# The de Jongh property for bounded constructive Zermelo-Fraenkel set theory

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**Abstract** The theory BCZF is obtained from constructive Zermelo-Fraenkel set theory CZF by restricting the collection schemes to bounded formulas. We prove that BCZF has the de Jongh property with respect to every intermediate logic that is characterised by a class of Kripke frames.

**Keywords** de Jongh property · Intuitionistic logic · Constructive set theory

**Mathematics Subject Classification (2010)** 03C90 · 03B20 · 03F50 · 03E70

## 1 Introduction

De Jongh's classical theorem states that the propositional logic of Heyting arithmetic is intuitionistic logic (for a precise mathematical statement of de Jongh's theorem, cf. Theorem 8). The de Jongh property is a generalisation of this theorem (Definition 10; for an extensive exposition of the history of the de Jongh property, cf. [8, section 2]).

Inspired by the Kripke model constructions used for independence proofs in constructive or intuitionistic set theory (e.g., Lubarsky [11, 12], Lubarsky and Rathjen [14], Lubarsky and Diener [13]), Iemhoff [5] introduced a class of models for subtheories of constructive set theory CZF based on classical models of ZF set theory and their generic extensions.

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In this work, we will use Iemhoff's construction to give a model-theoretic proof of the fact that the set theory satisfied by the class of Iemhoff models, namely, bounded constructive Zermelo-Fraenkel set theory BCZF, has the de Jongh property. We cannot strengthen this result to full CZF with the method presented in this paper as the Iemhoff models that involve forcing non-trivially cannot be models of the exponentiation axiom (a constructive consequence of the axiom of subset collection; cf. Corollary 32). As an aside, we will also define an Iemhoff model that forces the negation of the exponentiation axiom.

The paper is organised as follows: In the second section, we will introduce Kripke models for set theory and show how to use them to prove the de Jongh property in a general setting. We present Iemhoff's construction in the third section and provide a failure of the exponentiation axiom. The final section is devoted to proving the main result.

## 2 Kripke Models for set theory and the de Jongh property

The language of propositional logic consists of the logical symbols $\vee$, $\wedge$, $\rightarrow$ and $\bot$, and a fixed countable set Prop of propositional variables. The negation of a formula $\neg\varphi$ is defined as an abbreviation for $\varphi \rightarrow \bot$. Furthermore, the language of predicate logic extends the language of propositional logic with the logical symbols $\forall$ and $\exists$, and a countable set of variables, but has no propositional variables. The language $\mathcal{L}_\in$ of set theory extends the language of predicate logic with a binary relation symbol $\in$. Given any language $\mathcal{L}$, we denote the set of $\mathcal{L}$-sentences (i.e., $\mathcal{L}$-formulas without free variables) by $\mathcal{L}^{\mathsf{sent}}$.

As usual, we denote intuitionistic propositional logic with **IPC** and classical propositional logic with **CPC**. An intermediate logic **J** is a logic with $\mathbf{IPC} \subseteq \mathbf{J} \subseteq \mathbf{CPC}$. Note that propositional logics are sets of propositional formulas. For axiomatisations and descriptions of proof-calculi of these logics, we refer the reader to [18].

We work in ZFC as our meta-theory and, additionally, we assume the existence of a countable transitive set $M$ such that $(M, \in) \models \mathsf{ZFC}$.

### 2.1 Kripke models for set theory

A *Kripke frame* $(K, \leq)$ is a set $K$ equipped with a partial order $\leq$. A *Kripke model for* **IPC** is a triple $(K, \leq, V)$ such that $(K, \leq)$ is a Kripke frame and $V : \mathsf{Prop} \rightarrow \mathcal{P}(K)$ a valuation that is persistent, i.e., if $w \in V(p)$ and $w \leq v$, then $v \in V(p)$. We can then define, by induction on propositional formulas, the forcing relation for propositional logic at a node $v \in K$ in the following way:

(i) $K, V, v \Vdash p$ if and only if $v \in V(p)$,
(ii) $K, V, v \Vdash \varphi \wedge \psi$ if and only if $K, V, v \Vdash \varphi$ and $K, V, v \Vdash \psi$,
(iii) $K, V, v \Vdash \varphi \vee \psi$ if and only if $K, V, v \Vdash \varphi$ or $K, V, v \Vdash \psi$,
(iv) $K, V, v \Vdash \varphi \rightarrow \psi$ if and only if for all $w \geq v$, $K, V, w \Vdash \varphi$ implies $K, V, w \Vdash \psi$,
(v) $K, V, v \Vdash \bot$ holds never.

We will write $v \Vdash \varphi$ instead of $K, V, v \Vdash \varphi$ if the Kripke frame or the valuation are clear from the context. We will write $K, V \Vdash \varphi$ if $K, V, v \Vdash \varphi$ holds for all $v \in K$. A formula $\varphi$ is *valid in* $K$ if $K, V, v \Vdash \varphi$ holds for all valuations $V$ on $K$ and $v \in K$, and $\varphi$ is *valid* if it is valid in every Kripke frame $K$.

We can now define the logic of a Kripke frame and of a class of Kripke frames.

**Definition 1** If $(K, \leq)$ is a Kripke frame for **IPC**, we define the *propositional logic* $\mathbf{L}(K, \leq)$ to be the set of all propositional formulas that are valid in $K$. For a class $\mathcal{K}$ of Kripke frames, we define the *propositional logic* $\mathbf{L}(\mathcal{K})$ to be the set of all propositional formulas that are valid in all Kripke frames $(K, \leq)$ in $\mathcal{K}$. Given an intermediate logic $\mathbf{J}$, we say that $\mathcal{K}$ *characterises* $\mathbf{J}$ if $\mathbf{L}(\mathcal{K}) = \mathbf{J}$.

We will sometimes write $\mathbf{L}(K)$ for $\mathbf{L}(K, \leq)$. The next result is proved by induction on the complexity of formulas; it shows that persistence of the propositional variables transfers to all formulas.

**Proposition 2** *Let $(K, \leq, V)$ be a Kripke model for* **IPC***, $v \in K$ and $\varphi$ be a propositional formula such that $K, v \Vdash \varphi$ holds. Then $K, w \Vdash \varphi$ holds for all $w \geq v$.* □

By extending the Kripke models introduced above, we can obtain models for intuitionistic predicate logic. Instead of developing this theory in full generality, we will focus on the subcase of *Kripke models for set theory*.

