Instituut voor Taal, Logica en Informatie
Institute for Language, Logic and Information

Faculteit der Wiskunde en Informatica
(Department of Mathematics and Computer Science)
Plantage Muidergracht 24
1018TV Amsterdam

Faculteit der Wijsbegeerte
(Department of Philosophy)
Nieuwe Doelenstraat 15
1012CP Amsterdam

THE ADEQUACY PROBLEM FOR SEQUENTIAL PROPOSITIONAL LOGIC

Heinrich Wansing
Gruppe für Logik, Wissenschaftstheorie und Information
Freie Universität Berlin

Received October 1989

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Heinrich Wansing*
Gruppe für Logik, Wissenstheorie und Information an der Freien Universität Berlin

October 2, 1989

Abstract. In this note functional completeness for certain subsystems of intuitionistic propositional logic, notably Lambek propositional logic and intuitionistic linear propositional logic, is established w.r.t. a 'sequential' interpretation.

1 Introduction

In [13] Zucker and Tragesser present a solution to the adequacy problem for what they call 'inferential logic'. The label "inferential logic" here covers logical systems "with an 'intuitionistic' natural deduction formalization, and an *inferential* interpretation" ([13], p. 501 f.). An interpretation of a natural deduction formalization is *inferential*, if

($\star$) the meaning of each logical operation c is given by its set $R_I(c)$ of introduction rules.

Zucker and Tragesser show that under certain assumptions inferential propositional logic is functionally complete w.r.t. the set $S_0 = \{\supset, \wedge, \vee, \perp\}$ of logical operations and inferential predicate logic is functionally complete w.r.t. $S_1 = S_0 \cup \{\forall, \exists, =\}$. Prawitz [8] proves functional completeness of S_0 w.r.t. a semantics which slightly differs from the inferential interpretation. A proof of the adequacy of S_0 w.r.t. Kripke's semantics for intuitionistic propositional logic **IPL** can be found in [7]. The adequacy of S_0 w.r.t. an extended natural deduction framework that allows for assumptions of arbitrary finite level has been proved by Schroeder-Heister [9]. This result is extended to S_1 in [10]. In what follows, the adequacy problem for

*Address: Warthestraße 69, 1000 Berlin 44, West Berlin.

certain subsystems of **IPL** will be considered, starting from 'Lambek propositional logic' alias intuitionistic 'sequential' propositional logic **ISPL**, i.e. intuitionistic propositional logic without structural rules in a sequent calculus presentation. The analysis is carried out in a semantical framework which may be called *sequential* interpretation.

2 A few remarks on strategy

An intuitionistic system of natural deduction is formulated in terms of deductions from finite premiss-*sets*, assuming *monotonicity* of inference:

(C1) If $\Gamma \vdash A$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash A$.

Besides (C1), *inferential* logic is subject to the following conditions:

(C2) For every logical constant c there is a set $R_I(c)$ of introduction rules (I-rules) for c, and a set $R_E(c)$ of elimination rules (E-rules) for c. There are no other inference rules.

(C3) (explicitness) Every I-rule for c contains (one occurrence of) 'c' in the conclusion only. Every E-rule for c contains (one occurrence of) 'c' in the premisses only.

(C4) (separation) The rules in $R_I(c)$, $R_E(c)$ do not refer to any constant other than c.

(C5) Every connective is p-ary for some fixed p ($0 \leq p < \omega$).

Now, Zucker and Tragesser proceed as follows. They claim to present the most general form, say F, of an (intuitionistic) I-rule for a propositional connective c. By conditions (C3), (C4), instantiations of F are explicit and separated. From the shape of F and (C5), it can be concluded that for every connective c, the set $R_I(c)$ is finite. The core of the argument then consists in deriving from the shape of F the general syntactic form of the meaning of a propositional connective c. It turns out that F itself calls for connectives of two kinds, a conjunctive and an implicational connective. The need for a disjunctive connective and for absurdity $\perp$ stems from providing meanings by (finite) sets of rules.

Although (C1) is assumed, it is not explicitly used in Zucker and Tragesser's argument. Giving up (C1) will, however, affect F and the derivation of the general form of the meaning of a propositional connective c. In addressing the adequacy problem for weaker formalizations than intuitionistic natural deduction, a basic sequent calculus for **ISPL** comprising the 'sequential' conjunctive and implicational connectives of Categorial Grammar will serve as a starting-point. From an interpretational point of view, the shift from natural deductions to sequent calculi appears not to be very dramatic: I-rules for c do not essentially differ from ($\vdash c$)-rules. However, there is a point in distinguishing between E-rules for c and ($c \vdash$)-rules. Moreover, sequent calculi are extremely perspicious. In a sequent format conditions like (C1) take the form of *structural rules*. The addition of certain such structural rules to our base logic will transform some connectives into different operations and thereby guide the interpretation of the most general introduction-scheme (now looked at as the most general ($\vdash c$)-scheme) and the derivation of the general syntactic form of a propositional connective's meaning from this scheme. In the presence of a structural rule of permutation e.g., the Categorial Grammar implications $/$, $\backslash$ collaps into intuitionistic linear implication $\multimap$ (for the latter cf. e.g. [1]).

