THE BOLZANO-WEIERSTRASS THEOREM IN GENERALISED ANALYSIS

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Abstract. Let κ be an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$. We consider two totally ordered fields κ-$\mathbb{R}$ and $\mathbb{R}_\kappa$, due to Sikorski and the second author, respectively, that serve as the κ-analogues of the real line and consider generalisations of the Bolzano-Weierstraß theorem for them, showing that for $\mathbb{R}_\kappa$, the weak κ-Bolzano-Weierstraß theorem is closely related to the tree property of κ.

1. Introduction

The study of the set theoretic properties of the real numbers $\mathbb{R}$ was one of the driving forces for the development of set theory and is currently one of set theory's most important subfields. Set theorists often do not study the topological space $\mathbb{R}$, but rather *Baire space* ω^ω, the space of functions from ω to ω, which is similar, yet different: it is homeomorphic to $\mathbb{R}\backslash\mathbb{Q}$ which means that it differs from $\mathbb{R}$ only by a countable set, but removing this countable dense set from $\mathbb{R}$ creates gaps that make Baire space totally disconnected. This means that certain set theoretic properties easily transfer from ω^ω to $\mathbb{R}$, but others do not.

A property that does not transfer between $\mathbb{R}$ and ω^ω is the *Bolzano-Weierstraß theorem* BWT, i.e., "every sequence with bounded range has a cluster point". The property BWT concerns the interplay between boundedness and sequential compactness, i.e., the relation between the order and the topology. Hence, the validity of BWT is not a purely topological property: it is not preserved by homeomorphisms and, moreover, BWT fails on ω^ω.[1] Another fundamental property of the real line is the *Heine-Borel theorem* HBT, i.e., "for every subset X of $\mathbb{R}$ we have that X is compact if and only if X is closed and bounded". The BWT and the HBT are closely related: for ordered fields K, K is Dedekind-complete if and

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[1] Let $x^{(n)}$ be the sequence $(0n0\ldots)$. The sequence $(x^{(n)}\,;\,n\in\omega)$ is bounded in ω^ω, but has no cluster point.

only if K satisfies BWT if and only if K satisfies HBT (cf., e.g., [22, Chapter 5, Theorem 7.6]). Like BWT, the HBT is a property which is not preserved by homeomorphism and, in particular, it does not transfer from $\mathbb{R}$ to ω^ω.[2]

In recent years, set theorists have become increasingly interested in *generalised Baire spaces* κ^κ, the set of functions from κ to κ for an uncountable cardinal κ (for a survey and a list open questions, cf. [19]). Some of the classical results for Baire space generalise to the uncountable case, but others do not. These failures of generalisation are particularly interesting, as they shed light on structures and properties hidden in the classical setting. In this paper, we consider generalisations of BWT and HBT.

As mentioned, BWT and HBT both fail on Baire space, so the natural setting for uncountable generalisations of these theorems would not be κ^κ, but an appropriate space which can play the role of the real line for κ^κ. We shall consider two of these spaces, introduced by Sikorski [25] and the second author [13], respectively.

In the classical setting, the Bolzano-Weierstraß theorem is closely related to Weak Kőnig's lemma WKL, i.e., "every infinite binary tree has an infinite branch". This relationship was made precise by Brattka, Gherardi, and Marcone [2] in the setting of *computable analysis* where the computational strength of theorems is studied with the notion of *Weihrauch reducibility*: $\forall\exists$-theorems are interpreted as partial multi-valued functions, and Weihrauch reducibility, denoted by $\leq_{\mathrm{W}}$, is a pre-order whose corresponding equivalence relation $\equiv_{\mathrm{W}}$ measures the theorem's strength. In this paper, we are not dealing with a Weihrauch analysis of the theorems discussed and no knowledge of Weihrauch reducibility is needed anywhere in this paper; we refer the reader to [3, 17] for more information.

Brattka, Gherardi, and Marcone replace the term "bounded" by "relatively compact" in BWT and obtain a (weaker) purely topological version of Bolzano-Weierstraß, $\mathsf{BWT}^{\mathrm{top}}$, i.e., "every sequence with relatively compact range has a cluster point". In contrast to BWT, $\mathsf{BWT}^{\mathrm{top}}$ holds in Baire space (the failure of BWT in ω^ω corresponds to the fact that not all bounded subsets of ω^ω are relatively compact). Writing $\mathsf{BWT}^{\mathrm{top}}_X$ for the statement "every sequence in X with relatively compact range has a cluster point in X", there is a proper hierarchy of principles

$$\mathsf{BWT}^{\mathrm{top}}_2 <_{\mathrm{W}} \mathsf{BWT}^{\mathrm{top}}_3 <_{\mathrm{W}} \ldots <_{\mathrm{W}} \mathsf{BWT}^{\mathrm{top}}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{BWT}^{\mathrm{top}}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{BWT}^{\mathrm{top}}_{\omega^\omega} \equiv_{\mathrm{W}} \mathsf{WKL}',$$

[2]For $s \in \omega^{<\omega}$, the basic clopen set $[s]$ is bounded but not compact.

where WKL' denotes the jump of WKL.[3]

In this paper, we shall discuss generalisations of BWT to uncountable cardinals κ. For one of these, called the weak κ-Bolzano-Weierstraß theorem, we prove that if κ is inaccessible, then the weak κ-Bolzano-Weierstraß theorem holds for the generalised reals if and only if κ has the tree property (which is the generalisation WKL to the κ-setting; cf. Corollary 4.23).

The paper is organised as follows: in § 2, we discuss totally ordered sets, groups and fields, and their properties; in § 3, we shall introduce the two generalisations of the real line κ-$\mathbb{R}$ and $\mathbb{R}_\kappa$ due to Sikorski and the second author, respectively, and discuss their topological properties; in § 4, we shall study generalisations of the Bolzano-Weierstraß theorem on κ-$\mathbb{R}$ and $\mathbb{R}_\kappa$; finally, in § 5, we shall study of a generalised version of the Heine-Borel theorem.

2. Basic definitions

2.1. **Totally ordered sets.** Let $(X, \leq)$ be any totally ordered set; as usual, we use $<$ for the irreflexive relation associated to $\leq$ ($x < y$ if and only if $x \leq y$ and $x \neq y$). We define sets $(y, z) := \{x \in X\,;\, y < x < z\}$, $(-\infty, z) := \{x \in X\,;\, x < z\}$, and $(z, \infty) := \{x \in X\,;\, z < x\}$. We call these sets *(open) intervals*; we topologise totally ordered sets by taking the topology generated by the open intervals. Intervals of the form (y, z) for $y, z \in X$ are called *proper intervals*; as usual, we define closed intervals $[y, z] := (y, z) \cup \{y, z\}$, and half-open intervals $(x, z] := (x, z) \cup \{z\}$ and $[y, x) := (y, x) \cup \{y\}$ for $x \in X \cup \{-\infty, \infty\}$. A subset $Z \subseteq X$ is *bounded* if it is contained in a proper interval. As usual, if $Y, Z \subseteq X$, we write $Y < Z$ if for all $y \in Y$ and all $z \in Z$, we have $y < z$. In order to reduce the number of braces, we write $y < Z$ for $\{y\} < Z$ and $Y < z$ for $Y < \{z\}$. A subset $Z \subseteq X$ is called *convex* if for any $z, z' \in Z$ and x such that $z \leq x \leq z'$, we have that $x \in Z$. Clearly, every interval is convex.

We call $Z \subseteq X$ *cofinal* if for every $x \in X$ there is a $z \in Z$ such that $x \leq z$; similarly, we call $Z \subseteq X$ *coinitial* if for every $x \in X$ there is a $z \in Z$ such that $z \leq x$. The *coinitiality* and the *cofinality* of a totally ordered set $(X, \leq)$ are the sizes of coinitial or cofinal sets minimal in cardinality, respectively, and we write

[3] In the Weihrauch setting, the jump corresponds to an application of the monotone convergence theorem which allows us to do a transition from the subsequence produced by WKL to the cluster point needed by BWT; WKL is not sufficient to do that transition (in other words, $\mathsf{WKL} <_{\mathrm{W}} \mathsf{BWT}$). Note that $\mathsf{BWT}^{\mathrm{top}}_X$ and BWT_X are not in general Weihrauch equivalent for arbitrary ordered spaces X; however, they are in the case $X = \mathbb{R}$ (because of HBT). Therefore, $\mathsf{BWT}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{WKL}'$.

$\mathrm{coi}(X, \leq)$ and $\mathrm{cof}(X, \leq)$ for them. If the order $\leq$ is implicitly clear, we omit it from the notation.

A pair (L, R) of non-empty subsets of X is called a *Dedekind cut* in $(X, \leq)$ if $L \neq \varnothing \neq R$, L has no maximum, R has no minimum, $L \cup R = X$ and $L < R$.

Let λ be a cardinal. We say that $(X, \leq)$ is an *η_λ-set*[4] if for any $L, R \subseteq X$ such that $L < R$ and $|L| + |R| < \lambda$, there is $x \in X$ such that $L < x < R$.

The property of η_λ-ness relates to the model theoretic property of saturation: any densely ordered set $(X, \leq)$ without endpoints is λ-saturated in the sense of model theory if and only if it is an η_λ-set [4, Proposition 5.4.2]. In the following, we often refer to η_λ-ness as "saturation".

We now introduce the notion of spherical completeness which is a weakening of saturation and known from the theory of ultrametrics (cf., e.g., [23, § 20]). Let $\mathcal{I} = \{I_\gamma\,;\,\gamma < \alpha\}$ be a family of closed intervals. We call such a family *nested* if for $\gamma < \gamma'$, we have $I_\gamma \supseteq I_{\gamma'}$. Let $(X, \leq)$ be a totally ordered set, λ be a regular cardinal. Then $(X, \leq)$ is *λ-spherically complete* iff for every $\alpha < \lambda$ and for every nested family $\mathcal{I} = \{I_\gamma\,;\,\gamma < \alpha\}$ of closed intervals, we have that $\bigcap \mathcal{I} \neq \varnothing$.