**Definition 3** A *Kripke model $(K, \leq, D, e)$ for set theory* is a Kripke frame $(K, \leq)$ for **IPC** with a collection of domains $D = \{D_v \mid v \in K\}$ and a collection of set-membership relations $e = \{e_v \mid v \in K\}$, such that the following hold:

(i) $e_v$ is a binary relation on $D_v$ for every $v \in K$, and,
(ii) $D_v \subseteq D_w$ and $e_v \subseteq e_w$ for all $w \geq v \in K$.

Examples of Kripke models for set theory are not only the Iemhoff models that we will introduce in Section 3, but also the Kripke models introduced in [11, 12] by Lubarsky, [13] by Diener and Lubarsky and [14] by Lubarsky and Rathjen.

We can now extend the forcing relation to Kripke models for set theory, interpreting the language of set theory $\mathcal{L}_\in$. For the following definition, we tacitly enrich the language of set theory with constant symbols for every element of the domains of the Kripke model at hand.

**Definition 4** Let $(K, \leq, D, e)$ be a Kripke model for set theory. We define, by induction on $\mathcal{L}_\in$-formulas, the forcing relation at every node of a Kripke frame in the following way, where $\varphi$ and $\psi$ are formulas with all free variables shown, and $\bar{y} = y_0, \ldots, y_{n-1}$ are elements of $D_v$ for the node $v$ considered on the left side:

(i) $(K, \leq, D, e), v \Vdash a \in b$ if and only if $(a, b) \in e_v$,
(ii) $(K, \leq, D, e), v \Vdash a = b$ if and only if $a = b$,
(iii) $(K, \leq, D, e), v \Vdash \exists x \varphi(x, \bar{y})$ if and only if there is some $a \in D_v$ with $(K, \leq, D, e), v \Vdash \varphi(a, \bar{y})$,
(iv) $(K, \leq, D, e), v \Vdash \forall x \varphi(x, \bar{y})$ if and only if for all $w \geq v$ and $a \in D_w$ we have $(K, \leq, D, e), w \Vdash \varphi(a, \bar{y})$.

The cases for $\to$, $\wedge$, $\vee$ and $\perp$ are analogous to the ones in the above definition of the forcing relation for Kripke models for **IPC**. We will write $v \Vdash \varphi$ (or $K, v \Vdash \varphi$) instead of $(K, \leq, D, e), v \Vdash \varphi$ if the Kripke model is clear from the context. An $\mathcal{L}_\in$-formula $\varphi$ is *valid in* $K$ if $v \Vdash \varphi$ holds for all $v \in K$, and $\varphi$ is *valid* if it is valid in every Kripke frame $K$. Finally, we will call $(K, \leq)$ the *underlying Kripke frame* of $(K, \leq, D, e)$.

Persistence also holds in Kripke models for set theory.

**Proposition 5** *Let $(K, \leq, V)$ be a Kripke model for set theory, $v \in K$ and $\varphi$ be a formula in the language of set theory such that $K, v \Vdash \varphi$ holds. Then $K, w \Vdash \varphi$ holds for all $w \geq v$.* □

**Theorem 6** *A propositional formula $\varphi$ is derivable in* **IPC** *if and only if it is valid in all Kripke models for* **IPC**. *In particular, a propositional formula $\varphi$ is derivable in* **IPC** *if and only if it is valid in all finite Kripke models for* **IPC**. *Moreover, if a formula $\varphi$ of predicate logic is derivable in* **IQC**, *then it is valid in all Kripke models for set theory.*

A detailed proof of a more general version of this theorem can be found in [17, Theorem 6.6], where also completeness of **IQC** is proved for a class of models that is not restricted to set theory.

## 2.2 The de Jongh property

In this section, we will introduce the de Jongh property and provide a framework for proving it.

**Definition 7** Let $\varphi$ be a propositional formula and let $\sigma : \mathsf{Prop} \to \mathcal{L}_\in^{\mathsf{sent}}$ an assignment of propositional variables to $\mathcal{L}_\in$-sentences. By $\varphi^\sigma$ we denote the $\mathcal{L}_\in$-sentence obtained from $\varphi$ by replacing each propositional variable $p$ with the sentence $\sigma(p)$.

The de Jongh property is a generalisation of de Jongh's classical result [7] concerning Heyting arithmetic $\mathsf{HA}$ and intuitionistic propositional logic **IPC**.

**Theorem 8 (de Jongh)** *Let $\varphi$ be a formula of propositional logic. Then $\mathsf{HA} \vdash \varphi^\sigma$ for all $\sigma : \mathsf{Prop} \to \mathcal{L}_{\mathsf{HA}}^{\mathsf{sent}}$ if and only if* **IPC** $\vdash \varphi$.

Given a theory based on intuitionistic logic, we may consider its propositional logic, i.e., the set of propositional formulas that are derivable after substituting the propositional letters by arbitrary sentences in the language of the theory.

**Definition 9** Let $\mathsf{T}$ be a theory in intuitionistic predicate logic, formulated in a language $\mathcal{L}$. A propositional formula $\varphi$ will be called $\mathsf{T}$-*valid* if and only if $\mathsf{T} \vdash \varphi^\sigma$ for all $\sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}$. The *propositional logic* $\mathbf{L}(\mathsf{T})$ is the set of all $\mathsf{T}$-valid formulas.

Given a theory $\mathsf{T}$ and an intermediate logic $\mathbf{J}$, we denote by $\mathsf{T}(\mathbf{J})$ the theory obtained by closing $\mathsf{T}$ under $\mathbf{J}$.

**Definition 10** We say that a theory $\mathsf{T}$ has the *de Jongh property* if $\mathbf{L}(\mathsf{T}) = \mathbf{IPC}$. The theory $\mathsf{T}$ has the *de Jongh property with respect to an intermediate logic* $\mathbf{J}$ if $\mathbf{L}(\mathsf{T}(\mathbf{J})) = \mathbf{J}$.

De Jongh's theorem is equivalent to the assertion that Heyting arithmetic has the de Jongh property.

In [15], we introduced the notions of loyalty and faithfulness to capture the agreement or discrepancy between the propositional logic of (a class of) models of set theory and the logic of their underlying Kripke frames. Here, we will only focus on their connections with the de Jongh property.

First, we need to introduce some notation. Given a class $C$ of Kripke models for set theory, let $\mathcal{K}_C$ be the class of all underlying Kripke frames of $C$. Then, if $K \in \mathcal{K}_C$, we let $C_K \subseteq C$ consist of all Kripke models in $C$ with underlying Kripke frame $K$. If $\mathcal{K} \subseteq \mathcal{K}_C$ is a class of Kripke frames, let $C_{\mathcal{K}}$ consist of all Kripke models in $C$ with underlying frame $K$ such that $K \in \mathcal{K}$.

