recall that the map $\varepsilon : \mathsf{Id}_{\mathsf{Stone}} \to (-)_* \circ (-)^*$ associates with each

Stone space X an arrow $\varepsilon_X : X \to (X^*)_*$ by $\varepsilon_X(x) := \{V \in \mathsf{Clop}(X) \mid x \in V\}$. In [6, Lem. 3.15] it has been show that for all $x, y \in X$ we have xRy iff $\varepsilon_X(x)R\varepsilon_X(y)$. Hence, (X, R) is isomorphic to $((X, R)^*)_*$ via ε_X. Then the map $\varepsilon : \mathsf{Id}_{\mathsf{StR}} \to (-)_* \circ (-)^*$ also defines a natural isomorphisms. Together with the functors $(-)_*$ and $(-)^*$ described above, η and ε yield the following duality result.

Theorem 2.2.8 ([6, Thm. 3.16])**.** Sub *is dually equivalent to* StR.

We now present a second functor from StR to Sub as given in [3].

Define $(-)^+ : \mathsf{StR} \to \mathsf{Sub}$ as follows.

(i) For $(X, R) \in \mathsf{StR}$ let $(X, R)^+ := (B_X, \prec_R^+)$, where B_X is the Boolean algebra of clopen subsets of X and $U \prec_R^+ V$ if and only if there exists a clopen upset W of X such that $U \subseteq W \subseteq V$.

(ii) For $f : (X_2, R_2) \to (X_1, R_1)$ define $f^+ : (X_1, R_1)^+ \to (X_2, R_2)^+$ by $f^+ = f^{-1}$.

Lemma 2.2.9. *Let* $(X, R) \in \mathsf{StR}$. *Then* $(X, R)^+ \in \mathsf{Sub}$.

Proof. From Stone duality it follows that B_X is a Stone space. Since the least element 0_X of B_X is $\emptyset$, and the greatest element 1_X is X, by the reflexivity of $\subseteq$ it immediately follows that (B1) holds, that is, $0_X \prec_R^+ 0_X$ and $1_X \prec_R^+ 1_X$. Pick $U, V_1, V_2 \in B_X$ such that $U \prec_R^+ V_1, V_2$. Then there are clopen upsets W_1, W_2 such that $U \subseteq W_1 \subseteq V_1$ and $U \subseteq W_2 \subseteq V_2$. The set $W_1 \cap W_2$ is clopen and $U \subseteq W_1 \cap W_2 \subseteq V_1 \cap V_2$, hence $U \prec_R^+ V_1 \cap V_2$, and so axiom (B2) is satisfied. That $(B_X, \prec_R^+)$ satisfies (B3) is shown analogously. Now pick $V_1, V_2, V_3, V_4 \in B_X$ such that $V_1 \subseteq V_2 \prec_R^+ V_3 \subseteq V_4$. Then there exists clopen upset W such that $V_1 \subseteq V_2 \subseteq W \subseteq V_3 \subseteq V_4$. Clearly it follows that $V_1 \prec_R^+ V_4$ and hence, $(B_X, \prec_R^+)$ satisfies (B4). $\square$

Lemma 2.2.10. *Let* (X, R) *be a subordinated Stone space and* U *and* V *be clopen subsets of* X. *Then* $U \prec_R^+ V$ *implies* $U \prec_R^* V$, *i.e, the relation* $\prec_R^+$ *is a subset of the relation* $\prec_R^*$.

Proof. Suppose that $U \prec_R^+ V$ holds. By definition of $\prec_R^+$, this means that there exists a clopen upset W such that $U \subseteq W \subseteq V$. Now, pick $y \in R[U]$. Then, for some $x \in U$, we have xRy. Since $x \in U$ and $U \subseteq W$, it must be $x \in W$. Now, since W is an upset, it follows that $y \in W$. Then $y \in V$ and hence, $R[U] \subseteq V$. $\square$

The converse of Lemma 2.2.10 does not hold in general. Consider the counterexample in Table 2.1 of a three-element subordinated Stone space (X, R) consisting of elements x, y, z with the discrete topology and $R \coloneqq \{(x, y), (y, z)\}$. Let $U = \{x\}$ and $V = \{y\}$. Then $R[U] \subseteq V$ but, there is no clopen upset W such that $U \subseteq W \subseteq V$. We show that the converse of Lemma 2.2.10 holds in case that R is a preorder and X satisfies the Priestley axiom. In what follows, we make use of the fact that the closed subsets Y of a compact space X are compact subsets of X (see, for instance [33, Thm. 26.2]).

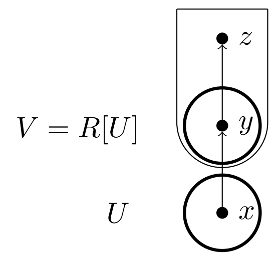


Table 2.1: The space (X, R)

Lemma 2.2.11 ([34, Thm. 4])**.** *Let (X, R) be a pair consisting of a Stone space X and a preorder R on X satisfying the Priestley separation axiom. Then, for each pair of closed subsets U and V of X, if $R[U] \cap R^{-1}[V] = \emptyset$ then there exists a clopen upset W such that $U \subseteq W$ and $V \cap W = \emptyset$.*

Proof. Let U and V be closed subsets of X such that $R[U] \cap R^{-1}[V] = \emptyset$. First, observe that for all $x \in U$, for all $y \in V$, we have $\neg(xRy)$. Towards a contradiction, suppose otherwise. Then there would exist $x \in U$ and $y \in V$ such that xRy, hence $y \in R[U]$. Now, since $y \in V$ and (by the reflexivity of R) yRy, it follows that $y \in R^{-1}[V]$. Then $y \in R[U] \cap R^{-1}[V]$, but this contradicts the assumption that $R[U] \cap R^{-1}[V] = \emptyset$.

Then, for all $x \in U$ and $y \in V$, from $\neg(xRy)$ by the Priestley separation axiom it follows that there exists a clopen upset W_{xy} such that $x \in W_{xy}$ but $y \notin W_{xy}$. For all $y \in V$, we let $\mathcal{W}_y$ denote the collection of clopen upsets W_{xy} such that there exists $x \in U$ with $x \in W_{xy}$ and $y \notin W_{xy}$.

Observe that, given any $y \in V$, for all $x \in U$ there exists such a clopen upset W_{xy} and thus $U \subseteq \bigcup \mathcal{W}_y$. And, since $\mathcal{W}_y$ is a collection of clopen (and thus open) sets, by the compactness of U it follows that there exists a finite subset $\mathcal{W}_y^{\mathsf{Fin}} \subseteq \mathcal{W}_y$ such that $U \subseteq \bigcup \mathcal{W}_y^{\mathsf{Fin}}$. We will denote the union $\bigcup \mathcal{W}_y^{\mathsf{Fin}}$ by $\mathcal{W}_y^*$.

Clearly, $y \notin \mathcal{W}_y^*$ and so, $y \in X \setminus \mathcal{W}_y^*$. Note that $\mathcal{W}_y^*$ is the finite union of clopen upsets and thus, itself a clopen upset. It then follows that $X \setminus \mathcal{W}_y^*$ is a clopen downset. Now, let $\mathcal{C}$ denote the collection of all clopen downsets $X \setminus \mathcal{W}_y^*$, for $y \in V$. Since, for all $y \in V$ we have $y \in \bigcup \mathcal{C}$, it holds that $V \subseteq \bigcup \mathcal{C}$ and thus, by compactness of V, there exists a finite subset $\mathcal{C}^{\mathsf{Fin}}$ of $\mathcal{C}$ such that

$V \subseteq \bigcup \mathcal{C}^{\mathsf{Fin}}$. Since $\mathcal{C}^{\mathsf{Fin}}$ is a finite collection of clopen downsets, $\bigcup \mathcal{C}^{\mathsf{Fin}}$ is also a clopen downset. Let W^* denote $X \setminus \bigcup \mathcal{C}^{\mathsf{Fin}}$. Then, W^* is a clopen upset such that $V \cap W^* = \emptyset$. Now, since, for all $y \in V$ we have $U \subseteq W_y^*$, it must be that $U \cap X \setminus \mathcal{W}_y^* = \emptyset$. Then also $U \cap \bigcup \mathcal{C}^* = \emptyset$ and so $U \subseteq W^*$, which is what we wanted to show. □

Lemma 2.2.12. *Let (X, R) be a subordinated Stone space such that R is a preorder and X satisfies the Priestley axiom. Let U and V be subsets of X. Then $U \prec_R^* V$ if and only if $U \prec_R^+ V$, i.e, we have $\prec_R^* = \prec_R^+$.*

Proof. The right-to-left direction follows immediately from Lemma 2.2.10. For the other direction, pick clopen subsets $U, V \subseteq X$ such that $U \prec_R^* V$, in other words, so that $R[U] \subseteq V$ holds. Since V is clopen, its complement $X \setminus V$ is also clopen (and thus closed). And, since $R[U] \subseteq V$, we have $R[U] \cap X \setminus V = \emptyset$. Then we also have $R[U] \cap R^{-1}[X \setminus V] = \emptyset$. Otherwise, there would exist $y \in R[U] \cap R^{-1}[X \setminus V]$. Then, $y \in R[U]$ and $y \in R^{-1}[X \setminus V]$. This means that there are $x \in U$ and $z \in X \setminus V$ so that xRy and yRz. From the transitivity of R it follows that xRz and thus, $z \in U$. But this contradicts that $R[U] \cap X \setminus V = \emptyset$. Hence, it must be that $R[U] \cap R^{-1}[X \setminus V] = \emptyset$. From Lemma 2.2.11 stated above, it follows that there exists a clopen upset W such that $U \subseteq W$ and $W \cap X \setminus V = \emptyset$. Then $W \subseteq V$, and so we have $U \prec_R^+ V$. □

Lemma 2.2.13 ([6, Lem. 6.1])**.** *Let $(B, \prec) \in \mathsf{Sub}$ and $(B, \prec)_* := (X_B, R)$.*

(i) $(B, \prec)$ satisfies axiom (B5) if and only if R is reflexive;
(ii) $(B, \prec)$ satisfies axiom (B6) if and only if R is transitive.

Proof.
(i) Suppose that $(B, \prec)$ satisifies (B5), pick $x \in X_B$, and consider $b \in \twoheaduparrow x$. By definition there exists $a \in x$ such that $a \prec b$. From axiom (B5) it follows that $a \leq b$. Now, since x is a filter, $a \in x$ and $a \leq b$ imply that $b \in x$. Thus $\twoheaduparrow x \subseteq x$ and so xRx. Hence, R is reflexive.

Conversely, assume that R is reflexive and pick $a, b \in B$ such that $a \prec b$. Then $\phi(a) \prec \phi(b)$ in $((B, \prec)_*)^*$, which means that $R[\phi(a)] \subseteq \phi(b)$. By reflexivity of R it follows that $\phi(a) \subseteq R[\phi(a)]$ and hence $\phi(a) \subseteq \phi(b)$. Thus, $a \leq b$ and so $(B, \prec)$ satisfies (B5).

(ii) Now suppose that $(B, \prec)$ satisfies (B6), pick $x, y, z \in X_B$ such that xRy and yRz, and pick $b \in \twoheaduparrow x$. Then, there exists $a \in x$ with $a \prec b$. By axiom (B6) it follows that there exists $c \in B$ with $a \prec c \prec b$. Then $c \in \twoheaduparrow x$ and so $c \in y$.

This means that $b \in {\Uparrow}y$ and hence $b \in z$. Thus ${\Uparrow}x \subseteq z$ and so xRz. Hence, R is transitive.

Suppose R is transitive and pick $a, b \in B$ such that $a \prec b$. Then $\phi(a) \prec \phi(b)$ in $((B, \prec)_*)^*$ and so $R[\phi(a)] \subseteq \phi(b)$, which means $R[\phi(a)] \cap X_B \setminus \phi(b) = \emptyset$. By transitivity of R, it must be that $R[\phi(a)]$ and $R^{-1}[X_B \setminus \phi(b)]$ are also disjoint. Hence, there exists a clopen $U \subseteq X_B$ such that $R[\phi(a)] \subseteq U$ and $U \cap R^{-1}[X \setminus \phi(b)] = \emptyset$. The latter implies that $R[U] \subseteq \phi(b)$. This shows that, whenever $a \prec b$, there exists U in $((B, \prec)_*)^*$ such that $\phi(a) \prec U \prec \phi(b)$. Then, since $((B, \prec)_*)^*$ and $(B, \prec)$ are isomorphic, it must be that there exists a $c \in B$ for which $a \prec c \prec b$. Hence, $(B, \prec)$ satisfies (B6). $\square$

2.3 Lattice subordinations on Boolean algebras

In this section, following the results of [3] we define a special kind of subordinations on Boolean algebras, the so-called lattice subordinations. Thereafter, we generalise Priestley spaces to quasi-ordered Priestley spaces and show that the category of Boolean algebras with a lattice subordination is dually equivalent to the category of quasi-ordered Priestley spaces.

Definition 2.3.1 (Lattice Subordination [3, Def. 2.1])**.** Let B be a Boolean algebra. A subordination $\prec$ on B is called a *lattice subordination* if, for all $a, b \in B$, it satisfies the additional axiom,

(QP) $a \prec b$ implies there exists $c \in B$ such that $c \prec c$ and $a \leq c \leq b$. $\dashv$

Equivalently, we can characterize lattice subordinations in terms of the $\rightsquigarrow$ operator by adding the following axiom to (I1) - (I3).

(QP$'$) $a \rightsquigarrow b = 1$ implies there exists $c \in B$ such that $c \rightsquigarrow c = 1$ and $a \leq c \leq b$.

We will now prove some additional facts about lattice subordinations that will be of use for the duality proof to come.

Lemma 2.3.2 ([3, Lem. 2.2])**.** *Let B be a Boolean algebra and $\prec$ a lattice subordination on B. For $a, b, d \in B$,*

(i) $a \prec b$ *implies* $a \leq b$*, i.e.,* $(B, \prec)$ *satisfies (B5);*
(ii) $a \prec b$ *implies there exists* $c \in B$ *with* $a \prec c \prec b$*,* $(B, \prec)$ *satisfies (B6);*
(iii) $a \prec b$ *if and only if there exists* $c \in B$ *such that* $c \prec c$ *and* $a \leq c \leq b$*;*
(iv) $a \prec d \prec b$ *implies* $a \prec b$*.*

Proof.
(i) Suppose $a \prec b$. From (QP) it follows that there exists $c \in B$ such that $c \prec c$ and $a \leq c \leq b$. Thus we have $a \leq b$.

(ii) Suppose again $a \prec b$. By (QP) there exists $c \in B$ with $c \prec c$ and $a \leq c \leq b$. Since $\leq$ is reflexive, we know that $c \leq c$. Then we have $a \leq c \prec c \leq c$. Hence, by (B4) also $a \prec c$. Similarly we deduce $c \prec b$ from $c \leq c \prec c \leq b$. Thus, $a \prec c \prec b$ follows.

(iii) Given (QP), we only need to show the right-to-left direction. Suppose there exists $c \in B$ such that $c \prec c$ and $a \leq c \leq b$. Then $a \leq c \prec c \leq b$. By (B4) we derive $a \prec b$.

(iv) Suppose $a \prec d \prec b$. By (QP) it follows from $a \prec d$ that there exists c such that $c \prec c$ and $a \leq c \leq d$. By (i) of this lemma we know that $d \prec b$ entails $d \leq b$. Thus, we have $a \leq c \prec c \leq b$. By (B4) it follows that $a \prec b$. □

Henceforth, by BLS we will refer to the full subcategory of Sub whose objects are pairs $(B, \prec)$ that consist of a Boolean algebra B and a lattice subordination $\prec$ on B.

In this section and Section 2.2, in definitions 2.2.1, 2.3.1, and 2.2.4 we have presented conditions (B1)$-$(B6) and (QP) for a binary relation $\prec$ on a Boolean algebra, as well as corresponding conditions (I1)$-$(I3), (I5), (I6), and (QP$'$) for a characteristic function $\rightsquigarrow$ on B. As we will frequently refer to these conditions throughout this thesis, for the convenience of the reader these axioms are recollected in Table 2.2.

(B1)	$0 \prec 0$ and $1 \prec 1$	(I1)	$0 \rightsquigarrow a = a \rightsquigarrow 1 = 1$.
(B2)	$a \prec b, c$ implies $a \prec b \wedge c$.	(I2)	$(a \vee b) \rightsquigarrow c = (a \rightsquigarrow c) \wedge (b \rightsquigarrow c)$.
(B3)	$a, b \prec c$ implies $a \vee b \prec c$.	(I3)	$a \rightsquigarrow (b \wedge c) = (a \rightsquigarrow b) \wedge (a \rightsquigarrow c)$.
(B4)	$a \leq b \prec c \leq d$ implies $a \prec d$.		
(B5)	$a \prec b$ implies $a \leq b$.	(I5)	$a \rightsquigarrow b \leq a \rightarrow b$.
(B6)	$a \prec b$ implies there is $c' \in B$ such that $a \prec c' \prec b$.	(I6)	$a \rightsquigarrow b = 1$ implies there exists $c \in B$ with $a \rightsquigarrow c = 1$ and $c \rightsquigarrow b = 1$.
(QP)	$a \prec b$ implies there exists $c \in B$ with $c \prec c$ and $a \leq c \leq b$.	(QP$'$)	$a \rightsquigarrow b = 1$ implies there exists $c \in B$ with $c \rightsquigarrow c = 1$ and $a \leq c \leq b$.

(B1)-(B4): Subordination	Sub
(B1)-(B4) + (B5): Reflexive subordination.	RSub
(B1)-(B4) + (B6): Transitive subordination	
(B1)-(B4) + (QP): Lattice subordination	BLS

Table 2.2: Axiom list $\prec$ and $\rightsquigarrow$.

2.3.1 Duality for Boolean algebras with a lattice subordination

Given a set X, we call a binary relation R on X a *Priestley quasi-order* if it is a quasi-order and it satisfies the Priestley separation axiom. We call a pair (X, R) a *quasi-ordered Priestley space* if X is a Stone space and R a Priestley quasi-order on X. By QPS we refer to the full subcategory of StR whose objects are quasi-Priestley spaces. In the context of QPS, we will refer to continuous stable maps between quasi-ordered Priestley spaces as *continuous order-preserving maps*.

The contravariant functors $(-)_*$ and $(-)^+$ as defined in 2.2.1 restrict to the categories BLS and QPS. For convenience, we will denote $(-)_*$ in this setting by $(-)_+$.

Define $(-)_+ : \mathsf{BLS} \to \mathsf{QPS}$ as follows.

(i) For $(B, \prec) \in \mathsf{BLS}$, let $(B, \prec)_+ := (X_B, R)$, where X_B is the Stone dual of B and xRy if and only if $\twoheaduparrow x \subseteq y$;

(ii) For BLS-morphism $h : (B, \prec_B) \to (A, \prec_A)$, let $h_+ : (A, \prec_A)_+ \to (B, \prec_B)_+$ be defined by $h_+ := h^{-1}$, that is, for each ultrafilter $F \in (X_A, R)$ let $h_+(F) = h^{-1}(F)$.

Lemma 2.3.3. *Let* $(B, \prec) \in \mathsf{BLS}$.

(i) For all $S \subseteq B$ *we have* $\Uparrow S = \Uparrow\Uparrow S$. *That is,* $\Uparrow : \mathcal{P}(B) \to \mathcal{P}(B)$ *is idempotent;*

(ii) For all $S_1, S_2 \subseteq B$, *if* $S_1 \subseteq S_2$ *then* $\Uparrow S_1 \subseteq \Uparrow S_2$. *That is,* $\Uparrow : \mathcal{P}(B) \to \mathcal{P}(B)$ *is monotone.*

Proof.
(i) Pick $S \subseteq B$ and consider $b \in \Uparrow S$. Since $b \in \Uparrow S$, there exists $a \in S$ such that $a \prec b$. By Lemma 2.3.2 item (ii) it follows that there exists $c \in B$ such that $a \prec c \prec b$. Since $a \prec c$ we have $c \in \Uparrow S$. Since $c \prec b$ we have $b \in \Uparrow\Uparrow S$. Thus, $\Uparrow S \subseteq \Uparrow\Uparrow S$. Now pick $b \in \Uparrow\Uparrow S$. This means that there exists $c \in \Uparrow S$ such that $c \prec b$. Then there exists $a \in S$ such that $a \prec c$. By Lemma 2.3.2 (iv) it follows that $a \prec b$. So $b \in \Uparrow S$ and thus, $\Uparrow\Uparrow S \subseteq \Uparrow S$. Hence, $\Uparrow S = \Uparrow\Uparrow S$.

(ii) Pick $S_1, S_2 \subseteq B$ such that $S_1 \subseteq S_2$ and consider $b \in \Uparrow S_1$. Since $b \in \Uparrow S_1$, there exists $a \in S_1$ such that $a \prec b$. Since $S_1 \subseteq S_2$ we have $a \in S_2$, so also $b \in \Uparrow S_2$. Thus, $\Uparrow$ is monotone. $\square$

Observe that condition (ii) of Lemma 2.3.3 does not necessarily hold for pairs $(B, \prec)$ where B is a Boolean algebra and $\prec$ a subordination on B that is not transitive. Consider the four-element chain $C_4 = \{0, a, b, 1\}$ with a partial order $\leq$ such that $0 \leq a \leq b \leq 1$. Let $\prec$ be a reflexive subordination on B defined by $0 \prec x$ and $x \prec 1$ for all $x \in C_4$ and $a \prec b$. It is readily seen that $(C_4 \prec)$ satisfies (B1)-(B5) and thus defines a reflexive subordination on C_4. However, we have that $\Uparrow\{a\} = \{b, 1\} \neq \{1\} = \Uparrow\Uparrow\{a\}$, therefore $\Uparrow : \mathcal{P}(B) \to \mathcal{P}(B)$ is not idempotent for all $(B, \prec) \in \mathsf{RSub}$.

Lemma 2.3.4 ([3, Cor. 5.3])**.** *Let* $(B, \prec) \in \mathsf{BLS}$. *Then* $(B, \prec)_+ \in \mathsf{QPS}$.

Proof. From Stone duality we know that X_B is a Stone space. We show that R is a Priestley quasi-order on X_B. Pick $x \in X_B$ and consider $\Uparrow x$. For arbitrary $b \in \Uparrow x$ there exists $a \in x$ such that $a \prec b$. By Lemma 2.3.2 (i) we know that $a \prec b$ implies $a \leq b$. Since x is a filter and we have $a \in x$ and $a \leq b$, we also have $b \in x$. Then $\Uparrow x \subseteq x$ and thus xRx. Hence, R is reflexive.

Now pick $x, y, z \in X_B$ such that xRy and yRz. Then $\Uparrow x \subseteq y$ and $\Uparrow y \subseteq z$. By 2.3.3 (ii), it follows from $\Uparrow x \subseteq y$ that $\Uparrow\Uparrow x \subseteq \Uparrow y$. Since also $\Uparrow y \subseteq z$, it follows that $\Uparrow\Uparrow x \subseteq z$. Moreover, since $\Uparrow$ is idempotent by 2.3.3 (i), $\Uparrow x = \Uparrow\Uparrow x$. Hence, R is transitive and so R is a quasi-order.