3 Lambek propositional logic

Our base logic will be Lambek propositional logic **ISPL**. A sequent-style presentation of **ISPL** in $\{\mathbf{t}, \bot, 1, /, \backslash, \bullet, \oplus, \sqcap\}$ is:

axiomscheme: $A \vdash A$;

operational rules:

$$(\vdash \mathbf{t})\ \frac{}{X \vdash \mathbf{t}}, \qquad (\bot \vdash)\ \frac{}{\bot \vdash A};$$

$$(\vdash 1)\ \frac{}{\vdash 1}, \qquad (1 \vdash)\ \frac{XY \vdash A}{X1Y \vdash A};$$

$$(\vdash /)\ \frac{XA \vdash B}{X \vdash (B/A)}, \qquad (/ \vdash)\ \frac{Y \vdash A \quad XBZ \vdash C}{X(B/A)YZ \vdash C};$$

$$(\vdash \backslash)\ \frac{AX \vdash B}{X \vdash (A \backslash B)}, \qquad (\backslash \vdash)\ \frac{Y \vdash A \quad XBZ \vdash C}{XY(A \backslash B)Z \vdash C};$$

$$(\vdash \sqcap)\ \frac{XAY \vdash C}{XA \sqcap BY \vdash C}, \qquad (\sqcap \vdash)\ \frac{X \vdash A \quad X \vdash B}{X \vdash A \sqcap B};$$

$$\frac{XBY \vdash C}{XA \sqcap BY \vdash C},$$

$$(\vdash \bullet)\ \frac{X \vdash A \quad Y \vdash B}{XY \vdash (A \bullet B)}, \qquad (\bullet \vdash)\ \frac{XABY \vdash C}{X(A \bullet B)Y \vdash C};$$

$$(\vdash \oplus)\ \frac{X \vdash A}{X \vdash (A \oplus B)}, \qquad (\oplus \vdash)\ \frac{XAY \vdash C \quad XBY \vdash C}{X(A \oplus B)Y \vdash C};$$

$$\frac{X \vdash A}{X \vdash (B \oplus A)},$$

structural rules:

$$(cut) \quad \frac{Y \vdash A \quad XAZ \vdash B}{XYZ \vdash B}.$$

The operations \, / are order-sensitive implications, $\bullet$ is conjunction in the sense of concatenation, and $\oplus$ (in **ISPL**) is disjunction on lists. Note that **t** is definable as e.g. $(\bot/\bot)$. The correctness of this definition can be seen as follows:

$$\frac{\bot \vdash \bot}{x_1(x_1 \backslash \bot) \vdash \bot}$$

$$\vdots$$

$$\frac{\dfrac{\bot \vdash (x_n \backslash \ldots \backslash (x_1 \backslash \bot)\ldots) \quad x_1 \ldots x_n(x_n \backslash \ldots \backslash (x_1 \backslash \bot)\ldots) \vdash \bot}{x_1 \ldots x_n \bot \vdash \bot}}{x_1 \ldots x_n \vdash \bot/\bot.}$$

The connectives $\bullet$, $\oplus$, and $\sqcap$ are associative, $\oplus$ and $\sqcap$ are also commutative. $\bullet$ distributes with $\oplus$ as follows: $A \bullet (B \oplus C) \vdash (A \bullet B) \oplus (A \bullet C)$ and, conversely, $(A \bullet B) \oplus (A \bullet C) \vdash A \bullet (B \oplus C)$ are provable in **ISPL**. $\sqcap$ and $\oplus$ are semi-distributive w.r.t. each other: $A \oplus (B \sqcap C) \vdash (A \oplus B) \sqcap (A \oplus C)$ and $(A \sqcap B) \oplus (A \sqcap C) \vdash A \sqcap (B \oplus C)$ are provable in **ISPL**. The $\{/, \backslash, \bullet\}$-fragment of **ISPL** is known as the bidirectional, associative Lambek Calculus of Categorial Grammar (cf. [5]). By reduction on the complexity of (*cut*), it can be schown that (*cut*) is an *admissible* rule of **ISPL**, i.e. applications of (*cut*) can be eliminated from derivations in **ISPL** (cf. e.g. [2], [5]). Since (*cut*) is eliminable and every operational rule is complexity-introducing, bottom-up proof search constitutes a decision procedure for **ISPL**.

It should be emphasized that starting with a base logic does not imply advocating a *formalistic* view of the nature of proofs, according to which the notion of proof depends on a given logical calculus (this viewpoint is criticized in [8], p. 25). There is a *general* notion of syntactic consequence underlying **ISPL**. At the level of sequents, we have 'intuitionistic' deductions

$$\begin{array}{c}\Gamma\\ \vdots\\ s,\end{array}$$

where Γ is a finite (possibly empty) set of sequents, s is a sequent, and monotonicity of inference is assumed. Actually, we are interested not in *deducibility over sequents*, but in provable sequents, i.e. in the relation $\vdash$. This treatment, at the level of formulas, will be considered in section 6.