**Definition 11** Let $(K, \leq, D, e)$ be a Kripke model for set theory. *The propositional logic* $\mathbf{L}(K, \leq, D, e)$ *of* $(K, \leq, D, e)$ is the set of all propositional formulas $\varphi$ such that for all substitutions $\sigma : \mathsf{Prop} \to \mathcal{L}_{\in}^{\mathsf{sent}}$ we have that $(K, \leq, D, e), v \Vdash \varphi^{\sigma}$ for all $v \in K$. If $C$ is a class of Kripke models for set theory, we let $\mathbf{L}(C)$ be *the propositional logic of* $C$, i.e., the set of all formulas $\varphi$ such that $\varphi \in \mathbf{L}(K, \leq, D, e)$ for all $(K, \leq, D, e) \in C$.

**Proposition 12** *The propositional logic* $\mathbf{L}(K, \leq, D, e)$ *is an intermediate logic for any Kripke model for set theory* $(K, \leq, D, e)$*, and so is* $\mathbf{L}(C)$ *for any class* $C$ *of Kripke models for set theory.* □

Let us call $[\![\varphi]\!]^{(K,\leq,D,e)} = \{v \in K \mid v \Vdash \varphi\}$ the *truth set* of a sentence in the language of set theory in a Kripke model for set theory $(K, \leq, D, e)$. When the model is clear from the context, we will also write $[\![\varphi]\!]^K$.

**Proposition 13** *Let* $(K, \leq, D, e)$ *be a Kripke model for set theory, then* $\mathbf{L}(K, \leq) \subseteq \mathbf{L}(K, \leq, D, e)$.

*Proof* Given any translation $\sigma : \mathsf{Prop} \to \mathcal{L}_{\in}^{\mathsf{sent}}$, we define a valuation $V : \mathsf{Prop} \to \mathcal{P}(K)$ by taking $V(p) = [\![\sigma(p)]\!]$ for all $p \in \mathsf{Prop}$. In this situation, a straightforward induction on the complexity of propositional formulas $\varphi$ shows that $(K, \leq, V), v \Vdash \varphi$ if and only if $(K, \leq, D, e), v \Vdash \varphi^{\sigma}$. This shows that if $(K, \leq) \Vdash \varphi$, then $(K, \leq, D, e) \Vdash \varphi^{\sigma}$ for all translations $\sigma$. Hence, if $\varphi \in \mathbf{L}(K, \leq)$, then $\varphi \in \mathbf{L}(K, \leq, D, e)$. □

The notion of loyalty is obtained by strengthening this inclusion to an equality.

**Definition 14** Let $(K, \leq, D, e)$ be a Kripke model for set theory. We will say that $(K, \leq, D, e)$ is *loyal* if $\mathbf{L}(K, \leq, D, e) = \mathbf{L}(K, \leq)$. A class $C$ of Kripke models for set theory is called *loyal* if every model in $C$ is loyal; $C$ is called *weakly loyal* if $\mathbf{L}(C) = \mathbf{L}(\mathcal{K}_C)$.

**Definition 15** Let $(K, \leq, D, e)$ be a Kripke model for set theory. We will say that $(K, \leq, D, e)$ is *faithful* if for every valuation $V : \mathsf{Prop} \to \mathcal{P}(K)$, and every propositional letter $p \in \mathsf{Prop}$, there is a sentence $\varphi$ in the language of set theory such that

$[\![\varphi]\!]^{(K,\leq,D,e)} = V(p)$. A class $C$ of Kripke models for set theory is called *faithful* if every model in $C$ is faithful.

We will say that $C$ *is finitely faithful* if for every valuation $V : \mathsf{Prop} \to \mathcal{P}(K)$, where $(K, \leq) \in \mathcal{K}_C$, and every finite collection $\{p_i \mid i < n\}$ of propositional variables, there are some $(K, \leq, D, e) \in C_K$ and a collection $\{\varphi_i \in \mathcal{L}_\in^{\mathsf{sent}} \mid i < n\}$ of $\mathcal{L}_\in$-sentences such that $[\![\varphi_i]\!]^{(K,\leq,D,e)} = V(p_i)$ for all $i < n$.

Note that these definitions of loyalty and faithfulness are slightly stronger, but more natural, than the notions introduced in [15].

**Proposition 16** *If a Kripke model for set theory $(K, \leq, D, e)$ is faithful, then it is loyal.*

*Proof* We have to show that $\mathbf{L}(K, \leq, D, e) = \mathbf{L}(K, \leq)$. The inclusion from right to left holds by Proposition 13 without making use of the assumption of faithfulness. For the converse direction, let $\varphi \in \mathbf{L}(K, \leq, D, e)$ be given, i.e., $(K, \leq, D, e) \Vdash \varphi^\sigma$ for all translations $\sigma : \mathsf{Prop} \to \mathcal{L}_\in^{\mathsf{sent}}$. Now, take any valuation $V : \mathsf{Prop} \to \mathcal{P}(K)$ and let $\{p_i \mid i < n\}$ be the collection of propositional variables appearing in $\varphi$. By faithfulness, there is a collection of sentences $\{\varphi_i \mid i < n\}$ such that $V(p) = [\![\varphi_i]\!]^{(K,\leq,D,e)}$. Let $\sigma : \mathsf{Prop} \to \mathcal{L}_\in^{\mathsf{sent}}$ be any translation with $\sigma(p_i) = \varphi_i$. An easy induction on the complexity of formulas $\psi$ with propositional variables among $\{p_i \mid i < n\}$ shows that $(K, \leq, D, e), v \Vdash \psi^\sigma$ if and only if $(K, \leq), v \Vdash \psi$. In this situation, it follows from our assumption $(K, \leq, D, e) \Vdash \varphi^\sigma$ that $(K, \leq) \Vdash \varphi$. □

Note that the the loyalty of a class of Kripke frames is strictly weaker than its faithfulness (cf. [15] for a broader introduction to these notions). A study of algebra-valued models of intuitionistic and paraconsistent set theories that violate these conditions was conducted by Löwe, Passmann and Tarafder in [10].

