DYNAMIC SET THEORY

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written by

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Abstract

This thesis develops a novel approach to a problem in the philosophy of mathematics, namely the problem of how to model the constantly evolving universe of mathematical objects. In the foundations of mathematics as formulated by set theory, this problem manifests itself as the question *why* the universe of sets is not a set itself, referred to as the *why*-question. Potentialism is a view in the philosophy of mathematics aiming to answer this question by interpreting the hierarchy of sets as not actually but potentially existing. This view has recently gained renewed interest from both philosophers and mathematicians. However, Potentialism as currently implemented is not able to do what it purports to do. In this thesis, a dynamic formulation of set theory (DST) will be developed in order to overcome this deficiency. Dynamic logic fits the philosophical view that Potentialism aims to capture since it can model the growth of information, which in this case refers to the expansion of the hierarchy of sets. Based on this dynamic formulation of set theory, a formal comparison between ZF and DST and a consideration on whether DST is able to answer the *why*-question is made. This formal comparison between ZF and DST shows that the two theories are interpretable in ZF given the right choice of semantics (either both classical, both intuitionistic or intuitionistic DST into classical ZF). However, DST is not able to give an in-depth answer to the *why*-question that is not open to existing criticism. The use of dynamic logic does allow for an independent motivation for this method of modelling the universe of sets. Moreover, it is argued that there are good reasons to think that a fully satisfactory, in-depth answer to the *why*-question might be too high an aim.

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Introduction

This thesis presents an approach to modelling the continuously developing universe of mathematical objects by capturing an old and intuitive idea of infinity with a method new to the foundations of mathematics. The conception of infinity that will be used here can already be found in ancient philosophy:

> Infinity turns out to be the opposite of what people say it is. It is not "that which has nothing beyond itself" that is infinite, but "that which always has something beyond itself". (Aristotle, *Physics*, 3.6)

The conception that Aristotle sketches of infinity fits our intuitive idea of it, as a process that never stops, something that goes on and on without termination. For instance, if I would ask you to count, you would (assuming you will not protest) start listing the numbers in the natural number sequence in its order. However, if I would ask you to count to infinity, you can only imagine yourself continuing to count without ever stopping. All the numbers you manage to list would be finite, so never at any point would you reach your destination. Our conception of infinity, it seems, is something that simply goes on and on, without bounds. In this thesis, we will exploit these intuitions of infinity, by taking a potentialist perspective on the foundations of mathematics and using a dynamic approach to implement this perspective in set theory.

This idea of infinity as a never-ending process was the standard view of infinity for a long time, already present in ancient philosophy. However, Georg Cantor changed the views of many mathematicians and philosophers on infinity with his work on the transfinite; distinguishing between actual and potential infinity. This work also started the development of set theory as a foundation of mathematics.

Set theory, in general, is the mathematical theory on collections of well-defined objects, namely sets and their elements. Currently, the standard form of set theory is ZF(C), formulated by Zermelo and Fraenkel. Their axiomatic system is so construed that all objects of mathematics can be represented as sets. Due to this feature, set theory can be seen as a foundation of mathematics. Seeing ZF(C) as a foundation for mathematics does raise some important questions; are the axioms just mathematically successful in representing the objects of mathematics or do the axioms of ZF(C) capture something more fundamental about the nature of mathematics and its objects? In order to answer this question, a conception of sets is needed.

The axioms of ZF(C) are most commonly motivated by a specific conception of sets called the iterative conception. This conception of sets gives a specific structure to the universe of sets. However, there are two opposing views on how to interpret the iterative conception of sets, namely, Actualism and Potentialism. Actualism regards the hierarchy of sets as actually existing, all at once. This can be seen as problematic since it raises the question why the hierarchy of sets itself is not a set. Generally, it is thought that, following Cantor, potential infinities cannot form sets, while actual infinities can. However, if the entire hierarchy is actual, why can the universe of sets still not be a set itself (all paradoxes aside)? Potentialism regards the hierarchy of sets as merely potentially existing. This position has its own problems, mostly regarding the interpretation of *potential existence*.