4 Sequential interpretation

Why at all should one interpret logical constants in terms of operational rules in a sequent calculus rather than in terms of (sets of) I-rules in a system of natural deduction? The reason is that whereas ($\vdash c$)-rules can directly be viewed as I-rules for c written into meta-notation, there is in general no such correspondence between ($c \vdash$)-rules and E-rules for c. In particular, in the case of **ISPL** there simply are no E-rules for $\bullet$ and $\oplus$ which correspond to ($\bullet \vdash$), ($\oplus \vdash$), respectively. To see this, suppose that

$$\begin{array}{c}XABY\\ \vdots\\ C.\end{array}$$

If one wants naturally to deduce C from $X(A \bullet B)Y$, one must assume

$$\begin{array}{c}(A \bullet B)\\ \vdots\\ A\,B,\end{array}$$

the latter being no single-conclusion inference. If

$$\begin{array}{ccc}XAY & & XBY\\ \vdots & \text{and} & \vdots\\ C & & C,\end{array}$$

in order naturally to derive C from $X(A \oplus B)Y$, one must either assume

$$\begin{array}{ccc} (A \oplus B) & & (A \oplus B) \\ \vdots & \text{or} & \vdots \\ A & & B. \end{array}$$

However, $(A \oplus B) \vdash A$ and $(A \oplus B) \vdash B$ are obviously not provable in Lambek propositional logic. The problem is that, since **ISPL** comprises none of the usual structural rules of **IPL** apart from the (*cut*)-rule, one has to consider natural deductions from *sequences of premiss-occurrences.* Therefore, also the following E-rule for $\oplus$, which takes its pattern from the E-rule for intuitionistic $\vee$, cannot capture $(\oplus \vdash)$:

$$\frac{\begin{array}{c} Z \\ \vdots \\ A \oplus B \end{array} \quad \begin{array}{c} X[A]Y \\ \vdots \\ C \end{array} \quad \begin{array}{c} [XBY] \\ \vdots \\ C \end{array}}{C.}$$

These observations concerning $\bullet$ and $\oplus$ are rather important, because they prevent one reformulating **ISPL** in the style of Schroeder-Heister's analysis in [9].

To begin with, besides adopting (C5) the following assumptions concerning operational rules in sequent calculi will be made:

(C6) (explicitness) Every $(\vdash c)$- rule contains (one occurence of) 'c' on the right-hand side of '$\vdash$' in the conclusion only. Every $(c \vdash)$-rule contains (one occurence of) 'c' on the left-hand side of '$\vdash$' in the conclusion only.

(C7) (separation) The $(\vdash c)$-rules and $(c \vdash)$-rules do not refer to any constant other than c.

For the time being, let us also assume:

(C8) There are no structural rules besides the (*cut*)-rule.

In analogy to the inferential interpretation (i.e. $(\star)$), the basic assumption of the sequential interpretation should be:

$(\star\star)$ The meaning of every constant c is given by its set $R_{(\vdash c)}$ of $(\vdash c)$-rules.

The inferential approach to meaning is, however, not properly formulated. Given that the meaning of a constant c is provided by $R_I(c)$, (C2) implies that *every* constant c has a meaning (in the worst case, the empty set of I-rules for c). Whereas Zucker and Trageser claim to deal with logical operations, i.e. interpreted constant symbols, they in fact consider every constant as a logical operation. In particular, even constants c for which there are neither I-rules nor E-rules become meaningful. Yet, if Zucker and Tragesser claimed that meanings are given by I-rules rather than (finite) sets of I-rules, $\bot$ would become void of meaning, since $R_I(\bot) = \emptyset$. Concerning sequent calculi, instead of ($\star\star$) we assume:

($\star\star\star$) The meaning of every constant c is given by $R_{(\vdash c)}$, if $R_{(\vdash c)} \neq \emptyset$. If $R_{(\vdash c)} = \emptyset$ and $c \vdash \bot$ is not provable, then c is meaningless.

Thus, a constant symbol c without ($\vdash c$)-rules is meaningful, only if c and $\bot$ are interdeducible. ($\star\star\star$) still needs some further elaboration, because so far nothing has been said about the function of the ($c \vdash$)-rules (if there are also ($\vdash c$)-rules).

5 The role of the ($c \vdash$)-rules

Intuitively the ($c \vdash$)-rules *do* contribute to the meaning of c. If e.g. one premiss in the ($\oplus \vdash$)-rule of **ISPL** is dropped, one would insist that the meaning of the constant symbol '$\oplus$' has changed. (In fact, the '`tonk`-like' situation arises that $A \vdash B$ is provable for arbitrary A, B.)