We now show that (X_B, R) satisfies the Priestley axiom. Pick $x, y \in X_B$ such that $\neg(xRy)$. Then $\Uparrow x \not\subseteq y$, so there exists $b \in \Uparrow x$ with $b \notin y$, which means there exists $a \in x$ such that $a \prec b$. From $a \prec b$, by (QP) it follows that there exists $c \in B$ such that $c \prec c$ and $a \leq c \leq b$. Now, consider $\phi(c)$. Observe that that $\phi(c)$ belongs to the clopen subbasis of X_B. We show that $\phi(c)$ is also an upset. Pick $x \in \phi(c)$ and consider $y \in X_B$ such that $\Uparrow x \subseteq y$. From $c \in x$ and $c \prec c$ it readily follows that $c \in \Uparrow x$ and thus $c \in y$. Moreover, $\phi(c)$ separates x and y. Since $a \in B$, $a \leq c$, and the fact that x is a filter, we have $c \in x$ and so, $x \in \phi(c)$. Also, since $b \notin y$, $c \leq b$, and the fact that y is a filter, it cannot be that $c \in y$, thus $y \notin \phi(c)$. Thus, $\phi(c)$ is a clopen upset that contains x but not y. Hence, (X_B, R) satisfies the Priestley axiom. $\square$

Lemma 2.3.5. *Let* $(X, R) \in \mathsf{QPS}$*. Then* $(X, R)^+ \in \mathsf{BLS}$*.*

Proof. Let $(X, R) \in \mathsf{QPS}$ and consider $(B_X, \prec_R^+)$. From Stone duality and Lemma 2.2.9 it follows that B_X is a Boolean algebra that satisfies the axioms (B1)-(B4). Now, pick $U, V \in X_B$ such that $U \prec_R^+ V$. Then there exists a clopen upset W such that $U \subseteq W \subseteq V$. Thus, we have $U \leq W \leq V$ in B_X. Moreover, from the reflexivity of $\subseteq$ it follows that $W \prec_R^+ W$. Hence, $(B_X, \prec_R^+)$ also satisfies (QP) and so $(X, R)^+ \in \mathsf{BLS}$. $\square$

Moreover, from the preceding section 2.2.1 we know that, since h_+ and f^+ are defined in the same way as h_* and f^*, the map h_+ defines an order-preserving continuous map between quasi-ordered Priestley spaces and f^+ defines a subordination homomorphism between Boolean algebras. Thus, the maps $(-)_+$ and $(-)^+$ restricted to the categories of BLS and QPS are well-defined functors. Now, note that from Lemma 2.2.12 it follows that the functors $(-)^*$ and $(-)^+$ restricted to quasi-ordered Priestley spaces are equal. Hence, from section 2.2.1 it follows that, for $(B, \prec) \in \mathsf{BLS}$, the map $\eta_B : (B, \prec) \to ((B, \prec)_+)^+$ defined by $\eta_B(a) \coloneqq \phi(a)$ is an isomorphism between $(B, \prec)$ and $((B, \prec)_+)^+$. And, for each $(X, R) \in \mathsf{QPS}$, the map $\varepsilon_X : (X, R) \to ((X, R)^+)_+$ defined by by $\varepsilon_X(x) \coloneqq \{V \in \mathsf{Clop}(X) \mid x \in V\}$ is an isomorphism between (X, R) and $((X, R)^+)_+$. In other words, we can define two natural isomorphisms $\eta' : \mathsf{Id}_{\mathsf{BLS}} \to (-)^+ \circ (-)_+$ and $\varepsilon' : \mathsf{Id}_{\mathsf{QPS}} \to (-)_+ \circ (-)^+$ in the same way as we have defined the maps $\eta : \mathsf{Id}_{\mathsf{Sub}} \to (-)^* \circ (-)_*$ and $\varepsilon : \mathsf{Id}_{\mathsf{StR}} \to (-)_* \circ (-)^*$. Hence, we obtain the theorem below.

Theorem 2.3.6 ([3, Cor. 5.3])**.** BLS *is dually equivalent to* QPS*.*

2.4 Boolean algebras with a bounded sublattice

In this section, we show a correspondence between lattice subordinations $\prec$ on a Boolean algebra B and bounded sublattices D of B. We define the category BDA of Boolean algebras with a bounded sublattice and establish a dual equivalence between the categories BLS and BDA. Furthermore, we identify a subcatergory GBDA of BDA that is equivalent to the category of bounded distributive lattices BDL.

Definition 2.4.1 ([3, Def. 2.3])**.** Given a pair $(B, \prec)$ where B is a Boolean algebra and $\prec$ a lattice subordination on B, we define $D_\prec$ to be the set of reflexive elements under $\prec$. That is,

$$D_\prec := \{a \in B : a \prec a\}. \qquad \dashv$$

Lemma 2.4.2 ([3, Lem. 2.4])**.** *Let B be a Boolean algebra and $\prec$ a lattice subordination on B. Then $D_\prec$ is a bounded sublattice of B.*

Proof. Let $(B, \prec) \in$ BLS and consider $D_\prec$. Then $D_\prec \neq \emptyset$ since, by (B1), $0, 1 \in D_\prec$. Now, pick $a, b \in D_\prec$. Since $a \wedge b \leq a \prec a \leq a$, by (B4) we have $a \wedge b \prec a$. Analogously we obtain $a \wedge b \prec b$. Thus, by (B2) we have $a \wedge b \prec a \wedge b$. Furthermore, since $a \leq a \prec a \leq a \vee b$ we have $a \prec a \vee b$ and, analogously obtained, $b \prec a \vee b$. Thus, by (B3) we have $a \vee b \prec a \vee b$. Hence, $D_\prec$ is a sublattice of B. Moreover, since $0, 1 \in D_\prec$, $D_\prec$ is a bounded sublattice of B. □

Definition 2.4.3 ([3, Def. 2.5])**.** Let D be a bounded sublattice of a Boolean algebra B. We define a binary relation $\prec_D$ on B as follows. For $a, b \in B$,

$$a \prec_D b \text{ if and only if there exists } c \in D \text{ with } a \leq c \leq b. \qquad \dashv$$

Lemma 2.4.4 ([3, Lem. 2.6])**.** *If D is a bounded sublattice of a Boolean algebra B, then $\prec_D$ as defined above is a lattice subordination on B.*

Proof. Let B be a Boolean algebra, D a bounded sublattice of B, and $\prec_D$ as defined above. Since D is a bounded sublattice of B we have $0, 1 \in D$. Then, from reflexivity of $\leq$ it immediately follows that $0 \prec_D 0$ and $1 \prec_D 1$ and thus $(B, \prec_D)$ satisfies (B1). Now, pick $a, b, c \in B$ such that $a \prec_D b, c$. Then there exists $c_1, c_2 \in D$ such that $a \leq c_1 \leq b$ and $a \leq c_2 \leq c$. Since D is a sublattice of B, from $c_1, c_2 \in D$ it follows that $c_1 \wedge c_2 \in D$. $a \leq c_1 \wedge c_2 \leq b \wedge c$. So $a \prec_D b \wedge c$ and hence, $(B, \prec_D)$ satisfies (B2). That $(B, \prec_D)$ satisfies (B3) is shown similarly. Assume $a, b, c, d \in B$ such that $a \leq b \prec_D c \leq d$. Then there exists $c' \in D$ such that $b \leq c' \leq c$. Thus, $a \leq c' \leq d$ and so $a \prec_D d$, which means that $(B, \prec_D)$ satisfies (B4). Lastly, suppose $a \prec_D b$ for $a, b \in B$. Then there exists $c \in D$ such that $a \leq c \leq b$. By reflexivity we have $c \prec_D c$. Hence, $(B, \prec_D)$ also satisfies (B5) and thus $D_\prec$ defines a lattice subordination on B. $\square$

Observe that for each $D \in \mathsf{BDL}$ we have $D = D_{\prec_D}$, and for each $(B, \prec) \in \mathsf{BLS}$, $\prec = \prec_{D_\prec}$. Let BDA denote the category whose objects are pairs (B, D) where B is a Boolean algebra and D a bounded sublattice of B and whose morphisms are Boolean homomorphisms $h : A \to B$ which satisfy the condition that if $a \in D_A$ then $h(a) \in D_B$. We describe the two functors Φ and Ψ between BLS and BDA that establish a categorical isomorphism as defined by [3] below. For detailed proofs thereof, we refer the reader to [3].

Define $\Phi : \mathsf{BLS} \to \mathsf{BDA}$ by (i) for $(B, \prec) \in \mathsf{BLS}$, let $\Phi(B, \prec) \coloneqq (B, D_\prec)$, where $D_\prec$ is as in Definition 2.4.1; (ii) for a BLS-morphism $h : (B, \prec_B) \to (A, \prec_A)$, let $\Phi(h) \coloneqq h$.

Let $\Psi : \mathsf{BDA} \to \mathsf{BLS}$ be defined as follows, (i) for an object $(B, D) \in \mathsf{BDA}$, let $\Psi(B, D) \coloneqq (B, \prec_D)$, where $B_\prec$ is as in Definition 2.4.3; (ii) for BDA-morphism $h : (B, D) \to (A, D)$, define $\Psi(h) \coloneqq h$.

Theorem 2.4.5 ([3, Thm. 2.10])**.** BLS *is isomorphic to* BDA*.*

Corollary 2.4.6 ([3, Thm. 5.2])**.** BDA *is dually equivalent to* QPS*.*

In this section, we have seen that, for each $(B, \prec) \in \mathsf{BLS}$, the set of elements $D_\prec \coloneqq \{a \in B \mid a \prec a\}$ forms a bounded distributive lattice. Conversely, as is shown in [3], for each bounded distributive lattice $D \in \mathsf{BDL}$ there exists a suitable $(B, \prec) \in \mathsf{BLS}$ such that D is isomorphic to $D_\prec$ which we identify as follows. Let $(B, D) \in \mathsf{BDA}$. We say that B is *generated by* D or that B is D*-generated* if B is the smallest Boolean subalgebra of B that contains D and call B the *Boolean envelope of* D [3, Def. 6.1]. We let GBDA denote the full subcategory of BDA that has as objects pairs (B, D) where B is D-generated.

Theorem 2.4.7 ([3, Thm. 6.3])**.** GBDA *is equivalent to* BDL*.*

Thus, the category BDL can be embedded into the category of BLS. In the following section, we look at their dual categories Priest and QPS respectively, and how they relate.

2.5 Quasi-ordered Priestley spaces and Priestley spaces

From the previous sections we know that the category BLS is dually equivalent to the category QPS and that BLS is isomorphic to the category BDA. Now, we also know that the full subcategory GBDA of BDA is isomorphic to the category BDL and hence, dually equivalent to the category Priest. This means that, for a given $(B, \prec) \in$ BLS we can form the dual space $(X_B, R_\prec) \in$ QPS and the isomorphic structure $(B, D_\prec) \in$ BDA. From $D_\prec$ we can construct a dual space $(X_{D_\prec}, R_{D_\prec}) \in$ Priest. In this section, we will show how, for any $(B, \prec) \in$ BLS, the spaces $(X_B, R_\prec)$ and $(X_{D_\prec}, R_{D_\prec})$ relate.

Definition 2.5.1. Let R be a pre-order on a set X. We let $\sim_R$ be the equivalence relation on X defined by,

$$x \sim_R y \text{ iff } xRy \text{ and } yRx, \text{ for all } x, y \in X.$$

For all $x \in X$, by the *cluster of x under R* we mean the equivalence class,

$$[x]_{\sim_R} \coloneqq \{y \in X \mid x \sim_R y\}.$$

We denote the collection of clusters of X under R by $X^\sim$. Given a cluster $[x]_{\sim_R}$, we will drop the subscript $\sim_R$ when this is clear from the context. We define a binary relation $R^\sim$ on $X^\sim$ by,

$$CR^\sim C' \text{ iff there exist } x \in C \text{ and } y \in C' \text{ such that } xRy, \text{ for all } C, C' \in X^\sim.$$

Let (X, τ) be a topological space and $\sim$ an equivalence relation on X. Denote by $X^\sim$ the set of equivalence classes of X. Recall that the *quotient mapping* $q : X \to X^\sim$ is the map defined by $x \mapsto [x]$ and the *quotient topology* $\tau^\sim$ on $X^\sim$ is the family of sets U such that $q^{-1}(U) \in \tau$. Furthermore, we recall from [23, p. 361] that, given a Stone space X and an equivalence relation $\sim$ on X, the relation $\sim$ is called *Boolean* if, for any two distinct equivalence classes C, C' of $\sim$, there exists a clopen subset $V \subseteq X$ that is the union of a collection of equivalence classes of $\sim$ and includes either C or C'.

Lemma 2.5.2 ([23, Lemm. 37.1])**.** *Let X be a Stone space and $\sim$ an equivalence relation on X. The quotient space $X^\sim$ is a Stone space iff the relation $\sim$ is Boolean.*

Lemma 2.5.3 ([22, Prop. 8])**.** *Let (X, R) be a quasi-ordered Priestley space. The set $X^\sim$ ordered by $R^\sim$ together with the quotient topology forms a Priestley space.*

Proof. We first show that the relation $\sim_R$ is Boolean. Pick $C, C' \in X^{\sim}$ such that $C \neq C'$. Then $C \nsubseteq C'$ or $C' \nsubseteq C$. Without loss of generality, assume $C \nsubseteq C'$. Then there exist $x \in C$ with $x \notin C'$. Now pick $y \in C'$. Note that $x \not\sim_R y$, so $x \not R y$ or $y \not R x$. Without loss of generality, assume $x \not R y$. By the Priestley separation axiom, there exists a clopen upset $W \subseteq X$ with $x \in W$ but $y \notin W$. It follows that $C \subseteq W$ but $C' \nsubseteq W$. Now, suppose W is not the union of a collection of equivalence classes of $X^{\sim}$. Then there exists an equivalence class D of X under $\sim_R$ such that $D \nsubseteq W$ and $D \cap W \neq \emptyset$. Pick $z_1 \in D$ with $z_1 \notin W$ and $z_2 \in D \cap W$. Since $z_1, z_2 \in D$, we have $z_1 \sim_R z_2$ and so $z_2 R z_1$. Since W is an upset, it follows that $z_1 \in W$, but this cannot be. Hence, $\sim_R$ is Boolean and by Lemma 2.5.2, $X^{\sim}$ is a Stone space.

We now show that $X^{\sim}$ satisfies the Priestley separation axiom. Pick $C, C' \in X^{\sim}$ with $C \not R^{\sim} C'$ and pick $x \in C$ and $y \in C'$. Then $x \not R y$, so, by the Priestley separation axiom, there exists a clopen upset $W \subseteq X$ with $x \in W$ but $y \notin W$. We already know that W can is the union of a collection of equivalences classes $W^{\sim}$ of X under $\sim_R$. Observe that $q^{-1}(W^{\sim}) = W$, so $W^{\sim}$ is clopen. Clearly $C \in W^{\sim}$ but $C' \notin W^{\sim}$ and, moreover, $W^{\sim}$ is an upset. Hence, $(X^{\sim}, R^{\sim})$ is satisfies the Priestley separation axiom and thus is a Priestley space.

$\square$

In what follows, for the dual space $(X_B, R_{\prec})$ of any $(B, \prec) \in \mathsf{BLS}$, for all $x \in X_B$, we define $x_{\prec} := \{a \in x \mid a \prec a\}$. We recall that for $S \subseteq B$ we have defined set $\Uparrow S$ is to be the upset of S with respect to $\prec$, that is,

$$\Uparrow S := \{b \in B \mid \exists a \in S : a \prec b\}.$$

Lemma 2.5.4. *Let $(B, \prec) \in \mathsf{BLS}$ and $(X_B, R_{\prec})$ denote its dual space. Then, for all $x, y \in X_B$,*

$$\Uparrow x \subseteq y \text{ if and only if } x_{\prec} \subseteq y_{\prec}.$$

Proof.
($\Rightarrow$): Assume that $x, y \in X_B$ are such that $\Uparrow x \subseteq y$ and consider $b \in x_{\prec}$. Then $b \prec b$ and thus $b \in \Uparrow x \subseteq y$. Hence $b \in y_{\prec}$.

($\Leftarrow$): Now chose $x, y \in X_B$ such that $x_{\prec} \subseteq y_{\prec}$ and pick $b \in \Uparrow x$. Then there exists $a \in x$ such that $a \prec b$. From axiom (QP) it follows that there exists $c \in B$ such that $c \prec c$ and $a \leq c \leq b$. Now, x is a filter, so it must be that $c \in x$ and thus $c \in x_{\prec} \subseteq y_{\prec}$. Then also $c \in y$, and thus $b \in y$ since y is a filter and $c \leq b$. $\square$

Corollary 2.5.5. *Let $(B, \prec) \in \mathsf{BLS}$ and $(X_B, R_\prec)$ denote its dual space. For all $x, y \in X_B$,*

$$\Uparrow x \subseteq y \text{ and } \Uparrow y \subseteq x \text{ if and only if } x_\prec = y_\prec.$$

Lemma 2.5.6. *Let $(B, \prec) \in \mathsf{BLS}$. For every ultrafilter $x \in X_B$ we have $x_\prec \in X_{D_\prec}$.*

Proof. Pick $x \prec\in X_B$. Clearly, $x_\prec$ is a filter in $D_\prec$ and, since x is a prime filter, for any $a, b \in D_\prec$ with $a \vee b \in x$, it must be that $a \in x$ or $b \in x$, so, $x \in X_{D_\prec}$. □

Lemma 2.5.7. *Let $(B, \prec) \in \mathsf{BLS}$. For every prime filter $x \in X_{D_\prec}$, there is an ultrafilter $y \in X_B$ for which $x = y_\prec$.*

Proof. Let x be a prime filter in $D_\prec$. Observe that $\uparrow x$ defines a filter in B and that $\downarrow(D_\prec \setminus x)$ defines an ideal in B such that $\uparrow x \cap \downarrow(D_\prec \setminus x) = \emptyset$. By the prime filter theorem (see e.g., [2, Thm. III.4.1]), there exists a maximal filter y in B such that $\uparrow x \subseteq y$ and $y \cap \downarrow(D \setminus x) = \emptyset$. Then, it must be that $y_\prec = x$. □

Theorem 2.5.8. *Let $(B, \prec) \in \mathsf{BLS}$. We define a map $(-)_\times : X_{D_\prec} \to X_B^\sim$ from the dual space of $D_\prec$ to the set of clusters of X_B by,*

$$(x)_\times = \{y \in X_B \mid x = y_\prec\}.$$

Then, $(-)_\times : X_{D_\prec} \to X_B^\sim$ is a well-defined bijection from $X_{D_\prec}$ to $X_B^\sim$.

Proof. Pick $x \in X_D$. By Lemma 2.5.7 we know that there exists $y \in (x)_\times$ with $x = y_\prec$ and $y \in X_B$. Now, for any $y, z \in (x)_\times$ we have $y_\prec = z_\prec$. Then, by Corollary 2.5.5 it follows that xR_By and yR_Bx. Hence, $(x)_\times$ is a cluster of X_B under R_B and the map $(-)_\times$ is well-defined. Now pick $x, y \in X_{D_\prec}$ such that $(x)_\times = (y)_\times$. Then for all $z \in X_B$ we have $x = z_\prec$ if and only if $y = z_\prec$, so $x = y$ and thus $(-)_\times$ is injective. Furthermore, $(-)_\times$ is surjective, since, from Lemma 2.5.6 we know that for any cluster $[x]$, $x_\prec$ is a prime filter in $D_\prec$ and so $(x_\prec)_\times = [x]$. □

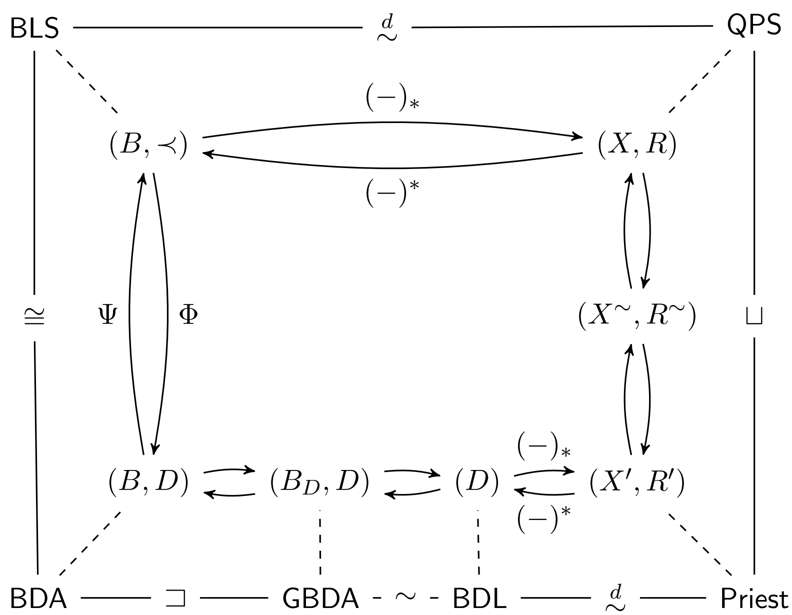


Table 2.3: Categorical isomorphisms ($\cong$), equivalences ($\sim$), dual equivalences ($\overset{d}{\sim}$), and full subcategories ($\sqsubset$).

Category	Objects	Section
Priest	Priestley spaces	2.1.1
Stone	Stone spaces	2.1.2
StR	Stone spaces with a closed relation	2.2
QPS	Quasi-ordered Priestley spaces	2.3

We summarize the categorical correspondences that have been presented from section 2.1.1 onwards in the following scheme. The outer square depicts the correspondences between categories and the inner square the maps between their objects.

Category	Objects	Axioms	Section
BDL	Bounded distributive lattices		2.1.1
Bool	Boolean algebras		2.1.2
Sub	Boolean algebras with a subordination	(B1)-(B4)	2.2
RSub	Boolean algebras with a reflexive subordination	(B1)-(B5)	2.2
BLS	Boolean algebras with a lattice subordination	(B1)-(B4, (QP)	2.3
BDA	Boolean algebras with a bounded sublattice		2.4
GBDA	D-generated objects of BDA		2.4

2.6 Heyting lattice subordinations

In this section we present Heyting lattice subordinations introduced in [3] and show that the category of Boolean algebras with a Heyting lattice subordination is isomorphic to the category of S4-algebras.

Definition 2.6.1 (Heyting lattice Subordination [3, Def. 2.1])**.** Let B be a Boolean algebra. A lattice subordination $\prec$ on B is called a *Heyting lattice subordination* if, for all $a \in B$, the set $\{b \in B \mid b \prec a\}$ has a largest element, denoted by $\boxminus a$. ⊣

Lemma 2.6.2 ([3, Lem. 4.2])**.** *Let $\prec$ be a Heyting lattice subordination on a Boolean algebra B. Then, for all $a, b \in B$,*

(i) $a \prec a$ *if and only if* $\boxminus a = a$*;*
(ii) $\boxminus a \prec \boxminus a$*;*
(iii) $a \prec b$ *implies* $\boxminus a \prec \boxminus b$*;*
(iv) $\boxminus a \leq a$*;*
(v) $a \prec b$ *if and only if* $a \leq \boxminus b$*.*

Proof.
(i) Pick $a \in B$ such that $a \prec a$. Since $\prec$ is a lattice subordination $(B, \prec)$ satisfies (B5) and so, for all b with $b \prec a$, we have $b \leq a$. Then since $a \prec a$, a must be the largest element of $\{b \in B \mid b \prec a\}$, which means that $\boxminus a = a$.

(ii) Pick $a \in B$. By definition, $\boxminus a \prec a$. From axiom (B6) it follows that there exists $c \in B$ with $\boxminus a \prec c \prec a$. Since $\boxminus a$ is the largest element of $\{b \in B \mid b \prec a\}$, it must be that $\boxminus a = c$ and so $\boxminus a \prec \boxminus a$. The converse is immediate.

(iii) Pick $a, b \in B$ such that $a \prec b$. By axiom (QP), it follows that there exists $c \in B$ with $c \prec c$ and $a \leq c \leq b$. By item (ii) of this lemma we have $\boxminus c \prec \boxminus c$ and from $a \leq c \leq b$ it follows that $\boxminus a \leq \boxminus c \leq \boxminus b$. Hence, $\boxminus a \prec \boxminus b$.

(iv) Since $\boxminus a \prec a$ and the fact that lattice subordinations are reflexive it holds that $\boxminus a \leq a$.

(v) Pick $a, b \in B$ such that $a \prec b$. Then $a \leq \boxminus b$, since $\boxminus b$ is by definition the greatest element with this property. Conversely, suppose $a \leq \boxminus b$. By items (ii) and (iv) of this lemma we know that $a \leq \boxminus b \prec \boxminus b \leq b$. By axiom (B4) herefrom it follows that $a \prec b$. $\square$

Let BLH denote the category whose objects are Boolean algebras with a Heyting lattice subordination and whose morphisms are subordination homomorphisms $h : A \to B$ that satisfy for all $a \in A$, if there exists $e \in B$ such that $e \prec_B \boxminus h(a)$ then there exists $b \in A$ with $b \prec_A a$ and $e \prec_B h(b)$. We will refer to the morphisms of BLH as BLH-morphisms.

Definition 2.6.3. An S4-*algebra* (or *interior* or *closure algebra*) is a pair $(B, \Box)$ where B is a Boolean algebra and $\Box : B \to B$ a unary function on B such that for all $a, b \in B$,

(1) $\Box 1 = 1$;
(2) $\Box(a \wedge b) = \Box a \wedge \Box b$;
(3) $\Box a \leq \Box\Box a$;
(4) $\Box a \leq a$. $\dashv$

Given a Boolean algebra homomorphism h between S4-algebras $(A, \Box_A)$ and $(B, \Box_B)$, we call h a modal algebra homomorphism whenever, for all $a \in A$, $h(\Box_A a) = \Box_B h(a)$. We let S4 denote the category of S4-algebras and modal algebra homomorphisms.