Proposition 2.1. *Let $(X, \leq)$ be a totally ordered set and λ be a regular cardinal. If X is an η_λ-set, then X is λ-spherically complete.*

Proof. Let $\mathcal{I} = \{I_\gamma\,;\,\gamma < \alpha\}$ be a nested family of closed intervals with $I_\gamma = [x_\gamma, y_\gamma]$ for some $\alpha < \lambda$. Then apply saturation to the pair $(\{x_\gamma\,;\,\gamma < \alpha\}, \{y_\gamma\,;\,\gamma < \alpha\})$ to obtain an element in the intersection of $\mathcal{I}$. □

Note on the other hand that there are λ-spherically complete ordered sets which are not η_λ-sets: e.g., the real line $\mathbb{R}$ is $\aleph_1$-spherically complete, but not an $\eta_{\aleph_1}$-set. (Let $L = \mathbb{N} \subseteq \mathbb{R}$ and $R = \varnothing$.)

2.2. **Totally ordered groups and fields.** Let $(G, +, 0, \leq)$ be a totally ordered group. We denote the *positive part of G* as $G^+ := \{x \in G\,;\,x > 0\}$. Moreover, following [8, Definition 1.19], we call $\mathrm{bn}(G) := \mathrm{coi}(G^+)$ the *base number of G*.[5]

The following definition is due to Sikorski [26]: let X be a set and $(G, +, 0, \leq)$ be a totally ordered abelian group; a function $d : X \times X \to G^+$ is a *G-metric* if for all $x, y, z \in X$, we have

(1) $d(x, y) \geq 0$,
(2) $d(x, y) = 0$ if and only if $x = y$,

[4] Note that our notation differs from Hausdorff's: his η_α would correspond to our $\eta_{\aleph_\alpha}$.

[5] This number was called the *degree of G*, in symbols $\deg(G)$, in [12, 13, 14]. Sikorski says that G has *character κ* if (in our notation) $\mathrm{bn}(G) \leq \kappa$ [25]. The term *base number* is due to Dales and Woodin who use the notation $\delta(G)$ for our $\mathrm{bn}(G)$.

(3) $d(x,y) \leq d(x,z) + d(z,y)$,
(4) $d(x,y) = d(y,x)$.

As in the case of a metric, we can define *open balls* with respect to a G-metric d:

$$B_d(c,r) = \{x \in X \,;\, d(c,x) < r\}$$

where $c \in X$ and $r \in G^+$. If λ is a regular cardinal, then we will say that a topological space (X,τ) is λ*-metrisable* if there is a totally ordered abelian group $(G,+,0,\leq)$ with $\mathrm{bn}(G) = \lambda$ and a G-metric $d : X \times X \to G$ such that $\{B_d(c,r)\,;\, c \in X \wedge r \in G^+\}$ is a base for τ.

Using the fact that G is totally ordered, we can measure the distance of elements in G by

$$|x - y| := \begin{cases} x - y & \text{if } x - y \in G^+ \text{ and} \\ y - x & \text{otherwise,} \end{cases}$$

and this is a G-metric. Therefore, every totally ordered abelian group $(G,+,0,\leq)$ with $\mathrm{bn}(G) = \lambda$ is λ-metrisable. If C and C' are two convex sets, we say that C and C' are *separated by a distance of at least* $\varepsilon \in K^+$ if for all $x \in C$ and all $y \in C'$, we have that $|x - y| > \varepsilon$.

Now let $(K,+,\cdot,0,1,\leq)$ be a totally ordered field. As usual, we identify the element

$$\underbrace{1 + \ldots + 1}_{n \text{ times}}$$

with the natural number n and thus assume that $\mathbb{N} \subseteq K$. The field K is called *Archimedean* if $\mathbb{N}$ is cofinal in K.

The field operations ensure that the order structure of K is homogeneous as order-theoretic phenomena can be shifted around in the field. E.g., if one considers subsets of K^+, the map $x \mapsto x^{-1}$ transforms sets that are cofinal in K^+ into sets that are coinitial in K^+ and vice versa; therefore $\mathrm{bn}(K) = \mathrm{coi}(K^+) = \mathrm{cof}(K)$.

Also, if (a,b) and (c,d) are any proper intervals in K, then the map $\pi : z \mapsto \frac{d-c}{b-a}(z-a)+c$ is a linear transformation of the one-dimensional K-vector space K such that the interval (a,b) is bijectively and order-preservingly mapped to (c,d). Clearly, this map translates subsets of (a,b) into subsets of (c,d) while preserving properties such as convergence and divergence:

Lemma 2.2. *Let $(K,+,\cdot,0,1,\leq)$ be a totally ordered field and (a,b) and (c,d) proper intervals in K. If $s : \alpha \to (a,b)$ is a convergent or divergent sequence, then so is $\pi \circ s : \alpha \to (c,d)$.*

The following results (Lemmas 2.3, 2.4, 2.5, 2.7, 2.8 and Corollary 2.6) are explaining how the properties of a field relates to the existence of divergent and convergent sequences of a given length. These will prove to be the main tools for our later proofs.

Lemma 2.3. *Let $(K, +, \cdot, 0, 1, \leq)$ be a non-Archimedean totally ordered field and C be a convex subset of K with at least two elements. Then there are strictly increasing and strictly decreasing ω-sequences inside C.*

Proof. We only construct the strictly decreasing sequence, the existence of a strictly increasing sequence follows by symmetry. Let $x, y \in C$ be such that $x < y$. Since K is non-Archimedean, we find $k \in K$ such that $\mathbb{N} < k$. Set $\varepsilon := \frac{y-x}{k}$ and define $y_n := y - n \cdot \varepsilon \in C$ for each $n \in \omega$. Clearly, this is a strictly decreasing ω-sequence in C. □

Lemma 2.4. *Let λ be an uncountable regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field such that $\mathrm{bn}(K) = \lambda$. Then the following are equivalent:*

(1) *K is λ-spherically complete,*
(2) *for every $\alpha < \lambda$, every nested family $\mathcal{I} = \{I_\gamma\, ; \gamma < \alpha\}$ of non-empty open intervals has non-empty intersection.*

Proof. Clearly, (2) implies (1). For the other implication, fix $\mathcal{I} = \{I_\gamma\, ; \gamma < \alpha\}$ with $I_\gamma =: (x_\gamma, y_\gamma)$. We only have to consider the case $\alpha \geq \omega$. By (1), we have that $\bigcap_{\gamma<\alpha}[x_\gamma, y_\gamma] \neq \varnothing$, so pick $x \in \bigcap_{\gamma<\alpha}[x_\gamma, y_\gamma]$. Since $\mathrm{bn}(K) = \lambda > \alpha$ and λ is regular, there is $\varepsilon > 0$ such $(x - \varepsilon, x + \varepsilon) \subseteq \bigcap_{\gamma<\alpha}[x_\gamma, y_\gamma]$ Note that $(x - \varepsilon, x + \varepsilon) \subseteq \bigcap_{\gamma<\alpha}(x_\gamma, y_\gamma)$ which proves the claim. □

Clearly, if K is an η_λ-set, then $\mathrm{bn}(K) \geq \lambda$. Having large base number provides us with a weaker version of η_λ-ness that is sometimes sufficient for our arguments:

Lemma 2.5. *Let λ be a regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field with $\mathrm{bn}(K) = \lambda$. Let $F \subseteq K$ be finite and $X \subseteq K$ be such that $|X| < \lambda$. Then if $X < F$, there is some $x \in K$ such that $X < x < F$. Similarly, if $F < X$, then there is some $x \in K$ such that $F < x < X$.*

Proof. Since the proofs are similar, we only deal with the case $X < F$. The case $F = \varnothing$ follows directly from $\mathrm{bn}(K) = \lambda$. Let $F = \{x_0, ..., x_n\}$ with $x_0 < x_1 < ... < x_n$, let $\mu := \mathrm{cof}(X) \leq |X| < \lambda$, and let $s : \mu \to X$ be strictly increasing and cofinal in X. If $\gamma < \lambda$, let $\varepsilon_\gamma := x_0 - s(\gamma)$. Since $\mathrm{bn}(K) = \lambda > \mu$, we find $\varepsilon \in K^+$ such that for all $\gamma < \mu$, we have $x_0 - \varepsilon > s(\gamma)$. But then $X < x_0 - \varepsilon < F$. □

Corollary 2.6. *Let λ be a regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field with* $\mathrm{bn}(K) = \lambda$.

(i) *If I is an open interval in dK, then* $\mathrm{cof}(I) = \mathrm{coi}(I) = \mathrm{bn}(K)$.
(ii) *If $\mu < \lambda$, then every μ-sequence is bounded and it is either eventually constant or divergent.*
(iii) *Every infinite convex set C contains strictly descending and strictly increasing λ-sequences bounded in C; in particular, it contains bounded and divergent μ-sequences for every $\mu < \lambda$.*

Proof. Statement (i) and (ii) are obvious from Lemma 2.5. For statement (iii), find $x, y \in C$ and apply (i) to (x, y) to find coinitial and cofinal sequences of length λ; apply (ii) to see that the initial segments of these of length μ are divergent. $\square$

The *weight* of a totally ordered field $(K, +, \cdot, 0, 1, \leq)$ is the size of the smallest dense subset of K and is denoted by $\mathrm{w}(K)$. Since every dense set is cofinal, we have that $\mathrm{bn}(K) \leq \mathrm{w}(K)$.

Lemma 2.7. *Let λ be a regular uncountable cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field such that $\mathrm{w}(K) = \lambda$ and K is an η_λ-set. Then every interval $(x, y) \subseteq K$ contains a convex bounded subset $B \subseteq (x, y)$ without least upper or greatest lower bound such that* $\mathrm{coi}(B) = \mathrm{cof}(B) = \lambda$.

Proof. Clearly, the assumptions imply that K is non-Archimedean. Pick $z \in (x, y)$ and use Lemma 2.3 to find a strictly increasing sequence $s : \omega \to (x, z)$ with $S := \mathrm{ran}(s)$ and a strictly decreasing sequence $s' : \omega \to (z, y)$ with $S' := \mathrm{ran}(s')$; in particular, $S < S'$. By Corollary 2.6 (ii), both s and s' are bounded and divergent; also, z is both an upper bound for S and a lower bound for S'. Let $B := \{b \in (x, y)\,;\, S < b < S'\}$ be the set of these elements. Clearly, B is convex; a greatest lower bound for B would be a least upper bound for S and a least upper bound for B would be a greatest lower bound for S', but since s and s' are divergent, these do not exist, so B has neither greatest lower nor least upper bound.

We shall now show that $\mathrm{coi}(B) = \mathrm{cof}(B) = \lambda$. The two proofs are similar, so let us just discuss the proof for coinitiality.

