Coalgebraic Geometric Logic

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Abstract

Using the theory of coalgebra, we introduce a uniform framework for adding modalities to the language of propositional geometric logic. Models for this logic are based on coalgebras for an endofunctor T on some full subcategory of the category **Top** of topological spaces and continuous functions. We compare the notions of modal equivalence, behavioural equivalence and bisimulation on the resulting class of models, and we provide a final object for the corresponding category. Furthermore, we specify a method of lifting an endofunctor on **Set**, accompanied by a collection of predicate liftings, to an endofunctor on the category of topological spaces.

1 Introduction

Propositional geometric logic arose at the interface of (pointfree) topology, logic and theoretical computer science as the logic of *finite observations* [1, 27]. Its language is constructed from a set of proposition letters by applying finite conjunctions and arbitrary disjunctions, these being the propositional operations preserving the property of finite observability. Through an interesting topological connection, formulas of geometric logic can be interpreted in the frame of open sets of a topological space. Central to this connection is the well-known adjunction between the category **Frm** of frames and frame morphisms and the category **Top** of topological spaces and continuous maps, which restricts to several interesting Stone-type dualities [15].

Coalgebraic logic is a framework in which generalised versions of modal logics are developed parametric in the signature of the language and a functor $\mathsf{T} : \mathbf{C} \to \mathbf{C}$ on some base category $\mathbf{C}$. With classical propositional logic as base logic, two natural choices for the base category are **Set**, the category of sets and functions, and **Stone**, the category of Stone spaces and continuous functions, i.e. the topological dual to the algebraic category of Boolean algebras. Coalgebraic logic for endofunctors on **Set** has been well investigated and still is an active area of research, see e.g. [8, 20]. In this setting, modal operators can be defined using the notion of relation lifting [22] or predicate lifting [23]. Coalgebraic logic in the category of Stone coalgebras has been studied in [19, 13, 9], and there is a fairly extensive literature on the design of a coalgebraic modal logic based on a general Stone-type duality (or adjunction), see for instance [7] and references therein.

In this paper we investigate some links between coalgebraic logic and geometric logic. That is, we shall use methods from coalgebraic logic to introduce modal operators to the language of geometric logic, with the intention of studying interpretations of these logics in certain topological coalgebras. Note that extensions of geometric logic with the basic modalities $\Box$ and $\Diamond$, which are closely related to the topological Vietoris construction, have received much attention in the literature, see [27] for some early history. A first step towards developing coalgebraic geometric logic was taken in [26], where a method is explored to lift a functor on

Set to a functor on the category **KHaus** of compact Hausdorff spaces, and the connection is investigated between the lifted functor and a relation-lifting based "cover" modality.

Our aim here is to develop a framework for the coalgebraic geometric logics that arise if we extend geometric logic with modalities that are induced by appropriate *predicate liftings*. Guided by the connection between geometric logic and topological spaces, we choose the base category of our framework to be **Top** itself, or one of its full subcategories such as **Sob** (sober spaces), **KSob** (compact sober spaces) or **KHaus** (compact Hausdorff spaces). On this base category **C** we then consider an arbitrary endofunctor T which serves as the type of our topological coalgebras. Furthermore, we shall see that if we want our formulas to be interpreted as *open sets* of the coalgebra carrier, we need the predicate liftings that interpret the modalities of the language to satisfy some natural *openness* condition. Summarizing, we shall study the coalgebraic geometric logic induced by (1) a functor $\mathsf{T} : \mathbf{C} \to \mathbf{C}$, where **C** is a full subcategory of Top, and (2) a set Λ of open predicate liftings for T. As running examples we take the combination of the basic modalities for the Vietoris functor, and that of the monotone box and diamond modalities for various topological manifestations of the monotone neighborhood functor on **Set**. The structures providing the semantics for our coalgebraic geometric logics are the T-*models* consisting of a T-coalgebra together with a valuation mapping proposition letters to open sets in the coalgebra carrier.

The main results that we report on here are the following:

- Section 4 contains a detailed description of a *monotone neighbourhood functor* on **KHaus**, which naturally extends the monotone functor on **Stone** that corresponds to monotone modal logic.
- In section 5 we adapt the method of [17], in order to lift a **Set**-functor together with a collection of predicate liftings to an endofunctor on **Top**. We obtain the Vietoris functor and monotone functor on **KHaus** as restrictions of such lifted functors.
- After that, in section 6, we construct a final object in the category of T-models, where T is an endofunctor on **Top** which preserves sobriety.
- Finally, in section 7 we transfer the notion of Λ-bisimilarity from [10, 2] to our setting, and we compare this to geometric modal equivalence, behavioural equivalence and Aczel-Mendler bisimilarity. Our main finding is that on the categories **Top**, **Sob** and **KSob**, the first three notions coincide, provided Λ and T meet some reasonable conditions.

We finish the paper with listing some questions for further research.

2 Preliminaries

We briefly fix notation and review some preliminaries.

Categories and functors We use a **bold font** for categories. We assume familiarity with the following categories and functors:

- **Set** is the category of sets and functions;
- **Top** is the category of topological spaces and continuous functions;
- **KHaus** and **Stone** are the full subcategories of **Top** whose objects are compact Hausdorff spaces and Stone spaces respectively;

- BA is the category of Boolean algebras and Boolean algebra morphisms.

Categories can be connected by functors. We use this font for functors. In particular, the following functors are regularly used in this paper:

- $\mathsf{U} : \mathbf{Top} \to \mathbf{Set}$ is the forgetful functor sending a topological space to its underlying set. The functor U restricts to every subcategory of $\mathbf{Top}$, in which case we shall abuse notation and also call it U;

- $\mathsf{P} : \mathbf{Set} \to \mathbf{Set}$ and $\breve{\mathsf{P}} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ are the covariant and contravariant powerset functor respectively;

- $\mathsf{Q} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{BA}$ sends a set to its powerset Boolean algebra and a function to the inverse image map viewed as morphism in $\mathbf{BA}$;

- $\Omega : \mathbf{Top} \to \mathbf{Set}$ sends a topological space to the set of opens.

Note that $\breve{\mathsf{P}} = \mathsf{U} \circ \mathsf{Q}$. More categories and functors will be defined along the way.

Coalgebra Let $\mathbf{C}$ be a category and T an endofunctor on $\mathbf{C}$. A T-*coalgebra* is a pair (X, γ) where X is an object in $\mathbf{C}$ and $\gamma : X \to \mathsf{T}X$ is a morphism in $\mathbf{C}$. A T-*coalgebra morphism* between two T-coalgebras (X, γ) and (X', γ') is a morphism $f : X \to X'$ in $\mathbf{C}$ satisfying $\gamma' \circ f = \mathsf{T}f \circ \gamma$. The collection of T-coalgebras and T-coalgebra morphisms forms a category, which we shall denote by $\mathbf{Coalg}(\mathsf{T})$. The category $\mathbf{C}$ is called the *base category* of $\mathbf{Coalg}(\mathsf{T})$.

Example 2.1 (Kripke frames)**.** Kripke frames correspond 1-1 with P-coalgebras. For a Kripke frame (X, R) define $\gamma_R : X \to \mathsf{P}X : x \mapsto \{y \mid xRy\}$. Then (X, γ_R) is a P-coalgebra. Conversely, for a P-coalgebra (X, γ) define R_γ by $xR_\gamma y$ iff $y \in \gamma(x)$. Then (X, R_γ) is a Kripke frame. It is not hard to see that $R_{\gamma_R} = R$ and $\gamma_{R_\gamma} = \gamma$, so we obtain a bijection between Kripke frames and P-coalgebras. Moreover, bounded morphisms between Kripke frames are precisely P-coalgebra morphisms. Thus, we have

$$\mathbf{Krip} \cong \mathbf{Coalg}(\mathsf{P}),$$

where $\mathbf{Krip}$ is the category of Kripke frames and bounded morphisms.

Example 2.2 (Monotone neighbourhood frames)**.** Let $\mathsf{D} : \mathbf{Set} \to \mathbf{Set}$ be the functor given on objects by

$$\mathsf{D}X = \{W \subseteq \mathsf{P}X \mid \text{ if } a \in W \text{ and } a \subseteq b \text{ then } b \in W\},$$

for X a set. For a morphism $f : X \to X'$ define

$$\mathsf{D}f : \mathsf{D}X \to \mathsf{D}X' : W \mapsto \{a' \in \mathsf{P}X' \mid f^{-1}(a') \in W\}.$$

Then the category of monotone frames a bounded morphisms is isomorphic to $\mathbf{Coalg}(\mathsf{D})$ [6, 12, 13].

Coalgebraic logic for Set-coalgebras Let T be a $\mathbf{Set}$-functor and Φ a set of proposition letters. A T-*model* is a triple (X, γ, V) where (X, γ) is a T-coalgebra and $V : \Phi \to \mathsf{P}X$ is a valuation of the proposition letters. An *n-ary predicate lifting for* T is a natural transformation

$$\lambda : \breve{\mathsf{P}}^n \to \breve{\mathsf{P}} \circ \mathsf{T}.$$

Let Λ be a set of predicate liftings for T. Define the *language* $\mathrm{ML}(\Lambda)$ by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n),$$

where $p \in \Phi$ and $\lambda \in \Lambda$ is n-ary. The *semantics* of $\varphi \in \mathrm{ML}(\Lambda)$ on a T-model $\mathfrak{X} = (X, \gamma, V)$ is given recursively by

$$\llbracket p \rrbracket^{\mathfrak{X}} = V(p), \quad \llbracket \varphi_1 \wedge \varphi_2 \rrbracket^{\mathfrak{X}} = \llbracket \varphi_1 \rrbracket^{\mathfrak{X}} \cap \llbracket \varphi_2 \rrbracket^{\mathfrak{X}}, \quad \llbracket \neg\varphi \rrbracket^{\mathfrak{X}} = X \setminus \llbracket \varphi \rrbracket^{\mathfrak{X}},$$
$$\llbracket \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n) \rrbracket^{\mathfrak{X}} = \gamma^{-1}(\lambda(\llbracket \varphi_1 \rrbracket^{\mathfrak{X}}, \ldots, \llbracket \varphi_n \rrbracket^{\mathfrak{X}})),$$

where $p \in \Phi$ and λ ranges over Λ.

Example 2.3 (Kripke models)**.** Consider for P-models the predicate lifting $\lambda : \breve{\mathsf{P}} \to \breve{\mathsf{P}} \circ \mathsf{P}$ given by

$$\lambda_X(a) = \{b \in \mathsf{P}X \mid b \subseteq a\}.$$

Then λ gives the usual box semantics of modal logic, that is, for every P-model $\mathfrak{X} = (X, \gamma, V)$ and state $x \in X$ we have $\mathfrak{X}, x \Vdash \Box\varphi$ iff $\mathfrak{X}, x \Vdash \heartsuit^{\lambda}(\varphi)$.

We refer to [20] for many more examples of coalgebraic logic for **Set**-functors.

Frames and spaces A *frame* is a complete lattice F in which for all $a \in F$ and $S \subseteq F$ the infinite distributive law holds:

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}.$$

A *frame homomorphism* is a function between frames that preserves finite meets and arbitrary joins.

For $a, b \in F$ we say that a is *well inside* b, notation: $a \lessdot b$, if there is a $c \in F$ such that $c \wedge a = \bot$ and $c \vee b = \top$. An element $a \in F$ is called *regular* if $a = \bigvee \{b \in F \mid b \lessdot a\}$ and a frame is called *regular* if all of its elements are regular. The *negation* of $a \in F$ is defined as ${\sim}a = \bigvee \{b \in F \mid a \wedge b = \bot\}$. A frame is said to be *compact* if $\bigvee S = \top$ implies that there is a finite subset $S' \subseteq S$ such that $\bigvee S' = \top$.

Lemma 2.4. *For all elements a, b in a frame F we have $a \lessdot b$ iff ${\sim}a \vee b = \top$.*

Proof. See III1.1 in [15]. □

Lemma 2.5. *Finite meets and arbitrary joins of regular elements are regular.*

Proof. It is known that $d \leq c \lessdot a \leq b$ implies $d \lessdot b$. We first show that $c \lessdot a$ and $d \lessdot b$ implies $c \wedge d \lessdot a \wedge b$. It is clear that $c \wedge d \lessdot a$ and $c \wedge d \lessdot b$. Since ${\sim}(c \wedge d) \vee (a \wedge b) = ({\sim}(c \wedge d) \vee a) \wedge ({\sim}(c \wedge d) \vee b) = \top \wedge \top = \top$ we know $c \wedge d \lessdot a \wedge b$.

Now suppose a and b are regular elements, then

$$a \wedge b = \bigvee \{c \mid c \lessdot a\} \wedge \bigvee \{d \mid d \lessdot b\} = \bigvee \{c \wedge d \mid c \lessdot a, d \lessdot b\} \leq \bigvee \{c \mid c \lessdot a \wedge b\} \leq a \wedge b,$$

so $a \wedge b$ is regular. If a_i is regular for all i in some index set I, then

$$\bigvee_{i \in I} a_i = \bigvee_{i \in I} \Big(\bigvee \{c \mid c \lessdot a_i\} \Big) \leq \bigvee \Big\{ c \mid c \lessdot \bigvee_{i \in I} a_i \Big\} \leq \bigvee_{i \in I} a_i,$$

so an arbitrary join of regular elements is regular. □

Frames can be presented by generators and relations.

Definition 2.6. A *presentation* is a pair $\langle G, R\rangle$ where G is a set of generators and R is a collection of relations between expressions constructed from the generators using arbitrary joins and finite meets.

Let F be a frame and $\mathsf{Z}F$ its underlying set. We say that $\langle G, R\rangle$ *presents* F if there is an assignment $f : G \to \mathsf{Z}F$ of the generators such that (i), (ii) and (iii) hold:

(i) The set $\{f(g) \mid g \in G\}$ generates F.

The assignment f can be extended to an assignement $\widetilde{f}$ for any expression x build from the generators in G using $\wedge$ and $\bigvee$. We require:

(ii) If $x = x'$ is a relation in R, then $\widetilde{f}(x) = \widetilde{f}(x')$ in F.

(iii) For any F' and assignment $f' : G \to \mathsf{Z}F'$ satisfying property (ii) there exists a frame homomorphism $h : F \to F'$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "f'"'] & \mathsf{Z}F \arrow[d, "\mathsf{Z}h"] \\ & \mathsf{Z}F' \end{tikzcd}$$

commutes.

The frame homomorphism from (iii) is necessarily unique, because the image of the generating set $\{f(g) \mid g \in G\}$ under h is determined by the diagram. A detailed account of frame presentations may be found in chapter 4 of [27].

Remark 2.7. We will regularly want to define a frame homomorphism $F \to F'$ from a frame F presented by $\langle G, R\rangle$ to some frame F'. By definition 2.6 it suffices to give an assignment $f' : G \to F'$ such that (ii) holds, because this yields a unique frame homomorphism $F \to F'$. By abuse of notation, we will denote the unique frame homomorphism $F \to F'$ such that the diagram in (iii) commutes with f' as well.

The next fact allows us to define a frame by specifying generators and relations. A proof can be found in [15, Proposition II2.11].

Fact 2.8. *Any presentation by generators and relations presents a unique frame.*

The collection of open sets of a topological space $\mathcal{X}$ forms a frame, denoted $\mathsf{opn}\mathcal{X}$. A continuous map $f : \mathcal{X} \to \mathcal{X}'$ induces $\mathsf{opn}f = f^{-1} : \mathsf{opn}\mathcal{X}' \to \mathsf{opn}\mathcal{X}$ and with this definition opn is a contravariant functor $\mathbf{Top} \to \mathbf{Frm}$. A frame is called *spatial* if it isomorphic to $\mathsf{opn}\mathcal{X}$ for some topological space $\mathcal{X}$.