Define a functor $(-)_\star : \mathsf{BLH} \to \mathsf{S4}$ as follows.

> For $(B, \prec) \in \mathsf{BLH}$, let $(B, \prec)_\star := (B, \boxed{\prec})$, where $\boxed{\prec} : B \to B$ denotes the unary operator on B that associates with every $a \in B$ the element $\boxed{\prec}a$, the largest element of the set $\{b \in B \mid b \prec a\}$. For BLH-morphism h, define $h_\star := h$.

Lemma 2.6.4. *Let* $(B, \prec) \in \mathsf{BLH}$*. Then* $(B, \prec)_\star \in \mathsf{S4}$*.*

Proof. Given that 1 is the largest element of B and $1 \prec 1$, it must be that $\boxed{\prec}1 = 1$, so $(B, \boxed{\prec})$ satisfies $\mathsf{S4}$-axiom 1. To see that $(B, \boxed{\prec})$ satisfies $\mathsf{S4}$-axiom (2), observe that from $a \wedge b \leq a, b$ it follows that $\boxed{\prec}(a \wedge b) \leq \boxed{\prec}a, \boxed{\prec}b$ and thus also $\boxed{\prec}(a \wedge b) \leq \boxed{\prec}a \wedge \boxed{\prec}b$. Furthermore, since for all $a, b \in B$ we have $\boxed{\prec}a \prec a$ and $\boxed{\prec}b \prec b$ it must be that $\boxed{\prec}a \wedge \boxed{\prec}b \prec a \wedge b$. Now, since $\boxed{\prec}(a \wedge b)$ is the largest element in the set $\{c \in B \mid c \prec a \wedge b\}$ and $\boxed{\prec}(a \wedge b) \leq \boxed{\prec}a \wedge \boxed{\prec}b$, it must be that $\boxed{\prec}(a \wedge b) = \boxed{\prec}a \wedge \boxed{\prec}b$. Hence, $(B, \boxed{\prec})$ satisfies $\mathsf{S4}$-axiom (2). To see that $(B, \boxed{\prec})$ satisfies $\mathsf{S4}$-axiom (3), recall from Lemma 2.6.2 item (ii) that for all $a \in B$ we have $\boxed{\prec}a \prec \boxed{\prec}a$. Then, by item (v) of Lemma 2.6.2, it immediately follows that $\boxed{\prec}a \leq \boxed{\prec}\boxed{\prec}a$. Hence, $(B, \boxed{\prec})$ satisfies $\mathsf{S4}$-axiom (3). That $(B, \boxed{\prec})$ satisfies $\mathsf{S4}$-axiom (4) follows from 2.6.2 (v). $\square$

Define $(-)^\star : \mathsf{S4} \to \mathsf{BLH}$ as follows.

> For $(B, \Box) \in \mathsf{S4}$, let $(B, \Box)^\star := (B, \prec_\Box)$, where $\prec_\Box$ is a binary relation on B defined by $a \prec_\Box b$ if and only if $a \leq \Box b$, for all $a, b \in B$. For modal algebra homomorphism h, define $h^\star := h$.

Lemma 2.6.5. *Let* $(B, \Box) \in \mathsf{S4}$*. Then* $(B, \Box)^\star \in \mathsf{BLH}$*.*

Proof. It is a routine check to see that $(B, \prec_\Box)$ satisfies (B1)-(B4). To see that $(B, \prec_\Box)$ satisfies (QP), pick $a, b \in B$ such that $a \prec_\Box b$, i.e., $a \leq \Box b$. From $\mathsf{S4}$-axiom (3), we know that $\Box b \leq \Box\Box b$, hence $\Box b \prec_\Box \Box b$. By $\mathsf{S4}$-axiom (4) we also have $\Box b \leq b$. Since $\Box b \prec_\Box \Box b$ and $a \leq \Box b \leq b$, we have that $(B, \prec_\Box)$ satisfies (QP). Thus, $\prec_\Box$ is a lattice subordination on B. Moreover, by reflexivity of $\leq$ for any $a \in B$ we have $\Box a \leq \Box a$, so the set $\{b \in B \mid b \prec_\Box a\}$ has a largest element. Thus, $(B, \prec_\Box) \in \mathsf{BLH}$. $\square$

Remark 2.6.6. Observe that for all Boolean algebras B, given a Heyting lattice subordination $\prec$ on B, from Lemma 2.6.2 item (iv) it follows that for all $a, b \in B$ we have $a \prec b$ if and only if $a \prec_{\boxed{\prec}} b$. Moreover, given a unary operator $\Box$ on B that satisfies the $\mathsf{S4}$-axioms, for all a, the largest element of the set $\{b \in B \mid b \prec_\Box a\}$ is $\Box a$, so $\Box a = \boxed{\prec}_\Box a$.

Lemma 2.6.7 ([3, Thm. 4.8])**.** *Let $(A, \Box_A)$ and $(B, \Box_B)$ be* S4*-algebras and $\prec_A$ and $\prec_B$ denote $\prec_{\Box_A}$ and $\prec_{\Box_B}$ respectively. Let $h : A \to B$ be a Boolean algebra homomorphism. Then $h(\Box_A a) = \Box_B h(a)$ for all $a \in A$ iff (i) for all $a, b \in A$ we have that $a \prec_A b$ implies $h(a) \prec_B h(b)$ and, (ii) for all $a \in A$, $c \prec_B \boxminus h(a)$ implies there exists $b \in A$ with $b \prec_A a$ and $c \prec_B h(b)$.*

Proof. Suppose that $h(\Box_A a) = \Box_B h(a)$, for all $a \in A$. Pick $a, b \in A$ such that $a \prec_A b$, i.e., $a \leq \boxminus_A b$. It follows that $h(a) \leq h(\boxminus_A b) = \boxminus_B(h(b))$ and hence $h(a) \prec_B h(b)$. Now pick $a \in A$ such that $c \prec_B \boxminus_B h(a)$ for some $c \in B$ and let $b \coloneqq \boxminus_A a$. Then $b \prec_A a$ and since $h(b) = h(\boxminus_A a) = \boxminus_B h(a)$, also $c \prec_B h(b)$.

Conversely, assume h satisfies condition (i) and (ii). Pick $a \in A$. From $\Box_A a \prec a$ it follows that $h(\Box_A a) \prec h(a)$. By Lemma 2.6.2 item (iv) this means that $h(\Box_A a) \leq \Box_B h(a)$. Now recall that by Lemma 2.6.2 item (ii) we have $\Box_A a \prec_A \Box_A a$ and hence $h(\Box_A a) \prec_B h(\Box_A a)$. By condition (ii) stated above, it follows that there exists $b \in A$ with $b \prec_A a$ and $h(\Box_A a) \prec_B h(b)$. From $b \prec_A a$ we obtain $b \leq \Box_A a$ and so $h(b) \leq h(\Box_A a)$. By S4-axiom (4), $\Box_B h(b) \leq h(b)$, so $\Box_B h(b) \leq h(\Box_A a)$. Hence, $h(\Box_A a) = \Box_B h(a)$. $\Box$

Theorem 2.6.8 ([3, Cor. 4.9])**.** BLH *is isomorphic to* S4*.*

Proof. Lemmas 2.6.4, 2.6.5, and 2.6.7 show that $(-)_\star$: BLH $\to$ S4 and $(-)^\star$: S4 $\to$ BLH are well-defined functors. From Remark 2.6.6 it follows that for all $(B, \prec) \in$ BLH we have $(B, \prec) = ((B, \prec)_\star)^\star$ and for all $(B, \Box) \in$ S4 we have $(B, \Box) = ((B, \Box)^\star)_\star$. Thus, BLH is isomorphic to S4. $\Box$

2.6.1 Heyting algebras and Heyting lattice subordinations

In section 2.4 we have seen that, given a lattice subordination $\prec$ on a Boolean algebra B, we can identify a corresponding bounded sublattice $D_\prec$ of B and, vice versa, by means of a bounded sublattice D of a Boolean algebra B, we can define a lattice subordination $\prec_D$ on B. In this section we will see that for Heyting lattice subordinations, the corresponding bounded distributive lattice $D_\prec$ is in fact a Heyting algebra. Conversely, for every bounded sublattice D of B, if D is a Heyting algebra, then $\prec_D$ defines a Heyting lattice subordination.

Definition 2.6.9. A *Heyting algebra* $(H, \vee, \wedge, \to, 0, 1)$ is a bounded distributive lattice endowed with a binary operation $\to$ called *Heyting implication* such that for every $a, b \in H$ there exists an element $a \to b$ such that, for all $c \in H$,

$$c \leq a \to b \text{ if and only if } a \wedge c \leq b. \qquad \dashv$$

By Heyt, we denote the category that has as objects Heyting algebras and morphisms Heyting algebra homomorphisms. Let $(B, \Box)$ be an S4-algebra. We call the elements $a \in B$ that are such that $\Box a = a$ the *fixed points* or *open elements of* B and, define $B_\Box := \{a \in B \mid \Box a = a\}$.

Lemma 2.6.10 (see e.g.,[13, Prop. 8.31])**.** *For all* S4*-algebras* $(B, \Box)$*, the set of fixed points* $B_\Box := \{a \in B \mid \Box a = a\}$ *is a sublattice of* B *with a Heyting implication given by* $a \to b := \Box(\neg a \vee b)$*.*

Lemma 2.6.11 (see e.g.,[13, Cor. 8.35])**.** *For every Heyting algebra* H*, there exists an* S4*-algebra* $(B, \Box)$ *such that* $B_\Box \cong H$*.*

Lemma 2.6.12 (Cf., [3, Lem. 4.5])**.** *Let* B *be a Boolean algebra and* $\prec$ *a Heyting lattice subordination on* B*. Then* $D_\prec$ *is a bounded sublattice of* B *that is a Heyting algebra.*

Proof. From 2.6.4 we know that $(B, \boxtimes)$ is an S4-algebra and by Lemma 2.6.11 that $B_\boxtimes$ defines a Heyting algebra. By Lemma 2.6.2 item (i) it immediately follows $D_\prec = B_\boxtimes$. $\Box$

Lemma 2.6.13 (Cf., [3, Lem. 4.5])**.** *If a bounded sublattice* D *of a Boolean algebra* B *is such that* D *it has a Heyting implication, then* $\prec_D$ *is a Heyting lattice subordination on* B*.*

In this chapter, we have seen how the category BDL of bounded distributive lattices relates to the category BLS of Boolean algebras with a lattice subordination. Specifically, for every Boolean algebra with a lattice subordination, the lattice of reflexive elements is a bounded distributive lattice and conversely, each bounded distributive lattice is isomorphic to the bounded distributive lattice of reflexive elements of a suitable Boolean algebra with a lattice subordination. Moreover, we have looked at the specific case of Boolean algebras with a Heyting lattice subordination and Heyting algebras. In what follows we will look at the syntactic analogue of the correspondence between BDL and BLS. We will first present a positive hypersequent calculus that is sound and complete with respect to the class of bounded distributive lattices and introduce a calculus that is sound and complete with respect to the class of Boolean algebras with a lattice subordination. Thereafter, we will define a translation that embeds the positive calculus in the latter one and show that it is full and faithfull.

Chapter 3

Positive calculus

In this chapter, we introduce a calculus $\mathbf{PC}_+$ of hypersequent rules and prove that it is sound and complete with respect to the class BDL of bounded distributive lattices. First, we introduce our formal language (a positive logic) $\mathcal{L}_+$ and a semantics for this language. Thereafter, we present the axioms and rules of the system $\mathbf{PC}_+$ and prove strong soundness and completeness with respect to BDL. As we will see, the basic hypersequent system $\mathbf{PC}_+$ is in fact equivalent to a sequent calculus $\mathbf{SC}_+$ with respect to the derivation of single-component hypersequents (i.e., regular sequents). That is, for all sequents S, $\mathbf{PC}_+$ derives the sequent S if and only if the sequent calculus $\mathbf{SC}_+$ derives S. Of course $\mathbf{PC}_+$ derives many hypersequents with multiple components, yet these do not make sense in the context of $\mathbf{SC}_+$. However, we will see that the additional hypersequent context is a sensible choice. Similar to the results of [14], we show that every positive sequent rule that is consistent (a positive rule is consistent if there exists a non-trivial bounded distributive lattice that validates the rule), then this positive sequent rule is already derivable in the calculus $\mathbf{SC}_+$. Thus, we would not be able to identify proper subclasses of BDL by extensions of $\mathbf{SC}_+$.

3.1 Syntax and semantics

Let Prop be a countably infinite set of propositional variables. We generate the *positive language* $\mathcal{L}_+$ from Prop using the connectives $\wedge$ and $\vee$ and constants $\top$ and $\bot$. The well-formed formulas φ of $\mathcal{L}_+$ are called *positive formulas* or *terms* and are given by the grammar:

$$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi, \qquad p \in \mathsf{Prop}.$$

A *positive sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite (possibly empty) multisets of positive formulas. A *positive hypersequent* is a finite multiset of positive sequents,

$$\Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$$

where, for all $i \leq n$, $\Gamma_i \Rightarrow \Delta_i$ is an ordinary positive sequent called a *component* of the hypersequent.

A *substitution* is a function $\sigma : \mathsf{Prop} \to \mathcal{L}_+$. We extend this function recursively to a map $(-)^\sigma : \mathcal{L}_+ \to \mathcal{L}_+$ from formulas to formulas in the usual way. Then, given a substitution σ and a multiset of formulas Γ, we let Γ^σ denote the multiset $\{\varphi^\sigma \mid \varphi \in \Gamma\}$ with the convention that the multiplicity of φ^σ in Γ^σ is that of φ in Γ. Furthermore, given a sequent $\Gamma \Rightarrow \Delta$, we let $(\Gamma \Rightarrow \Delta)^\sigma$ denote the sequent $\Gamma^\sigma \Rightarrow \Delta^\sigma$. And, similarly, given a hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$, we let $(\Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n)^\sigma$ denote the hypersequent $(\Gamma_1 \Rightarrow \Delta_1)^\sigma \mid ... \mid (\Gamma_n \Rightarrow \Delta_n)^\sigma$.

We interpret the formulas of the language above in bounded distributive lattices D, where D is regarded as an algebra $(D, \vee, \wedge, 0, 1)$. Then, a *valuation* v is a map $v : \mathsf{Prop} \to D$. We extend this map to a map $[\![-]\!]_v : \mathcal{L}_+ \to D$ that assigns a valuation to positive formulas in the usual recursive fashion,

$$\begin{array}{rcl} [\![\varphi \wedge \psi]\!]_v & \coloneqq & [\![\varphi]\!]_v \wedge [\![\psi]\!]_v, \\ [\![\varphi \vee \psi]\!]_v & \coloneqq & [\![\varphi]\!]_v \vee [\![\psi]\!]_v, \\ [\![\top]\!]_v & \coloneqq & 1, \\ [\![\bot]\!]_v & \coloneqq & 0. \end{array}$$

For a multiset of positive formulas Γ we follow the convention that if $\Gamma = \emptyset$ then $[\![\bigwedge \Gamma]\!]_v = 1$, and $[\![\bigvee \Gamma]\!]_v = 0$ for any valuation v.

We say that a positive sequent $\Gamma \Rightarrow \Delta$ is *true* in a distributive lattice D under a valuation v (or, in other words, that D *satisfies* $\Gamma \Rightarrow \Delta$ under v) iff $[\![\bigwedge \Gamma]\!]_v \leq [\![\bigvee \Delta]\!]_v$, and write $(D, v) \vDash \Gamma \Rightarrow \Delta$. We extend this to positive hypersequents by saying that a positive hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$ is true in D under a valuation v if it satisfies a component $\Gamma_i \Rightarrow \Delta_i$ of the hypersequent under v. Furthermore, a positive (hyper)sequent H is said to be *valid* in D if it is true under all valuations v on D, in which case we write $D \vDash H$.

We derive the notion of truth of a positive formula φ by saying that φ is true in D under a valuation v iff the sequent $\top \Rightarrow \varphi$ is true in D under that valuation

v. And, we say that a positive formula φ is valid if the corresponding sequent $\top \Rightarrow \varphi$ is.

Definition 3.1.1 (Cf., [8])**.** A *positive hypersequent rule* is a pair $\mathrm{M} = \langle \mathcal{H}, S \rangle$, where $\mathcal{H}$ is a finite set of positive hypersequents and S a single positive hypersequent. We write $\langle \mathcal{H}, S \rangle$ as $\mathcal{H}/S$ or, if $\mathcal{H} = \{H_0, H_1, ..., H_n\}$ we write,

$$\frac{H_0 \qquad H_1 \qquad ... \qquad H_n}{S}\ (\mathrm{M})$$

We call $\mathcal{H}$ and S the *premises* of the rule and the *conclusion* respectively. If $\mathcal{H} = \emptyset$ we call (M) an *axiom*. ⊣

We say that $D \in \mathsf{BDL}$ validates a positive hypersequent rule $\mathcal{H}/S$ if, for all valuations v, the conclusion S is true under v whenever all the premises in the set $\mathcal{H}$ are true under that valuation v, and write $D \vDash \mathcal{H}/S$ or $\mathcal{H} \vDash_D S$. Given an arbitrary subclass K of BDL, we write $\mathcal{H} \vDash_K S$ if for all $D \in K$, we have $\mathcal{H} \vDash_D S$.

3.2 Positive hypersequent calculi $\mathbf{HC}_+$

In this section, we present the hypersequent calculus $\mathbf{PC}_+$ for the positive language $\mathcal{L}_+$ and show algebraic soundness and completeness concerning the derivability of positive hypersequent rules with respect to bounded distributive lattices.

3.2.1 The calculus $\mathbf{PC}_+$

Definition 3.2.1. Let $\varphi, \varphi_1, \varphi_2, \psi_1, \psi_2$ be positive formulas and $\Gamma, \Delta, \Gamma', \Delta'$ be finite multisets of positive formulas. The *hypersequent calculus* $\mathbf{PC}_+$ consists of the following rules.

Axioms

$$G \mid \Gamma, \bot \Rightarrow \Delta \quad G \mid \Gamma \Rightarrow \top, \Delta \quad G \mid \Gamma, \varphi \Rightarrow \varphi, \Delta$$

Cut rule

$$\frac{G \mid \Gamma \Rightarrow \varphi, \Delta \qquad G \mid \varphi, \Gamma' \Rightarrow \Delta'}{G \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (\mathrm{cut})$$

Logical rules

$$\frac{G \mid \Gamma, \varphi_1, \varphi_2 \Rightarrow \Delta}{G \mid \Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta}\ (\wedge\mathrm{l}) \qquad \frac{G \mid \Gamma \Rightarrow \psi_1, \Delta \quad G \mid \Gamma \Rightarrow \psi_2, \Delta}{G \mid \Gamma \Rightarrow \psi_1 \wedge \psi_2, \Delta}\ (\wedge\mathrm{r})$$

$$\frac{G \mid \Gamma, \psi_1 \Rightarrow \Delta \quad G \mid \Gamma, \psi_2 \Rightarrow \Delta}{G \mid \Gamma, \psi_1 \vee \psi_2 \Rightarrow \Delta}\ (\vee\mathrm{l}) \qquad \frac{G \mid \Gamma \Rightarrow \psi_1, \psi_2, \Delta}{G \mid \Gamma \Rightarrow \psi_1 \vee \psi_2, \Delta}\ (\vee\mathrm{r})$$

Internal structural rules

$$\frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (\mathrm{iw}) \qquad \frac{G \mid \Gamma', \Gamma, \Gamma \Rightarrow \Delta}{G \mid \Gamma', \Gamma \Rightarrow \Delta}\ (\mathrm{icl}) \qquad \frac{G \mid \Gamma \Rightarrow \Delta, \Delta, \Delta'}{G \mid \Gamma \Rightarrow \Delta, \Delta'}\ (\mathrm{icr})$$

External structural rules

$$\frac{G}{G \mid \Gamma \Rightarrow \Delta}\ (\mathrm{ew}) \qquad \frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ (\mathrm{ec})$$

By a *positive hypersequent calculus* we mean any collection of positive hypersequent rules extending the calculus $\mathbf{PC}_+$. Now, let $\{H, H_1, ..., H_n\}$ be a set of positive hypersequents and let

$$\frac{G_0 \quad G_1 \quad ... \quad G_n}{G}\ (\mathrm{M})$$

be a positive hypersequent rule. Following [8], we say that a hypersequent H *is obtained from* $H_1, ..., H_n$ *by an application of the rule (M)*, if there exist a substitution σ and a hypersequent G' such that H is of the form $G' \mid G\sigma$ and H_i is of the form $G' \mid G_i\sigma$ for $i \leq n$. Let $\mathcal{H} \cup \{S\}$ be a set of hypersequents and $\mathbf{HC}_+$ a positive hypersequent calculus. We say that S is *derivable (or provable) from* $\mathcal{H}$ *over* $\mathbf{HC}_+$ and write $\mathcal{H} \vdash_{\mathbf{HC}_+} S$, if there exists a finite sequence of hypersequents $H_1, ..., H_m$ such that H_m is the hypersequent S and for all $i < m$ either H_i belongs to $\mathcal{H}$ or H_i is an axiom or H_i is obtained by applying a rule from $\mathbf{HC}_+$ to some subset of $\{H_j \mid j < i\}$. If S is not derivable from $\mathcal{H}$ over $\mathbf{HC}_+$ we write $\mathcal{H} \nvdash_{\mathbf{HC}_+} S$. Furthermore, we say that a positive hypersequent rule $\mathcal{H}/S$ is derivable in a calculus $\mathbf{HC}_+$ if we have $\mathcal{H} \vdash_{\mathbf{HC}_+} S$. We will sometimes write $\vdash_{\mathbf{HC}_+} \mathcal{H}/S$ or $\mathbf{HC}_+ \vdash \mathcal{H}/S$ to indicate that we are referring to the derivability of a rule. Finally, for sets of positive hypersequent rules $\mathcal{R}_1$ and $\mathcal{R}_2$, we say that $\mathcal{R}_2$ *is derivable from* $\mathcal{R}_1$ *over* $\mathbf{HC}_+$ if all the rules of $\mathcal{R}_2$ are derivable in $\mathbf{HC}_+ \cup \mathcal{R}_1$ and express this by $\mathcal{R}_1 \vdash_{\mathbf{HC}_+} \mathcal{R}_2$. The sets $\mathcal{R}_1$ and $\mathcal{R}_2$ are said to be *equivalent over* $\mathbf{HC}_+$ if we have both $\mathcal{R}_1 \vdash_{\mathbf{HC}_+} \mathcal{R}_2$ and $\mathcal{R}_2 \vdash_{\mathbf{HC}_+} \mathcal{R}_1$.

Observe that any hypersequent rule is equivalent to a finite set of hypersequent rules the premises of which are all single-component hypersequents (i.e., sequents).