As far as E-rules in natural deduction are concerned, Zucker and Tragesser are rather vague about the role played by those rules. According to them ([13], p. 506), the E-rules:

> "*stabilize* or *delimit* the meaning of the logical constant concerned, by saying, in effect, of the given I-rules: "*These are the only ways in which this constant can be introduced*". Without such E-rules, one would have the option (so to speak) of *changing* the meaning of a constant by adding new I-rules for it .... The E-rules prevent this (i.e., remove this option), and it is in this sense that they stabilize the meaning."

Zucker and Tragesser also approvingly cite Gentzen saying that the "introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions" ([4], p. 80). As Prawitz ([8], p. 37) describes it, the E-rules for c should be the strongest rules whose correctness can be seen merely from the meaning of c, i.e. from the (set of) I-rules for c. It is in this sense that E-rules delimit meanings.

Prawitz then claims that the E-rules for the intuitionistic connectives $\wedge$, $\vee$, and $\supset$ are indeed the strongest such E-rules. There are E-rules corresponding to $(\backslash\ \vdash)$, $(/\ \vdash)$, respectively. Following Prawitz, these directional versions of *modus ponens* obviously are the strongest E-rules for $\backslash, /$. Prawitz's specification reminds one of Tennant's *Principle of Harmony* for intuitionistic natural deduction ([11], p. 74), which excludes connectives like Prior's **tonk**. Let us define the proposition expressed by A as $\{B \mid A \vdash B, B \vdash A$ are provable in **ISPL**$\}$, and let us identify formulas with the propositions they express. The *strongest* proposition which has property P is the one proposition with property P from which every proposition with property P is derivable. The *weakest* proposition with property P is the one proposition with property P which is derivable from every proposition with this property. Tennant's principle can now be reformulated as follows:

> *Principle of Harmony*: If both $R_{(\vdash c)}$ and $R_{(c\vdash)}$ are non-empty, the $(\vdash\ c)$-rules and $(c\ \vdash)$-rules must be formulated so that a sentence A with c dominant expresses the strongest proposition such that $X \vdash A$ is provable when the conditions for appying a $(\vdash\ c)$-rule are satisfied; while it expresses the weakest proposition such that $XAY \vdash B$ is provable when the conditions for applying a $(c\ \vdash)$-rule are satisfied.

In addition to the earlier constraints, we shall make the sequential interpretation subject to the *Principle of Harmony*:

(C9) If $R_{(\vdash c)}$ and $R_{(c\vdash)}$ are non-empty, the $(\vdash c)$-rules and $(c \vdash)$-rules harmonize.

It can easily be shown that the operational rules for 1, $/$, $\backslash, \sqcap, \bullet$, and $\oplus$ harmonize:

(1): (i) 1 is the strongest proposition such that $\vdash 1$ is provable,
(ii) 1 is the weakest proposition such that $X1Y \vdash A$ is provable if $XY \vdash A$ is provable;

(/): (i) (B/A) is the strongest proposition such that $X \vdash (B/A)$ is provable if $XA \vdash B$ is provable;
(ii) (B/A) is the weakest proposition such that $Y(B/A)AZ \vdash C$ is provable if $YBZ \vdash C$ is provable;

(\): (i) $(A \backslash B)$ is the strongest proposition such that $X \vdash (A \backslash B)$ is provable if $AX \vdash B$ is provable;
(ii) $(A \backslash B)$ is the weakest proposition such that $YA(A \backslash B)Z \vdash C$ is provable if $YBZ \vdash C$ is provable;

($\sqcap$): (i) $(A \sqcap B)$ is the strongest proposition such that $X \vdash (A \sqcap B)$ is provable

if $X \vdash A$ and $X \vdash B$ are provable,
(ii) $(A \sqcap B)$ is the weakest proposition such that $X(A \sqcap B)Y \vdash C$ is provable if $XAY \vdash C$ or if $XBY \vdash C$ is provable;

($\bullet$): (i) $(A \bullet B)$ is the strongest proposition such that $XY \vdash (A \bullet B)$ is provable if $X \vdash A$ and $Y \vdash B$ are provable;
(ii) $(A \bullet B)$ is the weakest proposition such that $X(A \bullet B)Y \vdash C$ is provable if $XABY \vdash C$ is provable;

($\oplus$): (i) $(A \oplus B)$ is the strongest proposition such that $X \vdash (A \oplus B)$ is provable if $X \vdash A$ or if $X \vdash B$ is provable;
(ii) $(A \oplus B)$ is the weakest proposition such that $X(A \oplus B)Y \vdash C$ is provable if $XAY \vdash C$ and $XBY \vdash C$ are provable.

Here is a proof of ($/$): (i) Suppose that D is any proposition such that $X \vdash D$ is provable if $XA \vdash B$ is provable. Since $(B/A)A \vdash B$ is provable, $(B/A) \vdash D$ is provable by ($/ \vdash$). (ii) Suppose that D is any proposition such that if $YBZ \vdash C$ is provable, $YDAZ \vdash C$ is provable. Then, since $B \vdash B$ is provable, $DA \vdash B$ is provable, from which $D \vdash (B/A)$ is provable by ($\vdash /$). The remaining cases are similar.