A *point* of a frame F is a frame homomorphism $p : F \to 2$, with $2 = \{\top, \bot\}$ the two-element frame. Let $\mathsf{pt}F$ be the collection of points of F endowed with the topology $\{\widetilde{a} \mid a \in F\}$, where $\widetilde{a} = \{p \in \mathsf{pt}F \mid p(a) = \top\}$. For a frame homomorphism $f : F \to F'$ define $\mathsf{pt}f : \mathsf{pt}F' \to \mathsf{pt}F$ by $p \mapsto p \circ f$. The assignment pt defines a functor $\mathbf{Frm} \to \mathbf{Top}$. A topological space that arises as the space of points of a lattice is called *sober*. The *sobrification* of a topological space $\mathcal{X}$ is $\mathsf{pt}(\mathsf{opn}\mathcal{X})$.

We denote by $\mathbf{Sob}$ and $\mathbf{KSob}$ the full subcategories of $\mathbf{Top}$ whose objects are sober spaces and compact sober spaces, respectively. Where $\mathbf{Frm}$ is the category of frames and frame homomorphisms, $\mathbf{SFrm}$, $\mathbf{KSFrm}$ and $\mathbf{KRFrm}$ are the full subcategories of $\mathbf{Frm}$ whose objects are spatial frames, compact spatial frames and compact regular frames, respectively. The functor $\mathsf{Z} : \mathbf{Frm} \to \mathbf{Set}$ is the forgetful functor sending a frame to the underlying set, and restricts to every subcategory of $\mathbf{Frm}$. Note that $\Omega = \mathsf{Z} \circ \mathsf{opn}$.

Fact 2.9. *The functor* pt *is a right adjoint to* opn. *This adjunction restricts to a duality between the category of spatial frames and the category of sober spaces,*

$$\mathbf{SFrm} \equiv \mathbf{Sob}^{\mathrm{op}}.$$

This duality restricts to the dualities

$$\mathbf{KSFrm} \equiv \mathbf{KSob}^{\mathrm{op}}$$

and

$$\mathbf{KRFrm} \equiv \mathbf{KHaus}^{\mathrm{op}}.$$

For a more thorough exposition of frames and spaces, and a proof of the statements in fact 2.9 we refer to section C1.2 of [16]. We explicitly mention one isomorphism which is part of this duality, for we will encounter it later on.

Remark 2.10. Let $\mathcal{X}$ be a sober space. Then fact 2.9 entails that there is an isomorphism $\mathcal{X} \to \mathsf{pt}(\mathsf{opn}\mathcal{X})$. This isomorphism is given by $x \mapsto p_x$, where p_x is the point given by

$$p_x : \mathsf{opn}\mathcal{X} \to 2 : \begin{cases} a \mapsto \top & \text{if } x \in a \\ a \mapsto \bot & \text{if } x \notin a \end{cases}$$

for all $x \in \mathcal{X}$ and $a \in \Omega\mathcal{X}$.

3 Logic for topological coalgebras

Although not all of our results can be shown for every full subcategory of $\mathbf{Top}$, we will give the basic definitions in full generality. To this end, we let $\mathbf{C}$ be some full subcategory of $\mathbf{Top}$ and define coalgebraic logic over base category $\mathbf{C}$. In particular $\mathbf{C} = \mathbf{KHaus}$ and $\mathbf{C} = \mathbf{Sob}$ will be of interest. Throughout this section T is an arbitrary endofunctor on $\mathbf{C}$. Recall that Φ is an arbitrary but fixed set of proposition letters. We begin with defining the topological version of a predicate lifting, called an open predicate lifting.

Definition 3.1. An *open predicate lifting for* T is a natural transformation

$$\lambda : \Omega^n \to \Omega \circ \mathsf{T}.$$

A collection of open predicate liftings for T is called a *geometric modal signature* for T. An open predicate lifting is called *monotone in its i-th argument* if for all $a_1, \ldots, a_n, b$ we have $\lambda_{\mathcal{X}}(a_1, \ldots, a_i, \ldots, a_n) \subseteq \lambda_{\mathcal{X}}(a_1, \ldots, a_i \cup b, \ldots, a_n)$ and *monotone* if it is monotone in every argument. A geometric modal signature for a functor T is called *monotone* if every open predicate lifting in it is monotone, and *characteristic* if for every topological space $\mathcal{X}$ in $\mathbf{C}$ the collection

$$\{\lambda_{\mathcal{X}}(a_1, \ldots, a_n) \mid \lambda \in \Lambda \ n\text{-ary}\ , a_i \in \Omega\mathcal{X}\}$$

is a sub-base for the topology on $\mathsf{T}\mathcal{X}$.

Definition 3.2. The *language* induced by a geometric modal signature Λ is the collection $\mathrm{GML}(\Lambda)$ of formulas defined by the grammar

$$\varphi ::= \top \mid p \mid \varphi_1 \wedge \varphi_2 \mid \bigvee_{i \in I} \varphi_i \mid \heartsuit^\lambda(\varphi_1, \dots, \varphi_n),$$

where $p \in \Phi$, I is some index set and $\lambda \in \Lambda$ is n-ary. Abbreviate $\bot := \bigvee \varnothing$.

The language $\mathrm{GML}(\Lambda)$ is interpreted in so-called geometric T-models.

Definition 3.3. A *geometric* T*-model* is a triple $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ where $(\boldsymbol{\mathcal{X}}, \gamma)$ is a T-coalgebra and $V : \Phi \to \Omega\boldsymbol{\mathcal{X}}$ is a valuation of the proposition letters. A map $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ is a *geometric* T*-model morphism* from $(\boldsymbol{\mathcal{X}}, \gamma, V)$ to $(\boldsymbol{\mathcal{X}}', \gamma', V')$ if f is a coalgebra morphism between the underlying coalgebras and $f^{-1} \circ V' = V$. The collection of geometric T-models and geometric T-model morphisms forms a category, which we denote by $\mathbf{Mod}(\mathsf{T})$.

Definition 3.4. The *semantics* of $\varphi \in \mathrm{GML}(\Lambda)$ on a geometric T-model $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ is given recursively by

$$\begin{aligned}
&[\![\top]\!]^{\mathfrak{X}} = X, \quad [\![p]\!]^{\mathfrak{X}} = V(p), \quad [\![\varphi \wedge \psi]\!]^{\mathfrak{X}} = [\![\varphi]\!]^{\mathfrak{X}} \cap [\![\psi]\!]^{\mathfrak{X}}, \quad [\![\bigvee_{i \in I} \varphi_i]\!]^{\mathfrak{X}} = \bigcup_{i \in I} [\![\varphi_i]\!]^{\mathfrak{X}}, \\
&[\![\heartsuit^\lambda(\varphi_1, \dots, \varphi_n)]\!]^{\mathfrak{X}} = \gamma^{-1}(\lambda_{\boldsymbol{\mathcal{X}}}([\![\varphi_1]\!]^{\mathfrak{X}}, \dots, [\![\varphi_n]\!]^{\mathfrak{X}})).
\end{aligned}$$

We write $\mathfrak{X}, x \Vdash \varphi$ iff $x \in [\![\varphi]\!]^{\mathfrak{X}}$. Two states x and x' are called *modally equivalent* if they satisfy the same formulas, notation: $x \equiv_\Lambda x'$.

The following proposition shows that morphisms preserve truth. Its proof is similar to the proof of theorem 6.17 in [25].

Proposition 3.5. *Let Λ be a geometric modal signature for T. Let $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ and $\mathfrak{X}' = (\boldsymbol{\mathcal{X}}', \gamma', V')$ be geometric T-models and let $f : \mathfrak{X} \to \mathfrak{X}'$ be a geometric T-model morphism. Then for all $\varphi \in \mathrm{GML}(\Lambda)$ and $x \in \boldsymbol{\mathcal{X}}$ we have*

$$\mathfrak{X}, x \Vdash \varphi \quad \textit{iff} \quad \mathfrak{X}', f(x) \Vdash \varphi.$$

We state the notion of behavioural equivalence for future reference.

Definition 3.6. Let $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ and $\mathfrak{X}' = (\boldsymbol{\mathcal{X}}', \gamma', V')$ be two geometric T-models and $x \in \boldsymbol{\mathcal{X}}$, $x' \in \boldsymbol{\mathcal{X}}'$ two states. We say that x and x' are behaviourally equivalent in $\mathbf{Mod}(\mathsf{T})$, notation: $x \simeq_{\mathbf{Mod}(\mathsf{T})} x'$, if there exists a geometric T-model $\mathfrak{Y}$ and T-model morphisms

$$\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xleftarrow{f'} \mathfrak{X}'$$

such that $f(x) = f'(x')$.

As an immediate consequence of proposition 3.5 we find that behavioural equivalence implies modal equivalence.

Let us give some concrete examples of functors.

Example 3.7 (Trivial functor)**.** Let $\mathbf{2} = \{0, 1\}$ be the topological space with two elements and the trivial topology, that is, the only opens are the empty set and the space itself. Define the functor $\mathsf{F} : \mathbf{Top} \to \mathbf{Top}$ by $\mathsf{F}\boldsymbol{\mathcal{X}} = \mathbf{2}$ for every topological space $\boldsymbol{\mathcal{X}}$ and $\mathsf{F}f = \mathrm{id}_{\mathbf{2}}$, the identity map on $\mathbf{2}$, for every continuous function f. This clearly defines a functor.

As an example of a natural transformation consider $\lambda : \Omega \to \Omega \circ \mathsf{F}$ given by

$$\lambda_{\boldsymbol{\mathcal{X}}}(a) = \begin{cases} \mathsf{U}\mathbf{2} & \text{if } a = \mathsf{U}\boldsymbol{\mathcal{X}} \\ \varnothing & \text{otherwise.} \end{cases}$$

For a F-model $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ we then have $\mathfrak{X}, x \Vdash \heartsuit^{\lambda}\varphi$ iff $\gamma(x) \in \lambda(\llbracket\varphi\rrbracket^{\mathfrak{X}})$ iff $\llbracket\varphi\rrbracket^{\mathfrak{X}} = \mathsf{U}\boldsymbol{\mathcal{X}}$. So $\heartsuit^{\lambda}$ is the universal modality.

Next we have a look at the Vietoris functor on **KHaus**. Coalgebras for this functor have also been studied in [3].

Example 3.8 (Vietoris functor)**.** For a compact Hausdorff space $\boldsymbol{\mathcal{X}}$, let $\mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}$ be the collection of closed subsets of $\boldsymbol{\mathcal{X}}$ topologized by the subbase

$$\boxdot a := \{b \in \mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}} \mid b \subseteq a\}, \quad \diamonddot a := \{b \in \mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}} \mid a \cap b \neq \varnothing\},$$

where a ranges over $\Omega\boldsymbol{\mathcal{X}}$. For a continuous map $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ define $\mathsf{V}_{\mathsf{kh}}f : \mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}} \to \mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}'$ by $\mathsf{V}_{\mathsf{kh}}f(a) = f[a]$. If $\boldsymbol{\mathcal{X}}$ is compact Hausdorff, then so is $\mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}$ [21, Theorem 4.9], and if $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ is a continuous map between compact Hausdorff spaces, then $\mathsf{V}_{\mathsf{kh}}f$ is well defined and continuous [19, Lemma 3.8], so V_{kh} defines an endofunctor on **KHaus**.

Let $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ be a V_{kh}-model. If we set

$$\lambda^{\Box}_{\boldsymbol{\mathcal{X}}} : \Omega\boldsymbol{\mathcal{X}} \to \Omega(\mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}) : a \mapsto \{b \in \mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}} \mid b \subseteq a\},$$

where $\boldsymbol{\mathcal{X}} \in \mathbf{Top}$, then we have $\mathfrak{X}, x \Vdash \Box\varphi$ iff $\gamma(x) \in \lambda^{\Box}_{\boldsymbol{\mathcal{X}}}(\llbracket\varphi\rrbracket^{\mathfrak{X}})$ iff $\gamma(x) \subseteq \llbracket\varphi\rrbracket^{\mathfrak{X}}$ iff every successor of x satisfies φ. Similarly $\lambda^{\Diamond}_{\boldsymbol{\mathcal{X}}} : \Omega\boldsymbol{\mathcal{X}} \to \Omega \circ \mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}$, given by $\lambda^{\Diamond}_{\boldsymbol{\mathcal{X}}}(a) = \diamonddot a$ yields the usual semantics of the diamond modality.

In some cases it turns out to be useful to have a slightly stronger notion of open predicate liftings, which we shall call strong open predicate liftings. Recall that $\mathsf{U} : \mathbf{Top} \to \mathbf{Set}$ is the forgetful functor.

Whereas the action of open predicate liftings is defined only on open subsets of a topological space, a strong open predicate lifting acts on *every* subset of elements of a topological space. Evidently, every strong open predicate lifting restricts to an open predicate lifting, and it is only this weaker notion of open predicate lifting that has an effect on the semantics.

Definition 3.9. A *strong open predicate lifting* for T is a natural transformation

$$\lambda : (\breve{\mathsf{P}} \circ \mathsf{U})^n \to \breve{\mathsf{P}} \circ \mathsf{U} \circ \mathsf{T}$$

such that for all topological spaces $\boldsymbol{\mathcal{X}}$ and $a_1, \ldots, a_n \in \Omega\boldsymbol{\mathcal{X}}$ the set $\lambda_{\boldsymbol{\mathcal{X}}}(a_1, \ldots, a_n)$ is open in $\mathsf{T}\boldsymbol{\mathcal{X}}$. Monotonicity of strong open predicate liftings is defined as expected.

We call an open predicate lifting (from definition 3.2) *strong* if it is the restriction of some strong open predicate lifting and *strongly monotone* if it is the restriction of a monotone strong open predicate lifting.

Example 3.10. The predicate lifting corresponding to the box modality from example 3.8 is strong, for it is the restriction of $\lambda^s : \mathsf{U} \to \mathsf{U} \circ \mathsf{V}_{\mathsf{kh}}$ given by

$$\lambda^s_{\boldsymbol{\mathcal{X}}}(u) = \{b \in \mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}} \mid b \subseteq u\}.$$

The same holds for the diamond modality from example 3.8.

In some cases we are assured that our predicate liftings are strong.

Proposition 3.11. *Let* T *be an endofunctor on* KHaus *and* Λ *a monotone geometric modal signature for* T*. Then* Λ *is strongly monotone.*

Proof. Let $\lambda \in \Lambda$. We need to show that λ is the restriction of some strong monotone predicate lifting. Define

$$\widetilde{\lambda}_{\boldsymbol{\mathcal{X}}} : \breve{\mathsf{P}}^n\mathsf{U}\boldsymbol{\mathcal{X}} \to \breve{\mathsf{P}}\mathsf{U}\mathsf{T}\boldsymbol{\mathcal{X}} : (b_1, \dots, b_n) \mapsto \bigcap\{\lambda_{\boldsymbol{\mathcal{X}}}(a_1, \dots, a_n) \mid a_i \in \Omega\boldsymbol{\mathcal{X}} \text{ and } a_i \supseteq b_i\}.$$

Monotonicity of $\lambda_{\boldsymbol{\mathcal{X}}}$ ensures that we have $\widetilde{\lambda}_{\boldsymbol{\mathcal{X}}}(a) = \lambda_{\boldsymbol{\mathcal{X}}}(a)$ for all $a \in \Omega\boldsymbol{\mathcal{X}}$. Besides, $\widetilde{\lambda}$ is monotone by construction. So we only need to show that $\widetilde{\lambda}$ is indeed a strong open predicate lifting, i.e. that it is a natural transformation $\breve{\mathsf{P}}^n\mathsf{U}\boldsymbol{\mathcal{X}} \to \breve{\mathsf{P}}\mathsf{U}\mathsf{T}\boldsymbol{\mathcal{X}}$. We prove this for unary λ, the general case being similar.