Definition 3.2.2. let ρ be an arbitrary positive hypersequent rule consisting of the premises $\alpha_{11} \mid ... \mid \alpha_{1m}$, ..., $\alpha_{k1} \mid ... \mid \alpha_{kn}$ and conclusion G. We define a corresponding set $\mathcal{R}_\rho$ of positive hypersequent rules as follows,

$\mathcal{R}_\rho \coloneqq \{\{\alpha_{1i} \ ... \ \alpha_{kj}\}/G \mid \alpha_{lp}$ is a component of $\alpha_{11} \mid ... \mid \alpha_{1m}$, ..., and α_{kj} of $\alpha_{k1} \mid ... \mid \alpha_{kn}\}$. $\dashv$

Lemma 3.2.3 ([8, Lem. 7]). *Let* $\mathbf{HC}_+$ *be a positive hypersequent calculus,* $\mathcal{H}$ *a set of positive hypersequents,* α *a positive sequent, and* G *a positive hypersequent. Then,*

$$\mathcal{H} \cup \{\alpha\} \vdash_{\mathbf{HC}_+} G \textit{ and } \mathcal{H} \vdash_{\mathbf{HC}_+} \alpha \mid G \textit{ imply } \mathcal{H} \vdash_{\mathbf{HC}_+} G.$$

Proof. Assume that $\mathcal{H} \vdash_{\mathbf{HC}_+} \alpha \mid G$. Then, for any hypersequent G', if $\mathcal{H} \cup \{\alpha\} \vdash_{\mathbf{HC}_+} G'$, by an induction on the derivation of G' it follows that $\mathcal{H} \vdash_{\mathbf{HC}_+} G' \mid G$. In particular, this means that whenever $\mathcal{H} \cup \{\alpha\} \vdash_{\mathbf{HC}_+} G$ it must also be that $\mathcal{H} \vdash_{\mathbf{HC}_+} G \mid G$. By an application of the rule external weakening, it follows that $\mathcal{H} \vdash_{\mathbf{HC}_+} G$. □

Corollary 3.2.4. *Let* $\mathbf{HC}_+$ *be a positive hypersequent calculus,* $\mathcal{H}$ *a set of positive hypersequents,* $\{\alpha_1, ..., \alpha_n\}$ *a finite set of positive sequents, and* G *a positive hypersequent. Then,*

$$\mathcal{H} \vdash_{\mathbf{HC}_+} G \mid \alpha_1 \mid ... \mid \alpha_n \textit{ and } \mathcal{H} \cup \{\alpha_i\} \vdash_{\mathbf{HC}_+} G, \textit{ for all } i \leq n, \textit{ imply } \mathcal{H} \vdash_{\mathbf{HC}_+} G.$$

Proof. Assum that $\mathcal{H} \vdash_{\mathbf{HC}_+} G \mid \alpha_1 \mid ... \mid \alpha_n$ and $\mathcal{H} \cup \{\alpha_i\} \vdash_{\mathbf{HC}_+} G$, for all $i \leq n$. By (ew), $\mathcal{H} \cup \{\alpha_1\} \vdash_{\mathbf{HC}_+} G$ implies $\mathcal{H} \cup \{\alpha_1\} \vdash_{\mathbf{HC}_+} G \mid \alpha_2 \mid ... \mid \alpha_n$. Together with $\mathcal{H} \vdash_{\mathbf{HC}_+} G \mid \alpha_1 \mid ... \mid \alpha_n$, by Lemma 3.2.3 this entails $\mathcal{H} \vdash_{\mathbf{HC}_+} G \mid \alpha_2 \mid ... \mid \alpha_n$. Applying this reasoning $n-1$ times, we obtain $\mathcal{H} \vdash_{\mathbf{HC}_+} G$. □

Theorem 3.2.5. *Let* ρ *be an arbitrary positive hypersequent rule and let* $\mathcal{R}_\rho$ *be as defined above. Then, for any positive hypersequent calculus* $\mathbf{HC}_+$ *we have (i)* $\rho \vdash_{\mathbf{HC}_+} \mathcal{R}_\rho$ *and, (ii)* $\mathcal{R}_\rho \vdash_{\mathbf{HC}_+} \rho$.

Proof.
(i) Let $\vdash$ denote $\vdash_{\mathbf{HC}_+ \cup \{\rho\}}$. We show that for each rule $\alpha_{1i} \ ... \ \alpha_{kj}/G \in \mathcal{R}_\rho$ it holds that $\{\alpha_{1i}, ..., \alpha_{kj}\} \vdash G$. Observe that, by applying the rule (ew) $m-1$ times on the premise α_{1i}, ..., and $n-1$ times on the premise α_{kj}, we can derive the all premises of ρ, that is, $\alpha_{11} \mid ... \mid \alpha_{1m}$, ..., $\alpha_{k1} \mid ... \mid \alpha_{kn}$, from α_{1i}, ...,

and α_{kj}. Then, by the application of ρ, we derive G. Thus, each rule α_{1i} ... $\alpha_{kj}/G \in \mathcal{R}_\rho$ is derivable in $\mathbf{HC}_+ \cup \{\rho\}$. Hence the set of rules $\mathcal{R}_\rho$ is derivable from ρ over $\mathbf{HC}_+$.

(ii) Let $\vdash$ denote $\vdash_{\mathbf{HC}_+ \cup \mathcal{R}_\rho}$ and $\mathcal{H}$ the set $\{\alpha_{11} \mid ... \mid \alpha_{1m}, ..., \alpha_{k1} \mid ... \mid \alpha_{kn}\}$, consisting of the premises of ρ. We show that $\mathcal{H} \vdash G$. Observe that if $k = 1$, what needs to be shown follows immediately from Corollary 3.2.4. Assume that $k > 1$. From $\{\alpha_{11}, \alpha_{21}, ..., \alpha_{k1}\}/G \in \mathcal{R}_\rho$ and (ew) it follows that $\mathcal{H} \cup \{\alpha_{11}, \alpha_{21}, ..., \alpha_{k1}\} \vdash G \mid \alpha_{12} \mid ... \mid \alpha_{1m}$. Since $\alpha_{11} \mid \alpha_{12} \mid ... \mid \alpha_{1m} \in \mathcal{H}$, we have $\mathcal{H} \cup \{\alpha_{21}, ..., \alpha_{k1}\} \vdash G \mid \alpha_{11} \mid \alpha_{12} \mid ... \mid \alpha_{1m}$. By Lemma 3.2.3, this implies that $\mathcal{H} \cup \{\alpha_{21}, ..., \alpha_{k1}\} \vdash G \mid \alpha_{12} \mid ... \mid \alpha_{1m}$. By Lemma 3.2.3 and given that $\mathcal{H} \cup \{\alpha_{21}, ..., \alpha_{k1}\} \cup \{\alpha_{12}\} \vdash G \mid \alpha_{13} \mid ... \mid \alpha_{1m}$, thereby it follows that we have $\mathcal{H} \cup \{\alpha_{21}, ..., \alpha_{k1}\} \vdash G \mid \alpha_{13} \mid ... \mid \alpha_{1m}$.
Applying the argument $m - 2$ more times, we obtain $\mathcal{H} \cup \{\alpha_{21}, ..., \alpha_{k1}\} \vdash G$. By the same argument, for $\alpha_{21} \mid ... \mid \alpha_{2j} \in \mathcal{H}$, from $\{\alpha_{11}, \alpha_{2i}, \alpha_{31}, ..., \alpha_{k1}\}/G \in \mathcal{R}_\rho$, it follows that $\mathcal{H} \cup \{\alpha_{2_i}, \alpha_{31}, ..., \alpha_{k1}\} \vdash G$, for all $i \leq j$. By corollary 3.2.4, from $\mathcal{H} \cup \{\alpha_{31}, ..., \alpha_{k1}\} \vdash G \mid \alpha_{21} \mid ... \mid \alpha_{2j}$, it follows that $\mathcal{H} \cup \{\alpha_{31}, ..., \alpha_{k1}\} \vdash G$. Iterating this process $k - 2$ times, we obtain $\mathcal{H} \vdash G$. □

Theorem 3.2.5 will prove useful in Chapter 5 where we translate positive hypersequent rules into strict implication hypersequent rules. By Theorem 3.2.5, it suffices to define a translation for positive rules with only single-component premises, since, every rule with premises of a higher complexity corresponds to a finite set of such rules with single-component premises.

Remark 3.2.6. Observe that since we interpret positive sequents as inequalities in bounded distributive lattices, they correspond to equations in the language of bounded distributive lattices. From this, we see that positive hypersequents $\alpha_1 \mid ... \mid \alpha_n$ correspond to disjunctions of equations e_1 or ... or e_n, where e_i is the equation corresponding to α_i, and that any positive hypersequent rule $\{H_1, ..., H_m\}/H$ corresponds to a universal formulas of the form $(\mathsf{AND}_{i=1}^m E_i) \implies E$, where E_i denotes the disjunction corresponding to H_i. As a special case, we obtain that sequent rules correspond to *quasi-equations*, that is, expressions of the form $(\mathsf{AND}_{i=1}^n e_i) \implies e$.

The following derivations provide two concrete examples of derivations that will be of use for the completeness proof below.

Derivation 3.2.1.

$$\cfrac{\cfrac{\cfrac{\cfrac{\varphi,\psi\Rightarrow\varphi,\varphi\wedge\chi \quad \varphi,\psi\Rightarrow\psi,\varphi\wedge\chi}{\varphi,\psi\Rightarrow\varphi\wedge\psi,\varphi\wedge\chi}\,(\wedge\mathrm{r}) \quad \cfrac{\varphi,\chi\Rightarrow\varphi\wedge\psi,\varphi \quad \varphi,\chi\Rightarrow\varphi\wedge\psi,\chi}{\varphi,\chi\Rightarrow\varphi\wedge\psi,\varphi\wedge\chi}\,(\wedge\mathrm{r})}{\varphi,\psi\vee\chi\Rightarrow\varphi\wedge\psi,\varphi\wedge\chi}\,(\vee\mathrm{l})}{\varphi\wedge(\psi\vee\chi)\Rightarrow\varphi\wedge\psi,\varphi\wedge\chi}\,(\wedge\mathrm{l})}{\varphi\wedge(\psi\vee\chi)\Rightarrow(\varphi\wedge\psi)\vee(\varphi\wedge\chi)}\,(\vee\mathrm{r})$$

Derivation 3.2.2.

$$\cfrac{\cfrac{\cfrac{\cfrac{\varphi,\psi\Rightarrow\psi,\chi}{\varphi,\psi\Rightarrow\psi\vee\chi}\,(\vee\mathrm{r}) \quad \varphi,\psi\Rightarrow\varphi}{\varphi,\psi\Rightarrow\varphi\wedge(\psi\vee\chi)}\,(\wedge\mathrm{r})}{\varphi\wedge\psi\Rightarrow\varphi\wedge(\psi\vee\chi)}\,(\wedge\mathrm{l}) \quad \cfrac{\cfrac{\cfrac{\varphi,\chi\Rightarrow\psi,\chi}{\varphi,\chi\Rightarrow\psi\vee\chi}\,(\vee\mathrm{r}) \quad \varphi,\chi\Rightarrow\varphi}{\varphi,\chi\Rightarrow\varphi\wedge(\psi\vee\chi)}\,(\wedge\mathrm{r})}{\varphi\wedge\chi\Rightarrow\varphi\wedge(\psi\vee\chi)}\,(\wedge\mathrm{l})}{(\varphi\wedge\psi)\vee(\varphi\wedge\chi)\Rightarrow\varphi\wedge(\psi\vee\chi)}\,(\vee\mathrm{l})$$

Lemma 3.2.7. *Let Γ and Δ be finite sets of positive formulas. Then,*

$$\bigwedge\Gamma\Rightarrow\bigvee\Delta\vdash_{\mathbf{HC}_+}\Gamma\Rightarrow\Delta.$$

Proof. Let $\Gamma \coloneqq \{\gamma_1,...,\gamma_n\}$ and $\Delta \coloneqq \{\delta_1,...,\delta_m\}$. Note that, for all $i\leq n$ and all $j\leq m$, the sequents $\gamma_1,...,\gamma_n\Rightarrow\gamma_i$ and $\delta_j\Rightarrow\delta_1,...,\delta_m$ are instances of the axiom $G\mid\Gamma,\varphi\Rightarrow\varphi,\Delta$. By applying the $(\wedge\mathrm{r})$-rule $n-1$ times, and the $(\vee\mathrm{l})$-rule $m-1$ times, we derive the sequents $\gamma_1,...,\gamma_n\Rightarrow\bigwedge_{i=1}^n\gamma_i$ and $\bigvee_{j=1}^m\delta_j\Rightarrow\delta_1,...,\delta_m$. Then, from the assumption $\bigwedge_{1=i}^n\gamma_i\Rightarrow\bigvee_{1=j}^m\delta_j$ by applying (cut) twice, we obtain the sequent $\gamma_1,...,\gamma_n\Rightarrow\delta_1,...,\delta_m$.

Note that this also follows if Γ or Δ is empty. We know that, if $\Gamma=\emptyset$ then $\bigwedge\Gamma=\top$ and, if $\Delta=\emptyset$ then $\bigvee\Delta=\bot$. By the same argument as above, from the assumption $\top\Rightarrow\bigvee\Delta$ and $\bigvee\Delta\Rightarrow\Delta$ we have $\top\Rightarrow\Delta$. Similarly, from $\bigwedge\Gamma\Rightarrow\bot$ we obtain the sequent $\Gamma\Rightarrow\bot$. □

3.2.2 Soundness and completeness

In this section, we establish soundness and completeness of derivability of positive hypersequent rules with respect to bounded distributive lattices.

Lemma 3.2.8 (Algebraic soundness, cf., [8, Thm. 2.5])**.** *Let $\mathbf{HC}_+$ be a positive hypersequent calculus and let $\mathcal{H}/S$ be a hypersequent rule. If the rule $\mathcal{H}/S$ is derivable in $\mathbf{HC}_+$ then all distributive lattices validating $\mathbf{HC}_+$ also validate $\mathcal{H}/S$.*

Proof. Let $\mathcal{C}(\mathbf{HC}_+)$ be the class of bounded distributive lattices that validate $\mathbf{HC}_+$. We show by induction on derivation length that for all positive hypersequent rules $\mathcal{H}/S$ and for all $D \in \mathcal{C}(\mathbf{HC}_+)$,

$$\mathcal{H} \vdash_{\mathbf{HC}_+} S \text{ implies } \mathcal{H} \vDash_D S.$$

This amounts to showing that all the axioms of $\mathbf{PC}_+$ are valid and all the rules of $\mathbf{PC}_+$ preserve validity in all $D \in \mathcal{C}(\mathbf{HC}_+)$. We spell out the proof that (cut) preserves validity. The other cases are proved in the same vain.

Observe that, for our soundness proof, we can disregard all side hypersequents G that occur in the premises as well as in the conclusion. Since, if a side hypersequent G in the premises is true, then, for any hypersequent G', the hypersequent $G \mid G'$ is also true. Now, pick arbitrary $D \in \mathcal{C}(\mathbf{HC}_+)$ such that the sequents $\Gamma \Rightarrow \varphi, \Delta$ and $\varphi, \Gamma' \Rightarrow \Delta'$ are valid in D. Then, for all valuations v on D we have $[\![\bigwedge \Gamma]\!]_v \leq [\![\varphi \vee \bigvee \Delta]\!]_v$ and $[\![\varphi \wedge \bigwedge \Gamma']\!]_v \leq [\![\bigvee \Delta']\!]_v$. We show that $[\![\bigwedge \Gamma \wedge \bigwedge \Gamma']\!]_v \leq [\![\bigvee \Delta \vee \bigvee \Delta']\!]_v$ follows by showing that, for all distributive lattices D, for all $a, a', b, c, c' \in D$,

$$a \leq b \vee c \text{ and } b \wedge a' \leq c' \text{ implies } a \wedge a' \leq c \vee c'.$$

Let D be an arbitrary distributive lattice and pick $a, a', b, c, c' \in D$ such that $a \leq b \vee c$ and $b \wedge a' \leq c'$. It follows that $a \wedge a' \leq (b \vee c) \wedge a'$ and $(b \wedge a') \vee c \leq c' \vee c$. Note that we also have $(b \vee c) \wedge a' \leq ((b \vee c) \wedge a') \vee c$. Now, since we assumed D to be distributive and by absorption, the following equality holds,

$$((b \vee c) \wedge a') \vee c = (b \wedge a') \vee (c \wedge a') \vee c = (b \wedge a') \vee c.$$

This means that, $(b \vee c) \wedge a' \leq (b \wedge a') \vee c$. Then, by transitivity of $\leq$ it follows that $a \wedge a' \leq c \vee c'$. And so, in particular, $[\![\bigwedge \Gamma \wedge \bigwedge \Gamma']\!]_v \leq [\![\bigvee \Delta \vee \bigvee \Delta']\!]_v$ follows from $[\![\bigwedge \Gamma]\!]_v \leq [\![\varphi \vee \bigvee \Delta]\!]_v$ and $[\![\varphi \wedge \bigwedge \Gamma']\!]_v \leq [\![\bigvee \Delta']\!]_v$, which is what we wanted to show. Thus, $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is valid in D and hence, (cut) preserves validity. □

We now proceed to establish algebraic completeness. In what follows, for a set of positive hypersequents $\mathcal{H} \cup \{S\}$, let the set $\mathsf{P}(\mathcal{H}, S)$ denote the propositional variables occurring in $\mathcal{H} \cup \{S\}$ and let $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))$ denote the set of positive formulas over $\mathsf{P}(\mathcal{H}, S)$.

Definition 3.2.9. Let $\mathbf{HC}_+$ be a positive hypersequent calculus and $\mathcal{H} \cup \{S\}$ a set of positive hypersequents. We say that $\mathcal{H}$ is *maximal with respect to S in $\mathbf{HC}_+$* if (i) we have $\mathcal{H} \nvdash_{\mathbf{HC}_+} S$ and (ii) for all positive hypersequents S' over the set $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))$ for which $S' \notin \mathcal{H}$ holds, we have $\mathcal{H}, S' \vdash_{\mathbf{HC}_+} S$. ⊣

Lemma 3.2.10 (Lindenbaum Lemma)**.** *Let $\mathcal{H}$ be a set of positive hypersequents and S a single hypersequent such that $\mathcal{H} \nvdash_{\mathbf{HC}_+} S$. Let $\mathsf{P}(\mathcal{H}, S)$ denote the propositional variables occurring in $\mathcal{H} \cup \{S\}$ and $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))$ the set of positive formulas over $\mathsf{P}(\mathcal{H}, S)$. Then there exists a set of positive hypersequents $\widetilde{\mathcal{H}}$ based on $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))$ such that $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}$ is maximal with respect to S.*

Proof. Let $\mathcal{G} \coloneqq \{G_m\}_{m\in\mathbb{N}}$ be an enumeration of all hypersequents based on $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))$. We construct an increasing chain $\Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq ...$ of sets of positive hypersequents that do not derive S as follows,

$$\Delta_0 \coloneqq \mathcal{H}$$
$$\Delta_{n+1} \coloneqq \begin{cases} \Delta_n \cup \{G_n\}, & \text{if } \Delta_n, G_n \nvdash_{\mathbf{HC}_+} S \\ \Delta_n, & \text{otherwise.} \end{cases}$$

Let $\widetilde{\mathcal{H}} = \bigcup_{m\in\mathbb{N}} \Delta_m$. Clearly, we have $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$. We show that $\widetilde{\mathcal{H}}$ is maximal with respect to S. Observe that, for all $n \in \mathbb{N}$ we have $\Delta_n \nvdash_{\mathbf{HC}_+} S$. Now, if we have $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} S$, then there exists a finite sequence of hypersequents $H^* = H_1, ..., H_k$ such that $H_k = S$ and for all $i < k$ either $H_i \in \widetilde{\mathcal{H}}$ or H_i is obtained by applying a rule from $\mathbf{HC}_+$ to some subset of $\{H_j \mid j < i\}$. Note that each H_i from H^* occurs indexed in $\mathcal{G}$. Let j be the highest index assigned to the hypersequents H^*. Then, at Δ_j, all the hypersequents of H^* that belong to $\widetilde{\mathcal{H}}$ are in Δ_j. So it must be that $\Delta_j \vdash_{\mathbf{HC}_+} S$, which leads to a contradiction. Thus, it must be that $\widetilde{\mathcal{H}} \nvdash_{\mathbf{HC}_+} S$.

Now, pick a hypersequent S_j for some $j \in \mathbb{N}$ such that $S_j \notin \widetilde{\mathcal{H}}$. Then in the increasing chain constructed above, at stage j the hypersequent S_j is not added and we have $\Delta_j = \Delta_{j-1}$, so it must be that $\Delta_j, S_j \vdash_{\mathbf{HC}_+} S$. Since $\Delta_j \subseteq \widetilde{\mathcal{H}}$, by weakening it follows that $\widetilde{\mathcal{H}}, S_j \vdash_{\mathbf{HC}_+} S$. We may conclude that $\widetilde{\mathcal{H}}$ is maximal with respect to S. □

In what follows, given a set of hypersequents $\mathcal{H} \cup \{S\}$, let $\mathsf{P}(\mathcal{H}, \mathsf{S})$ denote the set of propositional variables that occur in $\mathcal{H} \cup \{S\}$ and $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))$ the set of positive formulas over $\mathsf{P}(\mathcal{H}, S)$. If additionally $\mathcal{H} \nvdash_{\mathbf{HC}_+} S$ for some hypersequent calculus $\mathbf{HC}_+$, let $\widetilde{\mathcal{H}}$ denote a set of positive hypersequents based on $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))$ that extends $\mathcal{H}$ and is maximal with respect to S in $\mathbf{HC}_+$ (observe that, by Lemma 3.2.10, such a set exists).

Definition 3.2.11. Let $\mathbf{HC}_+$ be a positive hypersequent calculus and $\mathcal{H} \cup \{S\}$ a set positive hypersequents such that $\mathcal{H} \nvdash_{\mathbf{HC}_+} S$. We define a relation $\approx_+$ on $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))$ as follows,

$$\varphi \approx_+ \psi \quad \text{iff} \quad \widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \varphi \Rightarrow \psi \text{ and } \widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \psi \Rightarrow \varphi.$$

For all positive formulas φ, let $[\varphi]$ denote the class of formulas $\{\psi \mid \varphi \approx_+ \psi\}$. $\dashv$

Lemma 3.2.12. *Let $\mathbf{HC}_+$ be a positive hypersequent calculus and $\mathcal{H} \cup \{S\}$ a set positive hypersequents such that $\mathcal{H} \nvdash_{\mathbf{HC}_+} S$.*

(i) The relation $\approx_+$ defines a congruence relation on $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))$.

(ii) The quotient $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))/{\approx_+}$ with constants $[\bot]$ and $[\top]$ and binary operations $\wedge$ and $\vee$ defined by $[\varphi] \wedge [\psi] \coloneqq [\varphi \wedge \psi]$ and $[\varphi] \vee [\psi] \coloneqq [\varphi \vee \psi]$ is a bounded distributive lattice.

(iii) Let $\leq$ denote the partial order associated with the lattice $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))/{\approx_+}$ and let Γ and Δ be finite sets of positive formulas. Then,

$$[\textstyle\bigwedge \Gamma] \leq [\bigvee \Delta] \quad \textit{iff} \quad \widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \Gamma \Rightarrow \Delta.$$

(iv) Let $\Gamma \Rightarrow \Delta$ be a positive sequent based on $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))$ and pick an arbitrary valuation $v : \mathsf{P}(\mathcal{H}, S) \to \mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))/{\approx_+}$. Let σ_v denote the substitution determined by the valuation v. Then,

$$\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \Gamma^{\sigma_v} \Rightarrow \Delta^{\sigma_v} \textit{ if and only if } [\![\textstyle\bigwedge \Gamma]\!]_v \leq [\![\bigvee \Delta]\!]_v.$$

Proof.

(i) It is easy to see that $\approx_+$ defines an equivalence relation on $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))$ since, $\approx_+$ is reflexive, symmetric, and from the cut rule it follows that $\approx_+$ is also transitive. Now, pick $\varphi, \psi, \varphi', \psi' \in \mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))$ such that $\varphi \approx_+ \psi$ and $\varphi' \approx_+ \psi'$. This means that the sequents $\varphi \Rightarrow \psi$, $\psi \Rightarrow \varphi$, $\varphi' \Rightarrow \psi'$, and $\psi' \Rightarrow \varphi'$ are derivable from $\widetilde{\mathcal{H}}$ over $\mathbf{HC}_+$. Then, for some finite subset $\{H_1, ..., H_m, H'_1, ..., H'_m\} \subseteq \widetilde{\mathcal{H}}$ we can construct the following derivations,

$$\dfrac{\dfrac{\dfrac{[H_1]...[H_i]}{\varphi \Rightarrow \psi}\vdots}{\varphi, \varphi' \Rightarrow \psi}(\text{iw}) \quad \dfrac{\dfrac{[H_j]...[H_m]}{\varphi' \Rightarrow \psi'}\vdots}{\varphi, \varphi' \Rightarrow \psi'}(\text{iw})}{\dfrac{\varphi, \varphi' \Rightarrow \psi \wedge \psi'}{\varphi \wedge \varphi' \Rightarrow \psi \wedge \psi'}(\wedge\text{l})}(\wedge\text{r}) \qquad \dfrac{\dfrac{\dfrac{[H'_1]...[H'_i]}{\psi \Rightarrow \varphi}\vdots}{\psi, \psi' \Rightarrow \varphi}(\text{iw}) \quad \dfrac{\dfrac{[H'_j]...[H'_m]}{\psi' \Rightarrow \varphi'}\vdots}{\psi, \psi' \Rightarrow \varphi'}(\text{iw})}{\dfrac{\psi, \psi' \Rightarrow \varphi \wedge \varphi'}{\psi \wedge \psi' \Rightarrow \varphi \wedge \varphi'}(\wedge\text{l})}(\wedge\text{r})$$

Thus, it follows that the sequents $\varphi \wedge \varphi' \Rightarrow \psi \wedge \psi'$ and $\psi \wedge \psi' \Rightarrow \varphi \wedge \varphi'$ are derivable from $\widetilde{\mathcal{H}}$ over $\mathbf{HC}_+$ and so we have $\varphi \wedge \varphi' \approx_+ \psi \wedge \psi'$. The argument for $\varphi \vee \varphi' \approx_+ \psi \vee \psi'$ is analogous. Hence, $\approx_+$ is a congruence relation on $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))$.