**Proposition 17** *Let $C$ be a class of Kripke models for set theory, $\mathsf{T}$ an $\mathcal{L}_\in$-theory, and suppose that $A \models \mathsf{T}$ holds for all $A \in C$. Then $\mathsf{T}$ has the de Jongh property with respect to the intermediate logic $\mathbf{L}(C)$.*

*Proof* Let $\mathbf{J} = \mathbf{L}(C)$. We need to show that $\mathbf{L}(\mathsf{T}(\mathbf{J})) = \mathbf{J}$, and the inclusion from right to left is clear. To prove the other direction, assume that $\varphi \notin \mathbf{J} = \mathbf{L}(C)$. Then there is $A \in C$ and $\sigma : \mathsf{Prop} \to \mathcal{L}_\in^{\mathsf{sent}}$ such that $A \nVdash \varphi^\sigma$. Hence, $\mathsf{T} \nvdash \varphi^\sigma$, i.e., $\varphi \notin \mathbf{L}(\mathsf{T}(\mathbf{J}))$. □

To prove that a particular theory has the de Jongh property, it will be enough to construct finitely faithful classes of models.

**Proposition 18** *If a class $C$ of Kripke models for set theory is finitely faithful and $\mathcal{K} \subseteq \mathcal{K}_C$ is a class of Kripke frames, then $C_\mathcal{K} \subseteq C$ is finitely faithful.*

*Proof* When restricting from $C$ to $C_\mathcal{K}$, we keep all Kripke models that are based on a frame in $\mathcal{K}$. Hence, all witnesses for finite faithfulness required by Definition 15 are still available. □

**Proposition 19** *If a class $C$ of Kripke models for set theory is finitely faithful, then it is weakly loyal.*

*Proof* Assume that $C$ is finitely faithful. We need to show that $\mathbf{L}(C) = \mathbf{L}(\mathcal{K}_C)$. To do so, it suffices to prove $\mathbf{L}(C) \subseteq \mathbf{L}(\mathcal{K}_C)$ by Proposition 13. Given $\varphi \notin \mathbf{L}(\mathcal{K}_C)$, there is a Kripke frame $(K, \leq)$ and a valuation $V : \mathsf{Prop} \to K$ such that $(K, \leq), V \nVdash \varphi$. By finite faithfulness, we can obtain set-theoretical sentences $\psi_i$ and a Kripke model for set theory $(K, \leq, D, e) \in C$ such that $[\![\psi_i]\!]^{(K,\leq,D,e)} = [\![p_i]\!]^{(K,\leq)}$ for all propositional variables $p_i$ appearing in $\varphi$. By essentially the same induction as in the proof of Proposition 16, we obtain that $[\![\varphi]\!]^{(K,\leq,D,e)} = [\![\varphi^\sigma]\!]^{(K,\leq)}$, where $\sigma$ is the assignment $p_i \mapsto \psi_i$. Therefore, $(K, \leq, D, e) \nVdash \varphi$ and $\varphi \notin \mathbf{L}(C)$. □

**Proposition 20** *If a class $C$ of Kripke models for set theory is weakly loyal and $C \vDash \mathsf{T}$ for an $\mathcal{L}_\in$-theory $\mathsf{T}$, then $\mathsf{T}$ has the de Jongh property with respect to $\mathbf{L}(\mathcal{K}_C)$.*

*Proof* By Proposition 17, $\mathsf{T}$ has the de Jongh property with respect to $\mathbf{L}(C)$. Because $C$ is weakly loyal, we have $\mathbf{L}(C) = \mathbf{L}(\mathcal{K}_C)$ by Definition 14. □

Now, we can derive our main tool for proving the de Jongh property.

**Corollary 21** *If a class $C$ of Kripke models for set theory is finitely faithful with $C \vDash \mathsf{T}$ for an $\mathcal{L}_\in$-theory $\mathsf{T}$, and $\mathcal{K} \subseteq \mathcal{K}_C$ a class of Kripke frames, then $\mathsf{T}$ has the de Jongh property with respect to $\mathbf{L}(\mathcal{K})$.*

*Proof* If the class $C$ is finitely faithful with $C \vDash \mathsf{T}$ and $\mathcal{K} \subseteq \mathcal{K}_C$ a class of Kripke frames, then, by Proposition 18, $C_\mathcal{K}$ is finitely faithful. Hence, $C_\mathcal{K}$ is weakly loyal by Proposition 19. Note that $\mathcal{K}_{C_\mathcal{K}} = \mathcal{K}$. Therefore, Proposition 20 implies that $\mathsf{T}$ has the de Jongh property with respect to $\mathbf{L}(\mathcal{K})$. □

In fact, all of the above can be straightforwardly generalised to arbitrary theories and languages (cf. [15]). Let us remark here that the last propositions encapsulate one of the typical methods of proving the de Jongh property: For example, many of the results of de Jongh, Verbrugge, and Visser [8] that establish results concerning the de Jongh property of Heyting arithmetic $\mathsf{HA}$ can be construed as providing instances of finitely faithful or weakly loyal classes of models for $\mathsf{HA}$. Similarly, Ardeshir and Mojtahedi [2] construct a finitely faithful class of models for basic arithmetic to prove that basic arithmetic has the de Jongh property with respect to the basic propositional calculus.

## 3 Kripke models for bounded constructive Zermelo-Fraenkel set theory

In this section, we will introduce bounded constructive set theory $\mathsf{BCZF}$, present the Iemhoff models and exhibit a failure of the axiom of exponentiation in models that involve forcing non-trivially.

### 3.1 Bounded constructive Zermelo-Fraenkel set theory

To begin with, let us introduce *bounded constructive Zermelo-Fraenkel set theory* $\mathsf{BCZF}$ as well as *constructive Zermelo-Fraenkel set theory* $\mathsf{CZF}$. To do so, we list the relevant axioms and axiom schemes. As usual, the bounded quantifiers $\forall x \in a\ \varphi(x)$ and $\exists x \in a\ \varphi(x)$ are abbreviations for $\forall x(x \in a \to \varphi(x))$ and $\exists x(x \in a \wedge \varphi(x))$, respectively.

$$\forall a \forall b(\forall x(x \in a \leftrightarrow x \in b) \to a = b) \qquad \text{(Extensionality)}$$
$$\exists a\, \forall x \in a\ \bot \qquad \text{(Empty Set)}$$
$$\forall a \forall b \exists y \forall x(x \in y \leftrightarrow (x = a \vee x = b)) \qquad \text{(Pairing)}$$
$$\forall a \exists y \forall x(x \in y \leftrightarrow \exists u(u \in a \wedge x \in u)) \qquad \text{(Union)}$$
$$(\forall a(\forall x \in a\ \varphi(x) \to \varphi(a))) \to \forall a \varphi(a) \qquad \text{(Set Induction)}$$