For a continuous map $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ between compact Hausdorff spaces we need to show that the following diagram commutes:

$$\begin{tikzcd}
\breve{\mathsf{P}}\mathsf{U}\boldsymbol{\mathcal{X}} \arrow[r, "\widetilde{\lambda}_{\boldsymbol{\mathcal{X}}}"] & \breve{\mathsf{P}}\mathsf{U}\mathsf{T}\boldsymbol{\mathcal{X}} \\
\breve{\mathsf{P}}\mathsf{U}\boldsymbol{\mathcal{X}}' \arrow[u, "f^{-1}"] \arrow[r, "\widetilde{\lambda}_{\boldsymbol{\mathcal{X}}'}"'] & \breve{\mathsf{P}}\mathsf{U}\mathsf{T}\boldsymbol{\mathcal{X}}' \arrow[u, "(\mathsf{T}f)^{-1}"']
\end{tikzcd}$$

That is, for all $b' \in \breve{\mathsf{P}}\mathsf{U}\boldsymbol{\mathcal{X}}'$ we claim that

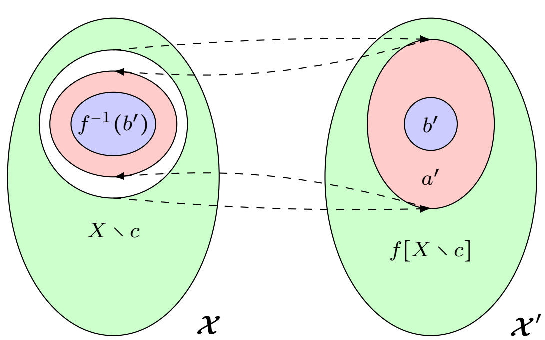
$$\bigcap\{\lambda_{\boldsymbol{\mathcal{X}}}(c) \mid c \in \Omega\boldsymbol{\mathcal{X}} \text{ and } f^{-1}(b') \subseteq c\} = (\mathsf{T}f)^{-1}(\bigcap\{\lambda_{\boldsymbol{\mathcal{X}}'}(a') \mid a' \in \Omega\boldsymbol{\mathcal{X}}' \text{ and } b' \subseteq a'\}).$$

By naturality of λ the right hand side is equal to $\bigcap\{\lambda_{\boldsymbol{\mathcal{X}}}(f^{-1}(a')) \mid a' \in \Omega\boldsymbol{\mathcal{X}}' \text{ and } b' \subseteq a'\}$, so it suffices to show

$$\bigcap\{\lambda_{\boldsymbol{\mathcal{X}}}(c) \mid c \in \Omega\boldsymbol{\mathcal{X}} \text{ and } f^{-1}(b') \subseteq c\} = \bigcap\{\lambda_{\boldsymbol{\mathcal{X}}}(f^{-1}(a')) \mid a' \in \Omega\boldsymbol{\mathcal{X}}' \text{ and } b' \subseteq a'\}. \tag{1}$$

If a' is an open superset of b' then clearly $f^{-1}(b') \subseteq f^{-1}(a')$. So every element in the intersection of the right hand side is contained in the one on the left hand side and therefore we have $\subseteq$ in (1).

For the converse, we will show that for every $c \in \Omega\boldsymbol{\mathcal{X}}$ with $f^{-1}(b') \subseteq c$ there exists $a' \in \Omega\boldsymbol{\mathcal{X}}'$ with $b' \subseteq a'$ and $f^{-1}(a') \subseteq c$, because then $f^{-1}(a') \cap c = f^{-1}(a')$, so the element c does not make the intersection on the left hand side smaller. This entails that elements in the left intersection that are not of the form $\lambda_{\boldsymbol{\mathcal{X}}}(f^{-1}(a))$ can be deleted without changing the outcome of the intersection. So take some $c \in \Omega\boldsymbol{\mathcal{X}}$ with $f^{-1}(b') \subseteq c$. Then $X \setminus c$ is closed and because both $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{X}}'$ are compact Hausdorff the set $f[X \setminus c]$ is closed in $\boldsymbol{\mathcal{X}}'$. Let $a' = X' \setminus f[X \setminus c]$, then a' is open, contains b', and we have $f^{-1}(b') \subseteq f^{-1}(a') \subseteq c$. See the illustration below, the red parts contain the purple parts:



This proves the proposition. □

4 The monotone functor on KHaus

This section is devoted to the monotone functor $\mathsf{D_{kh}}$ on **KHaus**. It serves as an example for the developed theory. The functor $\mathsf{D_{kh}}$ generalises the monotone functor on **Stone**, whose coalgebras are descriptive monotone frames [13]. Below, we will define it, prove preservation properties and give a dual functor of $\mathsf{D_{kh}}$ on **KRFrm**.

Definition 4.1. Let $\boldsymbol{\mathcal{X}} = (X, \tau)$ be a compact Hausdorff space. Let $\mathsf{D_{kh}}\boldsymbol{\mathcal{X}}$ be the collection of sets $W \subseteq \mathsf{P}X$ such that $u \in W$ iff there exists a closed $c \subseteq u$ such that every open superset of c is in W. Endow $\mathsf{D_{kh}}\boldsymbol{\mathcal{X}}$ with the topology generated by the subbase

$$\boxplus a \coloneqq \{W \in \mathsf{D_{kh}}\boldsymbol{\mathcal{X}} \mid a \in W\}, \quad \otimes a \coloneqq \{W \in \mathsf{D_{kh}}\boldsymbol{\mathcal{X}} \mid X \setminus a \notin W\},$$

where a ranges over $\Omega\boldsymbol{\mathcal{X}}$. For continuous functions $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ define $\mathsf{D_{kh}}f : \mathsf{D_{kh}}\boldsymbol{\mathcal{X}} \to \mathsf{D_{kh}}\boldsymbol{\mathcal{X}}' : W \mapsto \{a \in \mathsf{P}X \mid f^{-1}(a) \in W\}$.

Lemma 4.2. *If $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ is a morphism in* **Stone***, then $\mathsf{D_{kh}}f$ is a well-defined continuous function from $\mathsf{D_{kh}}\boldsymbol{\mathcal{X}}$ to $\mathsf{D_{kh}}\boldsymbol{\mathcal{X}}'$.*

Proof. $\mathsf{D_{kh}}f$ *is well-defined.* Let $W \in \mathsf{D_{kh}}\boldsymbol{\mathcal{X}}$. We need to show that $\mathsf{D_{kh}}f(W) \in \mathsf{D_{kh}}\boldsymbol{\mathcal{X}}'$. Suppose $a' \in \mathsf{D_{kh}}f(W)$. Then $f^{-1}[a'] \in W$, so there exists a closed $c \subseteq f^{-1}[a']$ such that $c \in W$. Since $\boldsymbol{\mathcal{X}}$ is compact and $\boldsymbol{\mathcal{X}}'$ is Hausdorff, $f[c]$ is a closed set in $\boldsymbol{\mathcal{X}}'$. In addition we have $f[c] \subseteq a'$. Suppose $f[c] \subseteq b$ for some open $b \in \Omega\boldsymbol{\mathcal{X}}'$, then $c \subseteq f^{-1}[b]$ so $f^{-1}[b] \in W$ and hence $b \in \mathsf{D_{kh}}f(W)$. So all open supersets of $f[c]$ are in $\mathsf{D_{kh}}f(W)$, and therefore $f[c] \in \mathsf{D_{kh}}f(W)$. Thus, for $a' \in \mathsf{D_{kh}}f(W)$, there exists a closed subset (in this case $f[c]$) of a' with the property that every clopen superset is in $\mathsf{D_{kh}}f(W)$.

$\mathsf{D_{kh}}f$ *is continuous.* For continuity we need to show that both $(\mathsf{D_{kh}}f)^{-1}[\boxplus a']$ and $(\mathsf{D_{kh}}f)^{-1}[\otimes a']$ are open in $\mathsf{D_{kh}}\boldsymbol{\mathcal{X}}$, whenever $a' \in \Omega(\boldsymbol{\mathcal{X}}')$. It follows from a straightforward computation that $(\mathsf{D_{kh}}f)^{-1}(\boxplus a') = \boxplus f^{-1}(a')$, which is open in $\mathsf{D_{kh}}\boldsymbol{\mathcal{X}}$ by definition, and similarly $(\mathsf{D_{kh}}f)^{-1}(\otimes a') = \otimes f^{-1}[a'] \in \Omega\mathsf{D_{kh}}\boldsymbol{\mathcal{X}}$. □

Shortly, we will show that $\mathsf{D_{kh}}$ is indeed a functor on compact Hausdorff spaces. Besides, it is worth noting that, when restricted to **Stone**, this functor is the same as the monotone functor defined in [9], which is in turn an equivalent description of the monotone functor in [13]. For details we refer to section 2.3 in [11].

Lemma 4.3. *If $\boldsymbol{\mathcal{X}}$ is a compact space then so is $\mathsf{D_{kh}}\boldsymbol{\mathcal{X}}$.*

Proof. By the Alexander subbasis theorem it suffices to show that any cover of the form

$$\bigcup_{i \in I} \boxplus a_i \cup \bigcup_{j \in J} \otimes b_j$$

has a finite subcover. So suppose the above covers $\mathsf{D_{kh}}\boldsymbol{\mathcal{X}}$. Since $\varnothing \in \mathsf{D_{kh}}\boldsymbol{\mathcal{X}}$ and $\varnothing \notin \boxplus a$ for any $a \in \Omega\boldsymbol{\mathcal{X}}$, we must have $|J| \geq 1$. Furthermore, we must have $k \in I$ such that a_k is a superset of $X \setminus b_j$ for some $j \in J$, because otherwise the up-set $\uparrow\{X \setminus b_j \mid j \in J\}$, where each member is a superset of at least one of the b_j, is not in the cover.

Let j be such that $X \setminus b_j \subseteq a_k$ and let $W \in \mathsf{D_{kh}}\boldsymbol{\mathcal{X}}$. If $X \setminus b_j \notin W$ then $W \in \otimes b_j$ and if $X \setminus b_j \in W$ then $W \in \boxplus a_k$. This shows that $\boxplus a_k \cup \otimes b_j$ is a finite subcover. □

We will now give a dual functor of $\mathsf{D_{kh}}$, i.e. a functor M on **Frm** such that for every compact Hausdorff space $\boldsymbol{\mathcal{X}}$ we have

$$\mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}}) \cong \mathsf{opn}(\mathsf{D_{kh}}\boldsymbol{\mathcal{X}}).$$

Definition 4.4. Let F be a frame. Let $\mathsf{M}F$ be the frame generated by $\Box a, \Diamond a$, where a ranges over F, subject to the relations

(M_1) $\Box(a \wedge b) \leq \Box a$

(M_2) $\Box a \wedge \Diamond b = \bot$ whenever $a \wedge b = \bot$

(M_3) $\Box \bigvee^{\uparrow} A = \bigvee^{\uparrow}\{\Box a \mid a \in A\}$

(M_4) $\Diamond a \leq \Diamond(a \vee b)$

(M_5) $\Box a \vee \Diamond b = \top$ whenever $a \vee b = \top$

(M_6) $\Diamond \bigvee^{\uparrow} A = \bigvee^{\uparrow}\{\Diamond a \mid a \in A\}$,

where $a, b \in F$ and A is a directed subset of F. For a homomorphism $f : F \to F'$ define $\mathsf{M}f : \mathsf{M}F \to \mathsf{M}F'$ on generators by $\Box a \mapsto \Box f(a)$ and $\Diamond a \mapsto \Diamond f(a)$. The assignment M defines a functor on **Frm**.

The proof of the following proposition closely resembles that of proposition III4.3 in [15].

Proposition 4.5. *If F is a regular frame, then so is $\mathsf{M}F$.*

Proof. We need to show that for all $c \in \mathsf{M}F$ we have $c = \bigvee\{d \in \mathsf{M}F \mid d \lessdot c\}$. It follows from lemma 2.5 that it suffices to focus on the generators of $\mathsf{M}F$. Let $a \in F$, then we know $\bigvee\{d \in \mathsf{M}F \mid d \lessdot \Box a\} \leq \Box a$. Suppose $b \lessdot a$ in F, then by lemma 2.4 ${\sim}b \vee a = \top$ and hence $\Diamond{\sim}b \vee \Box a \geq \top$. Also ${\sim}b \wedge b = \bot$ so it follows from (M_2) that $\Diamond{\sim}b \wedge \Box b = \bot$. This proves $\Box b \lessdot \Box a$, because the element $\Diamond{\sim}b$ is such that $\Diamond{\sim}b \vee \Box a = \top$ and $\Diamond{\sim}b \wedge \Box b = \bot$. Since F is regular and $\{b \in F \mid b \lessdot a\}$ is directed, it follows that

$$\Box a = \Box \bigvee^{\uparrow}\{b \in F \mid b \lessdot a\} = \bigvee^{\uparrow}\{\Box b \in \mathsf{M}F \mid b \lessdot a\} \leq \bigvee\{d \in \mathsf{M}F \mid d \lessdot \Box a\}$$

so $\Box a = \bigvee\{d \in \mathsf{M}F \mid d \lessdot \Box a\}$. In a similar fashion one may show that $\Diamond a = \bigvee\{d \in \mathsf{M}F \mid d \lessdot \Diamond a\}$. This proves the lemma. ◻

As desired, we have:

Theorem 4.6. *If $\boldsymbol{\mathcal{X}}$ is a compact Hausdorff space then*

$$\mathsf{pt}(\mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}})) \cong \mathsf{D_{kh}}\boldsymbol{\mathcal{X}}.$$

Proof. Define a map

$$\varphi : \mathsf{D_{kh}}\boldsymbol{\mathcal{X}} \to \mathsf{pt}(\mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}})) : W \mapsto p_W,$$

where we define p_W on generators by

$$p_W : \mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}}) \to 2 : \begin{cases} \Box a \mapsto \top & \text{iff } a \in W \\ \Diamond a \mapsto \bot & \text{iff } X \setminus a \in W \end{cases}.$$

Conversely, for a point $p \in \mathsf{pt}(\mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}}))$ let

$$W_p := {\uparrow}\{X \setminus a \mid p(\Diamond a) = \bot\}.$$

This gives rise to a map $\psi : \mathsf{pt}(\mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}})) \to \mathsf{D_{kh}}\boldsymbol{\mathcal{X}}$. It is clear that $W_p \in \mathsf{D_{kh}}\boldsymbol{\mathcal{X}}$ because it is the up-set of a collection of closed sets; indeed, for each $b \in W_p$ there exists a closed subset $X \setminus a \subseteq b$ with $p(\Diamond a) = \bot$ and by definition all open supersets of $X \setminus a$ are in W_p.

We will show that the p_W are well-defined, that φ is a bijection with inverse ψ, and that φ is continuous.

Claim 4.6.A. *If $W \in \mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}$ then $p_W : \mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}}) \to 2$ is a point.*

Proof of claim. Since p_W is a frame homomorphisms defined on generators, it suffices to check that the $p_W(\Box a)$ and $p_W(\Diamond a)$ (where the a range over $\Omega\boldsymbol{\mathcal{X}}$) satisfy (M_1) through (M_6) from definition 4.4. Let us check (M_1), (M_2) and (M_3), items (M_4), (M_5) and (M_6) being similar.

(M_1) If $p_W(\Box(a \cap b)) = \top$ then $a \cap b \in W$. Since W is upward closed $a \in W$, so $p_W(\Box a) = \top$.

(M_2) If $a \cap b = \varnothing$ then $a \subseteq X \smallsetminus b$. Suppose $p_W(\Box a) = \top$ then $a \in W$ so $X \smallsetminus b \in W$ so $p_W(\Diamond b) = \bot$ hence $p_W(\Box a) \wedge p_W(\Diamond b) = \bot$.

(M_3) We claim that for all $W \in \mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}$ and directed sets $A \subseteq \Omega\boldsymbol{\mathcal{X}}$ we have $\bigcup^{\uparrow} A \in W$ iff there is $a \in A$ with $a \in W$. The direction from right to left follows from the fact that W is upwards closed. Conversely, suppose $\bigcup^{\uparrow} A \in W$, then there is a closed set $k \subseteq \bigcup^{\uparrow} A$ with $k \in W$. The elements of A now cover the closed therefore compact set k, so there is a finite $A' \subseteq A$ with $k \subseteq \bigcup A'$ and since A is directed there is $a \in A$ with $\bigcup A' \subseteq a$. As $k \in W$ and $k \subseteq a$ it follows that $a \in W$.