Clearly, if $X \subseteq B$ with $|X| < \lambda$, then X cannot be coinitial by saturation. So $\mathrm{coi}(B) \geq \lambda$. We shall now construct a coinitial set of size λ. For this, let D be a dense set of size $\mathrm{w}(K) = \lambda$, let $B' := B \cap D$ and let $\sigma : \lambda \to B'$ be a surjection. We construct a strictly decreasing coinitial λ-sequence $t : \lambda \to B$: Pick any element $t(0) \in B$. Suppose $\alpha < \lambda$ and assume that $t{\restriction}\alpha$ has been defined and is a strictly descending sequence. Then $B^* := \operatorname{ran}(t{\restriction}\alpha) \cup \operatorname{ran}(\sigma{\restriction}\alpha)$ has size $|B^*| \leq |\alpha \times 2| < \lambda$. By saturation, we find b such that $S < b < B^*$; then let $t(\alpha) := b$.

We claim that t is coinitial: if $b \in B$ is arbitrary, then by saturation, we find some $z \in B$ such that $S < z < b$. Now density of D means that we find some $d \in D$ with $S < z < d < b$. Clearly, $d \in B'$. Find α such that $\sigma(\alpha) = d$. Then $t(\alpha + 1) < d < b$. □

Lemma 2.8. *Let λ be an uncountable regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ is a λ-spherically complete totally ordered field. Let $\mu < \lambda$, $s : \mu \to K$ be a bounded divergent strictly decreasing sequence with $S := \operatorname{ran}(s)$, and $L := \{b \in K\,;\, b < S\}$ be the set of lower bounds of S. Then $\operatorname{cof}(L) \geq \mu^+$. (The same is true for a strictly increasing sequence and its set of upper bounds.)*

Proof. Suppose $t : \mu \to L$ is a cofinal sequence in L with $T := \operatorname{ran}(T)$. For $\gamma < \mu$, let $I_\gamma := [t(\gamma), s(\gamma)]$. Then λ-spherical completeness implies that there is some x with $T < x < S$, but that contradicts the fact that s was divergent. □

The final technical result of this section will be the core of our constructions in the main proofs, allowing us to split intervals:

Lemma 2.9. *Let λ be an uncountable regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field with $\operatorname{bn}(K) = \lambda$. If $I = (x, y)$ is an open interval in K with half-way point $\frac{x+y}{2}$ and $\mu < \lambda$ is a cardinal, then there is a family $\{I_\alpha\,;\, \alpha < \mu\}$ of pairwise disjoint non-empty subintervals of I with union $U := \bigcup_{\alpha<\mu} I_\alpha$ such that*

(1) *there is an $\varepsilon_0 \in K^+$ such that for all $z \in U$, we have $|z - x| > \varepsilon_0$ and $|z - y| > \varepsilon_0$,*
(2) *$\frac{x+y}{2} \notin U$, and*
(3) *there is $\varepsilon_1 \in K^+$ such that for all $\alpha \neq \beta < \mu$, I_α and I_β are separated by a distance of at least ε_1 (i.e., for all $x_\alpha \in I_\alpha$, and $x_\beta \in I_\beta$, we have that $|x_\alpha - x_\beta| > \varepsilon_1$).*

Proof. Pick any $x', y' \in (x, \frac{x+y}{2})$ and work inside $I' := (x', y')$. Clearly, any family of subintervals contained in I' will trivially satisfy (1) and (2). By Corollary 2.6 (i), $\operatorname{cof}(I') = \lambda$, so let $s : \lambda \to I'$ be a strictly increasing sequence cofinal in I'. Suppose that $\nu < \mu$ is a limit ordinal and $n \in \mathbb{N}$. We define

$$I_{\nu+n} := (s(\nu + 2n + 1), s(\nu + 2n + 2))$$

and claim that this sequence of intervals satisfies (3). If $\alpha < \beta = \nu + n < \lambda$, then the distance between I_α and I_β is at least

$$\delta_\beta := s(\nu + 2n + 1) - s(\nu + 2n) > 0.$$

Apply Lemma 2.5 to the sets $\{0\}$ and $\{\delta_\beta\,;\,\beta < \mu\}$ to find $\varepsilon_1 > 0$ as required by (3). □

2.3. **Completeness.** Given a totally ordered field $(K, +, \cdot, 0, 1, \leq)$, a Dedekind cut (L, R) in K, is called a *Veronese cut*[6] if for each $\varepsilon \in K^+$ there are $\ell \in L$ and $r \in R$ such that $r < \ell + \varepsilon$.

A totally ordered field $(K, +, \cdot, 0, 1, \leq)$ is called *Dedekind complete* if there are no Dedekind cuts in K and it is called *Veronese complete* if there are no Veronese cuts in K. Clearly, Dedekind completeness implies Veronese completeness, but the converse is not true in general. In fact, a totally ordered field is Dedekind complete if and only if it is isomorphic to $\mathbb{R}$ (cf. [5, Corollary 8.7.4] or [27, Theorem 2.4]).

We need to generalise the standard definitions from real analysis to accommodate transfinite sequences:

Definition 2.10 (Cauchy sequences)**.** Let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field and α be an ordinal. A sequence $(x_i)_{i\in\alpha}$ of elements of K is called *Cauchy* if

$$\forall\varepsilon \in K^+ \exists\beta < \alpha\forall\gamma, \gamma' \geq \beta(|x_{\gamma'} - x_\gamma| < \varepsilon).$$

The sequence is *convergent* if there is $x \in K$ such that

$$\forall\varepsilon \in K^+ \exists\beta < \alpha\forall\gamma \geq \beta(|x_\gamma - x| < \varepsilon).$$

In this case, we shall say that x is the *limit of the sequence*. The field K is called *Cauchy complete* if every Cauchy sequence of length $\mathrm{bn}(K)$ converges.

Theorem 2.11 (Folklore)**.** *A totally ordered field is Veronese complete if and only if it is Cauchy complete.*

Proof. Cf., e.g., [8, Proposition 3.5]. □

By Theorem 2.11 and [16, Theorem 3.11], if K is Archimedean, then Dedekind completeness and Veronese completeness coincide.

In light of Theorem 2.11, we shall from now on only use the more common term "Cauchy completeness" (even though we shall be using the property of Veronese completeness in our proofs).

Lemma 2.12. *Let $(K, +, \cdot, 0, 1, \leq)$ be a Cauchy complete totally ordered field. For every convex set $C \subseteq K$ the following hold:*

[6]The term "Veronese cut" is used by Ehrlich to honour the pioneering contributions of Giuseppe Veronese in the late XIXth century to theory of infinity and infinitesimals; the same concept has various other names in the literature, e.g., Cauchy cut or Scott cut.

(1) *If C has no supremum, there is $\varepsilon \in K^+$ such that for every $x \in C$ we have $x+\varepsilon \in C$.*
(2) *If I has no infimum, there is $\varepsilon \in K^+$ such that for every $x \in C$ we have $x-\varepsilon \in C$.*
(3) *If C has neither infimum nor supremum, then there is $\varepsilon \in K^+$ such that for every $x \in C$ the interval $(x-\varepsilon, x+\varepsilon)$ is a subinterval of C.*

Proof. Clearly, (2) follows from (1) by considering $\{-c\,;\, c \in C\}$ and (3) follows from (1) and (2). We now prove (1). Since C is convex with no supremum $\langle C, \{y \in K\,;\, C < y\}\rangle$ is not a Veronese cut. Therefore there is ε such that for every $x \in C$ we have $x+\varepsilon < \{y \in K\,;\, C < y\}$. $\square$

3. Generalising the real line

The symbol κ will be reserved for an uncountable regular cardinal κ such that $\kappa^{<\kappa} = \kappa$. There are many different approaches for generalising the real line for very different purposes (for an overview, cf. [10]) of which we shall introduce two in §§ 3.1 & 3.3.

3.1. **The real ordinal numbers κ-$\mathbb{R}$.** The real ordinal numbers were introduced by Sikorski [25], studied by Klaua [20], and recently independently re-discovered by Asperó and Tsaprounis [1].

The underlying idea is to do the classical set theoretic construction of the reals, but instead of starting with the natural numbers $\mathbb{N}$, we start with an ordinal δ, considered as a total order $(\delta, \leq)$. Since ordinal addition and multiplication are not commutative, we use instead the *Hessenberg operations* $\oplus$ and $\otimes$ (also called *natural sum* and *natural product*) which are commutative. If δ is a delta number (i.e., an ordinal number closed under multiplication, or, equivalently, $\delta = \omega^{\omega^\beta}$ for some β), then $(\delta, \oplus, \otimes, 0, 1, \leq)$ is a commutative ordered semi-ring. As in the standard construction of $\mathbb{Q}$ from $\mathbb{N}$, one can define $\delta\text{-}\mathbb{Z} := \delta \cup \{-\alpha\,;\, 0 < \alpha < \delta\}$ and δ-$\mathbb{Q}$ as the $\sim$-equivalence classes of $\delta\text{-}\mathbb{Z} \times (\delta \setminus \{0\})$ where $(\pm\alpha, \beta) \sim (\pm\alpha', \beta')$ if and only if $\alpha \otimes \beta' = \alpha' \otimes \beta$; with the usual operations of addition and multiplication defined on δ-$\mathbb{Z}$ and δ-$\mathbb{Q}$ (cf., e.g., [20]). Furthermore, we let δ-$\mathbb{R}$ be the Cauchy completion of δ-$\mathbb{Q}$.

Theorem 3.1 (Sikorski)**.** *If δ is a delta number, then δ-$\mathbb{Z}$ is a totally ordered ring, δ-$\mathbb{Q}$ is a totally ordered field, and δ-$\mathbb{R}$ is a Cauchy complete totally ordered field with* $\mathrm{bn}(\delta\text{-}\mathbb{Q}) = \mathrm{bn}(\delta\text{-}\mathbb{R}) = \mathrm{cof}(\delta)$.

Furthermore, if δ is a regular cardinal, then δ-$\mathbb{Q}$ is Cauchy complete, and therefore $\delta\text{-}\mathbb{Q} = \delta\text{-}\mathbb{R}$ and $\mathrm{w}(\delta\text{-}\mathbb{R}) = \delta$.

Proof. The usual proof in which ω is substituted by κ works, see, e.g., [25, 20]. $\square$

This result was further extended by Asperó and Tsaprounis in [1], where they showed that for every delta number δ with uncountable cofinality, δ-$\mathbb{Q} = \delta$-$\mathbb{R}$.