Moreover, we have the axiom scheme of bounded separation, where $\varphi$ ranges over the bounded formulas:

$$\forall a \exists y \forall x(x \in y \leftrightarrow x \in a \wedge \varphi(x)) \quad (\varphi \text{ is } \Delta_0\text{-formula}) \qquad \text{(Bounded Separation)}$$

In the following strong infinity axiom, $\mathrm{Ind}(a)$ is the formula denoting that $a$ is an inductive set: $\mathrm{Ind}(a)$ abbreviates $\emptyset \in a \wedge \forall x \in a \exists y \in a\ y = \{x\}$.

$$\exists a(\mathrm{Ind}(a) \wedge \forall b(\mathrm{Ind}(b) \to \forall x \in a(x \in b))) \qquad \text{(Strong Infinity)}$$

Finally, we have the schemes of strong collection and subset collection for all formulas $\varphi(x, y)$ and $\psi(x, y, u)$, respectively.

$$\begin{aligned} &\forall a(\forall x \in a \exists y\ \varphi(x, y) \to \\ &\qquad \exists b(\forall x \in a \exists y \in b\ \varphi(x, y) \wedge \forall y \in b \exists x \in a\ \varphi(x, y))) \qquad \text{(Strong Collection)} \\ &\forall a \forall b \exists c \forall u(\forall x \in a \exists y \in b\ \psi(x, y, u) \to \\ &\qquad \exists d \in c(\forall x \in a \exists y \in d\ \psi(x, y, u) \wedge \forall y \in d \exists x \in a\ \psi(x, y, u))) \\ &\qquad\qquad \text{(Subset Collection)} \end{aligned}$$

**Definition 22** The theory $\mathsf{CZF}$ of *constructive Zermelo-Fraenkel set theory* consists of the axioms and rules of intuitionistic predicate logic for the language $\mathcal{L}_\in$ extended by the axioms of extensionality, empty set, pairing, union and strong infinity as well as the axiom schemes of set induction, bounded separation, strong collection and subset collection.

In the statement of the following axiom of exponentiation, $f : x \to y$ is an abbreviation for the $\Delta_0$-formula $\varphi(f, x, y)$ stating that $f$ is a function from $x$ to $y$.

$$\forall x\, \forall y\, \exists z\, \forall f(f \in z \;\leftrightarrow\; f : x \to y) \qquad \text{(Exponentiation, } \mathsf{Exp}\text{)}$$

The axiom of exponentiation is a constructive consequence of the axiom of subset collection over $\mathsf{CZF}$ (cf. [1, Theorem 5.1.2]). Hence, a failure of exponentiation implies a failure of subset collection. We will see in Section 3.3 that the Iemhoff models do not satisfy the axiom of exponentiation in general, and therefore, they cannot satisfy full $\mathsf{CZF}$. Hence, we need to consider a weakened version of $\mathsf{CZF}$.

The axiom scheme obtained from strong collection when restricting $\varphi$ to range over $\Delta_0$-formulas only will be called *Bounded Strong Collection*. Similarly, we obtain the axiom scheme of *Set-bounded Subset Collection* from the axiom scheme of subset collection when restricting $\psi$ to $\Delta_0$-formulas such that $z$ is set-bounded in $\psi$ (i.e., it is possible to intuitionistically derive $z \in t$ for some term $t$ that appears in $\psi$ from $\psi(x, y, z)$).

**Definition 23** The theory BCZF of *bounded constructive Zermelo-Fraenkel set theory* consists of the axioms and rules of intuitionistic predicate logic for the language $\mathcal{L}_\in$ extended by the axioms of extensionality, empty set, pairing, union and strong infinity as well as the axiom schemes of set induction, bounded separation, bounded strong collection and set-bounded subset collection.

## 3.2 Iemhoff models

The idea is to obtain models of set theory by assigning classical models of ZF set theory to every node of a Kripke model for intuitionistic predicate logic. We will closely follow Iemhoff's [5], but give up on some generality that we do not need for our purposes. We will start by giving a condition for when an assignment of models to nodes is suitable for our purposes.

**Definition 24** Let $(K, \leq)$ be a Kripke frame. An assignment $\mathcal{M} : K \to V$ of nodes to transitive models of ZF set theory is called *sound for $K$* if for all nodes $i, j \in K$ with $i \leq j$ we have that $\mathcal{M}(i) \subseteq \mathcal{M}(j)$, and the inclusion map is a homomorphism of models of set theory (i.e., it preserves $\in$ and $=$).

For convenience, we will write $\mathcal{M}_v$ for $\mathcal{M}(v)$. Of course, this could be readily generalised to homomorphisms of models of set theory that are not necessarily inclusions, but we will not need this level of generality here.

**Definition 25** Given a Kripke model $(K, \leq)$ and a sound assignment $\mathcal{M} : K \to V$, we define the *Iemhoff model $K(\mathcal{M})$* to be the Kripke model for set theory $(K, \leq, \mathcal{M}, e)$ where $e_v = \in \upharpoonright (\mathcal{M}_v \times \mathcal{M}_v)$.

Persistence for Iemhoff models is a special case of persistence for Kripke models for set theory.

**Proposition 26** *If $K(\mathcal{M})$ is an Iemhoff model with nodes $v, w \in K$ such that $v \leq w$, then for all formulas $\varphi$, $K(\mathcal{M}), v \Vdash \varphi$ implies $K(\mathcal{M}), w \Vdash \varphi$.* □

We will now analyse the set theory satisfied by these models.

**Definition 27** We say that a set-theoretic formula $\varphi(x_0, \ldots, x_{n-1})$ is *evaluated locally* if for all Iemhoff models $K(\mathcal{M})$, where $\mathcal{M}$ is a sound assignment, we have that $K(\mathcal{M}), v \Vdash \varphi(a_0, \ldots, a_{n-1})$ if and only if $\mathcal{M}_v \models \varphi(a_0, \ldots, a_{n-1})$ for all $a_0, \ldots, a_{n-1} \in \mathcal{M}_v$.