Now we have $p_W(\Box \bigcup^{\uparrow} A) = \top$ iff $\bigcup^{\uparrow} A \in W$ iff there is $a \in A$ with $a \in W$ iff $\bigvee^{\uparrow}\{p_W(\Box a) \mid a \in A\} = \top$. ◇

Claim 4.6.B. *For all $p \in \mathsf{pt}(\mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}}))$ we have $X \smallsetminus a \in W_p$ iff $p(\Diamond a) = \bot$ and $a \in W$ iff $p(\Box a) = \top$.*

Proof of claim. If $p(\Diamond a) = \bot$ then $X \smallsetminus a \in W_p$. Conversely, Suppose $X \smallsetminus a \in W_p$, then there is some b with $p(\Diamond b) = \bot$ and $X \smallsetminus b \subseteq X \smallsetminus a$. Therefore $a \subseteq b$ and $p(\Diamond a) \leq p(\Diamond b) = \bot$. This proves $X \smallsetminus a \in W_p$ iff $p(\Diamond a) = \bot$.

If $a \in W_p$ then there is $X \smallsetminus b \subseteq a$ in W_p, so $p(\Diamond b) = \bot$. Then $a \cup b = X$, so it follows from (M_5) of 4.4 that $p(\Box a) = \top$. If $a \notin W_p$ and $a' \lessdot a$, then there exists b with $b \cap a' = \varnothing$ and $b \cup a = \boldsymbol{\mathcal{X}}$. Since $X \smallsetminus b \subseteq a$, set set $X \smallsetminus b$ is not in W_p and hence we must have $p(\Diamond b) = \top$. As $a' \cap a = \varnothing$ it follows from (M_2) that $p(\Box a') = p(\varnothing) = \bot$. Now we use (M_3) and the fact that $a = \bigvee^{\uparrow}\{a' \mid a' \lessdot a\}$ (this is true because $\boldsymbol{\mathcal{X}}$ is assumed to be compact Hausdorff so $\mathsf{opn}\boldsymbol{\mathcal{X}}$ is compact regular) to find

$$p(\Box a) = \bigvee\nolimits^{\uparrow}\{p(\Box a') \mid a' \lessdot a\} = \bigvee\nolimits^{\uparrow}\{\bot \mid a' \lessdot a\} = \bot.$$

It follows that $a \in W_p$ iff $p(\Box a) = \top$. ◇

Claim 4.6.C. *The maps φ and ψ define a bijection between $\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}$ and $\mathsf{pt}(\mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}}))$.*

Proof of claim. For $p \in \mathsf{pt}(\mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}}))$ and $W \in \mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}$ we will show that $p_{W_p} = p$ and $W_{p_W} = W$. In order to prove that (the frame homomorphisms) p and p_{W_p} coincide, it suffices to show that they coincide on the generators of $\mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}})$. By definition and claim 4.6.B have

$$p(\Box a) = \top \quad \text{iff} \quad a \in W_p \quad \text{iff} \quad p_{W_p}(\Box a) = \top$$

and

$$p(\Diamond a) = \bot \quad \text{iff} \quad X \smallsetminus a \notin W_p \quad \text{iff} \quad p_{W_p}(\Diamond a) = \bot.$$

In order to show that $W = W_{p_W}$ it suffices to show that $X \smallsetminus a \in W$ iff $X \smallsetminus a \in W_{p_W}$ for all open sets a, because elements of $\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}$ are uniquely determined by the closed sets they contain. This follows immediately from the definitions and claim 4.6.B,

$$X \smallsetminus a \in W \quad \text{iff} \quad p_W(\Diamond a) = \bot \quad \text{iff} \quad X \smallsetminus a \in W_{p_W}.$$

This proves the claim. ◇

Claim 4.6.D. *The maps* $\varphi : \mathsf{D}_{\mathsf{kh}}\mathcal{X} \to \mathsf{pt}(\mathsf{M}(\mathsf{opn}\mathcal{X}))$ *and* $\psi : \mathsf{pt}(\mathsf{M}(\mathsf{opn}\mathcal{X})) \to \mathsf{D}_{\mathsf{kh}}\mathcal{X}$ *are continuous.*

Proof of claim. The opens of $\mathsf{pt}(\mathsf{M}(\mathsf{opn}\mathcal{X}))$ are generated by $\widetilde{\Box a} = \{p \mid p(\Box a) = \top\}$ and $\widetilde{\Diamond a} = \{p \mid p(\Diamond a) = \top\}$, for $a \in \Omega\mathcal{X}$. We have

$$\varphi^{-1}(\widetilde{\Box a}) = \varphi^{-1}(\{p \mid p(\Box a) = \top\}) = \{W \in \mathsf{D}_{\mathsf{kh}}\mathcal{X} \mid a \in W\} = \boxplus a$$

and similarly $\varphi^{-1}(\widetilde{\Diamond a}) = \diamondtimes a$. Since $\boxplus a$ and $\diamondtimes a$ are open in $\mathsf{D}_{\mathsf{kh}}\mathcal{X}$, this proves continuity of φ.

The opens of $\mathsf{D}_{\mathsf{kh}}\mathcal{X}$ are generated by $\boxplus a$ and $\diamondtimes a$, where a ranges over $\Omega\mathcal{X}$. Using the proof of claim 4.6.C, it is routine to see that $\psi^{-1}(\boxplus a) = \widetilde{\Box a}$ and $\psi^{-1}(\diamondtimes a) = \widetilde{\Diamond a}$. This proves continuity of ψ. $\Diamond$

We showed that φ is a continuous function with continuous inverse ψ, hence a homeomorphism. This completes the proof of the theorem. $\Box$

As a corollary of theorem 4.6, lemma 4.3 and proposition 4.5 it follows that D_{kh} is indeed an endofunctor on **KHaus**. Besides, M restricts to an endofunctor on **KRFrm**. Denote by M_{kr} the restriction of M to **KRFrm**. Theorem 4.6 yields a map $\mathsf{M}_{\mathsf{kr}}(\mathsf{opn}\mathcal{X}) \to \mathsf{opn}(\mathsf{D}_{\mathsf{kh}}\mathcal{X})$ for a compact Hausdorff space $\mathcal{X}$ given by

$$\mathsf{M}_{\mathsf{kr}}(\mathsf{opn}\mathcal{X}) \longrightarrow \mathsf{opn}(\mathsf{pt}(\mathsf{M}_{\mathsf{kr}}(\mathsf{opn}\mathcal{X}))) \xrightarrow{\mathsf{opn}\varphi} \mathsf{opn}(\mathsf{D}_{\mathsf{kh}}\mathcal{X}).$$

Unravelling the definitions shows that, on generators, it is given by $\Box a \mapsto \boxplus a$ and $\Diamond a \mapsto \diamondtimes a$.

Definition 4.7. For every compact Hausdorff space $\mathcal{X}$ define $\eta_{\mathcal{X}} : \mathsf{M}_{\mathsf{kr}}(\mathsf{opn}\mathcal{X}) \to \mathsf{opn}(\mathsf{D}_{\mathsf{kh}}\mathcal{X})$ on generators by $\eta_{\mathcal{X}}(\Box a) = \boxplus a$ and $\eta_{\mathcal{X}}(\Diamond a) = \diamondtimes a$. By the preceding discussion $\eta_{\mathcal{X}}$ is a well-defined frame isomorphism.

It turns out that the maps $\eta_{\mathcal{X}}$ constitute a natural isomorphism.

Proposition 4.8. *The collection* $\eta = (\eta_{\mathcal{X}})_{\mathcal{X} \in \mathsf{KHaus}}$ *forms a natural isomorphism.*

Proof. It follows from proposition 4.6 that each of the $\eta_{\mathcal{X}}$ is an isomorphism, so we only need to show naturality. That is, for any morphism $f : \mathcal{X} \to \mathcal{X}'$ in **KHaus**, the following diagram commutes,

$$\begin{tikzcd}
\mathsf{M}_{\mathsf{kr}}(\mathsf{opn}\mathcal{X}) \arrow[d, "\eta_{\mathcal{X}}"'] & \mathsf{M}_{\mathsf{kr}}(\mathsf{opn}\mathcal{X}') \arrow[l, "\mathsf{M}_{\mathsf{kr}}(\mathsf{opn}f)"'] \arrow[d, "\eta_{\mathcal{X}'}"] \\
\mathsf{opn}(\mathsf{D}_{\mathsf{kh}})\mathcal{X} & \mathsf{opn}(\mathsf{D}_{\mathsf{kh}}\mathcal{X}') \arrow[l, "\mathsf{opn}(\mathsf{D}_{\mathsf{kh}}f)"]
\end{tikzcd}$$

(Since opn is a contravariant functor, the horizontal arrows are reversed.) For this, suppose $\Box a'$ is a generator of $\mathsf{M}_{\mathsf{kr}}(\mathsf{opn}\mathcal{X}')$. Then

$$\begin{aligned}
\mathsf{opn}(\mathsf{D}_{\mathsf{kh}}f) \circ \eta_{\mathcal{X}'}(\Box a) &= \mathsf{opn}(\mathsf{D}_{\mathsf{kh}}f)(\boxplus a) && \text{(definition 4.7)} \\
&= (\mathsf{D}_{\mathsf{kh}}f)^{-1}(\boxplus a) && \text{(definition of } \mathsf{opn}) \\
&= \boxplus f^{-1}(a) && \text{(lemma 4.2)} \\
&= \eta_{\mathcal{X}}(\Box f^{-1}(a)) && \text{(definition 4.7)} \\
&= \eta_{\mathcal{X}} \circ \mathsf{M}_{\mathsf{kr}}(f^{-1}(\Box a)) && \text{(definition of } \mathsf{M}) \\
&= \eta_{\mathcal{X}} \circ \mathsf{M}_{\mathsf{kr}}(\mathsf{opn}f)(\Box a). && \text{(definition of } \mathsf{opn})
\end{aligned}$$

and by analogous reasoning $\Omega\mathsf{D}_{\mathbf{kh}}f \circ \eta_{\boldsymbol{\mathcal{X}}'}(\Diamond a) = \eta_{\boldsymbol{\mathcal{X}}} \circ \mathsf{M}_{\mathbf{kr}}(\mathsf{opn}f)(\Diamond a)$. This proves that the diagram commutes. □

Corollary 4.9. *There is a dual equivalence*

$$\mathbf{Alg}(\mathsf{M}_{\mathbf{kr}}) \equiv^{\mathrm{op}} \mathbf{Coalg}(\mathsf{D}_{\mathbf{kh}}).$$

Example 4.10 (Monotone functor)**.** Consider $\mathsf{D}_{\mathbf{kh}}$ and the box lifting for $\mathsf{D}_{\mathbf{kh}}$ by

$$\lambda^{\Box}_{\boldsymbol{\mathcal{X}}} : \Omega\boldsymbol{\mathcal{X}} \to \Omega(\mathsf{D}_{\mathbf{kh}}\boldsymbol{\mathcal{X}}) : a \mapsto \boxplus a.$$

Let $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ be a $\mathsf{D}_{\mathbf{kh}}$-model. Then we have $\mathfrak{X}, x \Vdash \Box a$ iff $[\![\varphi]\!]^{\mathfrak{X}} \in \gamma(x)$, which gives the usual box semantics for monotone modal logic. One may define $\lambda^{\Diamond}_{\boldsymbol{\mathcal{X}}}$ in a similar way.

Remark 4.11. We will see in example 5.6 that the functor $\mathsf{D}_{\mathbf{kh}}$ on **KHaus** can be generalised to an endofunctor of **Top** which restricts to **Sob**.

5 Lifting functors from Set to Top

In [18, Section 4] the authors give a method to lift a **Set**-functor $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$, together with a collection of predicate liftings Λ for T, to an endofunctor on **Stone**. In this section we adapt their approach to obtain an endofunctor $\widehat{\mathsf{T}}_\Lambda$ on **Top**. To define the action of $\widehat{\mathsf{T}}_\Lambda$ on objects, for a topological space $\boldsymbol{\mathcal{X}}$ we take the following steps:

Step 1. Construct a frame $\dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}$ of the images of predicate liftings applied to the open sets of $\boldsymbol{\mathcal{X}}$ (viewed simply as subsets of $\mathsf{T}(\mathsf{U}\boldsymbol{\mathcal{X}})$);

Step 2. Quotient $\dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}$ with a suitable relation that ensures $\bigvee^{\uparrow}_{b\in B}\lambda(b) = \lambda(\bigvee^{\uparrow} B)$ whenever λ is monotone;

Step 3. Employ the functor $\mathsf{pt} : \mathbf{Frm} \to \mathbf{Top}$ to obtain a (sober) topological space.

This is the content of definitions 5.1, 5.3 and 5.5.

Recall that U is the forgetful functor which sends a topological space to its underlying set, and that Q is the contravariant functor sending a set to its powerset, viewed as a Boolean algebra.

Definition 5.1. Let $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ be a functor and Λ a collection of predicate liftings for T. We define a contravariant functor $\dot{\mathsf{F}}_\Lambda : \mathbf{Top} \to \mathbf{Frm}$. For a topological space $\boldsymbol{\mathcal{X}}$ define $\dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}$ to be the subframe of $\mathsf{Q}(\mathsf{T}(\mathsf{U}\boldsymbol{\mathcal{X}}))$ generated by the set

$$\{\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}(a_1, \ldots, a_n) \mid \lambda \in \Lambda \text{ } n\text{-ary, } a_1, \ldots, a_n \in \Omega\boldsymbol{\mathcal{X}}\}.$$

That is, we close this set under finite intersections and arbitrary unions in $\mathsf{Q}(\mathsf{T}(\mathsf{U}\boldsymbol{\mathcal{X}}))$. For a continuous function $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ let $\dot{\mathsf{F}}_\Lambda f : \dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}' \to \dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}$ be the restriction of $\mathsf{Q}(\mathsf{T}(\mathsf{U}f))$ to $\dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}'$.

Lemma 5.2. *The map $\dot{\mathsf{F}}_\Lambda$ defines a contravariant functor.*

Proof. We need to show that $\dot{\mathsf{F}}_\Lambda$ is well defined on morphisms and that it is functorial. To show that the action of $\dot{\mathsf{F}}_\Lambda$ on morphisms is well-defined, it suffices to show that

$(\dot{\mathsf{F}}_\Lambda f)(\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}(a_1',\dots,a_n')) \in \dot{\mathsf{F}}_\Lambda(\boldsymbol{\mathcal{X}})$ for all generators $\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}(a_1',\dots,a_n')$ of $\dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}'$, because frame homomorphisms preserve finite meets and all joins. This holds by naturality of λ:

$$(\dot{\mathsf{F}}_\Lambda f)(\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}'}(a_1,\dots,a_n)) = (\mathsf{T}f)^{-1}(\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}'}(a_1,\dots,a_n)) = \lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}(f^{-1}(a_1),\dots,f^{-1}(a_n)).$$

By continuity of f we have $f^{-1}(a_i) \in \Omega\boldsymbol{\mathcal{X}}$ so the latter is indeed in $\dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}$. Functoriality of $\dot{\mathsf{F}}_\Lambda$ follows from functoriality of $\mathsf{Q}\circ\mathsf{T}\circ\mathsf{U}$. □

Definition 5.3. Let Λ be a collection of predicate liftings for a set functor T and let $\boldsymbol{\mathcal{X}}$ be a topological space. Let $\widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}$ be the quotient of $\dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}$ with respect to the congruence $\sim$ generated by

$$\textstyle\bigvee^{\uparrow}_{b\in B}\lambda(a_1,\dots,a_{i-1},b,a_{i+1},\dots,a_n) \sim \lambda(a_1,\dots,a_{i-1},\bigvee^{\uparrow}B,a_{i+1},\dots,a_n)$$

for all $a_i \in \Omega\boldsymbol{\mathcal{X}}$, $B \subseteq \Omega\boldsymbol{\mathcal{X}}$ directed, and $\lambda \in \Lambda$ monotone in its i-th argument. Write $q_{\boldsymbol{\mathcal{X}}} : \dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}} \to \widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}$ for the quotient map and $[x]$ for the equivalence class in $\widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}$ of an element $x \in \dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}$. For a continuous function $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ define $\widehat{\mathsf{F}}_\Lambda f : \widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}' \to \widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}} : [\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}(a_1,\dots,a_n)] \mapsto [\dot{\mathsf{F}}_\Lambda(\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}(a_1,\dots,a_n))]$.