Sikorski also observed that the real ordinal numbers are very non-saturated:

Theorem 3.2 (Sikorski). *If δ is an infinite cardinal, then δ-$\mathbb{R}$ is not an $\eta_{\aleph_1}$-set.*

Proof. This result was proved in [25], a more recent proof for delta ordinals can be found in [1, Corollary 4.4]. $\square$

3.2. **Surreal numbers.** The second construction that we shall give in § 3.3 uses Conway's *surreal numbers*. For details, we refer to [6, 15] and give a basic sketch of the construction:

A *surreal number* is a function from an ordinal $\alpha \in \mathrm{On}$ to $\{+, -\}$, i.e., a sequence of pluses and minuses of ordinal length. For notational purposes, it is sometimes useful to introduce a third value $\uparrow$ representing "undefined", and we formally order the set $\{+, -, \uparrow\}$ as follows: $- < \uparrow < +$. In this notation, surreal numbers are class-functions defined on the class of all ordinals with the property that $\mathrm{dom}(x) := \{\beta \in \mathrm{On}\,;\, x(\beta) \neq \uparrow\}$ is an ordinal. We shall denote the class of surreal numbers by No. The *length* of a surreal number is its domain, and for $x \in \mathrm{No}$, we write $\ell(x) := \mathrm{dom}(x)$. We write

$$\mathrm{No}_{<\alpha} := \{x \in \mathrm{No}\,;\, \ell(x) < \alpha\} \text{ and}$$
$$\mathrm{No}_{\leq\alpha} := \{x \in \mathrm{No}\,;\, \ell(x) \leq \alpha\}.$$

We order the surreal numbers lexicographically, i.e., if $x \neq y$ and β is least such that $x(\beta) \neq y(\beta)$, then $x < y$ if $x(\beta) < y(\beta)$.

Theorem 3.3 (Conway's Simplicity Theorem). *Let L and R be two sets of surreal numbers such that $L < R$. Then there is a unique surreal z of minimal length such that $L < \{z\} < R$. We say in this case that the pair of sets (L, R)* represents *z and write $z = [L, R]$.*

Proof. Cf. [15, Theorem 2.1]. $\square$

The Simplicity Theorem 3.3 allows us to switch back and forth between surreal numbers and their representations. Exploiting this, one can define addition and multiplication on No and show that No with these operations satisfies the axioms of real closed fields and that $\mathbb{R}$ is a subfield of $\mathrm{No}_{\leq\omega}$. Moreover, Ehrlich proved that it is the universal real closed field in the sense that every real closed field is isomorphic to a subfield of No; cf., e.g., [9].

Theorem 3.4 (van den Dries & Ehrlich). *If ε is an epsilon number (i.e., an ordinal closed under exponentiation or, equivalently, $\varepsilon = \omega^\varepsilon$), then $\mathrm{No}_{<\varepsilon}$ is a real closed field. In particular, for every cardinal λ, $\mathrm{No}_{<\lambda}$ is a real closed field.*

Proof. Cf. [28, Proposition 4.7]. □

Proposition 3.5 (Folklore). *Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Then $|\mathrm{No}_{<\kappa}| = \mathrm{bn}(\mathrm{No}_{<\kappa}) = \mathrm{w}(\mathrm{No}_{<\kappa}) = \kappa$ and $\mathrm{No}_{<\kappa}$ is an η_κ-set.*

Proof. Cf., e.g., [12, Propositions 3.4.3 & 3.4.4]. □

3.3. **The generalised real line $\mathbb{R}_\kappa$.** As before, κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. We shall now use the theory of surreal numbers from § 3.2 to define the second generalisation of the real number continuum which is due to the second author [12, 13, 14]. Let us call a field $K \supseteq \mathbb{R}$ a *super dense κ-real extension of* $\mathbb{R}$ if it has the following properties:

(1) K is a real closed field,
(2) $\mathrm{w}(K) = \kappa$,
(3) K is an η_κ-set,
(4) K is Cauchy complete, and
(5) $|K| = 2^\kappa$.

Since the theory of real closed fields is complete [21, Corollary 3.3.16], any super dense κ-real extension of $\mathbb{R}$ has the same first order properties as $\mathbb{R}$. In [12, 13], the second author argues why being a super dense κ-real extension of $\mathbb{R}$ is an adequate demand for being an appropriate κ-analogue of $\mathbb{R}$.

Theorem 3.4 and Proposition 3.5 tell us that $\mathrm{No}_{<\kappa}$ has almost all the properties that we want from $\mathbb{R}_\kappa$ except for (4) and (5) (for the failure of (4), cf., e.g., [11, Lemma 1.32]). Therefore, we define

$$\mathbb{R}_\kappa := \mathrm{No}_{<\kappa} \cup \{x\,;\, x = [L|R] \text{ where } \langle L, R\rangle \text{ is a Veronese cut on } \mathrm{No}_{<\kappa}\}.$$

Theorem 3.6 (Galeotti). *Let κ be an uncountable cardinal with $\kappa^{<\kappa} = \kappa$. Then $\mathbb{R}_\kappa$ is the unique super dense κ-real extension of $\mathbb{R}$. Moreover, $\mathrm{bn}(\mathbb{R}_\kappa) = \kappa$.*

Proof. Cf. [13, Theorem 4]. □

4. The Bolzano-Weierstrass theorem

4.1. **The classical Bolzano-Weierstraß theorem.** Let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. Then the *Bolzano-Weierstraß theorem for* K, abbreviated as BWT_K, is the statement “every bounded sequence of elements of K has a convergent subsequence”. In this statement, by “sequence” we mean a sequence of length ω.

Theorem 4.1. *Let $(K,+,\cdot,0,1,\leq)$ be a totally ordered field. Then* BWT_K *holds if and only if K is Dedekind complete.*

PROOF. Cf. [22, Theorem 7.6]. □

We had seen in § 2 that this means that up to isomorphism, $\mathbb{R}$ is the only field satisfying the Bolzano-Weierstraß theorem.

Corollary 4.2. *Let κ be an uncountable regular cardinal such that $\kappa^{<\kappa} = \kappa$. Then* $\mathsf{BWT}_{\kappa\text{-}\mathbb{R}}$ *and* $\mathsf{BWT}_{\mathbb{R}_\kappa}$ *do not hold.*

The reason for this is that the statement of BWT_K talks only about sequences of length ω, but $\mathrm{bn}(\kappa\text{-}\mathbb{R}) = \mathrm{bn}(\mathbb{R}_\kappa) = \kappa$, so these sequences are simply too short to have convergent subsequences (using Corollary 2.6 (ii)).

4.2. **The generalised Bolzano-Weierstraß theorem.** We identified the problem with BWT to be the length of the sequences; consequently, the following generalisation due to Sikorski is natural:

Let $(K,+,\cdot,0,1,\leq)$ be a totally ordered field and λ be a regular cardinal. Then the *λ-Bolzano-Weierstraß theorem for K*, abbreviated as λ-BWT_K, is the statement "every bounded λ-sequence of elements of K has a convergent λ-subsequence".

The property λ-BWT_K was studied by several authors, cf., [25, 24, 7]; clearly, $\aleph_0$-BWT_K is the same as BWT_K.

We say that a totally ordered field $(K,+,\cdot,0,1,\leq)$ is *λ-divergent* if and only if every interval contains a strictly monotone divergent λ-sequence.

Observation 4.3. If λ is an uncountable regular cardinal and $(K,+,\cdot,0,1,\leq)$ is a λ-divergent totally ordered field then λ-BWT_K fails.

A cardinal λ is called *weakly compact* if $\lambda \to (\lambda)^2_2$ holds, i.e., if for every partition of $\lambda \times \lambda$ into two sets there is a subset H of λ of cardinality λ such that all the pairs of elements of H are all in the same set of the partition. For λ weakly compact we can reformulate the λ-Bolzano-Weierstraß theorem in terms of λ-divergence.

Theorem 4.4. *Let $\lambda > \omega$ be a weakly compact cardinal and $(K,+,\cdot,0,1,\leq)$ be a totally ordered field. Then the following are equivalent:*

(1) *the field K is λ-divergent and*
(2) λ-BWT_K *does not hold.*

Proof. By Observation 4.3, we only need to prove "(2)⇒(1)". By Lemma 2.2, it is enough to show that there is an interval with a monotone divergent λ-subsequence. Let s be a bounded λ-sequence which has no convergent λ-subsequence. We will show that s has a monotone subsequence. Define the following partition of $\lambda \times \lambda$: $f(\alpha, \beta) := 1$ if $\alpha < \beta$ and $s(\alpha) < s(\beta)$, $f(\alpha, \beta) := 0$ otherwise. Since κ is weakly compact there is $H \subseteq \lambda$ such that either for all $h \in H \times H$, $f(h) = 1$ or for all $h \in H \times H$, $f(h) = 0$. Without loss of generality assume the former. Now, we define recursively a subsequence s' of s. Assume we that have already defined $s \restriction \alpha$, we define: $s'(\alpha) := s(\beta)$ where β is the least ordinal in $H \setminus \{s'(\gamma) \mid \gamma \in \alpha\}$. It is easy to see that s' is strictly increasing. Indeed, if $\alpha < \beta$ then $s'(\alpha) = s(\gamma)$ and $s'(\beta) = s(\gamma')$ for some $\gamma, \gamma' \in H$ such that $\gamma < \gamma'$, but then $f(\gamma, \gamma') = 1$ which implies $s(\gamma) < s(\gamma')$ as desired. □

We do not know whether there is a non λ-divergent field K such that $\lambda\text{-}\mathsf{BWT}_K$ fails.

In some cases, we can prove or refute $\lambda\text{-}\mathsf{BWT}_K$ using elementary arguments:

Theorem 4.5. *Let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field.*

(1) *If $\lambda > |K|$, then $\lambda\text{-}\mathsf{BWT}_K$ holds.*
(2) *If $\lambda < \mathrm{bn}(K)$, then $\lambda\text{-}\mathsf{BWT}_K$ does not hold.*
(3) *If $\mathrm{w}(K) < \lambda$, then every convergent sequence of elements of K of length λ is eventually constant. Consequently, if $\mathrm{w}(K) < \lambda \leq |K|$, $\lambda\text{-}\mathsf{BWT}_K$ does not hold.*

Proof. (1) follows from the pigeonhole principle: every λ-sequence in K contains a constant λ-subsequence. For (2), observe that by Corollary 2.6 (ii) & (iii), if $\lambda < \mathrm{bn}(K)$, then K is λ-divergent. Then Observation 4.3 implies the claim.