**Proposition 28** *If $\varphi$ is a $\Delta_0$-formula, then $\varphi$ is evaluated locally.*

*Proof* This statement can be shown by actually proving a stronger statement by induction on $\Delta_0$-formulas, simultaneously for all $v \in K$. Namely, we can show that for all $w \geq v$ it holds that $w \Vdash \varphi(a_0, \dots, a_n)$ if and only if $\mathcal{M}_v \vDash \varphi(a_0, \dots, a_n)$. To prove the case of the bounded universal quantifier and the case of implication, we need that the quantifier is outside in the sense that our induction hypothesis will be:

$$\forall w \geq v(w \Vdash \varphi(a_0, \dots, a_n) \iff \mathcal{M}_v \vDash \varphi(a_0, \dots, a_n)).$$

With this setup, the induction follows straightforwardly. □

**Theorem 29 (Iemhoff, [5, Corollary 4])** *Let $K(\mathcal{M})$ be an Iemhoff model. Then* $K(\mathcal{M}) \Vdash \mathsf{BCZF}$.

Let us conclude this section with the following curious observation.

**Proposition 30** *If $K(\mathcal{M})$ is an Iemhoff model such that every $\mathcal{M}_v$ is a model of the axiom of choice, then the axiom of choice holds in $K(\mathcal{M})$.*

*Proof* Recall that the axiom of choice is the following statement:

$$\forall a((\forall x \in a \forall y \in a\ (x \neq y \to x \cap y = \emptyset)) \to \exists b \forall x \in a \exists! z \in b\ z \in x). \tag{AC}$$

Let $v \in K$ and $a \in \mathcal{M}_v$ such that $v \Vdash \forall x \in a \forall y \in a\ (x \neq y \to x \cap y = \emptyset)$. This is a $\Delta_0$-formula, so we can apply Proposition 28 to derive that $\mathcal{M}_v \vDash \forall x \in a \forall y \in a\ (x \neq y \to x \cap y = \emptyset)$. As $\mathcal{M}_v \vDash \mathsf{AC}$, there is some $b \in \mathcal{M}_v$ such that $\mathcal{M}_v \vDash \forall x \in a \exists! z \in b\ z \in x$. Again, this is a $\Delta_0$-formula, so it holds that $v \Vdash \forall x \in a \exists! z \in b\ z \in x$. As $b \in \mathcal{M}_v$, we have $v \Vdash \exists b \forall x \in a \exists! z \in b\ z \in x$. But this shows that $v \Vdash \mathsf{AC}$. □

As BCZF contains the bounded separation axiom, it follows that AC implies the law of excluded middle for bounded formulas in the models of the proposition (cf. [1, Chapter 10.1]).

### 3.3 A failure of exponentiation

In this section, we will exhibit a failure of the axiom of exponentiation in particular Iemhoff models.

**Proposition 31** *Let $K(\mathcal{M})$ be an Iemhoff model such that there are $v, w \in K$ with $v < w$. If $a, b \in \mathcal{M}_v$ and $g : a \to b$ is a function contained in $\mathcal{M}_w$ but not in $\mathcal{M}_v$, then* $K(\mathcal{M}) \nVdash \mathsf{Exp}$.

*Proof* Assume, for a contradiction, that $K(\mathcal{M}) \Vdash \mathsf{Exp}$. Further, assume that $a, b \in \mathcal{M}_v$ and $g : a \to b$ is a function contained in $\mathcal{M}_w$ but not in $\mathcal{M}_v$. Then,

$$K(\mathcal{M}), v \Vdash \forall x\ \forall y\ \exists z\ \forall f(f \in z \leftrightarrow f : x \to y),$$

and by the definition of our semantics this just means that there is some $c \in \mathcal{M}_v$ such that $K(\mathcal{M}), v \Vdash \forall f(f \in c \leftrightarrow f : a \to b)$. By the semantics of universal quantification, this means that $K(\mathcal{M}), w \Vdash g \in c \leftrightarrow g : a \to b$. Since $g$ is indeed a function from $a \to b$, it follows that $K(\mathcal{M}), w \Vdash g \in c$. As $c$ is a member of $\mathcal{M}_v$ by assumption, we have $g \in c \in \mathcal{M}_v$. Hence, by transitivity, $g \in \mathcal{M}_v$. But this is a contradiction to our assumption that $g$ is not contained in $\mathcal{M}_v$. □

Of course, when adding a generic filter for a non-trivial forcing notion, we always add such a function, namely the characteristic function of the generic filter. Therefore, Proposition 31 yields:

**Corollary 32** *Let $K(\mathcal{M})$ be an Iemhoff model. If there are nodes $v < w \in K$ such that $\mathcal{M}_w$ is a non-trivial generic extension of $\mathcal{M}_v$ (i.e., $\mathcal{M}_w = \mathcal{M}_v[G]$ for some generic $G \notin \mathcal{M}_v$), then it is not a model of* $\mathsf{CZF}$. □

In Kripke semantics for intuitionistic logic, $K(\mathcal{M}) \Vdash \neg\varphi$ is strictly stronger than $K(\mathcal{M}) \nVdash \varphi$. The above results give an instance of the latter, now we will provide an example of the former.

**Proposition 33** *There is an Iemhoff model $K(\mathcal{M})$ that forces the negation of the exponentiation axiom, i.e., $K(\mathcal{M}) \Vdash \neg\mathsf{Exp}$.*

*Proof* Consider the Kripke frame $K = (\omega, <)$ where $<$ is the standard ordering of the natural numbers. Construct the assignment $\mathcal{M}$ as follows: Choose $\mathcal{M}_0$ to be any countable and transitive model of $\mathsf{ZFC}$. If $\mathcal{M}_i$ is constructed, let $\mathcal{M}_{i+1} = \mathcal{M}_i[G_i]$ where $G_i$ is generic for Cohen forcing over $\mathcal{M}_i$ (actually, every non-trivial forcing notion does the job). Clearly, $\mathcal{M}$ is a sound assignment of models of set theory. Now, we want to show that for every $i \in \omega$ we have that $i \Vdash \neg\mathsf{Exp}$, i.e., for all $j \geq i$ we need to show that $j \Vdash \mathsf{Exp}$ implies $j \Vdash \bot$. This, however, is done exactly as in the proof of Proposition 31, where the witnesses are the characteristic functions $\chi_{G_i}$ of the generic filters $G_i$. □

## 4 Bounded constructive Zermelo-Fraenkel set theory has the de Jongh property

The aim of this section is to prove that the class of Iemhoff models is finitely faithful. By Corollary 21 this is sufficient to prove that $\mathsf{BCZF}$ has the de Jongh property with respect to all logics characterised by a class of Kripke frame.

### 4.1 Technical preliminaries

We define the relativisation $\varphi \mapsto \varphi^{\mathrm{L}}$ of a formula of set theory to the constructible universe L in the usual way. Note, however, that in our setting the evaluation of universal quantifiers and implications is in general not local (in contrast to classical models of set theory). Nevertheless, we will now show that—under mild assumptions—statements about the constructible universe can be evaluated locally.