Lemma 5.4. *The assignment $\widehat{\mathsf{F}}_\Lambda$ defines a contravariant functor.*

Proof. We need to prove functoriality of $\widehat{\mathsf{F}}_\Lambda$ and that $\widehat{\mathsf{F}}_\Lambda f$ is well defined for every continuous map $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$.

In order to show that $\widehat{\mathsf{F}}_\Lambda$ is well defined, it suffices to show that $\dot{\mathsf{F}}_\Lambda f$ is invariant under the congruence $\sim$. If $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ is a continuous, then

$$\begin{aligned}
&\textstyle\bigvee^{\uparrow}_{b'\in B}(\dot{\mathsf{F}}_\Lambda f)(\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}'}(a_1',\dots,a_{i-1}',b',a_{i+1}',\dots,a_n'))\\
&\qquad= \textstyle\bigvee^{\uparrow}_{b'\in B}(\mathsf{T}f)^{-1}(\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}'}(a_1',\dots,a_{i-1}',b',a_{i+1}',\dots,a_n'))\\
&\qquad= \textstyle\bigvee^{\uparrow}_{b'\in B}\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}(f^{-1}(a_1'),\dots,f^{-1}(a_{i-1}'),f^{-1}(b'),f^{-1}(a_{i+1}'),\dots,f^{-1}(a_n'))\\
&\qquad\sim \lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}(f^{-1}(a_1'),\dots,f^{-1}(a_{i-1}'),f^{-1}(\textstyle\bigvee^{\uparrow}B),f^{-1}(a_{i+1}'),\dots,f^{-1}(a_n'))\\
&\qquad= \dot{\mathsf{F}}_\Lambda f(\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}(a_1',\dots,a_{i-1}',\textstyle\bigvee^{\uparrow}B,a_{i+1}',\dots,a_n'))
\end{aligned}$$

so $\dot{\mathsf{F}}_\Lambda f$ is indeed invariant under the congruence. In the $\sim$-step, we use the fact that $\{f^{-1}(b') \mid b' \in B\}$ is directed in $\Omega\boldsymbol{\mathcal{X}}$. Functoriality of $\widehat{\mathsf{F}}_\Lambda f$ follows from functoriality of $\mathsf{Q}\circ\mathsf{T}\circ\mathsf{U}$. So $\widehat{\mathsf{F}}_\Lambda : \mathbf{Top} \to \mathbf{Frm}$ is a functor. □

We are now ready to define the topological Kupke-Kurz-Pattinson lift of a functor on **Set** together with a collection of predicate liftings, to a functor on **Top**.

Definition 5.5. Define the *topological Kupke-Kurz-Pattinson lift* (KKP lift for short) of T with respect to Λ to be the functor

$$\widehat{\mathsf{T}}_\Lambda = \mathsf{pt}\circ\widehat{\mathsf{F}}_\Lambda.$$

This is a functor **Top** → **Top**. (Note that since pt lands in **Sob** it restricts to an endofunctor on **Sob**.)

Let us put our theory into action. As stated in section 4 we can generalise the monotone functor $\mathsf{D}_{\mathbf{kh}}$ on **KHaus** from definition 4.1. We will show that lifting the monotone set-functor D with respect to the usual box and diamond lifting gives a functor which restricts to $\mathsf{D}_{\mathbf{kh}}$.

Example 5.6 (The monotone functor)**.** Recall the set functor D from example 2.2: $\mathsf{D} : X \to \{W \in \breve{\mathsf{P}}\breve{\mathsf{P}}X \mid W \text{ is up-closed under inclusion order}\}$. The box and diamond are given by the predicate liftings $\lambda^{\Box}, \lambda^{\Diamond} : \breve{\mathsf{P}} \to \breve{\mathsf{P}} \circ \mathsf{D}$ defined by

$$\lambda^{\Box}_X(a) \coloneqq \{W \in \mathsf{D}X \mid a \in W\}, \quad \lambda^{\Diamond}_X(a) \coloneqq \{W \in \mathsf{D}X \mid (X \setminus a) \notin W\},$$

where $X \in \mathbf{Set}$. Furthermore recall from definition 4.1 that for a compact Hausdorff space $\boldsymbol{\mathcal{X}}$ the space $\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}$ is the subset of $\mathsf{D}(\mathsf{U}\boldsymbol{\mathcal{X}})$ of collections of sets W satisfying for all $u \subseteq \mathsf{U}\boldsymbol{\mathcal{X}}$ that $u \in W$ iff there exists a closed $c \subseteq u$ such that every open superset of c is in W. So $\mathsf{U}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}) \subseteq \mathsf{D}(\mathsf{U}\boldsymbol{\mathcal{X}})$. The set $\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}$ is topologised by the subbase

$$\boxplus a \coloneqq \{W \in \mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}} \mid a \in W\}, \quad \diamondtimes a \coloneqq \{W \in \mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}} \mid (\boldsymbol{\mathcal{X}} \setminus a) \notin W\}.$$

By theorem 4.6 the functor $\mathsf{M} : \mathbf{Frm} \to \mathbf{Frm}$ from definition 4.4 is such that $\mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}}) \cong \mathsf{opn}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}})$ whenever $\boldsymbol{\mathcal{X}}$ is a compact Hausdorff space.

We claim that

$$\mathsf{D}_{\mathsf{kh}} = (\widehat{\mathsf{D}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}})_{\restriction \mathbf{KHaus}} \tag{2}$$

and to prove this we will show that $\widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}} = \mathsf{opn}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}})$ for every compact Hausdorff space $\boldsymbol{\mathcal{X}}$. Define a map $\varphi : \mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}}) \to \widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}}$ on generators by $\Box a \mapsto [\lambda^{\Box}(a)]$ and $\Diamond a \mapsto [\lambda^{\Diamond}(a)]$. This is well-defined because the $[\lambda^{\Box}(a)], [\lambda^{\Diamond}(a)]$ satisfy relations $(M_1) - (M_6)$ from definition 4.4 and it is surjective because the image of φ contains the generators of $\widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}}$.

So we only need to show injectivity of φ. Our strategy to prove this is to define a map $\psi : \widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}} \to \mathsf{opn}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}})$ and show that it is inverse to φ on the level of sets. Since a set-theoretic inverse suffices we do not need to prove that ψ is a homomorphism; we just want it to be well defined. Instead of defining $\psi : \widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}} \to \mathsf{opn}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}})$ directly, we will give a well-defined map $\psi' : \dot{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}} \to \mathsf{opn}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}})$ whose kernel contains the kernel of the quotient map $q_{\boldsymbol{\mathcal{X}}} : \dot{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}} \to \widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}}$. This in turn yields the map ψ we require. In a diagram:

$$\begin{tikzcd}
\dot{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}} \arrow[rr, "\psi'"] \arrow[dr, "q_{\boldsymbol{\mathcal{X}}}"'] & & \mathsf{opn}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}) \\
& \widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}} \arrow[ur, dashed, "\psi"'] &
\end{tikzcd} \tag{3}$$

Define $\psi' : \dot{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}} \to \mathsf{M}(\mathsf{opn}\boldsymbol{\mathcal{X}})$ on generators by $\lambda^{\Box}(a) \mapsto \boxplus a$ and $\lambda^{\Diamond}(a) \mapsto \diamondtimes a$. In order to show that this assignments yields a well-defined map (hence extends to a frame homomorphism by remark 2.7) we need to show that the presentation of an element in $\dot{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}}$ does not affect its image under ψ'. That is, if

$$\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^{\Box}(a_{i,j}) \cap \bigcap_{j'\in J'_i}\lambda^{\Diamond}(a_{i,j'})\Big) = \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^{\Box}(a_{k,\ell}) \cap \bigcap_{\ell'\in L'_k}\lambda^{\Diamond}(a_{k,\ell'})\Big), \tag{4}$$

where J_i, J'_i, L_k and L'_k are finite index sets, then

$$\bigcup_{i\in I}\Big(\bigcap_{j\in J}\psi'(\lambda^{\Box}(a_{i,j})) \cap \bigcap_{j'\in J'}\psi'(\lambda^{\Diamond}(a_{i,j'}))\Big) = \bigcup_{k\in K}\Big(\bigcap_{\ell\in L}\psi'(\lambda^{\Box}(a_{k,\ell})) \cap \bigcap_{\ell'\in L'}\psi'(\lambda^{\Diamond}(a_{k,\ell'}))\Big).$$

As stated we have $\mathsf{U}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}) \subseteq \mathsf{D}(\mathsf{U}\boldsymbol{\mathcal{X}})$. Observe

$$\psi'(\lambda^{\Box}(a)) = \boxplus a = \{W \in \mathsf{D}(\mathsf{U}\boldsymbol{\mathcal{X}}) \mid a \in W\} \cap \mathsf{U}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}) = \lambda^{\Box}(a) \cap \mathsf{U}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}),$$

and similarly $\psi'(\lambda^{\Diamond}(a)) = \lambda^{\Diamond}(a) \cap \mathsf{U}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}})$. Suppose the identity in (4) holds, then we have

$$\begin{aligned}
&\bigcup_{i\in I}\Big(\bigcap_{j\in J}\psi'(\lambda^{\Box}(a_{i,j})) \cap \bigcap_{j'\in J'}\psi'(\lambda^{\Diamond}(a_{i,j'}))\Big)\\
&\quad= \bigcup_{i\in I}\Big(\bigcap_{j\in J}(\lambda^{\Box}(a_{i,j}) \cap \mathsf{U}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}})) \cap \bigcap_{j'\in J'}(\lambda^{\Diamond}(a_{i,j'}) \cap \mathsf{U}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}))\Big)\\
&\quad= \bigcup_{i\in I}\Big(\mathsf{U}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}) \cap \bigcap_{j\in J}\lambda^{\Box}(a_{i,j}) \cap \bigcap_{j'\in J'}\lambda^{\Diamond}(a_{i,j'})\Big)\\
&\quad= \mathsf{U}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}) \cap \bigcup_{i\in I}\Big(\bigcap_{j\in J}\lambda^{\Box}(a_{i,j}) \cap \bigcap_{j'\in J'}\lambda^{\Diamond}(a_{i,j'})\Big)\\
&\quad= \mathsf{U}(\mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}) \cap \bigcup_{k\in K}\Big(\bigcap_{\ell\in L}\lambda^{\Box}(a_{k,\ell}) \cap \bigcap_{\ell'\in L'}\lambda^{\Diamond}(a_{k,\ell'})\Big)\\
&\quad= \bigcup_{k\in K}\Big(\bigcap_{\ell\in L}\psi'(\lambda^{\Box}(a_{k,\ell})) \cap \bigcap_{\ell'\in L'}\psi'(\lambda^{\Diamond}(a_{k,\ell'}))\Big).
\end{aligned}$$

So ψ' is well defined.

It is easy to see that $\bigvee^{\uparrow}{}_{b\in B}\lambda(b) \sim \lambda(\bigvee^{\uparrow}B)$ implies $\big(\bigvee^{\uparrow}{}_{b\in B}\lambda(b), \lambda(\bigvee^{\uparrow}B)\big) \in \ker(\psi)$ for $\lambda \in \{\lambda^{\Box}, \lambda^{\Diamond}\}$. Since these pairs generate the congruence of definition 5.3, we have $\sim\, = \ker(q_{\boldsymbol{\mathcal{X}}}) \subseteq \ker(\psi')$ and hence there exists a map $\psi : \widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}} \to \mathsf{opn}(\widehat{\mathsf{T}}\boldsymbol{\mathcal{X}})$ such that the diagram in (3) commutes. Therefore ψ is (well) defined on generators by $[\lambda^{\Box}(a)] \mapsto \Box a$ and $[\lambda^{\Diamond}(a)] \mapsto \Diamond a$. One can easily check that $\psi \circ \varphi = \mathrm{id}$ and $\varphi \circ \psi = \mathrm{id}$ by looking at the action on the generators. It follows that φ is injective.

This entails that for compact Hausdorff spaces $\boldsymbol{\mathcal{X}}$,

$$\widehat{\mathsf{D}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\boldsymbol{\mathcal{X}} = \mathsf{D}_{\mathsf{kh}}\boldsymbol{\mathcal{X}},$$

Furthermore, it can be seen that for continuous maps $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ we have $\mathsf{F}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}f = \mathsf{opn}(\mathsf{D}_{\mathsf{kh}}f)$. As a consequence, when restricted to **KHaus** we have (2) indeed. That is, lifting the monotone functor on **Set** with respect to the box/diamond lifting yields a generalisation of the monotone functor on **KHaus** from definition 4.1.

Example 5.7. Using similar techniques as in the previous example, one can show that, when restricted to **KHaus**, the topological Kupke-Kurz-Pattinson lift of P with respect to the usual box and diamond lifting coincides with the Vietoris functor. (An algebraic description similar to the one in theorem 4.6 is given in proposition III4.6 of [15].)

Example 5.8. Not every endofunctor on **Top** can be obtained as the lift of a **Set**-functor with respect to a (cleverly) chosen set of predicate liftings in the sense of definition 5.5. A trivial counterexample is the functor $\mathsf{F} : \mathbf{Top} \to \mathbf{Top}$ from example 3.7. For every topological space $\boldsymbol{\mathcal{X}}$ we have $\mathsf{F}\boldsymbol{\mathcal{X}} = \mathbf{2}$, which is not a T_0 space, hence not a sober space. Therefore F does not preserve sobriety, while every lifted functor automatically preserves sobriety. Thus F is not the lift of a **Set**-functor.

The next definition and lemma describe how to lift a predicate lifting to an open predicate lifting. Recall that $\mathsf{Z} : \mathbf{Frm} \to \mathbf{Set}$ is the forgetful functor which sends a frame to its underlying set.

Definition 5.9. Let Λ be a collection of predicate liftings for a functor $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$. A predicate lifting $\lambda : \breve{\mathsf{P}}^n \to \breve{\mathsf{P}} \circ \mathsf{T}$ in Λ induces an open predicate lifting $\widehat{\lambda} : \Omega^n \to \Omega \circ \widehat{\mathsf{T}}$ for $\widehat{\mathsf{T}}$ via

$$\Omega^n\boldsymbol{\mathcal{X}} \xrightarrow{\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}} \mathsf{Z}(\dot{\mathsf{F}}_{\Lambda}\boldsymbol{\mathcal{X}}) \xrightarrow{q_{\boldsymbol{\mathcal{X}}}} \mathsf{Z}(\widehat{\mathsf{F}}_{\Lambda}\boldsymbol{\mathcal{X}}) \xrightarrow{\mathsf{Z}k_{\widehat{\mathsf{F}}_{\Lambda}\boldsymbol{\mathcal{X}}}} \Omega(\mathsf{pt}(\widehat{\mathsf{F}}_{\Lambda}\boldsymbol{\mathcal{X}})) = \Omega(\widehat{\mathsf{T}}\boldsymbol{\mathcal{X}}).$$

By $\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}$ we actually mean the restriction of $\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}$ to $\Omega^n\boldsymbol{\mathcal{X}} \subseteq \breve{\mathsf{P}}(\mathsf{U}\boldsymbol{\mathcal{X}})$. The map $k_{\mathsf{F}\boldsymbol{\mathcal{X}}}$ is the frame homomorphism given by $a \mapsto \{p \in \mathsf{pt}(\mathsf{F}_\Lambda\boldsymbol{\mathcal{X}}) \mid p(a) = 1\}$. Define $\widehat{\Lambda} = \{\widehat{\lambda} \mid \lambda \in \Lambda\}$ to be the collection of lifted predicate liftings. Then $\widehat{\Lambda}$ is a geometric modal signature for $\widehat{\mathsf{T}}_\Lambda$.

