R.e. prime powers and total rigidity

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Synopsis. We introduce *r.e. prime powers* as the least common multiple of the recursive ultrapowers of $\mathbb{N}$ from Hirschfeld [16] and the r.e. ultrapowers of $\mathbb{N}$ from Hirschfeld & Wheeler [18]. R.e. prime powers help us with establishing that r.e. ultrapowers admit no non-identity self-embeddings, settling an issue raised by Hirschfeld & Wheeler. This parallels an earlier theorem by McLaughlin [34] for recursive ultrapowers.
The road to solution takes us through a number of variants of recursive/online forest colouring tasks. Along the way we also take a look at a Rudin–Keisler-like category of prime filters in the lattice of r.e. sets and discover some r.e. prime powers that do admit non-trivial self-embeddings.

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0. Introduction

Recursive ultrapowers of the natural numbers first saw the light of day at the hand of Hirschfeld [15] and [16]. A *recursive ultrapower* (a.k.a. *Nerode semiring*) $\mathbb{N}[u]$ is formed by unary total recursive functions reduced by agreement modulo u, a fixed non-principal ultrafilter in the algebra $\mathcal{R}$ of recursive subsets of ω. Hirschfeld & Wheeler [18, Chapter 9] supplemented this with r.e. ultrapowers. Where p is a maximal (hence prime) non-principal filter in the lattice $\mathcal{E}$ of r.e. sets, an *r.e. ultrapower* (a.k.a. *simple model*) $\mathbb{N}[p]$ is the collection of unary partial recursive functions with domain in p reduced by agreement modulo p. R.e. ultrapowers are exactly those existentially closed models of the true $\forall\exists$ arithmetic TA_2 which are finitely generated w.r.t. partial recursive functions.

Hirschfeld & Wheeler [18, 9.6(iii)] show that each r.e. ultrapower $\mathbb{N}[u]$ is *rigid*, that is, the only automorphism of $\mathbb{N}[u]$ is the identity. McLaughlin [30, Theorem 3.7] observes that rigidity also holds for recursive ultrapowers. On top of that, McLaughlin [34] establishes that, much like most ultrapower-like consructions, recursive ultrapowers are *totally rigid*, which means that they admit no non-identity self-embeddings.

The aim of this paper is twofold.

First, we present and study *r.e. prime powers*, a class of restricted powers of the natural numbers that straddles the divide between the recursive ultrapowers on the one hand and the r.e. ultrapowers on the other. Both of the latter classes are included among r.e. prime powers. Our definition of r.e. prime powers merely replaces the maximal non-principal filter p in the Hirschfeld–Wheeler definition of r.e. ultrapowers by an arbitrary non-principal prime filter in $\mathcal{E}$ — we call these filters *(r.e.) primes* following Shavrukov [43]. This results in structures isomorphic to *cohesive powers* of $\mathbb{N}$ from Dimitrov [7], who studies powers of arbitrary computable structures without specific focus on powers of $\mathbb{N}$. Our interest in r.e. prime powers is in part motivated by the general principle, confirmed in various earlier settings, that power structures can offer a useful second perspective on properties of the underlying ultra-, or, in our case, prime filters.

Second, we address the question of total rigidity for r.e. prime powers. Non-trivial self-embeddability, especially self-embeddability as a proper initial segment, is a well-studied (Kaye [22, Chapter 12]) and almost ubiquitous (see e.g. Dimitracopoulos & Paris [6]) property in countable models of arithmetic theories with stronger collection footing than TA_2. While initial self-embeddabiity for r.e. prime powers is out of the question (Corollary 2.12), the finitely generated aspect of power structures imparts a slightly different flavour to embeddability properties of r.e. prime powers — the role of Σ_1 types of elements of the power becomes decisive (Lemma 2.15). Each self-embedding ι is uniquely determined by the image of the power's generator $\boldsymbol{x}$. The element $\iota(\boldsymbol{x})$ is represented by some partial recursive function f, which, if $\iota(\boldsymbol{x}) \neq \boldsymbol{x}$, can be assumed fixed-point-free. For such a function f to correspond to a self-embedding, it is necessary that the graph of f evade (partial) recursive colourings with finitely many colours. Thus constructing a recursive colouring of the graph of each candidate f will prove total rigidity. A popular approach to recursive colourability consists in *on-line colouring* games where in each successive round the Builder player adds new vertices and/or edges to the graph while the Painter player responds by assigning colours to (some of the) recently added vertices.

We devise a recursive strategy for colouring a sufficiently large portion of the graph of any sufficiently nice partial recursive function, which turns out to suffice for establishing the total rigidity of each r.e. ultrapower (Corollary 6.2). On the other hand, a different set of rules leads to Forester, a younger cousin of Builder, winning in a series of on-line forest colouring games, and we use this to construct an r.e. prime power which does admit a non-identity self-embedding (Theorem 3.1), revealing just how lucky the r.e. ultrapowers are in enjoying total rigidity.

0.A. Contents

Section 1 reviews recursive ultrapowers and McLaughlin's theorem on their total rigidity. We present an alternative argument for (a version of the graph colouring lemma involved in) this theorem. Subsection 1.B points out the connection of McLaughlin's theorem to a rigidity-like property of the complements of r-maximal r.e. sets first observed by Lerman [27] — we show that this property is a consequence of total rigidity.

In section 2 we present the definition and first properties of r.e. prime powers including a restricted version of Łoś' Lemma and criteria for embeddability and Σ_1-elementary embeddability.

Section 3 witnesses the defeat of Painter in a forest colouring game where vertices only have to be assigned a colour once they reach a fixed depth in the forest being constructed. We show how this leads to the construction of a non-trivial self-embedding of an appropriate r.e. prime power.

The study of r.e. prime powers resumes in section 4 with the introduction of new tools such as total recursive skies in models of TA_2, a version of Rudin–Keisler ordering on primes where only partial recursive reductions are allowed, and the corresponding category $\mathbf{rk}_\Sigma$. We also detail the connection between said reductions among r.e. primes and Σ_1 types of elements in models of TA_2 and explain how *hinged* primes, introduced in Shavrukov [43], relate to recursive ultrapowers.

In section 5 we focus on general properties of non-trivial self-embeddings of r.e. prime powers linking them to non-identity $\mathbf{rk}_\Sigma$-endomorphisms of primes and to inclusion-downward morphisms in $\mathbf{rk}_\Sigma$. This helps us force an arbitrary partial recursive function inducing a non-trivial self-embedding to conform to a stringent standard, which sets the scene for the final section.

A technical lemma on large diagonal intersections of uniformly r.e. families opens section 6. For a class of r.e. prime powers that includes r.e. ultrapowers, that lemma helps to tilt the battlefield in Painter's favour, and we describe a colouring strategy under the new auspicious conditions. This entails total rigidity for r.e. ultra- and some other prime powers.

0.B. Notation, terminology, conventions

For a function f, its restriction to a set X is denoted $f|_X$, and $f[X] = \{\, f(x) \mid x \in X \,\}$. The set of fixed points of f is $\operatorname{fix} f$. Superscripts to functional symbols denote iteration.

For $X \subseteq \omega$, the complement $\omega - X$ is denoted $\overline{X}$. *Almost inclusion* $X \subseteq^* Y$ means that $Y - X$ is finite, and $X =^* Y$ stands for almost inclusion in both directions.

An *r.e. splitting* of an r.e. set X is a partition of X in two r.e. pieces. When we say that the finite set $\alpha(x)$ is a (partial) recursive function of x, we mean that the code of $\alpha(x)$ is recursive in x.

We consider the natural numbers $\mathbb{N}$ as a 1st order structure in the language $\mathcal{L} = (0, 1, +, \times)$, so that $x \le y$ is an abbreviation for $\exists z\,(x + z = y)$. We prefer a minimalist purely functional language because it makes embeddability arguments shorter. Let $\mathcal{L}^*$ denote the language $\mathcal{L}$ expanded by a relation symbol $*$. The $\Delta_0^{\le}$ *formulas* of $\mathcal{L}^{\le}$ are those in which each quantifier is bounded by an $\mathcal{L}$-term without occurrences of the quantified variable. The Δ_0 *formulas* of $\mathcal{L}$ are translations into $\mathcal{L}$ of the $\Delta_0^{\le}$ formulas. The classes Σ_n and Π_n of formulas have their usual definition. When you start with the class of quantifier-free formulas instead of Δ_0 ones, you get the classes $\exists_n$ and $\forall_n$ in place of Σ_n and Π_n respectively. $\forall_n^*$ *formulas* are the $\forall_n$ formulas of $\mathcal{L}^*$, and similarly for $\exists_n$. For a class Γ of formulas, an embedding ι between structures is Γ-*elementary* if Γ formulas are absolute for ι. A *sentence* is a formula without free variables.

The $\mathcal{L}$-theory TA_2 is axiomatized by the Π_2 sentences true in $\mathbb{N}$. Hirschfeld [16, Corollary 1.7.1] shows that its $\mathcal{L}^{<}$-variant $\mathrm{TA}_2^{<} = \mathrm{Th}_{\Pi_2^{<}}\,\mathbb{N}$ also has a smaller axiom set consisting of the true $\forall_2^{<}$ sentences. Since $<$ is definable in $\mathrm{TA}_2^{<}$ by both an $\exists_1$ and by a $\forall_1$ formula, every $\forall_2^{<}$ sentence is equivalent in $\mathrm{TA}_2^{<}$ to a $\forall_2$ sentence. Hence $\mathrm{Th}_{\forall_2^{<}}\,\mathbb{N}$ is a definitional extension of $\mathrm{Th}_{\forall_2}\,\mathbb{N}$ in the sense of Hodges [19, subsection 2.6.2]. Therefore Hirschfeld's conclusion implies $\mathrm{TA}_2 = \mathrm{Th}_{\forall_2}\,\mathbb{N}$.

A formula $\varphi(\vec{x})$ is Δ_1 *in* TA_2 if $\varphi(\vec{x})$ is equivalent in TA_2 both to a Σ_1 and to a Π_1 formula — Hirschfeld [16, 1.2] calls Δ_1 in TA_2 formulas *recursive*.

Under normal circumstances, the theories $\mathrm{I}\Delta_0$ and $\mathrm{I}\Delta_0$+exp (see Gaifman & Dimitracopoulos [12] or Hájek & Pudlák [14, V.1(a)]) speak the language $\mathcal{L}^{\le}$. We shall nevertheless treat these theories as $\mathcal{L}$-theories with the understanding that they are axiomatized by the $\mathcal{L}$-translations of their usual $\mathcal{L}^{\le}$ axioms. Just like with TA_2, of which both $\mathrm{I}\Delta_0$ and $\mathrm{I}\Delta_0$+exp are subtheories, the traditional $\mathcal{L}^{\le}$-versions of the latter theories are definitional extensions of our $\mathcal{L}$-versions. All homomorphisms between models of $\mathrm{I}\Delta_0$ are embeddings.

We fix a quaternary $\mathcal{L}$-formula $t : \{e\}(x) = y$ to represent a variant of Kleene's T-predicate, "t is the computation protocol witnessing that the eth computing device outputs y on input x". Smullyan [44, Theorem IV.9] shows that one can select a Δ_0 formula with the properties of the T-predicate, so we will assume that $t : \{e\}(x) = y$ is Δ_0. As usual, $\{e\}(x) = y$ is $\exists t\ t : \{e\}(x) = y$, and $\{e\}(x)\!\downarrow$ is $\exists y\,\{e\}(x) = y$.

1. McLaughlin's Theorem

Recursive ultrapowers were introduced by Hirschfeld [15] and [16]. A certain subclass of those had been previously studied by Lerman [27] — we take a closer look at it in subsection 1.B.

1.1. Definition (Hirschfeld [16, section 2]). A *recursive ultrafilter* is a non-principal ultrafilter in the Boolean algebra $\mathcal{R}$ of recursive subsets of ω.

Let u be a recursive ultrafilter. For functions $f, g : \omega \to \omega$, let $f \equiv_u g$ stand for agreement on some set in u. Then $\equiv_u$ is a congruence on the semiring $\mathcal{T}$ of unary total recursive functions $\omega \to \omega$ with pointwise addition and multiplication. Define the *recursive ultrapower* $\mathbb{N}[u]$ corresponding to u as $\mathcal{T}/{\equiv_u}$.

We denote by $[f]$ the $\equiv_u$-equivalence class of f and let $\boldsymbol{x} = [\mathrm{id}]$.

1.2. Fact (Hirschfeld [16, Corollary 2.4]). *For any recursive ultrafilter u one has* $\mathbb{N}[u] \models \mathrm{TA}_2$.
In particular, $\mathbb{N}[u] \models \mathrm{I}\Delta_0{+}\mathrm{exp}$. ∎

1.3. Convention. In (any model M of) TA_2, we use the expressions $f(x) = y$ and $z \in X$, where f is a (partial) recursive function and X is an r.e. set, as abbreviations for the Σ_1 formulas $\{e\}(x) = y$ and $\{d\}(z){\downarrow}$ respectively, where e is an index for f and d is an r.e. index for X. No matter which index we choose for a given recursive function f, we end up with formulas which are equivalent in TA_2, for $\forall x, y\,(\{e\}(x) = y \leftrightarrow \{c\}(x) = y)$ is a true Π_2 sentence provided both e and c are recursive indices for f. Similarly for r.e. sets. This allows us to treat (partial) recursive functions and r.e sets as virtual elements of the 1st order language.

The same understanding remains in force even when f or X are compound expressions such as $g \circ h$ or $g[Y] \cup Z$. The next lemma will say that, in simpler situations, the distinction does not really matter.

When we say that f is a (partial) recursive function, we always imply that, even though f may operate in a non-standard model, f is standard, that is, it has a standard index. Similarly for r.e. sets.

1.4. Lemma. *Let* $M \models \mathrm{TA}_2$*, let* $i \in \omega$ *be a standard number, X and Y r.e. sets, R a recursive one, and f and g partial recursive functions. Then the following hold in M:*

(a) $\forall x\,(f(g(x)) = (f \circ g)(x))$;

(b) $\forall x\,(x \in X \wedge x \in Y \leftrightarrow x \in X \cap Y)$, *and similarly for* $\cup$;

(c) $\forall x\,(x \in R \leftrightarrow x \notin \overline{R})$;

(d) $\forall x\,(x = i \leftrightarrow x \in \{i\})$;

(e) $\forall x\,(f(x) \in X \leftrightarrow x \in f^{-1}[X])$.

Proof. This follows at once from Fact 1.2, as each of the clauses is Π_2 and true. ∎

The following fact completes an emergency kit of basic principles intended to last us through the present section. Facts 1.2 and 1.5 will be (re-)established in section 2 in greater generality.

1.5. Fact (Hirschfeld [16]). *Let u be a recursive ultrafilter. Then*

(a) ([16, Lemma 2.5]) $\mathbb{N}[u] \models f(\boldsymbol{x}) = [f]$ *for each total recursive f;*

(b) ([16, Theorem 2.3]) *For each recursive set R,* $\mathbb{N}[u] \models \boldsymbol{x} \in R$ *iff* $R \in u$. ∎

Clause (a) tells us, among other things, that each element of $\mathbb{N}[u]$ is a total recursive value of $\boldsymbol{x}$.

1.A. A proof of McLaughlin's Theorem

In this subsection we present an alternative argument for

1.6. McLaughlin's Theorem ([34]). *Let u be a recursive ultrafiler.*

(a) *The generator $\boldsymbol{x}$ is the only element a of $\mathbb{N}[u]$ satisfying $\mathbb{N}[u] \models \boldsymbol{x} \in R \to a \in R$ for each recursive set R.*

(b) *$\mathbb{N}[u]$ is totally rigid.*

Towards the proof of the theorem, we first fix some graph colouring terminology.

1.7. Definition. Let Γ be a graph whose vertex set is (a subset of) ω, and let $k \in \omega$. The graph Γ is *highly recursive* (Bean [1]) if it is locally finite and, given a vertex $x \in \omega$, one can effectively compute the finite set of all vertices Γ-adjacent to x.

A *Γ-colouring of* a set X *with k colours* is a colour assignment $\chi : X \to k$ such that $\chi(x) \neq \chi(y)$ whenever $x, y \in X$ and a Γ-edge between x and y is present. When Γ is the graph of a function f, we speak of *f-colourings*.

We invoke a particular instance of a theorem by Schmerl [40] which bounds the recursive chromatic number of a highly recursive graph in terms of its classical chromatic number:

1.8. Fact (Schmerl [40, Theorem 1, $n = 2$]). *Suppose Γ is a highly recursive forest. Then there exists a recursive Γ-colouring (of ω) with 3 colours.* ∎

Schmerl also shows that one cannot generally do better than three colours.

The classical forebears of the following Lemma concerned the existence of f-colourings of the whole of the graph of an arbitrary fixed-point-free function f with finitely many colours. Katětov [21] shows that 3 colours generally suffice (see also Blass [2, proof of Theorem I.5] or Comfort & Negrepontis [5, Lemma 9.1]). Katětov's theorem is a consequence of a more general argument by De Bruijn & Erdős [3, Theorem 3, $k = 1$]. Later studies subjected both f and the f-colouring to topological restrictions — see e.g. Krawczyk & Steprāns [25]. Applications in the present paper follow McLaughlin [34] in considering recursive functions f and recursive f-colourings.

1.9. Lemma. *For each total recursive function f there is a recursive f-colouring of $\overline{\operatorname{fix} f}$ with 5 colours.*

Proof. We first partition $\overline{\operatorname{fix} f}$ in two recursive pieces, $D = \{\, x \in \omega \mid f(x) < x \,\}$ and $U = \{\, x \in \omega \mid f(x) > x \,\}$ (compare with the proof of III.A.3 in Rudin [39]).

We use two colours for elements of D. Assume that χ has already been defined on $D \cap \{0, \ldots, x-1\}$. For $x \in D$, define $\chi(x) \neq \chi(f(x))$ if $f(x) \in D$, or select $\chi(x)$ arbitrarily in the opposite case.

Observe next that the restriction of the graph of f to U is a highly recursive forest, for U cannot contain any f-cycles, and each vertex in U adjacent to $x \in U$ is contained in the finite computable set $\{0, \ldots, x-1\} \cup \{f(x)\}$. Therefore by Fact 1.8 there is a recursive f-colouring of U with just the three remaining colours. ∎

McLaughlin [34] proves a stronger version of Lemma 1.9 that uses four rather than five colours. His proof is longer and does not rely on Fact 1.8. It will be clear from the proof of Theorem 1.6 that any finite number of colours would suffice for the intended application. McLaughlin [34] however also claims that Lemma 1.9 holds true with just three colours at the cost of a more sophisticated construction. Corollary 3.10 will confirm McLaughlin's claim.

A significant portion of the present paper will investigate variants — or failures thereof — of Lemma 1.9 for partial recursive colourings of (parts of) the graph of a partial recursive function.

We recall one more external fact based on the provability in $\mathrm{I}\Delta_0$+exp of instances of the Matiyasevich–Robinson–Davis–Putnam Theorem:

1.10. Fact (Gaifman & Dimitracopoulos [12]). (a) ([12, Theorem 4.1]) *Any Δ_0 formula is equivalent in* $\mathrm{I}\Delta_0$+exp *to a $\exists_1$ formula.*

(b) ([12, Proposition 5.1]) *Any embedding between models of* $\mathrm{I}\Delta_0$+exp *is Δ_0-elementary. Hence Σ_1 formulas persist from sources to targets of such embeddings.*

(c) ([12, Proposition 5.2, $n = 0$]) *Any cofinal embedding between models of* $\mathrm{I}\Delta_0$+exp *is Σ_1-elementary.* ∎

The remaining part of the proof reproduces McLaughlin's argument:

1.11. Proof of Theorem 1.6 concluded. (a) Assume f is total recursive and $f(\boldsymbol{x})$ satisfies $\mathbb{N}[u] \models \boldsymbol{x} \in R \to f(\boldsymbol{x}) \in R$ for each recursive set R.

Suppose $\mathbb{N}[u] \models \chi(\boldsymbol{x}) = i$, where χ is an f-colouring of $\overline{\mathrm{fix}\, f}$ with 5 colours as in Lemma 1.9 and $i < 5$. Then $\mathbb{N}[u] \models \chi(f(\boldsymbol{x})) = i$ by Lemma 1.4, for $\chi^{-1}[\{i\}]$ is a recursive set. On the other hand, $\mathbb{N}[u] \models \forall x\, (\chi(x) = i \to \chi(f(x)) \neq i)$, the r.h.s. being a true Π_2 sentence by Lemma 1.9. Therefore, in $\mathbb{N}[u]$, $\boldsymbol{x}$ is left uncoloured by χ, hence $\mathbb{N}[u] \models f(\boldsymbol{x}) = \boldsymbol{x}$ because the Π_2 sentence $\forall x\, (\chi(x){\downarrow} \vee f(x) = x)$ is true.

(b) If ι is a self-embedding of $\mathbb{N}[u]$ with $\iota(\boldsymbol{x}) = [f]$ for some total recursive f, then $\mathbb{N}[u] \models \boldsymbol{x} \in R \to f(\boldsymbol{x}) \in R$ for each recursive set R by Facts 1.5(a) and 1.10(b). By clause (a), $\mathbb{N}[u] \models f(\boldsymbol{x}) = \boldsymbol{x}$, thus $\iota(\boldsymbol{x}) = \boldsymbol{x}$.

