Constructing illoyal algebra-valued models of set theory

BENEDIKT LÖWE

Institute for Logic, Language and Computation, Universiteit van Amsterdam
Fachbereich Mathematik, Universität Hamburg
Churchill College, University of Cambridge

and

ROBERT PASSMANN

Institute for Logic, Language and Computation, Universiteit van Amsterdam

and

SOURAV TARAFDER

Department of Mathematics, St. Xavier's College, Kolkata

Abstract. An algebra-valued model of set theory is called loyal to its algebra if the model and its algebra have the same propositional logic; it is called faithful if all elements of the algebra are truth values of a sentence of the language of set theory in the model. We observe that non-trivial automorphisms of the algebra result in models that are not faithful and apply this to construct three classes of illoyal models: the tail stretches, the transposition twists, and the maximal twists. (Version 7; 2 October 2018)

§1. Background The construction of *algebra-valued models of set theory* starts from an algebra $\mathbb{A}$ and a model V of set theory and forms an $\mathbb{A}$-valued model of set theory that reflects both the set theory of V and the logic of $\mathbb{A}$. This construction is the natural generalisation of Boolean-valued models, Heyting-valued models, lattice-valued models, and orthomodular-valued models (Bell, 2005; Grayson, 1979; Ozawa, 2009; Titani, 1999) and was developed by Löwe and Tarafder (2015).

Löwe and Tarafder (2015, § 6) used an algebra $\mathbb{PS}_3$ of paraconsistent logic to construct a $\mathbb{PS}_3$-valued model of set theory that exhibits the paraconsistency inherited from the algebra $\mathbb{PS}_3$. For more on the algebra $\mathbb{PS}_3$, cf. (Chakraborty and Tarafder, 2016); for more on the set theory in the $\mathbb{PS}_3$-valued model, cf. (Tarafder, 2015).

Recently, Passmann (2018) introduced the terms "loyalty" and "faithfulness" while studying the precise relationship between the logic of the algebra $\mathbb{A}$ and the logical phenomena witnessed in the $\mathbb{A}$-valued model of set theory. A model is called *loyal* to its algebra if the propositional logic in the model is the same as the logic of the algebra from which it was constructed and *faithful* if every element of the algebra is the truth value of a sentence in the model. The model constructed by Löwe and Tarafder (2015) is both loyal and faithful to $\mathbb{PS}_3$. In this paper, we shall give elementary constructions to produce illoyal models by stretching and twisting Boolean algebras.

This research was partially supported by the Marie Skłodowska-Curie fellowship REGPROP (706219) funded by the European Commission at the Universität Hamburg. The authors would like to thank Nick Bezhanishvili and Lorenzo Galeotti for various discussions about Heyting algebras and their logics.

After we give the basic definitions in §2., we remind the reader of the construction of algebra-valued models of set theory in §3.. In §4., we introduce our main technique: a non-trivial automorphisms of an algebra $\mathbb{A}$ excludes values from being truth values of sentences in the $\mathbb{A}$-valued model of set theory (Corollary 7). Finally, in §5., we apply this technique to produce three classes of models: tail stretches (§ 5.2.), transposition twists (§ 5.3.), and maximal twists (§ 5.4.).

§2. Basic definitions

Algebras. Let Λ be a set of logical connectives; in this paper, we shall assume that

$$\{\wedge, \vee, \mathbf{0}, \mathbf{1}\} \subseteq \Lambda \subseteq \{\wedge, \vee, \rightarrow, \neg, \mathbf{0}, \mathbf{1}\}.$$

An algebra $\mathbb{A}$ with underlying set A is called a Λ-*algebra* if it has one operation for each of the logical connectives in Λ such that $(A, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a distributive lattice;[1] we can define $\leq$ on $\mathbb{A}$ by $x \leq y$ if and only if $x \wedge y = x$. An element $a \in A$ is an *atom* if it is $\leq$-minimal in $A\backslash\{\mathbf{0}\}$; we write $\mathrm{At}(\mathbb{A})$ for the set of atoms in $\mathbb{A}$. If $\Lambda = \{\wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1}\}$, we call $\mathbb{A}$ an *implication algebra* and if $\Lambda = \{\wedge, \vee, \rightarrow \neg, \mathbf{0}, \mathbf{1}\}$, we call $\mathbb{A}$ an *implication-negation algebra*.

We call a Λ-algebra $\mathbb{A}$ with underlying set A *complete* if for every $X \subseteq A$, the $\leq$-supremum and $\leq$-infimum exist; in this case, we write $\bigvee X$ and $\bigwedge X$ for these elements of $\mathbb{A}$. A complete Λ-algebra $\mathbb{A}$ is called *atomic* if for every $a \in A$, there is an $X \subseteq \mathrm{At}(\mathbb{A})$ such that $a = \bigvee X$.

Boolean algebras, complementation, & Heyting algebras. An algebra $\mathbb{B} = (B, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1})$ is called a *Boolean algebra* if for all $b \in B$, we have that $b \wedge \neg b = \mathbf{0}$ and $b \vee \neg b = \mathbf{1}$. As usual, we can define an implication by

$$x \rightarrow y := \neg x \vee y; \tag{\#}$$

using this definition, we can consider Boolean algebras as implication algebras or implication-negation algebras. An implication algebra $(B, \wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1})$ is called a *Boolean implication algebra* if there is a Boolean algebra $(B, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1})$ such that $\rightarrow$ is defined by (#) from $\vee$ and $\neg$ or, equivalently, if the negation defined by $\neg_* x := x \rightarrow \mathbf{0}$ satisfies $\neg_* b \wedge b = \mathbf{0}$ and $\neg_* b \vee b = \mathbf{1}$.

On an atomic distributive lattice $\mathbb{A} = (A, \wedge, \vee, \mathbf{0}, \mathbf{1})$, we have a canonical definition for a negation operation, the *complementation negation*: since $\mathbb{A}$ is atomic, every element $a \in A$ is uniquely represented by a set $X \subseteq \mathrm{At}(\mathbb{A})$ such that $a = \bigvee X$. Then we define the complementation negation by

$$\neg_{\mathrm{c}}(\bigvee X) := \bigvee\{t \in \mathrm{At}(\mathbb{A})\,;\, t \notin X\}.$$

In this situation, $(A, \wedge, \vee, \neg_{\mathrm{c}}, \mathbf{0}, \mathbf{1})$ is an atomic Boolean algebra. Moreover, if $(A, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1})$ is an atomic Boolean algebra and $\neg_{\mathrm{c}}$ is the complementation negation of the atomic distributive lattice $(A, \wedge, \vee, \mathbf{0}, \mathbf{1})$, then $\neg = \neg_{\mathrm{c}}$. Of course, for

[1] As usual, we use the same notation for the syntactic logical connectives and the operations on $\mathbb{A}$ interpreting them. In the rare cases where proper marking of these symbols improves readability, we attach a subscript $\mathbb{A}$ to the algebra operations in $\mathbb{A}$, e.g., $\wedge_{\mathbb{A}}$, $\vee_{\mathbb{A}}$, $\bigwedge_{\mathbb{A}}$, or $\bigvee_{\mathbb{A}}$.

every set X, the power set algebra $(\wp(X), \cap, \cup, \varnothing, X)$ forms an atomic distributive lattice and, with the set complementation operator, a Boolean algebra.