For (3), let D be a dense subset of K of cardinality $\kappa < \lambda$. Towards a contradiction, let $s : \lambda \to K$ be a convergent sequence with limit $\ell \in K$ that is not eventually constant. Without loss of generality, we can assume that for each $\alpha < \lambda$, $s(\alpha) \neq s(\alpha + 1)$ and furthermore that $\ell \notin \mathrm{ran}(s)$. Thus, since D is dense, for each $\alpha < \lambda$, we find some $d_\alpha \in (D \cap (s(\alpha), s(\alpha+1))) \cup (D \cap (s(\alpha+1), s(\alpha)))$ such that $d_\alpha \neq \ell$. We define $\widehat{s} : \lambda \to K$ by $\widehat{s}(\alpha) := d_\alpha$.

By construction s and $\widehat{s}$ both converge to the same limit ℓ. Since $|D| < \lambda$ there is an element $d \in D$ which appears λ many times in $\widehat{s}$. Hence, $\widehat{s}$ has a subsequence of length λ which is eventually constant (and converges to the same limit as $\widehat{s}$, i.e., ℓ). But this is a contradiction since ℓ is not an element of $\mathrm{ran}(\widehat{s})$. □

Theorem 4.5 covers all cases except for $\mathrm{bn}(K) \leq \lambda \leq \mathrm{w}(K)$. It turns out that in this case, the answer depends on the saturation properties of K. We shall now have a closer look at this case:

Theorem 4.6 (Sikorski)**.** *Let λ be an uncountable regular cardinal. Then λ-$\mathsf{BWT}_{\lambda\text{-}\mathbb{R}}$ holds.*

Proof. This result was proved by Sikorski in [25]. We give a sketch of the proof. Let $s : \lambda \to \lambda\text{-}\mathbb{R}$ be a bounded λ-sequence. Without loss of generality, by regularity of λ, we can assume s to be injective. By using the fact that elements of λ-$\mathbb{R}$ can be represented as finite sequences of ordinals and rational numbers, see, e.g., [1, Theorem 3.4], it is not hard to see that s has a monotone bounded λ-subsequence $b : \lambda \to \lambda\text{-}\mathbb{R}$. By [1, Proposition 4.2] every monotone bounded λ-sequence in λ-$\mathbb{R}$ is Cauchy. Therefore, b is Cauchy. Finally, since λ-$\mathbb{R}$ is by definition Cauchy complete, b is a convergent subsequence of s as desired. $\square$

Theorem 4.6 heavily relies on the fact that λ-$\mathbb{R}$ is not saturated (Theorem 3.2). Saturated fields behave very differently, as the following observation shows.

Theorem 4.7. *Let λ be a regular uncountable cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. If $\mathrm{w}(K) = \lambda$ and K is an η_λ-set, then K is λ-divergent.*

Proof. Fix any interval I; by Lemma 2.7, we find a convex set $B \subseteq I$ without least upper bound and $\mathrm{cof}(B) = \lambda$. Any cofinal λ-sequence in B must be divergent since B has no least upper bound. $\square$

Corollary 4.8. *Let λ be a regular uncountable cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. If $\mathrm{w}(K) = \lambda$ and K is an η_λ-set, then λ-BWT_K does not hold.*

Proof. Follows directly from Observation 4.3 and Theorem 4.7. $\square$

In fact, we do not need full saturation for this: for successor cardinals, spherical completeness is sufficient:

Lemma 4.9. *Let λ be any cardinal and let $(K, +, \cdot, 0, 1, \leq)$ be a λ^+-spherically complete totally ordered field with $\mathrm{w}(K) = \mathrm{bn}(K) = \lambda^+$. Then K is λ^+-divergent.*

Proof. The proof is a variant of that of Lemma 2.7: Fix any interval I. Corollary 2.6 (ii) & (iii) gives us a strictly decreasing divergent λ-sequence $s : \lambda \to I$ bounded in I; as usual, we write $S := \mathrm{ran}(s)$. Let $L := \{b \in I\,;\, b < S\}$ be the set of lower bounds of S. By Lemma 2.8, we know that $\mathrm{cof}(L) \geq \lambda^+$. We use $\mathrm{w}(K) = \lambda^+$, exactly as in the proof of Lemma 2.7, to get that $\mathrm{cof}(L) = \lambda^+$.

	κ-$\mathbb{R}$	$\mathbb{R}_\kappa$
$\lambda < \kappa$	No	No
$\lambda = \kappa$	Yes	No
$\kappa < \lambda \leq 2^\kappa$	Yes	No
$2^\kappa < \lambda$	Yes	Yes

Table 1. Does λ-BWT_K hold for $K = \kappa$-$\mathbb{R}$ and $K = \mathbb{R}_\kappa$?

Any cofinal sequence of length λ^+ must be divergent, thus witnessing that K is λ^+-divergent (since I was arbitrary). □

Corollary 4.10. *Let λ be any cardinal and let $(K, +, \cdot, 0, 1, \leq)$ be a λ^+-spherically complete totally ordered field with* $\mathrm{w}(K) = \mathrm{bn}(K) = \lambda^+$. *Then λ^+-BWT_K does not hold.*

Proof. Follows directly from Observation 4.3 and Lemma 4.9. □

For an uncountable cardinal κ with $\kappa^{<\kappa} = \kappa$, we shall summarise the results of this section concerning the fields κ-$\mathbb{R}$ and $\mathbb{R}_\kappa$ in Table 1. In the table, we are using Theorems 4.5 & 4.6 and Corollary 4.8, as well as the facts that $|\kappa\text{-}\mathbb{R}| = \kappa < 2^\kappa = |\mathbb{R}_\kappa|$ and that $\mathrm{bn}(\kappa\text{-}\mathbb{R}) = \mathrm{w}(\kappa\text{-}\mathbb{R}) = \mathrm{bn}(\mathbb{R}_\kappa) = \mathrm{w}(\mathbb{R}_\kappa) = \kappa$ and that $\mathbb{R}_\kappa$ is an η_κ-set (Theorems 3.1 & 3.6).

4.3. **Weakening the generalised Bolzano-Weierstraß theorem, part I: a first step.** In § 4.2, we have seen that the failure of the λ-Bolzano-Weierstraß theorem is closely related to the existence of bounded convex sets that are not intervals; their cofinal or coinitial sequences provide potential counterexamples to the Bolzano-Weierstraß theorem. This suggests a rather natural weakening of the Bolzano-Weierstraß theorem by restricting our attention to sequences that avoid this situation.

In this section, we shall define this natural weakening. As we will see, this weakened principle, *the intermediate version of Bolzano-Weierstraß*, is still too strong to hold in $\mathbb{R}_\kappa$. Moreover, we will show that, for κ weakly compact, the intermediate version of Bolzano-Weierstraß theorem and the κ-Bolzano-Weierstraß theorem are equivalent.

Definition 4.11. Let λ be a regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. Let $s : \lambda \to K$ be a λ-sequence in K and $S := \mathrm{ran}(s)$. We say that

s is *weakly interval witnessed* if for every bounded convex set C in K such that $|S \cap C| = \lambda$, there is an interval $(x, y) = I \subseteq C$ such that $|S \cap I| = \lambda$.

We then say that K satisfies the *intermediate λ-Bolzano Weierstraß theorem* if every bounded weakly interval witnessed λ-sequence in K has a convergent λ-subsequence. We abbreviate this statement with λ-iBWT_K.

Theorem 4.12. *Let λ be an uncountable regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a λ-divergent totally ordered field. Then λ-iBWT_K fails.*

Proof. Fix a bounded strictly increasing λ-sequence $t : \lambda \to K$ which exists by the assumption. Let $\mathcal{S}$ be the set of strictly increasing λ-sequences in K and $T := \lambda^{<\omega}$ be the full tree of finite sequences of ordinals in λ; this is a λ-branching tree of height ω. We now recursively assign elements of $\mathcal{S}$ to the nodes of T by a function $L : T \to \mathcal{S}$. For each $p \in T$, we write $T_p := \mathrm{ran}(L(p))$ and also write $T_n := \bigcup_{p \in \lambda^n} T_p$.

We let $L(\varnothing) := t$. If $p \in \lambda^n$ and $L(p)$ is already defined, then for each $\gamma < \lambda$, $L(p)(\gamma) < L(p)(\gamma + 1)$, so $(L(p)(\gamma), L(p)(\gamma + 1))$ is a non-empty open interval. By the assumption, we find a strictly increasing divergent λ-sequence $t_{p,\gamma}$ in this interval and let $L(p^\frown\gamma) := t_{p,\gamma}$.

By construction, it is clear that if $x = L(p)(\gamma)$ and $y = L(p')(\gamma')$, then

$$(*) \qquad x < y \text{ if and only if } p <_{\mathrm{lex}} p' \text{ or } (p = p' \text{ and } \gamma < \gamma'),$$

where $<_{\mathrm{lex}}$ is the lexicographic order.

Now fix a bijection $f : \lambda \to \lambda^{<\omega} \times \lambda$ with $f(\gamma) = (f_0(\gamma), f_1(\gamma))$ and define $s : \lambda \to K$ by

$$s(\gamma) = L(f_0(\gamma))(f_1(\gamma));$$

as usual, we write $S := \mathrm{ran}(s)$.

We claim that s is weakly interval witnessed. For this, let C be a bounded convex set such that $|S \cap C| = \lambda$. Pick any $x, y \in S \cap C$ with $L(p)(\gamma) = x < y$ for some $p \in T$ and $\gamma < \lambda$. By $(*)$ and by the construction of L, we know that $t_{p,\gamma}$ is a λ-sequence all of whose elements lie strictly between x and y, and so $|S \cap (x, y)| = \lambda$.

Finally, we claim that every λ-subsequence of s is divergent. Consider any injective $s' : \lambda \to S$ with $S' := \mathrm{ran}(s')$ and observe that since $S = \bigcup_{n \in \omega} T_n$ and λ is regular, there is some $n \in \omega$ such that $|S' \cap T_n| = \lambda$.

Case 1. There is some $p \in \lambda^n$ such that $|T_p \cap S'| = \lambda$. Then s' is a subsequence of $L(p)$ which, by construction, is a strictly increasing divergent λ-sequence and hence has no convergent subsequences.

Case 2. If that is not the case, then for every $p \in \lambda^n$, $|T_p \cap S'| < \lambda$. Define $W := \{p \in \lambda^n\,;\, 0 < |T_p \cap S'|\}$ and for each $q \in T$, $W_q := \{p \in W\,;\, q \subseteq p\}$. We say that q is *sparse* if $|W_q| < \lambda$ and we say that q is *cofinal* if $\{\gamma\,;\, W_{q^\frown\gamma} \neq \varnothing\}$ is cofinal in λ.