**Proposition 34** *There is a $\Sigma_1$-formula $\varphi(x)$ such that in any model $M \vDash \mathsf{ZFC}$, we have $M \vDash \varphi(x) \leftrightarrow x \in \mathrm{L}$.*

*Proof* See [6, Lemma 13.14]. □

From now on, let $x \in \mathrm{L}$ be an abbreviation for $\varphi(x)$, where $\varphi$ is the $\Sigma_1$-formula from Proposition 34.

**Proposition 35** *Let $K$ be a Kripke frame and $\mathcal{M}$ a sound assignment of nodes to transitive models of* $\mathsf{ZFC}$*. Then $K(\mathcal{M}), v \Vdash x \in \mathrm{L}$ if and only if $\mathcal{M}_v \vDash x \in \mathrm{L}$, i.e., the formula $x \in \mathrm{L}$ is evaluated locally.*

*Proof* Recall that the existential quantifier is defined locally, i.e., the witness for the quantification must be found within the domain associated to the current node in the Kripke model. Then, the statement of the proposition follows from the fact that $\Delta_0$-formulas are evaluated locally by Proposition 28. ◻

The crucial detail of the following technical Lemma 37 is the fact that the constructible universe is absolute between inner models of set theory. We will therefore need to strengthen the notion of a sound assignment.

**Definition 36** Let $K$ be a Kripke frame. We say that a sound assignment $\mathcal{M} : K \to V$ *agrees on* L if there is a transitive model $N \vDash \mathsf{ZFC} + V = \mathrm{L}$ such that $N$ is an inner model of $\mathcal{M}_v$ for every $v \in K$.

In particular, if $K$ is a Kripke frame and $\mathcal{M} : K \to V$ agrees on L, then we are justified in referring to the constructible universe L from the point of view of all models in $\mathcal{M}$.

**Lemma 37** *Let $K$ be a Kripke frame and $\mathcal{M}$ be a sound assignment that agrees on* L*. Then the following are equivalent for any formula $\varphi(x)$ in the language of set theory, and all parameters $a_0, \ldots, a_{n-1} \in \mathrm{L}$:*

*(i) for all $v \in K$, we have $K(\mathcal{M}), v \Vdash (\varphi(a_0, \ldots, a_{n-1}))^{\mathrm{L}}$,*
*(ii) for all $v \in K$, we have $\mathcal{M}_v \vDash (\varphi(a_0, \ldots, a_{n-1}))^{\mathrm{L}}$,*
*(iii) there is a $v \in K$ such that $\mathcal{M}_v \vDash (\varphi(a_0, \ldots, a_{n-1}))^{\mathrm{L}}$, and,*
*(iv)* $\mathrm{L} \vDash \varphi(a_0, \ldots, a_{n-1})$.

*Proof* By our assumption, $a_0, \ldots, a_{n-1} \in \mathcal{M}_v$ for all $v \in K$ as $\mathrm{L} \subseteq \mathcal{M}_v$ for all $v \in K$. The equivalence of (ii), (iii) and (iv) follows directly from the fact that L is absolute between inner models of $\mathsf{ZFC}$.

The equivalence of (i) and (ii) can be proved by an induction on set-theoretic formulas simultaneously for all nodes in $K$ with the induction hypothesis as in the proof of Proposition 28. For the case of the universal quantifier, we make use of the fact that $\mathcal{M}$ agrees on L (hence, that L is absolute between all models $\mathcal{M}_v$ for $v \in K$), and apply Proposition 35. ◻

Friedman, Fuchino and Sakai [3] presented the family of sentences that we are going to use to imitate the behaviour of propositional variables in a valuation of a Kripke frame. Consider the following statements $\psi_i$:

$$\text{There is an injection from } \aleph_{i+2}^{\mathrm{L}} \text{ to } \mathcal{P}(\aleph_i^{\mathrm{L}}).$$

There are different ways of formalising these statements that are classically equivalent, but differ in the way they are evaluated in a Kripke model. For our purposes, we choose to define the sentence $\psi_i$ like this:

$$\begin{aligned}\exists x \exists y \exists g((x = \aleph_{i+2})^{\mathrm{L}} \wedge (y = \aleph_i)^{\mathrm{L}} \\ &\wedge g \text{ “is an injective function”} \\ &\wedge \operatorname{dom}(g) = x \\ &\wedge \forall \alpha \in x \forall z \in g(\alpha)\, z \in y)\end{aligned}$$

The main reason for this choice of formalisation is that the semantics of the existential quantifier is local, which will allow us to prove the following crucial observation.

**Proposition 38** *Let $K$ be a Kripke frame and $\mathcal{M}$ a sound assignment that agrees on* L. *Then $K(\mathcal{M}), v \Vdash \psi_i$ if and only if $\mathcal{M}_v \vDash \psi_i$, i.e., the sentences $\psi_i$ are evaluated locally.*

*Proof* This follows from Lemma 37, Proposition 28 and the fact that the semantics of the existential quantifier is local, i.e., the sets $x$, $y$ and $g$ of the above statement must (or may not) be found within $\mathcal{M}_v$. In this situation, it suffices to argue that the following conjunction is evaluated locally:

$$\begin{aligned}(x = \aleph_{i+2})^{\mathrm{L}} \wedge (y = \aleph_i)^{\mathrm{L}} &\wedge g \text{ “is an injective function”} \\ &\wedge \operatorname{dom}(g) = x \\ &\wedge \forall \alpha \in x \forall z \in g(\alpha)\, z \in y.\end{aligned}$$

It suffices to argue that every conjunct is evaluated locally. For the first two conjuncts of the form $\varphi^{\mathrm{L}}$ this holds by Lemma 37. The final three conjuncts are $\Delta_0$-formulas. So we can apply Proposition 28 and the desired result follows. □

## 4.2 The construction of the Iemhoff model

We will now obtain a collection of models of set theory using the forcing notions from Friedman, Fuchino and Sakai in [3]. From this collection, we define Iemhoff models by constructing sound assignments that agree on L.