6 Functional completeness of Lambek propositional logic w. r. t. $\{/, \backslash, \bullet, \sqcap, \oplus, 1, \bot\}$

In order to make use of something like Zucker and Tragesser's most general introduction scheme for a propositional connective c, I shall recast **ISPL** as a system which brings out ($\vdash c$)-rules as introduction rules. The general notion of deduction here is that of single conclusion inferences from *sequences* of premiss-occurrences, *without* assuming monotonicity of inference (in one form or another). In [12], in the context of a formalistic definition, inferences of this kind are called 'relevant quasi-deductions'. Let "$D(\Pi, A, X)$" abbreviate "Π is a relevant quasi-deduction of A from the finite (possibly empty) sequence X of premiss-occurrences". The notion of relevant quasi-deduction in Lambek propositional logic is inductively defined as follows.

1. $D(A; A; A)$;
2. ($\mathbf{t} - right$) $D(\frac{X}{\mathbf{t}}, \mathbf{t}, X)$;
3. ($\bot - left$) $D(\frac{\bot}{A}, A; \bot)$;
4. ($1 - right$) $D(\frac{}{1}, 1, \emptyset)$;

5. $(1-left)$ If $D(\frac{XY}{A}, A, XY)$, then $D(\frac{X1Y}{A}, A; X1Y)$;

6. $(/-right)$ If $D(\frac{XA}{B}, B, XA)$, then $D(\frac{X}{(B/A)}, (B/A), X)$;

7. $(/-left)$ If $D(\frac{YBZ}{C}, C, YBZ)$, then $D(\frac{Y(B/A)AZ}{C}, C, Y(B/A)AZ)$;

8. $(\backslash-right)$ If $D(\frac{AX}{B}, B, AX)$, then $D(\frac{X}{(A\backslash B)}, (A\setminus B), X)$;

9. $(\backslash-left)$ If $D(\frac{YBZ}{C}, C, YBZ)$, then $D(\frac{YA(A\backslash B)Z}{C}, C, YA(A\setminus B)Z)$;

10. $(\sqcap-right)$ If $D(\frac{X}{A}, A, X)$ and $D(\frac{X}{B}, B, X)$, then $D(\frac{X}{A\sqcap B}, A\sqcap B, X)$;

11. $(\sqcap-left)$ If $D(\frac{XAY}{C}, C, XAY)$, then $D(\frac{X(A\sqcap B)Y}{C}, C, X(A\sqcap B)Y)$,

    If $D(\frac{XBY}{C}, C, XBY)$, then $D(\frac{X(A\sqcap B)Y}{C}, C, X(A\sqcap B)Y)$;

12. $(\bullet-right)$ If $D(\frac{X}{A}, A, X)$ and $D(\frac{Y}{B}, B, Y)$, then $D(\frac{XY}{(A\bullet B)}, (A\bullet B), XY)$;

13. $(\bullet-left)$ If $D(\frac{XABY}{C}, C, XABY)$, then $D(\frac{X(A\bullet B)Y}{C}, C, X(A\bullet B)Y)$;

14. $(\oplus-right)$ If $D(\frac{X}{A}, A, X)$, then $D(\frac{X}{(A\oplus B)}, (A\oplus B), X)$,

    If $D(\frac{X}{A}, A, X)$, then $D(\frac{X}{(B\oplus A)}, (B\oplus A), X)$;

15. $(\oplus-left)$ If $D(\frac{XAY}{C}, C, XAY)$ and $D(\frac{XBY}{C}, C, XBY)$, then

    $D(\frac{X(A\oplus B)Y}{C}, C, X(A\oplus B)Y)$;

16. $(cumulativity)$ If $D(\frac{Y}{A}, A, Y)$ and $D(\frac{XAZ}{B}, B, XAZ)$, then $D(\frac{XYZ}{B}, B, XYZ)$;

17. relevant quasi-deductions in **ISPL** are obtained by 1. - 16. only.

We may now consider the most general form of a $(c-right)$-rule (i.e. $(\vdash c)$-rule) for a propositional connective c (cf. [13], p. 504]):

$(\vdash c)_G$

$$\frac{\begin{array}{ccc} X\ [A_{1_1}\dots A_{1_{k1}}]X_1[A_{1_{k1+1}}\dots A_{1_{k1+m1}}] & & [A_{n_1}\dots A_{n_{kn}}]X_n[A_{n_{kn+1}}\dots A_{n_{kn+mn}}] \\ \vdots & \dots & \vdots \\ B_1 & & B_n \end{array}}{c(A_{1_1},\dots,A_{1_{k1+m1}},\dots,A_{n_1},\dots,A_{n_{kn+mn}},B_1,\dots,B_n,C_1,\dots,C_m).}$$