Lemma 5.10. *The assignment $\widehat{\lambda}$ is a natural transformation.*

Proof. Let $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ be a continuous function, then the following diagram commutes in **Set**:

$$\begin{tikzcd}
\Omega^n\boldsymbol{\mathcal{X}}' \arrow[r, "\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}'}"] \arrow[d, "(f^{-1})^n"'] & \mathsf{Z}(\dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}') \arrow[r, "\mathsf{Z}q_{\boldsymbol{\mathcal{X}}'}"] \arrow[d, "(\mathsf{T}f)^{-1}"] & \mathsf{Z}(\widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}') \arrow[r, "\mathsf{Z}k_{\widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}'}"] \arrow[d, "(\mathsf{T}f)^{-1}"] & \Omega(\mathsf{pt}(\widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}')) \arrow[d, "\Omega(\mathsf{pt}((\mathsf{T}f)^{-1}))"] \\
\Omega^n\boldsymbol{\mathcal{X}} \arrow[r, "\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}"'] & \mathsf{Z}(\dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}) \arrow[r, "\mathsf{Z}q_{\boldsymbol{\mathcal{X}}}"'] & \mathsf{Z}(\widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}) \arrow[r, "\mathsf{Z}k_{\widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}}"'] & \Omega(\mathsf{pt}(\widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}))
\end{tikzcd}$$

Commutativity of the left square follows from naturality of λ, commutativity of the middle square follows from the proof of lemma 5.4 and commutativity of the right square can be seen as follows: let $a_1', \dots, a_n' \in \Omega\boldsymbol{\mathcal{X}}'$, then

$$\begin{aligned}
\Omega(\mathsf{pt}((\mathsf{T}f)^{-1})) \circ \mathsf{Z}k_{\mathsf{F}_\Lambda\boldsymbol{\mathcal{X}}'}(\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}'}(a_1', \dots, a_n')) &= \{q \in \mathsf{pt}(\mathsf{F}_\Lambda\boldsymbol{\mathcal{X}}) \mid q \circ (\mathsf{T}f)^{-1}(\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}'}(a_1', \dots, a_n')) = 1\} \\
&= \mathsf{Z}k_{\mathsf{F}_\Lambda\boldsymbol{\mathcal{X}}}((\mathsf{T}f)^{-1}(\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}'}(a_1', \dots, a_n'))).
\end{aligned}$$

So $\widehat{\lambda}$ is an open predicate lifting. □

The nature of the definitions of $\widehat{\mathsf{T}}$ and $\widehat{\Lambda}$ yields the following desirable result.

Proposition 5.11. *Let $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ be a functor and Λ a collection of predicate liftings for T. Then $\widehat{\Lambda}$ is characteristic for $\widehat{\mathsf{T}}_\Lambda$.*

Proof. Let $\boldsymbol{\mathcal{X}}$ be a topological space. We need to show that the collection

$$\{\widehat{\lambda}(a_1, \dots, a_n) \mid \lambda \in \Lambda \ n\text{-ary}, a_i \in \Omega\boldsymbol{\mathcal{X}}\} \tag{5}$$

forms a subbase for the topology on $\widehat{\mathsf{T}}_\Lambda\boldsymbol{\mathcal{X}}$. An arbitrary nonempty open set of $\widehat{\mathsf{T}}_\Lambda\boldsymbol{\mathcal{X}}$ is of the form $\widetilde{x} = \{p \in \mathsf{pt}(\widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}) \mid p(x) = 1\}$, for $x \in \widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}$. An arbitrary element of $\widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}$ is the equivalence class of an arbitrary union of finite intersections of elements of the form $\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}(a_1, \dots, a_n)$, for $\lambda \in \Lambda$ and $a_1, \dots, a_n \in \Omega\boldsymbol{\mathcal{X}}$. So we may write

$$x = \bigcup_{i \in I}\Big(\bigcap_{j \in J_i}[\lambda^{i,j}_{\mathsf{U}\boldsymbol{\mathcal{X}}}(a_1^{i,j}, \dots, a_{n_{i,j}}^{i,j})]\Big)$$

for some possibly infinite index set I, finite index sets J_i, $\lambda^{i,j} \in \Lambda$ and open sets $a_k^{i,j} \in \Omega\boldsymbol{\mathcal{X}}$. We get

$$\begin{aligned}
\widetilde{x} &= \bigcup_{i \in I}\Big(\bigcap_{j \in J_i}\widetilde{[\lambda^{i,j}_{\mathsf{U}\boldsymbol{\mathcal{X}}}(a_1^{i,j}, \dots, a_{n_{i,j}}^{i,j})]}\Big) && \text{(obvious)} \\
&= \bigcup_{i \in I}\Big(\bigcap_{j \in J_i}\widehat{\lambda}^{i,j}_{\boldsymbol{\mathcal{X}}}(a_1^{i,j}, \dots, a_{n_{i,j}}^{i,j})\Big). && \text{(definition 5.9)}
\end{aligned}$$

This shows that the open sets in (5) indeed form a subbase for the open sets of $\widehat{\mathsf{T}}_\Lambda\boldsymbol{\mathcal{X}}$. □

6 A final coalgebra

In this section we investigate the possibility of constructing a final modal in $\mathbf{Mod}(\mathsf{T})$ for a functor T. Our construction relies on the assumption that either T is an endofunctor on $\mathbf{Sob}$, or T is an endofunctor on $\mathbf{Top}$ which preserves sobriety. This need not be problematic: If a functor on $\mathbf{Top}$ does not preserve sobriety we can look at its sobrification. This only affects those images $\mathsf{T}\boldsymbol{\mathcal{X}}$ that are not sober. Furthermore, topological functors which arise as lifts from set functors using the procedure in section 5 automatically preserve sobriety.

Throughout this section, we fix an endofunctor $\mathsf{T} : \mathbf{Top} \to \mathbf{Top}$ which preserves sobriety, and a characteristic geometric modal signature Λ for T. Recall that Φ denotes a set of proposition letters.

Definition 6.1. Call two formulas φ and ψ *equivalent* on $\mathbf{Mod}(\mathsf{T})$ with respect to Λ, notation: $\varphi \equiv_{\mathsf{T},\Lambda} \psi$, if $\mathfrak{X}, x \Vdash \varphi$ iff $\mathfrak{X}, x \Vdash \psi$ for all geometric T-models $\mathfrak{X}$ and states $x \in \mathfrak{X}$. Denote the equivalence class of φ in $\mathrm{GML}(\Lambda)$ by $[\varphi]$. Let $\mathbf{Equiv}(\mathsf{T}, \Lambda, \Phi)$ be the collection of formulas modulo $\equiv_{\mathsf{T},\Lambda}$ and define disjunction and arbitrary conjunction by

$$[\varphi] \wedge [\psi] \coloneqq [\varphi \wedge \psi]$$
$$\bigvee_{i \in I} [\varphi_i] \coloneqq \Big[\bigvee_{i \in I} \varphi_i \Big].$$

It is easy to check that this is well-defined and satisfies the infinite distributive law, hence $\mathbf{Equiv}(\mathsf{T}, \Lambda, \Phi)$ forms a frame, called the *equivalence frame* of T with respect to Λ and Φ.

Since, in this section, T, Λ and Φ are fixed, we may abbreviate $\mathbf{E} = \mathbf{Equiv}(\mathsf{T}, \Lambda, \Phi)$ without any confusion. The theory of a point x in a geometric T-model $\mathfrak{X}$ is the collection of formulas that are true at x. The theory of x defines a completely prime filter in $\mathbf{E}$. This motivates the next definition.

Definition 6.2. Let $\boldsymbol{\mathcal{Z}} = \mathsf{pt}\mathbf{E}$. For every geometric T-model $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ define the theory map by

$$\mathrm{th}_{\mathfrak{X}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}} : x \mapsto \{[\varphi] \in \mathbf{E} \mid \mathfrak{X}, x \Vdash \varphi\}.$$

The space $\boldsymbol{\mathcal{Z}}$ will turn out to be the state space of a final model in $\mathbf{Mod}(\mathsf{T})$ and we will see in proposition 6.9 that the theory maps are T-model morphisms.

Definition 6.3. Define $\mathsf{L} : \mathbf{Frm} \to \mathbf{Frm}$ by

$$\mathsf{L} = \mathsf{opn} \circ \mathsf{T} \circ \mathsf{pt}.$$

Obviously, this functor restricts to an endofunctor on $\mathbf{SFrm}$, which is dual to the restriction of T to $\mathbf{Sob}$.

We will now endow $\mathbf{E}$ with an L-algebra structure. Since Λ is characteristic, the frame $\mathsf{L}\mathbf{E}$ is generated by the sets $\{\lambda_{\boldsymbol{\mathcal{X}}}(a_1, \ldots, a_n) \mid \lambda \in \Lambda, a_i \in \Omega\boldsymbol{\mathcal{Z}}\}$. Therefore we can define an assignment on these generators and use remark 2.7 to extend this to a frame homomorphism $\mathsf{L}\mathbf{E} \to \mathbf{E}$. By definition all open sets of $\mathsf{pt}\mathbf{E}$ are of the form $\widetilde{[\varphi]}$ for some formula $\varphi \in \mathrm{GML}(\Lambda)$.

Definition 6.4. Endow $\mathbf{E}$ with an L-algebra structure $\delta : \mathsf{L}\mathbf{E} \to \mathbf{E}$, where δ is defined on generators by

$$\delta : \mathsf{L}\mathbf{E} \to \mathbf{E} : \lambda_{\boldsymbol{\mathcal{Z}}}(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n) \mapsto [\heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n)].$$

We need to show that δ is well defined. For this purpose it suffices to show that the images of the generators of $\mathbf{E}$ satisfy the same relations that they satisfy in $\mathsf{L}\mathbf{E}$. Recall $\mathcal{Z} = \mathsf{pt}\mathbf{E}$, then $\mathsf{L}\mathbf{E} = \mathsf{opn}(\mathsf{T}\mathcal{Z})$.

Lemma 6.5. *If*

$$\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^{i,j}_{\mathcal{Z}}(\widetilde{\varphi}^{i,j}_1,\dots,\widetilde{\varphi}^{i,j}_{n_{i,j}})\Big) = \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^{k,\ell}_{\mathcal{Z}}(\widetilde{\varphi}^{k,\ell}_1,\dots,\widetilde{\varphi}^{k,\ell}_{n_{k,\ell}})\Big) \tag{6}$$

then

$$\bigvee_{i\in I}\Big(\bigwedge_{j\in J_i}\heartsuit^{\lambda^{i,j}}(\varphi^{i,j}_1,\dots,\varphi^{i,j}_{n_{i,j}})\Big) \equiv_{\mathsf{T},\Lambda} \bigvee_{k\in K}\Big(\bigwedge_{\ell\in L_k}\heartsuit^{\lambda^{k,\ell}}(\varphi^{k,\ell}_1,\dots,\varphi^{k,\ell}_{n_{k,\ell}})\Big), \tag{7}$$

where the J_i and L_k are finite index sets and I and K are index sets of arbitrary size.

Proof. We will see that this follows from naturality of λ. Our strategy is to show that the truth sets of the left hand side and right hand side of (7) coincide in every geometric T-model $\mathfrak{X} = (\mathcal{X},\gamma,V)$.

Observe that the map $\mathrm{th}_{\mathfrak{X}} : \mathcal{X} \to \mathcal{Z}$, which sends a point to its theory, is continuous because

$$\mathrm{th}^{-1}_{\mathfrak{X}}(\widetilde{\varphi}) = [\![\varphi]\!]^{\mathfrak{X}}, \tag{8}$$

which is open in $\mathcal{X}$ for all formulas φ. Compute

$$\begin{aligned}
&\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^{i,j}_{\mathcal{X}}([\![\varphi^{i,j}_1]\!]^{\mathfrak{X}},\dots,[\![\varphi^{i,j}_{n_{i,j}}]\!]^{\mathfrak{X}})\Big)\\
&\qquad= \bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^{i,j}_{\mathcal{X}}(\mathrm{th}^{-1}_{\mathfrak{X}}(\widetilde{\varphi}^{i,j}_1),\dots,\mathrm{th}^{-1}_{\mathfrak{X}}(\widetilde{\varphi}^{i,j}_{n_{i,j}}))\Big) && \text{(by (8))}\\
&\qquad= \bigcup_{i\in I}\Big(\bigcap_{j\in J_i}(\mathsf{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}\big(\lambda^{i,j}_{\mathcal{Z}}(\widetilde{\varphi}^{i,j}_1,\dots,\widetilde{\varphi}^{i,j}_{n_{i,j}})\big)\Big) && \text{(naturality of } \lambda)\\
&\qquad= (\mathsf{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}\Big(\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^{i,j}_{\mathcal{Z}}(\widetilde{\varphi}^{i,j}_1,\dots,\widetilde{\varphi}^{i,j}_{n_{i,j}})\Big)\Big) && (\star)\\
&\qquad= (\mathsf{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}\Big(\bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^{k,\ell}_{\mathcal{Z}}(\widetilde{\psi}^{k,\ell}_1,\dots,\widetilde{\psi}^{k,\ell}_{n_{k,\ell}})\Big)\Big) && \text{(assumption (6))}\\
&\qquad= \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}(\mathsf{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}\big(\lambda^{k,\ell}_{\mathcal{Z}}(\widetilde{\psi}^{k,\ell}_1,\dots,\widetilde{\psi}^{k,\ell}_{n_{k,\ell}})\big)\Big) && (\star)\\
&\qquad= \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^{k,\ell}_{\mathcal{X}}(\mathrm{th}^{-1}_{\mathfrak{X}}(\psi^{k,\ell}_1),\dots,\mathrm{th}^{-1}_{\mathfrak{X}}(\psi^{k,\ell}_{n_{k,\ell}}))\Big) && \text{(naturality of } \lambda)\\
&\qquad= \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^{k,\ell}_{\mathcal{X}}([\![\psi^{k,\ell}_1]\!]^{\mathfrak{X}},\dots,[\![\psi^{k,\ell}_{n_{k,\ell}}]\!]^{\mathfrak{X}})\Big). && \text{(by (8))}
\end{aligned}$$

The steps with $(\star)$ hold because inverse images of maps preserve all unions and intersections. This entails that for all geometric T-models and all states x in $\mathfrak{X}$ we have

$$\mathfrak{X},x \Vdash \bigvee_{i\in I}\Big(\bigwedge_{j\in J_i}\heartsuit^{\lambda^{i,j}}(\varphi^{i,j}_1,\dots,\varphi^{i,j}_{n_{i,j}})\Big) \quad\text{iff}\quad \mathfrak{X},x \Vdash \bigvee_{k\in K}\Big(\bigwedge_{\ell\in L_k}\heartsuit^{\lambda^{k,\ell}}(\varphi^{k,\ell}_1,\dots,\varphi^{k,\ell}_{n_{k,\ell}})\Big),$$

and hence (7) holds. Therefore δ is well defined. □

Since we defined δ on generators of $\mathsf{L}\mathbf{E}$, by remark 2.7 it extends to a frame homomorphism which, by abuse of notation, we shall also call δ. The algebra structure on $\mathbf{E}$ entails a coalgebra structure on $\mathcal{Z}$.

Definition 6.6. Let $\zeta : \mathcal{Z} \to \mathsf{T}\mathcal{Z}$ be the composition

$$\mathsf{pt}\mathbf{E} \xrightarrow{\mathsf{pt}\delta} \mathsf{pt}(\mathsf{L}\mathbf{E}) = = \mathsf{pt}(\mathsf{opn}(\mathsf{T}(\mathsf{pt}\mathbf{E}))) \xrightarrow{k^{-1}_{\mathsf{T}(\mathsf{pt}\mathbf{E})}} \mathsf{T}(\mathsf{pt}\mathbf{E})$$

Here $k_{\mathsf{T}(\mathsf{pt}\mathbf{E})} : \mathsf{T}(\mathsf{pt}\mathbf{E}) \to \mathsf{pt}(\mathsf{opn}(\mathsf{T}(\mathsf{pt}\mathbf{E})))$ is the isomorphism given in remark 2.10. Since $\mathcal{Z} = \mathsf{pt}\mathbf{E}$ this indeed defines a map $\mathcal{Z} \to \mathsf{T}\mathcal{Z}$.