For an arbitary recursive g, Fact 1.10(b) ensures

$$\mathbb{N}[u] \models \forall x, y\, \Big(g(x) = y \to g\big(\iota(x)\big) = \iota(y) = \iota\big(g(x)\big)\Big)$$

because the formula $g(x) = y$ is Σ_1 and ι is a self-embedding. Hence, in particular, $\mathbb{N}[u] \models g(\boldsymbol{x}) = g(\iota(\boldsymbol{x})) = \iota(g(\boldsymbol{x}))$. Since each element of $\mathbb{N}[u]$ is of the form $g(\boldsymbol{x})$ for an appropriate total recursive g (Fact 1.5(a)), the only self-embedding of $\mathbb{N}[u]$ is the identity. ∎

1.B. An application to r-maximal sets

Recall that an r.e. set A is *r-maximal* if $\overline{A}$ is infinite and *r-cohesive*, that is, no recursive set splits $\overline{A}$ into two infinite pieces. The history of the following Proposition begins with Lerman [27, Proposition 2.1], with successive versions appearing in Kobzev [23, Предложение 2] and Omanadze [37, Lemma 3.5].

1.12. Proposition (Lerman, Kobzev, and Omanadze). *If A is an r-maximal r.e. set and f is a total recursive function such that $\overline{A} \cap f[\overline{A}]$ is infinite, then $\overline{A} \subseteq^* \mathrm{fix}\, f$.*

We shall see that Proposition 1.12 is a manifestation of a more general phenomenon:

1.13. Lemma. *Suppose u is a recursive ultrafilter and f is a total recursive function.*
Then either there exists a recursive $R \in u$ such that $R \cap f[R] = \varnothing$, or $\mathrm{fix}\, f \in u$.

PROOF. If there is a recursive set S such that $\mathbb{N}[u] \models \boldsymbol{x} \in S \not\ni f(\boldsymbol{x})$, then $\mathbb{N}[u] \models \boldsymbol{x} \in f^{-1}[\overline{S}]$ (Lemma 1.4). Let $R = S \cap f^{-1}[\overline{S}]$. Then $R \in u$ by Fact 1.5(b) and

$$R \cap f[R] = S \cap f^{-1}[\overline{S}] \cap f\big[S \cap f^{-1}[\overline{S}]\big] \subseteq S \cap f^{-1}[\overline{S}] \cap f[S] \cap \overline{S} = \varnothing.$$

If $\mathbb{N}[u] \models \boldsymbol{x} \in S \to f(\boldsymbol{x}) \in S$ for each recursive S, then $\mathbb{N}[u] \models f(\boldsymbol{x}) = \boldsymbol{x}$ by Theorem 1.6(a). Hence $f \equiv_u \mathrm{id}$, so $u \ni \mathrm{fix}\, f$. ∎

1.14. LEMMA (Hirschfeld [16, 4.1]). *Suppose A is an r-maximal r.e. set. Then*

$$u_A = \big\{ \text{recursive } R \,\big|\, R \supseteq^* \overline{A} \big\}$$

is a recursive ultrafilter.

PROOF. It is clear that u_A is a filter. For each recursive set R, either $R \supseteq^* \overline{A}$ or $\overline{R} \supseteq^* \overline{A}$ by the r-cohesion of $\overline{A}$. As $\overline{A}$ is infinite, u_A cannot be principal. ∎

The ultrafilter u_A is essentially the same thing as the *preference function for A* from Lerman & al. [28, Definition 1.1] (for an r-maximal set, the preference function is unique). Lerman [27] investigates ultrapowers $\mathbb{N}[u_A]$ with r-maximal A, defining the congruence $\equiv^A$ on $\mathcal{T}$ as agreement a.e. on $\overline{A}$. By Lemma 1.14, Lerman's $\equiv^A$ coincides with $\equiv_{u_A}$.

1.15. PROOF OF PROPOSITION 1.12 CONCLUDED. If there existed a recursive $R \in u_A$ with $R \cap f[R] = \varnothing$, then, as $R \supseteq^* \overline{A}$, the intersection $\overline{A} \cap f[\overline{A}]$ would be finite, contrary to assumption. By Lemma 1.13 it follows that $\mathrm{fix}\, f \in u_A$, hence $\mathrm{fix}\, f \supseteq^* \overline{A}$. ∎

2. R.e. prime powers

2.1. DEFINITION. An *(r.e.) prime p* is a non-principal proper prime filter in the lattice $\mathcal{E}$ of r.e. subsets of ω (equivalently, p is a proper prime filter in $\mathcal{E}^* = \mathcal{E}/{=^*}$).

R.e. primes are to r.e. prime powers what ultrafilters are to ultrapowers.

Shavrukov [43] studies the collection of r.e. primes ordered by inclusion, which together with an appropriate topology forms the dual space $(\mathcal{E}^*)^\star$ of $\mathcal{E}^*$. The following is an easy consequence of the Reduction Principle for r.e. sets.

2.2. FACT (Shavrukov [43, Corollary 1.4]). *The inclusion ordering on r.e. primes is forest-like: if $q \subseteq p$ and $r \subseteq p$, then $q \subseteq r$ or $r \subseteq q$.* ∎

In this section, p stands for an arbitrary prime.

2.3. DEFINITION. Let $\mathcal{P}_p$ be the collection of all unary partial recursive functions $f : \omega \to \omega$ with $\mathrm{dom}\, f \in p$. For $f, g \in \mathcal{P}_p$, write $f \equiv_p g$ if f and g agree on a set in p. Clearly, $\equiv_p$, *agreement modulo p*, is an equivalence relation on $\mathcal{P}_p$. Furthermore, $\equiv_p$ is a congruence for the pointwise $+$ and $\times$ on $\mathcal{P}_p$ (where for $(f \cdot g)(x)$ to be defined one requires the convergence of both $f(x)$ and $g(x)$). We can therefore define the quotient *r.e. prime power* $\mathbb{N}[p] = (\mathcal{P}_p, +, \times, 0, 1)/{\equiv_p}$.

We write $[f]_p$ (or just $[f]$ when confusion is unlikely) for the $\equiv_p$-equivalence class of $f \in \mathcal{P}_p$. Let us also fix the notation $\boldsymbol{x} = \boldsymbol{x}_p = [\mathrm{id}]_p$.

Our definition of r.e. prime powers is (equivalent to) a specialization of *reduced r.e. powers* from Hirschfeld [17, 1.4]. The definition of $\mathbb{N}[p]$ coincides with that of *r.e. ultrapowers* from Hirschfeld & Wheeler [18, 9.4] when the prime p is required to be maximal (w.r.t. inclusion). Lemma 2.6 will show that all recursive ultrapowers (Definition 1.1) are also contained among r.e. prime powers. In this section we shall see that many properties of recursive and r.e. ultrapowers generalise straightforwardly to r.e. prime powers.

2.4. Remark. An infinite set $C \subseteq \omega$ is *cohesive* if no r.e. set splits C into two infinite pieces. For a cohesive set C, let $\mathcal{P}_C$ be the collection of partial recursive functions f with $\operatorname{dom} f \supseteq^* C$, and let $\equiv_C$ stand for agreement a.e. on C among elements of $\mathcal{P}_C$. Dimitrov [7] defines *cohesive powers* (of arbitrary computable structures rather than just of $\mathbb{N}$) as $\mathcal{P}_p/{\equiv_C}$, with cohesive C and the usual pointwise operations. Any cohesive set C determines an r.e. prime $p_C = \{\, \text{r.e. } X \mid X \supseteq^* C \,\}$, and it is easily seen that $(\mathcal{P}_{p_C}, \equiv_{p_C})$ is identical to $(\mathcal{P}_C, \equiv_C)$. Conversely, given an r.e. prime p, let C be an infinite Hausdorff intersection of the (r.e.) sets in p together with the complements of all r.e. sets outside p (see e.g. Van Mill [35, Lemma 1.1.2]). Then C is cohesive, and $\equiv_C$ coincides with $\equiv_p$ on $\mathcal{P}_p = \mathcal{P}_C$. Thus r.e. prime powers are exactly the cohesive powers of $\mathbb{N}$. A cohesive set may however have further individual features that are not reflected in the corresponding r.e. prime nor in the power structure, whereas the r.e. prime is essentially visible in the corresponding r.e. prime power (Lemma 2.10(a)). This suggests that r.e. primes bear closer ties to the powers than cohesive sets do.

2.A. Basic properties of r.e. prime powers

2.5. Definition. Let u be a recursive ultrafilter, and let $\bar{u}$ be the least filter in $\mathcal{E}$ with $\bar{u} \supseteq u$. To see that $\bar{u}$ is prime, let X and Y be r.e. sets such that $X \cup Y \in \bar{u}$. Thus there is a recursive $R \in u$ with $X \cup Y \supseteq R$. By the Reduction Principle there are recursive $X' \subseteq X$ and $Y' \subseteq Y$ that partition R. Hence $X' \in u$ or $Y' \in u$. Therefore $X \in \bar{u}$ or $Y \in \bar{u}$, so $\bar{u}$ is prime. Primes of the form $\bar{u}$ are exactly the primes which are minimal w.r.t. inclusion (Shavrukov [43, Lemma 1.9]).

In the opposite direction, for a prime p we denote by p° the recursive ultrafiler $p \cap \mathcal{R}$. We clearly have $u = \bar{u}^\circ$ and $p \supseteq \overline{p^\circ}$.

The following lemma justifies our using the same notation for recursive ultrapowers and r.e. prime powers.

2.6. Lemma (Hirschfeld [17, 1.5]). *For any recursive ultrafilter u, the recursive ultrapower $\mathbb{N}[u]$ is canonically isomorphic to the r.e. prime power $\mathbb{N}[\bar{u}]$ via $[f]_u \mapsto [f]_{\bar{u}}$.*

The isomorphism is unique.

Proof. We show that $\iota : [f]_u \mapsto [f]_{\bar{u}}$ is an isomorphism. It is clearly correct and injective, for $f \equiv_u g$ is equivalent to $f \equiv_{\bar{u}} g$ for total recursive f and g. Since the operations in both $\mathbb{N}[u]$ and $\mathbb{N}[\bar{u}]$ are pointwise, ι is a homomorphism.

To show that ι is onto, consider an arbitrary $h \in \mathcal{P}_{\bar{u}}$. There is a recursive $R \in u$ such that $R \subseteq \operatorname{dom} h$. Define the total recursive function $\bar{h}(x) = \begin{cases} h(x) & \text{if } x \in R \\ 0 & \text{if } x \notin R \end{cases}$. Then $\bar{h} \equiv_{\bar{u}} h$, so $\iota([\bar{h}]_u) = [\bar{h}]_{\bar{u}} = [h]_{\bar{u}}$. Thus $\iota : \mathbb{N}[u] \to \mathbb{N}[\bar{u}]$ is an isomorphism.

The uniqueness of ι follows at once from McLaughlin's Theorem 1.6. ∎

The following restricted version of Łoś' Lemma is due to Hirschfeld [16, 2.3] for recursive ultrapowers, and to McLaughlin [31, Lemma 5.13] for r.e. ultrapowers. (Unlike our exposition, both

Hirschfeld and McLaughlin include < among the primitive symbols.) Dimitrov [7, Theorem 2.1.2] establishes Łoś' Lemma for $\exists_1$ formulas in cohesive powers of arbitrary computable structures — for us it is important to remember that Σ_1 formulas of L should not be confused with $\exists_1$ ones before Matiyasevich's theorem is available in the target structures, but see Remark 2.9. See also Hirschfeld [17, 1.4(g)], although Hirschfeld's version of Łoś' Lemma is not universally true in the more general setting of reduced r.e. powers.

2.7. Σ_1-Łoś Lemma. *For each Σ_1 formula $\sigma(x_1, \ldots, x_n)$ and $f_1, \ldots, f_n \in \mathcal{P}_p$ one has*

$$\mathbb{N}[p] \models \sigma\big([f_1], \ldots, [f_n]\big) \;\Leftrightarrow\; \Big\{ x \in \bigcap\nolimits_i \operatorname{dom} f_i \;\Big|\; \mathbb{N} \models \sigma\big(f_1(x), \ldots, f_n(x)\big) \Big\} \in p.$$

(Observe that the r.h.s. is not affected by adjunction of dummy variables to $\sigma(\vec{x})$.)

Proof. Say that a formula $\sigma(\vec{x})$ enjoys the *Łoś property* if $\sigma(\vec{x})$ satisfies the statement of the present Lemma.

Since operations in $\mathbb{N}[p]$ are pointwise, induction on the structure of the term $t(\vec{y})$ yields:

Claim 1. $\mathbb{N}[p] \models t([\vec{f}]) = [t(\vec{f})]$ *holds for each L-term $t(\vec{y})$ and each $\vec{f} \in \mathcal{P}_p$. Hence atomic formulas, i.e. those of the form $t(\vec{y}) = s(\vec{y})$, possess the Łoś property.* ◪

The following claim is straightforward, using that p is a filter:

Claim 2. *Suppose $\varphi(\vec{y})$ and $\psi(\vec{y})$ are Σ_1 and enjoy the Łoś property. Then so does $\varphi(\vec{y}) \wedge \psi(\vec{y})$.* ◪

The next claim is the only step in the proof that relies on the primality of p.

Claim 3. *Suppose $\delta(\vec{y})$ is Δ_1 in* TA_2 *and possesses the Łoś property. Then so does $\neg\delta(\vec{y})$.*

Proof. For $\vec{f} \in \mathcal{P}_p$, we have

$$\begin{aligned} \mathbb{N}[p] \models \neg\delta([\vec{f}]) &\Leftrightarrow \Big\{ x \in \bigcap\nolimits_i \operatorname{dom} f_i \;\Big|\; \mathbb{N} \models \delta\big(\vec{f}(x)\big) \Big\} \notin p && \text{(by assumption)} \\ &\Leftrightarrow \Big\{ x \in \bigcap\nolimits_i \operatorname{dom} f_i \;\Big|\; \mathbb{N} \models \neg\delta\big(\vec{f}(x)\big) \Big\} \in p, \end{aligned}$$

with the second equivalence holding because the two r.e. sets on the r.h.s. partition $\bigcap_i \operatorname{dom} f_i \in p$, and p is prime. ◪

Claim 4. *Suppose the formula $\varphi(y, \vec{z})$ is Σ_1 and has the Łoś property. Then so does the formula $\exists y\, \varphi(y, \vec{z})$.*

Proof. With $\vec{f} \in \mathcal{P}_p$,

$$\begin{aligned} \mathbb{N}[p] \models \exists y\, \varphi(y, [\vec{f}]) &\Leftrightarrow \exists g \in \mathcal{P}_p\; \mathbb{N}[p] \models \varphi([g], [\vec{f}]) \\ &\Leftrightarrow \exists g \in \mathcal{P}_p \Big\{ x \in \operatorname{dom} g \cap \bigcap\nolimits_i \operatorname{dom} f_i \;\Big|\; \mathbb{N} \models \varphi\big(g(x), \vec{f}(x)\big) \Big\} \in p \quad \text{(by assumption)} \\ &\Leftrightarrow \Big\{ x \in \bigcap\nolimits_i \operatorname{dom} f_i \;\Big|\; \mathbb{N} \models \exists y\, \varphi\big(y, \vec{f}(x)\big) \Big\} \in p. \end{aligned}$$

For the $\Leftarrow$ direction of the last equivalence, note that y, when it exists, can be selected partial recursively in $x \in \bigcap_i \operatorname{dom} f_i$ because $\varphi(y, \vec{z})$ is Σ_1. ◪

According to our conventions, $x \leq y$ is short for $\exists z\, (x + z = y)$. A direct consequence of Claims 1 and 4 is

Claim 5. *The Łoś property holds for the formula* $x \leq y$. ◪

Claim 6. *Suppose* $\varphi(y, \vec{z})$ *is* Σ_1 *and has the Łoś property. Then so does the formula* $\exists y{\leq}t(\vec{z})\,\varphi(y, \vec{z})$, *where* $t(\vec{z})$ *is any* $\mathcal{L}$*-term.*

Proof. Observe that $\exists y{\leq}t(\vec{z})\,\varphi(y, \vec{z})$ rewrites as $\exists y\,(y \leq t(\vec{z}) \wedge \varphi(y, \vec{z}))$, and use Claims 1, 5, 2, and 4 (in that order). ◪

To complete the proof, we use induction on the structure of $\sigma(\vec{x})$. Claim 1 takes care of atomic formulas, Claims 2 and 3 account for Boolean connectives, and Claims 6 and 3 for (the $\mathcal{L}$-translations of) bounded quantifiers, so that Δ_0 formulas are covered. Existential quantifiers are strapped on with the help of Claim 4. ■

Corollary 2.8 and Lemma 2.10 are straightforward generalisations of the corresponding items in Hirschfeld [16] for recursive ultrapowers and in Hirschfeld & Wheeler [18] and McLaughlin [31] for r.e. ultrapowers.

2.8. Corollary. $\mathbb{N}[p] \models \mathrm{TA}_2$. *In particular,* $\mathrm{I}\Delta_0{+}\mathrm{exp}$ *holds in* $\mathbb{N}[p]$.

Proof. Suppose $\mathbb{N} \models \forall\vec{y}\,\sigma(\vec{y})$ where $\sigma(\vec{y})$ is Σ_1. Take any $\vec{f} \in \mathcal{P}_p$. Then

$$\Big\{\, x \in \bigcap\nolimits_i \operatorname{dom} f_i \;\Big|\; \mathbb{N} \models \sigma\big(\vec{f}(x)\big) \,\Big\} = \bigcap\nolimits_i \operatorname{dom} f_i \in p.$$

By the Σ_1-Łoś Lemma, this implies $\mathbb{N}[p] \models \sigma([\vec{f}])$. Since $\vec{f} \in \mathcal{P}_p$ are arbitrary, we have $\mathbb{N}[p] \models \forall\vec{y}\,\sigma(\vec{y})$ as required. ■

2.9. Remark. Just like Corollary 2.8 does, Dimitrov [7, Theorem 2.1.4] infers from his $\exists_1$-Łoś Lemma ([7, Theorem 2.1.2]) that the $\forall_2$ theory of an arbitrary computable structure is shared by each of its cohesive (=r.e. prime) powers (Remark 2.4). Recall that TA_2 is axiomatized by $\mathrm{Th}_{\forall_2}\,\mathbb{N}$ to see that our Corollary 2.8 is a consequence of Dimitrov's theorem. Furthermore, our Σ_1-Łoś Lemma 2.7 now follows from Dimitrov's $\exists_1$-Łoś Lemma, for the Σ_1 formulas of $\mathcal{L}$ are equivalent in TA_2 (hence both in $\mathbb{N}$ and in $\mathbb{N}[p]$) to $\exists_1$ formulas (Fact 1.10(a)).

2.10. Lemma. *Let* f *be a partial recursive function, let* X *be r.e., and* $g \in \mathcal{P}_p$.

(a) $\mathbb{N}[p] \models [g] \in X$ *iff* $g^{-1}[X] \in p$. *In particular,* $\mathbb{N}[p] \models \boldsymbol{x} \in X$ *iff* $X \in p$.

(b) $\mathbb{N}[p] \models f([g]){\downarrow}$ *iff* $\operatorname{dom} f \circ g \in p$. *In particular,* $\mathbb{N}[p] \models f(\boldsymbol{x}){\downarrow}$ *iff* $\operatorname{dom} f \in p$;

(c) *If conditions from clause (b) hold, then* $\mathbb{N}[p] \models f([g]) = [f \circ g]$. *In particular,* $\mathbb{N}[p] \models f(\boldsymbol{x}) = [f]$.

Proof. (a) By the Σ_1-Łoś Lemma, $\mathbb{N}[p] \models [g] \in X$ iff $p \ni \{\, x \in \operatorname{dom} g \mid g(x) \in X \,\} = g^{-1}[X]$.

(b) By the Σ_1-Łoś Lemma, $\mathbb{N}[p] \models f([g]){\downarrow}$ iff $p \ni \{\, x \in \operatorname{dom} g \mid f(g(x)){\downarrow} \,\} = \operatorname{dom} f \circ g$.

(c) also follows from Σ_1-Łoś, for $\{\, x \in \operatorname{dom} g \mid f(g(x)) = f \circ g(x) \,\} = \operatorname{dom} f \circ g \in p$. ■

2.B. The failure of Σ_1 collection

The collection schema

$$\forall x, \vec{w}\,\big(\forall z{<}x\,\exists u\,\gamma(z, u; \vec{w}) \rightarrow \exists t\,\forall z{<}x\,\exists u{<}t\,\gamma(z, u; \vec{w})\big)$$

restricted to Γ formulas $\gamma(\cdots)$ is denoted $\mathrm{B}\Gamma$. According to Lemma I.2.10 in Hájek & Pudlák [14], $\mathrm{B}\Sigma_1$ is equivalent to $\mathrm{B}\Delta_0$ modulo $\mathrm{I}\Delta_0$.