If $(H, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a complete distributive lattice, then an implication algebra $\mathbb{H} = (H, \wedge, \vee, \to, \mathbf{0}, \mathbf{1})$ is called a *complete Heyting algebra* if and only if for all $a, b \in H$, we have that

$$a \to b = \bigvee \{x \in H\,;\, a \wedge x \leq b\}.$$

In a Heyting algebra $\mathbb{H}$, we can define a negation $\neg_{\mathbb{H}}$ by $\neg_{\mathbb{H}} x := x \to \mathbf{0}$. Note that Boolean implication algebras are Heyting algebras.

A Heyting algebra is called *linear* if $(H, \leq)$ is a linear order; Horn (1969) showed that the formula $(p \to q) \vee (q \to p)$ characterises the variety of Heyting algebras generated by the linear Heyting algebras.

We shall later use the following linear three element Heyting algebra $\mathbf{3} := (\{\mathbf{0}, \mathbf{1/2}, \mathbf{1}\}, \wedge, \vee, \to, \mathbf{0}, \mathbf{1})$ with $\mathbf{0} \leq \mathbf{1/2} \leq \mathbf{1}$ and $\to$ defined by

$\to$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$

.

Languages. Fix a set S of non-logical symbols, a countable set P of propositional variables, and a countable set V of first-order variables. We denote the propositional logic with connectives Λ and propositional variable P by $\mathcal{L}_\Lambda$ and the first-order logic with connectives Λ, variables in V and constant, relation and function symbols in S by $\mathcal{L}_{\Lambda,S}$. The subset of sentences of $\mathcal{L}_{\Lambda,S}$ will be denoted by $\mathrm{Sent}_{\Lambda,S}$. Note that both $\mathcal{L}_\Lambda$ and $\mathrm{Sent}_{\Lambda,S}$ have the structure of a Λ-algebra and that the Λ-algebra $\mathcal{L}_\Lambda$ is generated by closure under the connectives in Λ from the set P.

For arbitrary sets Λ of logical connectives and S of non-logical symbols, we define NFF_Λ to be the closure of P under the logical connectives other than $\neg$ and $\mathrm{NFF}_{\Lambda,S}$ to be the closure of the atomic formulae of $\mathcal{L}_{\Lambda,S}$ under the logical connectives other than $\neg$. These formulas are called the *negation-free* Λ*-formulas*. Clearly, if $\neg \notin \Lambda$, then $\mathcal{L}_\Lambda = \mathrm{NFF}_\Lambda$ and $\mathcal{L}_{\Lambda,S} = \mathrm{NFF}_{\Lambda,S}$.

Homomorphisms, assignments, & translations. For any two Λ-algebras $\mathbb{A}$ and $\mathbb{B}$, a map $f : A \to B$ is called a Λ*-homomorphism* if it preserves all connectives in Λ; it is called a Λ*-isomorphism* if it is a bijective Λ-homomorphism; isomorphisms from $\mathbb{A}$ to $\mathbb{A}$ are called Λ*-automorphisms*.

If $\mathbb{A}$ and $\mathbb{B}$ are two complete Λ-algebras and $f : A \to B$ is a Λ-homomorphism, we call it *complete* if it preserves the operations $\bigvee$ and $\bigwedge$, i.e., $f(\bigvee_{\mathbb{A}}(X)) = \bigvee_{\mathbb{B}}(\{f(x)\,;\, x \in X\})$ and $f(\bigwedge_{\mathbb{A}}(X)) = \bigwedge_{\mathbb{B}}(\{f(x)\,;\, x \in X\})$ for $X \subseteq A$.

Since $\mathcal{L}_\Lambda$ is generated from P, we can think of any Λ-homomorphism defined on $\mathcal{L}_\Lambda$ as a function on P, homomorphically extended to all of $\mathcal{L}_\Lambda$. If $\mathbb{A}$ is a Λ-algebra with underlying set A, we say that Λ-homomorphisms $\iota : \mathcal{L}_\Lambda \to A$ are $\mathbb{A}$*-assignments*; if S is a set of non-logical symbols, we say that Λ-homomorphisms $T : \mathcal{L}_\Lambda \to \mathrm{Sent}_{\Lambda,S}$ are S*-translations*.

The propositional logic of an algebra. A set $D \subseteq A$ is called a *designated set* if the following three conditions hold: (i) $\mathbf{1} \in D$, (ii) $\mathbf{0} \notin D$, and (iii) if $x \in D$ and $x \leq y$, then $y \in D$. A designated set D is called a *filter* if, in addition, (iv) for

$x, y \in D$, we have $x \wedge y \in D$. For any designated set D, the *propositional logic of* $(\mathbb{A}, D)$ is defined as

$$\mathbf{L}(\mathbb{A}, D) := \{\varphi \in \mathcal{L}_\Lambda \,;\, \iota(\varphi) \in D \text{ for all } \mathbb{A}\text{-assignments } \iota\}.$$

Note that if $\mathbb{B}$ is a Boolean algebra and D is any filter, then $\mathbf{L}(\mathbb{B}, D) = \mathbf{CPC}$, the classical propositional calculus (Blackburn et al., 2001, Theorem 5.11).

Algebra-valued structures and their propositional logic. If $\mathbb{A}$ is a Λ-algebra and S is a set of non-logical symbols, then any Λ-homomorphism $[\![\cdot]\!] : \mathrm{Sent}_{\Lambda,S} \to A$ will be called an $\mathbb{A}$-*valued* S-*structure*. Note that if $S' \subseteq S$ and $[\![\cdot]\!]$ is an $\mathbb{A}$-valued S-structure, then $[\![\cdot]\!]{\upharpoonright}\mathrm{Sent}_{\Lambda,S'}$ is an $\mathbb{A}$-valued S'-structure.