In this thesis we will focus on the potentialist perspective on the iterative conception of sets. The potentialist account of the universe of sets as currently implemented using a standard modal logic is not able to do what it purports to do. For instance, it cannot give an answer to the question why the universe of sets is not a set itself that goes beyond any answer also available to the actualist. It will be argued that there are two main problems for the current view of Potentialism: (i) its system is static, which makes it impossible to model the process of set development, and (ii) its interpretation of the modalities is ontological, which makes it hard to define what it means to *potentially* exist. Therefore, in this thesis, we propose a novel approach to implementation of Potentialism that will solve these problems by using a logic that truly captures its *dynamic* component and has a natural interpretation of the modalities that is linguistic and epistemic rather than ontological.

The conception of sets is an important issue for the philosophy of mathematics since, in order for set theory to function as a foundation of mathematics, we want it to not only provide to be a basis from which the pursuit of mathematics can grow but also capture the nature of its objects. This means that a successful foundation of mathematics should lead us to answers about the nature of the objects it describes. Within set theory, this requires a conception of sets and the hierarchy of sets that also motivates the axioms of set theory.

Moreover, Potentialism specifically has recently gained renewed attention from both philosophers and mathematicians. Not only have there been various formally valuable results from modal set theory, it has also been shown to form a solution for some of the philosophical issues that come up for other views. Solving some of the open problems for Potentialism would further strengthen it as a view on the universe of mathematical objects and infinity.

The main objectives of this thesis are the following two: (i) formulating a form of set theory that is *truly* dynamic, and (ii) offering a solution to the question *why* the universe of sets is not a set itself.

The first objective will consist of combining set theory and dynamic semantics such that we can model the process of set formation, in a way which still allows us to recover the results of ZF. Thus, this thesis is not aiming to change or disprove any results of set theory as we know it. What this thesis aims to do is find a formulation of set theory that makes it more clear what the practices of the set-theorist consist of and how we can understand the hierarchy of sets. We aim to provide a formulation of set theory that matches our intuitions and philosophical considerations of the concept of sets and the set-theoretical hierarchy.

In order to reach this objective we will need to get a clear perspective on the problems with the current implementation of Potentialism. The hypothesis is that while the existing modal implementation of Potentialism does make it possible to compare potential and actual existence of sets, and pluralities and sets, it cannot model the process of set formation and the expansion of the expressive resources available in set theory. In order to do this, we need to use a logic that is able to model processes and change. To this end, we will replace the static modal operators of Potentialism as currently implemented with dynamic modal operators. Then, since we do not aim to revise set-theoretical practice, we need to recover the results of set theory. This will be done by carefully choosing axioms and principles that are in line with our philosophical and mathematical intuitions and make it possible to recover the axioms of ZF.

The second objective of this thesis will consist of finding an interpretation of the dynamic operators such that it can provide an explanation of *why* the universe of sets is not a set itself, more in-depth than that its existence would lead to a contradiction. In order to this, we need to investigate the concept of set and the set-theoretic hierarchy to find out what the difference between a set and the hierarchy of sets is. However, finding a satisfactory and in-depth answer to the *why*-question will prove to be too complicated not to involve a more general discussion of the *why*-question itself. This will consist of an exposition of what answering the *why*-question requires and what criteria for a satisfactory answer are.

To reach the second objective we will use a linguistic interpretation of Potentialism building on the version of Potentialism formulated by Uzquiano (2015). This contrasts with other versions of Potentialism that use a more ontological interpretation of the modalities involved. Instead of modelling how the mathematical ontology expands as the stages of the hierarchy progress, the proposal is to model how our expressive resources expand by step-wise assigning more inclusive meanings to the word "set". Based on this interpretation it will be argued that we can never assign an all-inclusive meaning to "set". It will also be argued that a completely satisfactory and in-depth answer to the *why*-question is not possible. This does not mean that we should dismiss the pursuit entirely since some answers are still better than others. To make this point, a few constraints on conceptions of sets will be formulated.

Note that, in order to reach both objectives we will need to use a combination of formal and philosophical methods. These two cannot remain entirely separate; we will use philosophical considerations to motivate the construction of a formal system, and use the formal system to both answer philosophical questions and raise new ones.

In the first chapter, we will introduce the necessary background to understand the problem that this thesis aims to solve. This will start with a historical introduction of set theory and the foundational crisis within mathematics. Then, the currently most standard version of set theory and foundation of mathematics will be introduced along with the leading interpretation of it. This interpretation, the iterative conception of sets, will be used throughout this thesis. In this first chapter, we will introduce the two main views on this conceptions of set, Actualism and Potentialism. We will discuss the different reasons researches have used to favour either perspective and what the problems with each are. Since the focus of this thesis will be the potentialist perspective on the iterative conception, the central point of this chapter will be which problems Potentialism has solved and which problems still remain for it. We will finish this chapter with introducing the proposed solution which will be elaborated throughout the rest of the thesis; a dynamic approach to Potentialism.