We now claim that there is a cofinal $q \in T$:

We first observe that if $q \in \lambda^n$, then W_q has either zero or one elements, so all sequences of length n are sparse. Also, since

$$\lambda = |S' \cap T_n| = |\bigcup_{p \in W} S' \cap T_p|,$$

we know that $|W| = |W_\varnothing| = \lambda$, so $\varnothing$ is not sparse. If all immediate successors of q are sparse, then (using the regularity of λ) either q is cofinal or q is sparse. Assume now towards a contradiction that there is no cofinal sequence, then by induction, we get that $\varnothing$ is sparse. Contradiction; so there is a cofinal sequence $q \in T$.

Towards a contradiction, let us assume that s' converges to a limit ℓ. Therefore, all of its subsequences converge to ℓ as well. We now construct recursively a subsequence s'' of s': suppose that $s''{\restriction}\alpha$ is already defined with the property that for all $\gamma < \alpha$, there is some $p_\gamma \in W_q$ such that $s''(\gamma) \in T_{p_\gamma}$. For each such p_γ, let $\widehat{\gamma}$ be the unique ordinal such that $p_\gamma \in W_{q^\frown\widehat{\gamma}}$. Since q was cofinal, find $\beta > \sup\{\widehat{\gamma}\,;\, \gamma < \alpha\}$ such that $W_{q^\frown\beta} \neq \varnothing$. Pick $p \in W_{q^\frown\beta}$ and $x \in S' \cap T_p$ and let $s''(\alpha) := x$. As usual, we let $S'' := \mathrm{ran}(s'')$.

By construction, $L(q)(\beta) < x < L(q)(\beta + 1)$, so S'' is cofinal in T_q, and therefore, $L(q)$ converges to ℓ as well. But by construction, $L(q)$ was a divergent sequence; contradiction! $\square$

Corollary 4.13. *Let λ be a regular uncountable cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. If $\mathrm{w}(K) = \lambda$ and K is an η_λ-set, then λ-iBWT_K does not hold.*

Proof. Follows from Theorems 4.7 & 4.12. $\square$

Therefore, for $\kappa > \omega$ such that $\kappa^{<\kappa} = \kappa$, κ-$\mathsf{iBWT}_{\mathbb{R}_\kappa}$ fails.

Corollary 4.14. *Let λ be any cardinal and let $(K, +, \cdot, 0, 1, \leq)$ be a λ^+-spherically complete totally ordered field with $\mathrm{w}(K) = \mathrm{bn}(K) = \lambda^+$. Then K does not satisfy the λ^+-iBWT.*

Proof. Follows from Lemma 4.9 and Theorem 4.12. $\square$

Corollary 4.15. *Let λ be a weakly compact cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. Then the following are equivalent:*

(1) λ-BWT_K *and*
(2) λ-iBWT_K.

Proof. The direction "(1)⇒(2)" is obvious, the other direction follows directly from Theorems 4.4 & 4.12. □

4.4. **Weakening the generalised Bolzano-Weierstraß theorem, part II: the main result.** In this section, we shall finally define the version of the Bolzano-Weierstraß theorem that can hold for $\mathbb{R}_\kappa$ and then characterise those κ for which it holds. Once more, κ is a regular uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

As usual, a tree is a partial order $(T, \leq)$ such that for each $t \in T$, the set $\mathrm{pred}_T(t) := \{s \in T\,;\, s < t\}$ is wellordered by $<$. The height of t in T, denoted by $\mathrm{ht}_T(t)$ is the order type of $\mathrm{pred}_T(t)$. We call $\mathrm{lvl}_T(\alpha) := \{t \in T\,;\, \mathrm{ht}_T(t) = \alpha\}$ the *αth level of the tree T*. The *height of the tree* is defined by $\mathrm{ht}(T) := \sup\{\alpha + 1\,;\, \mathrm{lvl}_T(\alpha) \neq \varnothing\}$. A *branch* of T is a maximal subset of T wellordered by $<$; the length of a branch is its ordertype. A tree $(T, <)$ is called *λ-tree* if $\mathrm{ht}(T) = \lambda$ and for all α, $|\mathrm{lvl}_T(\alpha)| < \lambda$. A cardinal λ has the *tree property* if every λ-tree has a branch of length λ. Note that if λ is strongly inaccessible then κ has the tree property if and only if λ is weakly compact; cf., e.g., [18, Theorem 7.8].

In § 4.2, we have studied counterexamples to the λ-Bolzano-Weierstraß theorem, and in the proof of Theorem 4.12, we saw how to produce a weakly interval witnessed counterexample. We implement the lessons learned from this construction and strengthen the requirement as follows:

Definition 4.16. Let λ be an uncountable regular cardinal, let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field, and let $s : \lambda \to K$ be a λ-sequence with $S := \mathrm{ran}(s)$. The sequence s is called *interval witnessed* if for every bounded convex set C in K such that $|S \cap C| = \lambda$ and every $\varepsilon \in K^+$, there is a $\mu < \lambda$ and a family of pairwise disjoint intervals of size μ, i.e., $\{I_\alpha\,;\, \alpha < \mu\} \subseteq \wp(C)$ such that

(1) for each $\alpha < \mu$, the diameter of I_α is $< \varepsilon$, and
(2) $|(S \cap C) \setminus \bigcup_{\alpha<\mu} I_\alpha| < \lambda$.

We say that K satisfies the *weak λ-Bolzano-Weierstraß theorem* if every bounded interval witnessed λ-sequence in K has a convergent λ-subsequence. We abbreviate this statement with λ-wBWT_K.

Theorem 4.17. *Let λ be an uncountable regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a Cauchy complete, λ-spherically complete totally ordered field with* $\mathrm{bn}(K) = \lambda$. *Then λ-wBWT_K implies that λ has the tree property.*

Proof. Fix a λ-tree $(T, \leq)$ and a strictly decreasing coinitial sequence $\delta : \lambda \to K^+$. For each $t \in T$, we shall assign an open interval $L(t)$ in K by recursion on the level of the node t:

If $\mathrm{lvl}_T(t) = 0$, we let $L(t) := (0, 1)$. Let us assume that we have assigned intervals $L(t)$ to all nodes of level α and assign intervals to their successors: suppose $\mathrm{lvl}_T(t) = \alpha$, then since T is a λ-tree, the set of immediate successors of t has size $\mu < \lambda$ and thus can be written as $\{t_\alpha\,;\, \alpha < \mu\}$. Apply Lemma 2.9 to $L(t)$ to obtain a family $\{I_\alpha\,;\, \alpha < \mu\}$ of pairwise disjoint intervals with the additional properties (1) to (3) and assign $L(t_\alpha) := I_\alpha$.

Now let α be a limit ordinal and assume that for all $t \in T$ of level less than α, an interval $L(t)$ has been assigned. Suppose $\mathrm{lvl}_T(s) = \alpha$ and let $b_s := \mathrm{pred}_T(s)$ be the branch leading to s, a sequence of nodes of the tree of length $\alpha < \lambda$. For $\gamma < \alpha$, if $t_\gamma \in b_s$ is the uniquely defined node of level γ, we write $I_\gamma := (x_\gamma, y_\gamma)$ and $L(t_\gamma) := I_\gamma$ for the interval assigned to it. Clearly, $C := \bigcap_{\gamma<\alpha} I_\gamma$ is a convex set, and since K is λ-spherically complete, we can apply Lemma 2.4 to find $c \in C$ and then apply Lemma 2.5 to the pair $(\{c\}, \{y_\gamma\,;\, \gamma < \alpha\})$ to find a non-empty open interval (c, d) contained in C. Without loss of generality, we can find c and d such that $|d - c| < \delta(\alpha)$.

Note that two different nodes $s \neq s'$ of level α might have the same predecessors $b_s = b_{s'}$, however, since T was a λ-tree, the number of nodes sharing the same branch must be some $\mu < \lambda$. Apply Lemma 2.9 to obtain a pairwise disjoint family of subintervals that can be assigned to each of the nodes sharing the same branch.

This completes the assignment of intervals $t \mapsto L(t)$ to the nodes $t \in T$. Note that if $t < t'$, then $L(t) \supseteq L(t')$.

Claim 4.18. *For every $\alpha < \lambda$ there is $\varepsilon \in K^+$ such that if $t, t' \in \mathrm{lvl}_T(\alpha)$ and $t \neq t'$ then $L(t)$ and $L(t')$ are separated by a distance of at least ε (i.e., for every $x \in L(t)$ and $y \in L(t')$ we have $|x - y| > \varepsilon$).*

Proof. We show the claim by induction on α. For $\alpha = 0$, there is nothing to show. Fix $\alpha > 0$ and assume that for all $\beta < \alpha$, there is some ε_β such that for any $s \neq s' \in \mathrm{lvl}_T(\beta)$, the intervals $L(s)$ and $L(s')$ are separated by a distance of at least ε_β.

For each pair $(t, t') \in \mathrm{lvl}_T(\alpha)^2$ with $t \neq t'$, we shall assign an $\varepsilon_{t,t'}$ such that $L(t)$ and $L(t')$ are separated by a distance of at least $\varepsilon_{t,t'}$.

Case 1. There is a $\gamma < \alpha$ with $s, s' \in \mathrm{lvl}_T(\gamma)$, $s < t$, $s' < t'$, and $s \neq s'$. Then by induction hypothesis, $L(s)$ and $L(s')$ are separated by a distance of at least ε_γ. Since $L(t) \subseteq L(s)$ and $L(t') \subseteq L(s')$, we can set $\varepsilon_{t,t'} := \varepsilon_\gamma$.

Case 2. Otherwise (i.e., the sets of predecessors of t and t' are the same). Then by construction, $L(t)$ and $L(t')$ were constructed by an application of Lemma 2.9. By property (3) in Lemma 2.9, there is some ε_1 such that $L(t)$ and $L(t')$ are separated by a distance of at least ε_1, so let $\varepsilon_{t,t'} := \varepsilon_1$.