We define the *propositional logic of* $([\![\cdot]\!], D)$ as

$$\mathbf{L}([\![\cdot]\!], D) := \{\varphi \in \mathcal{L}_\Lambda \,;\, [\![T(\varphi)]\!] \in D \text{ for all } S\text{-translations } T\}.$$

Note that if T is an S-translation and $[\![\cdot]\!]$ is an $\mathbb{A}$-valued S-structure, then $\varphi \mapsto [\![T(\varphi)]\!]$ is an $\mathbb{A}$-assignment, so

$$\mathbf{L}(\mathbb{A}, D) \subseteq \mathbf{L}([\![\cdot]\!], D). \tag{$\dagger$}$$

Clearly, $\mathrm{ran}([\![\cdot]\!]) \subseteq A$ is closed under all operations in Λ (since $[\![\cdot]\!]$ is a homomorphism) and thus defines a sub-Λ-algebra $\mathbb{A}_{[\![\cdot]\!]}$ of $\mathbb{A}$. The $\mathbb{A}$-assignments that are of the form $\varphi \mapsto [\![T(\varphi)]\!]$ are exactly the $\mathbb{A}_{[\![\cdot]\!]}$-assignments, so we obtain

$$\mathbf{L}([\![\cdot]\!], D) = \mathbf{L}(\mathbb{A}_{[\![\cdot]\!]}, D).$$

Loyalty & faithfulness. An $\mathbb{A}$-valued S-structure $[\![\cdot]\!]$ is called *loyal to* $(\mathbb{A}, D)$ if the converse of ($\dagger$) holds, i.e., $\mathbf{L}(\mathbb{A}, D) = \mathbf{L}([\![\cdot]\!], D)$; it is called *faithful to* $\mathbb{A}$ if for every $a \in A$, there is a $\varphi \in \mathrm{Sent}_{\Lambda,S}$ such that $[\![\varphi]\!] = a$; equivalently, if $\mathbb{A}_{[\![\cdot]\!]} = \mathbb{A}$. The two notions central for our paper were introduced by Passmann (2018) in a more general setting for classes of so-called *Heyting structures* in the sense of Fourman and Scott (1979) (cf. Passmann, 2018, Definitions 2.39 & 2.40). In this paper, we shall not need the level of generality provided in (Passmann, 2018) and stick to the above simplified definitions.

Lemma 1 *Let Λ be any set of propositional connectives, S be any set of non-logical symbols, $\mathbb{A}$ be a Λ-algebra, and $[\![\cdot]\!]$ be an $\mathbb{A}$-valued S-structure. Then, if $[\![\cdot]\!]$ is faithful to $\mathbb{A}$, then it is loyal to $(\mathbb{A}, D)$ for any designated set D.*

Proof. By ($\dagger$), we only need to prove one inclusion; if $\varphi \notin \mathbf{L}(\mathbb{A}, D)$, then let $p_1, \ldots, p_n$ be the propositional variables occurring in φ and let ι be an assignment such that $\iota(\varphi) \notin D$. By faithfulness, find sentences $\sigma_i \in \mathrm{Sent}_{\Lambda,S}$ such that $[\![\sigma_i]\!] = \iota(p_i)$ for $1 \leq i \leq n$. Let T be any translation such that $T(p_i) = \sigma_i$ for $1 \leq i \leq n$. Then $[\![T(\varphi)]\!] = \iota(\varphi) \notin D$, and hence T witnesses that $\varphi \notin \mathbf{L}([\![\cdot]\!], D)$. (Cf. Passmann, 2018, Proposition 2.50, for a proof in the more general setting for classes of Heyting structures.) □

Note that the notions of faithfulness and loyalty crucially depend on the choice of S. As mentioned above, if $S' \subseteq S$ and $[\![\cdot]\!]$ is an $\mathbb{A}$-valued S-structure, then $[\![\cdot]\!]' := [\![\cdot]\!]{\upharpoonright}\mathrm{Sent}_{\Lambda,S'}$ is an $\mathbb{A}$-valued S'-structure. Since $\mathrm{Sent}_{\Lambda,S'} \subseteq \mathrm{Sent}_{\Lambda,S}$, we have that if $[\![\cdot]\!]'$ is faithful to $\mathbb{A}$, then so is $[\![\cdot]\!]$.

$$\forall x\forall y[\forall z(z\in x\leftrightarrow z\in y)\to x=y] \qquad \text{(Extensionality)}$$
$$\forall x\forall y\exists z\forall w(w\in z\leftrightarrow(w=x\vee w=y)) \qquad \text{(Pairing)}$$
$$\exists x[\exists y(\forall z(z\in y\to \mathbf{0})\wedge y\in x)\wedge\forall w\in x\exists u\in x(w\in u)] \qquad \text{(Infinity)}$$
$$\forall x\exists y\forall z(z\in y\leftrightarrow\exists w\in x(z\in x)) \qquad \text{(Union)}$$
$$\forall x\exists y\forall z(z\in y\leftrightarrow\forall w\in z(w\in x)) \qquad \text{(Power Set)}$$
$$\forall p_0\cdots\forall p_n\forall x\exists y\forall z(z\in y\leftrightarrow z\in x\wedge\varphi(z,p_0,\ldots,p_n)) \qquad \text{(Separation}_\varphi\text{)}$$
$$\forall p_0\cdots\forall p_{n-1}\forall x[\forall y\in x\exists z\varphi(y,z,p_0,\ldots,p_{n-1}) \to \exists w\forall v\in x\exists u\in w\ \varphi(v,u,p_0,\ldots,p_{n-1})] \qquad \text{(Collection}_\varphi\text{)}$$
$$\forall p_0\cdots\forall p_n\forall x[\forall y\in x\ \varphi(y,p_0,\ldots,p_n)\to\varphi(x,p_0,\ldots,p_n)] \to\forall z\varphi(z,p_0,\ldots,p_n) \qquad \text{(Set Induction}_\varphi\text{)}$$

Fig. 1. The axioms of ZF formulated in $\mathcal{L}_{\{\wedge,\vee,\to,\mathbf{0},\mathbf{1}\},\{\in\}}$.

However, faithfulness ties $[\![\cdot]\!]$ very closely to the algebra $\mathbb{A}$: in particular, it cannot hold if the algebra $\mathbb{A}$ is bigger than the set $\mathrm{Sent}_{\Lambda,S}$, so for countable languages, no $\mathbb{A}$-valued S-structure can be faithful to an uncountable algebra $\mathbb{A}$.

Thus, if $\mathbb{A}$ is an uncountable algebra, S an uncountable set of non-logical symbols, $[\![\cdot]\!]$ is an $\mathbb{A}$-valued S-structure that is faithful to $\mathbb{A}$, and S' is a countable subset of S, then $[\![\cdot]\!]' := [\![\cdot]\!]\restriction\mathcal{L}_{\Lambda,S'}$ cannot be faithful to $\mathbb{A}$.

§3. Algebra-valued models of set theory In the following, we give an overview of general construction of an algebra-valued model of set theory following Löwe and Tarafder (2015). The original ideas go back to Boolean-valued models independently discovered by Solovay and Vopěnka (1965) and were further generalised to other classes of algebras (Grayson, 1979; Ozawa, 2007, 2009; Takeuti and Titani, 1992; Titani, 1999; Titani and Kozawa, 2003). Details can be found in (Bell, 2005).

In the following, we shall use the phrase "*V is a model of set theory*" to mean that V is a transitive set such that $(V,\in)\models$ ZF. Of course, the existence of sets like this cannot be proved in ZF and requires some (mild) additional metamathematical assumptions. The choice of ZF as the set theory in our base model is not relevant for the constructions of this paper and one can generalise the results to models of weaker or alternative set theories; however, we shall not explore this route in this paper.