For an object $\Gamma \in \mathcal{Z}$, the element $(\mathsf{pt}\delta)(\Gamma)$ is the completely prime filter

$$F = \{\lambda(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) \in \mathsf{pt}(\mathsf{opn}(\mathsf{T}(\mathsf{pt}\mathbf{E}))) \mid [\heartsuit^\lambda(\varphi_1, \dots, \varphi_n)] \in \Gamma\}$$

in $\mathsf{pt}(\mathsf{opn}(\mathsf{T}(\mathsf{pt}\mathbf{E})))$. The element $\zeta(\Gamma)$ is the unique element in $\mathsf{T}(\mathsf{pt}\mathbf{E})$ corresponding to F under the isomorphism $k_{\mathsf{T}(\mathsf{pt}\mathbf{E})}$. By definition of $k_{\mathsf{T}(\mathsf{pt}\mathbf{E})}$, this is the unique element in the intersection of

$$\{\lambda(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) \mid [\heartsuit^\lambda(\varphi_1, \dots, \varphi_n)] \in \Gamma\}.$$

Moreover, it follows from the definition of $k_{\mathsf{T}(\mathsf{pt}\mathbf{E})}$ that $[\heartsuit^\lambda(\varphi_1, \dots, \varphi_n)] \notin \Gamma$ implies $\zeta(\Gamma) \notin \lambda(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n)$.

Notation. If no confusion is likely to occur we will omit the square brackets that indicate equivalence classes of formulas in $\mathbf{E}$. That is, we shall write $\varphi \in \mathbf{E}$ instead of $[\varphi] \in \mathbf{E}$.

We can now endow the T-coalgebra $(\mathcal{Z}, \zeta)$ with a valuation. Thereafter we will show that $(\mathcal{Z}, \zeta)$ together with this valuation is final in $\mathbf{Mod}(\mathsf{T})$.

Definition 6.7. Let $V_{\mathcal{Z}} : \Phi \to \Omega\mathcal{Z}$ be the valuation $p \mapsto \widetilde{p}$.

The triple $\mathfrak{Z} = (\mathcal{Z}, \zeta, V_{\mathcal{Z}})$ is a geometric T-model, simply because it is a T-coalgebra with a valuation. We can prove a truth lemma for $\mathfrak{Z}$:

Lemma 6.8 (Truth lemma)**.** *We have* $\mathfrak{Z}, \Gamma \Vdash \varphi$ *iff* $\varphi \in \Gamma$.

Proof. Use induction on the complexity of the formula. The propositional case follows immediately from the definition of $V_{\mathcal{Z}}$. The cases $\varphi = \varphi_1 \wedge \varphi_2$ and $\varphi = \bigvee_{i \in i} \varphi_i$ are routine. So suppose $\varphi = \heartsuit^\lambda(\varphi_1, \dots, \varphi_n)$. We have

$$\begin{aligned}
\mathfrak{Z}, \Gamma \Vdash \heartsuit^\lambda(\varphi_1, \dots, \varphi_n) \quad &\text{iff} \quad \zeta(\Gamma) \in \lambda_{\mathcal{Z}}([\![\varphi_1]\!]^{\mathfrak{Z}}, \dots, [\![\varphi_n]\!]^{\mathfrak{Z}}) && \text{(definition of } \Vdash) \\
&\text{iff} \quad \zeta(\Gamma) \in \lambda_{\mathcal{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) && \text{(induction)} \\
&\text{iff} \quad \heartsuit^\lambda(\varphi_1, \dots, \varphi_n) \in \Gamma. && \text{(definition of } \zeta)
\end{aligned}$$

This proves the lemma. □

Proposition 6.9. *For every geometric* T*-model* $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ *the map* $\mathrm{th}_{\mathfrak{X}} : \mathcal{X} \to \mathcal{Z}$ *is a* T*-model morphism.*

Proof. We need to show that $\mathrm{th}_{\mathfrak{X}}$ is a T-coalgebra morphism and that $\mathrm{th}_{\mathfrak{X}}^{-1} \circ V_{\mathfrak{Z}} = V$. The latter follows from the fact that for every proposition letter p we have

$$V(p) = \{x \in X \mid \mathfrak{X}, x \Vdash p\} = \mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{p}) = \mathrm{th}_{\mathfrak{X}}^{-1}(V_{\mathfrak{Z}}(p)).$$

In order to show that $\mathrm{th}_{\mathfrak{X}}$ is a T-coalgebra morphism, we have to show that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\mathrm{th}_{\mathfrak{X}}"] \arrow[d, "\gamma"'] & \mathcal{Z} \arrow[d, "\zeta"] \\
\mathsf{T}\mathcal{X} \arrow[r, "\mathsf{T}\,\mathrm{th}_{\mathfrak{X}}"'] & \mathsf{T}\mathcal{Z}
\end{tikzcd}$$

Let $x \in \boldsymbol{\mathcal{X}}$. Since $\mathsf{T}\boldsymbol{\mathcal{Z}}$ is sober, hence T_0, it suffices to show that $\mathsf{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x))$ and $\zeta(\mathrm{th}_{\mathfrak{X}}(x))$ are in precisely the same opens of $\mathsf{T}\boldsymbol{\mathcal{Z}}$. Moreover, we know that the open sets of $\mathsf{T}\boldsymbol{\mathcal{Z}}$ are generated by the sets $\lambda_{\boldsymbol{\mathcal{Z}}}(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$, so it suffices to show that for all $\lambda \in \Lambda$ and $\varphi_i \in \mathrm{GML}(\Lambda)$ we have

$$\mathsf{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x)) \in \lambda_{\boldsymbol{\mathcal{Z}}}(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n) \quad \text{iff} \quad \zeta(\mathrm{th}_{\mathfrak{X}}(x)) \in \lambda_{\boldsymbol{\mathcal{Z}}}(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n).$$

This follows from the following computation,

$$\begin{aligned}
&\mathsf{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x)) \in \lambda_{\boldsymbol{\mathcal{Z}}}(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n) && \\
&\qquad \text{iff} \quad \gamma(x) \in (\mathsf{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}(\lambda_{\boldsymbol{\mathcal{Z}}}(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)) && \\
&\qquad \text{iff} \quad \gamma(x) \in \lambda_{\boldsymbol{\mathcal{X}}}(\mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}_1), \ldots, \mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}_n)) && \text{(naturality of } \lambda) \\
&\qquad \text{iff} \quad \gamma(x) \in \lambda_{\boldsymbol{\mathcal{X}}}([\![\varphi_1]\!]^{\mathfrak{X}}, \ldots, [\![\varphi_n]\!]^{\mathfrak{X}}) && \text{(by (8))} \\
&\qquad \text{iff} \quad \mathfrak{X}, x \Vdash \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n) && \text{(definition of } \Vdash) \\
&\qquad \text{iff} \quad \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n) \in \mathrm{th}_{\mathfrak{X}}(x) && \text{(definition of } \mathrm{th}_{\mathfrak{X}}) \\
&\qquad \text{iff} \quad \zeta(\mathrm{th}_{\mathfrak{X}}(x)) \in \lambda_{\boldsymbol{\mathcal{Z}}}(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n) && \text{(definition of } \zeta)
\end{aligned}$$

This proves the proposition. □

The developed theory results in the following theorem.

Theorem 6.10. *Let* T *be an endofunctor on* **Top** *which preserves sobriety, and* Λ *a characteristic geometric modal signature for* T*. Then the geometric* T*-model* $\mathfrak{Z} = (\boldsymbol{\mathcal{Z}}, \zeta, V_{\boldsymbol{\mathcal{Z}}})$ *is final in* **Mod**(T)*.*

Proof. Proposition 6.9 states that for every geometric T-model $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ there exists a T-coalgebra morphism $\mathrm{th}_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{Z}$, so we only need to show that this morphism is unique.

Let $f : \mathfrak{X} \to \mathfrak{Z}$ be any coalgebra morphism. We know from proposition 3.5 that coalgebra morphisms preserve truth, so for all $x \in \boldsymbol{\mathcal{X}}$ we have $\varphi \in f(x)$ iff $\mathfrak{Z}, f(x) \Vdash \varphi$ iff $\mathfrak{X}, x \Vdash \varphi$. Therefore we must have $f(x) = \mathrm{th}_{\mathfrak{X}}(x)$. □

As an immediate corollary we obtain the following theorem. Recall from definition 3.6 that two states x and x' are behaviourally equivalent in **Mod**(T) if there are T-model morphisms f and f' with $f(x) = f'(x')$.

Theorem 6.11. *Let* $\mathfrak{X}$ *and* $\mathfrak{X}'$ *be two geometric* T*-models and* x, x' *two states in* $\mathfrak{X}$ *and* $\mathfrak{X}'$ *respectively. Then* x *and* x' *are modally equivalent if and only if they are behaviourally equivalent.*

Proof. If x and x' are behaviourally equivalent, then they are modally equivalent by proposition 3.5. Conversely, if they are modally equivalent, then $\mathrm{th}_{\mathfrak{X}}(x) = \mathrm{th}_{\mathfrak{X}'}(x')$ by construction, so they are behaviourally equivalent. □

Remark 6.12. If T is an endofunctor on **Sob** instead of **Top**, then the same procedure yields a final model in **Mod**(T). In particular, T need not be the restriction of a **Top**-endofunctor. However, if T is an endofunctor on **KSob** or **KHaus** the procedure above does not guarantee a final coalgebra in **Mod**(T); indeed the state space $\boldsymbol{\mathcal{Z}}$ of the final coalgebra $\mathfrak{Z}$ we just constructed need not be compact sober nor compact Hausdorff.

Of course, there may be a different way to attain similar results for **KSob** or **KHaus**. We leave this as an interesting open question. In theorem 7.9 we prove an analog of theorem 6.11 for endofunctors on **KSob**.

7 Bisimulations

This section is devoted to bisimulations and bisimilarity between coalgebraic geometric models. We compare two notions of bisimilarity, modal equivalence (from definition 3.4) and behavioural equivalence (definition 3.6). Again, where $\mathbf{C}$ is be a full subcategory of $\mathbf{Top}$ and T an endofunctor on $\mathbf{C}$, we give definitions and propositions in this generality where possible. When necessary, we will restrict our scope to particular instances of $\mathbf{C}$.

Definition 7.1. Let $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ and $\mathfrak{X}' = (\boldsymbol{\mathcal{X}}', \gamma', V')$ be two geometric T-models. Let $\boldsymbol{\mathcal{B}}$ be an object in $\mathbf{C}$ such that $\mathsf{U}\boldsymbol{\mathcal{B}} \subseteq \mathsf{U}\boldsymbol{\mathcal{X}} \times \mathsf{U}\boldsymbol{\mathcal{X}}'$, with projections $\pi : \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{X}}$ and $\pi' : \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{X}}'$. Then $\boldsymbol{\mathcal{B}}$ is called an *Aczel-Mendler bisimulation* between $\mathfrak{X}$ and $\mathfrak{X}'$ if for all $(x, x') \in \boldsymbol{\mathcal{B}}$ we have $x \in V(p)$ iff $x' \in V'(p)$ and there exists a transition map $\beta : \boldsymbol{\mathcal{B}} \to \mathsf{T}\boldsymbol{\mathcal{B}}$ that makes π and π' coalgebra morphisms. That is, β is such that the following diagram commutes:

$$\begin{tikzcd}
\boldsymbol{\mathcal{X}} \arrow[d, "\gamma"'] & \boldsymbol{\mathcal{B}} \arrow[l, "\pi"'] \arrow[r, "\pi'"] \arrow[d, "\beta"] & \boldsymbol{\mathcal{X}}' \arrow[d, "\gamma'"] \\
\mathsf{T}\boldsymbol{\mathcal{X}} & \mathsf{T}\boldsymbol{\mathcal{B}} \arrow[l, "\mathsf{T}\pi"] \arrow[r, "\mathsf{T}\pi'"'] & \mathsf{T}\boldsymbol{\mathcal{X}}'
\end{tikzcd}$$

Two states $x \in \mathsf{U}\boldsymbol{\mathcal{X}}, x' \in \mathsf{U}\boldsymbol{\mathcal{X}}'$ are called *bisimilar*, notation $x \leftrightarroweq x'$, if they are linked by a coalgebra bisimulation.

It follows from proposition 3.5 that bisimilar states satisfy the same formulas. Consequently, if T is an endofunctor on $\mathbf{Top}$ which preserves sobriety and Λ is characteristic, it follows from the previous lemma combined with theorem 6.11 that Aczel-Mendler bisimilarity implies behavioural equivalence. If moreover T preserves weak pullbacks, the converse holds as well. The proof of this is similar to theorem 4.3 and the preceding discussion in [24].

However, we do not wish to make this assumption on topological spaces, since few functors seem to preserve weak pullbacks. For example, the Vietoris functor does not preserve weak pullbacks [4, Corollary 4.3] and neither does the monotone functor from definition 4.1. (To see the latter statement, consider the example given in section 4 of [13] to show that UpP does not preserve weak pullbacks. This uses only finite frames. Endowing these frames with the discrete topology yields the result for $\mathsf{D}_{\mathbf{kh}}$.) Therefore we define Λ-bisimulations for $\mathbf{Top}$-coalgebras as an alternative to Aczel-Mendler bisimulations. This notion is an adaptation of ideas in [2, 10]. Under some conditions on Λ, Λ-bisimilarity coincides with behavioural equivalence.

In the next definition we need the concept of coherent pairs: If X and X' are two sets and $B \subseteq X \times X'$ is a relation, then a pair $(a, a') \in \mathsf{P}X \times \mathsf{P}X'$ is called *B-coherent* if $B[a] \subseteq a'$ and $B^{-1}[a'] \subseteq a$. For details and properties see section 2 in [14].

Definition 7.2. Let T be an endofunctor on $\mathbf{C}$, Λ a geometric modal signature for T and $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ and $\mathfrak{X}' = (\boldsymbol{\mathcal{X}}', \gamma', V')$ two geometric T-models. A *Λ-bisimulation* between $\mathfrak{X}$ and $\mathfrak{X}'$ is a relation $B \subseteq \mathsf{U}\boldsymbol{\mathcal{X}} \times \mathsf{U}\boldsymbol{\mathcal{X}}'$ such that for all $(x, x') \in B$ and $p \in \Phi$ and all tuples of B-coherent pairs of opens $(a_i, a_i') \in \Omega\boldsymbol{\mathcal{X}} \times \Omega\boldsymbol{\mathcal{X}}'$ we have

$$x \in V(p) \quad \text{iff} \quad x' \in V'(p)$$

and

$$\gamma(x) \in \lambda_{\boldsymbol{\mathcal{X}}}(a_1, \ldots, a_n) \quad \text{iff} \quad \gamma'(x') \in \lambda_{\boldsymbol{\mathcal{X}}'}(a_1', \ldots, a_n'). \tag{9}$$

Two states are called Λ-bisimilar if there is a Λ-bisimulation linking them, notation: $x \leftrightarroweq_\Lambda x'$.

As desired, Λ-bisimilar states satisfy the same formulas.

Proposition 7.3. *Let* T *be an endofunctor on* C *and* Λ *a geometric modal signature for* T. *Then* $\leftrightarroweq_\Lambda \subseteq \equiv_\Lambda$.