Since T was a λ-tree, we have that $|\mathrm{lvl}_T(\alpha)| < \lambda$, and thus we can apply Lemma 2.5 to the pair $(\{0\}, \{\varepsilon_{t,t'}\,;\, t \neq t' \in \mathrm{lvl}_T(\alpha)\})$ to obtain some ε that works as a uniform bound for all intervals assigned to nodes in $\mathrm{lvl}_T(\alpha)$. □

We write $L(t) = (x_t, y_t)$ and define $r_t := \frac{x_t+y_t}{2}$. Since T was a λ-tree, there is a bijection $\pi : \lambda \to T$, and we can define a λ-sequence $r : \lambda \to K$ by $r(\alpha) := r_{\pi(\alpha)}$. Note that by construction (using Lemma 2.9 (2)), the function r is injective. As usual, we let $R := \mathrm{ran}(r)$.

Claim 4.19. *The sequence r is interval witnessed.*

Proof. Let $C \subseteq (0, 1)$ be a bounded convex set such that $|C \cap R| = \lambda$ and let $\varepsilon_0 \in K^+$ be arbitrary. Without loss of generality, let us assume that C has neither a supremum nor an infimum; apply Lemma 2.12 to obtain $\varepsilon_1 \in K^+$ such that for all $x \in C$, $(x - \varepsilon_1, x + \varepsilon_1) \subseteq C$. Now let $\varepsilon := \min\{\varepsilon_0, \varepsilon_1\}$.

Since δ was coinitial in K^+, find a limit ordinal $\alpha < \lambda$ such that $\delta(\alpha) < \varepsilon$. By construction, if t is a node of level α or higher, then the interval $L(t)$ assigned to t has diameter $<\delta(\alpha) < \varepsilon$. We claim that for a node t of level α, the following are equivalent:

(i) $L(t) \subseteq C$,
(ii) $r(t) \in C$, and
(iii) $L(t) \cap C \neq \varnothing$.

The directions (i)⇒(ii)⇒(iii) are obvious. The direction (iii)⇒(i) follows from the choice of ε and the fact that $L(t)$ has diameter $<\varepsilon$. Note that if t is a node of level α and $t' > t$, then $r(t') \in L(t)$. The above equivalence therefore shows that

$$\text{if } r(t') \in C, \text{ then } r(t) \in C. \tag{†}$$

Let $X := \{t \in \mathrm{lvl}_T(\alpha)\,;\, r(t) \in C\}$. Since T was a λ-tree, the set $\mathrm{lvl}_T(\alpha)$ has size $<\lambda$ and so, there is some $\mu < \lambda$ such that $|X| = \mu$; write $X = \{t_\gamma\,;\, \gamma < \mu\}$ and write $I_\gamma := L(t_\gamma)$. By construction, each I_γ is a subset of C and the diameter of I_γ is less than ε.

We still need to show property (2) of Definition 4.16: by (†) and the above equivalence, if t' is any node of level at least α and t its predecessor of level α, then $r(t') \in C$ if and only if there is a γ such that $t = t_\gamma$. In particular, $r(t') \in I_\gamma$

by construction. This means that

$$(R \cap C) \setminus \bigcup_{\gamma<\mu} I_\gamma \subseteq \{r(t)\,;\, \exists\beta < \alpha(t \in \mathrm{lvl}_T(\beta)\}$$
$$= \bigcup_{\beta<\alpha} \{r(t)\,;\, t \in \mathrm{lvl}_T(\beta)\}.$$

Because T was a λ-tree and λ was regular, this shows that the size of this set is less than λ. □

Using Claim 4.19, we can apply λ-wBWT_K to r and obtain a convergent λ-subsequence v with $V := \mathrm{ran}(v)$. Since r was injective, we have that $|V| = \lambda$ and $|T_V| = \lambda$ for $T_V := \{t \in T\,;\, r(t) \in V\}$. We write ℓ for the limit of v, so in particular, for every ε, we have that

$$(\ddagger) \qquad |\{t \in T_V\,;\, |\ell - r(t)| > \varepsilon\}| < \lambda.$$

Claim 4.20. *For every $\alpha < \lambda$, there is exactly one $t \in \mathrm{lvl}_T(\alpha)$ such that $\ell \in L(t)$.*

Proof. Note that since the intervals assigned to the nodes of level α are disjoint, there can be at most one such $t \in \mathrm{lvl}_T(\alpha)$. We shall show by induction that each level contains such a t. By construction, we have $\ell \in (0, 1)$, which resolves the case $\alpha = 0$.

Let $\alpha > 0$ be arbitrary and assume that for each $\beta < \alpha$, there is a node $t \in \mathrm{lvl}_T(\beta)$ such that $\ell \in L(\beta)$. Note that these nodes must form a branch b through the tree of height α. Since T is a λ-tree, we let $\mathrm{lvl}_T(\alpha) = \{t_\gamma\,;\, \gamma < \mu\}$ for some $\mu < \lambda$. We write $T_{<\alpha} := \bigcup_{\beta<\alpha} \mathrm{lvl}_T(\beta)$ and $T_{\downarrow\gamma} := \{t \in T\,;\, t_\gamma \leq t\}$ and observe that we can write T as a disjoint union

$$T = T_{<\alpha} \cup \bigcup_{\gamma<\mu} T_{\downarrow\gamma}.$$

Clearly, by the fact that T was a λ-tree and by regularity of λ, $|T_{<\alpha}| < \lambda$.

We shall consider $T_V \cap T_{\downarrow\gamma}$ for each $\gamma < \mu$ and observe that there are three possible cases:

Case 1: the set of predecessors of t_γ is not the branch b. That means that there is some level $\beta < \alpha$ where the path to s diverged from the branch b. Let ε_1 be the separation bound for the intervals assigned to nodes of level β. Then for every element $x \in L(t_\gamma)$ (and thus for every $x \in L(s)$ where s is a successor of t_γ), we have that $|\ell - x| > \varepsilon_1$. By ($\ddagger$), we see that $|T_V \cap T_{\downarrow\gamma}| < \lambda$.

Case 2: the set of predecessors of t_γ is the branch b, but $\ell \notin L(t_\gamma)$. The intervals assigned to the immediate successors of t_γ are constructed using Lemma 2.9, and so there is an ε_2 such that for each successor s of t_γ and each $x \in L(s)$, we have $|\ell - x| > \varepsilon_2$. Once more, by (‡), we see that $|T_V \cap T_{\downarrow\gamma}| < \lambda$.

Case 3: the set of predecessors of t_γ is the branch b and $\ell \in L(t_\gamma)$. In the induction step, we need to show that there is a γ such that this case occurs.

If we now suppose towards a contradiction that *Case 3* never occurs, then

$$\begin{aligned} T_V &= T_V \cap \left(T_{<\alpha} \cup \bigcup_{\gamma<\mu} T_{\downarrow\gamma} \right) \\ &= (T_V \cap T_{<\alpha}) \cup \bigcup_{\gamma<\mu} (T_V \cap T_{\downarrow\gamma}), \end{aligned}$$

where by *Cases 1 & 2* each of the summands has size smaller than λ, so by regularity of λ, we obtain $|T_V| < \lambda$. Contradiction! □

Claim 4.20 directly gives us a branch of length λ through the tree T. □

Theorem 4.21. *Let κ be an uncountable strongly inaccessible cardinal and $(K, +, \cdot, 0, 1, \leq)$ a Cauchy complete ordered field with $\mathrm{bn}(K) = \kappa$. If κ has the tree property then K satisfies the κ-wBWT property.*

Proof. Let $s : \kappa \to K$ be an interval witnessed bounded κ-sequence (without loss of generality, s is an injective function), $S := \mathrm{ran}(s)$ and $\delta : \kappa \to K^+$ be a strictly decreasing sequence coinitial in K^+. Let (x^*, y^*) be any interval in K containing S. For each $\alpha < \kappa$, we define a set of pairwise disjoint intervals T_α by recursion. The construction will guarantee that

(1) for each $\alpha < \kappa$, $|T_\alpha| < \kappa$,
(2) for each $\alpha < \kappa$ and each $I \in T_\alpha$, we have that $|S \cap I| = \kappa$, and
(3) for each $\alpha < \beta < \kappa$ and every $I \in T_\beta$, there is a $J \in T_\alpha$ such that I is a subinterval of J,

so in particular

$$\bigcup\{I\,;\, I \in T_\beta\} \subseteq \bigcup\{I\,;\, I \in T_\alpha\}.$$

We define $S_\alpha := S \cap \bigcup\{I\,;\, I \in T_\alpha\}$ and $M_\alpha := S \backslash S_\alpha$. Property (2) implies that for each $\alpha < \kappa$, $|S_\alpha| = \kappa$. We shall furthermore check that

(4) for each $\alpha < \kappa$, we have $|M_\alpha| < \kappa$.

Case $\alpha = 0$. We let $T_0 := \{(x^*, y^*)\}$. Properties (1), (2), and (3) are obviously satisfied. Note that by choice of (x^*, y^*), we have that $S_0 = S$ and so $M_0 = \varnothing$, whence (4) is satisfied as well.

Case $\alpha = \beta + 1$. If $(x, y) \in T_\beta$, define $L_{x,y} := (x, \frac{x+y}{2})$, $R_{x,y} := (\frac{x+y}{2}, y)$, and $T_\alpha := \{L_{x,y}\,;\, (x, y) \in T_\beta \text{ and } |L_{x,y} \cap S| = \kappa\} \cup \{R_{x,y}\,;\, (x, y) \in T_\beta \text{ and } |R_{x,y} \cap S| =$

$\kappa\}$. Clearly, $|T_\alpha| \leq |2 \times T_\beta| < \kappa$, so (1) is satisfied. Properties (2) and (3) are satisfied by construction. Since $|T_\beta| < \kappa$ and κ is regular, we know that both $L_\alpha := \bigcup\{S \cap L_{x,y}\,;\, |S \cap L_{x,y}| < \kappa\}$ and $R_\alpha := \bigcup\{S \cap R_{x,y}\,;\, |S \cap R_{x,y}| < \kappa\}$ have size less than κ. Thus

$$S_\beta = S_\alpha \cup \left\{ \frac{x+y}{2}\,;\, (x,y) \in T_\beta \right\} \cup L_\alpha \cup R_\alpha,$$

so using inductively property (4) for M_β, we have that $|M_\alpha| = |M_\beta| + |T_\beta| + |L_\alpha| + |R_\alpha| < \kappa$ and thus (4) is satisfied.

Case α limit ordinal. Consider the tree $T_{<\alpha} := \bigcup_{\beta<\alpha} T_\beta$ ordered by reverse inclusion and let $\mathcal{B}$ be the set of branches through this tree. The strong inaccessibility of κ implies that $|\mathcal{B}| < \kappa$. For $b \in \mathcal{B}$, the set $C_b := \bigcap\{I\,;\, I \in b\}$ is a convex set.