In the second chapter, we will work out the details of this approach to Potentialism. We will start by discussing the intuitions on why it fits the conception of sets that Potentialism endorses. Based on these intuitions, the formal system, called dynamic set theory, will be worked out. We will start by introducing the model of dynamic set theory, the appropriate modal logic involved, the plural logic that will be adopted, the semantics of dynamic set theory and the nature of sets. Since there have been arguments for the adoption of intuitionistic logic for set theory based on potentialist considerations, we will introduce two sets of semantics; classical and intuitionistic. Based on these considerations and the formulation of the formal account of dynamic set theory, we will then recover the axioms of ZF. This highlights that this thesis does not aim to reform set theory as a mathematical practice, but rather aims to reformulate existing set theory in such a way that the underlying concept of set becomes explicit. We will end this chapter with a brief comparison of the formal systems resulting from adopting classical and intuitionistic semantics.

In chapter three, we will prove some results that compare dynamic set theory, in both its classical and intuitionistic version, with ZF. This will show that a formula is deducible in classical dynamic set theory if and only if it is deducible in ZF. The same result will be proven between intuitionistic dynamic set theory and intuitionistic ZF. A weaker result, only the direction from dynamic set theory to ZF, will be shown to apply to intuitionistic dynamic set theory and classical ZF. Based on these results we will further prove that both classical and intuitionistic dynamic set theory are interpretable in classical ZF, and that intuitionistic dynamic set theory is interpretable in intuitionistic ZF.

In the final chapter, we will discuss the philosophy of dynamic set theory. The first step will be to see whether the dynamic version of Potentialism can still recover the philosophical results of Potentialism as currently implemented. To this end, we will outline how dynamic set theory solves the problems that Potentialism was originally designed to solve and captures the open-ended nature of sets. The second step consists of the study of the concept of set and the interpretation of the dynamic modalities in dynamic set theory. In order to do this we will first introduce a different perspective on Potentialism that interprets its endeavour as more linguistic, rather than ontological. We will combine this view of Potentialism with the version of it discussed in the first chapter to formulate an interpretation of what dynamic set theory is modelling. Based on these considerations we will also compare the classical and intuitionistic version of dynamic set theory from a philosophical point of view. This will show which version of it best fits with the concept of sets and the interpretation of the hierarchy of sets that Potentialism aims to capture. Finally, the last step is to return to one of the central problems for Potentialism, namely, its trouble to providing an in-depth answer to the question *why* the universe of sets is not a set itself. We will discuss to what extend dynamic set theory is able to give a more satisfactory answer, that there is an independent reason to adopt dynamic operators in set theory and why answering the *why*-question in a satisfactory way might not be achievable in general. We will finish this last chapter with some considerations on what a conception of sets can and cannot be and what any conception should do.

Together these chapters will outline a version of set theory that is better able to capture the dynamic nature of sets inherent to a potential view of the iterative conception of sets by a dynamic implementation of Potentialism in set theory.

1

Potentialism

In this chapter, we will introduce the necessary background to understand the main subject of this thesis, Potentialism, as well as the view itself. We will begin from a historical perspective, starting from the foundational crisis within mathematics. This will lead us to ZF(C), the most common form of axiomatic set theory and foundation for mathematics nowadays, and the iterative conception of sets, which is usually used to motivate ZF(C). Even though ZF(C) is the most commonly used foundation for mathematics, there are some arguments against using it as such, some of which will be briefly discussed. Against this background we can consider Potentialism, which argues for a specific view on the iterative conception of sets and endorses some of the criticism to ZF(C). We will look at this theory from both a philosophical and a formal perspective; considering its critique on ZF(C), its conception of sets and the formal system resulting from it. Potentialism itself is not without problems either. So, directly after the outline of this theory, we will discuss some of its issues. Finally, we will look at what the options for Potentialism to circumvent these problems are, which will serve as a bridge to the next chapter.

1.1 The Foundations of Mathematics

At the end of the nineteenth century, mathematicians were searching for a foundation for mathematics. This search for foundation turned into a foundational crisis when the attempts ran into a variety of difficulties. In this section, we will briefly describe the route to modern set theory to put the Potentialist view of the foundations of mathematics and set theory into perspective. This route to modern set theory starts with Cantor and Dedekind and will lead us to modern set theory as formulated by Zermelo and Fraenkel. This section will finish by discussing why ZF(C) is sometimes said to be ad hoc and is not always seen as a fully satisfactory foundation for mathematics.