Basing on the argument in Hirschfeld & Wheeler [18, 9.8], McLaughlin [33, Theorems 1.1 and 1.4] shows that $\mathrm{B}\Sigma_1$ fails in each recursive ultrapower and in each r.e. ultrapower. (See also Paris & Kirby [38, proof of Proposition 7] for a similar technique.) The gist of the matter is that finite generation is the nemesis of collection. Essentially the same reasoning extends to all r.e. prime powers:

2.11. Proposition. $\mathbb{N}[p] \not\models \mathrm{B}\Sigma_1$.

Proof. Recall that the formula $u : \{y\}(\boldsymbol{x}) = z$ is Δ_0 according to our conventions. We are going to show that $\mathbb{N}[p]$ violates the following substitution instance of $\mathrm{B}\Delta_0$:

$$(*)\quad \forall z{\le}\boldsymbol{x}\, \exists u\, \exists y{<}\boldsymbol{x}\ u : \{y\}(\boldsymbol{x}) = z \longrightarrow \exists t\, \forall z{\le}\boldsymbol{x}\, \exists u{<}t\, \exists y{<}\boldsymbol{x}\ u : \{y\}(\boldsymbol{x}) = z.$$

To show that the antecedent of (*) holds, consider an arbitary element z of $\mathbb{N}[p]$. Then $z = [f]$ for some total recursive f with index $y \in \omega$ (in particular, $y < \boldsymbol{x}$), and $\mathbb{N}[p] \models \{y\}(\boldsymbol{x}) = [f] = z$ by Lemma 2.10(c). Hence there is some u in $\mathbb{N}[p]$ with $u : \{y\}(\boldsymbol{x}) = z$.

Fix an arbitrary t. For any x and y, there is at most one z with $\exists u{<}t\ u : \{y\}(x) = z$. This consideration, being Π_1 and true, persists from $\mathbb{N}$ to $\mathbb{N}[p]$ (Lemma 2.8). Thus the conclusion of (*) says that $y \mapsto z$ is a partial function (with the Δ_0 graph $\exists u{<}t\ u : \{y\}(\boldsymbol{x}) = z$) from $\boldsymbol{x}$ onto $\boldsymbol{x}+1$. Yet the (true) Pigeonhole Principle for Δ_0-definable partial functions

$$\forall x,t\left(\forall y{<}x\,\forall z_0,z_1{\le}x\,\big(\delta(y,z_0;x,t)\wedge\delta(y,z_1;x,t)\rightarrow z_0=z_1\big)\longrightarrow \exists z{\le}x\,\forall y{<}x\ \neg\delta(y,z;x,t)\right)$$

is also Π_1, so it must hold in $\mathbb{N}[p]$. This contradiction shows that the conclusion of (*) fails in $\mathbb{N}[p]$. ∎

Recall that an extension $M \supseteq K$ between models of $\mathrm{I}\Delta_0$ is a (*proper*) *end-extension* if K is a (proper) initial segment of M. An embedding ι is (*properly*) *initial* if the target model is a (proper) end-extension of the range of ι.

The following corollary extends Corollary 4.13(2) in McLaughlin [31].

2.12. Corollary. $\mathbb{N}[p]$ *cannot be properly end-extended to any model of* $\mathrm{I}\Delta_0$.
In particular, $\mathbb{N}[p]$ *admits no properly initial self-embeddings.*

Reference. Any model of $\mathrm{I}\Delta_0$ with a proper end-extension to a model of $\mathrm{I}\Delta_0$ must satisfy $\mathrm{B}\Sigma_1$ — see e.g. Hájek & Pudlák [14, Theorem IV.1.22, $k = 0$] or Kaye [22, Proposition 10.5] while recalling that initial embeddings between $\mathrm{I}\Delta_0$-models are Δ_0-elementary. ∎

2.C. Σ_1 and Δ_1 types and embeddability

2.13. Definition. Let M be a model of TA_2 and a an element of M. We identify the Σ_1 type of a with the collection of r.e. sets to which a belongs in M:

$$\mathrm{tp}^M_{\Sigma_1}\, a = \{\, \text{r.e. } X \subseteq \omega \mid M \models a \in X \,\}.$$

The identification is correct because when i and j are indices of the same r.e. set, the sentence $\forall x\,(\{i\}(x)\!\downarrow \leftrightarrow \{j\}(x)\!\downarrow)$, being Π_2 and true, holds in M. Similarly,

$$\mathrm{tp}^M_{\Delta_1}\, a = \{\, \text{recursive } R \subseteq \omega \mid M \models a \in R \,\}.$$

The following lemma is immediate:

2.14. Lemma. *If an element a of $M \models \mathrm{TA}_2$ is nonstandard, then $\mathrm{tp}^M_{\Sigma_1}\, a$ is a prime and $\mathrm{tp}^M_{\Delta_1}\, a$ is a recursive ultrafilter.*

If a is standard, then $\mathrm{tp}^M_{\Sigma_1}\, a$ is a maximal principal prime filter in $\mathcal{E}$, and $\mathrm{tp}^M_{\Delta_1}\, a$ is a principal ultrafilter in $\mathcal{R}$. ∎

Embeddability criteria for r.e. and recursive ultrapowers are implicit in papers by Hirschfeld and McLaughlin. Here is the version for r.e. prime powers:

2.15. Lemma. *Suppose $\mathbb{N} < a \in M \models \mathrm{TA}_2$.*

(a) *An embedding $\iota : \mathbb{N}[p] \to M$ with $\iota(\boldsymbol{x}) = a$ exists if and only if $p \subseteq \mathrm{tp}^M_{\Sigma_1}\, a$, in which case we have $\iota([f]) = f(a)$ for each $f \in \mathcal{P}_p$, so that the condition $\iota(\boldsymbol{x}) = a$ uniquely determines ι.*

(b) *If conditions from clause (a) are met, then ι is Σ_1-elementary if and only if $p = \mathrm{tp}^M_{\Sigma_1}\, a$. In this case $\iota[\mathbb{N}[p]]$ is the smallest Σ_1-elementary submodel of M containing a.*

Proof. (a) Assume ι exists. Suppose $X \in p$, so that $\mathbb{N}[p] \models \boldsymbol{x} \in X$ by Lemma 2.10(a). Then $M \models a \in X$ by Σ_1 persistence. Thus $X \in \mathrm{tp}^M_{\Sigma_1}\, a$.

In the opposite direction, assume $p \subseteq \mathrm{tp}^M_{\Sigma_1}\, a$. Let us verify that $\iota : [f] \mapsto f(a)$ for $f \in \mathcal{P}_p$ is an embedding. If $f \in \mathcal{P}_p$ then $\operatorname{dom} f \in \mathrm{tp}^M_{\Sigma_1}\, a$, so $M \models f(a)\!\downarrow$. If $[f] = [g]$, that is, f and g agree on a set $Y \in p$, then $M \models a \in Y$, and $M \models f(a) = g(a)$ because $\forall x{\in}Y\ f(x) = g(x)$ is a true Π_2 statement. Further, in M,

$$\iota([f] + [g]) = \iota([f + g]) = (f + g)(a) = f(a) + g(a) = \iota([f]) + \iota([g]),$$

and similarly for 0, 1, and $\times$.

Uniqueness holds because we have $\mathbb{N}[p] \models [f] = f(\boldsymbol{x})$ for $f \in \mathcal{P}_p$ by Lemma 2.10(c), and, $y = f(x)$ being a Σ_1 formula, we must have $M \models \iota([f]) = f(\iota(\boldsymbol{x})) = f(a)$.

(b) Suppose ι is Σ_1-elementary and $M \models a \in X$ where X is r.e. By the Σ_1 elementarity of ι one has $\mathbb{N}[p] \models \boldsymbol{x} \in X$, so $X \in p$ by Lemma 2.10(a). Thus $\mathrm{tp}^M_{\Sigma_1}\, a \subseteq p$.

Conversely, suppose $p = \mathrm{tp}^M_{\Sigma_1}\, a$. With the pairing/projection functions available in both $\mathbb{N}[p]$ and M, it suffices to consider the single-parameter case. In view of the Σ_1-Łoś Lemma 2.7, an arbitrary Σ_1 formula with at most the variable x free is equivalent to a formula of the form $x \in Y$ for an appropriate r.e. set Y. So let $\mathbb{N}[p] \models [f] \in Y$ with Y r.e. and $f \in \mathcal{P}_p$. We then have $\mathbb{N}[p] \models [f] = f(\boldsymbol{x})$, so $\mathbb{N}[p] \models \boldsymbol{x} \in f^{-1}[Y]$ by Lemma 1.4(e), hence $f^{-1}[Y] \in p$. By our assumption, $M \models a \in f^{-1}[Y]$, which implies $M \models \iota([f]) = f(a) \in Y$. Thus ι is Σ_1-elementary.

If $M \models f(a)\!\downarrow$ with partial recursive f, then $\operatorname{dom} f \in \mathrm{tp}^M_{\Sigma_1}\, a = p$, and $f(a) = \iota([f])$ has to belong to any Σ_1-elementary submodel of M that contains a because the formula $f(x) = y$ is Σ_1. Thus $\iota[\mathbb{N}[p]] \subseteq K$. ∎

In view of Fact 1.10(c), we can draw the following

2.16. Corollary. *If $q \subseteq p$ are primes, then the inclusion $\mathcal{P}_q \subseteq \mathcal{P}_p$ gives rise to an embedding $\nu : \mathbb{N}[q] \to \mathbb{N}[p]$ with $\iota(\boldsymbol{x}_p) = \boldsymbol{x}_q$.*

The embedding ν is Σ_1-elementary if and only if ν is cofinal if and only if $p = q$. ∎

The following corollary is implicit in McLaughlin [34].

2.17. Corollary. *Suppose $\mathbb{N} < a \in M \models \mathrm{TA}_2$, and let u be a recursive ultrafilter. Then $\boldsymbol{x} \mapsto a$ extends to an embedding $\iota : \mathbb{N}[u] \to M$ if and only if $u = \mathrm{tp}^M_{\Delta_1}\, a$. The extension is unique.*

Proof. Suppose $u = \mathrm{tp}^M_{\Delta_1}\, a$. Recall that $\bar{u}$ is the minimal prime containing $u = \mathrm{tp}^M_{\Delta_1}\, a \subset \mathrm{tp}^M_{\Sigma_1}\, a$, hence $\bar{u} \subseteq \mathrm{tp}^M_{\Sigma_1}\, a$. Lemma 2.6 says that $\mathbb{N}[u]$ is isomorphic to $\mathbb{N}[\bar{u}]$ with $\boldsymbol{x}_u$ corresponding to $\boldsymbol{x}_{\bar{u}}$, and Lemma 2.15(a) supplies a unique embedding $\varepsilon : \mathbb{N}[\bar{u}] \to M$ with $\varepsilon(\boldsymbol{x}_{\bar{u}}) = a$.

In the other direction, the embedding ι together with Lemmas 2.10(a), 2.6 and 2.15(a) entail $\bar{u} = \mathrm{tp}^{\mathbb{N}[u]}_{\Sigma_1}\, \boldsymbol{x}_u = \mathrm{tp}^{\mathbb{N}[\bar{u}]}_{\Sigma_1}\, \boldsymbol{x}_{\bar{u}} \subseteq \mathrm{tp}^M_{\Sigma_1}\, a$, hence $u = \mathrm{tp}^{\mathbb{N}[u]}_{\Delta_1}\, \boldsymbol{x}_u \subseteq \mathrm{tp}^M_{\Delta_1}\, a$. But then $u = \mathrm{tp}^M_{\Delta_1}\, a$ because both are ultrafilters. ∎

R.e. prime powers are fully representative of all finitely generated models of TA_2:

2.18. Proposition. *Suppose M is a non-standard model of* TA_2.

(a) *If M is finitely generated w.r.t. partial recursive functions, then M is isomorphic to an r.e. prime power.*

(b) *If M is finitely generated w.r.t. total recursive functions, then M is isomorphic to a recursive ultrapower.*

Proof. In view of the TA_2-availability of pairing/projection functions, it suffices to consider the case when M is generated by a single non-standard element a.

(a) Let $p = \mathrm{tp}^M_{\Sigma_1}\, a$. Lemma 2.15(a) provides an embedding $\iota : \mathbb{N}[p] \to M$ with $\iota(\boldsymbol{x}) = a$. If f is a partial recursive function and $M \models f(a)\!\downarrow$, then $\operatorname{dom} f \in p$, so $f \in \mathcal{P}_p$, hence $[f]$ is an element of $\mathbb{N}[p]$, and $\iota([f]) = f(a)$ by Lemma 2.15(a). Thus ι is an isomorphism.

(b) is established analogously, using Corollary 2.17 to see that M is isomorphic to the recursive ultrapower $\mathbb{N}[u]$, where $u = \mathrm{tp}^M_{\Delta_1}\, a$. ∎

2.D. Rigidity

Rigidity (i.e., the absence of non-identity automorphisms) was shown by Hirschfeld & Wheeler [18, 9.6(iii)] for r.e. ultrapowers and by McLaughlin [30, Theorem 3.7] for recursive ones (see also McLaughlin [31, Theorem 2.11]). All these arguments made use of

2.19. Lemma. *Suppose f is a one-to-one partial recursive function. Then there exists a partial recursive f-colouring of* $\operatorname{dom} f - \operatorname{fix} f$ *with* 3 *colours.*

Proof. Let us first note that $X = \operatorname{dom} f - \operatorname{fix} f$ is r.e. The situation where X is finite is straightforward, so assume that X is infinite. Let $(x_i)_{i\in\omega}$ be an effective repetition-free listing of X.

We calculate $\chi(x_i)$ by recursion on i: Compute $f(x_0), \ldots, f(x_i)$. Find out if there are $j < i$ and/or $k < i$ such that $f(x_j) = x_i$ and $f(x_i) = x_k$ — there can be at most one of each since f is a one-to-one function. Select $\chi(x_i)$ distinct from $\chi(x_j)$ and $\chi(x_k)$ if any of j, k are indeed present.

Clearly, the colouring $\chi : X \to 3$ instantiates the conclusion of the lemma because $f|_X$ is fixed-point-free. ∎

2.20. Proposition. *Each r.e. prime power* $\mathbb{N}[p]$ *is rigid.*

Proof. Consider an arbitrary automorphism ι of $\mathbb{N}[p]$. Let $f \in \mathcal{P}_p$ be such that $\iota(\boldsymbol{x}) = [f]$. Since ι is an automorphism, $[f]$ generates $\mathbb{N}[p]$ w.r.t. partially recursive functions. So there is a partial recursive g with $\mathbb{N}[p] \models g(f(\boldsymbol{x})) = g([f]) = \boldsymbol{x}$. By the Σ_1-Łoś Lemma, the set $Z = \{\, i \in \operatorname{dom} g \circ f \mid g(f(i)) = i \,\}$ is an element of p. In $\mathbb{N}$, $f|_Z$ must be one-to-one. By Lemma 2.19, there is a partial recursive f-colouring χ of $Z - \operatorname{fix} f$ with 3 colours. Since $p \ni Z$ is prime, either $\operatorname{fix} f$ or one of the sets $\chi^{-1}(i)$ with $i < 3$ must be an element of p. Hence $\mathbb{N}[p] \models f(\boldsymbol{x}) \neq \boldsymbol{x} \to \chi(\boldsymbol{x})\!\downarrow$.

In $\mathbb{N}[p]$, assume $f(\boldsymbol{x}) \neq \boldsymbol{x}$. Then $\chi(\boldsymbol{x})$ is defined. On the one hand, $f(\boldsymbol{x}) \notin \operatorname{fix} f$, $\chi(f(\boldsymbol{x}))\!\downarrow$, and $\chi(\boldsymbol{x}) = \chi(f(\boldsymbol{x}))$ because the automorphism ι maps $\boldsymbol{x}$ to $f(\boldsymbol{x})$. On the other hand, $\chi(f(\boldsymbol{x})) \neq \chi(\boldsymbol{x})$ by Lemmas 2.19 and 2.8, for $\forall x\, (\{x, f(x)\} \subseteq Z - \operatorname{fix} f \to \chi(f(x)) \neq \chi(x))$ is a true Π_2 statement.

The contradiction shows $\mathbb{N}[p] \models \boldsymbol{x} \in \operatorname{fix} f$. Hence $\mathbb{N}[p] \models \boldsymbol{x} = f(\boldsymbol{x}) = \iota(\boldsymbol{x})$. By Lemma 2.15(a), ι is the identity automorphism. ∎

3. A self-embeddable r.e. prime power

In this section we show that, in contrast to McLaughlin's Theorem 1.6(b), self-embeddings of r.e. prime powers can exist:

3.1. Theorem. *There is an r.e. prime p such that* $\mathbb{N}[p]$ *admits a non-identity self-embedding.*

The partial recursive function f from the following proposition diguises a non-trivial self-embedding of an appropriate r.e. prime power. The proposition can be seen as the failure of a sequence of attempts to adapt Lemma 1.9 to partial recursive functions and colourings.

3.2. Proposition. *There exists a fixed-point-free partial recursive function f such that for no integer $\ell \geq 1$ is there a recursive f-colouring of* $\operatorname{dom} f^\ell$ *with finitely many colours.*

3.A. From uncolourable functions to self-embeddings

Before constructing the function f of Proposition 3.2, let us see how it helps with procuring the prime p of Theorem 3.1.

3.3. Definition. Let f be a fixed-point-free partial recursive function. An r.e. subset $X \subseteq \operatorname{dom} f$ is *chromatic (w.r.t. f)* if for some $\ell \geq 1$ the set $\bigcap_{0 \leq i < \ell} f^{-i}[X]$ admits a recursive f-colouring with finitely many colours.

3.4. Lemma. *In the setup of Definition 3.3, the collection of chromatic r.e. subsets of* $\operatorname{dom} f$ *forms an ideal in the lattice of r.e. subsets of* $\operatorname{dom} f$*. This ideal contains all finite subsets.*

Proof. It is clear that all finite subsets of $\operatorname{dom} f$ are chromatic, and that chromaticity is inherited by subsets. The variables i and j below stand for non-negative integers.

Suppose that r.e. subsets X and Y of $\operatorname{dom} f$ are both chromatic. We aim to show the chromaticity of $X \cup Y$. In view of the Reduction Principle, we may assume that X and Y are disjoint. There are $\ell \geq 1$ and recursive f-colourings of $\bigcap_{i<\ell} f^{-i}[X]$ and $\bigcap_{i<\ell} f^{-i}[Y]$ in finitely many colours. Let us assume these use disjoint sets of colours. We are going to f-colour the set $\bigcap_{i<\ell} f^{-i}[X \cup Y]$ while keeping the existing colourings of $\bigcap_{i<\ell} f^{-i}[X]$ and $\bigcap_{i<\ell} f^{-i}[Y]$ and using $2^\ell - 2$ fresh colours that are thought of as non-constant ℓ-tuples of 0's and 1's.

Since X and Y are disjoint, $f^{-i}[X \cup Y]$ is the disjoint union of $f^{-i}[X]$ and $f^{-i}[Y]$. An element x of the set

$$E = \bigcap_{i<\ell} f^{-i}[X \cup Y] - \Big(\bigcap_{i<\ell} f^{-i}[X] \cup \bigcap_{i<\ell} f^{-i}[Y]\Big)$$

of elements not yet coloured is assigned the colour $(\varepsilon_0, \ldots, \varepsilon_{\ell-1})$ where

$$\varepsilon_i = \begin{cases} 0 & \text{if } f^i(x) \in X, \\ 1 & \text{if } f^i(x) \in Y. \end{cases}$$

Note that E is r.e., the colour assignment is recursive, and that all ε_i cannot coincide for a fixed x, for in that case $x \in \bigcap_{i<\ell} f^{-i}[X] \cup \bigcap_{i<\ell} f^{-i}[Y]$, so $x \notin E$. To see that this defines an f-colouring of $\bigcap_{i<\ell} f^{-i}[X \cup Y]$, it suffices to show that the colours $(\varepsilon_0, \ldots, \varepsilon_{\ell-1})$ and $(\delta_0, \ldots, \delta_{\ell-1})$ assigned to x and $f(x)$ respectively are distinct whenever $x, f(x) \in E$. In this case we have $\varepsilon_i = \delta_{i-1}$ for $0 < i < \ell$ because $f^i(x) = f^{i-1}(f(x))$. Let $j < \ell - 1$ be such that $\varepsilon_j \neq \varepsilon_{j+1}$. Then $\delta_j = \varepsilon_{j+1} \neq \varepsilon_j$, so x and $f(x)$ acquire distinct colours. ∎

3.5. Definition. We say that a partial recursive function $f \in \mathcal{P}_p$ *induces* a self-embedding of the r.e. prime power $\mathbb{N}[p]$ when $\boldsymbol{x} \mapsto f(\boldsymbol{x})$ extends to a self-embedding of $\mathbb{N}[p]$. (Recall from Lemma 2.15(a) that in this case the extension ι is unique.)

3.6. Proposition. *Let f be a fixed-point-free partial recursive function. Suppose F is a filter in $\mathcal{E}$ with $F \ni$ dom f but such that no chromatic subset of* dom f *belongs to F.*

Then there exists a prime $p \supseteq F$ such that f induces a non-identity self-embedding of $\mathbb{N}[p]$.

Proof. We construct a $\subseteq$-descending chain $(X_i)_{i\in\omega}$ of r.e. sets which is going to serve as a base for the required prime $p = \{\, \text{r.e. } Y \mid \exists i{\in}\omega\ Y \supseteq X_i \,\}$. For each i and each $V \in F$, the set $X_i \cap V$ will be achromatic w.r.t. f — we use this as inductive hypothesis.