(ii) The proof that $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))/{\approx_+}$ is a bounded distributive lattice, is a matter of routine checking that $\mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))/{\approx_+}$ satisfies the equations (L1)-(L5) and (D1) or (D2). We spell out the (D1)-case. Pick arbitrary positive formulas φ, ψ, and χ. Observe that $[\varphi] \wedge ([\psi] \vee [\chi]) = [\varphi \wedge (\psi \vee \chi)]$ and $([\varphi] \wedge [\psi]) \vee ([\varphi] \wedge [\chi]) = [(\varphi \wedge \psi) \vee (\varphi \wedge \chi)]$. We want to show that $[\varphi \wedge (\psi \vee \chi)] = [(\varphi \wedge \psi) \vee (\varphi \wedge \chi)]$ holds. In order to do so, we will show that $\varphi \wedge (\psi \vee \chi) \approx_+ (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$. From Derivation 3.2.1 and Derivation 3.2.2 we know that both $\vdash_{\mathbf{HC}_+} \varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$ and $\vdash_{\mathbf{HC}_+} (\varphi \wedge \psi) \vee (\varphi \wedge \chi) \Rightarrow \varphi \wedge (\psi \vee \chi)$ hold. By weakening, herefrom it follows that we also have $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$ and $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} (\varphi \wedge \psi) \vee (\varphi \wedge \chi) \Rightarrow \varphi \wedge (\psi \vee \chi)$. By the definition of $\approx_+$, this means that we have $\varphi \wedge (\psi \vee \chi) \approx_+ (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$.

(iii) ($\Rightarrow$): Let Γ and Δ be finite sets of positive formulas so that $[\bigwedge \Gamma] \leq [\bigvee \Delta]$. This means that the $[\bigwedge \Gamma] = [\bigwedge \Gamma \wedge \bigvee \Delta]$ and thus, $\bigwedge \Gamma \wedge \bigvee \Delta \approx_+ \bigwedge \Gamma$. Then, by definition of $\approx_+$ we have $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \bigwedge \Gamma \wedge \bigvee \Delta \Rightarrow \bigwedge \Gamma$ and $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \bigwedge \Gamma \Rightarrow \bigwedge \Gamma \wedge \bigvee \Delta$. The following derivation shows that from $\bigwedge \Gamma \Rightarrow \bigwedge \Gamma \wedge \bigvee \Delta$ we can derive $\bigwedge \Gamma \Rightarrow \bigvee \Delta$.

'

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\bigvee \Delta \Rightarrow \bigvee \Delta}\,(\mathrm{ax})}{\bigwedge \Gamma, \bigvee \Delta \Rightarrow \bigvee \Delta}\,(\mathrm{iw})}{\bigwedge \Gamma \wedge \bigvee \Delta \Rightarrow \bigvee \Delta}\,(\wedge \mathrm{l}) \qquad \bigwedge \Gamma \Rightarrow \bigwedge \Gamma \wedge \bigvee \Delta}{\bigwedge \Gamma \Rightarrow \bigvee \Delta}\,(\mathrm{cut})$$

From Lemma 3.2.7 we know that if $\bigwedge \Gamma \Rightarrow \bigvee \Delta$ is derivable, then so is $\Gamma \Rightarrow \Delta$, which is what we wanted to show.

($\Leftarrow$): Now suppose that $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \Gamma \Rightarrow \Delta$. We show that $[\bigwedge \Gamma] \leq [\bigvee \Delta]$. This amounts to showing that $[\bigwedge \Gamma] \wedge [\bigvee \Delta] = [\bigwedge \Gamma]$. That is, that $\bigwedge \Gamma \wedge \bigvee \Delta \approx_+ \bigwedge \Gamma$ holds. Observe that, for $|\Gamma| = n$ and $|\Delta| = m$, by applying ($\wedge$l) n times, and ($\vee$r) m times, we derive $\bigwedge \Gamma \Rightarrow \bigvee \Delta$ from

$\Gamma \Rightarrow \Delta$. Since we also have $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \Gamma \Rightarrow \Delta$, for some finite subset $\{H_1, ..., H_m\} \subseteq \widetilde{\mathcal{H}}$ we can construct the following derivations,

$$\dfrac{\dfrac{\dfrac{}{\bigwedge\Gamma \Rightarrow \bigwedge\Gamma}\,(\mathrm{ax})}{\bigwedge\Gamma, \bigvee\Delta \Rightarrow \bigwedge\Gamma}\,(\mathrm{iw})}{\bigwedge\Gamma \wedge \bigvee\Delta \Rightarrow \bigwedge\Gamma}\,(\wedge\mathrm{l}) \qquad \dfrac{\dfrac{}{\bigwedge\Gamma \Rightarrow \bigwedge\Gamma}\,(\mathrm{ax}) \quad \begin{array}{c}[H_1]...[H_m] \\ \vdots \\ \Gamma \Rightarrow \Delta \\ \vdots \\ \bigwedge\Gamma \Rightarrow \bigvee\Delta\end{array}}{\bigwedge\Gamma \Rightarrow \bigwedge\Gamma \wedge \bigvee\Delta}\,(\wedge\mathrm{r})$$

Hence, we have $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \bigwedge\Gamma \wedge \bigvee\Delta \Rightarrow \bigwedge\Gamma$, and $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \bigwedge\Gamma \Rightarrow \bigwedge\Gamma \wedge \bigvee\Delta$ and thus $\bigwedge\Gamma \wedge \bigvee\Delta \approx_+ \bigwedge\Gamma$.

(iv) ($\Rightarrow$): Suppose $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \Gamma^{\sigma_v} \Rightarrow \Delta^{\sigma_v}$. Observe that, since Γ and Δ are finite, we have $\Gamma = \{\varphi_1, ..., \varphi_n\}$ and $\Delta = \{\psi_1, ..., \psi_m\}$ for some $n, m \in \mathbb{N}$ and hence, $\Gamma^{\sigma_v} = \{\varphi_1^{\sigma_v}, ..., \varphi_n^{\sigma_v}\}$ and $\Delta^{\sigma_v} = \{\psi_1^{\sigma_v}, ..., \psi_m^{\sigma_v}\}$. By (iii) of this lemma, it follows that $[\bigwedge_{i=1}^n \varphi_i^{\sigma_v}] \leq [\bigvee_{j=1}^m \psi_j^{\sigma_v}]$. By an argument of induction on the complexity of φ, it is seen that for all $\varphi \in \Gamma \cup \Delta$ we have $[\varphi^{\sigma_v}] = [\![\varphi]\!]_v$. From (ii) of this lemma we know that $[\bigwedge_{i=1}^n \varphi_i^{\sigma_v}] = \bigwedge_{i=1}^n [\varphi_i^{\sigma_v}]$ and $[\bigvee_{j=1}^m \psi_j^{\sigma_v}] = \bigvee_{j=1}^m [\psi_j^{\sigma_v}]$ and from the definition of a valuation that $\bigwedge_{i=1}^n [\![\varphi_i]\!]_v = [\![\bigwedge_{i=1}^n \varphi_i]\!]_v$ and $\bigvee_{j=1}^m [\![\psi_j]\!]_v = [\![\bigvee_{j=1}^m \psi_j]\!]_v$. Herefrom it follows that $[\![\bigwedge_{i=1}^n \varphi_i]\!]_v \leq [\![\bigvee_{j=1}^m \psi_j]\!]_v$ and so $[\![\bigwedge\Gamma]\!]_v \leq [\![\bigvee\Delta]\!]_v$.

($\Leftarrow$): Now suppose $\widetilde{\mathcal{H}} \nvdash_{\mathbf{HC}_+} \Gamma^{\sigma_v} \Rightarrow \Delta^{\sigma_v}$. By an analogous argument as above, it follows that $[\![\bigwedge\Gamma]\!]_v \nleq [\![\bigvee\Delta]\!]_v$.

□

Lemma 3.2.13 ([8, Thm. 2.5]). *Let* $\mathbf{HC}_+$ *be a positive hypersequent calculus and* $\mathcal{H}/S$ *a positive hypersequent rule such that* $\mathcal{H} \nvdash_{\mathbf{HC}_+} S$. *There exists a bounded distributive lattice* $\mathcal{L}_+(\mathcal{H}, S)$ *that validates all the rules and axioms of* $\mathbf{HC}_+$ *but does not validate* $\mathcal{H}/S$.

Proof. Define a bounded distributive $\mathcal{L}_+(\mathcal{H}, S) := \mathsf{Form}_+(\mathsf{P}(\mathcal{H}, S))/\approx_+$, as constructed above. From Lemma 3.2.12 (ii) we know that $\mathcal{L}_+(\mathcal{H}, S)$ is a bounded distributive lattice. We show that it validates $\mathbf{HC}_+$ and that there exists a valuation v that makes all the hypersequents $H \in \mathcal{H}$ true but not S.

First, observe that for all positive hypersequents $\alpha_1 \mid ... \mid \alpha_n$ it holds that,

$$\text{If } \widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \alpha_1 \mid ... \mid \alpha_n \mid S \text{ then } \widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \alpha_i, \text{ for some } i \leq n.$$

Suppose this would not be the case. That is, that both $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \alpha_1 \mid ... \mid \alpha_n \mid S$ and $\widetilde{\mathcal{H}} \nvdash_{\mathbf{HC}_+} \alpha_i$, for all $i \leq n$ hold. This means that we would have $\alpha_i \notin \widetilde{\mathcal{H}}$, and thus by the maximality of $\widetilde{\mathcal{H}}$, also $\widetilde{\mathcal{H}}, \alpha_i \vdash_{\mathbf{HC}_+} S$, for all $i \leq n$. From $\widetilde{\mathcal{H}}, \alpha_1 \vdash_{\mathbf{HC}_+} S$ it follows by (ew) that $\widetilde{\mathcal{H}}, \alpha_1 \vdash_{\mathbf{HC}_+} \alpha_2 \mid ... \mid \alpha_n \mid S$. Together with $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \alpha_1 \mid \alpha_2 \mid ... \mid \alpha_n \mid S$ by Lemma 3.2.3 this implies $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \alpha_2 \mid ... \mid \alpha_n \mid S$. By applying this argument $n-1$ more times, we obtain $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} S$, which contradicts the maximality of $\widetilde{\mathcal{H}}$ with respect to S. Hence, it must be that $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \alpha_i$ for some $i \leq n$.

Now, let $\mathcal{G}/G$ be an arbitrary rule in $\mathbf{HC}_+$ with $\mathcal{G} \coloneqq \{G_1, ..., G_n\}$ and v a valuation that makes all $G_i \in \mathcal{G}$ true. Then, for all $i \leq n$ we have $[\![\bigwedge \Gamma]\!]_v \leq [\![\bigvee \Delta]\!]_v$ for some component $\Gamma \Rightarrow \Delta$ of G_i. Let σ_v denote the substitution determined by v. From Lemma 3.2.12 (iv) it follows that $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \Gamma^{\sigma_v} \Rightarrow \Delta^{\sigma_v}$. This means that for all $G_i \in \mathcal{G}$ we have $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} G_i^{\sigma_v}$. Now, since $\mathcal{G}/G$ is a rule from $\mathbf{HC}_+$, it must be that also $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} G^{\sigma_v}$. Then, by applying weakening it follows that $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} G^{\sigma_v} \mid S$ and thus, from our observation above, that $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \Gamma^{\sigma_v} \Rightarrow \Delta^{\sigma_v}$ for some component $\Gamma^{\sigma_v} \Rightarrow \Delta^{\sigma_v}$ of G^{σ_v}. By Lemma 3.2.12 (iv) again, it follows that $[\![\bigwedge \Gamma]\!]_v \leq [\![\bigvee \Delta]\!]_v$ and thus, since v makes one of the components of the hypersequent G true, v also makes G true. Hence, $\mathcal{L}_+(\mathcal{H}, S)$ validates $\mathbf{HC}_+$.

We now construct a valuation v on $\mathcal{L}_+(\mathcal{H}, S)$ such that all the hypersequents $H \in \mathcal{H}$ are true in $\mathcal{L}_+(\mathcal{H}, S)$ under v but the hypersequent S is not. Let the valuation $v : \mathsf{P}(\mathcal{H}, S) \to \mathcal{L}_+(\mathcal{H}, S)$ be defined by $p \mapsto [p]$. Then, since $\approx_+$ is a congruence, v extends to positive formulas φ by $[\![\varphi]\!]_v = [\varphi]$. Pick $\alpha_1 \mid ... \mid \alpha_n \in \mathcal{H}$. Since $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$, we have $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \alpha_1 \mid ... \mid \alpha_n$. By (ew) we obtain $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \alpha_1 \mid ... \mid \alpha_n \mid S$ and thus, by our observation above we have $\widetilde{\mathcal{H}} \vdash_{\mathbf{HC}_+} \alpha_i$, for some $i \leq n$. Now, the since α_i is a positive sequent, it is of the form $\Gamma \Rightarrow \Delta$ for some finite sets of positive formulas Γ and Δ. By Lemma 3.2.12 (iii) it follows that $[\bigwedge \Gamma] \leq [\bigvee \Delta]$ and, by our definition of v this means that $[\![\bigwedge \Gamma]\!]_v \leq [\![\bigvee \Delta]\!]_v$. Thus, the sequent α_i is true under v and so is the hypersequent $\alpha_1 \mid ... \mid \alpha_n$. Hence, v makes all $H \in \mathcal{H}$ true. Now, since $\widetilde{\mathcal{H}} \nvdash_{\mathbf{HC}_+} S$, for all components $\Gamma \Rightarrow \Delta$ of S we have that $\widetilde{\mathcal{H}} \nvdash_{\mathbf{HC}_+} \Gamma \Rightarrow \Delta$. By Lemma 3.2.12 (iii) it must be that $[\bigwedge \Gamma] \not\leq [\bigvee \Delta]$. This means that no component of S is true under v, so v does not make the hypersequent S true. □

Theorem 3.2.14 (Algebraic soundness and completeness [8, Thm. 2.5])**.** *Let* $\mathbf{HC}_+$ *be a positive hypersequent calculus and let* $\mathcal{H}/S$ *be a positive hypersequent rule. Then,*

$$\mathcal{H} \vdash_{\mathbf{HC}_+} S \textit{ iff } \mathcal{H} \vDash_{\mathbf{HC}_+} S$$

Proof. The left-to-right direction is established by 3.2.8. Its converse follows immediately from Lemma 3.2.13. □

3.3 The sequent calculus $\mathbf{SC}_+$

Let $\mathbf{SC}_+$ denote the sequent calculus consisting of the axioms, logical rules, internal structural rules, and the cut rule of $\mathbf{PC}_+$, all with the side-hypersequents dropped. We show that $\mathbf{SC}_+$ and $\mathbf{PC}_+$ are equivalent with respect to derivability of sequent rules. That is, that $\mathbf{PC}_+$ derives a sequent S, (i.e., a hypersequent with only one component), if and only if the sequent calculus $\mathbf{SC}_+$ derives S. Thereafter, we motivate the adoption of $\mathbf{PC}_+$ by showing that every consistent sequent rule is already derivable in $\mathbf{SC}_+$ and hence, that there are no non-trivial extensions of the calculus $\mathbf{SC}_+$.

Theorem 3.3.1 (Cf., [14, Prop. 5.3])**.** *For all positive sequent rules (r), (r) is derivable in $\mathbf{PC}_+$ if and only if (r) is derivable in $\mathbf{SC}_+$.*

Proof. By induction on derivation length. □

Corollary 3.3.2 (Algebraic soundness and completeness of $\mathbf{SC}_+$ for sequents)**.** *Let (r) be a positive sequent. Then,*

$$\vdash_{\mathbf{SC}_+} (r) \textit{ iff } \vDash_{\mathsf{BDL}} (r). \qquad \dashv$$

In order to show that, if a nontrivial bounded distributive lattice D validates a positive sequent rule, then all bounded distributive lattices do, we first recall some concepts and results from universal algebra (see e.g., [10]). A *congruence relation* θ on an algebra A is an equivalence relation that is compatible with the structure A. For a given algebra A, we let $\mathsf{Con}A$ denote the set of all congruences on A and $\triangle_A$ the diagonal relation $\{\langle a, a\rangle \mid a \in A\}$ on A. The *direct product of a family* $(A_i)_{i\in\mathcal{I}}$ *of algebras of type* $\mathcal{F}$ is the algebra that has as underlying set the Cartesian product $A = \prod_{i\in\mathcal{I}} A_i$, and for each n and each n-ary operation symbol $f \in \mathcal{F}$, for all $a_1, ..., a_n \in A$, $f^A(a_1, ..., a_n)$ is defined $f^{A_i}(a_1(i), ..., a_n(i))$ for $i \in \mathcal{I}$, i.e., f^A in A is defined coordinate-wise. For each $i \in \mathcal{I}$, we define the i^{th} *projection map* $\pi_i : A \to A_i$ by $\pi_i(a) = a(i)$. An algebra A is called a *subdirect product* of an indexed family $(A_i)_{i\in\mathcal{I}}$ of algebras if A is a subalgebra of $\Pi_{i\in\mathcal{I}} A_i$ and $\pi_i(A) = A_i$ for each $i \in \mathcal{I}$. Recall the following characterization of subdirectly irreducible algebras.

Lemma 3.3.3 (See e.g. [2, Thm. I.9.3] and [10, Thm. 8.4])**.** *an algebra A is* subdirectly irreducible *if and only if A is trivial or there is a minimum congruence in* $\bigcap(\mathsf{Con}A \setminus \{\triangle_A\})$.

Let **2** denote the two-element bounded distributive lattice $\{0, 1\}$. Given $D \in \mathsf{BDL}$, for $a \in D$, we note that both $\theta(\uparrow a) \coloneqq \{(b, c) \in D^2 \mid b \wedge a = c \wedge a\}$ and $\theta(\downarrow a) \coloneqq \{(b, c) \in D^2 \mid b \vee a = c \vee a\}$ are well-defined congruence relations on D. The following lemma shows that **2** determines all and only the subdirectly irreducible members of BDL.

Lemma 3.3.4 ([2, Thm. II.10.1])**.** *Every nontrivial $D \in \mathsf{BDL}$ can be represented as a subdirect product of copies of* **2**.

Proof. Since **2** has only two congruence relations, clearly, **2** is subdirectly irreducible. For the converse, towards a contradiction, suppose that D is subdirectly irreducible and that $D > 2$. Then there exist distinct elements $a < c < b \in D$. Observe that $\theta(\uparrow c) \cap \theta(\downarrow c) = \triangle$. For any $(d_1, d_2) \in \theta(\uparrow c) \cap \theta(\downarrow c)$ by distributivity of D it follows that $d_1 = d_2$, thus $(d_1, d_2) \in \triangle$. Since we assumed D to be subdirectly irreducible by Theorem 3.3.3, the set $\bigcap(\mathsf{Con}D \setminus \{\triangle_D\})$ has a minimum congruence and thus, it must be that $\theta(\uparrow c) = \triangle$ or $\theta(\downarrow c) = \triangle$. This cannot be, since for distinct a, b, c we have $(a, c) \in \theta(\uparrow c)$ and $(b, c) \in \theta(\downarrow c)$. We therefore see that for every nontrivial $D \in \mathsf{BDL}$, D is subdirectly irreducible if and only if $D = \mathbf{2}$. By Birkhoff's Theorem (see e.g., [10, Thm. 8.6] and [2, Thm. I.10.4]) we know that every algebra D is isomorphic to a subdirect product of subdirectly irreducible algebras that are homomorphic images of D . For bounded distributive lattices in particular, it hereby follows that every bounded distributive lattice can be represented as a subdirect product of copies of **2**. □

Thus, in order to show that a property holds for every $D \in \mathsf{BDL}$, it suffices to show that the property holds for the subdirectly irreducible members of BDL, and that it is preserved under the formation of subalgebras and direct products. We recall from Remark 3.2.6 that positive sequent rules correspond to *quasi-equations*, expressions of the form $(s_1 \approx t_1 \text{ and ... and } s_n \approx t_n) \implies s_0 \approx t_0$, where s_i and t_i are terms, for all $i \leq n$. Hence, to show that a sequent rule (r) is valid in a nontrivial $D \in \mathsf{BDL}$ if and only if it is valid in all $D \in \mathsf{BDL}$, it suffices to show that quasi-equations are preserved under the formation of subalgebras (by which **2** validates (r)) and direct products (since **2** validates (r), by which all $D \in \mathsf{BDL}$ validate (r)).

Lemma 3.3.5 (See e.g. [37, Thm. 1.2.19])**.** *Quasi-equations are preserved under the formation of subalgebras and direct products.*

Corollary 3.3.6. *For every positive sequent rule (r), either the rule (r) is inconsistent or (r) is valid on all bounded distributive lattices.*

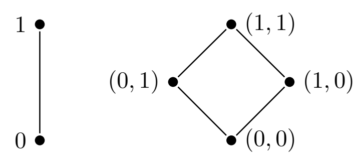
Proof. Assuming (r) is not inconsistent, there exists a non-trivial $D \in \mathsf{BDL}$ validating it. Since $\mathbf{2}$ is a sublattice of D, by Lemma 3.3.5 it follows that $\mathbf{2}$ validates (r). Since quasi-equations are preserved under the formation of direct products, every power of $\mathbf{2}$ also validates (r). By Lemma 3.3.4, any bounded distributive lattice is a subdirect product of two-element algebras and so in particular, a bounded sublattice of a power of $\mathbf{2}$. Therefore, every bounded distributive lattice validates (r). $\square$

Theorem 3.3.7 (Cf., [14, Cor. 7.2])**.** *For every positive sequent rule (r), either the rule (r) is derivable in* $\mathbf{SC}_+$ *or* $\mathbf{SC}_+ \cup \{(r)\}$ *derives every positive sequent in* $\mathbf{SC}_+$*.*

Proof. Let (r) be a positive sequent rule. By Corollary 3.3.6 we know that (r) is inconsistent or (r) is valid on all bounded distributive lattices. Observe that by completeness (Theorem 3.3.2), if the rule (r) is inconsistent then $\mathbf{SC}_+ \cup \{(r)\}$ derives all sequent rules. If (r) is not inconsistent, it is valid on all distributive lattices, which means that $\vDash_{\mathsf{BDL}} (r)$. By completeness of $\mathbf{SC}_+$ with respect to BDL it follows that $\vdash_{\mathbf{SC}_+} (r)$. $\square$

Observe that, unlike sequent rules, proper hypersequent rules are not preserved under the formation of direct products, hence, we cannot conclude that every bounded distributive lattice validates a positive hypersequent rule whenever $\mathbf{2}$ does. Consider the hypersequent rule (ρ_{lc}) and the bounded distributive lattice product $\mathbf{2} \times \mathbf{2}$ given in Table 3.1. A bounded distributive lattice D validates the rule (ρ_{lc}) if and only if $\forall a_1, b2, a_2, b_1 \in D$ $a_1 \leq b_2$ and $a_2 \leq b_1$ implies that $a_1 \leq b_1$ or $a_2 \leq b_2$. Then, clearly $\mathbf{2}$ validates ρ_{lc} but the product $\mathbf{2} \times \mathbf{2}$ does not since $(0,1) \leq (0,1)$ and $(1,0) \leq (1,0)$ but $(0,1) \nleq (1,0)$ and $(1,0) \nleq (0,1)$.
Hence, we can have non-trivial positive hypersequent rules even though we cannot have any such sequent rules.

$$(\rho_{\mathrm{lc}})\ \frac{G \mid \phi_1 \Rightarrow \psi_2 \qquad G \mid \phi_2 \Rightarrow \psi_1}{G \mid \phi_1 \Rightarrow \psi_1 \mid \phi_2 \Rightarrow \psi_2}$$



$\boxed{\mathbf{2}}$ $\boxed{\mathbf{2} \times \mathbf{2}}$

Table 3.1: The positive hypersequent rule (ρ_{lc}) and the product lattice $\mathbf{2} \times \mathbf{2}$.