Since we are sometimes working in languages without negation, we need to formulate the axioms of ZF in a negation-free context given in Figure 1, following Löwe and Tarafder (2015, § 3).[2] If V is a model of set theory and A is any set, then

[2] We should like to stress that the negation-free axioms given are classically equivalent to what is usually called ZF, but not exactly the same axioms: e.g., we use Collection and Set Induction in lieu of Replacement and Foundation. Many authors call this axiom system IZF.

we construct a universe of *names* by transfinite recursion:

$$\begin{aligned}\mathrm{Name}_\alpha(V,A) &:= \{x\,;\ x \text{ is a function and } \mathrm{ran}(x) \subseteq A \\ &\qquad \text{and there is } \xi < \alpha \text{ with } \mathrm{dom}(x) \subseteq \mathrm{Name}_\xi(V,A)\} \text{ and} \\ \mathrm{Name}(V,A) &:= \{x\,;\ \exists\alpha(x \in \mathrm{Name}_\alpha(V,A))\}.\end{aligned}$$

We let $S_{V,A}$ be the set of non-logical symbols consisting of the binary relation symbol $\in$ and a constant symbol for every name in $\mathrm{Name}(V,A)$ (as usual, we use the name itself as the constant symbol). The language $\mathcal{L}_{\Lambda,S_{V,A}}$ is usually called the *forcing language*.

If $\mathbb{A}$ is a Λ-algebra with underlying set A, we can now define a map $[\![\cdot]\!]^{\mathbb{A}}$ assigning to each $\varphi \in \mathcal{L}_{\Lambda,S_{V,A}}$ a truth value in $\mathbb{A}$ by recursion (the definition of $[\![u \in v]\!]^{\mathbb{A}}$ and $[\![u = v]\!]^{\mathbb{A}}$ is recursion on the hierarchy of names; the rest is a recursion on the complexity of φ):

$$\begin{aligned}[\![\mathbf{0}]\!]^{\mathbb{A}} &= \mathbf{0}, \\ [\![\mathbf{1}]\!]^{\mathbb{A}} &= \mathbf{1}, \\ [\![u \in v]\!]^{\mathbb{A}} &= \bigvee_{x \in \mathrm{dom}(v)} (v(x) \wedge [\![x = u]\!]^{\mathbb{A}}), \\ [\![u = v]\!]^{\mathbb{A}} &= \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \to [\![x \in v]\!]^{\mathbb{A}}) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \to [\![y \in u]\!]^{\mathbb{A}}), \\ [\![\varphi \wedge \psi]\!]^{\mathbb{A}} &= [\![\varphi]\!]^{\mathbb{A}} \wedge [\![\psi]\!]^{\mathbb{A}}, \\ [\![\varphi \vee \psi]\!]^{\mathbb{A}} &= [\![\varphi]\!]^{\mathbb{A}} \vee [\![\psi]\!]^{\mathbb{A}}, \\ [\![\varphi \to \psi]\!]^{\mathbb{A}} &= [\![\varphi]\!]^{\mathbb{A}} \to [\![\psi]\!]^{\mathbb{A}}, \\ [\![\neg\varphi]\!]^{\mathbb{A}} &= \neg[\![\varphi]\!]^{\mathbb{A}} \\ [\![\forall x\varphi(x)]\!]^{\mathbb{A}} &= \bigwedge_{u \in \mathrm{Name}(V,A)} [\![\varphi(u)]\!]^{\mathbb{A}}, \text{ and} \\ [\![\exists x\varphi(x)]\!]^{\mathbb{A}} &= \bigvee_{u \in \mathrm{Name}(V,A)} [\![\varphi(u)]\!]^{\mathbb{A}}.\end{aligned}$$

By construction, it is clear that $[\![\cdot]\!]^{\mathbb{A}}$ is an $\mathbb{A}$-valued $S_{V,A}$-structure and hence, by restricting it to $\mathrm{Sent}_{\Lambda,\{\in\}}$, we can consider it as an $\mathbb{A}$-valued $\{\in\}$-structure. Usually, it is the restriction to $\mathrm{Sent}_{\Lambda,\{\in\}}$ that set theorists are interested in: as a consequence, we shall use the notation $[\![\cdot]\!]_{\mathbb{A}}$ to refer to the $\mathbb{A}$-valued $\{\in\}$-structure and the notation $[\![\cdot]\!]^{\mathrm{Name}}_{\mathbb{A}} := [\![\cdot]\!]^{\mathbb{A}}$ to refer to its extension to $\mathrm{Sent}_{\Lambda,S_{V,A}}$.

The results for algebra-valued models of set theory were originally proved for Boolean algebras, then extended to Heyting algebras:

Theorem 2 *If V is a model of set theory, $\mathbb{B} = (B, \wedge, \vee, \to, \neg, \mathbf{0}, \mathbf{1})$ is a Boolean algebra or Heyting algebra, and φ is any axiom of* ZF*, then $[\![\varphi]\!]_{\mathbb{B}} = \mathbf{1}$.*

Proof. Cf. (Bell, 2005, Theorem 1.33 & pp. 165–166). □

Lemma 3 *Let $\mathbb{H} = (H, \wedge, \vee, \to, \mathbf{0}, \mathbf{1})$ be a Heyting algebra and V be a model of set theory. Then $[\![\cdot]\!]^{\mathrm{Name}}_{\mathbb{H}}$ is faithful to $\mathbb{H}$ (and hence, loyal to $(\mathbb{H}, D)$ for every designated set D on $\mathbb{H}$ by Lemma 1).*

$\wedge$	**0**	**1/2**	**1**
0	**0**	**0**	**0**
1/2	**0**	**1/2**	**1/2**
1	**0**	**1/2**	**1**

$\vee$	**0**	**1/2**	**1**
0	**0**	**1/2**	**1**
1/2	**1/2**	**1/2**	**1**
1	**1**	**1**	**1**

$\rightarrow$	**0**	**1/2**	**1**
0	**1**	**1**	**1**
1/2	**0**	**1**	**1**
1	**0**	**1**	**1**

$\neg$	
0	**1**
1/2	**1/2**
1	**0**

Fig. 2. Connectives for $\mathbb{PS}_3$

Proof. Consider $u := \{(\varnothing, a)\} \in \mathrm{Name}(V, H)$ and $\varphi := \exists x(x \in u) \in \mathrm{Sent}_{\Lambda, S_{V,H}}$. It is easy to check that $[\![\varphi]\!]^{\mathrm{Name}}_{\mathbb{H}} = a$. □

In order to formulate results for implication algebras, Löwe and Tarafder (2015, p. 197) introduced NFF-ZF, the axiom system of all ZF-axioms where the two axiom schemata are restricted to instances of negation-free formulas. They introduced a three-element algebra $\mathbb{PS}_3$ (Löwe and Tarafder, 2015, Fig. 2 & § 6) and proved the following result (for the sake of completeness, we give the definition of $\mathbb{PS}_3$ in Figure 2):

Theorem 4 *If V is a model of set theory and φ is any axiom of* NFF-ZF*, then $[\![\varphi]\!]_{\mathbb{PS}_3} = \mathbf{1}$. Furthermore, $[\![\cdot]\!]_{\mathbb{PS}_3}$ is faithful to $\mathbb{PS}_3$ and hence loyal to $(\mathbb{A}, D)$ for every designated set D by Lemma 1.*