Put $X_0 = \operatorname{dom} f \in F$. By assumption, X_0 is achromatic when intersected with any element of F. Next proceed in stages.

At odd stages i, consider a next pair Y, Z of r.e. sets such that $Y \cup Z \supseteq X_i \cap W$ for some $W \in F$ (we assume an exhaustive infinitely repetitive enumeration of all r.e. pairs). Suppose there existed $U, V \in F$ such that both $X_i \cap Y \cap U$ and $X_i \cap Z \cap V$ were chromatic. Then $(X_i \cap Y \cap U) \cup (X_i \cap Z \cap V) \supseteq X_i \cap (U \cap V \cap W)$ would also be chromatic by Lemma 3.4, contrary to the inductive hypothesis. Thus at least one of the choices $X_{i+1} = X_i \cap Y$ or $X_{i+1} = X_i \cap Z$ satisfies the inductive hypothesis. Odd stages ensure that p is going to be a prime filter.

At even stages, we put $X_{i+1} = X_i \cap f^{-1}[X_i]$. Suppose $X_{i+1} \cap V$ were chromatic for some $V \in F$. Then for some ℓ, the set $\bigcap_{j<\ell} f^{-j}[X_{i+1} \cap V]$ would admit a recursive f-colouring with finitely many colours. But since

$$\bigcap_{j<\ell} f^{-j}[X_{i+1} \cap V] = \bigcap_{j<\ell} f^{-j}\big[X_i \cap f^{-1}[X_i] \cap V\big] = \bigcap_{j<\ell}\big(f^{-j}[X_i \cap V] \cap f^{-(j+1)}[X_i]\big)$$
$$\supseteq \bigcap_{j<\ell}\big(f^{-j}[X_i \cap V] \cap f^{-(j+1)}[X_i \cap V]\big) = \bigcap_{j<\ell+1} f^{-j}[X_i \cap V],$$

the set $X_i \cap V$ would then have to be chromatic as well contradicting the inductive assumption.

Since all elements of p are achromatic, the prime filter p is non-principal. If $V \in F$, then at some odd stage, our construction handles the pair $V, \varnothing$ resulting in $V \in p$, so $p \supseteq F$. Even stages ensure the property $Y \in p \Rightarrow f^{-1}[Y] \in p$ for all r.e. Y.

Since $p \ni \operatorname{dom} f$, we have $\mathbb{N}[p] \models f(\boldsymbol{x})\downarrow$. Furthermore, $\mathbb{N}[p] \models f(\boldsymbol{x}) \neq \boldsymbol{x}$ because f is fixed-point-free. Finally, for an r.e. Y,

$$\mathbb{N}[p] \models \boldsymbol{x} \in Y \Leftrightarrow Y \in p \Rightarrow f^{-1}[Y] \in p \Leftrightarrow \mathbb{N}[p] \models \boldsymbol{x} \in f^{-1}[Y] \Leftrightarrow \mathbb{N}[p] \models f(\boldsymbol{x}) \in Y.$$

Thus $\operatorname{tp}_{\Sigma_1}^{\mathbb{N}[p]} \boldsymbol{x} \subseteq \operatorname{tp}_{\Sigma_1}^{\mathbb{N}[p]} f(\boldsymbol{x})$. By Lemma 2.15(a), there is then an embedding $\iota : \mathbb{N}[p] \to \mathbb{N}[p]$ with $\iota(\boldsymbol{x}) = [f] = f(\boldsymbol{x})$. As $\boldsymbol{x} \neq f(\boldsymbol{x})$, one has $\iota \neq \mathrm{id}$. ∎

The converse to the argument of Proposition 3.6 is also clear: if, for some ℓ, the set $\operatorname{dom} f^{\ell}$ can be recursively f-coloured in a finite number of colours, then $\boldsymbol{x}$ and $f(\boldsymbol{x})$ have incomparable Σ_1 types in any r.e. prime power where $f(\boldsymbol{x})$ converges — this is similar to applications of Katětov-like lemmas as in, e.g., the proof of Theorem 1.6.

3.7. Proof of Theorem 3.1 (modulo Proposition 3.2). Requisition the partial recursive function f from Proposition 3.2, and note that f together with the principal filter determined by $\operatorname{dom} f$ satisfy the premisses of Proposition 3.6. The latter Proposition then provides the required prime p such that f induces a non-identity self-embedding of $\mathbb{N}[p]$. ∎

3.B. Three colours suffice

In this subsection we show that three colours always suffice for any chromatic subset of $\operatorname{dom} f$. The argument consists in a re-colouring technique which we borrow from Krawczyk & Steprāns [25]. While not a deal-breaker for our purposes, the sufficiency of three colours is an interesting fact which will also help the rest of this section save a little space on subscripts.

3.8. Lemma (after Krawczyk & Steprāns [25, Lemma 2.1]). *Let f be a partial recursive function, let $X \subseteq \operatorname{dom} f$ be an r.e. set, $k \geq 3$ and $\ell \geq 1$. Suppose $\bigcap_{0 \leq i < \ell} f^{-i}[X]$ admits a partial recursive f-colouring with $k+1$ colours.*

Then $\bigcap_{0 \leq i < \ell+2} f^{-i}[X]$ admits a partial recursive f-colouring with k colours.

Proof. Let $\chi : \bigcap_{0 \leq i < \ell} f^{-i}[X] \to k+1$ be a partial recursive f-colouring with $k+1$ colours. Define a new colour assignment ψ with k colours for $x \in \bigcap_{0 \leq i < \ell+2} f^{-i}[X]$ by

$$\psi(x) = \begin{cases} \chi(f(x)) & \text{if } \chi(f(x)) < k, \\ \min(k - \{\chi(x), \chi(f^2(x))\}) & \text{if } \chi(f(x)) = k. \end{cases}$$

Note that $x, f(x), f^2(x) \in \bigcap_{0 \leq i < \ell} f^{-i}[X]$, so $\chi(x), \chi(f(x)), \chi(f^2(x))$ are defined. The set $k - \{\chi(x), \chi(f^2(x))\}$ is never empty because $k \geq 3$. The colouring ψ is partial recursive because χ is.

Let us verify that ψ is an f-colouring. Suppose $f(x) \in \bigcap_{0 \leq i < \ell+2} f^{-i}[X]$, so that $\chi(x), \ldots, \chi(f^3(x))$ are defined.

It cannot be the case that $\chi(f(x)) = \chi(f^2(x)) = k$ because χ is an f-colouring.

If $\chi(f(x))$ and $\chi(f^2(x))$ are both smaller than k, then $\psi(x) = \chi(f(x)) \neq \chi(f^2(x)) = \psi(f(x))$.

If $\chi(f(x)) = k > \chi(f^2(x))$, then $\psi(x) = \min(k - \{\chi(x), \chi(f^2(x))\}) \neq \chi(f^2(x)) = \psi(f(x))$.

If $\chi(f(x)) < k = \chi(f^2(x))$, then $\psi(x) = \chi(f(x)) \neq \min(k - \{\chi(f(x)), \chi(f^3(x))\}) = \psi(f(x))$.

Thus ψ always assigns distinct colours to x and $f(x)$. ∎

3.9. Corollary. *If f is a partial recursive function and* $\operatorname{dom} f^\ell$ *is f-colourable with finitely many colours for some $\ell \geq 1$, then there is a $k \in \omega$ such that* $\operatorname{dom} f^k$ *admits an f-colouring with* 3 *colours.*

Hint. $\operatorname{dom} f^{i+1} = f^{-i}[\operatorname{dom} f]$. ∎

We can now strengthen both Lemma 3 in McLaughlin [34] and our Lemma 1.9:

3.10. Corollary. *For any total recursive f there is a recursive f-colouring of* $\overline{\operatorname{fix} f}$ *with* 3 *colours.*

Proof. If $f[\overline{\operatorname{fix} f}] \subseteq \operatorname{fix} f$, then a single colour siffices to f-colour $\overline{\operatorname{fix} f}$. So suppose there is an $s \in \overline{\operatorname{fix} f}$ with $f(s) \notin \operatorname{fix} f$. Define the function $\hat{f} : \overline{\operatorname{fix} f} \to \overline{\operatorname{fix} f}$ by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } f(x) \notin \operatorname{fix} f, \\ s & \text{if } f(x) \in \operatorname{fix} f. \end{cases}$$

Observe that any $\hat{f}$-colouring of $\overline{\operatorname{fix} f}$ is also an f-colouring of $\overline{\operatorname{fix} f}$. Furthermore, $\hat{f}$ is fixed-point-free because $\hat{f}(s) = f(s) \neq s$. We have an $\hat{f}$-colouring of $\overline{\operatorname{fix} f}$ with 5 colours from Lemma 1.9, hence there is also one with 3 colours by Corollary 3.9 as $\operatorname{dom} \hat{f}^i = \overline{\operatorname{fix} f}$ for each $i > 0$. ∎

3.C. Painter vs Forester

We now describe a game of the type that presumably first appeared in Bean [1, proof of Theorem 2] and Gyárfás & Lehel [13, proof of Theorem 2.5]. In our case, we use the game as the lazy person's means to avoid the explicit description of a brute-force winning strategy for one of the players.

3.11. Construction. Let $\ell \geq 1$. Consider a Gale–Stewart win/lose game G_ℓ between Player F and Player P. A position $\mathbf{p}$ in G_ℓ is a finite directed graph (no loops, no multiple edges) with some vertices assigned one of 3 colours. Vertices of the position $\mathbf{p}$ graph that are incident on no outbound edges are called *(*$\mathbf{p}$*-)open*. The game starts with an empty graph.

In each round of G_ℓ, Player F moves first, adding a single vertex to the graph, possibly together with edges from some of the existing open vertices to the new vertex, thereby relieving the former of their open status. Thus Player F's move is determined by the choice of a subset of the currently open vertices.

It is clear that each vertex can only ever be incident on at most one outbound edge, so after each round we are left with a forest where each edge is directed away from leaves and towards one of the roots, the roots being exacly the open vertices.

Player P has a palette of 3 colours, and Player P's response move consists in colouring each vertex v such that there is a directed path $v = v_0, \ldots, v_\ell$ of length ℓ (i.e., with ℓ many edges) starting at v, provided v has not been assigned a colour in the course of one of the preceding rounds. For each vertex v such a path, if it exists, is unique. Clearly, only the startpoints of directed length ℓ paths ending at the newly added vertex require attention, all the others having been coloured in one of the previous rounds.

Any edge whose start- and endpoint are assigned the same colour signifies an immediate win for Player F. Any infinite play (without monochromatic edges) is a win for Player P.

Player F's win-set being open, one of the players must have a winning strategy. Also note that the tree of legal positions is finitely branching, with any infinite branch corresponding to a win for Player P. In view of Kőnig's lemma, this means that any winning strategy for Player F ensures victory in boundedly many rounds, so that strategy is described by a finite function. Given such a finite candidate strategy, one can effectively determine whether it is a winning strategy for Player F via exhaustive search through Player P's legal moves. Thus there exists a recursive function which, given ℓ, will output a winning strategy for Player F in G_ℓ provided such a strategy exists.

The closest relative of the game G_ℓ in existing literature may be the task of 1-*inductive on-line graph colouring with lookahead* as described by Irani [20, section 5]. 1-*inductive* means that all but at most one edge connect any vertex v to vertices that appear earlier than v in a given enumeration. The vertex v is always enumerated together with all edges connecting it to the vertices enumerated earlier. This situation is parallelled in G_ℓ. *On-line with lookahead* ℓ means that we are allowed to have a look at the next ℓ vertices to be enumerated before assigning a colour to v. In our case, Player P waits for v to acquire a chain of ℓ successors — which may or may not eventually happen — before deciding on a colour for v.

3.12. Definition. We define an *ℓ-seedling* to be a finite directed tree with edges oriented away from the leaves towards the root where all leaves have *depth* ℓ (i.e. the path from any leaf to the root has length ℓ). Furthermore, each leaf of an ℓ-seedling is assigned one of the 3 colours.

Let γ be the graph corresponding to a position after some round in a play of G_ℓ and let v be an open vertex of γ. Suppose the subtree of γ consisting of all vertices at edge-distance at most ℓ from v together with leaf colouring induced by that play forms an ℓ-seedling. Then that ℓ-seedling is said to be *associated with* v. (Some open vertices may fail to have an associated seedling because there may be leaves of γ at a smaller distance from v, but this will not affect our arguments.)

Given an $(\ell+1)$-seedling s and a vertex u in s such that there is an edge from u to the root of s, the *immediate (ℓ-)subseedling of* s determined by u is the subtree of s consisting of vertices that are startpoints of directed paths to u including the zero-length path. Leaf colouring is inherited from s.

The *strain of* a 1-seedling s is the set of colours of the leaves of s. The *strain of* an $(\ell+1)$-seedling t is the set of strains of all immediate (ℓ-)subseedlings of t. An *ℓ-strain* is the strain of some ℓ-seedling. Equivalently, a 1-strain is a non-empty subset of the 3 colours, while an $(\ell+1)$-strain is a non-empty set of ℓ-strains. Clearly, for each $\ell \geq 1$ the number of distinct ℓ-strains is finite.

A *homomorphism* between directed graphs is a vertex mapping that preserves directed edge presence from source to target. Any homomorphism between ℓ-seedlings must map leaves to leaves and root to root. Such a homomorphism is a *chromomorphism* if it preserves leaf colours.

3.13. Lemma. *For each $\ell \geq 1$ and any ℓ-seedlings s and t of the same strain, there is a chromomorphism $\varphi : s \longrightarrow t$.*

Proof. This is clear enough for $\ell = 1$.

Suppose the statement holds for ℓ and consider $(\ell+1)$-seedlings s and t. Since the strains of s and t coincide, to each immediate subseedling s' of s there corresponds at least one immediate subseedling t' of t of the same ℓ-strain as s'. Select one of those arbitrarily and use the induction hypothesis to find a chromomorphism $\varphi_{s'} : s' \longrightarrow t'$. Let φ map the root of s to that of t and put $\varphi|_{s'} = \varphi_{s'}$ for each immediate subseedling s' of s. ∎

The conclusion of Lemma 3.13 can be derived from assumptions weaker than equality of strains, but its current form will already suffice for our purposes.

The $\ell = 0$ case of the following proposition would, if present, have some similarity to Theorem 2.5 in Gyárfás & Lehel [13].

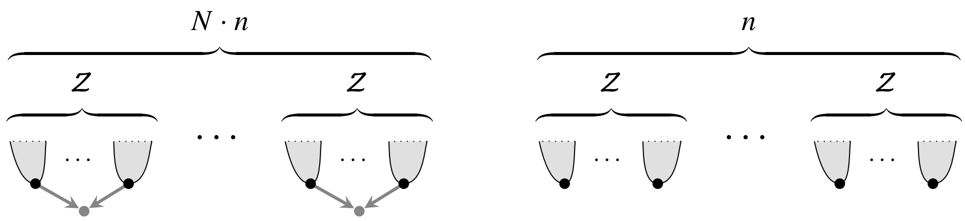
3.14. Proposition. *For each $\ell \geq 1$ Player F has a winning strategy for G_ℓ.*

Proof. We fix an $\ell \geq 1$ and show that Player P cannot have a winning strategy for G_ℓ. Towards contradiction, suppose π were such a winning strategy.

Call a set $\mathcal{W}$ of ℓ-strains *widespread* if for each $n > 0$, there is a position $\mathbf{p}$ in a play of G_ℓ in which Player P follows π such that for each element $e \in \mathcal{W}$ there are at least n many $\mathbf{p}$-open vertices with associated ℓ-seedlings of strain e. To see that non-empty widespread sets exist, let Player F progressively construct more and more disjoint single-leaf ℓ-seedlings.

Now let $\mathcal{Z}$ be $\subseteq$-maximal among widespread sets. Suppose there are N many ℓ-strains in total (not just in $\mathcal{Z}$).

Consider an $n > 0$. Since $\mathcal{Z}$ is widespread, there is a position $\mathbf{p}_n$ in a π-play of G_ℓ with, for each $e \in \mathcal{Z}$, at least $(N + 1) \cdot n$ many open vertices with associted ℓ-seedlings of strain e. Starting from $\mathbf{p}_n$, let Player F successively introduce $N \cdot n$ new vertices v with an edge to v from a $\mathbf{p}_n$-open vertex associated with an ℓ-seedling of strain e, one for each $e \in \mathcal{Z}$. This still leaves n many $\mathbf{p}_n$-open vertices for each element of $\mathcal{Z}$ which remain open in the new position:

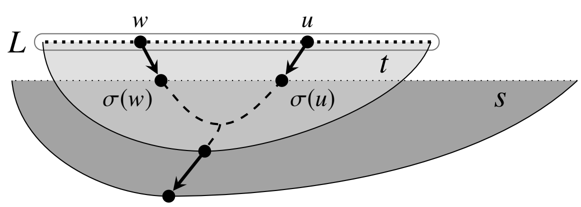


Since $\mathcal{Z} \neq \emptyset$, Player P's π-responses associate ℓ-seedlings to the $N \cdot n$ newly introduced vertices, bringing the play to position $\mathbf{q}_n$. By the Pigeonhole principle, there are n many among the new vertices with associated ℓ-seedlings of the same strain f_n.

Select an ℓ-strain f such that $f = f_n$ for n from an infinite set $I \subseteq \omega$. For $n \in I$, in position $\mathbf{q}_n$ there are n many open vertices with associated ℓ-seedlings of strain e for each $e \in \mathcal{Z}$, as well as n many open vertices with associated ℓ-seedlings of strain f. Since I is infinite, the set $\mathcal{Z} \cup \{f\}$ is widespread. By the maximality assumption on $\mathcal{Z}$, we must have $f \in \mathcal{Z}$.

Thus for $n \in I$, the partially coloured graph corresponding to position $\mathbf{q}_n$ has a subtree which is an ℓ-seedling s of strain f while one of the vertices of s adjacent to the root of s was in position $\mathbf{p}_n$ associated with an ℓ-seedling t of the same strain.

Let $\varphi : s \longrightarrow t$ be the chromomorphism from Lemma 3.13. Let σ be the mapping from the set L of leaves of t to the set of leaves of s defined by $\sigma(u) = z$ if (u, z) is an edge in t. Note that σ is well-defined as all elements of L are located at depth ℓ in t.



Let d be the edge-distance function (disregarding edge orientation). For all $u, w \in L$ we have

$$d(u,w) = \begin{cases} 0 & \text{if } u = w, \\ d(\sigma(u), \sigma(w)) + 2 & \text{otherwise.} \end{cases}$$

Furthermore, $d(u,w) \leq 2\ell$ (down to the root of t and back at worst). Hence $d(u,w) - 2 \leq \frac{\ell-1}{\ell} \cdot d(u,w)$. Therefore

$$d\Big(\varphi\big(\sigma(u)\big), \varphi\big(\sigma(w)\big)\Big) \leq d\big(\sigma(u), \sigma(w)\big) = \max\big\{\, 0, d(u,w) - 2 \,\big\} \leq \frac{\ell - 1}{\ell} \cdot d(u,w)$$

with the leftmost inequality holding because φ is a homomorphism. Thus $\varphi \circ \sigma : L \to L$ is a contraction mapping. Being finite, (L, d) is a complete metric space. By Banach's contraction principle, $\varphi \circ \sigma$ has a fixed point $v = \varphi(\sigma(v)) \in L$. The colour of $v = \varphi(\sigma(v))$ coincides with that of $\sigma(v)$ since φ preserves leaf colours. But v and $\sigma(v)$ are adjacent by definition. The monochromatic edge $(v, \sigma(v))$ contradicts the victoriousness of π, which brings the proof to its conclusion. ∎

For the final stretch of the argument, we follow in the slipstream of the proof of Theorem 2 in Bean [1].

3.15. Proof of Proposition 3.2 concluded. Let $(\langle e \rangle)_{e \in \omega}$ be a uniformly recursive indexing of all partial recursive functions with values in $\{0, 1, 2\}$ — these are the three colours. Fix a uniformly r.e. family $(R_{\ell,e})_{\ell \geq 1, e \in \omega}$ of pairwise disjoint infinite sets. $R_{\ell,e}$ will be viewed as a playboard for applying Player F's winning strategy τ_ℓ for G_ℓ against $\langle e \rangle$. Recall that τ_ℓ exists for each $\ell \geq 1$ by Proposition 3.14, and that τ_ℓ can be chosen recursively in ℓ as explained in Construction 3.11. We describe an effective procedure for enumerating a directed graph Γ whose vertex set is a subset of $\bigcup_{\ell \geq 1, e \in \omega} R_{\ell,e}$, and such that the start- and endpoint of each edge belong to the same $R_{\ell,e}$:

The graph $\Gamma_{\ell,e}$, the restriction of the graph Γ to $R_{\ell,e}$, is to correspond to a play of G_ℓ where Player F follows the winning strategy τ_ℓ. Each new vertex introduced by Player F is identified with a fresh element of $R_{\ell,e}$ in some recursive manner. Player P's moves are given by $\langle e \rangle$: before moving on to the next round, Player F waits for $\langle e \rangle(x)$ to converge for all $x \in R_{\ell,e}$ that, according to the rules of G_ℓ, need to acquire a color value in the current round — if this never happens, this counts as Player P's failure to make a move, so no further moves within $R_{\ell,e}$ will be made by Player F either.