Chapter 4

Strict implication logic

In this chapter we introduce a formal language $\mathcal{L}_{\rightsquigarrow}$ (a strict implication language) that is an augmentation of the classical language by a binary operator $\rightsquigarrow$. We present a semantics where the language $\mathcal{L}_{\rightsquigarrow}$ is interpreted in Boolean algebras with strict implications that correspond to lattice subordinations. Our aim thereafter, is to define a calculus $\mathbf{BC}_{\rightsquigarrow}$ that is sound an complete with respect to the class BLS of Boolean algebras with lattice subordinations. In order to do so, we build on the results from [5], that introduces a calculus $\mathbf{RC}_{\rightsquigarrow}$[1] that is sound and complete with respect to RSub and shows that the extensions of $\mathbf{RC}_{\rightsquigarrow}$ enhanced with a particular kind of nonstandard rules are sound and complete with respect to subclasses of RSub axiomatised by universal-existential statements.[2] In light hereof, and given that BLS is such a subclass of RSub axiomatised by universal-existential statements, we define a Π_2-rule that we call ρ_{qp} and show that the calculus $\mathbf{BC}_{\rightsquigarrow} = \mathbf{RC}_{\rightsquigarrow} \cup \{\rho_{qp}\}$ is sound and complete with respect to BLS.

4.1 Syntax and semantics

Let Prop be a countably infinite set of propositional variables. We generate the *strict implication language* $\mathcal{L}_{\rightsquigarrow}$ from Prop using the unary connective $\neg$ and the

[1]In [5], the authors use SIC to denote the respective calculus. We refer to SIC as $\mathbf{RC}_{\rightsquigarrow}$ in this thesis to keep a consistent terminology.

[2]We recall that a *universal-existential formula* (also known as Π_2*-statements*, $\forall\exists$*-statements*, or $\forall_2$*-statements*, are first-order formulas of the form $\forall v_1, ...\forall v_n \exists w_1 ...\exists w_m \phi(v, w)$, where ϕ is a quantifier-free formula [28, Sec. 2.4]. Note that, if $m = 0$, then $\forall v_1, ...\forall v_n \exists w_1 ...\exists w_m \phi(v, w)$ is a universal statement. Thus, universal statements are particular universal-existential statements.

binary connectives $\vee$ and $\rightsquigarrow$. The well-formed formulas φ of $\mathcal{L}_{\rightsquigarrow}$, called *strict implication formulas*, are defined as follows,

$$\varphi ::= p \mid \bot \mid \varphi \vee \varphi \mid \neg\varphi \mid \varphi \rightsquigarrow \varphi, \qquad\qquad p \in \mathsf{Prop}.$$

We make use of the standard abbreviations $\top := \neg\bot$, $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$, and $\varphi \to \psi := \neg\varphi \vee \psi$. Additionally, we let $\boxdot\varphi$ abbreviate the formula $\top \rightsquigarrow \varphi$. A *strict implication sequent*, *strict implication hypersequent*, and *strict implication hypersequent rule*, are defined analogous to positive sequents and hypersequents, but build from finite (possibly empty) multisets of strict implication formulas.[3] Now that we have the connective $\to$, we can consider the meaning of a strict implication sequent to be the formula $\bigwedge\Gamma \to \bigvee\Delta$, that is, the conjunction of all the formulas in Γ implies the disjunction of all the formulas in Δ (with the convention that, if Γ is empty, then $\bigwedge\Gamma = \top$ and $\bigvee\Gamma = \bot$). We will drop 'strict implication' in the usage of our terminology where it is clear from the context that we are working with the strict implication language. This should not give rise to confusion.

In the strict implication context, a *substitution* is a function $\sigma : \mathsf{Prop} \to \mathcal{L}_{\rightsquigarrow}$. In the same way as we have done for the positive language, we extend this function to a map $(-)^\sigma : \mathcal{L}_{\rightsquigarrow} \to \mathcal{L}_{\rightsquigarrow}$ from formula to formula, to sets of formulas Γ, and to (hyper)sequents.

We interpret formulas in Boolean algebras with a lattice subordination $(B, \prec)$, where we regard $(B, \prec)$ as an algebra $(B, 1, \vee, \neg, \rightsquigarrow)$. Recall from section 2.2 that we define the binary operator $\rightsquigarrow$ as,

$$a \rightsquigarrow b := \begin{cases} 1 & \text{if } a \prec b, \\ 0 & \text{otherwise.} \end{cases}$$

Then, a *valuation* v is a map $[\![-]\!]_v : \mathsf{Prop} \to B$. This map is extended recursively to formulas as follows,

$$\begin{array}{rcl} [\![\top]\!]_v & := & 1, \\ [\![\neg\varphi]\!]_v & := & \neg[\![\varphi]\!]_v, \\ [\![\varphi \vee \psi]\!]_v & := & [\![\varphi]\!]_v \vee [\![\psi]\!]_v, \\ [\![\varphi \rightsquigarrow \psi]\!]_v & := & [\![\varphi]\!]_v \rightsquigarrow [\![\psi]\!]_v. \end{array}$$

[3] In this chapter, we will not yet make use of the strict implication hypersequent setting. In the following chapter however, we will translate positive hypersequent rules into strict implication hypersequent rules, which motivates the introduction of strict implication hypersequents in this section.

Let $(B, \prec) \in \mathsf{BLS}$. Analogous to the case of a positive sequent, we say that a strict implication sequent $\Gamma \Rightarrow \Delta$ is *true* in $(B, \prec)$ under a valuation v or that v on $(B, \prec)$ *satisfies* a sequent iff $[\![\bigwedge \Gamma]\!]_v \leq [\![\bigvee \Delta]\!]_v$. We derive the notions of truth of a strict implication formula and hypersequent in the same way as we have done for the positive logic. Likewise, the notion of validity for strict implication formulas, (hyper)sequents, and hypersequent rules, is defined in the same way as in the case of positive logic. We define a *strict implication hypersequent rule* $\mathcal{H}/S$ as we have defined a positive hypersequent rule, with the exception that the set $\mathcal{H} \cup \{S\}$ is a set of strict implication hypersequents. Similar to the case of positive hypersequent rules, we say that $(B, \prec) \in \mathsf{BLS}$ validates a strict implication hypersequent rule $\mathcal{H}/S$ if, for all valuations v, the conclusion S is true under v whenever all the premises in the set $\mathcal{H}$ are true under that valuation v, and write $(B, \prec) \vDash \mathcal{H}/S$ or $\mathcal{H} \vDash_{(B,\prec)} S$. Given an arbitrary subclass K of BLS, we write $\mathcal{H} \vDash_K S$ if for all $(B, \prec) \in K$, we have $\mathcal{H} \vDash_{(B,\prec)} S$.

4.2 The strict implication calculus $\mathbf{BC}_{\rightsquigarrow}$

In this section, we present a deductive system $\mathbf{BC}_{\rightsquigarrow}$ that is strongly sound and complete with respect to the class BLS. We first recall the required concepts and results from [5], which presents the calculus $\mathbf{RC}_{\rightsquigarrow}$ and shows strong soundness and completeness with respect to the universal class RSub and will use $\mathbf{RC}_{\rightsquigarrow}$ as our base calculus.[4] Standard extensions of $\mathbf{RC}_{\rightsquigarrow}$, that is, $\mathbf{RC}_{\rightsquigarrow}$ together with a set of rules $\{\Gamma_i/\phi_i\}_{i\in\mathcal{I}}$ (where $\Gamma \cup \{\phi\}$ is a set of strict implication formulas and Γ possibly empty) correspond to universal subclasses of RSub. However, we are interested in subclasses of RSub that are axiomatised by universal-existential statements, and in particular the subclass BLS axiomatised by adding (QP). By a further result in [5], it is known that the calculus $\mathbf{RC}_{\rightsquigarrow}$ together with so-called Π_2-rules (which are non-standard rules that correspond, as we will see, to Π_2-statements, hence the name) is sound and complete with respect to the inductive subclass K of RSub that validates these corresponding Π_2-statements.[5] After having recalled the required concepts and aforementioned results, we present a specific Π_2-rule that we call ρ_{qp}, and show that the calculus $\mathbf{RC}_{\rightsquigarrow} \cup \{\rho_{qp}\}$ denoted

[4] In [5], the authors use SIC to denote the respective calculus. We will refer to SIC as $\mathbf{RC}_{\rightsquigarrow}$, so as to keep a consistent terminology in this thesis.

[5] A class of algebras is said to be an *inductive class* if it is closed under taking the union of chains (see e.g., [27, Sect. 8.2]). A famous theorem known as the Chang-Łoś-Suszko Theorem, establishes that inductive subclasses are exactly those that can be axiomatised by universal-existential statements (see for instance, [27, Thm. 6.5.9]).

by $\mathbf{BC}_{\rightsquigarrow}$ is sound and complete with respect to BLS.

4.2.1 The calculus $\mathbf{RC}_{\rightsquigarrow}$

Definition 4.2.1 ([5, Def. 4.1])**.** The *strict implication calculus* $\mathbf{RC}_{\rightsquigarrow}$ is the smallest set that contains all the axioms of CPC and the strict implication axioms (A1)-(A8) listed hereafter,

(A1) $(\bot \rightsquigarrow \varphi) \wedge (\varphi \rightsquigarrow \top)$;
(A2) $(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi) \leftrightarrow (\varphi \rightsquigarrow \psi \wedge \chi)$;
(A3) $(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \leftrightarrow (\varphi \vee \psi \rightsquigarrow \chi)$;
(A4) $(\varphi \rightsquigarrow \psi) \to (\varphi \to \psi)$;
(A5) $\boxplus(\varphi \to \psi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \rightsquigarrow \chi)$;
(A6) $(\varphi \rightsquigarrow \psi) \wedge \boxplus(\psi \to \chi) \to (\psi \rightsquigarrow \chi)$;
(A7) $(\varphi \rightsquigarrow \psi) \to (\chi \rightsquigarrow (\varphi \rightsquigarrow \psi))$;
(A8) $\neg(\varphi \rightsquigarrow \psi) \to (\chi \rightsquigarrow \neg(\varphi \rightsquigarrow \chi))$,

and is closed under the following inference rules,

$$\text{(MP)}\ \frac{\varphi \qquad \varphi \to \psi}{\psi} \qquad\qquad \text{(R)}\ \frac{\varphi}{\boxplus\varphi} \qquad\qquad \dashv$$

In [5, Thm. 4.4] it is established that the derivation system $\mathbf{RC}_{\rightsquigarrow}$ is strongly sound and complete with respect to the class RSub considered as a universal class of strict implication algebras. That is, for a set of formulas Γ and a formula ϕ,

$$\Gamma \vdash_{\mathbf{RC}_{\rightsquigarrow}} \phi \text{ if and only if } \Gamma \vDash_{\mathsf{RSub}} \phi.$$

Thus, for any set of strict implication inference rules $\{\Gamma_i/\phi_i\}_{i\in\mathcal{I}}$ (with Γ_i possibly empty), the extension of the calculus $\mathbf{RC}_{\rightsquigarrow}$ by $\{\Gamma_i/\phi_i\}_{i\in\mathcal{I}}$ corresponds to the universal subclass K of RSub, defined $K := \{D \in \mathsf{RSub} \mid \Gamma_i \vDash_D \phi_i, i \in \mathcal{I}\}$.

4.2.2 Adding Π_2-rules to $\mathbf{RC}_{\rightsquigarrow}$

In this section, following [5], we present so-called Π_2-rules and soundness and completeness of $\mathbf{RC}_{\rightsquigarrow}$ extended by Π_2-rules with respect to inductive subclasses of RSub. We define a particular Π_2-rule that we call ρ_{qp} and show that $\mathbf{RC}_{\rightsquigarrow}$ together with ρ_{qp} corresponds to the inductive subclass BLS.

Definition 4.2.2 (Π_2-rule [5, Def. 5.1])**.** Let F and G be formulas, χ be a formula, $\overline{\varphi}$ a tuple of formulas, and $\overline{p}$ a tuple of propositional letters. A derivation rule is called a Π_2-rule if it is of the form,

$$(\rho)\ \frac{F(\overline{\varphi}, \overline{p}) \to \chi}{G(\overline{\varphi}) \to \chi}$$

We associate the following universal-existential first-order formula Φ_ρ with the rule ρ,

$$\Phi_\rho \coloneqq \forall \overline{x}, z(G(\overline{x}) \not\leq z \to \exists \overline{y} : F(\overline{x}, \overline{y}) \not\leq z). \qquad \dashv$$

The Π_2-rules defined above are non-standard in virtue of their application. Namely, they are subject to the side-condition that the propositional letters $\overline{p}$ cannot occur in the history of the derivation, a notion made precise in the definition of proof given below. The rule ρ_{qp} together with the corresponding Φ_{qp} formula given in Table 4.1 is a particular example of a Π_2-rule that we will use to extend the calculus $\mathbf{RC}_{\rightsquigarrow}$ later on in this chapter.

$(\rho_{qp})\ \dfrac{((p \rightsquigarrow p) \wedge (\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi)) \to \chi}{\varphi \rightsquigarrow \psi \to \chi}$
$\Phi_{qp} \coloneqq \forall abd(a \rightsquigarrow b \not\leq d \to \exists c((c \rightsquigarrow c) \wedge (a \rightsquigarrow c) \wedge (c \rightsquigarrow b) \not\leq d))$

Table 4.1: The rule ρ_{qp} and formula Φ_{qp}.

Let $\{\rho_i\}_{i \in \mathcal{I}}$ denote a collection of Π_2-rules, such that, for all $i \in \mathcal{I}$,

$$(\rho_i)\ \frac{F_i(\overline{\varphi}, \overline{p}) \to \chi}{G_i(\overline{\varphi}) \to \chi}$$

whereof the tuples $\overline{\varphi}$ and $\overline{p}$ may vary in length. By an extended strict implication calculus $\mathbf{RC}_{\rightsquigarrow} + \{\rho_i\}_{i \in \mathcal{I}}$ we denote the calculus $\mathbf{RC}_{\rightsquigarrow}$ together with $\{\rho_i\}_{i \in \mathcal{I}}$. Then, a $\mathbf{RC}_{\rightsquigarrow} + \{\rho_i\}_{i \in \mathcal{I}}$ *proof* is a finite sequence of strict implication formulas $\psi_1, ..., \psi_n$. And, a formula ψ_k for $k \leq n$ is said to be an assumption of the proof unless one of the following conditions holds,

(i) there exists a substitution σ such that $\psi_k = \varphi^\sigma$, for some axiom φ of $\mathbf{RC}_{\rightsquigarrow}$;

(ii) ψ_k follows from ψ_i, ψ_j for $i, j < k$ by an application of the rule (MP);

(iii) ψ_k follows from ψ_i for $i < k$ by and application of the rule (R);

(iv) for some tuple of formulas $\overline{\varphi}$ and formula χ, for some $i \in \mathcal{I}$, we have that $\psi_k := G_i(\overline{\varphi}) \to \chi$ such that there exists $j < k$ for which $\psi_j := F_i(\overline{\varphi}, \overline{p}) \to \chi$, satisfying the condition that the propositional letters $\overline{p}$ do not occur in $\overline{\varphi}$, χ, nor in any of the assumptions φ_m for $m \leq k$.

If Γ' denotes the set of assumptions of a proof $\psi_1, ..., \psi_n$, then, for all sets of formulas Γ such that $\Gamma \subseteq \Gamma'$ we say that $\psi_1, ..., \psi_n$ *is a proof over* $\mathbf{RC}_{\rightsquigarrow} + \{\rho_i\}_{i\in\mathcal{I}}$ *for* $\Gamma \vdash_{\{\rho_i\}_{i\in\mathcal{I}}} \psi_n$. If $\psi_1, ..., \psi_n$ has no assumptions we say that ψ_n is a theorem of $\mathbf{RC}_{\rightsquigarrow} + \{\rho_i\}_{i\in\mathcal{I}}$, and write $\vdash_{\{\rho_i\}_{i\in\mathcal{I}}} \psi_n$. We say that a formula ϕ is *derivable or provable from* Γ *over* $\mathbf{RC}_{\rightsquigarrow} + \{\rho_i\}_{i\in\mathcal{I}}$ if there exists a proof for $\Gamma \vdash_{\{\rho_i\}_{i\in\mathcal{I}}} \varphi$.

Theorem 4.2.3 ([5, Thm. 5.5])**.** *Let* $\{\rho_i\}_{i\in\mathcal{I}}$ *be a set of* Π_2*-rules. Then* $\mathbf{RC}_{\rightsquigarrow} + \{\rho_i\}_{i\in\mathcal{I}}$ *is strongly sound and complete with respect to the inductive subclass* K *of* RSub *axiomatised by the statements* $\{\Phi_{\rho_i}\}_{i\in\mathcal{I}}$.

Lemma 4.2.4. *Let* $(B, \prec) \in \mathsf{RSub}$*. Then* $(B, \prec)$ *satisfies the axiom (QP) if and only if it satisfies the first-order formula* Φ_{qp}.

Proof.
($\Rightarrow$): Suppose that $(B, \prec)$ satisfies (QP) and pick $a, b, d \in B$ such that $a \rightsquigarrow b \nleq d$. Then it must be that $d \neq 1$ and $a \rightsquigarrow b \neq 0$ and so $a \rightsquigarrow b = 1$. Thus, $a \prec b$ and by (QP) there exists $c \in B$ for which $c \prec c$ and $a \leq c \leq b$. Now, since $(B, \prec)$ satisfies (B5), from $c \prec c$ it follows that $c \leq c$. Then, from $a \leq c \prec c \leq c$ and $c \leq c \prec c \leq b$ by (B4) we obtain $a \prec c$ and $c \prec b$. Then $c \rightsquigarrow c = a \rightsquigarrow c = c \rightsquigarrow b = 1$ and so $(c \rightsquigarrow c) \wedge (a \rightsquigarrow c) \wedge (c \rightsquigarrow b) = 1 \nleq d$. Hence, $(B, \prec)$ satisfies Φ_{qp}.

($\Leftarrow$): Suppose that $(B, \prec)$ satisfies Φ_{qp} and pick $a, b \in B$ such that $a \prec b$. Then $a \rightsquigarrow b = 1$. Now, either $1 \nleq 0$, or $(B, \prec)$ is the trivial RSub-algebra and $1 = 0$. If $1 \nleq 0$, then by Φ_{qp} there exists $c \in B$ such that $(c \rightsquigarrow c) \wedge (a \rightsquigarrow c) \wedge (c \rightsquigarrow b) \nleq 0$. Then it must be that $c \rightsquigarrow c = a \rightsquigarrow c = c \rightsquigarrow b = 1$, otherwise the meet would equal zero. This means that we also have $c \prec c$, $a \prec c$, and $c \prec b$. Now, since $(B, \prec)$ satisfies (B5), it follows that $a \leq c$ and $c \leq b$. Thus, $(B, \prec)$ satisfies (QP). If $(B, \prec)$ is the trivial RSub-algebra, then $a = 1 = b$ and so $a \leq 1 \leq b$. Since by (B1) we have $1 \prec 1$, in this case $(B, \prec)$ also satisfies (QP). $\square$

Definition 4.2.5. We let $\mathbf{BC}_{\rightsquigarrow}$ denote the calculus $\mathbf{RC}_{\rightsquigarrow} \cup \{\rho_{qp}\}$. $\dashv$

By Lemma 4.2.4, we see that the RSub-algebras that validate ρ_{qp} are precisely the BLS-algebras. Therefore, from Theorem 4.2.3 and Lemma 4.2.4, we conclude that $\mathbf{BC}_{\rightsquigarrow}$ is sound and complete with respect to BLS.

Theorem 4.2.6. *Let Γ be a set of formulas and ϕ a formula,*

$$\Gamma \vdash_{\mathbf{BC}_{\rightsquigarrow}} \phi \text{ if and only if } \Gamma \vDash_{\mathsf{BLS}} \phi.$$

In this chapter, we have introduced the calculus $\mathbf{BC}_{\rightsquigarrow}$ and showed that it is sound and complete with respect to the class of Boolean algebras with a lattice subordination. In the previous chapter, we have introduced a positive hypersequent calculus $\mathbf{PC}_{+}$ which is sound and complete with respect to the class of bounded distributive lattices. Moreover, we have seen that there is a correspondence between bounded distributive lattices and Boolean algebra with a lattice subordination. Specifically, for each Boolean algebra with a lattice subordination the set of reflexive elements forms a bounded distributive lattice and conversely, for each bounded distributive lattice there exists a suitable Boolean algebra with a lattice subordination whereof the set of reflexive elements forms an isomorphic bounded distributive lattice. In the following chapter, we present the syntactic counterpart of this correspondence by translation $\mathrm{Tr}(-)$ from positive hypersequent rules to strict implication ones and show that it is full and faithfull.

Chapter 5

A translation from $\mathbf{PC}_+$-rules to $\mathbf{BC}_{\rightsquigarrow}$-formulas

In this chapter, we define a translation from positive hypersequent rules to strict implication formulas. In order to do so, we first show that every strict implication hypersequent rule is in fact equivalent to a strict implication formula. That is, we show that for all $(B, \prec) \in \mathsf{BLS}$, we have that $(B, \prec)$ validates a strict implication hypersequent rule if and only if it validates the corresponding strict implication formula. Next, we define a translation $\mathrm{Tr}(-)$ from positive hypersequent rules with single-component premises into strict implication hypersequent rules. We then show the main theorem of this chapter, namely that a positive hypersequent rule is derivable in the calculus $\mathbf{PC}_+$ if and only if the formula that corresponds to the translation of the rule is derivable in the calculus $\mathbf{BC}_{\rightsquigarrow}$. In other words, $\mathrm{Tr}(-)$ defines an embedding of the positive logic into a strict implication system based on classical logic. We spell out the relation between the translation $\mathrm{Tr}(-)$ and the Gödel-McKinsey-Tarski translation. We conclude this chapter by presenting so-called strict implication companions for extensions of the calculus $\mathbf{PC}_+$, as an analogue of modal companions, and present two examples thereof.

5.1 From strict implication rules to strict implication formulas

In this section, we show that every strict implication hypersequent rule corresponds to a strict implication formula. Recall from Theorem 3.2.5 that every positive hypersequent rule is equivalent to a finite set of positive hypersequent rules the premises of which are all single-component hypersequents (i.e., sequents). This means that, for a translation from positive hypersequent rules to strict implication ones, it will suffice to consider hypersequent rules of the aforementioned simpler kind. Moreover, we will see that the premises of a translated hypersequent rule do not have more components then the premises of the initial rule. Thus, given that we will only encounter strict implication rules that come from positive ones, in defining corresponding formulas it also suffices to consider the simpler case of hypersequent rules with single-component premises.

Definition 5.1.1. Let ρ be a strict implication hypersequent rule consisting of the single-component premises $\Gamma_1 \Rightarrow \Delta_1$, ..., $\Gamma_m \Rightarrow \Delta_m$ and conclusion $\Gamma_{m+1} \Rightarrow \Delta_{m+1} \mid ... \mid \Gamma_n \Rightarrow \Delta_n$. We define a strict implication formula $\phi_\rho^{\rightsquigarrow}$ that corresponds to ρ as follows.

$$\phi_\rho^{\rightsquigarrow} \coloneqq \bigwedge_{i=1}^{m} \boxdot(\bigwedge \Gamma_i \to \bigvee \Delta_i) \to \bigvee_{j=m+1}^{n} \boxdot(\bigwedge \Gamma_j \to \bigvee \Delta_j) \qquad \dashv$$

Lemma 5.1.2. *Let ρ and $\phi_\rho^{\rightsquigarrow}$ be as in Definition 5.1.1. For all $(B, \prec) \in \mathsf{BLS}$,*

$$(B, \prec) \vDash \rho \text{ if and only if } (B, \prec) \vDash \phi_\rho^{\rightsquigarrow}.$$

Proof.
($\Rightarrow$): Suppose $(B, \prec) \in \mathsf{BLS}$ validates ρ and pick a valuation v on $(B, \prec)$. Observe that for any formula ϕ, the formula $\boxdot\phi$ evaluates to 1 if and only if ϕ evaluates to 1 and that $\boxdot\phi$ evaluates to 0 otherwise. Thus, for each $i \le m$ it must be that $[\![\boxdot(\bigwedge \Gamma_i \to \bigvee \Delta_i)]\!]_v \in \{0, 1\}$. Then clearly for their meet, the antecedent of $\phi_\rho^{\rightsquigarrow}$, it also holds that $[\![\bigwedge_{i=1}^m \boxdot(\bigwedge \Gamma_i \to \bigvee \Delta_i)]\!]_v \in \{0, 1\}$.