Proof. Cf. (Löwe and Tarafder, 2015, Corollary 5.2) for the first claim. Löwe and Tarafder (2015, Theorem 6.2) give a sentence $\varphi \in \mathrm{Sent}_{\Lambda,\{\in\}}$ such that $[\![\varphi]\!]_{\mathbb{PS}_3} = \mathbf{1/2}$ which establishes faithfulness. □

§4. Automorphisms and algebra-valued models of set theory Given a model of set theory V and any Λ-algebras $\mathbb{A}$ and $\mathbb{B}$ and a Λ-homomorphism $f : \mathbb{A} \to \mathbb{B}$, we can define a map $\widehat{f} : \mathrm{Name}(V, \mathbb{A}) \to \mathrm{Name}(V, \mathbb{B})$ by $\in$-recursion via

$$\mathrm{dom}(\widehat{f}(u)) := \{\widehat{f}(v)\,;\, v \in \mathrm{dom}(u)\} \text{ and}$$
$$\widehat{f}(u)(\widehat{f}(v)) := f(u(v)).$$

Proposition 5 *Suppose that V is a model of set theory, $\mathbb{A}$ and $\mathbb{B}$ are complete Λ-algebras and $f : \mathbb{A} \to \mathbb{B}$ is a complete Λ-isomorphism. Let $\varphi \in \mathcal{L}_{\Lambda,\{\in\}}$ with n free variables and $u_1, \ldots, u_n \in \mathrm{Name}(V, \mathbb{A})$. Then*

$$f([\![\varphi(u_1, \ldots, u_n)]\!]_{\mathbb{A}}) = [\![\varphi(\widehat{f}(u_1), \ldots, \widehat{f}(u_n))]\!]_{\mathbb{B}}.$$

Proof. For atomic formulas, this is easily proved by induction on the rank of the names involved. For non-atomic formulas, the claim follows by induction on the complexity of the formula (where the quantifier cases need the fact that f is a bijection). □

Corollary 6 *Suppose that V is a model of set theory, $\mathbb{A}$ and $\mathbb{B}$ are complete Λ-algebras and $f : \mathbb{A} \to \mathbb{B}$ is a complete Λ-isomorphism. Let $\varphi \in \mathrm{Sent}_{\Lambda,\{\in\}}$. Then*

$$f([\![\varphi]\!]_{\mathbb{A}}) = [\![\varphi]\!]_{\mathbb{B}}.$$

Corollary 7 *Suppose that V is a model of set theory, $\mathbb{A}$ is a complete Λ-algebra with underlying set A, $a \in A$, and that $f : \mathbb{A} \to \mathbb{A}$ is a complete Λ-automorphism with $f(a) \neq a$. Then there is no $\varphi \in \mathrm{Sent}_{\Lambda,\{\in\}}$ such that $[\![\varphi]\!]_{\mathbb{A}} = a$.*

Proof. By Corollary 6, if $[\![\varphi]\!]_{\mathbb{A}} = a$, then $f(a) = a$. □

Proposition 8 *If* $\mathbb{A} = (A, \wedge, \vee, \mathbf{0}, \mathbf{1})$ *is an atomic distributive lattice and* $a \in A \backslash \{\mathbf{0}, \mathbf{1}\}$*, then there is a* $\{\wedge, \vee, \neg_{\mathrm{c}}, \mathbf{0}, \mathbf{1}\}$*-automorphism* f *of* $\mathbb{A}$ *such that* $f(a) \neq a$.

Proof. Note that the assumptions imply that $A \neq \{\mathbf{0}, \mathbf{1}\}$ and hence $\mathrm{At}(\mathbb{A}) \neq \varnothing$. By atomicity, every permutation $\pi : \mathrm{At}(\mathbb{A}) \to \mathrm{At}(\mathbb{A})$ induces an automorphism of $\mathbb{A}$ preserving $\wedge$, $\vee$, $\neg_{\mathrm{c}}$, $\mathbf{0}$, and $\mathbf{1}$ by $f_\pi(\bigvee X) = \bigvee\{\pi(t)\,;\, t \in X\}$ for $X \subseteq \mathrm{At}(\mathbb{A})$. Let $a = \bigvee X_a$. Since $a \neq \mathbf{0}$, we have $X_a \neq \varnothing$; since $a \neq \mathbf{1}$, we have $X_a \neq \mathrm{At}(\mathbb{A})$. So, pick $t_0 \in X_a$ and $t_1 \in \mathrm{At}(\mathbb{A}) \backslash X_a$ and let π be the transposition that interchanges t_0 and t_1. Then

$$t_0 \leq \bigvee X_a = a, \text{ but}$$
$$t_0 \not\leq \bigvee\{\pi(t)\,;\, t \in X_a\} = f_\pi(\bigvee X_a) = f_\pi(a),$$

whence $a \neq f_\pi(a)$. □

Corollary 9 *If* V *is a model of set theory,* $\mathbb{B}$ *is an atomic Boolean (implication) algebra with more than two elements, and* D *is any filter on* $\mathbb{B}$*, then* $[\![\cdot]\!]_{\mathbb{B}}$ *is loyal, but not faithful to* $(\mathbb{B}, D)$.

Proof. By Proposition 8, all elements except for $\mathbf{0}$ and $\mathbf{1}$ are moved by some automorphism of an atomic Boolean (implication) algebra and hence by Corollary 7, for each sentence $\varphi \in \mathcal{L}_{\Lambda,\{\in\}}$, we have that $[\![\varphi]\!]_{\mathbb{B}} \in \{\mathbf{0}, \mathbf{1}\}$. In particular, this means that $\mathbf{L}([\![\cdot]\!]_{\mathbb{B}}, D) = \mathbf{L}(\{\mathbf{0}, \mathbf{1}\}, \{\mathbf{1}\}) = \mathbf{CPC} = \mathbf{L}(\mathbb{B}, D)$. □

Clearly, atomicity is not a necessary condition for the conclusion of Corollary 9: the Boolean algebra of infinite and co-infinite subsets of $\mathbb{N}$ is atomless and hence non-atomic, but every nontrivial element is moved by an automorphism, so Corollary 7 applies. We do not know whether this result extends to Boolean algebras without this property, e.g., rigid Boolean algebras (cf. van Douwen et al., 1980, § 2):

Question 10 *Are there (necessarily countable) Boolean algebras* $\mathbb{B}$ *such that* $[\![\cdot]\!]_{\mathbb{B}}$ *is faithful to* $\mathbb{B}$ *for some designated set* D*?*

§5. Stretching and twisting the loyalty of Boolean algebras

5.1. What can be considered a negation? In this section, we start from an atomic, complete Boolean algebra $\mathbb{B}$ and modify it, to get an algebra $\mathbb{A}$ that gives rise to an illoyal $[\![\cdot]\!]_{\mathbb{A}}$. The first construction is the well-known construction of tail extensions of Boolean algebras to obtain a Heyting algebra. The other two constructions are *negation twists*: in these, we interpret $\mathbb{B}$ as a Boolean implication algebra via the definition $a \to b := \neg a \vee b$, and then add a new, twisted negation to it that changes its logic.