As in [13], c here is a p-ary connective, where $p = n+k_1+m_1+\ldots+k_n+m_n+m$. There are n (≥ 0) occurrences $B_1, \ldots, B_n$, in this order from left to right. Each occurrence B_i ($1 \leq i \leq n$) is deducible from the sequence $A_{i_1} \ldots A_{i_{k_i}} X_i A_{i_{k_i+1}} \ldots A_{i_{k_i+m_i}}$ of assumption-occurrences. The $A_{i_j} (0 \leq j \leq k_i + m_i)$ are discharged at this inference so that $c(\ldots)$ is deducible from Y, where $Y = X$ if $n = 0$, and where $Y = X_1 \ldots X_n$ or $Y =$ the result of deleting any repetitions of any X_i's ($1 \leq i \leq n$) from $X_1 \ldots X_n$ ('identification of contexts'), if $n > 0$ (cf. e.g. ($\sqcap - right$)). The propositions C_s ($0 \leq s \leq m$) are extra 'dummy' arguments of c.

By conditions (C6), (C7), instantiations of $(\vdash c)_G$ are explicit and separated. From (C5) and the shape of $(\vdash c)_G$ one can infer the *finiteness* condition

(Fin) The set of $(c - right)$-rules (i.e. $R_{(\vdash c)}$) is finite for each connective c.

Following Zucker and Tragesser's strategy, one now has to derive from the shape of $(\vdash c)_G$ the general syntactic form of the meaning of $c(\ldots A_{i_j} \ldots B_i \ldots C_s)$. We may distinguish three cases.

CASE 1: The connective c has just *one* $(c - right)$-rule.

Then the meaning of $c(\ldots A_{i_j} \ldots B_i \ldots C_s)$ is expressed by f:

$$\underset{i=1}{\overset{n}{\sqcap\!\sqcap}} \times \underset{i=1}{\overset{n}{\bullet\!\bullet}} \left(\underset{1 \leq l \leq k_i}{(A_{i_l} \setminus B_i)} \times \underset{k_i+1 \leq r \leq k_i+m_i}{(B_i / A_{i_r})} \right).$$

This notation is to be understood as follows. If $i_l = i_r = 0$ (for some i), then '$\underset{1 \leq l \leq k_i}{(A_{i_l} \setminus B_i)} \times \underset{k_i+1 \leq r \leq k_i+m_i}{(B_i / A_{i_r})}$' reduces to '$B_i$'. If $n = 0$, then 'f' reduces to '1' if $X = \emptyset$; if $n = 0$ and $X \neq \emptyset$, 'f' reduces to '$\top$'. If $i_l > 0$ or $i_r > 0$ (for some i) and there is no identification of contexts, then 'f' is an n-fold $\bullet$-conjunction and is built up according to the following examples.

EXAMPLE 1:

If $l = 1$ and $r = 0$, then $\underset{1 \leq l \leq k_i}{(A_{i_l} \setminus B_i)} \times \underset{k_i+1 \leq r \leq k_i+m_i}{(B_i / A_{i_r})} = (A_{i_1} \setminus B_i)$.

EXAMPLE 2:

If $l = 0$ and $r = 1$, then $\underset{1 \le l \le k_i}{(A_{i_l} \setminus B_i)} \times \underset{k_i+1 \le r \le k_i+m_i}{(B_i / A_{i_r})} = (B_i/A_{i_1})$.

EXAMPLE 3:

If $l = r = 1$, then $\underset{1 \le l \le k_i}{(A_{i_l} \setminus B_i)} \times \underset{k_i+1 \le r \le k_i+m_i}{(B_i / A_{i_r})} = ((A_{i_1} \setminus B_i)/A_{i_2})$. The latter is equivalent to $(A_{i_1} \setminus (B_i/A_{i_2}))$.

If there occurs identification of contexts, for each such context Z involved there is a leftmost occurrence of Z, say X_Z, in Y. In order to obtain f, conjoin all conclusions of deleted occurrences of Z with the conclusion of X_Z by means of $\sqcap$. Thus, if e.g. $X_1 = X_3 = X_5$,

$$\frac{\begin{array}{ccc} X_1 & & X_5 \\ \vdots & \dots & \vdots \\ B_1 & & B_5 \end{array}}{c(B_1 \dots B_5)},$$

and $Y = X_2X_3X_4X_5$, then $f = B_2 \bullet (B_1 \sqcap B_3) \bullet B_4 \bullet B_5$.

CASE 2: c has *more than one* $(c - right)$-rule.

By (Fin), the $(c - right)$-rules can be arranged into a finite list. Suppose that c has t (> 1) $(c - right)$-rules. In CASE 1 syntactic forms $f_1, \dots f_t$ were provided corresponding to these $(c - right)$-rules. The meaning of $c(\dots)$ then becomes

$$\bigoplus_{h=1}^{t} f_h.$$

CASE 3: c has no $(c - right)$-rule.

Since we deal with interpreted symbols, by $(\star\star\star)$, c is $\perp$.

Now, for $\diamond \in \{/, \setminus, \bullet, \oplus, \sqcap\}$ the meaning of $(A \diamond B)$ is just $(A \diamond B)$; the meaning of $\mathbf{t}$, $1, \perp$ is $\mathbf{t}$, $1, \perp$, respectively. One simple example may suffice to illustrate that the above interpretation of $c(\dots)$ is adequate.