The function f is defined by putting $f(x) = y$ each time there is a Γ-edge from x to y — recall that each vertex is incident on at most one outbound edge. Note that f is partial recursive because the enumeration procedure for the graph is effective, and f is fixed-point-free because Player F never introduces loops.

Suppose for some $\ell \geq 1$, the function $\langle e \rangle$ effected a recursive f-colouring of $\operatorname{dom} f^\ell$ with 3 colours, and proceed towards contradiction. Let us focus on $\Gamma_{\ell,e}$. If $x \in R_{\ell,e}$ and there is a directed Γ-path of length ℓ starting at x, then $x \in \operatorname{dom} f^\ell$, so $\langle e \rangle$ assigns a colour to x. In other words, $\langle e \rangle$ makes all the moves required of Player P by G_ℓ. On the other hand, since Player F follows the winning strategy τ_ℓ for the construction of $\Gamma_{\ell,e}$, there will be a monochromatic edge between two vertices of $\Gamma_{\ell,e}$, both of which lie in $\operatorname{dom} f^\ell$. Hence $\langle e \rangle$ cannot represent a recursive f-colouring of $\operatorname{dom} f^\ell$.

Thus $\operatorname{dom} f^\ell$ cannot be recursively f-coloured with 3 colours for any $\ell \geq 1$. By Corollary 3.9, no finite number of colours suffices either. ∎

3.16. Remark. For each ℓ, the strategy τ_ℓ sees Player F to victory in finitely, say n_ℓ, many rounds, as explained in Construction 3.11. Hence we could have chosen each set $R_{\ell,e}$ in the proof of Theorem 3.1 to contain n_ℓ rather than infinitely many elements. Player F, if desired, can choose fresh vertices in $R_{\ell,e}$ in ascending or descending order. The two possibilities would lead to $x < f(x)$ or $f(x) < x$ respectively for all $x \in \operatorname{dom} f$. For the self-embedding ι of Theorem 3.1 one can therefore arrange for either one of $\mathbb{N}[p] \models \iota(\boldsymbol{x}) = f(\boldsymbol{x}) > \boldsymbol{x}$ and $\mathbb{N}[p] \models \iota(\boldsymbol{x}) = f(\boldsymbol{x}) < \boldsymbol{x}$ to hold. In contrast to this, Proposition 5.12(b) will show that $\boldsymbol{x}$ must be much smaller than the *time* it takes to compute $f(\boldsymbol{x})$.

3.17. Remark. A previously studied partial recursive function can take the place of the function f constructed in the proof of Theorem 3.1. A partial recursive j is a *unary universal* function if there exists a binary total recursive function s satisfying $\{d\}(x) = j(s(d,x))$ for all d and x. According to e.g. Ershov [9, Lemma II.10] or [10, Лемма 2.5.9], the graph of any partial recursive function, in particular, that of our f, can be effectively and faithfully embedded into the graph of j. In other words, there exists a one-to-one total recursive e such that

$$f(x) = y \iff j\big(e(x)\big) = e(y) \quad \text{for all } x, y.$$

It follows that there exists a prime $q \ni e[\operatorname{dom} f]$ such that j induces a non-identity self-embedding of $\mathbb{N}[q]$.

4. Σ_1 types, restricted RK, and hinged primes

In this section, p and q are arbitrary r.e. primes on which individual lemmas may or may not place additional assumptions.

4.A. An RK-like ordering of primes and Σ_1 types

4.1. Definition. Let f be a partial recursive function with $\operatorname{dom} f \in p$. Put

$$f_*(p) = \big\{\text{r.e. } X \subseteq \omega \bigm| f^{-1}[X] \in p\big\}.$$

The following lemma is a miniaturization of Lemma 1(a) in Ng & Render [36].

4.2. Lemma. *If* $\mathbb{N} < a \in M \models \mathrm{TA}_2$ *and* f *is partial recursive with* $\operatorname{dom} f \in p = \operatorname{tp}^M_{\Sigma_1} a$*, then* $f_*(p) = \operatorname{tp}^M_{\Sigma_1} f(a)$*. In particular,* $f_*(p) = \operatorname{tp}^{\mathbb{N}[p]}_{\Sigma_1} f(\boldsymbol{x})$*.*

Thus $f_*(p)$ *is a prime if* $M \models f(a) > \mathbb{N}$*, and* $f_*(p)$ *is a principal maximal filter in* $\mathcal{E}$ *if* $M \models f(a) \in \mathbb{N}$*.*

Hint. Lemmas 1.4(e) and 2.14. ∎

4.3. Lemma. *Let* $f \in \mathcal{P}_p$*. The following are equivalent:*

(i) $f_*(p) = q$;

(ii) $\operatorname{tp}^{\mathbb{N}[p]}_{\Sigma_1} f(\boldsymbol{x}_p) = q$;

(iii) $\boldsymbol{x}_q \mapsto f(\boldsymbol{x}_p)$ *extends to a* Σ_1*-elementary embedding* $\mathbb{N}[q] \to \mathbb{N}[p]$*.*

PROOF. (i)⇒(ii) is immediate from Lemma 4.2, and (ii)⇒(iii) follows from Lemma 2.15(b).

For (iii)⇒(i), use Σ_1 elementarity and Lemmas 2.10(a) and 4.2 to show $q = \mathrm{tp}^{\mathbb{N}[q]}_{\Sigma_1} \boldsymbol{x}_q = \mathrm{tp}^{\mathbb{N}[p]}_{\Sigma_1} f(\boldsymbol{x}_p) = f_*(p)$. ∎

4.4. DEFINITION (Shavrukov [43, Definition 5.8]).

$$q \leq_{\mathrm{rk}} p \Leftrightarrow \exists f \in \mathcal{P}_p \;\; f_*(p) = q.$$

We call a function f as above an *(rk-)reduction*. That $\leq_{\mathrm{rk}}$ is a preorder among r.e. primes follows e.g. from (i)⇔(iii) of Corollary 4.5 to Lemma 4.3.

The definition of $\leq_{\mathrm{rk}}$ is obviously modelled on the classical Rudin–Keisler ordering of ultrafilters on, say, ω — see e.g. Rudin [39], Comfort & Negrepontis [5, § 9], or Ng & Render [36]. A definable variant of Rudin–Keisler obtains by considering complete non-principal (1-)types of a theory such as PA together with the ordering determined by (definable) Skolem term reductions between the types — see Schmerl [41, Section 4], who explains how the classical setup can be subsumed as a particular case, or Lascar [26] for an even more general setting where the ordering is defined in terms of elementary embeddability of models. In our circumstances, the appropriate analogue is Σ_1-elementary embeddability between r.e. prime powers. The classical vesion of the following corollary is found in e.g. Cherlin & Hirschfeld [4, Theorem 2.6].

4.5. COROLLARY. *The following are equivalent:*

(i) $q \leq_{\mathrm{rk}} p$;

(ii) *there is an element a of* $\mathbb{N}[p]$ *with* $\mathrm{tp}^{\mathbb{N}[p]}_{\Sigma_1} a = q$;

(iii) $\mathbb{N}[q]$ *embeds* Σ_1*-elementarily into* $\mathbb{N}[p]$. ∎

The r.e. primes underlying our version of $\leq_{\mathrm{rk}}$ represent a departure from the Boolean nature of traditional variants of Rudin–Keisler in that now we have a meaningful inclusion ordering to contend with. Lemma 4.7 lists the first simple facts about the interplay between reductions and inclusion.

4.6. FACT. *Suppose* $\mathbb{N} < a \in M \models \mathrm{TA}_2$ *and the prime q satisfies* $q \supseteq \mathrm{tp}^{M}_{\Sigma_1} a$. *Then M extends to a* TA_2*-model K such that* $\mathrm{tp}^{K}_{\Sigma_1} a = q$.

COMMENT. Proposition 2.13 in Shavrukov [43] establishes this with M and K being models of full 1st order aritmetic TA. Its proof however holds verbatim with TA_2 in place of TA. (Alternatively, observe that any model of TA_2 embeds Σ_1-elementarily into some model of TA.) ∎

4.7. LEMMA. (a) *Suppose $p \subseteq q$ and $f \in \mathcal{P}_p$ (so that $f_*(p)$ is defined). Then $f_*(q)$ is defined and $f_*(p) \subseteq f_*(q)$.*

(b) *If $(q_i)_{i \in I}$ is a chain of primes and $f \in \mathcal{P}_{q_i}$ for all $i \in I$, then $f_*(\bigcup_{i \in I} q_i) = \bigcup_{i \in I} f_*(q_i)$.*

(c) *Suppose $f \in \mathcal{P}_p$ is such that $f_*(p) \subseteq q$. Then there exists a prime $r \supseteq p$ satisfying $f_*(r) = q$.*

Proof. Clauses (a) and (b) are immediate from the definitions.

(c) Since $q \supseteq f_*(p) = \mathrm{tp}_{\Sigma_1}^{\mathbb{N}[p]} f(\boldsymbol{x})$ (Lemma 4.2), Fact 4.6 supplies a TA_2-extension K of $\mathbb{N}[p]$ such that $q = \mathrm{tp}_{\Sigma_1}^{K} f(\boldsymbol{x})$. Put $r = \mathrm{tp}_{\Sigma_1}^{K} \boldsymbol{x}$. Then $r \supseteq \mathrm{tp}_{\Sigma_1}^{\mathbb{N}[p]} \boldsymbol{x} = p$ because formulas of the form $\boldsymbol{x} \in X$ with X r.e. are Σ_1 and therefore persist from $\mathbb{N}[p]$ to K. Finally, $f_*(r) = \mathrm{tp}_{\Sigma_1}^{K} f(\boldsymbol{x}) = q$ by Lemma 4.2. ∎

4.B. The category $\mathbf{rk}_\Sigma$

4.8. Definition. The small category $\mathbf{rk}_\Sigma$ has as objects all r.e. primes $p \in (\mathcal{E}^*)^\star$. The arrows $p \to q$ are represented by partial recursive functions $f \in \mathcal{P}_p$ such that $f_*(p) = q$. Two such functions f and g represent the same arrow when $f \equiv_p g$ — this identification is easily seen to be correct w.r.t. compositon. The relation of isomorphism in $\mathbf{rk}_\Sigma$ is denoted by $\sim_{\mathrm{rk}}$.

Blass [2, § 2] studied a somewhat similar category of ultrafilters (in full powersets) with arrows represented by arbitrary functions.

4.9. Lemma. *One has $p \sim_{\mathrm{rk}} q$ if and only if $\mathbb{N}[p] \cong \mathbb{N}[q]$.*

Proof. Follows at once from (i)⇔(iii) of Lemma 4.3. ∎

4.10. Exercise. *Suppose the primes p and q are $\mathbf{rk}_\Sigma$-isomorphic. Then*

(a) *if $f : p \to q$ and $g : q \to p$ are partial recursive functions witnessing the isomorphism between p and q, then there are r.e. sets $X \in p$ and $Y \in q$ with $X \subseteq \operatorname{dom} f$ and $Y \subseteq \operatorname{dom} g$ such that $f|_X$ and $g|_Y$ are mutually inverse bijections between X and Y;*

(b) *the quotient lattices $\mathcal{E}/p$ and $\mathcal{E}/q$ are isomorphic.* ∎

The classical analogue of Exercise 4.10(a) is found in Blass [2, Proposition I.7] or Comfort & Negrepontis [5, Theorem 9.2(b)], and an analogue for recursive ultrapowers in McLaughlin [30, Lemma 3.2].

4.11. Observation. *For each prime p, the identity is the only $\mathbf{rk}_\Sigma$-automorphism $p \to p$.*

Proof. If f and g are partially recursive functions representing mutually inverse $\mathbf{rk}_\Sigma$-arrows $p \to p$, then $\boldsymbol{x}_p \mapsto f(\boldsymbol{x}_p)$ and $\boldsymbol{x}_p \mapsto g(\boldsymbol{x}_p)$ induce mutually inverse self-embeddings (hence automorphisms) of $\mathbb{N}[p]$ by Lemma 4.3. By Proposition 2.20, both automorphisms are the identity, so $f \equiv_p g \equiv_p \mathrm{id}$. ∎

4.C. $\lessapprox$, skies, and hinged primes

4.12. Definition. In a model M of TA_2, $a \ll b$ means that $f(a) < b$ for all total recursive f. Accordingly, $a \lessapprox b$ means that $f(b) \geq a$ for some total recursive f. $\lessapprox$ is a total preorder. The corresponding equivalence relation is denoted $\approx$ and its classes are called *(total recursive) skies*. Skies are convex subsets of M. The standard numbers $\mathbb{N}$ form the lowermost sky in M.

Observe that $a \lessapprox b$ is absolute for embeddings between models of TA_2 because so are the formulas $f(b) \geq a$.

4.13. Convention. Recall that a *(recursive) enumeration* of an r.e. set X is an increasing recursive sequence $(X_t)_{t\in\omega}$ of finite sets such that $X = \bigcup_{t\in\omega} X_t$ and $\max X_t < t$ whenever $X_t \neq \varnothing$. We shall always silently assume that a recursive enumeration of each r.e. set we consider is chosen and stays the same throughout an argument. The same accord extends to partial recursive functions f, where $(f_t)_{t\in\omega}$ is understood to recursively enumerate the graph of f, and one has $f_t(x) = y \Rightarrow x, y < t$.

4.14. Definition (Shavrukov [43, Definition 3.9 and Proposition 3.11]). For r.e. sets X and Y and a total recursive function f, define the r.e. set

$$X \setminus_f Y = \left\{ x \in Y \;\middle|\; \mu t\,[x \in X_t] \leq f\big(\mu t\,[x \in Y_t]\big) \right\}.$$

Note that $X \setminus_f Y \subseteq X \cap Y$ and that $X \setminus_f Y$ depends on the choice of individual enumerations of X and Y rather than just on the sets X and Y.

A prime p is *hinged on* Y if $Y \in p$ and p is minimal among the primes containing Y. Equivalently, $Y \in p$ and for each $X \in p$ there is a total recursive f with $X \setminus_f Y \in p$. The set Y is then called a *hinge for* Y. A prime is *hinged* if it has a hinge. For example, each inclusion-minimal prime is hinged (on, say, ω).

Given any $Z \in p$, there is a prime $q \subseteq p$ which hinges on Z (Shavrukov [43, Lemma 3.14]). The prime $q \subseteq p$ with this property is unique by Fact 2.2, and $q = \bigcap\{\text{ primes } r \subseteq p \mid r \ni Z\}$. We denote this q by $p\lceil_Z$.

4.15. Lemma. *If f is total recursive and $X \setminus_f Y \in p \supseteq q \ni Y$, then $X \setminus_f Y \in q$.*

Proof. Since $\{ X \setminus_f Y, Y - X \setminus_f Y \}$ is an r.e. splitting of $Y \in q$ (see Shavrukov [43, Lemma 3.8]), exactly one of its pieces must belong to q. As $X \setminus_f Y \in p \supseteq q$, one cannot have $q \ni Y - X \setminus_f Y$. ∎

The next lemma tells us that, in a TA_2-model, the sky witnessing the entrance of a given element into a given r.e. set is uniquely determined.

4.16. Lemma. *Suppose M is a model of TA_2 and $a \in M \models a \in X$ where X is r.e. If $(X_t)_{t\in\omega}$ and $(\hat{X}_t)_{t\in\omega}$ are two enumerations of X, then $M \models \mu t[a \in X_t] \approx \mu t\,[a \in \hat{X}_t]$.*

Hint. The function $t \mapsto \mu s[X_t \subseteq \hat{X}_s]$ is total recursive. ∎

4.17. Definition. For $X, Y \in p$, write $X \lessapprox_p Y$ if $\mathbb{N}[p] \models \mu t[\boldsymbol{x} \in X_t] \lessapprox \mu t[\boldsymbol{x} \in Y_t]$, and similarly for $\ll_p$ and $\approx_p$.

Observe that in view of Lemma 4.16, $X \lessapprox_p Y$ does not depend on the choice of enumerations for X and Y. Neither do the relations $\ll_p$ and $\approx_p$. Therefore $\lessapprox_p$ is a (total) preordering on p.

Define the ordering $\mathcal{S}_p$ as $(p/{\approx_p}, \lessapprox_p)$ where, strictly speaking, $\lessapprox_p$ stands in for $\lessapprox_p/{\approx_p}$. (We shall also freely confuse individual elements $X \in p$ with the $\approx_p$-equivalence classes they represent.)

The pre-ordering $\lessapprox_p$ records a coarse chronology of $\boldsymbol{x}$ entering the r.e. sets that it belongs to in $\mathbb{N}[p]$.

It should be noted that $\mathbf{rk}_\Sigma$-isomorphism between primes p and q does not generally imply the isomorphism between $\mathcal{S}_p$ and $\mathcal{S}_q$, except for the case when both directions of the $\mathbf{rk}_\Sigma$-isomorphism are effected by total recursive functions.

The following lemma holds for the same reason as Lemma 1.4.

4.18. Lemma. *Suppose $a \in M \models \mathrm{TA}_2$, X and Y are r.e., and f is total recursive. Then*

$$M \models a \in X \setminus_f Y \leftrightarrow \mu t\,[a \in X_t] \leq f\big(\mu t\,[a \in Y_t]\big). \qquad \blacksquare$$

4.19. Lemma. *For $X, Y \in p$, the following are equivalent:*

(i) *$X \lessdot_p Y$;*

(ii) *There is a total recursive f such that $X \setminus_f Y \in p$;*

(iii) *$X \in p\lceil_Y$.*

Proof. (i)⇔(ii) For a total recursive f, $\mathbb{N}[p] \models \mu t[\boldsymbol{x} \in X_t] \leq f(\mu t[\boldsymbol{x} \in X_t])$ iff $\mathbb{N}[p] \models \boldsymbol{x} \in X \setminus_f Y$ iff $X \setminus_f Y \in p$ by Lemmas 4.18 and 2.10(a).

(ii)⇒(iii) Since $Y \in p\lceil_Y$ and $X \setminus_f Y \in p \supseteq p\lceil_Y$, we must by Lemma 4.15 have $X \setminus_f Y \in p\lceil_Y$, hence $X \in p\lceil_Y$.

(iii)⇒(ii) Since Y is a hinge for $p\lceil_Y \ni X$, $X \setminus_f Y \in p\lceil_Y$ holds for some total recursive f. $\blacksquare$

The equivalence (i)⇔(ii) of Lemma 4.19 allows us to rephrase Lemma 4.15 as

4.20. Corollary. *If $p \supseteq q \ni Y$ and $X \lessdot_p Y$, then $X \in q$ and $X \lessdot_q Y$.*
In particular, $\lessdot_q$ is the restriction of $\lessdot_p$ to q when $q \subseteq p$. $\blacksquare$

4.21. Lemma. (a) *If $p \ni X \subseteq^* Y$, then $Y \lessdot_p X$.*

(b) *If $R \in p$ is recursive, then $R \lessdot_p X$ for all $X \in p$.*

(c) *$Y \lessdot_p X$ for all $X \in p$ iff $Y \lessdot_p \omega$ iff $Y \in \overline{p^\circ}$* (Definition 2.5).
Hence $\overline{p^\circ}$ is the $\approx_p$-equivalence class of ω is the least element of $\mathcal{S}_p$.

(d) *If $X, Y \in p$, then $X \cap Y \lessdot_p X$ or $X \cap Y \lessdot_p Y$.*

(e) *If $X \cup Y \in p$, then $X \lessdot_p X \cup Y$ or $Y \lessdot_p X \cup Y$.*

Proof. Clause (a) is left to the reader.

(b) Since $R \in p$ is recursive, we have $q \ni R$ for all $q \subseteq p$. In particular, $R \in p\lceil_X$, so $R \lessdot_p X$ for each $X \in p$ by Lemma 4.19.

(c) If $Y \lessdot_p X$ for all $X \in p$, then $Y \lessdot_p \omega$. Since $\omega \in \overline{p^\circ} \subseteq p$, one has $Y \in \overline{p^\circ}$ by Corollary 4.20.
In the opposite direction, $Y \in \overline{p^\circ}$ says that $Y \supseteq R \in p$ for some recursive R, hence $Y \lessdot_p R \lessdot_p X$ for each $X \in p$ by clauses (a) and (b).

(d) Since $\lessdot_p$ is total, we may assume $X \lessdot_p Y$ by symmetry. Then $X \in p\lceil_Y$ by Lemma 4.19, so $\mathbb{N}[p\lceil_Y] \models \boldsymbol{x} \in X \wedge \boldsymbol{x} \in Y$. Therefore $\mathbb{N}[p\lceil_Y] \models \boldsymbol{x} \in X \cap Y$, hence $X \cap Y \in p\lceil_Y$, yielding $X \cap Y \lessdot_p Y$.