When twisting the negation, we need to pay attention to the fact that not every unary function on an implication algebra is a sensible negation. In his survey of varieties of negation, Dunn (1995) lists Hazen's *subminimal negation* as the bottom of his *Kite of Negations*: only the rule of contraposition, i.e., $a \leq b$ implies $\neg b \leq \neg a$, is required. In the following, we shall use this as a necessary requirement to be a reasonable candidate for negation.

5.2. Tail stretches Let $\mathbb{B} = (B, \wedge, \vee, \to, \neg, \mathbf{0}, \mathbf{1})$ be a Boolean algebra, and $\mathbf{1}^* \notin B$ be an additional element that we add to the top of $\mathbb{B}$ to form the *tail stretch* $\mathbb{H}$ as follows: $H := B \cup \{\mathbf{1}^*\}$, the complete lattice structure of $\mathbb{H}$ is the order sum of $\mathbb{B}$ and the one element lattice $\{\mathbf{1}^*\}$, and $\to^*$ is defined as follows:[3]

$$a \to^* b := \begin{cases} a \to b & \text{if } a, b \in B \text{ such that } a \not\leq b, \\ \mathbf{1}^* & \text{if } a, b \in B \text{ with } a \leq b \text{ or if } b = \mathbf{1}^*, \\ b & \text{if } a = \mathbf{1}^*. \end{cases}$$

Lemma 11 *The tail stretch* $\mathbb{H} = (H, \wedge, \vee, \to^*, \mathbf{0}, \mathbf{1}^*)$ *is a Heyting algebra with* $p \vee \neg p \notin \mathbf{L}(\mathbb{H}, \{\mathbf{1}^*\})$*, so in particular,* $\mathbf{L}(\mathbb{H}, \{\mathbf{1}^*\}) \neq \mathbf{CPC}$.

Proof. If $b \neq \mathbf{0} \in B$, then by definition $b \to^* \mathbf{0} = \neg b$ where $\neg$ refers to the negation in $\mathbb{B}$. In particular, $b \vee \neg_{\mathbb{H}} b = b \vee \neg b = \mathbf{1} \neq \mathbf{1}^*$. □

Lemma 12 *If* $f : B \to B$ *is an automorphism of the Boolean algebra* $\mathbb{B}$*, then* $f^* : H \to H$ *defined by*

$$f^*(b) := \begin{cases} f(b) & \textit{if } b \in B \textit{ and} \\ \mathbf{1}^* & \textit{if } b = \mathbf{1}^*. \end{cases}$$

is an automorphism of $\mathbb{H}$.

Proof. Easy to check. □

Theorem 13 *Let* V *be a model of set theory,* $\mathbb{B}$ *an atomic Boolean algebra with more than two elements, and* $\mathbb{H}$ *be the tail stretch of* $\mathbb{B}$ *as defined above. Then the* $\mathbb{H}$*-valued model of set theory* $V^{\mathbb{H}}$ *is not faithful to* $\mathbb{H}$*. Furthermore, we have that*

$$(p \to q) \vee (q \to p) \in \mathbf{L}([\![\cdot]\!]_{\mathbb{H}}, \{\mathbf{1}^*\}) \backslash \mathbf{L}(\mathbb{H}, \{\mathbf{1}^*\}).$$

Consequently, $V^{\mathbb{H}}$ *is illoyal to* $(\mathbb{H}, \{\mathbf{1}^*\})$.

Proof. Since $\mathbb{B}$ is atomic with more than two elements, each of the non-trivial elements of B is moved by an automorphism of $\mathbb{B}$ by Proposition 8. By Lemma 12, these remain automorphisms of $\mathbb{H}$. As a consequence, we can apply Corollary 6 to get that $\operatorname{ran}([\![\cdot]\!]_{\mathbb{H}}) \subseteq \{\mathbf{0}, \mathbf{1}, \mathbf{1}^*\}$ which is isomorphic to the linear Heyting algebra $\mathbf{3}$ and thus the range is a linear Heyting algebra. As mentioned, Horn (1969) proved that $(p \to q) \vee (q \to p)$ characterises the variety generated by the linear Heyting algebras, so $(p \to q) \vee (q \to p) \in \mathbf{L}([\![\cdot]\!]_{\mathbb{H}}, \{\mathbf{1}^*\})$. However, since $\mathbb{B}$ has more than two elements, we can pick imcomparable $a, b \in B$. Then $a \to^* b$ and $b \to^* a$ are both elements of B, and thus $(p \to q) \vee (q \to p) \notin \mathbf{L}(\mathbb{H}, \{\mathbf{1}^*\})$. □

We remark that $\operatorname{ran}([\![\cdot]\!]_{\mathbb{H}}) = \{\mathbf{0}, \mathbf{1}, \mathbf{1}^*\}$, as can be seen by checking that the $[\![\cdot]\!]_{\mathbb{H}}$-value of the sentence formalising the statement "every subset of $\{\varnothing\}$ is either $\varnothing$ or $\{\varnothing\}$" is $\mathbf{1}$.

---

[3] In $\mathbb{H}$, we use the (Heyting algebra) definition $\neg_{\mathbb{H}} h := h \to^* \mathbf{0}$ to define a negation; note that if $\mathbf{0} \neq b \in B$, $\neg_{\mathbb{H}} b = \neg b$, but $\neg_{\mathbb{H}} \mathbf{0} = \mathbf{1}^* \neq \mathbf{1} = \neg \mathbf{0}$.

5.3. Transposition twists Let $\mathbb{B} = (B, \wedge, \vee, \rightarrow, \neg, \mathbf{0}, \mathbf{1})$ be an atomic Boolean algebra, $a, b \in \mathrm{At}(\mathbb{B})$ with $a \neq b$, and π be the transposition that transposes a and b. Since $\mathbb{B}$ is an atomic Boolean algebra, $\neg = \neg_{\mathrm{c}}$. Then f_π as defined in the proof of Proposition 8 is a $\{\wedge, \vee, \rightarrow, \neg, \mathbf{0}, \mathbf{1}\}$-automorphism of $\mathbb{B}$. We now define a twisted negation by

$$\neg_\pi(\bigvee X) := \bigvee\{\pi(t) \in \mathrm{At}(\mathbb{B})\,;\, t \notin X\}$$

and let the π-*twist* of $\mathbb{B}$ be $\mathbb{B}_\pi := (B, \wedge, \vee, \rightarrow, \neg_\pi, \mathbf{0}, \mathbf{1})$.[4] We observe that the twisted negation $\neg_\pi$ satisfies the rule of contraposition.