EXAMPLE 4: Suppose that

$$\frac{\begin{array}{cc}[A_1]X_1[A_{1_1}] & [A_2]X_2[A_{2_1}] \\ \vdots & \vdots \\ B_1 & B_2\end{array}}{c(A_1, A_{1_1}, A_2, A_{2_1}, B_1, B_2)}$$

and $Y = X_1X_2$. The appropriate instantiation of 'f' can be relevantly quasi-deduced as follows:

$$\frac{\dfrac{\dfrac{\begin{array}{c}[A_1]X_1[A_{1_1}] \\ \vdots \\ B_1\end{array}}{(A_1 \setminus B_1)}}{((A_1 \setminus B_1)/A_{1_1})} \quad \dfrac{\dfrac{\begin{array}{c}[A_2]X_2[A_{2_1}] \\ \vdots \\ B_2\end{array}}{(A_2 \setminus B_2)}}{((A_2 \setminus B_2)/A_{2_1})}}{(((A_1 \setminus B_1)/A_{1_1}) \bullet ((A_2 \setminus B_2)/A_{2_1}))}.$$

If $X_1 = X_2$ and identification of contexts occurs, then one obtains $(((A_1 \setminus B_1)/A_{1_1}) \sqcap ((A_2 \setminus B_2)/A_{2_1}))$ instaed of $(((A_1 \setminus B_1)/A_{1_1}) \bullet ((A_2 \setminus B_2)/A_{2_1}))$.

Thus, we may conclude that, if $R_{(\vdash c)} \neq \emptyset$, then $c(\ldots)$ and f have the same set of introduction rules. According to $(\star\star\star)$, they therefore also have the same meaning. Altogether, every sequentially meaningful connective is explicitly definable in terms of $\{1, \perp, /, \setminus, \bullet, \sqcap, \oplus\}$.

REMARK. The above 'most general' ($\vdash c$)-scheme apparently imposes an additional restriction on the sequential interpretation: 'infix-operators' are to be excluded. However, allowing for infix-operators leads to a violation of (C8). Cf. e.g. the following explicit, separated, and harmonic infix-operator $\leadsto$:

$$(\vdash\leadsto) \quad \frac{XAY \vdash B}{XY \vdash (A \leadsto B)},$$

$$(\leadsto\vdash) \quad \frac{YBZ \vdash C}{YA(A \leadsto B)Z \vdash C}, \qquad \frac{YBZ \vdash C}{Y(A \leadsto B)AZ \vdash C}.$$

It can easily be seen that in the presence of $\leadsto$ permutation of premiss-occurrences becomes derivable as a structural rule.

7 Functional completeness for systems intermediate between ISPL and IPL

In the presence of the structural rule

permutation (P):
$$\frac{XABY \vdash C}{XBAY \vdash C},$$

relevant quasi-deductions become *linear* deductions (cf. [1]), and our basic connectives translate into connectives of intuitionistic linear propositional logic **ILPL**. As already remarked, the implications /, \ collaps into intuitionistic linear (or undirectional Categorial Grammar) implication $\multimap$. The connective $\bullet$ translates into $\otimes$; $\sqcap$, $\oplus$, **t**, 1 and $\perp$ are not affected in the sense that for the respective fragments the set of provable sequents remains the same.

Note that *there are* E-rules for $\otimes$, $\oplus$ corresponding to $(\otimes\vdash)$, $(\oplus\vdash)$, respectively:

$$\frac{\begin{matrix} X & [A,B]Y \\ \vdots & \vdots \\ (A\otimes B) & C, \end{matrix}}{C} \qquad \frac{\begin{matrix} X & [A]Y & [BY] \\ \vdots & \vdots & \vdots \\ (A\oplus B) & C & C. \end{matrix}}{C}$$

With **P** present, the most general scheme $(\vdash c)_G$ receives a different interpretation. It can now be written as:

$(\vdash c)_G{}'$

$$\frac{\begin{matrix} X \quad [A_{1_1}\ldots A_{1_{k1}}]X_1 & & [A_{n_1}\ldots A_{n_{kn}}]X_n \\ \vdots & \cdots & \vdots \\ B_1 & & B_n \end{matrix}}{c(A_{1_1},\ldots,A_{1_{k1}},\ldots,A_{n_1},\ldots,A_{n_{kn}},B_1,\ldots,B_n,C_1,\ldots,C_m)},$$

where $p = n + k_1 + \ldots + k_n + m$. In the case of exactly one $(\vdash c)$-rule, the meaning of $c(\ldots A_{i_j}\ldots B_i\ldots C_s)$ becomes f':

$$\prod_{i=1}^{n} \times \bigotimes_{i=1}^{n} \Big((A_{i_1} \multimap (A_{i_2} \multimap \ldots \multimap (A_{i_{k_i}} \multimap B_i)\ldots) \Big).$$

Due to the presence of **P**, the order of the A_{i_j} is irrelevant. f' is equivalent to

$$\prod_{i=1}^{n} \times \bigotimes_{i=1}^{n} \left(\left(\bigotimes_{j=1}^{k_i} A_{i_j} \right) -\!\circ B_i \right).$$

The remaining two cases call for disjunction $\oplus$ (look at the (finite) list of ($\vdash$ c)-rules as a multi-set) and $\bot$; **t** is definable as ($\bot -\!\circ \bot$). Thus, sequentially interpreted **ILPL** is functionally complete w.r.t. $\{1, -\!\circ, \otimes, \oplus, \sqcap, \bot\}$.