Proof. Let B be a Λ-bisimulation between two geometric T-models $\mathfrak{X}$ and $\mathfrak{X}'$, and suppose xBx'. We will show that for every formula φ we have $\mathfrak{X}, x \Vdash \varphi$ iff $\mathfrak{X}', x' \Vdash \varphi$, using induction on the complexity of the formula. The propositional case is by definition. If φ is a finite meet or an arbitrary join of formulas then the lemma is routine. Suppose $\mathfrak{X}, x \Vdash \heartsuit^\lambda(\varphi_1, \ldots, \varphi_n)$, then $\gamma(x) \in \lambda_{\boldsymbol{\mathcal{X}}}([\![\varphi_1]\!]^{\mathfrak{X}}, \ldots, [\![\varphi_n]\!]^{\mathfrak{X}})$. By the induction hypothesis, $([\![\varphi_i]\!]^{\mathfrak{X}}, [\![\varphi_i]\!]^{\mathfrak{X}'})$ is a B-coherent pair for all i, so by definition of a Λ-bisimulation we find $\gamma'(x') \in \lambda_{\boldsymbol{\mathcal{X}}'}([\![\varphi_1]\!]^{\mathfrak{X}'}, \ldots, [\![\varphi_n]\!]^{\mathfrak{X}'})$ and hence $\mathfrak{X}', x' \Vdash \heartsuit^\lambda(\varphi_1, \ldots, \varphi_n)$. The converse direction is proven symmetrically. □

Proposition 7.4. *Let* T *be an endofunctor on* C *and* Λ *a geometric modal signature for* T. *Then* $\leftrightarroweq \subseteq \leftrightarroweq_\Lambda$.

Proof. It suffices to show that every Aczel-Mendler bisimulation is a Λ-bisimulation. Suppose B is an Aczel-Mendler bisimulation and let β be the map that turns B into a coalgebra, then the following diagram commutes:

$$\begin{tikzcd}
\boldsymbol{\mathcal{X}} \arrow[d, "\gamma"'] & B \arrow[l, "\pi"'] \arrow[r, "\pi'"] \arrow[d, "\beta"] & \boldsymbol{\mathcal{X}}' \arrow[d, "\gamma'"] \\
\mathsf{T}\boldsymbol{\mathcal{X}} & \mathsf{T}B \arrow[l, "\mathsf{T}\pi"] \arrow[r, "\mathsf{T}\pi'"'] & \mathsf{T}\boldsymbol{\mathcal{X}}'
\end{tikzcd} \tag{10}$$

We will show that B is a Λ-bisimulation. By definition $x \in V(p)$ iff $x' \in V'(p)$ whenever xBx'. We prove the forth condition from definition 7.2. Let $\lambda \in \Lambda$ and $(x, x') \in B$. Suppose $(a_1, a_1'), \ldots, (a_n, a_n')$ are B-coherent pairs of opens and $\gamma(x) \in \lambda_{\boldsymbol{\mathcal{X}}}(a_1, \ldots, a_n)$. Then we have

$$\begin{aligned}
\beta(x, x') &\in (\mathsf{T}\pi)^{-1}(\lambda_{\boldsymbol{\mathcal{X}}}(a_1, \ldots, a_n)) && \text{(follows from (10))} \\
&= \lambda_B(\pi^{-1}(a_1), \ldots, \pi^{-1}(a_n)) && \text{(naturality of } \lambda) \\
&\subseteq \lambda_B\big((\pi')^{-1} \circ \pi'[\pi^{-1}(a_1)], \ldots, (\pi')^{-1} \circ \pi'[\pi^{-1}(a_n)]\big) && \text{(monotonicity of } \lambda) \\
&= \lambda_B\big((\pi')^{-1}(B[a_1]), \ldots, (\pi')^{-1}(B[a_n])\big) && (B[a] = \pi' \circ \pi^{-1}(a)) \\
&\subseteq \lambda_B((\pi')^{-1}(a_1'), \ldots, (\pi')^{-1}(a_n')) && \text{(monotonicity of } \lambda) \\
&= (\mathsf{T}\pi')^{-1}(\lambda_{\boldsymbol{\mathcal{X}}'}(a_1', \ldots, a_n')). && \text{(naturality of } \lambda)
\end{aligned}$$

Therefore

$$\gamma'(x') = (\mathsf{T}\pi')(\beta(x, x')) \in \lambda_{\boldsymbol{\mathcal{X}}'}(a_1', \ldots, a_n'),$$

as desired. □

The collection of Λ-bisimulations between two models enjoys the following interesting property.

Proposition 7.5. *Let* Λ *be a geometric modal signature of a functor* $\mathsf{T} : \mathsf{Top} \to \mathsf{Top}$ *and let* $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ *and* $\mathfrak{X}' = (\boldsymbol{\mathcal{X}}', \gamma', V')$ *be two geometric* T*-models. The collection of* Λ*-bisimulations between* $\mathfrak{X}$ *and* $\mathfrak{X}'$ *forms a complete lattice.*

Proof. It is obvious that the collection of Λ-bisimulations is a poset. We will show that this collection is closed under taking arbitrary unions; the result then follows from theorem 4.2 in [5].

Let J be some index set and for all $j \in J$ let B_j be Λ-bisimulations between $\mathfrak{X}$ and $\mathfrak{X}'$ and set $B = \bigcup_{j \in J} B_j$. We claim that B is a Λ-bisimulation.

Let (a_i, a_i') be B-coherent pairs of opens. Suppose xBx' and $\gamma(x) \in \lambda_{\boldsymbol{\mathcal{X}}}(a_1, \dots, a_n)$. Then there is $j \in J$ with xB_jx' hence $x \in V(p)$ iff $x' \in V'(p)$. As $B_j[a_i] \subseteq B[a] \subseteq a_i'$ and $B_j^{-1}[a'] \subseteq B^{-1}[a'] \subseteq a_i$, all B-coherent pairs (a_i, a_i') are also B_j-coherent. Since B_j is a Λ-bisimulation we get $\gamma'(x') \in \lambda_{\boldsymbol{\mathcal{X}}'}(a_1', \dots, a_n')$. The converse direction is proven symmetrically. □

We know by now that Λ-bisimilarity implies modal equivalence. Furthermore, if T is an endofunctor on **Top** which preserves sobriety, modal equivalence implies behavioural equivalence. In order to prove a converse, i.e. that behavioural equivalence implies Λ-bisimilarity, we need to assume that the geometric modal signature is strong.

Recall that two elements x, x' in two models are behaviourally equivalent in $\mathbf{Mod}(\mathsf{T})$, notation: $\simeq_{\mathbf{Mod}(\mathsf{T})}$, if there exist morphisms f, f' in $\mathbf{Mod}(\mathsf{T})$ such that $f(x) = f'(x')$.

Proposition 7.6. *Let T be an endofunctor on $\mathbf{C}$ and Λ a strongly monotone geometric modal signature for T. Let $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ and $\mathfrak{X}' = (\boldsymbol{\mathcal{X}}', \gamma', V')$ be two geometric T-models. Then $\simeq_{\mathbf{Mod}(\mathsf{T})} \subseteq \;\underline{\leftrightarrow}_\Lambda$.*

Proof. Suppose x and x' are behaviourally equivalent. Then there are some geometric T-model $\mathfrak{Y} = (\boldsymbol{\mathcal{Y}}, \nu, V_{\boldsymbol{\mathcal{Y}}})$ and T-model morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ and $f' : \mathfrak{X}' \to \mathfrak{Y}$ such that $f(x) = f'(x')$. We will define a Λ-bisimulation B linking x and x'.

Let B be the pullback of f and f' in $\mathbf{C}$,

$$B = \{(u, u') \in X \times X' \mid f(u) = f'(u')\}. \tag{11}$$

Then clearly xBx'. It follows from proposition 3.5 that u and u' satisfy precisely the same formulas whenever $(u, u') \in B$.

Suppose $\lambda \in \Lambda$ is n-ary and for $1 \le i \le n$ let (a_i, a_i') be a B-coherent pair of opens. Suppose uBu' and $\gamma(u) \in \lambda_{\boldsymbol{\mathcal{X}}}(a_1, \dots, a_n)$. We will show that $\gamma'(u') \in \lambda_{\boldsymbol{\mathcal{X}}'}(a_1', \dots, a_n')$, the converse direction is similar. By monotonicity and naturality of λ we obtain

$$\gamma(u) \in \lambda_{\boldsymbol{\mathcal{X}}}(a_1, \dots, a_n) \subseteq \lambda_{\boldsymbol{\mathcal{X}}}(f^{-1}(f[a_1]), \dots, f^{-1}(f[a_n])) = (\mathsf{T}f)^{-1}(\lambda_{\boldsymbol{\mathcal{Y}}}(f[a_1], \dots, f[a_n])), \tag{12}$$

so $(\mathsf{T}f)(\gamma(u)) \in \lambda_{\boldsymbol{\mathcal{Y}}}(f[a_1], \dots, f[a_n])$. (Note that the $f[a_i]$ need not be open in $\boldsymbol{\mathcal{Y}}$, but $\lambda_{\boldsymbol{\mathcal{Y}}}(f[a_1], \dots, f[a_n])$ is defined because λ is assumed to be strong.) Since f and f' are coalgebra morphisms and $f(u) = f'(u')$ we have $(\mathsf{T}f)(\gamma(u)) = \nu(f(u)) = \nu(f'(u')) = (\mathsf{T}f')(\gamma'(u'))$. Coherence of (a_i, a_i') and monotonicity and naturality of λ yield

$$\begin{aligned}
\gamma'(u') &\in (\mathsf{T}f')^{-1}(\lambda_{\boldsymbol{\mathcal{Y}}}(f[a_1], \dots, f[a_n])) \\
&= \lambda_{\boldsymbol{\mathcal{X}}'}((f')^{-1}(f[a_1]), \dots, (f')^{-1}(f[a_n])) && \text{(naturality of } \lambda) \\
&= \lambda_{\boldsymbol{\mathcal{X}}'}(B[a_1], \dots, B[a_n]) && \text{(strong monotonicity of } \lambda) \\
&\subseteq \lambda_{\boldsymbol{\mathcal{X}}'}(a_1', \dots, a_n'). && \text{(coherence of } (a_i, a_i'))
\end{aligned}$$

This proves the proposition. □

Remark 7.7. If $\mathbf{C} = \mathbf{KHaus}$ in the proposition above, then proposition 3.11 allows us to drop the assumption that Λ be strong.

Let T be an endofunctor on $\mathbf{Top}$ and let Λ be a geometric modal signature for T. The following diagram summarises the results from propositions 7.3 and 7.6 and theorem 6.11. The arrows indicate that one form of equivalence implies the other. Here (1) holds if T preserves weak pullbacks, (2) is true when Λ is characteristic and T preserves sobriety, and (3) holds when Λ is strongly monotone. Note that the converse of (2) always holds, because morphisms preserve truth (proposition 3.5).

$$\begin{tikzcd} \Leftrightarrow \arrow[r] & \Leftrightarrow_\Lambda \arrow[r] & \equiv_\Lambda \arrow[r, shift left, "(2)"] & \simeq_{\mathsf{Mod}(\mathsf{T})} \arrow[l, shift left] \arrow[lll, bend right, "(1)"'] \arrow[ll, bend left, "(3)"] \end{tikzcd} \tag{13}$$

This yields the following theorem.

Theorem 7.8. *Let T be an endofunctor on $\mathbf{Top}$ which preserves sobriety and Λ a characteristic strongly monotone geometric modal signature for T. If x and x' are two states in two geometric T-models, then*

$$x \Leftrightarrow_\Lambda x' \quad \textit{iff} \quad x \equiv_\Lambda x' \quad \textit{iff} \quad x \simeq_{\mathsf{Mod}(\mathsf{T})} x'.$$

As stated in the introduction we are not only interested in endofunctors on $\mathbf{Top}$, but also in endofunctors on full subcategories of $\mathbf{Top}$, in particular $\mathbf{KHaus}$.

The previous theorem holds for endofunctors on $\mathbf{Sob}$ as well (use remark 6.12). Moreover, with some extra effort it can be made to work for endofunctors on $\mathbf{KSob}$ as well. In order to achieve this, we have to redo the proof for the bi-implication between modal equivalence and behavioural equivalence. This is the content of the following theorem.

Theorem 7.9. *Let T be an endofunctor on $\mathbf{KSob}$, Λ a characteristic geometric modal signature for T and $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ and $\mathfrak{X}' = (\boldsymbol{\mathcal{X}}', \gamma', V')$ two geometric T-models. Then $x \in \boldsymbol{\mathcal{X}}$ and $x' \in \boldsymbol{\mathcal{X}}'$ are modally equivalent if and only if they are behaviourally equivalent.*

Proof. If x and x' are behaviourally equivalent then they are modally equivalent by proposition 3.5. The converse direction can be proved using similar reasoning as in section 6. The major difference is the following: We define an equivalence relation $\equiv_2$ on $\mathrm{GML}(\Lambda)$ by $\varphi \equiv_2 \psi$ iff $[\![\varphi]\!]^{\mathfrak{X}} = [\![\psi]\!]^{\mathfrak{X}}$ and $[\![\varphi]\!]^{\mathfrak{X}'} = [\![\psi]\!]^{\mathfrak{X}'}$. (Note that $\mathfrak{X}$ and $\mathfrak{X}'$ are now fixed.) That is, $\varphi \equiv_2 \psi$ iff φ and ψ are satisfied by precisely the same states in $\mathfrak{X}$ and $\mathfrak{X}'$ (compare definition 6.1). The frame $\mathbf{E}_2 \coloneqq \mathrm{GML}(\Lambda)/\equiv_2$ can then be shown to be a compact frame and hence $\boldsymbol{\mathcal{Z}}_2 \coloneqq \mathsf{pt}\mathbf{E}_2$ is a compact sober space. The remainder of the proof is analogous to section 6. A detailed proof can be found in [11, theorem 3.34]. □

We summarise the results for two full subcategories of $\mathbf{Top}$:

Theorem 7.10. *Let T be an endofunctor on $\mathbf{Sob}$ or $\mathbf{KSob}$ and Λ a characteristic strongly monotone geometric modal signature for T. If x and x' are two states in two geometric T-models, then*

$$x \Leftrightarrow_\Lambda x' \quad \textit{iff} \quad x \equiv_\Lambda x' \quad \textit{iff} \quad x \simeq_{\mathsf{Mod}(\mathsf{T})} x'.$$

We cannot claim the same for coalgebras over base category $\mathbf{KHaus}$. However, we do have:

Theorem 7.11. *Let T be an endofunctor on $\mathbf{KHaus}$ which is the restriction of an endofunctor on $\mathbf{Sob}$ and let Λ be a characteristic strongly monotone geometric modal signature for T. Then*

$$x \Leftrightarrow_\Lambda x' \quad \textit{iff} \quad x \equiv_\Lambda x'.$$

8 Conclusion

We have started building a framework for coalgebraic *geometric* logic and we have investigated examples of concrete functors. There are still many unanswered and interesting questions. We outline possible directions for further research.

Modal equivalence versus behavioural equivalence From theorems 7.8 and 7.10 we know that modal equivalence and behavioural equivalence coincide in $\mathbf{Mod}(\mathsf{T})$ if T is an endofunctor on **KSob**, **Sob** or an endofunctor on **Top** which preserves sobriety. A natural question is whether the same holds when T is an endofunctor on **KHaus**.

When does a lifted functor restrict to KHaus? We know of two examples, namely the powerset functor with the box and diamond lifting, and the monotone functor on **Set** with the box and diamond lifting, where the lifted functor on **Top** restricts to **KHaus**. It would be interesting to investigate whether there are explicit conditions guaranteeing that the lift of a functor restricts to **KHaus**. These conditions could be either for the **Set**-functor one starts with, or the collection of predicate liftings for this functor, or both.

Bisimulations In [2] the authors define Λ-bisimulations (which are inspired by [10]) between **Set**-coalgebas. In this paper we define Λ-bisimulations between **C**-coalgebras. A similar definition yields a notion of Λ-bisimulation between **Stone**-coalgebras, where the interpretants of the proposition letters are clopen sets, see [11, Definition 2.19]. This raises the question whether a more uniform treatment of Λ-bisimulations is possible, which encompasses all these cases.

Modalities and finite observations Geometric logic is generally introduced as the logic of finite observations, and this explains the choice of connectives ($\wedge$, $\bigvee$ and, in the first-order version, $\exists$). We would like to understand to which degree *modalities* can safely be added to the base language, without violating the (semantic) intuition of finite observability. Clearly there is a connection with the requirement of *Scott continuity* (preservation of directed joins), and we would like to make this connection precise, specifically in the topological setting.

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