Lemma 14 *If either* $\neg_{\mathrm{c}} a = \bigvee\{t \in \mathrm{At}(\mathbb{B})\,;\, t \neq a\}$ *or* $\neg_{\mathrm{c}} b = \bigvee\{t \in \mathrm{At}(\mathbb{B})\,;\, t \neq b\}$ *is not in* D*, then* $\neg(p \wedge \neg p) \notin \mathbf{L}(\mathbb{B}_\pi, D)$*. In particular,* $\mathbf{L}(\mathbb{B}_\pi, D) \neq \mathbf{CPC}$.

Proof. Without loss of generality, $\bigvee\{t \in \mathrm{At}(\mathbb{B})\,;\, t \neq b\} = \neg_{\mathrm{c}} b = \neg_\pi a \notin D$. Since $a \leq \neg_\pi a$, we have that $a = \neg_\pi a \wedge a$, and so $\neg_\pi(\neg_\pi a \wedge a) = \neg_\pi a \notin D$. □

Lemma 15 *There is an automorphism* f *of* $\mathbb{B}_\pi$ *such that* $f(a) = b$*. In particular,* $[\![\cdot]\!]_{\mathbb{B}_\pi}$ *is not faithful to* $\mathbb{B}_\pi$.

Proof. We know that f_π is an automorphism of $\mathbb{B}$. Since π is a transposition, we have that $\pi^2 = \mathrm{id}$ and $\pi = \pi^{-1}$; using this, we observe that f_π still preserves $\neg_\pi$:

$$\begin{aligned} f_\pi(\neg_\pi(\bigvee X)) &= f_\pi(\bigvee\{\pi(t) \in \mathrm{At}(\mathbb{B})\,;\, t \notin X\}) \\ &= \bigvee\{\pi(\pi(t)) \in \mathrm{At}(\mathbb{B})\,;\, t \notin X\} \\ &= \bigvee\{t \in \mathrm{At}(\mathbb{B})\,;\, t \notin X\} \\ &= \neg_\pi(\bigvee\{\pi(t) \in \mathrm{At}(\mathbb{B})\,;\, t \notin X\}) \\ &= \neg_\pi(f_\pi(\bigvee X))). \end{aligned}$$

Thus, f_π is an automorphism of $\mathbb{B}_\pi$; clearly, $f_\pi(a) = b$. The second claim follows from Corollary 7. □

Now let V be a model of set theory and $[\![\cdot]\!]_{\mathbb{B}_\pi}$ the $\mathbb{B}_\pi$-valued $\{\in\}$-structure derived from V and $\mathbb{B}$.

Lemma 16 *If* $x \in \mathrm{ran}([\![\cdot]\!]_{\mathbb{B}_\pi})$*, then* $\neg_\pi x = \neg_{\mathrm{c}} x$.

Proof. Let $x = \bigvee X$ for some $X \subseteq \mathrm{At}(\mathbb{B})$. By Corollary 7 and Lemma 15, if $x \in \mathrm{ran}([\![\cdot]\!]_{\mathbb{B}_\pi})$, then $f_\pi(x) = x$. This means that either both $a, b \in X$ or both $a, b \notin X$. In both cases, it is easily seen that $\neg_\pi x = \neg_{\mathrm{c}} x$. □

Theorem 17 *For any filter* D, $\mathbf{L}([\![\cdot]\!]_{\mathbb{B}_\pi}, D) = \mathbf{CPC}$*. In particular, if either* $\neg_{\mathrm{c}} a$ *or* $\neg_{\mathrm{c}} b$ *is not in* D*, then* $[\![\cdot]\!]_{\mathbb{B}_\pi}$ *is not loyal to* $(\mathbb{B}_\pi, D)$.

Proof. As mentioned in §2., if we let $\mathbb{C} := \mathbb{B}_{[\![\cdot]\!]_{\mathbb{B}_\pi}} = (\mathrm{ran}([\![\cdot]\!]_{\mathbb{B}_\pi}, \wedge, \vee, \rightarrow, \neg_\pi, \mathbf{0}, \mathbf{1})$, then $\mathbf{L}([\![\cdot]\!]_{\mathbb{B}_\pi}, D) = \mathbf{L}(\mathbb{C}, D)$. But Lemma 16 implies that $\mathbb{C} = (\mathrm{ran}([\![\cdot]\!]_{\mathbb{B}_\pi}, \wedge, \vee, \rightarrow, \neg_{\mathrm{c}}, \mathbf{0}, \mathbf{1})$ which is a Boolean algebra (as a subalgebra of $\mathbb{B}$). Thus, $\mathbf{L}([\![\cdot]\!]_{\mathbb{B}_\pi}, D) = \mathbf{L}(\mathbb{C}, D) = \mathbf{CPC}$. The second claim follows from Lemma 14. □

---

[4] Note that we do not twist the implication $\rightarrow$ which remains the implication of the original Boolean algebra $\mathbb{B}$ defined by $x \rightarrow y := \neg_{\mathrm{c}} x \vee y$.

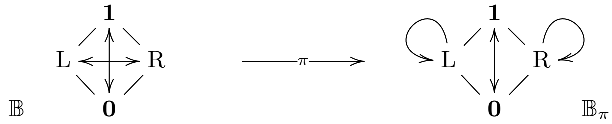

Fig. 3. The four-element Boolean algebra and its transposition twist. Negations are indicated by arrows.

As the simplest possible special case, we can consider the Boolean algebra $\mathbb{B}$ generated by two atoms L and R; then, there is one nontrivial transposition $\pi(\mathrm{L}) = \mathrm{R}$ and all nontrivial elements of $\mathbb{B}$ are moved by the automorphism f_π. As a consequence of Corollary 7, all sentences will get either value $\mathbf{0}$ or value $\mathbf{1}$ under $[\![\cdot]\!]_{\mathbb{B}_\pi}$, and hence $\mathbf{L}([\![\cdot]\!]_{\mathbb{B}_\pi}, D)$ is classical (cf. Figure 3).

5.4. Maximal twists Again, let $\mathbb{B} = (B, \wedge, \vee, \to, \neg, \mathbf{0}, \mathbf{1})$ be an atomic Boolean algebra with more than two elements and define the maximal negation by

$$\neg_{\mathrm{m}} b := \begin{cases} \mathbf{1} & \text{if } b \neq \mathbf{1} \text{ and} \\ \mathbf{0} & \text{if } b = \mathbf{1} \end{cases}$$

for every $b \in B$. We let the *maximal twist* of $\mathbb{B}$ be $\mathbb{B}_{\mathrm{m}} := (B, \wedge, \vee, \to, \neg_{\mathrm{m}}, \mathbf{0}, \mathbf{1})$; once more observe that the maximal negation $\neg_{\mathrm{m}}$ satisfies the rule of contraposition.