Assume that $[\![\bigwedge_{i=1}^m \boxdot(\bigwedge \Gamma_i \to \bigvee \Delta_i)]\!]_v = 0$ is the case. Note that for any $a \in B$ we have $0 \to a = 1$. Thus, it follows that $[\![\top]\!]_v \le [\![\phi_\rho^{\rightsquigarrow}]\!]_v$, which means that $\phi_\rho^{\rightsquigarrow}$ is true on $(B, \prec)$ under v. Now suppose that $[\![\bigwedge_{i=1}^m \boxdot(\bigwedge \Gamma_i \to \bigvee \Delta_i)]\!]_v = 1$. Then for each $i \le m$, the equality $[\![\boxdot(\bigwedge \Gamma_i \to \bigvee \Delta_i)]\!]_v = 1$ holds, otherwise their meet would equal 0. And so for each $i \le m$, the equality $[\![\bigwedge \Gamma_i \to \bigvee \Delta_i]\!]_v = 1$ also holds. Now note that, for all $a, b \in B$, we have $a \to b = 1$ if and only if $a \le b$. Then, for all $i \le m$, from $[\![\bigwedge \Gamma_i \to \bigvee \Delta_i]\!]_v = 1$ it follows that $[\![\bigwedge \Gamma_i]\!]_v \le [\![\bigvee \Delta_i]\!]_v$. This means that all the premises of ρ are true under v. Since $(B, \prec)$ validates ρ, it must be that the conclusion of ρ is also true under v. Hence, for some j with $m + 1 \le j \le n$ we have $[\![\bigwedge \Gamma_j]\!]_v \le [\![\bigvee \Delta_j]\!]_v$ and so

$[\![\bigwedge\Gamma_j \to \bigvee\Delta_j]\!]_v = 1$. Then $[\![\boxminus(\bigwedge\Gamma_j \to \bigvee\Delta_j)]\!]_v$ also evaluates to 1 and therefrom it follows that $[\![\bigvee_{j=m+1}^{n}\boxminus(\bigwedge\Gamma_j \to \bigvee\Delta_j)]\!]_v = 1$. Since both the antecedent and consequent of $\phi_\rho^{\rightsquigarrow}$ evaluate to 1 under v, we have $[\![\top]\!]_v \leq [\![\phi_\rho^{\rightsquigarrow}]\!]_v$. Thus, if $(B, \prec)$ validates ρ, it also validates $\phi_\rho^{\rightsquigarrow}$.

($\Leftarrow$): Now suppose that $(B, \prec) \in \mathsf{BLS}$ is such that $(B, \prec) \vDash \phi_\rho^{\rightsquigarrow}$ and pick a valuation v on $(B, \prec)$ such that $[\![\bigwedge\Gamma_i]\!]_v \leq [\![\bigvee\Delta_i]\!]_v$ for all $i \leq m$. Then for all $i \leq m$ we have $[\![\bigwedge\Gamma_i \to \bigvee\Delta_i]\!]_v = 1$ and so, $[\![\bigwedge_{i=1}^{m}\boxminus(\bigwedge\Gamma_i \to \bigvee\Delta_i)]\!]_v = 1$. Thus, v makes the antecedent of $\phi_\rho^{\rightsquigarrow}$ true. Now, since $(B, \prec)$ validates $\phi_\rho^{\rightsquigarrow}$, it must be that its consequent $[\![\bigvee_{j=m+1}^{n}\boxminus(\bigwedge\Gamma_j \to \bigvee\Delta_j)]\!]_v = 1$ otherwise, $\phi_\rho^{\rightsquigarrow}$ would not evaluate to 1 under v. Recall that for every formula ϕ we have $[\![\boxminus\phi]\!]_v \in \{0, 1\}$. Then, since $[\![\bigvee_{j=m+1}^{n}\boxminus(\bigwedge\Gamma_j \to \bigvee\Delta_j)]\!]_v = 1$, at least one of the conjuncts should evaluate to 1. Hence, there must be a j with $m+1 \leq j \leq n$ such that $[\![\bigwedge\Gamma_j \to \bigvee\Delta_j]\!]_v = 1$ and so $[\![\bigwedge\Gamma_j]\!]_v \leq [\![\bigvee\Delta_j]\!]_v$. Thus, v makes the conclusion of ρ true and so $(B, \prec)$ validates ρ. □

5.2 From positive rules to strict implication formulas

In this section, we define a translation Tr(−) that translates positive hypersequent rules into strict implication sequent rules. Thereafter, we establish that a positive rule is derivable in $\mathbf{PC}_+$ if and only if the formula $\phi_{\mathrm{Tr}(\rho)}^{\rightsquigarrow}$ that corresponds to the translated rule Tr(ρ) is derivable in the calculus $\mathbf{BC}_{\rightsquigarrow}$.

Definition 5.2.1 (Rule Translation)**.** We define a translation Tr(−) from positive hypersequent rules with only single-sequent premises to strict implication sequent rules as follows. Let (ρ) be an arbitrary positive hypersequent with premises $\Gamma_1 \Rightarrow \Delta_1$,..., $\Gamma_m \Rightarrow \Delta_m$ and conclusion G and let $\{p_i\}_{i \leq n}$ denote the set of propositional letters occurring in ρ. Then Tr(ρ) is defined by,

$$\mathrm{Tr}(\rho)\ \frac{\top \Rightarrow \bigwedge_{i=1}^{n} p_i \rightsquigarrow p_i \qquad \Gamma_1 \Rightarrow \Delta_1 \qquad ... \qquad \Gamma_m \Rightarrow \Delta_m}{G}$$

⊣

Lemma 5.2.2. *Let $(B, \prec) \in \mathsf{BLS}$ and $D_\prec$ be the reflexive elements under $\prec$. Let ρ be an arbitrary D-rule. Then*

$$(B, \prec) \vDash \mathit{Tr}(\rho) \text{ if and only if } D_\prec \vDash \rho.$$

Proof.
($\Rightarrow$): Suppose $(B, \prec)$ validates the rule $\mathrm{Tr}(\rho)$ and pick a valuation $v : \mathsf{Prop} \to D_\prec$ such that $[\![\bigwedge \Gamma_j]\!]_v \leq [\![\bigvee \Delta_j]\!]_v$ for all $j \leq m$. Observe that v also defines a valuation on $(B, \prec)$ since all the elements of $D_\prec$ are contained in B. Since v maps each propositional variable $p \in \mathsf{Prop}$ on an element of $D_\prec$, for all $p \in \mathsf{Prop}$ we have $[\![p]\!]_v \prec [\![p]\!]_v$. In particular, this means that, for all propositional variables p_i occurring in ρ we have $[\![p_i]\!]_v \prec [\![p_i]\!]_v$. Herefrom it follows that $[\![\top]\!]_v \leq [\![p_i]\!]_v \rightsquigarrow [\![p_i]\!]_v = [\![p_i \rightsquigarrow p_i]\!]_v$. Then, v makes the premises of $\mathrm{Tr}(\rho)$ true in $(B, \prec)$ and so it must be that v also makes G true in $(B, \prec)$, and hence in $D_\prec$.

($\Leftarrow$) Suppose that $D_\prec$ validates ρ and pick a valuation v on $(B, \prec)$ such that $[\![\top]\!]_v \leq [\![p_i \rightsquigarrow p_i]\!]_v$ and $[\![\bigwedge \Gamma_j]\!]_v \leq [\![\bigvee \Delta_j]\!]_v$ for all $i \leq n$ and $j \leq m$. Since $[\![\top]\!]_v \leq [\![p_i \rightsquigarrow p_i]\!]_v$ for all propositional variables p_i occurring ρ, it must be that $[\![p_i]\!]_v \prec [\![p_i]\!]_v$ and thus $[\![p_i]\!]_v \in D_\prec$. Observe that for all $j \leq m$, the positive formulas $\bigwedge \Gamma_j$ and $\bigvee \Delta_j$ are build from elements of $D_\prec$, and thus $\bigwedge \Gamma_j, \bigvee \Delta_j \in D_\prec$. Then if we restrict v to $D_\prec$ we have $[\![\bigwedge \Gamma_j]\!]_v \leq [\![\bigvee \Delta_j]\!]_v$ and since $D_\prec$ validates ρ, it must be that v makes G true in $D_\prec$ and subsequently in $(B, \prec)$. □

Lemma 5.2.3. *Let $D \in \mathsf{BDL}$ and $(B_D, \prec_D) \in \mathsf{BLS}$ be such that B_D is the Boolean envelope of D and $\prec_D$ a subordination on B_D as described in Definition 2.4.3. Let ρ be an arbitrary D-rule. Then,*

$$(B_D, \prec_D) \vDash Tr(\rho) \text{ if and only if } D \vDash \rho.$$

Proof. Observe that from Lemma 5.2.2 it follows that $(B_D, \prec_D) \vDash \mathrm{Tr}(\rho)$ if and only if $D_{\prec_D} \vDash \rho$. Since $D_{\prec_D} = D$, it immediately follows that $(B_D, \prec_D) \vDash \mathrm{Tr}(\rho)$ if and only if $D \vDash \rho$. □

The following theorem establishes the relation between positive derivability and strict implication derivability, namely that the positive calculus $\mathbf{PC}_+$ can be embedded into the strict implication calculus $\mathbf{BC}_\rightsquigarrow$.

Theorem 5.2.4. *For any positive hypersequent rule ρ we have,*

$$\vdash_{\mathbf{PC}_+} \rho \text{ if and only if } \vdash_{\mathbf{BC}_\rightsquigarrow} \phi^{\rightsquigarrow}_{Tr(\rho)}.$$

Proof. For both directions, we will prove the contrapositive, namely that $\nvdash_{\mathbf{BC}_\rightsquigarrow} \phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho)}$ if and only if $\nvdash_{\mathbf{PC}_+} \rho$.

($\Rightarrow$): Suppose that $\nvdash_{\mathbf{BC}_\rightsquigarrow} \phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho)}$. By completeness of $\mathbf{BC}_\rightsquigarrow$ with respect to BLS (Theorem 4.2.6) it follows that there exists $(B, \prec) \in \mathsf{BLS}$ so that $(B, \prec) \nvDash \phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho)}$.

By Lemma 5.1.1 this means that $(B, \prec) \nvDash \mathrm{Tr}(\rho)$. Then, given Lemma 5.2.2 it must be that $D_\prec \nvDash \rho$. From soundness of $\mathbf{PC}_+$ with respect to BDL (Theorem 3.2.14) we have that $\nvdash_{\mathbf{PC}_+} \rho$.

($\Leftarrow$): Suppose that $\nvdash_{\mathbf{PC}_+} \rho$. By completeness of $\mathbf{PC}_+$ with respect to BDL (Theorem 3.2.14) it follows that there exists $D \in \mathsf{BDL}$ such that $D \nvDash \rho$. Let B_D denote the free Boolean algebra generated by D and let $\prec_D$ be as in in Definition 2.4.3. From Lemma 5.2.3 it follows that $(B_D, \prec_D) \nvDash \mathrm{Tr}(\rho)$. By Lemma 5.1.1 this means that $(B_D, \prec_D) \nvDash \phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho)}$. Then, by soundness of $\mathbf{BC}_{\rightsquigarrow}$ with respect to BLS (Theorem 4.2.6) we have that $\nvdash_{\mathbf{BC}_{\rightsquigarrow}} \phi^{\rightsquigarrow}_{\mathrm{Tr}}(\rho)$. $\square$

5.3 The connection with the Gödel translation

In this section, we recall the Gödel-McKinsey-Tarski translation, which embeds the intuitionistic propositional logic IPC into the modal expansion $\mathbf{S4}$ of classical propositional logic and show that the translation $\mathrm{Tr}(-)$, in the restricted case, is equivalent to the Gödel-McKinsey-Tarski translation with respect to positive rules.

5.3.1 Intuitionistic logic

By $\mathcal{L}$, we denote the propositional language generated from a countably infinite set of propositional variables Prop using the binary connectives $\wedge$, $\vee$, and $\rightarrow$, and constant $\top$. The well-formed formulas φ of $\mathcal{L}$ are given by the grammar,

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \phi \vee \varphi \mid \varphi \rightarrow \varphi, \qquad p \in \mathsf{Prop}.$$

We make use of the abbreviations $\neg\varphi := \varphi \rightarrow \bot$ and $\top := \neg\bot$. A *(hyper)sequent* and *hypersequent rule* are defined analogous to positive (hyper)sequents and hypersequent rules, but build from finite (possibly empty) multisets of formulas of $\mathcal{L}$. By a *substitution*, we refer to a function $\sigma : \mathsf{Prop} \rightarrow \mathcal{L}$. In the same way as we have done for the positive language, we extend this function to a map $(-)^\sigma : \mathcal{L} \rightarrow \mathcal{L}$ from formula to formula, to sets of formulas Γ, and to (hyper)sequents.

We interpret the formulas of the language $\mathcal{L}$ in Heyting algebras. Given a Heyting algebra H, we define a *valuation* v to be a map $[\![-]\!]_v : \mathsf{Prop} \rightarrow H$ and extend this map to formulas in the standard way. Analogous to the case of a positive sequent, we say that an sequent $\Gamma \Rightarrow \Delta$ is *true* in a Heyting algebra H under a valuation v or that v on H *satisfies* a sequent iff $[\![\bigwedge \Gamma]\!]_v \leq [\![\bigvee \Delta]\!]_v$. We derive the notions of truth of a formula and hypersequent and define the notion of validity for formulas, (hyper)sequents, and hypersequent rules in the same way as in the case of positive logic.

We denote the *intuitionistic propositional calculus* by IPC.

Theorem 5.3.1 (see e.g. [13, Thm. 7.21])**.** IPC *is sound and complete with respect to the class of Heyting algebras.*

5.3.2 Modal logic S4

We generate the *modal language* $\mathcal{ML}$ from a countably infinite set of propositional variables Prop using the binary connectives $\wedge$, $\vee$, and $\rightarrow$, the unary connective $\Box$, and the constant $\top$. The well-formed formulas φ of $\mathcal{ML}$, called *modal formulas*, are given by the grammar,

$$\varphi ::= p \mid \Box\varphi \mid \bot \mid \varphi \wedge \varphi \mid \phi \vee \varphi \mid \varphi \rightarrow \varphi, \qquad p \in \mathsf{Prop}.$$

We make use of the abbreviations $\neg\varphi := \varphi \rightarrow \bot$, $\Diamond\varphi := \neg\Box\neg\varphi$, and $\top := \neg\bot$. Again, *(hyper)sequents* and *hypersequent rules* are defined analogous to positive (hyper)sequents and hypersequent rules, but build from finite (possibly empty) multisets of formulas of $\mathcal{ML}$. A *substitution* is a function $\sigma : \mathsf{Prop} \rightarrow \mathcal{ML}$ that we extend, in the same way as we have done for the positive language, to a map $(-)^\sigma : \mathcal{L} \rightarrow \mathcal{ML}$ from formula to formula, to sets of formulas Γ, and to (hyper)sequents.

Formulas of the language $\mathcal{ML}$ are interpreted in $\mathsf{S4}$-algebras. For any $\mathsf{S4}$-algebra $(B, \Box)$, we define a *valuation* v to be a map $[\![-]\!]_v : \mathsf{Prop} \rightarrow B$. This map is extended to formulas in the usual recursive fashion. As in the case of a positive sequent, we say that a modal sequent $\Gamma \Rightarrow \Delta$ is *true* in an $\mathsf{S4}$-algebra $(B, \Box)$ under a valuation v or that v on $(B, \Box)$ *satisfies* a sequent iff $[\![\bigwedge \Gamma]\!]_v \leq [\![\bigvee \Delta]\!]_v$. We derive the notions of truth of a modal formula and hypersequent in the same way as we have done for the positive logic. Likewise, the notion of validity for modal formulas, (hyper)sequents, and hypersequent rules, is defined in the same way as in the case of positive logic.

Definition 5.3.2. The calculus **S4** is the smallest set that contains all the axioms of CPC and the following axioms,

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
(T) $\Box\varphi \to \varphi$;
(K4) $\Box\varphi \to \Box\Box\varphi$;
and is closed under the rules Modus Ponens and Necessitation. $\dashv$

Theorem 5.3.3. *The calculus* **S4** *is sound and complete with respect to the class of* $\mathsf{S4}$*-algebras.*

Definition 5.3.4 (see, e.g., [13, Sect. 3.9])**.** The *Gödel translation* associates with each well-formed formula φ of $\mathcal{L}$ the modal propositional formula $\mathrm{T}(\varphi)$ which is defined recursively as follows, for all $p \in \mathsf{Prop}$,

$$\begin{aligned} T(p) &\coloneqq \Box p; \\ T(\bot) &\coloneqq \bot; \\ T(\varphi \wedge \psi) &\coloneqq T(\varphi) \wedge T(\psi); \\ T(\varphi \vee \psi) &\coloneqq T(\varphi) \vee T(\psi); \\ T(\varphi \to \psi) &\coloneqq \Box(T(\varphi) \to T(\psi)). \end{aligned}$$ $\dashv$

Theorem 5.3.5 (Gödel-McKinsey-Tarski (see e.g., [13, Thm. 3.83])**.** *For any propositional formula φ, we have,*

$$\vdash_{\mathsf{IPC}} \varphi \textit{ if and only if } \vdash_{\mathsf{S4}} T(\varphi).$$

5.3.3 Relating translation Tr(−) and translation T

To spell out the connection between the translation $\mathrm{Tr}(-)$ and the translation T, we first observe that each positive hypersequent rule ρ is equivalent to a corresponding intuitionistic formula ψ_ρ when evaluated on a Heyting algebra. We recall that each Boolean algebra with a Heyting lattice subordination can be equivalently seen as an $\mathsf{S4}$-algebra. Next, we show that the translation $\mathrm{Tr}(\rho)$ of ρ, when evaluated on Boolean algebras with a Heyting lattice subordination, is equivalent to a a hypersequent rule $\mathrm{Tr}_\Box(\rho)$ based on the modal propositional language. We remark that such modal hypersequent rules $\mathrm{Tr}_\Box(\rho)$ correspond to modal propositional formulas $\psi^\Box_{\mathrm{Tr}(\rho)}$. Thereafter, by soundness and completeness for IPC and **S4** respectively, we conclude that IPC derives ψ_ρ if and only if **S4** derives $\psi^\Box_{\mathrm{Tr}(\rho)}$.

Definition 5.3.6. Let ρ be a positive hypersequent rule with single-component premises $\Gamma_1 \Rightarrow \Delta_1$, ..., $\Gamma_m \Rightarrow \Delta_m$ and conclusion $\Gamma_{m+1} \Rightarrow \Delta_{m+1} \mid ... \mid \Gamma_n \Rightarrow \Delta_n$.

We define a corresponding formula ψ_ρ in the intuitionistic propositional language as follows.

$$\psi_\rho \coloneqq \bigwedge_{i=1}^{m}(\bigwedge \Gamma_i \to \bigvee \Delta_i) \to \bigvee_{j=m+1}^{n}(\bigwedge \Gamma_j \to \bigvee \Delta_j) \qquad \dashv$$

We recall that a Heyting algebra H is said to be *well-connected* if, for all $a, b \in H$ we have that $a \vee b = 1$ implies $a = 1$ or $b = 1$.

Lemma 5.3.7. *Let ρ be a positive hypersequent rule that has only single-component premises and ψ_ρ be as defined above. For any well-connected $H \in \mathsf{Heyt}$,*

$$\vDash_H \rho \textit{ if and only if } \vDash_H \psi_\rho.$$

Proof. It is routine to check that ρ and ψ_ρ are equivalent on each well-connected Heyting algebra. $\square$

Definition 5.3.8. Let ρ be a positive hypersequent rule with premises $\Gamma_1 \Rightarrow \Delta_1$, ..., $\Gamma_m \Rightarrow \Delta_m$ and conclusion $\Gamma_{m+1} \Rightarrow \Delta_{m+1} \mid ... \mid \Gamma_n \Rightarrow \Delta_n$, let $\{p_i\}_{i \leq n}$ denote the set of propositional letters occurring in ρ, and let $\mathrm{Tr}(\rho)$ denote its translation. We define a modal propositional rule $\mathrm{Tr}_\Box(\rho)$ that corresponds to $\mathrm{Tr}(\rho)$ as follows.

$$\mathrm{Tr}(_\Box\rho)\ \frac{\top \Rightarrow \bigwedge_{i=1}^{n} \Box p_i \leftrightarrow p_i \qquad \Gamma_1 \Rightarrow \Delta_1 \qquad ... \qquad \Gamma_m \Rightarrow \Delta_m}{G} \qquad \dashv$$

Lemma 5.3.9. *For all $(B, \prec) \in \mathsf{BLH}$ and each $a \in B$,*

$$a \rightsquigarrow a = 1 \textit{ if and only if } \boxminus a \leftrightarrow a = 1.$$

Proof. Let $(B, \prec) \in \mathsf{BLH}$ and pick $a \in B$. Observe that $a \rightsquigarrow a = 1$ if and only if $a \prec a$. Recall Lemma 1.4.1 item (iv) that $a \prec a$ if and only if $a \leq \boxminus a$. Now, $a \leq \boxminus a$ if and only if $a \to \boxminus a = 1$. Note that for all $a \in B$ we have $\boxminus a \leq a$ and thus $\boxminus a \to a = 1$. Therefore $a \in B$, $a \rightsquigarrow a = 1$ if and only if $\boxminus a \leftrightarrow a = 1$. $\square$

Lemma 5.3.10. *Let ρ be a positive hypersequent rule with only single-component premises, let $Tr(\rho)$ denote its translation and $Tr_\Box(\rho)$ be as defined above. For each $(B, \prec) \in \mathsf{BLH}$ and for each $(B, \Box) \in \mathsf{S4}$,*

(i) $(B, \prec) \vDash Tr(\rho)$ *if and only if* $(B, \boxminus) \vDash Tr_\Box(\rho)$.

(ii) $(B, \Box) \vDash Tr_\Box(\rho)$ *if and only if* $(B, \Box_\prec) \vDash Tr(\rho)$.

Proof. Let $\{p_i\}_{i\leq n}$ denote the set of propositional letters occurring in ρ. Observe that, the rules $\mathrm{Tr}(\rho)$ and $\mathrm{Tr}_\Box(\rho)$ only differ with respect to the premises $\top \Rightarrow \bigwedge_{i=1}^n p_i \rightsquigarrow p_i$ and $\top \Rightarrow \bigwedge_{i=1}^n \Box p_i \leftrightarrow p_i$ in $\mathrm{Tr}(\rho)$ and $\mathrm{Tr}_\Box(\rho)$ respectively. Hence, both items (i) and (ii) of this lemma follow immediately from Lemma 5.3.9. $\square$

Definition 5.3.11. Let ρ be a hypersequent rule based on the modal propositional language with single-component premises $\Gamma_1 \Rightarrow \Delta_1$, ..., $\Gamma_m \Rightarrow \Delta_m$ and conclusion $\Gamma_{m+1} \Rightarrow \Delta_{m+1} \mid ... \mid \Gamma_n \Rightarrow \Delta_n$. We define a corresponding formula $\psi_\rho^\Box$ in the modal propositional language as follows.

$$\psi_\rho^\Box := \Box \bigwedge_{i=1}^{m} (\bigwedge \Gamma_i \to \bigvee \Delta_i) \to \Box \bigvee_{j=m+1}^{n} (\bigwedge \Gamma_j \to \bigvee \Delta_j) \qquad \dashv$$

We recall that an $\mathsf{S4}$-algebra $(B, \Box)$ is *well-connected* if, for all $a, b \in B$, whenever $\Box a \vee \Box b = 1$ it follows that $a = 1$ or $b = 1$ [37, Def. 1.10].

Lemma 5.3.12. *Let ρ be a hypersequent rule based on the modal propositional language with only single-component premises and $\psi_\rho^\Box$ be as defined above. For any well-connected $\mathsf{S4}$-algebra $(B, \Box)$,*

$$\vDash_{(B,\Box)} \rho \textit{ if and only if } \vDash_{(B,\Box)} \psi_\rho^\Box.$$

Proof. It is routine to check that ρ and $\psi_\rho^\Box$ are equivalent on each well-connected $\mathsf{S4}$-algebra. $\square$

Lemma 5.3.13. *For all $(B, \Box) \in \mathsf{S4}$ and all $H \in \mathsf{Heyt}$,*

(i) If $(B, \Box)$ is well-connected, then so is the Heyting algebra $H_{\prec_\Box}$.

(ii) If H is well-connected, then so is the $\mathsf{S4}$-algebra $(B_H, \boxminus_H)$.