Claim 4.22. *We have that $S \setminus \bigcup_{\beta<\alpha} M_\beta = S \cap \bigcup_{b\in\mathcal{B}} C_b$.*

Proof. "$\subseteq$": If x is not in any M_β, then for every $\beta < \alpha$, there is an $I_\beta \in T_\beta$ such that $x \in I_\beta$. By construction, these intervals form a branch $b := \{I_\beta\,;\, \beta < \alpha\}$ in the tree $T_{<\alpha}$ and $x \in C_b$. "$\supseteq$": If $x \in S \cap C_b$, then the elements of the branch b witness that $x \notin M_\beta$ for any $\beta < \alpha$. □

By regularity of κ and the inductive assumption that all earlier levels satisfy property (4), we know that $\bigcup_{\beta<\alpha} M_\beta$ has size less than κ, so by Claim 4.22, we know that $|S \cap \bigcup_{b\in\mathcal{B}} C_b| = \kappa$. Since $|\mathcal{B}| < \kappa$, we know that there are branches $b \in \mathcal{B}$ such that $|S \cap C_b| = \kappa$.

Consequently, we can apply the fact that s was interval witnessed to such a convex set C_b and find a set $\mathcal{I}_b$ of fewer than κ many subintervals of C_b with diameter $<\delta(\alpha)$ such that $|S \cap (C_b \setminus \bigcup \mathcal{I}_b)| < \kappa$. Now let $T_\alpha := \{I\,;$ there is a $b \in \mathcal{B}$ such that $|S \cap C_b| = \kappa$ and $I \in \mathcal{I}_b$ and $|S \cap I| = \kappa\}$.

Property (1) follows from the facts that κ is regular, $|\mathcal{B}| < \kappa$, and for each $b \in \mathcal{B}$, $|\mathcal{I}_b| < \kappa$. Property (2) and (3) are clear by construction. Let $W_0 := \bigcup\{S \cap C_b\,;\, |S \cap C_b| < \kappa\}$; once more, by regularity of κ and $|\mathcal{B}| < \kappa$, we get that $|W_0| < \kappa$. Furthermore, let $W_1 := \bigcup\{S \cap (C_b \setminus \bigcup \mathcal{I}_b)\,;\, |S \cap C_b| = \kappa\}$; again, regularity of κ and the choice of $\mathcal{I}_b$ implies that $|W_1| < \kappa$. But $M_\alpha = \bigcup_{\beta<\alpha} M_\beta \cup W_0 \cup W_1$, so it has size less than κ, and thus we checked that property (4) holds as well.

This finishes the recursive construction. From property (1), it follows that the resulting tree $T := \bigcup_{\alpha<\kappa} T_\alpha$ is a κ-tree, so by the tree property, T has a branch $b = \{I_\alpha\,;\, \alpha < \kappa\}$. For each $\alpha < \kappa$, pick some $r_\alpha \in S \cap I_\alpha$. By the choice of the diameter of the intervals at the limit levels, the sequence $\alpha \mapsto r_\alpha$ is a Cauchy subsequence of s, thus by Cauchy completeness of K, it is convergent. □

Corollary 4.23. *Let κ be an uncountable strongly inaccessible cardinal and let $(K, +, \cdot, 0, 1, \leq)$ be a Cauchy complete and κ-spherically complete totally ordered field with* $\mathrm{bn}(K) = \kappa$. *Then the following are equivalent:*

(1) *κ has the tree property and*
(2) *κ-wBWT_K holds.*

In particular, κ has the tree property if and only κ-$\mathsf{wBWT}_{\mathbb{R}_\kappa}$ holds.

As we have seen in § 4.3, if κ is weakly compact then the κ-Bolzano-Weierstraß theorem and the intermediate κ-Bolzano-Weierstraß theorem are equivalent. In this case, as Corollary 4.23 shows, the weak κ-Bolzano-Weierstraß theorem becomes a natural generalisation of the classical Bolzano-Weierstraß theorem.

5. The generalised Heine-Borel theorem

We end this paper by considering a generalised version of the Heine-Borel theorem. First recall that the Heine-Borel theorem for $\mathbb{R}$ can be stated as follows:

Theorem 5.1. *For every set $X \subseteq \mathbb{R}$, the following are equivalent:*

(i) *X is closed and bounded,*
(ii) *every open cover of X has a finite subcover, i.e., X is compact.*

In order to generalise the Heine-Borel theorem to uncountable cardinals, we remind the reader of the concept of κ-metrisability from § 2.2: since every totally ordered group of base number κ is trivially κ-metrisable, κ-$\mathbb{R}$ and $\mathbb{R}_\kappa$ are both κ-metrisable topological fields. Since κ-$\mathbb{R}$ is a totally ordered subgroup of $\mathbb{R}_\kappa$, every (κ-$\mathbb{R}$)-metrisable space is also $\mathbb{R}_\kappa$-metrisable, and thus κ-$\mathbb{R}$ is $\mathbb{R}_\kappa$-metrisable. Moreover, [26, Theorems viii & x] show that $\mathbb{R}_\kappa$ is (κ-$\mathbb{R}$)-metrisable. We do not know whether notions of (κ-$\mathbb{R}$)-metrisability and $\mathbb{R}_\kappa$-metrisability coincide.

Let (X, τ) be a topological space and λ be a cardinal. Then (X, τ) is *λ-compact* if every open cover of X of cardinality λ has a subcover of cardinality $<\lambda$; (X, τ) is *λ-sequentially compact* iff every λ-sequence has a convergent λ-subsequence.

Theorem 5.2. *Let (X, τ) be λ-metrisable. Then (X, τ) is λ-compact if and only if it is λ-sequentially compact.*

Proof. The standard proof of the equivalence of compactness and sequential compactness transfers directly to the case of G-metrics for a totally ordered group $(G, +, 0, \leq)$ with $\mathrm{bn}(G) = \lambda$. □

The following natural generalisation of the Heine-Borel theorem is due to Cowles and LaGrange [7]:

Definition 5.3. Let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field and λ be a cardinal. Then we shall say that K satisfies the *λ-Heine-Borel theorem* if for every $X \subseteq K$ the following are equivalent:

(1) X is closed and bounded,
(2) X is λ-compact.

We abbreviate this statement as $\lambda\text{-}\mathsf{HBT}_K$.

Theorem 5.4 (Cowles & LaGrange, 1983)**.** *Let K be ordered field with* $\mathrm{bn}(K) = \lambda$. *Then $\lambda\text{-}\mathsf{BWT}_K$ holds if and only if $\lambda\text{-}\mathsf{HBT}_K$ holds.*

Proof. Cf. [7, p. 136]. □

Corollary 5.5. *For every regular cardinal λ, we have that $\lambda\text{-}\mathsf{HBT}_{\lambda\text{-}\mathbb{R}}$ holds.*

Proof. Follows from Theorem 5.4 and Theorem 4.6. □

Corollary 5.6. *Let λ be an uncountable regular cardinal and let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field with* $\mathrm{w}(K) = \lambda$ *which is an η_λ-set. Then $\lambda\text{-}\mathsf{HBT}_K$ does not hold.*

Proof. Follows from Theorem 5.4 and Theorem 4.7. □

In particular, if κ is such that $\kappa^{<\kappa} = \kappa$, then $\mathbb{R}_\kappa$ does not satisfy the κ-Heine-Borel theorem. The underlying reason for this is that closed intervals in $\mathbb{R}_\kappa$ are not κ-compact.

Proposition 5.7. *Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Then closed intervals in $\mathbb{R}_\kappa$ are not κ-compact.*

Proof. Let I be a closed interval; we use the proof of Lemma 2.7 to find a strictly increasing ω-sequence $s : \omega \to I$ such that the set B of its upper bounds has coinitiality κ. Take a coinitial sequence $t : \kappa \to B$ and two elements $x < I$ and $y > I$. Then the family

$$\{(x, s(n))\,;\, n \in \omega\} \cup \{(t(\alpha), y)\,;\, \alpha < \kappa\}$$

is an open cover of I that has no subcover of size less than κ. □

In line with the definitions from § 4.4, we say that a topological space (X, τ) is called *interval witnessed κ-sequentially compact* if every interval witnessed κ-sequence in X has a convergent subsequence. If $(K, +, \cdot, 0, 1, \leq)$ is a totally ordered field and κ be a cardinal, we shall say that K satisfies the *weak κ-Heine-Borel theorem* (in symbols: $\kappa\text{-}\mathsf{wHBT}_K$) if for every $X \subseteq K$, the following are equivalent:

(1) X is closed and bounded,
(2) X is interval witnessed κ-sequentially compact.

As for the classical case it turns out that for ordered fields of base number κ, κ-wHBT and κ-wBWT are equivalent.

Theorem 5.8. *Let λ be an uncountable regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field with* $\mathrm{bn}(K) = \lambda$. *Then λ-wBWT_K holds if and only if λ-wHBT_K holds.*

Proof. Clearly, if λ-wHBT_K, then λ-wBWT_K. Also, if X is bounded and closed and λ-wBWT_K holds, then X is interval witnessed λ-sequentially compact.

So, let us now assume that λ-wBWT_K holds and that X is interval witnessed λ-sequentially compact. If $s : \lambda \to X$ is a sequence converging in K, then this is a Cauchy sequence, and hence interval witnessed. Thus by interval witnessed λ-sequential compactness, s must converge to an element of X; hence, X is closed. Finally, assume towards a contradiction that X is unbounded in K, so there is a strictly increasing sequence $t : \lambda \to X$ cofinal in K. But then, no bounded convex set contains λ many elements of $\mathrm{ran}(t)$ and therefore, t is trivially interval witnessed. By interval witnessed λ-sequential compactness, t converges contradicting the assumption that it is cofinal in K. □

We combine Corollary 4.23 with Theorem 5.8:

Corollary 5.9. *Let κ be an uncountable strongly inaccessible cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a Cauchy complete κ-spherically complete ordered field with* $\mathrm{bn}(K) = \kappa$. *Then the following are equivalent:*

(1) *κ has the tree property,*
(2) *κ-wBWT_K holds, and*
(3) *κ-wHBT_K holds.*

In particular, κ has the tree property if and only if κ-$\mathsf{wBWT}_{\mathbb{R}_\kappa}$ holds if and only if κ-$\mathsf{wHBT}_{\mathbb{R}_\kappa}$ holds.

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