In the second half of the nineteenth century, both Dedekind and Cantor were working on definitions of real numbers. In 1873, their work led to the exploration of the transfinite after a letter from Cantor to Dedekind. In this correspondence (see Ewald (1996) for a translation), Cantor poses the question whether there is a one-to-one correspondence between the natural numbers $\mathbb{N}$ and the real numbers $\mathbb{R}$. In response to this followed a proof by Dedekind of the one-to-one correspondence between the natural numbers and algebraic numbers, and a proof by Cantor showing $\mathbb{R}$ to be non-denumerable and, thereby, of a larger cardinality than $\mathbb{N}$. Both Cantor and Dedekind, based on this work, posited definitions of the real numbers built on set theory (Giaquinto, 2002).

In the following years, a variety of researchers were studying whether set theory could form a foundation for mathematics. Among these, was Frege (1903, §147) who with his logicist project aimed to show that the truths of arithmetic are logical truths by trying to derive the fundamental axioms and theorems of arithmetic by purely logical methods.

The early formulation of set theory is, what is now called, naive set theory based on the naive conception of sets. In the naive conception of sets, sets were taken to be collections of things falling under a certain concept, combined with the idea that any predicate is able to determine a set. This idea was captured by the axiom schema of comprehension: if $\varphi(x)$ is a formula in the language of first-order set theory, then there exists a set $Y = \{x : \varphi(x)\}$. (Giaquinto, 2002; Tiles, 2004)

This axiom schema was a crucial figure in Frege's Basic Law V, which he used in his attempts to derive the truths of arithmetic by logical means. However, both Frege's logicist project and his Basic Law V, and naive set theory more generally were soon found problematic by the discovery of multiple set-theoretic paradoxes starting with the Burali-Forti paradox and, specifically problematic for Frege's project, later Russell's paradox (see e.g. Ferreirós (2016); Giaquinto (2002)).

In a letter from 1902, Russell proves Frege's Basic Law V to be inconsistent by using the axiom schema of comprehension (see Van Heijenoort (1967) for a translation). To see how Russell's proof works, let $\varphi(x)$ stand for $x \in x$ and take $R = \{x : \neg\varphi(x)\}$. In other words, let R be the set such that its members are exactly those objects which are not a member of themselves. Now suppose that R is a member of itself. This means that it is not a member of itself since it must satisfy the condition $\neg\varphi(x)$. Suppose instead that R is not a member of itself. Then it follows that it is a member of itself since it must not satisfy the condition $\neg\varphi(x)$. Since classically we have that either R is a member of itself or it is not a member of itself, we have a contradiction. This proves that Frege's Basic Law V is inconsistent since this law states that the extensions of two predicates (F, G) are the same if and only if for all x, $F(x) \leftrightarrow G(x)$ (see e.g. Gianquinto (2002, 53-54)).

The discovery of the set-theoretic paradoxes led to a variety of attempts to develop a new system for the foundations of mathematics. In 1908, Zermelo published his system of axioms for set theory with an exposition of how this new system was able to prohibit the earlier found paradoxes from arising. From this grew the nowadays most common form of axiomatic set theory, formulated by Zermelo and Fraenkel (ZF or ZFC depending on, respectively, the exclusion or inclusion of the axiom of choice).

To eliminate paradoxes such as the paradox of the ordinals and Russell's paradox the axiom schema of comprehension is relinquished and instead the weaker axiom schema of separation is used (Ferreirós, 2016): if $\varphi(x)$ is a formula in the language of first-order set theory, $\mathcal{L}_\in$, then for any set X there exists a set $Y = \{x \in X : \varphi(x)\}$. In other words, for every set X and every φ expressible in $\mathcal{L}_\in$ there is a set consisting of all objects in X that satisfy φ. This removes the possibility of formulating sets such as Russell's set since we cannot take the set of all sets that are not a member of themselves, merely the set of all sets that are not a member of themselves within a certain set. More generally, this also eliminates the possibility to have a set of all sets (Ferreirós, 2016). Aside from the axiom schema of separation, ZF(C) includes further axioms to recover the desired mathematical properties of sets and further capture the concept of set (Bagaria, 2019).