We begin by setting up the forcing construction. We start from some countable transitive constructible universe L (that is, a countable transitive model of ZFC set theory satisfying the axiom $V = \mathrm{L}$). Let $\mathbb{Q}_n$ be the forcing notion[1] $\mathrm{Fn}(\aleph^{\mathrm{L}}_{n+2}, 2, \aleph^{\mathrm{L}}_n)$, defined within L. Given $A \subseteq \omega$, we define the following forcings:

$$\mathbb{P}^A_n = \begin{cases} \mathbb{Q}_n, & \text{if } n \in A, \\ \mathbb{1}, & \text{otherwise.} \end{cases}$$

Then let $\mathbb{P}^A = \prod_{n<\omega} \mathbb{P}^A_n$ be the full support product of the forcing notions $\mathbb{P}^A_n$. Recall that the ordering $<$ on $\mathbb{P}^A$ is defined by $(a_i)_{i\in\omega} < (b_i)_{i\in\omega}$ if and only if $a_i <_i b_i$ for

[1] The notation $\mathrm{Fn}(I, J, \lambda)$ is introduced by Kunen in [9, Definition 6.1] and denotes the set of all partial functions $p : I \to J$ of cardinality less than $\lambda$ ordered by reversed inclusion.

all $i \in \omega$. Now, let $G$ be $\mathbb{P}^\omega$-generic over L, and let $G_n = \pi_n[G]$ be the $n$-th projection of $G$. Let $H$ be the trivial generic filter on the trivial forcing $\mathbb{1}$. Now, for $A \subseteq \omega$ and $n \in \omega$ define the collection of filters:

$$G_n^A = \begin{cases} G_n, & \text{if } n \in A, \\ H, & \text{otherwise,} \end{cases}$$

and let $G^A = \prod_{n<\omega} G_n^A$.

**Proposition 39** *The filter $G^A$ is $\mathbb{P}^A$-generic over* L. □

**Proposition 40** *If $A \subseteq B \subseteq \omega$ and $A \in \mathrm{L}[G^B]$, then $\mathrm{L}[G^A] \subseteq \mathrm{L}[G^B]$. Indeed, $\mathrm{L}[G^A]$ is an inner model of $\mathrm{L}[G^B]$.* □

The additional assumption $A \in \mathrm{L}[G^B]$ is necessary because there are forcing extensions that cannot be amalgamated (see [4, Observation 35] for a discussion of this).

**Proposition 41 (Friedman, Fuchino and Sakai, [3, Proposition 5.1])** *Let $i \in \omega$ and $A \subseteq \omega$. Then $\mathrm{L}[G^A] \models \psi_i$ if and only if $i \in A$.* □

This concludes our preparatory work and we can now prove our main result.

**Theorem 42** *The class of Iemhoff models is finitely faithful.*

*Proof* Let $(K, \leq)$ be any Kripke frame. By the definition of finite faithfulness, we need to show that for any valuation $V : \mathsf{Prop} \to \mathcal{P}(K)$ on $K$, and every finite collection $p_i$, $i < n$ of propositional letters, there is a collection of set-theoretical sentences $\varphi_i$ and an Iemhoff model $K(\mathcal{M})$ such that $\{v \in K \mid K(\mathcal{M}), v \Vdash \varphi_i\} = V(p_i)$ for all $i < n$, i.e., the truth sets of all $\varphi_i$ and $v_i$ coincide.

Now, let $\bar{V}$ be the valuation with $\bar{V}(p_i) = V(p_i)$ for each $p_i$, $i < n$, and $\bar{V}(p) = \emptyset$ otherwise. Observe that $\bar{V}^{-1}(v)$ is finite for every $v \in K$ and we can define $A_v = \{i < \omega \mid v \in \bar{V}(p_i)\}$ for any $v \in K$. It holds that $A_v \in \mathrm{L}$ as it is a finite subset of $\omega$. Note that $v \leq w \in K$ implies that $A_v \subseteq A_w$ by monotonicity of the original valuation $V$. Therefore, we know by Proposition 40 that $\mathrm{L}[G^{A_v}]$ is an inner model of $\mathrm{L}[G^{A_w}]$ for all $v \leq w \in K$. Hence, the assignment $\mathcal{M} : K \to V$ with $\mathcal{M}_v = \mathrm{L}[G^{A_v}]$ is sound and agrees on L. This yields the Iemhoff model $K(\mathcal{M})$. Choose $\varphi_i = \psi_i$ for all $i < n$. Then:

$$\begin{aligned} K(\mathcal{M}), v \Vdash \varphi_i &\iff K(\mathcal{M}), v \Vdash \psi_i && \text{(by the definition of } \varphi_i) \\ &\iff \mathcal{M}_v \models \psi_i && \text{(by Proposition 38)} \\ &\iff \mathrm{L}[G^{A_v}] \models \psi_i && \text{(by definition of } \mathcal{M}) \\ &\iff i \in A_v && \text{(by Proposition 41)} \\ &\iff p_i \in \bar{V}^{-1}(v) && \text{(by definition of } A_v) \\ &\iff K, \bar{V}, v \Vdash p_i. \end{aligned}$$

This shows that $[\![\varphi_i]\!]^{K(\mathcal{M})} = \{v \in K \mid K(\mathcal{M}), v \Vdash \varphi_i\} = \bar{V}(p_i) = V(p_i)$ for all $i < n$, and this finishes the proof of the finite faithfulness of the class of Iemhoff models. □

**Corollary 43** *The theory* BCZF *has the de Jongh property with respect to every logic that is characterised by a class of Kripke frames.*

*Proof* By Theorem 42, we know that the class $C$ of Iemhoff models is faithful. Moreover, $C \models$ BCZF by Theorem 29. In this situation, Corollary 21 implies that BCZF has the de Jongh property with respect to $\mathbf{L}(\mathcal{K})$ for every class $\mathcal{K}$ of Kripke frames. □

For example, BCZF has the de Jongh property with respect to the logics discussed in [8], such as Dummett's logic **LC**, the logic of the weak excluded middle **KC**, the logic **KP** of Kreisel and Putnam, and the logics $\mathbf{BD}_n$ of bounded depth and Scott's logic **Sc**. Our limiting result Proposition 32 shows that we cannot easily push this method to obtain a proof of the de Jongh property of stronger set theories than BCZF, such as full CZF.

Let us remark here that there is an alternative way of proving Corollary 43 for logics that are characterised by a class of *finite* Kripke frames that is reminiscent of what is sometimes called *Smoryński's trick*: Given a collection of Kripke models for Heyting arithmetic HA, one can construct a new model of HA by adding a new root underneath the old models and attaching the standard model $\omega$ of HA to it (cf. [16]). In our case, given a collection of Iemhoff models that are compatible in the sense that they agree on L, one can obtain a new Iemhoff model by adding a new root to the disjoint union of the Kripke frames, attaching L to it and leaving the rest of the assignments unaltered.

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