(e) As $\mathbb{N}[p\lceil_{X\cup Y}] \models \boldsymbol{x} \in X \cup Y$, we have $\mathbb{N}[p\lceil_{X\cup Y}] \models \boldsymbol{x} \in X \vee \boldsymbol{x} \in Y$. Assume $\mathbb{N}[p\lceil_{X\cup Y}] \models \boldsymbol{x} \in X$ by symmetry. Then $X \in p\lceil_{X\cup Y}$, so $X \lessdot_p X \cup Y$ by Lemma 4.19. $\blacksquare$

4.22. Proposition. *The mapping $q \mapsto q/{\approx_p}$ is an isomorphism of the inclusion ordering on $\downarrow p = \{\text{primes } q \mid q \subseteq p\}$ to the collection of non-empty initial segments of $\mathcal{S}_p$ with inclusion ordering.*

Proof. By Corollary 4.20, $q/{\approx_p} = q/{\approx_q}$ is a (non-empty) initial segment of $\mathcal{S}_p$ for each prime $q \subseteq p$. Conversely, if I a non-empty initial segment of $\mathcal{S}_p$, then Lemma 4.21 guarantees that $\bigcup I \subseteq p$ is a prime. Inclusion is clearly preserved in either direction.

Finally, $q = \bigcup(q/{\approx_p})$ and $I = (\bigcup I)/{\approx_p}$ are straightforward. ∎

Observe that the isomorphism of Proposition 4.22 sends $\overline{p^\circ} = \min{\downarrow} p$ to the singleton initial segment $\{\overline{p^\circ}\}$ of $\mathcal{S}_p$ in view of Lemma 4.21(c).

4.23. Lemma. *The mapping $X \mapsto \mu t[\boldsymbol{x} \in X_t]$ is a cofinal order-embedding of $\mathcal{S}_p$ into $\mathbb{N}[p]/{\approx}$.*

Proof. Both correctness and the embedding property are clear from the definition of $\preccurlyeq_p$ and Lemma 4.16. The embedding is cofinal because, for an arbitrary element $[f]$ of $\mathbb{N}[p]$ where $f \in \mathcal{P}_p$ and $X = \operatorname{dom} f$, one has $\mathbb{N}[p] \models \mu t[\boldsymbol{x} \in X_t] \approx \mu t[f_t(\boldsymbol{x}){\downarrow}] \geq f(\boldsymbol{x}) = [f]$. ∎

Under the order-embedding from Lemma 4.23, the image of the least element of $\mathcal{S}_p$ is the sky of $\boldsymbol{x}$. The range of the embedding need not generally contain all of the non-standard skies of $\mathbb{N}[p]$, nor even all sufficietntly large ones.

Proposition 4.22 and Lemma 4.23 underscore the fact that, more than being an unstructured collection of r.e. sets, an r.e. prime can be said to possess a semblance of historical memory.

4.D. Topmost skies and recursive ultrapowers

4.24. Lemma. *The prime p is minimal if and only if the generator $\boldsymbol{x}_p$ belongs to the topmost sky of $\mathbb{N}[p]$.*

Proof. (only if) If p is minimal, then $p = \bar{u}$ for the recursive ultrafilter $u = p^\circ$ (Definition 2.5). The generator $\boldsymbol{x}_u$ belongs to the topmost sky of the recursive ultrapower $\mathbb{N}[u]$ because each element of $\mathbb{N}[u]$ is the value of some total recursive function at $\boldsymbol{x}_u$. By Lemma 2.6, $\boldsymbol{x}_u \mapsto \boldsymbol{x}_p$ extends to an isomorphism between $\mathbb{N}[u]$ and $\mathbb{N}[p]$.

(if) If p is not minimal, fix a prime $q \subsetneq p$. The embedding $\nu : \mathbb{N}[q] \to \mathbb{N}[p]$ given by $\boldsymbol{x}_q \mapsto \boldsymbol{x}_p$ is not Σ_1-elementary by Corollary 2.16, hence ν cannot be cofinal by Fact 1.10(c), so any element $t \in \mathbb{N}[p]$ with $\nu[\mathbb{N}[q]] < t$ lies in a higher sky of $\mathbb{N}[p]$ than $\boldsymbol{x}_p = \nu(\boldsymbol{x}_q)$ does. ∎

4.25. Proposition. *The following are equivalent:*

(i) *p is hinged;*

(ii) *$\mathbb{N}[p]$ is isomorphic to a recursive ultrapower;*

(iii) *p is $\mathbf{rk}_\Sigma$-isomorphic to a minimal prime;*

(iv) *$\mathbb{N}[p]$ possesses a topmost sky;*

(v) *The ordering $\mathcal{S}_p$ possesses a largest element.*

Proof. (i)⇒(ii) Suppose p hinges on X. Since $X \in p$, we have $\mathbb{N}[p] \models \boldsymbol{x} \in X$ (Lemma 2.10(a)). Consider $s = \langle \boldsymbol{x}, \mu t[\boldsymbol{x} \in X_t] \rangle$, where $\langle x, y \rangle$ is any of the conventional pairing functions with matching projections $x = (\langle x, y \rangle)_0$ and $y = (\langle x, y \rangle)_1$. We claim that s generates $\mathbb{N}[p]$ w.r.t. total recursive functions.

Let $[g]$ be an arbitrary element of $\mathbb{N}[p]$. We have $Y = \operatorname{dom} g \in p$. Since X is a hinge for p, one has $Y \setminus_f X \in p$ for some total recursive f. Then $\mathbb{N}[p] \models \boldsymbol{x} \in Y \setminus_f X$ so that $\mathbb{N}[p] \models \boldsymbol{x} \in Y_{f(\mu t[\boldsymbol{x} \in X_t])}$. Let k be a total recursive function with the Π_2 property $\forall x, u\, (x \in Y_u \to k(x, u) = g(x))$. Then

$\mathbb{N}[p] \models k(\boldsymbol{x}, f(\mu t[\boldsymbol{x} \in X_t])) = g(\boldsymbol{x}) = [g]$. Therefore $\mathbb{N}[p] \models k((s)_0, f((s)_1)) = [g]$, so every element $[g]$ of $\mathbb{N}[p]$ is a total recursive value of s. By Proposition 2.18(b), $\mathbb{N}[p]$ is isomorphic to a recursive ultrapower.

(ii)⇔(iii) holds thanks to Lemmas 2.6 and 4.9.

(ii)⇒(iv) is clear — the sky containing the recursive ultrapower generator is topmost (Lemma 4.24).

(iv)⇒(v) follows at once from Lemma 4.23.

(v)⇒(i) Let the r.e. set $X \in p$ represent the largest element of $\mathcal{S}_p$. We argue that X is a hinge for p. Indeed, for an arbitray $Y \in p$ we have $Y \lessdot_p X$ and hence $Y \searrow_f X \in p$ for some total recursive f by Lemma 4.19. ∎

From Lemma 4.9 and Proposition 4.25 we obtain

4.26. Corollary. $\mathbf{rk}_\Sigma$*-isomorphisms between primes preserve the property of being hinged.* ∎

4.27. Corollary. *The mapping* $q \mapsto \max(q/{\approx_p})$ *is an isomorphism of the subordering of hinged primes in* $\downarrow p$ *to* $\mathcal{S}_p$.

Proof. According to (i)⇔(v) of Lemma 4.25, the isomorphism $q \mapsto q/{\approx_p}$ from Proposition 4.22 restricts to an isomorphism between the collection of hinged primes in $\downarrow p$ and the collection of initial segements of $\mathcal{S}_p$ with a largest element, the latter collection being clearly isomorphic to $\mathcal{S}_p$. ∎

The next corollary settles question Q3 from McLaughlin [32].

4.28. Corollary. *Each r.e. prime power which embeds cofinally into a recursive ultrapower is itself isomorphic to a recursive ultrapower.*

Proof. If ι is a cofinal embedding of an r.e. prime power $\mathbb{N}[p]$ into a recursive ultrapower $\mathbb{N}[u]$, then any element of $\mathbb{N}[p]$ taken by ι to the topmost sky of $\mathbb{N}[u]$ must belong to the topmost sky of $\mathbb{N}[p]$ (this happens because formulas of the form $f(x) < y$ with f total recursive, being Δ_1 in TA_2, are absolute for embeddigs between TA_2-models). By (iv)⇒(ii) of Proposition 4.25, $\mathbb{N}[p]$ is isomorphic to a recursive ultrapower. ∎

Shavrukov [43, Proposition 3.23] shows that each hinged maximal prime is, in fact, minimal. It follows that r.e. ultrapowers $\mathbb{N}[p]$ with a topmost sky are exactly those corresponding to minimax primes p.

The following is immediate from (i)⇒(ii) of Proposition 4.25 and Theorem 1.6:

4.29. Corollary. *If p is hinged, then* $\mathbb{N}[p]$ *is totally rigid.* ∎

Proposition 4.25 tells us that each hinged prime p is $\mathbf{rk}_\Sigma$-isomorphic to some minimal prime. All minimal primes are hinged. Which minimal primes are isomorphic to some non-minimal (hinged) prime?

Say that a non-standard model $M \models \mathrm{TA}_2$ is *single-sky* if all the non-standard elements of M belong to one and the same sky. In the opposite case, M is *multi-sky*. Both single- and multi-sky recursive ultrapowers exist. Schmerl & Shavrukov [42] show that in each multi-sky model of TA_2 the ordering of skies is dense.

A prime p is *single-sky* if $\mathbb{N}[p]$ is. A single-sky prime must be minimal in view of Lemma 4.24 — the r.e. prime powers corresponding to non-minimal primes p possess skies higher than that of $\boldsymbol{x}_p$. For the same reason, no single-sky prime can be $\mathbf{rk}_\Sigma$-isomorphic to any non-minimal prime.

4.30. Question. Must each multi-sky minimal prime be $\mathbf{rk}_\Sigma$-isomorphic to a non-minimal one?

A negative answer would be equivalent to the existence of a multi-sky recursive ultrapower $\mathbb{N}[u]$ such that the initial segment $I = \{\, a \in \mathbb{N}[u] \mid \mathbb{N}[u] \models a \ll \boldsymbol{x}_u \,\}$ (i.e., I is $\mathbb{N}[u]$ with the topmost sky removed) is Σ_1-elementary in $\mathbb{N}[u]$.

5. Properties of non-trivial self-embeddings

In this section we analyse the structure of a non-identity self-embedding in order both to learn what we can about existing self-embeddings and to grease the wheels for the proof of total rigidity in the next section.

We keep the convention that p and q are arbitrary primes.

5.A. Self-embeddings and $\mathbf{rk}_\Sigma$-endomorphisms

In the proof of Proposition 3.6, we had the function f induce a non-trivial self-embedding of $\mathbb{N}[p]$ by virtue of the inclusion $\mathrm{tp}_{\Sigma_1}^{\mathbb{N}[p]} f(\boldsymbol{x}) \supseteq \mathrm{tp}_{\Sigma_1}^{\mathbb{N}[p]} \boldsymbol{x}$. We now argue that these two Σ_1 types must, in fact, coincide, for the application of any partial recursive function cannot properly inflate the Σ_1 type of an element.

5.1. Proposition. *If $q \supsetneq p$, then $\mathbb{N}[q]$ is not embeddable into $\mathbb{N}[p]$.*
(In view of Corollary 4.5, this is equivalent to $q \supsetneq p$ ruling out $q \leq_{\mathrm{rk}} p$.)

Proof. Suppose $X \in q - p$ and let $r = q{\restriction}_X \subseteq q$. The prime r is hinged. Since $X \in r - p$, we have $p \subsetneq r$ by Fact 2.2.

Suppose we had $\iota : \mathbb{N}[q] \to \mathbb{N}[p]$, and let $\nu_{rq} : \mathbb{N}[r] \to \mathbb{N}[q]$ and $\nu_{pr} : \mathbb{N}[p] \to \mathbb{N}[r]$ be the embeddings from Corollary 2.16. In view of Fact 1.10(c), the range of ν_{pr} is bounded in $\mathbb{N}[r]$, hence the composition

$$\mathbb{N}[r] \xrightarrow{\nu_{rq}} \mathbb{N}[q] \xrightarrow{\iota} \mathbb{N}[p] \xrightarrow{\nu_{pr}} \mathbb{N}[r]$$

cannot be the identity. The prime r being hinged, we have reached a contradiction with Corollary 4.29. This proves the non-existence of ι. ∎

5.2. Corollary. (a) *For a self-embedding $\iota : \mathbb{N}[p] \to \mathbb{N}[p]$, one has $\mathrm{tp}_{\Sigma_1}^{\mathbb{N}[p]} \iota(\boldsymbol{x}) = p$.*
In particular, any self-embedding is Σ_1-elementary.

(b) *Suppose $\mathbb{N} < a \in M \models \mathrm{TA}_2 + f(a){\downarrow}$ where f is partial recursive. Then $\mathrm{tp}_{\Sigma_1}^{M} f(a) \supsetneq \mathrm{tp}_{\Sigma_1}^{M} a$ cannot hold.*

Proof. (a) Since ι is an embedding, we have $q = \mathrm{tp}_{\Sigma_1}^{\mathbb{N}[p]} \iota(\boldsymbol{x}) \supseteq p$ by Lemma 2.15(a). By the same Lemma, there is an embedding $\mathbb{N}[q] \to \mathbb{N}[p]$. Now Proposition 5.1 tells us that $q = p$, so ι is Σ_1-elementary by Lemma 2.15(b).

(b) According to Lemma 2.15(b), the smallest Σ_1 elementary submodel $K \ni a$ of M is isomorphic to $\mathbb{N}[\mathrm{tp}^M_{\Sigma_1} a]$. If $\mathrm{tp}^M_{\Sigma_1} f(a) \supseteq \mathrm{tp}^M_{\Sigma_1} a$, then $\mathrm{tp}^K_{\Sigma_1} f(a) = \mathrm{tp}^M_{\Sigma_1} f(a) \supseteq \mathrm{tp}^M_{\Sigma_1} a = \mathrm{tp}^K_{\Sigma_1} a$, hence $a \mapsto f(a)$ extends to a self-embedding of K by Lemma 2.15(a). Therefore $\mathrm{tp}^M_{\Sigma_1} f(a) = \mathrm{tp}^M_{\Sigma_1} a$ by clause (a). ∎

5.3. Question. Can a non-identity self-embedding of $\mathbb{N}[p]$ be Σ_2-elementary? Can a self-embedding fail to be Σ_2-elementary?

5.4. Corollary. (a) *A partial recursive $f \in \mathcal{P}_p$ induces a non-identity self-embedding of $\mathbb{N}[p]$ if and only if f represents a non-identity $\mathbf{rk}_\Sigma$-arrow $p \to p$.*

(b) *There exists a prime p with a non-identity $\mathbf{rk}_\Sigma$-arrow $p \to p$.*

Proof. (a) (if) Suppose $f_*(p) = p$ and $f \not\equiv_p \mathrm{id}$. Then $\mathrm{tp}^{\mathbb{N}[p]}_{\Sigma_1} f(\boldsymbol{x}) = p$ by Lemma 4.3. Hence f induces a self-embedding ι of $\mathbb{N}[p]$ by Lemma 2.15(a), and $\iota \neq \mathrm{id}$ because $\mathbb{N}[p] \models f(\boldsymbol{x}) \neq \boldsymbol{x}$ in view of Lemma 2.10(c).

(only if) If f induces a non-identity self-embedding of $\mathbb{N}[p]$, then $f_*(p) = \mathrm{tp}^{\mathbb{N}[p]}_{\Sigma_1} f(\boldsymbol{x}) = p$ by Lemma 4.3 and Corollary 5.2(a). As $\mathbb{N}[p] \models f(\boldsymbol{x}) \neq \boldsymbol{x}$, we cannot have $f \equiv_p \mathrm{id}$.

(b) follows from Theorem 3.1 and clause (a). ∎

As counterbalance to Corollary 5.4(b), Proposition 2.20 tells us that identities are the only $\mathbf{rk}_\Sigma$-automorphisms of r.e. primes.

Corollary 5.4(b) marks a notable difference between $\mathbf{rk}_\Sigma$ and the traditional Rudin–Keisler variants where the only endomorphism of any object is the identity — see Blass [2, Theorem I.5] or Comfort & Negrepontis [5, Theorem 9.2(a)] for the classical version, and Ehrenfeucht [8], Gaifman [11, Theorem 4.1], or Kossak & Schmerl [24, Theorem 1.7.2] for types over 1st order arithmetic. McLaughlin's Theorem 1.6 establishes the analogous property for the category of recursive ultrafilters and total recursive reductions. Uniqueness of endomorphisms immediately implies that, between any two objects, the presence of arrows in both directions entails isomorphism. I do not know if this is the case in $\mathbf{rk}_\Sigma$:

5.5. Question. Does $p \leq_{\mathrm{rk}} q \leq_{\mathrm{rk}} p$ always imply $p \sim_{\mathrm{rk}} q$?

5.B. The anatomy of a self-embedding

The following proposition, combined with Theorem 3.1, shows that, in contrast to Proposition 5.1, pairs $p \subsetneq q$ of primes with $p \leq_{\mathrm{rk}} q$ do exist, and, furthermore, such pairs are indicative of proximity to non-trivially self-embeddable powers.

5.6. Proposition. (a) *If $f \in \mathcal{P}_p$ induces a non-trivial self-embedding of $\mathbb{N}[p]$, then for any hinged prime $q \subseteq p$ with $q \ni \mathrm{dom}\, f$ one has $f_*(q) \subsetneq q$.*

(b) *If $f \in \mathcal{P}_q$ satisfies $f_*(q) \subsetneq q$, then there is a prime $p \supseteq q$ such that f induces a non-trivial self-embedding of $\mathbb{N}[p]$.*

Proof. (a) We have $f_*(p) = p$ by Corollary 5.2(a), hence $f_*(q) \subseteq p$ by Lemma 4.7(a). Note that $f \equiv_q \mathrm{id}$ cannot hold, for that would imply $\mathbb{N}[q] \models f(\boldsymbol{x}) = \boldsymbol{x}$, and then $\mathbb{N}[p] \models f(\boldsymbol{x}) = \boldsymbol{x}$. By Fact 2.2, $f_*(q)$ is comparable with q w.r.t. $\subseteq$. We cannot have $f_*(q) \supseteq q$, as that eventuality would have f induce a non-trivial self-embedding of $\mathbb{N}[q]$ via Lemma 2.15(a), contradicting Corollary 4.29. Thus $f_*(q) \subsetneq q$.

(b) Let $q_0 = f_*(q)$ and $q_1 = q$. Suppose we have constructed $(q_i)_{i \le n}$ satisfying $q_{j+1} \supseteq q_j = f_*(q_{j+1})$ for all $j < n$. Since $f_*(q_n) = q_{n-1} \subseteq q_n$, Lemma 4.7(c) furnishes a prime $q_{n+1} \supseteq q_n$ with $f_*(q_{n+1}) = q_n$. Let $p = \bigcup_{n \in \omega} q_n$. Then

$$f_*(p) = f_*\left(\bigcup_{n\in\omega} q_n\right) = f_*(q) \cup \bigcup_{n\in\omega} q_n = p \supseteq q$$

because f_* commutes with unions of chains by Lemma 4.7(b). By Lemma 2.15(a), f induces a self-embedding ι of $\mathbb{N}[p]$. Since $f_*(q) \neq q$, one has $f \not\equiv_q \mathrm{id}$, hence $f \not\equiv_p \mathrm{id}$ so ι cannot be trivial by Corollary 5.4(a). ∎

5.7. Remark. In the situation of Proposition 5.6(a) with $f_*(q)$ properly included in q (as instantiated by Theorem 3.1), we have by Lemma 2.15 that $\boldsymbol{x}_{f_*(q)} \mapsto f(\boldsymbol{x}_q)$ extends to a Σ_1-elementary embedding $\mathbb{N}[f_*(q)] \to \mathbb{N}[q]$, whereas, according to Corollary 2.16, the generator-to-geneator embedding $\nu : \boldsymbol{x}_{f_*(q)} \mapsto \boldsymbol{x}_q$ is not Σ_1-elementary.

5.8. Lemma. *Suppose $M \ni a$ and $K \supseteq M$ both model* TA_2*, and X and Y are r.e. sets such that $M \models a \in X \wedge a \notin Y$ while $K \models a \in X \cap Y$.*

Then $K \models \mu t[a \in X_t] \ll \mu t[a \in Y_t]$.

Proof. Towards contradiction, suppose $K \models \mu t[a \in Y_t] \leqslant \mu t[a \in X_t]$. Then $K \models a \in Y \searrow_g X$ for an appropriate total recursive g. Now recall that $\{ Y \searrow_g X, X - Y \searrow_g X \}$ is an r.e. splitting of X. The possibility $M \models a \in Y \searrow_g X$ cannot materialize, for $M \models a \notin Y$. Hence $M \models a \in X - Y \searrow_g X$. But since $X - Y \searrow_g X$ is an r.e. set, this situation persists to K and contradicts $K \models a \in Y \searrow_g X$. ∎

When f induces a non-trivial self-embedding of $\mathbb{N}[p]$, we now show that the element $f(\boldsymbol{x})$ eventually falls behind $\boldsymbol{x}$ in the race to enter r.e. sets — remember that in $\mathbb{N}[p]$, $\boldsymbol{x}$ and $f(\boldsymbol{x})$ are elements of exactly the same r.e. sets (Corollary 5.2(a)).