Lemma 18 *If there is some $\mathbf{0} \neq b \notin D$, then $(p \wedge \neg p) \to q \notin \mathbf{L}(\mathbb{B}_{\mathrm{m}}, D)$. In particular, $\mathbf{L}(\mathbb{B}_{\mathrm{m}}, D) \neq \mathbf{CPC}$.*

Proof. Let $c := \neg_{\mathrm{c}} b$. Note that the assumption $b \neq \mathbf{0}$ implies $c \neq \mathbf{1}$. In particular, $\neg_{\mathrm{m}} c = \mathbf{1}$, and thus $c \wedge \neg_{\mathrm{m}} c = c$. Also

$$\begin{aligned} c \to b &= \neg_{\mathrm{c}} b \to b \\ &= \neg_{\mathrm{c}} \neg_{\mathrm{c}} b \vee b \\ &= b \vee b = b. \end{aligned}$$

Thus, the assignment ι with $p \mapsto c$ and $q \mapsto b$ yields $\iota((p \wedge \neg p) \to q) = b \notin D$. □

Lemma 19 *For any $b \notin \{\mathbf{0}, \mathbf{1}\}$, there is an automorphism f of $\mathbb{B}_{\mathrm{m}}$ such that $f(b) \neq b$. In particular, $[\![\cdot]\!]_{\mathbb{B}_{\mathrm{m}}}$ is not faithful to $\mathbb{B}_{\mathrm{m}}$.*

Proof. We claim that any automorphism f of $\mathbb{B}$ also preserves $\neg_{\mathrm{m}}$. Suppose f is an automorphism of $\mathbb{B}$. If $b = \mathbf{1}$, then clearly $f(\neg_{\mathrm{m}} \mathbf{1}) = f(\mathbf{0}) = \mathbf{0} = \neg_{\mathrm{m}} \mathbf{1} = \neg_{\mathrm{m}} f(\mathbf{1})$. Now let $b \neq \mathbf{1}$. Since f is bijective and $f(\mathbf{1}) = \mathbf{1}$, we have that $f(b) \neq \mathbf{1}$. So $f(\neg_{\mathrm{m}} b) = f(\mathbf{1}) = \mathbf{1} = \neg_{\mathrm{m}} f(b)$. The second claim follows from Corollary 7. □

Theorem 20 *For any designated set D, $\mathbf{L}([\![\cdot]\!]_{\mathbb{B}_{\mathrm{m}}}, D) = \mathbf{CPC}$. In particular, $[\![\cdot]\!]_{\mathbb{B}_{\mathrm{m}}}$ is not loyal to $(\mathbb{B}_{\mathrm{m}}, D)$.*

Proof. Lemma 19 gives us that every nontrivial element of $\mathbb{B}$ is moved by an automorphism, so we can apply the argument from the proof of Corollary 9: since for each $\varphi \in \mathcal{L}_{\Lambda,\{\in\}}$, we have that $[\![\varphi]\!]_{\mathbb{B}_{\mathrm{m}}} \in \{\mathbf{0}, \mathbf{1}\}$, we get that

$$\mathbf{L}([\![\cdot]\!]_{\mathbb{B}_{\mathrm{m}}}, D) = \mathbf{L}(\{\mathbf{0}, \mathbf{1}\}, \{\mathbf{1}\}) = \mathbf{CPC}.$$

The second claim follows from Lemma 18. □

As mentioned at the end of §2., our examples show that restricting the language can change faithful models into illoyal ones: for our twisted algebras $\mathbb{B}_\pi$ and $\mathbb{B}_{\rm m}$, the general faithfulness result Lemma 3 holds for $[\![\cdot]\!]^{\rm Name}_{\mathbb{B}_\pi}$ and $[\![\cdot]\!]^{\rm Name}_{\mathbb{B}_{\rm m}}$. However, Theorems 17 & 20 show that their restrictions $[\![\cdot]\!]_{\mathbb{B}_\pi}$ and $[\![\cdot]\!]_{\mathbb{B}_{\rm m}}$ are neither faithful nor loyal.

BIBLIOGRAPHY

Bell, J. L. (2005). *Set theory, Boolean-valued models and independence proofs*, volume 47 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, Oxford, third edition.

Blackburn, P., de Rijke, M., and Venema, Y. (2001). *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press.

Chakraborty, M. K. and Tarafder, S. (2016). A paraconsistent logic obtained from an algebra-valued model of set theory. In Béziau, J. Y., Chakraborty, M. K., and Dutta, S., editors, *New Directions in Paraconsistent Logic, 5th WCP, Kolkata, India, February 2014*, volume 152 of *Springer Proceedings in Mathematics & Statistics*, pages 165–183. Springer-Verlag.

Dunn, J. M. (1995). Generalized ortho negation. In Wansing, H., editor, *Negation. A Notion in Focus*, volume 7 of *Perspektiven der Analytischen Philosophie*, pages 3–26. De Gruyter.

Fourman, M. P. and Scott, D. S. (1979). Sheaves and logic. In Fourman, M. P., Mulvey, C. J., and Scott, D. S., editors, *Applications of sheaves. Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra and Analysis held at the University of Durham, Durham, July 9–21, 1977*, volume 753 of *Lecture Notes in Mathematics*, pages 302–401. Springer-Verlag.

Grayson, R. J. (1979). Heyting-valued models for intuitionistic set theory. In Fourman, M. P., Mulvey, C. J., and Scott, D. S., editors, *Applications of sheaves, Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra and Analysis held at the University of Durham, Durham, July 9–21, 1977*, volume 753 of *Lecture Notes in Mathematics*, pages 402–414. Springer, Berlin.

Horn, A. (1969). Logic with truth values in a linearly ordered heyting algebra. *Journal of Symbolic Logic*, 34:395–408.

Löwe, B. and Tarafder, S. (2015). Generalized algebra-valued models of set theory. *Review of Symbolic Logic*, 8(1):192–205.

Ozawa, M. (2007). Transfer principle in quantum set theory. *Journal of Symbolic Logic*, 72(2):625–648.

Ozawa, M. (2009). Orthomodular-valued models for quantum set theory. preprint, arXiv 0908.0367.

Passmann, R. (2018). Loyalty and faithfulness of model constructions for constructive set theory. Master's thesis, Universiteit van Amsterdam.

Takeuti, G. and Titani, S. (1992). Fuzzy logic and fuzzy set theory. *Archive for Mathematical Logic*, 32(1):1–32.

Tarafder, S. (2015). Ordinals in an algebra-valued model of a paraconsistent set theory. In Banerjee, M. and Krishna, S. N., editors, *Logic and Its Applications, 6th International Conference, ICLA 2015, Mumbai, India, January 8–10, 2015, Proceedings*, volume 8923 of *Lecture Notes in Computer Science*, pages 195–206. Springer-Verlag.

Titani, S. (1999). A lattice-valued set theory. *Archive for Mathematical Logic*, 38(6):395–421.

Titani, S. and Kozawa, H. (2003). Quantum set theory. *International Journal of Theoretical Physics*, 42(11):2575–2602.

van Douwen, E. K., Monk, J. D., and Rubin, M. (1980). Some questions about boolean algebras. *Algebra Universalis*, 11:220–243.

Vopěnka, P. (1965). The limits of sheaves and applications on constructions of models. *Bulletin de l'Académie polonaise des sciences. Série des sciences mathématiques, astronomiques, et physiques*, 13:189–192.