Proof. (i) Let $(B, \Box) \in \mathsf{S4}$ be well-connected and pick $a, b \in H_{\prec_\Box}$ such that $a \vee_H b = 1_H$. Since $H_{\prec_\Box}$ is a bounded sublattice of B, we also have $a, b \in B$ and it must be that $a \vee_B b = a \vee_H b = 1_H = 1_B$. Now, since $a, b \in H_{\prec_\Box}$, we have $a \prec_\Box a$ and $b \prec_\Box b$, which means that $a \leq \Box a$ and $b \leq \Box b$. It follows that $a \vee_B b \leq \Box a \vee_B \Box b$, thus $\Box a \vee_B \Box b = 1_B$. Then, since $(B, \Box)$ is well-connected, $a = 1_B = 1_H$ or $b = 1_B = 1_H$. Hence, $H_{\prec_\Box}$ is well-connected.

(ii) Let $H \in \mathsf{Heyt}$ be well-connected and pick $a, b \in (B_H, \boxminus_H)$ such that $\boxminus_H a \vee_{B_H} \boxminus_H b = 1_{B_H}$. Note that $\boxminus_H a \leq \boxminus_H \boxminus_H a$ and $\boxminus_H b \leq \boxminus_H \boxminus_H b$, thus we have $\boxminus_H a \prec_H \boxminus_H a$ and $\boxminus_H b \prec_H \boxminus b$. Recall that $H = H_{\prec_{B_H}}$, so $\boxminus_H a, \boxminus_H b \in H$. Then $\boxminus_H a \vee_H \boxminus_H b = 1_H$, and since H is well-connected, it follows that $\boxminus_H a = 1_H$ or $\boxminus_H b = 1_H$. W.l.o.g., assume that $\boxminus_H a = 1_H$. Then also $\boxminus_H a = 1_B$ and since $\boxminus_H a \leq a$ this means that $a = 1_{B_H}$. Hence, $(B_H, \boxminus_H)$ is well-connected.

□

Theorem 5.3.14. *Let ρ be a positive hypersequent rule. Then*

$$\vdash_{\mathsf{IPC}} \psi_\rho \textit{ if and only if } \vdash_{\mathbf{S4}} \psi^{\Box}_{Tr_\Box(\rho)}.$$

Proof. We show the contrapositive that $\nvdash_{\mathbf{S4}} \psi^{\Box}_{\mathrm{Tr}_\Box(\rho)}$ if and only if $\nvdash_{\mathsf{IPC}} \psi_\rho$.

($\Rightarrow$): Suppose that $\nvdash_{\mathbf{S4}} \psi^{\Box}_{\mathrm{Tr}_\Box(\rho)}$. By completeness of **S4** with respect to the class of well-connected $\mathsf{S4}$-algebras (Theorem 5.3.3) it follows that there exists $(B, \Box) \in \mathsf{S4}$ such that $(B, \Box) \nvDash \psi^{\Box}_{\mathrm{Tr}_\Box(\rho)}$. By Lemma 5.3.12 this means that $(B, \Box) \nvDash \mathrm{Tr}_\Box(\rho)$ and therefrom by Lemma 5.3.10 item (ii) that $(B, \prec_\Box) \nvDash \mathrm{Tr}(\rho)$. Then, given Lemma 5.2.2 it must be that $D_{\prec_\Box} \nvDash \rho$. Now, by Lemma 2.6.12 and Lemma 5.3.13 we know that $D_{\prec_\Box} \in \mathsf{Heyt}$ and that it is well-connected, so by Lemma 5.3.7 $D_{\prec_\Box} \nvDash \psi_\rho$. From soundness of IPC with respect to Heyt (Theorem 5.3.1) we have that $\nvdash_{\mathbf{IPC}} \psi_\rho$.

($\Leftarrow$): Suppose that $\nvdash_{\mathsf{IPC}} \psi_\rho$. By completeness of IPC with respect to the class of well-connected Heyting algebras (Theorem 5.3.1) it follows that there exists $H \in \mathsf{Heyt}$ such that $H \nvDash \rho$. Let B_H denote the free Boolean algebra generated by H and let $\prec_H$ be as in in Definition 2.4.3. From Lemma 2.6.13 it follows that $(B_H, \prec_H) \nvDash \mathrm{Tr}(\rho)$ and by Lemma 5.3.10 item (i) that $(B_H, \prec_{\boxminus_H}) \nvDash \mathrm{Tr}_\Box(\rho)$. By Lemma 5.3.13 and Lemma 5.3.12 this means that $(B_H, \prec_{\boxminus_H}) \nvDash \psi^{\Box}_{\mathrm{Tr}_\Box(\rho)}$. Then, by soundness of **S4** with respect to $\mathsf{S4}$ (Theorem 5.3.3) we have that $\nvdash_{\mathbf{S4}} \psi^{\Box}_{\mathrm{Tr}_\Box(\rho)}$. □

Theorem 5.3.14 above shows that the translation $\mathrm{Tr}(-)$, when restricted to positive rules that are valid on Heyting algebras, is equivalent to the Gödel translation T. Thus, at the syntactic level $\mathrm{Tr}(-)$ translates positive rules into strict implication rules and, whenever a positive rule ρ is valid in the class of Heyting algebras and hence the corresponding intuitionistic formula ψ_ρ is derivable IPC, then the modal formula $\psi^{\Box}_{\mathrm{Tr}_\Box(\rho)}$ that corresponds to $\mathrm{Tr}(\rho)$ is derivable in **S4**.

Vice versa, if **S4** derives the formula $\psi^{\Box}_{\mathrm{Tr}_{\Box}(\rho)}$ then IPC derives ψ_ρ. This connection reflected algebraically by the fact that the category Heyt is a subcategory of BDL and that the category $\mathsf{S4}$ is isomorphic to the subcategory BLH of BLS. We summarize this in the Table 5.1 below.

$$\mathbf{PC}_+ \xhookrightarrow{\mathrm{Tr}(-)} \mathbf{BC}_{\rightsquigarrow}$$

$$\begin{array}{ccccccccc} \mathsf{BDL} & \sim & \mathsf{GBDA} & \cdots & \mathsf{BDA} & \cong & \mathsf{BLS} & & \\ \cup & & \cup & & \cup & & \cup & & \\ \mathsf{Heyt} & \sim & \mathsf{GBHA} & \cdots & \mathsf{BDH} & \cong & \mathsf{BLH} & \cong & \mathsf{S4} \end{array}$$

$$\mathsf{IPC} \xhookrightarrow[\mathrm{T}(-)]{} \mathbf{S4}$$

Table 5.1: Categorical isomorphisms ($\cong$), equivalences ($\sim$), and full subcategories ($\subset$), and embedding of calculi $\hookrightarrow$.

5.4 Strict implication companions of positive calculi

In this chapter, we have seen that the calculus $\mathbf{PC}_+$ can be embedded into the calculus $\mathbf{BC}_{\rightsquigarrow}$. In this section, we look at extensions of both calculi $\mathbf{PC}_+$ and $\mathbf{BC}_{\rightsquigarrow}$ with the objective to develop strict implication analogues of modal companions. We show that every extension of $\mathbf{PC}_+$ is embedded via $\mathrm{Tr}(-)$ into some extension of $\mathbf{BC}_{\rightsquigarrow}$ and that, for every $\mathbf{BC}_{\rightsquigarrow}$ extension there exists a $\mathbf{PC}_+$ extension that is embeddable in it via $\mathrm{Tr}(-)$. We will conclude this section with two examples of such extensions.

By a *superpositive calculus* we mean a positive hypersequent calculus $\mathbf{C}_+$ that extends the positive calculus $\mathbf{PC}_+$.

Definition 5.4.1. A strict implication logic $\mathbf{L} \supseteq \mathbf{BC}_{\rightsquigarrow}$ is said to be a *strict implication companion of a superpositive calculus* $\mathbf{C}_+ \supseteq \mathbf{PC}_+$ if $\mathbf{L}$ is embedded in $\mathbf{C}_+$ by the translation $\mathrm{tr}(-)$. That is, if for every positive hypersequent rule we have,

$$\vdash_{\mathbf{C}_+} \rho \text{ if and only if } \vdash_{\mathbf{L}} \phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho)}.$$

Given a modal companion $\mathbf{L}$ of $\mathbf{C}_+$, we call $\mathbf{C}_+$ the *superpositive fragment of* $\mathbf{L}$ and denote $\mathbf{C}_+$ by $\varrho\mathbf{L}$. $\dashv$

In what follows, by $\mathsf{HypR}(\mathcal{L}_+)$ we denote the set of positive hypersequent rules generated by the set of well-formed positive formulas $\mathsf{Form}(\mathcal{L}_+)$.

Theorem 5.4.2. *For every* $\mathbf{L} \supseteq \mathbf{BC}_{\rightsquigarrow}$ *there exists a superpositive calculus* $\mathbf{C}_+$ *such that* $\mathbf{C}_+ = \varrho\mathbf{L}$.

Proof. Let $\mathbf{C}_+ \coloneqq \{\rho \in \mathsf{HypR}(\mathcal{L}_+) \mid \mathbf{L} \vdash \phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho)}\}$. Given that $\mathbf{L} \supseteq \mathbf{BC}_{\rightsquigarrow}$, by Theorem 5.2.4 we know that $\mathbf{L} \vdash \phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho)}$, for all rules $\rho \in \mathbf{PC}_+$. Hence, $\mathbf{C}_+$ is an extension of the calculus $\mathbf{PC}_+$ and thus defines a superpositive calculus for which $\mathbf{L}$ is a strict implication companion, in other words, $\mathbf{C}_+ = \varrho\mathbf{L}$. $\square$

By Theorem 5.4.2, we see that ϱ defines a function from the set of extensions of $\mathbf{BC}_+$ to the set of extensions of $\mathbf{PC}_+$. Given a set of positive hypersequent rules $\{\rho_i\}_{i\in I}$, we associate with every superpositive calculus $\mathbf{C}_+ = \mathbf{PC}_+ \cup \{\rho_i\}_{i\in I}$ the strict implication logic $\tau\mathbf{C}_+ \coloneqq \mathbf{BC}_{\rightsquigarrow} \cup \{\phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho_i)}\}_{i\in\mathcal{I}}$.

Theorem 5.4.3. *For every superpositive calculus* $\mathbf{C}_+$*,* $\tau\mathbf{C}_+$ *is a strict implication companion of* $\mathbf{C}_+$*, that is,*

$$\vdash_{\mathbf{C}_+} \rho \textit{ if and only if } \vdash_{\tau\mathbf{C}_+} \phi^{\rightsquigarrow}_{Tr(\rho)}.$$

Proof. For both directions, we prove the contrapositive, namely that $\nvdash_{\tau\mathbf{C}_+} \phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho)}$ if and only if $\nvdash_{\mathbf{C}_+} \rho$.

($\Rightarrow$): Suppose that $\nvdash_{\tau\mathbf{C}_+} \phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho)}$. Let $K(\tau\mathbf{C}_+)$ denote the subclass of BLS that validates all the formulas of $\{\phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho_i)}\}_{i\in\mathcal{I}}$. Observe that, since $\{\phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho_i)}\}_{i\in\mathcal{I}}$ is a set of Π_2-statements, $K(\tau\mathbf{C}_+)$ constitutes an inductive subclass of RSub. Hence, by Theorem 4.2.3, $\tau\mathbf{C}_+$, which equals $RC_{\rightsquigarrow} + \{\rho_{qp}\} + \{\phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho_i)}\}_{i\in\mathcal{I}}$, is strongly sound and complete with respect to the inductive subclass of RSub axiomatised by the statements $\{\Phi_{qp}\} \cup \{\phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho_i)}\}_{i\in\mathcal{I}}$. It follows that there exists a $(B, \prec) \in K(\tau\mathbf{C}_+)$ such that $(B, \prec) \nvDash \phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho)}$. By Lemma 5.1.1 we know that this entails $(B, \prec) \nvDash \mathrm{Tr}(\rho)$ and moreover, that $(B, \prec) \vDash \mathrm{Tr}(\rho_i)$, for all $i \in \mathcal{I}$. Then, given Lemma 5.2.2 it must be that $D_\prec \nvDash \rho$ and $D_\prec \vDash \rho_i$, for all $i \in \mathcal{I}$. From strong soundness of $\mathbf{PC}_+$ with respect to BDL (Theorem 3.2.14) we have that $\nvdash_{\mathbf{C}_+} \rho$.

($\Leftarrow$): Suppose that $\nvdash_{\mathbf{C}_+} \rho$. Let $K(\mathbf{C}_+)$ denote the subclass BDL that validates $\{\rho_i\}_{i\in\mathcal{I}}$. By completeness of $\mathbf{PC}_+$ with respect to BDL (Theorem 3.2.14) it follows that $\nvDash_{K(\mathbf{C}_+)} \rho$ hence, there exists $D \in K(\mathbf{C}_+)$ such that $D \nvDash \rho$. Let B_D denote the free Boolean algebra generated by D, and let $\prec_D$ be as in in Definition 2.4.3. From Lemma 5.2.3 it follows that $(B_D, \prec_D) \nvDash \mathrm{Tr}(\rho)$, and, given that $D \vDash \rho_i$, for each $i \in \mathcal{I}$, that $(B_D, \prec_D) \vDash \mathrm{Tr}(\rho_i)$, for each $i \in \mathcal{I}$. By Lemma 5.1.1 this means that $(B_D, \prec_D) \nvDash \phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho)}$ and $(B_D, \prec_D) \vDash \phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho_i)}$, for each $i \in \mathcal{I}$. Then $(B_D, \prec_D) \in K(\tau\mathbf{C}_+)$ and thus, by by Theorem 4.2.3 we have $\nvdash_{\tau\mathbf{C}_+} \phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho)}$. □

From Lemma 5.4.3 above it immediately follows that $\mathbf{C}_+ = \varrho\tau\mathbf{C}_+$, therefore, ϱ is in fact surjective.

Corollary 5.4.4. *For any superpositive calculus $\mathbf{C}_+$, $\tau\mathbf{C}_+$ is the smallest strict implication companion of $\mathbf{C}_+$. In other words, $\tau\mathbf{C}_+$ is the smallest element in $\varrho^{-1}(\mathbf{C}_+)$ with respect to $\subseteq$.*

5.4.1 Strict implication companions: examples

In this section, we spell out two positive hypersequent rules that characterise a subclass of bounded distributive lattices and define their respective strict implication companions.

$$(\rho_{\mathrm{kc}})\ \frac{G \mid \phi \wedge \psi \Rightarrow \bot}{G \mid \phi \Rightarrow \bot \mid \psi \Rightarrow \bot}$$

$$(\rho_{\mathrm{lc}})\ \frac{G \mid \phi_1 \Rightarrow \psi_2 \qquad G \mid \phi_2 \Rightarrow \psi_1}{G \mid \phi_1 \Rightarrow \psi_1 \mid \phi_2 \Rightarrow \psi_2}$$

Table 5.2: The positive hypersequent rules ρ_{kc} and ρ_{lc}.

Recall that, for a set X and a partial order R on X, we say that R is *directed* if for all $x, y \in X$ there is an $z \in X$ such that xRz and yRz. And, R is said to be *linear* if for all $x, y \in X$ we have xRy or yRx.

Theorem 5.4.5. *Let D be a bounded distributive lattice and (X_D, R) denote its dual Priestley space. Then,*

(i) *D validates the rule ρ_{kc} if and only if the partial order R is directed.*
(ii) *D validates the rule ρ_{lc} if and only if the partial order R is linear.*

Proof.
(i) Suppose that D validates ρ_{kc} and pick $x, y \in X_D$. Consider $z \coloneqq D \setminus \{0\}$. Since we assumed D to satisfy ρ_{kc}, if $a \neq 0$ and $b \neq 0$ we have that $a \wedge b \neq 0$. Thus, for all $a, b \in z$ it must be that $a \wedge b \in z$. From here it is easy to see that z is a prime filter with $x \leq z$ and $y \leq z$.

Now, for the right-to-left direction assume that D does not validate ρ_{kc}. Then there exists $a, b \in D$ with $a \wedge b = 0$ but $a \neq 0$ and $b \neq 0$. Consider the filter $\uparrow a$ and ideal $\downarrow b$. Since these are disjoint, from the prime ideal theorem for distributive lattices (see e.g., [2, Thm. III.4.1]) it follows that there exists a prime filter $x \in X_D$ such that $\uparrow a \subseteq x$ and $\downarrow b \cap x = \emptyset$. Analogously, for the filter $\uparrow b$ and ideal $\downarrow a$ there exists a prime filter $y \in X_D$ such that $\uparrow b \subseteq y$ and $\downarrow a \cap y = \emptyset$. Now observe that, if there would exist $z \in X_D$ with $x \subseteq z$ and $y \subseteq z$ then $a, b \in z$ and so also, $a \wedge b \in z$. But since $a \wedge b = 0$ and z is a proper filter, this cannot be. Hence, R on X_D is not directed.

(ii) Suppose that D satisfies ρ_{lc} and pick $x, y \in X_D$. Towards a contradiction, suppose that $x \not\subseteq y$ and $y \not\subseteq x$. Then, there exists $a \in x$ with $a \notin y$ and $b \in y$ such that $b \notin x$. Now D validates ρ_{lc} thus, for all a_1, a_2, b_1, b_2 from $a_1 \leq b_2$ and $a_2 \leq b_1$ it follows that $a_1 \leq b_1$ or $a_2 \leq b_2$. Then, since $a \leq a$ and $b \leq b$, it must be that $a \leq b$ or $b \leq a$. Now, x and y are filters, so it would follow that $b \in x$ or $a \in y$, which contradicts our initial assumption. Hence, it must be that $x \subseteq y$ or $y \subseteq x$. Thus, R on X_D is linear.

For the other direction, assume that D does not satisfy ρ_{lc}. Then there exist $a_1, a_2, b_1, b_2 \in D$ with $a_1 \leq b_2$ and $a_2 \leq b_1$ but $a_1 \not\leq b_1$ and $a_2 \not\leq b_2$. Consider the filter $\uparrow a_1$ and ideal $\downarrow b_2$. Since $\uparrow a_1 \cap \downarrow b_2 = \emptyset$, by the Prime filter theorem for distributive lattices, there exists a prime filter $x \in X_D$ such that $\uparrow a_1 \subseteq x$ but $\downarrow b_2 \cap x = \emptyset$. Similarly, for the filter $\uparrow a_2$ and ideal $\downarrow b_2$ there exists a prime filter $y \in X_D$ $\uparrow a_2 \cap \downarrow b_1 = \emptyset$. Now, since $a_1 \not\leq b_1$ and $a_2 \not\leq b_2$, it cannot be that $x \subseteq y$ or $y \subseteq x$ and thus, R on X_D is not linear. □

Before we proceed to present the respective companions of the rules ρ_{kc} and ρ_{lc}, we make some observations to simplify the literal translation of these rules. We let $\Diamond$ denote the dual operator of $\boxdot$ defined by $\Diamond\phi \coloneqq \phi \rightsquigarrow \bot$.

Lemma 5.4.6. *Let $(B, \prec) \in \mathsf{BLS}$ and let ϕ denote an arbitrary strict implication formula.*

(i) $(B, \prec) \vDash \Diamond\phi$ *if and only if* $(B, \prec) \vDash \Box\neg\phi$*;*

(ii) $(B, \prec) \vDash \Box\phi$ *if and only if* $(B, \prec) \vDash \Diamond\neg\phi$*.*

Proof. (i) Observe that, for all valuations v on $(B, \prec)$, $1 \leq [\![\Diamond\phi]\!]_v$ if and only if $1 \leq [\![\phi \rightsquigarrow \bot]\!]_v$ if and only if $[\![\phi]\!]_v \leq 0$ if and only if $1 \leq [\![\neg\phi]\!]_v$ if and only if $1 \leq [\![\top \rightsquigarrow \neg\phi]\!]_v$ if and only if $1 \leq [\![\Box\neg\phi]\!]_v$.

For item (ii), observe that $1 \leq [\![\Box\phi]\!]$ if and only if $1 \leq [\![\top \rightsquigarrow \phi]\!]_v$ if and only if $1 \leq [\![\phi]\!]_v$ if and only if $[\![\neg\phi]\!]_v \leq 0$ if and only if $1 \leq [\![\neg\phi \rightsquigarrow \bot]\!]_v$ if and only if $1 \leq [\![\Diamond\neg\phi]\!]_v$. □

This enables us to simplify the formulas $\phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho_{kc})}$ and $\phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho_{lc})}$ to equivalent formulas ϕ_{kc} and ϕ_{lc} given in Table 5.3 below.

For formulas ϕ and ψ, let $\{p_i\}_{i \leq n}$ denote the set of propositional variables occurring in ϕ and ψ.

$\phi_{kc} := ((\bigwedge_{i=1}^{n} p_i \rightsquigarrow p_i) \wedge \Diamond(\phi \wedge \psi)) \to (\Diamond\phi \vee \Diamond\psi)$.

$\phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho_{kc})} := (\Box(\top \to (\bigwedge_{i=1}^{n} p_i \rightsquigarrow p_i)) \wedge \Box(\phi \wedge \psi \to \bot)) \to (\Box(\phi \to \bot) \vee \Box(\psi \to \bot))$.

$\phi_{lc} := ((\bigwedge_{i=1}^{n} p_i \rightsquigarrow p_i) \wedge \Box(\phi_1 \to \psi_2) \wedge \Box(\phi_2 \to \psi_1)) \to (\Box(\phi_1 \to \psi_1) \vee \Box(\phi_2 \to \psi_2))$.

$\phi^{\rightsquigarrow}_{\mathrm{Tr}(\rho_{lc})} := (\Box(\top \to (\bigwedge_{i=1}^{n} p_i \rightsquigarrow p_i)) \wedge \Box(\phi_1 \to \psi_2) \wedge \Box(\phi_2 \to \psi_1)) \to (\Box(\phi_1 \to \psi_1) \vee \Box(\phi_2 \to \psi_2))$.

Table 5.3: The strict implication axiom schemes ϕ_{kc} and ϕ_{lc}.

Lemma 5.4.7. *Let* $(B, \prec) \in \mathsf{BLS}$*. Then,*

(i) $(B, \prec)$ *validates* $\phi^{\rightsquigarrow}_{Tr(\rho_{kc})}$ *if and only if it validates* ϕ_{kc}*;*

(ii) $(B, \prec)$ *validates* $\phi^{\rightsquigarrow}_{Tr(\rho_{lc})}$ *if and only if it validates* ϕ_{lc}*.*

Proof. It is readily checked that for any formula ϕ we have that $(B, \prec)$ validates $\Box(\top \to (\phi \rightsquigarrow \phi)$ if and only if it validates $\phi \rightsquigarrow \phi$. Herefrom it follows that item (ii) of this lemma holds. Item (i) follows from the additional observation that, by Lemma 5.4.6, for every formula ϕ we have that $\Box(\phi \to \bot) = \Diamond\neg(\neg\phi) = \Diamond\phi$. □

Chapter 6

Conclusion

In this thesis, we have seen that the hypersequent calculus $\mathbf{PC}_+$, which is sound and complete with respect to the class of bounded distributive lattices, can be embedded into the strict implication calculus $\mathbf{BC}_{\rightsquigarrow}$, which is sound and complete with respect to the class of Boolean algebras with a lattice subordination. Moreover, since the category BLS of Boolean algebras with a lattice subordination is dually equivalent to the category QPS of quasi-ordered Priestley spaces, the calculus $\mathbf{BC}_{\rightsquigarrow}$ can be used to reason about quasi-ordered Priestley spaces. Furthermore we have seen that for each extension $\mathbf{C}_+$ of $\mathbf{PC}_+$ there exists an extension $\mathbf{L}$ of $\mathbf{BC}_{\rightsquigarrow}$ such that $\mathbf{C}_+$ can be embedded in it, which we termed the strict implication companion of $\mathbf{C}_+$. Vice versa, for each $\mathbf{L}$ extending the calculus $\mathbf{BC}_{\rightsquigarrow}$, we have seen that there exists an extension $\mathbf{C}_+$ of $\mathbf{PC}_+$, termed the positive fragment of $\mathbf{L}$, such that $\mathbf{C}_+$ is embeddable in $\mathbf{L}$. We have provided two examples $\mathbf{PC}_+ \cup \{\rho_{kc}\}$ and $\mathbf{PC}_+ \cup \{\rho_{lc}\}$ of such extensions and their strict implication companions, which demonstrates the possibility for a theory of strict implication companions of positive calculi. Further points of research would be whether positive calculi have least and greatest companions and if an analogue of the Blok Esakia theorem can be established.

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