5.9. Lemma. *Suppose $f \in \mathcal{P}_p$ induces a non-identity self-embedding ι of $\mathbb{N}[p]$, $X \in p$, and $\mathbb{N}[p] \models \mu t[f_t(\boldsymbol{x})\downarrow] \leqslant \mu t[\boldsymbol{x} \in X_t]$. Then*

(a) *$\mathbb{N}[p] \models \mu t[\boldsymbol{x} \in X_t] \ll \mu t[f(\boldsymbol{x}) \in X_t] = \iota(\mu t[\boldsymbol{x} \in X_t])$;*

(b) *$f_*(p{\upharpoonright}_{f^{-1}[X]}) = p{\upharpoonright}_X$. (Note that $f^{-1}[X] \in p$ because $X \in p = f_*(p)$.)*

Proof. (a) $\mu t[f(\boldsymbol{x}) \in X_t] = \iota(\mu t[\boldsymbol{x} \in X_t])$ follows at once from Lemma 2.15(a).

Next observe that $\mathbb{N}[p] \models \boldsymbol{x} \in X$, $f_*(p) = p$ (Corollary 5.4(a)), $p{\upharpoonright}_X$ is hinged, and $\mathrm{dom}\, f \in p{\upharpoonright}_X$ because $\mathrm{dom}\, f \searrow_g X \in p$ for some total recursive g. Hence Proposition 5.6(a) yields $f_*(p{\upharpoonright}_X) \subsetneq p{\upharpoonright}_X$. Therefore $X \notin f_*(p{\upharpoonright}_X)$, or, equivalently, $\mathbb{N}[p{\upharpoonright}_X] \models \boldsymbol{x} \notin f^{-1}[X]$. Applying Lemma 5.8 with $K = \mathbb{N}[p]$, $M = \mathbb{N}[p{\upharpoonright}_X]$ embedded into K as in Corollary 2.16, $a = \boldsymbol{x}$, and $Y = f^{-1}[X]$, we obtain $\mathbb{N}[p] \models \mu t[\boldsymbol{x} \in X_t] \ll \mu t[\boldsymbol{x} \in Y_t] \approx \mu t[\boldsymbol{x} \in f_t^{-1}[X_t]]$ because $(f_t^{-1}[X_t])_{t\in\omega}$ is an enumeration of $f^{-1}[X] = Y$. In $\mathbb{N}[p]$ one has

$$(\dagger) \quad \mu t\left[\boldsymbol{x} \in f_t^{-1}[X_t]\right] = \max\left\{ \mu t\left[f_t(\boldsymbol{x})\downarrow\right], \mu t\left[f(\boldsymbol{x}) \in X_t\right] \right\} = \mu t\left[f(\boldsymbol{x}) \in X_t\right],$$

for $\mu t[f_t(\boldsymbol{x})\downarrow] \leqslant \mu t[\boldsymbol{x} \in X_t]$ by assumption and $\mu t[\boldsymbol{x} \in X_t] \ll \mu t[\boldsymbol{x} \in f_t^{-1}[X_t]]$ as we have just shown. All in all, $\mathbb{N}[p] \models \mu t[\boldsymbol{x} \in X_t] \ll \mu t[f(\boldsymbol{x}) \in X_t]$ as promised.

(b) For an arbitrary r.e. set Y, the following equvalences hold:

$$
\begin{aligned}
Y \in p\lceil_X \;&\Leftrightarrow\; Y \lesssim_p X && \text{(by Lemma 4.19)}\\
&\Leftrightarrow\; \mathbb{N}[p] \models \mu t\,[\boldsymbol{x} \in Y_t] \lesssim \mu t\,[\boldsymbol{x} \in X_t]\\
&\Leftrightarrow\; \mathbb{N}[p] \models \mu t\,\big[f(\boldsymbol{x}) \in Y_t\big] \lesssim \mu t\,\big[f(\boldsymbol{x}) \in X_t\big] && \text{(for } f(\boldsymbol{x}) = \iota(\boldsymbol{x}) \text{ and } \iota \text{ is } \Sigma_1 \text{ elementary)}\\
&\Leftrightarrow\; \mathbb{N}[p] \models \mu t\,\big[\boldsymbol{x} \in f_t^{-1}[Y_t]\big] \lesssim \mu t\,\big[f(\boldsymbol{x}) \in X_t\big]
\end{aligned}
$$

(as $\mu t[\boldsymbol{x} \in f_t^{-1}[Y_t]] = \max\{\mu t[f_t(\boldsymbol{x})\downarrow], \mu t[f(\boldsymbol{x}) \in Y_t]\}$ while $\mu t[f_t(\boldsymbol{x})\downarrow] \ll \mu t[f(\boldsymbol{x}) \in X_t]$ by (a))

$$
\begin{aligned}
&\Leftrightarrow\; \mathbb{N}[p] \models \mu t\,\big[\boldsymbol{x} \in f_t^{-1}[Y_t]\big] \lesssim \mu t\,\big[\boldsymbol{x} \in f_t^{-1}[X_t]\big] && \text{(by (†))}\\
&\Leftrightarrow\; f^{-1}[Y] \lesssim_p f^{-1}[X]
\end{aligned}
$$

(since $(f_t^{-1}[Y_t])_{t\in\omega}$ and $(f_t^{-1}[X_t])_{t\in\omega}$ are enumerations of $f^{-1}[Y]$ and $f^{-1}[X]$ resp.)

$$
\begin{aligned}
&\Leftrightarrow\; f^{-1}[Y] \in p\lceil_{f^{-1}[X]} && \text{(by Lemma 4.19)}\\
&\Leftrightarrow\; Y \in f_*\big(p\lceil_{f^{-1}[X]}\big). && \blacksquare
\end{aligned}
$$

The next corollary strengthens Corollary 5.2(a).

5.10. Corollary. *Each self-embedding of any r.e. prime power is cofinal.*

Proof. Let f induce a self-embedding ι of $\mathbb{N}[p]$ which we may assume to be non-trivial. Consider an arbitrary element $g(\boldsymbol{x}) \geq \mu t[f_t(\boldsymbol{x})\downarrow]$ of $\mathbb{N}[p]$, and let $X = \operatorname{dom} g$. Then by Lemma 5.9(a),

$$\mathbb{N}[p] \models g(\boldsymbol{x}) \leq \mu t\,\big[g_t(\boldsymbol{x})\downarrow\big] \approx \mu t\,[\boldsymbol{x} \in X_t] \ll \mu t\,\big[f(\boldsymbol{x}) \in X_t\big] = \iota\big(\mu t\,[\boldsymbol{x} \in X_t]\big). \qquad \blacksquare$$

Lemma 5.9(a) tells us that $\iota(a) \gg a$ for each a that belongs to a sufficiently large sky that witnesses the entrance of $\boldsymbol{x}$ into some r.e. set — these are exactly the $\mathbb{N}[p]$-skies in the range of the embedding of $\mathcal{S}_p$ from Lemma 4.23. This does not generally cover all sufficiently large elements of $\mathbb{N}[p]$.

5.11. Question. Suppose f induces a non-trivial self-embedding ι of $\mathbb{N}[p]$. Must $\mathbb{N}[p] \models \iota(a) \gg a$ hold for all sufficiently large $a \in \mathbb{N}[p]$?

5.12. Proposition. *Suppose $f \in \mathcal{P}_p$ induces a non-identity self-embedding of $\mathbb{N}[p]$. Then*

(a) $\mathbb{N}[p] \models f^n(\boldsymbol{x})\downarrow$ *for all $n \in \omega$;*

(b) $\mathbb{N}[p] \models \boldsymbol{x} \ll \mu t[f_t(\boldsymbol{x})\downarrow]$. *Hence f cannot be total recursive;*

(c) $\mathbb{N}[p] \models \mu t[f_t(f^n(\boldsymbol{x}))\downarrow] \ll \mu t[f_t(f^{n+1}(\boldsymbol{x}))\downarrow]$ *for all $n \in \omega$;*

(d) $\mathbb{N}[p] \models \mu t[f_t(f^n(\boldsymbol{x}))\downarrow] \approx \mu t[\boldsymbol{x} \in X_t]$ *for $X = \operatorname{dom} f^{n+1}$ and all $n \in \omega$;*

(e) $\operatorname{dom} f^n \ll_p \operatorname{dom} f^{n+1}$ *for all $n > 0$;*

(f) $f_*(p\lceil_{\operatorname{dom} f^{n+1}}) = p\lceil_{\operatorname{dom} f^n}$ *for all $n > 0$.*

Proof. Let ι be the self-embedding induced by f.

(a) Assuming $\mathbb{N}[p] \models f^n(\boldsymbol{x})\downarrow$, we have $\mathbb{N}[p] \models f^n(\iota(\boldsymbol{x}))\downarrow$ by Σ_1 persistence. As $\iota(\boldsymbol{x}) = f(\boldsymbol{x})$, the claim follows.

(b) If $\mathbb{N}[p] \models \mu t[f_t(\boldsymbol{x})\downarrow] \lesssim \boldsymbol{x}$, then $\mathbb{N}[p] \models f(\boldsymbol{x}) = g(\boldsymbol{x}) \neq \boldsymbol{x}$ for some total recursive g, so g also induces ι. By Corollary 5.2(a) we have $\mathrm{tp}_{\Sigma_1}^{\mathbb{N}[p]}\, g(\boldsymbol{x}) = p = \mathrm{tp}_{\Sigma_1}^{\mathbb{N}[p]}\, \boldsymbol{x}$. Therefore, since g is total recursive, $\mathrm{tp}_{\Delta_1}^{\mathbb{N}[p^\circ]}\, g(\boldsymbol{x}) = \mathrm{tp}_{\Delta_1}^{\mathbb{N}[p]}\, g(\boldsymbol{x}) = \mathrm{tp}_{\Delta_1}^{\mathbb{N}[p]}\, \boldsymbol{x} = \mathrm{tp}_{\Delta_1}^{\mathbb{N}[p^\circ]}\, \boldsymbol{x}$ (Definition 2.5). By Corollary 2.17, g induces a self-embedding ε of the recursive ultrapower $\mathbb{N}[p^\circ]$. Since $g(\boldsymbol{x}) \neq \boldsymbol{x}$ is absolute for the natural embedding $\mathbb{N}[p^\circ] \to \mathbb{N}[p]$ from Lemma 2.6 and Corollary 2.16, the embedding ε is not trivial. But this contradicts Theorem 1.6.

(c) Apply Lemma 5.9(a) with $X = \mathrm{dom}\, f$ to see that

$$\mathbb{N}[p] \models \mu t \left[f_t(\boldsymbol{x})\downarrow\right] \approx \mu t \left[\boldsymbol{x} \in X_t\right] \ll \mu t \left[f(\boldsymbol{x}) \in X_t\right] \approx \mu t \left[f_t\big(f(\boldsymbol{x})\big)\downarrow\right].$$

Assuming $\mathbb{N}[p] \models \mu t[f_t(f^n(\boldsymbol{x}))\downarrow] \ll \mu t[f_t(f^{n+1}(\boldsymbol{x}))\downarrow]$, the same must hold with $\iota(\boldsymbol{x}) = f(\boldsymbol{x})$ substituted for $\boldsymbol{x}$ since ι is a self-embedding. This yields $\mathbb{N}[p] \models \mu t[f_t(f^{n+1}(\boldsymbol{x}))\downarrow] \ll \mu t[f_t(f^{n+2}(\boldsymbol{x}))\downarrow]$.

(d) is immediate for $n = 0$ by Lemma 4.16. With $X_n = \mathrm{dom}\, f^n$ for $n > 0$, it follows by induction from clause (c) that

$$\mathbb{N}[p] \models \mu t \left[\boldsymbol{x} \in X_{n+1,t}\right] \approx \max\left\{ \mu t \left[\boldsymbol{x} \in X_{n,t}\right], \mu t \left[f_t\big(f^n(\boldsymbol{x})\big)\downarrow\right] \right\} = \mu t \left[f_t\big(f^n(\boldsymbol{x})\big)\downarrow\right].$$

(e) is a consequence of clauses (c) and (d).

(f) Let $X = \mathrm{dom}\, f^n$. In view of clause (c), one has $\mathbb{N}[p] \models \mu t[f_t(\boldsymbol{x})\downarrow] \lesssim \mu t[\boldsymbol{x} \in X_t]$. Hence Lemma 5.9(b) yields $f_*(p\lceil_{\mathrm{dom}\, f^{n+1}}) = f_*(p\lceil_{f^{-1}[\mathrm{dom}\, f^n]}) = p\lceil_{\mathrm{dom}\, f^n}$. ∎

5.13. Corollary. *If $f \in \mathcal{P}_p$ induces a non-identity self-emdedding of $\mathbb{N}[p]$, then f also induces a (non-identity) self-embedding of $\mathbb{N}[q]$, where $q = \bigcup_{n\in\omega} p\lceil_{\mathrm{dom}\, f^n} \subseteq p$.*

The sequence $(\mu t[f_t(f^n(\boldsymbol{x}))\downarrow])_{n\in\omega}$ is cofinal in $\mathbb{N}[q]$. The sequence $(\mathrm{dom}\, f^n)_{n>0}$ *is cofinal in $\mathcal{S}_q$* (Definition 4.17).

Proof. Just as in the proof of Proposition 5.6(b), we have $f_*(q) = q$ thanks to Proposition 5.12(f). Hence f induces a self-embedding of $\mathbb{N}[q]$.

Cofinality of $(\mu t[f_t(f^n(\boldsymbol{x}))\downarrow])_{n\in\omega}$ in $\mathbb{N}[q]$ follows from Proposition 5.12(c). This is equivalent to the cofinality of $(\mathrm{dom}\, f^n)_{n>0}$ in $\mathcal{S}_q$ by Proposition 5.12(d). ∎

We do not know if the situation $q \subsetneq p$ ever obtains in the setting of Corollary 5.13.

5.14. Question. Can there exist primes $q \subsetneq p$ and an $f \in \mathcal{P}_q$ inducing non-identity self-embeddings of both $\mathbb{N}[q]$ and $\mathbb{N}[p]$?

Equivalently, is there a prime p and $f \in \mathcal{P}_p$ inducing a non-identity self-embedding of $\mathbb{N}[p]$ such that $(\mu t[f_t(f^n(\boldsymbol{x}))\downarrow])_{n\in\omega}$ is not cofinal in $\mathbb{N}[p]$?

A negative answer to Question 5.14 would imply a positive one to Question 5.11.

5.C. A normal form for self-embedding-inducing functions

5.15. Notation. In this subsection, we write $f(x)\downarrow < f(y)\downarrow$ for $\mu t\,[f_t(x)\downarrow] < \mu t\,[f_t(y)\downarrow]$.

From Proposition 5.12(a) and (c) one immediately has

5.16. Corollary. *Suppose $f \in \mathcal{P}_p$ induces a non-identity self-embedding of $\mathbb{N}[p]$. Then $\mathbb{N}[p] \models f(f(\boldsymbol{x}))\downarrow \wedge f(\boldsymbol{x})\downarrow < f(f(\boldsymbol{x}))\downarrow$.* ∎

With the following definition we aim to bring an arbitrary partial recursive function inducing a non-identity self-embedding into a workable tamer form.

5.17. Definition. Let f be any partial recursive function. Define the partial recursive function $\tilde{f}$ by

$$\tilde{f}(x) = y \iff f(x) = y \;\&\; f(y)\downarrow \;\&\; f(x)\downarrow < f(y)\downarrow.$$

Note that $\tilde{f} \subseteq f$ and that the last condition implies $x \neq y$.

The following lemma, which follows immediately from Corollaries 5.16 and 2.8, says that $\tilde{f}$ is as good as f.

5.18. Lemma. *If $f \in \mathcal{P}_p$ induces a self-embedding of $\mathbb{N}[p]$, then $\mathbb{N}[p] \models \tilde{f}(\boldsymbol{x})\downarrow \wedge \tilde{f}(\boldsymbol{x}) = f(\boldsymbol{x})$, i.e. $\tilde{f}$ induces the same self-embedding of $\mathbb{N}[p]$ as f does.* ∎

The next lemma explains why $\tilde{f}$ is better than f. Clause (b) is inspired by the notion of highly recursive graph from Bean [1].

5.19. Lemma. *Let f be any partial recursive function. Then*

(a) *$\tilde{f}$ has no fixed points nor other cycles;*

(b) *for each $y \in \operatorname{dom} \tilde{f} \cup \operatorname{rng} \tilde{f}$, the set $\bar{\bar{y}} = \bigcup_{n\in\omega} \tilde{f}^{-n}[\{y\}]$ is finite and computable in y, as is the restriction of $\tilde{f}$ to $\bar{\bar{y}} - \{y\}$.*

Proof. (a) A fixed point or other cycle $\tilde{f}^n(x) = x$ $(n > 0)$ leads to $f(x)\downarrow < f(x)\downarrow$ according to the definition of $\tilde{f}$.

(b) Observe that $y \in \operatorname{dom} \tilde{f} \cup \operatorname{rng} \tilde{f}$ implies $f(y)\downarrow$ and that for $x \in \bar{\bar{y}} - \{y\}$ we must have $f(x)\downarrow < f(y)\downarrow$, so there are at most finitely many such x (see Convention 4.13). All these x together with the values of f can be found by bounded search below $\mu t\,[f_t(y)\downarrow]$. ∎

6. Relatively maximal r.e. prime powers are totally rigid

In this section we establish

6.1. Theorem. *Suppose P is r.e. and the prime q is maximal w.r.t. the property $P \notin q$. Then $\mathbb{N}[q]$ is totally rigid.*

The r.e. prime powers $\mathbb{N}[q]$ with q satisfying the premiss of the theorem for some r.e. P are the *relatively maximal* r.e. prime powers of the section title. With $P = \varnothing$, this includes r.e. ultrapowers as a particular case:

6.2. Corollary. *All r.e. ultrapowers are totally rigid.* ■

The case of single-sky r.e. ultrapowers — these correspond to minimax primes q — is already covered by McLaughlin's Theorem 1.6, because these are isomorphic to recursive ultrapowers (Lemma 2.6). The corollary fills in a blank pointed out by Hirschfeld & Wheeler [18, 9.6(iii)].

6.A. A large diagonal intersection

The following lemma presents a version of diagonal intersection of a uniformly r.e. family with a largeness property (ii) resembling simplicity (of Z in the "meet" of all $X_i - P$). The function I is a free by-product of the construction for Z, but its role in the proof of the key Proposition 6.7 below will be rather crucial.

6.3. Lemma. *Let P be an r.e. set and $(X_i)_{i\in\omega}$ a uniformly r.e. downward chain, i.e. $X_{i+1} \subseteq X_i$ for all i. There exists an r.e. set Z such that*

(i) *for all i, $Z - P \subseteq^* X_i - P$, and*

(ii) *for each r.e. set U, $U \cap Z - P$ is infinite if and only if $U \cap X_i - P$ is infinite for all i.*

Furthermore, there is a recursive function I with $\operatorname{dom} I = Z$ *such that*

(iii) *the restriction $I|_{Z-P}$ is one-to-one, and*

(iv) *if $x \in Z$, then $x \in X_{I(x)}$.*

Proof. Fix a conventional uniformly r.e. family $(W_i)_{i\in\omega}$ of all r.e. sets.

In the beginning, call each index i *open*. Simultaneously for all open i, look for the first (in a fixed uniform enumeration) element in $W_i \cap X_i$ which has not yet been enumerated into P. As soon as such an element x is found, *associate* x to i, call the index i *closed* (i.e., not open), enumerate x into Z, and, finally, put $I(x) = i$ unless $I(x)$ has been defined earlier. If the number x associated to i is enumerated into P at some later stage, we *dissociate* x from i, and i becomes open again, hungry for a fresh element of $W_i \cap X_i$.

The set Z is clearly r.e., and I is one-to-one on $Z - P$ because at most one number can stay forever associated to a given index. The assignment $I(x) = i$ is only possible when $x \in X_i$, so condition (iv) is also met. Each element x of $Z - X_i$ must satisfy $I(x) < i$, so by condition (iii) there are at most i many elements in $(Z - X_i) - P$ — this takes care of condition (i).

As for condition (ii), supposing $U \cap Z - P$ is infinite, $U \cap X_i - P$ must also be infinite for each i by condition (i). Conversely, suppose $U \cap Z - P$ is finite. Let i be such that $W_i = U - (Z - P) =^* U$. If $W_i \cap X_i - P$ were non-empty, then one of its elements would have gone into Z, contradicting the choice of i. Thus $W_i \cap X_i - P = \varnothing$, so $U \cap X_i - P$ is finite. ■

The lemma above can be adapted to uniformly r.e. families $(Y_i)_{i\in\omega}$ that are not chains by putting $X_i = Y_0 \cap \cdots \cap Y_i$. In this latter form, Lemma 6.3 is an analogue of Theorem 4 in Lindström [29] on Π_1-*conservative* Σ_1 sentences. In fact, we can rephrase conditions (i) and (ii) of our lemma as saying that the theory $\mathrm{TA}_2 + \boldsymbol{x} > \mathbb{N} + \boldsymbol{x} \notin P + \boldsymbol{x} \in Z$ is a $\Pi_1(\boldsymbol{x})$-conservative extension of $\mathrm{TA}_2 + \boldsymbol{x} > \mathbb{N} + \boldsymbol{x} \notin P + \{\boldsymbol{x} \in X_i\}_{i\in\omega}$. Lemma 6.4 below underscores the similarity. In our context, however, the proof of Lemma 6.3 is much less sophisticated than the one in [29].