Other structural rules that might be added to the earlier sequent-style presentation of **ISPL** are:

contraction (**C**):

$$\frac{XAAY \vdash B}{XAY \vdash B};$$

cancellation (**C'**):

$$\frac{XAYAZ \vdash B}{XAYZ \vdash B}, \quad \frac{XAYAZ \vdash B}{XYAZ \vdash B};$$

expansion (**E**):

$$\frac{XAY \vdash B}{XAAY \vdash B};$$

duplication (**E'**):

$$\frac{XAYZ \vdash B}{XAYAZ \vdash B}, \quad \frac{XYAZ \vdash B}{XAYAZ \vdash B};$$

monotonicity (**M**):

$$\frac{XY \vdash A}{XBY \vdash A}.$$

In the presence of **C**, intuitionistic linear implication $-\!\circ$ becomes *relevant* implication $-\!\bullet$. The connectives $\oplus$, $\sqcap$, **t**, 1, and $\bot$ are not affected; $\otimes$ becomes 'fusion' $\circ$. **t** can be defined as ($\bot -\!\bullet \bot$). In strict analogy to the reasoning in the linear case, one may now conclude that sequential propositional logic based on relevant implication (**IRPL**) is functionally complete w.r.t. $\{1, -\!\bullet, \sqcap, \circ, \oplus, \bot\}$.[1] It should by now be clear,

[1]This does not, however, imply that Anderson and Belnap's propositional relevance logic **R** (cf. e.g. [3]) is functionally complete w.r.t. $\{1, -\!\bullet, \sqcap, \circ, \oplus, \bot\}$. Although **IRPL** and **R** share the same implicational fragment, namely Church's *weak theory of implication*, **R**'s conjunction and disjunction e.g. are not captured by $\sqcap$, $\oplus$.

how functional completeness w.r.t. the sequential interpretation can be established for those extensions of Lambek propositional logic which are obtained by adding combinations of the above structural rules to our stock of operational rules. In each case the effects on the general scheme $(\vdash c)_G$ and on the basic connectives of Lambek propositional logic are to be considered. In particular, adding **P**, **C**, and **M** to **ISPL** will corroborate the results in [7], [13], [8], and [9] concerning **IPL**.

ISPL		**ILPL**		**IRPL**		**IPL**
$\oplus$	$\rightarrow$	$\oplus$	$\rightarrow$	$\oplus$	$\rightarrow$	$\vee$
$\bullet$	$\rightarrow$	$\otimes$	$\rightarrow$	$\circ$	$\searrow$	
						$\wedge$
$\sqcap$	$\rightarrow$	$\sqcap$	$\rightarrow$	$\sqcap$	$\nearrow$	
$\backslash$	$\searrow$					
		$\multimap$	$\rightarrow$	$-\!\bullet$	$\rightarrow$	$\supset$
$/$	$\nearrow$					
1	$\rightarrow$	1	$\rightarrow$	1	$\searrow$	
						$\top$
t	$\rightarrow$	t	$\rightarrow$	t	$\nearrow$	
$\perp$	$\rightarrow$	$\perp$	$\rightarrow$	$\perp$	$\rightarrow$	$\perp$

8 Outlook

It turned out that, starting with intuitionistic sequential propositional logic as a base logic, Zucker and Tragesser's approach to the adequacy problem for inferential propositional logic can be generalized so as to provide a uniform method to obtain functional completeness results for a spectrum of sublogics of **IPL**. Moreover, these results are themselves uniform: besides $\perp$ and 1 they involve certain implicational, conjunctive and disjunctive connectives.

What about the adequacy problem for sequential predicate logic? In [13] one can find a quite general argument for the case of inferential predicate logic, now using Martin-Löf's intuitionistic type theory [6]. One might expect that suitable modifications of this argument could provide functional completeness results for certain subsystems of intuitionistic predicate logic. Before this question can be tackled, however, something like a 'Lambek type theory' must be developed. The latter is, of course, in itself an interesting research task.

Acknowledgement. This note was prepared during a sojourn at the department of Mathematics and Computer Science of the University of Amsterdam. I am grateful to Johan van Benthem and the Studienstiftung des deutschen Volkes for making this visit possible. Moreover, I should like to thank Johan van Benthem, Dirk Roorda, and Peter Schroeder-Heister for helpful comments on an earlier version of this note.

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