6.4. Lemma. *Suppose P and $(X_i)_{i\in\omega}$ satisfy the assumptions of Lemma 6.3, Z is r.e., and the prime q is maximal w.r.t. the property $P \notin q$.*

(a) *If Z satisfies condition (i) of Lemma 6.3, then $Z \in q$ implies $X_i \in q$ for all i.*

(b) *If Z satisfies condition (ii) of Lemma 6.3 and $X_i \in q$ for each i, then $Z \in q$.*

Proof. (a) If $Z \in q$ then $X_i \in q$ as $Z \subseteq^* X_i$ by condition (i) of Lemma 6.3.

(b) Conversely, suppose $Z \notin q$. By the maximality property of q, there is $U \in q$ with $U \cap Z \subseteq P$. Then by condition (ii) of Lemma 6.3, $q \ni U \cap X_i \subseteq^* P$ for some i. Therefore $U \notin q$. The contradiction shows $Z \in q$. ∎

With $P = \varnothing$, Lemma 6.3 becomes

6.5. Corollary. *Let $(X_i)_{i \in \omega}$ be a uniformly r.e. downward chain. Then there exists an r.e. set Z such that*

(i) *for all i, $Z \subseteq^* X_i$, and*

(ii) *for each r.e. set U, $U \cap Z$ is infinite if and only if $U \cap X_i$ is infinite for all i.* ∎

Our application of Lemma 6.3 will consist in acquiring a colouring target.

6.6. Definition. Let A and B be sets and f a function. We say that an k-colour assignment $\chi : A \to k$ is an *f-colouring of A excepting B with k colours* when for each $a, b \in A - B$ satisfying $f(a) = b$ one has $\chi(a) \neq \chi(b)$. Observe that this coheres with Definition 1.7.

The proof of the following proposition is postponed until the next subsection. We are going to see first how it helps with Theorem 6.1.

6.7. Proposition. *Suppose P is r.e. and g is a partial recursive function without fixed points or other cycles with the property that for each $y \in \operatorname{dom} g \cup \operatorname{rng} g$, the set $\bar{\bar{y}} = \bigcup_{n \in \omega} g^{-n}[\{y\}]$ is finite and computable in y.*

Then there exists an r.e. set Q such that

(i) *$U \cap Q - P$ is infinite whenever U is r.e. and $U \cap \operatorname{dom} g^n - P$ is infinite for all $n \in \omega$, and*

(ii) *There is a recursive g-colouring χ of Q excepting P with 3 colours.*

6.8. Proof of Theorem 6.1 (modulo Proposition 6.7). Suppose f is a partial recursive function inducing a non-identity self-embeddig of $\mathbb{N}[q]$ with q maximal w.r.t. $P \notin q$. By Lemma 5.18, this self-embedding is also induced by $\tilde{f}$. By Lemma 5.19 and Proposition 6.7, there is a recursive $\tilde{f}$-colouring χ of an r.e. set Q excepting P with 3 colours s.t. $Q - P$ has infinite intersection with each r.e. set which has infinite intersections with each of $\operatorname{dom} \tilde{f}^n - P$. For all n, we have $q \ni \operatorname{dom} \tilde{f}^n$ by Proposition 5.12(a). By Lemma 6.4(b), $Q \in q$, so $\mathbb{N}[q] \models \{\boldsymbol{x}, \tilde{f}(\boldsymbol{x})\} \subseteq Q - P$ by Lemma 2.10(a). Since

$$\forall x, y \in Q - P \left(\tilde{f}(x) = y \to \chi(x)\!\downarrow \wedge\, \chi(y)\!\downarrow \wedge\, \chi(x) \neq \chi(y)\right)$$

is a true Π_2 statement, in $\mathbb{N}[q]$ the colour of $\boldsymbol{x}$ cannot coincide with the colour of $\tilde{f}(\boldsymbol{x})$, so $\tilde{f}$ cannot give rise to a self-embedding in view of Corollary 5.2(a). By Lemma 5.18, neither can f. We have therefore shown that no non-identity self-embeddings of $\mathbb{N}[q]$ exist. ∎

6.B. Painter's revanche

This subsection is devoted to a proof of Proposition 6.7. Our construction of the recursive colouring can be seen as a (recursive) winning strategy for Painter in an appropriate on-line game.

Assume that the partial recursive g satisfies the assumptions of that Proposition. We are going to talk about the graph of g using directed graph terminology. Consider the graph Γ on ω where there is a directed edge from x to y just in case $g(x) = y$. As g is a function, each vertex is incident on at most one outbound edge. Since g has no cycles, Γ is tree-like. Say that x lies *upstream of* y if $x \in \overleftarrow{y}$. In the same situation, y lies *downstream of* x. The *upstream closure* $\overleftarrow{X}$ of a set X is defined by $\overleftarrow{X} = \bigcup_{n\in\omega} g^{-n}[X]$ where $g^0 = \mathrm{id}$. Note that this coheres with the notation $\overleftarrow{x}$. The finiteness of $\overleftarrow{x}$ for each x enables downstream-inductive definitions and arguments by downstream induction. We also note that since g is partial recursive and the finite set $\overleftarrow{y}$ is recursive in $y \in \operatorname{dom} g \cup \operatorname{rng} g$, the restriction of g to $\overleftarrow{y} - \{y\}$, as well as that of Γ to $\overleftarrow{y}$, must also be recursive in these y. All *colourings* in this subsection are g-colourings (equivalently, Γ-colourings) with 3 colours.

6.9. Construction. Lemma 6.3 applied to the uniformly r.e. downward chain $(\operatorname{dom} g^{1+n})_{n\in\omega}$ supplies us with the r.e. set Q satisfying condition (i) of Proposition 6.7 together with a recursive function I with $\operatorname{dom} I = Q$ such that $x \in \operatorname{dom} g^{1+I(x)}$ for each $x \in Q$, and I is one-to-one on $Q - P$.

We fix effective enumerations $(Q_n)_{n\in\omega}$ and $(P_n)_{n\in\omega}$ of Q and P respectively as increasing chains of finite subsets satisfying the properties

(E1) $Q_0 = P_0 = \varnothing$;

(E2) $|Q_{n+1} - Q_n| + |P_{n+1} - P_n| \leq 1$ *for each* $n \in \omega$;

(E3) I *is one-to-one on* $Q_n - P_n$ *for each* $n \in \omega$.

Property (E3) is achieved by slowing down a given enumeration of Q as follows: When a number x with the same value of I as some number $y \neq x$ already present in $Q_n - P_n$ wishes to enter Q, we know that at least one of x and y is going to eventually enumerate into P because $I|_{Q-P}$ is one-to-one. So we wait for one of x and y to appear in P_m before enumerating x into Q_k for an appropriate $k > m$. Any other candidates for membership in Q with the same value of I await their turn in a first-in-first-out queue.

6.10. Construction. We define an increasing sequence $(F_i)_{i\in\omega}$ of finite sets by

$$F_0 = \varnothing \quad \text{and} \quad F_{n+1} = \begin{cases} \overleftarrow{g^{1+I(x)}(x)} \cup F_n & \text{if } \{x\} = Q_{n+1} - Q_n, \\ F_n & \text{if } Q_{n+1} = Q_n. \end{cases}$$

The r.h.s. is always well-defined thanks to the properties of I. Induction on n shows that F_n is upstream-closed. F_n is therefore a finite disjoint union of finite upstream-closed subtrees of Γ. Call the roots of those subtrees F_n*-sinks*. In other words, F_n-sinks are the downstream-most elements of F_n.

By the assumptions of Proposition 6.7 on g and since $g^{1+I(x)}(x) \in \operatorname{rng} g$, F_n is a recursive function of n, as is the restriction of Γ to F_n.

Summarizing informally, the latest addition x to Q_{n+1} is decorated by a downstream tail of edge-length $1 + I(x)$. The end of this tail is $g^{1+I(x)}(x)$. All the structure upstream of the tail's end is recursive in n and goes into F_{n+1}.

Note that $Q_n \subseteq F_n$ for all n as $x \in \overleftarrow{g^{1+I(x)}(x)}$.

We now define a numerical characteristic which will aid further construction:

6.11. Construction. For $X, Y \subseteq \omega$ and $y \in \omega$, define the *congestion grade* $c(y, X, Y)$ *of* X *at* y *excepting* Y by downstream recursion:

$$c(y,X,Y) = \begin{cases} 0 & \text{if } y \in Y, \\ 1 & \text{if } y \in X - Y, \\ \frac{1}{2} \cdot \sum_{g(x)=y} c(x,X,Y) & \text{otherwise.} \end{cases}$$

The values of c are always non-negative (induction). Note that $c(y, X, Y)$, when restricted to $y \in F_n$ and finite X, Y, is a recursive function of n, y, X, and Y.

6.12. Lemma. (a) $c(y, X \cup Z, Y) \le c(y, X, Y) + c(y, Z, Y)$.

(b) $c(y, X, Y) \le c(y, X - Y, \varnothing)$.

(c) *If* $\overleftarrow{y} \cap X = \overleftarrow{y} \cap Z$*, then* $c(y, X, Y) = c(y, Z, Y)$.

(d) *If* $y \notin \overleftarrow{X}$*, then* $c(y, X, Y) = c(y, \overleftarrow{X}, Y)$.

(e) *If* $c(z, X, Y) \ge 1$ *for all* $z \in Z$*, then* $c(y, X \cup Z, Y) \le c(y, X, Y)$.

(f) *If* $x \in \overleftarrow{y}$*, then* $c(y, \{x\}, \varnothing) = 2^{-d(x,y)}$*, where* d *is the number-of-edges distance.*

Proof. All clauses are established by straightforward (downstream) induction on y. We only handle clause (e) by way of example:

If $y \in Y$, then $c(y, X \cup Z, Y) = c(y, X, Y) = 0$.
If $y \in X - Y$, then $c(y, X \cup Z, Y) = c(y, X, Y) = 1$.
If $y \in Z - Y$, then $c(y, X \cup Z, Y) = 1 \le c(y, X, Y)$ by the property of Z.
Finally, when $y \notin X \cup Z \cup Y$, the induction hypothesis yields

$$c(y, X \cup Z, Y) = \frac{1}{2} \sum_{g(x)=y} c(x, X \cup Z, Y) \le \frac{1}{2} \sum_{g(x)=y} c(x, X, Y) = c(y, X, Y). \qquad \blacksquare$$

6.13. Lemma. *If* w *is an* F_n*-sink, then*

$$c(w, Q_n, P_n) \le \sum_{x \in \overleftarrow{w} \cap Q_n - P_n} 2^{-1-I(x)}.$$

Proof. When $x \in \overleftarrow{w} \cap Q_n$, one has $c(w, \{x\}, \varnothing) \le 2^{-1-I(x)}$ by Lemma 6.12(f) because $g^{1+I(x)} \in \overleftarrow{w}$, so $d(x, w) \ge 1 + I(x)$. Therefore

$$\begin{aligned} c(w, Q_n, P_n) &\le c(w, Q_n - P_n, \varnothing) && \text{(by Lemma 6.12(b))} \\ &= c(w, \overleftarrow{w} \cap Q_n - P_n, \varnothing) && \text{(by Lemma 6.12(c))} \\ &\le \sum_{x \in \overleftarrow{w} \cap Q_n - P_n} c\big(w, \{x\}, \varnothing\big) && \text{(by Lemma 6.12(a))} \\ &\le \sum_{x \in \overleftarrow{w} \cap Q_n - P_n} 2^{-1-I(x)}. && \blacksquare \end{aligned}$$

6.14. LEMMA. *Suppose ψ is a g-colouring of an finite upstream-closed set X excepting Y, $Z \supseteq X$ is a finite upstream-closed set, and $c(z, X, Y) < 1$ for all $z \in Z - X$.* (Recall that all colourings only use 3 colours.)

Then ψ extends to a g-colouring of Z excepting Y.

PROOF. We are going to show by downstream induction that for each $y \in Z - X$ there is an extension υ_y of ψ to a g-colouring of $\overleftarrow{y} \cup X$ excepting Y. To keep the induction going, we require a stronger conclusion, namely,

> *There are two extensions of ψ to g-colourings of $\overleftarrow{y} \cup X$ excepting Y, assigning distinct colours to y.*

As $c(y, X, Y) < 1$, either $y \in Y$, in which case any of the three colours can be assigned to y, or there is at most one $z \in X - Y$ with $g(z) = y$ (so $z \in \operatorname{dom} \psi$ and $c(z, X, Y) = 1$). Assign to y one of the colours distinct from $\psi(z)$ if such z is indeed present — note that this gives at least two choices. By the induction hypothesis, for each $u \notin X$ with $g(u) = y$ there is a g-colouring of $\overleftarrow{u} \cup X$ excepting Y extending ψ which assigns to u a colour distinct from the one that we have just assigned to y. Use it to patch together the extension υ_y of ψ to $\overleftarrow{y} \cup X$ excepting Y.

If S is the set of downstream-most elements of Z, then $\bigcup_{x \in S} \upsilon_x$ extends ψ to a g-colouring of Z excepting Y because the graph Γ is tree-like. ∎

6.15. CONSTRUCTION. We construct a recursive upward chain $(\chi_n)_{n\in\omega}$ where each χ_n is a colour assignment $\overleftarrow{Q}_n \to 3$. The intention is to have the desired g-colouring of Q excepting P equal to the restriction of $\bigcup_{n\in\omega} \chi_n$ to Q. We shall find it convenient to cultivate an auxiliary sequence $(\upsilon_n)_{n\in\omega}$ where $\upsilon_n \supseteq \chi_n$ is a colour assignment $F_n \to 3$. Accordingly, we maintain the inductive conditions

(I1(n)) *χ_n is a g-colouring of $\overleftarrow{Q}_n$ excepting P_n;*

(I2(n)) *$\chi_n \supseteq \chi_m$ for all $m < n$;*

(I3(n)) *υ_n is a g-colouring of F_n excepting P_n;*

(I4(n)) *$\upsilon_n \supseteq \chi_n$.*

The role of υ_n will be both to ensure a measure of consistency in χ_n and to serve as a possible sketch for (fragments of) χ_{n+1}. We do *not* require $\upsilon_{n+1} \supseteq \upsilon_n$.

Put $\chi_0 = \upsilon_0 = \varnothing$. Conditions (I1–4(0)) hold by virtue of (E1).

The construction of χ_{n+1} and υ_{n+1} distinguishes three cases:

Case 1: $Q_{n+1} = Q_n$.

In this case we have $F_{n+1} = F_n$. We put $\chi_{n+1} = \chi_n$ and $\upsilon_{n+1} = \upsilon_n$, and note that any colouring excepting P_n is also a colouring excepting $P_{n+1} \supseteq P_n$, so (I1–4($n + 1$)) hold.

Case 2: $Q_{n+1} \neq Q_n$ and $F_{n+1} = F_n$.

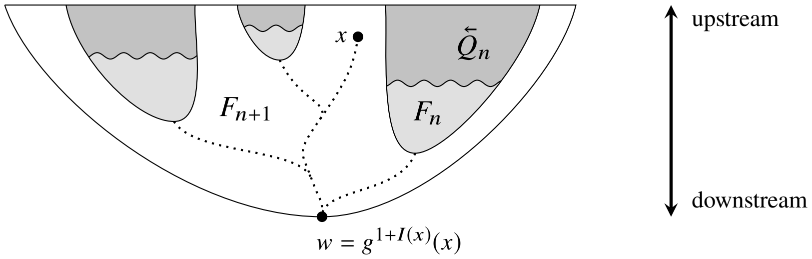
Note that $P_{n+1} = P_n$ by (E2).

Put $\chi_{n+1} = \upsilon_n|_{\overleftarrow{Q}_{n+1}}$ and $\upsilon_{n+1} = \upsilon_n$.

(I4($n + 1$)) is clearly met. Further, χ_{n+1} extends χ_n because, according to (I4(n)), υ_n does — this takes care of (I2($n + 1$)). Conditions (I1($n + 1$)) and (I3($n + 1$)) follow at once from (I3(n)).

Case 3: $Q_{n+1} - Q_n = \{x\}$ *and* $F_{n+1} \neq F_n$.

In this case we also have $P_{n+1} = P_n$ by (E2), there is a new F_{n+1}-sink $w = g^{1+I(x)}(x)$ which is not an F_n-sink, and $F_{n+1} = \overleftarrow{w} \cup F_n$.



Put

$$C_n = \left\{ z \in F_n \,\middle|\, c(z, Q_n, P_n) \geq 1 \right\} \quad \text{and} \quad \psi_n = \upsilon_n|_{\overleftarrow{C}_n \cup \overleftarrow{Q}_n} \supseteq \chi_n.$$

Then ψ_n is a g-colouring of $\overleftarrow{C}_n \cup \overleftarrow{Q}_n \subseteq F_n$ excepting P_n by (I3(n)).

Consider an arbitary $u \in F_{n+1} - (\overleftarrow{C}_n \cup \overleftarrow{Q}_n)$. Observe that

$$\begin{aligned} c\big(u, \overleftarrow{C}_n \cup \overleftarrow{Q}_n, P_n\big) &= c\big(u, C_n \cup \overleftarrow{Q}_n, P_n\big) && \text{(by Lemma 6.12(d) as } \overleftarrow{C}_n \cup \overleftarrow{Q}_n = \overleftarrow{C_n \cup \overleftarrow{Q}_n}\text{)} \\ &\leq c\big(u, \overleftarrow{Q}_n, P_n\big) && \text{(by Lemma 6.12(e))} \\ &= c(u, Q_n, P_n) && \text{(by Lemma 6.12(d)).} \end{aligned}$$

If $u \in F_n - (\overleftarrow{C}_n \cup \overleftarrow{Q}_n)$, then $c(u, Q_n, P_n) < 1$ as $u \notin C_n$. If $u \in F_{n+1} - F_n$, then

$$\begin{aligned} c(u, Q_n, P_n) \leq \frac{1}{2} \sum_{F_n\text{-sinks } v \in \overleftarrow{u}} c(v, Q_n, P_n) &\leq \sum_{F_n\text{-sinks } v \in \overleftarrow{u}} c(v, Q_n, P_n) \\ &\leq \sum_{z \in \overleftarrow{u} \cap Q_n - P_n} 2^{-1-I(z)} && \text{(by Lemma 6.13)} \\ &< 1 && \text{(by (E3)).} \end{aligned}$$

Thus $c(u, \overleftarrow{C}_n \cup \overleftarrow{Q}_n, P_n) < 1$ holds for all $u \in F_{n+1} - (\overleftarrow{C}_n \cup \overleftarrow{Q}_n)$. To accomodate (I3($n+1$)), we may therefore use Lemma 6.14 to produce the extension υ_{n+1} of ψ_n (and hence of χ_n) to a g-colouring of F_{n+1} excepting $P_{n+1} = P_n$. We put $\chi_{n+1} = \upsilon_{n+1}|_{\overleftarrow{Q}_{n+1}} \supseteq \chi_n$, which takes care of (I1($n+1$)), (I2($n+1$)), and (I4($n+1$)).

The construction is effective because the restriction of the graph of g to F_n, as well as the restriction of the function c to F_n and subsets of F_n, are recursive uniformly in n.

6.16. Proof of Proposition 6.7 concluded. Since $(\chi_n)_{n \in \omega}$ from Consruction 6.15 is a recursive chain of g-colourings of $\overleftarrow{Q}_n$ excepting P_n, its union $\chi = \bigcup_{n \in \omega} \chi_n$ is a recursive 3-colour assignment to $\overleftarrow{Q}$, and for all $x, y \in \overleftarrow{Q} - P$ with $g(x) = y$ there is an $m \in \omega$ such that $x, y \in \overleftarrow{Q}_m - P_m$, so $\chi(x) = \chi_m(x) \neq \chi_m(y) = \chi(y)$. Therefore the restriction $\chi|_Q$ is a g-colouring of Q excepting P which establishes the remaining condition (ii) of Proposition 6.7. ∎

We close with a couple of questions.

6.17. Question. Suppose the sequence $(P_i)_{i \in \omega}$ is uniformly r.e. and the prime q is maximal w.r.t. the property $P_i \notin q$ for all i. Must $\mathbb{N}[q]$ be totally rigid?

Σ_1 induction implies Σ_1 collection and therefore fails in each r.e. prime power (Proposition 2.11). Hirschfeld & Wheeler [18, Theorem 8.30] show that r.e. *ultra*powers $\mathbb{N}[p]$ nevertheless satisfy (parametric) Σ_1 *overspill for* $\mathbb{N}$: if $\sigma(x, y)$ is Σ_1 and $a \in \mathbb{N}[p] \models \sigma(n, a)$ for each $n \in \omega$, then $\mathbb{N}[p] \models \sigma(c, a)$ for some non-standard $c \in \mathbb{N}[p]$. Σ_1 overspill for $\mathbb{N}$ also holds in $\mathbb{N}[q]$ with relatively maximal q or even with primes q from Question 6.17.

6.18. Question. Do r.e. prime powers satisfying Σ_1 overspill for $\mathbb{N}$ have to be totally rigid?

In the opposite direction, it can be shown that each multi-sky recursive ultrapower, while totally rigid by Theorem 1.6, violates Σ_1 overspill for $\mathbb{N}$.

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