The most common way to interpret ZF(C), is by the iterative conception, first suggested by Gödel (1933). On the iterative conception of sets, sets are what one gets by iterating the power set operation on the well-ordered class of ordinals. We start with the empty set at stage V_0. Then, by applying the power set operation, we get $V_1 = \mathcal{P}(V_0)$. Following this further we get to the first limit ordinal, such that $V_\omega = \bigcup V_n$. From this follows the hierarchy, as shown in figure 1.1.

The iterative conception is not only seen as a way to interpret ZF(C) but even as an independent way to motivate it.

> ZF alone (together with its extensions and subsystems) is not only a consistent (apparently) but also an independently motivated theory of sets: there is, so to speak, a "thought behind it" about the nature of sets which might have been put forth even if, impossibly, naive set theory had been consistent. (Boolos, 1971, 219)

One of the benefits of this iterative conception of set is that it gives an intuitive way to counter to paradoxes such as Russell's by its hierarchical structure. It provides an interpretation of the nature of sets that directly prohibits the existence of sets that are members of themselves since each set is formed out of objects from the stages below it. This means that for a set to be a member of itself, it would already have had to exist at the level before it was formed, which is impossible. This also means that there is no set of all sets since such a set would contain itself, and since in the hierarchy only sets of all sets occurring at earlier stages exist it would also not occur in the hierarchy.

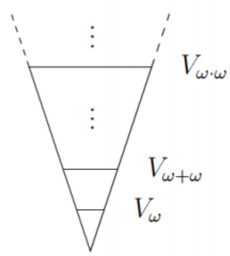

Figure 1.1: The cumulative hierarchy of sets

As seen, modern set theory as formulated by Zermelo and Fraenkel (ZF(C)) is designed to avoid the set-theoretic paradoxes by prohibiting the existence of a universal set and getting rid of the axiom schema of comprehension. But, while some argue that ZF(C) is independently motivated by the iterative conception, like is apparent from the quote from Boolos (1971) above, others argue that the choice of axioms in ZF(C) is ad hoc; guided to prohibit the set-theoretic paradoxes from arising without there being any further justification for them (see e.g. Hellman and Cook (2018)). As we have seen, in ZF(C), the axiom schema of comprehension is restricted to the axiom schema of separation but there are other ways to restrict the axiom schema of comprehension. For instance, in New Foundations (Quine, 1937) the axiom schema is restricted by assigning types to the set comprehensions and permitting only instances of the schema that are well-typed, and in Positive Set Theories the comprehension schema is used for all positive formulas Holmes (2017). In both these cases the paradoxes are prohobited by restricting the axiom schema of comprehension as well. Therefore, in order to claim that the axioms of ZF(C) are motivated by the concept of set, one needs to motivate why the way ZF(C) restricts comprehension fits the conception of sets better than the others.

Moreover, according to some, there is still a problem underlying these paradoxes. The claim is that prohibiting paradoxes from arising is not enough; we need to look at the roots of the problem, not just solve its symptoms. Michael Dummett, for instance, argued that the real problem lies in the indefinite extensibility or open-endedness of the concept of set (e.g. Dummett (1994a)).

Indefinite extensibility refers to concepts such that when we form a conception of the totality of the objects that fall under a specific concept we can, by reference to this totality, characterise a bigger totality of objects that fall under the concept. This process of extending can go on indefinitely. This is problematic according to Dummett because, since we are not able to fix a definite totality of objects falling under the concept "set", we are not able to fix the truth-values of statements quantifying over this totality. Moreover, since there is nothing else that can determine those truth-values, we are left with statements with no determinate truth-value. This is not acceptable in classical logic and cannot be circumvented, therefore we should adopt a different logic (see e.g. Dummett (1963, 1991, 1994a,b)).

Although Dummett's argument for intuitionistic logic based on indefinite extensibility is notoriously obscure (see e.g. Linnebo (2018, 36)), indefinite extensibility does seem to clearly relate to the paradoxes of early set theory. Russell argued that the paradoxes

> result from the fact that [...] there are what we may call self-reproductive processes and classes. That is, there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect all of the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property. (Russell, 1907)

Although this does not give us a much clearer perspective on what indefinite extensibility or self-reproductiveness is exactly, it does show that there is a relation between these phenomenena and the set-theoretic paradoxes. Note, however, that the problem does not lie in the conception of sets as in the iterative hierarchy, but rather in the way that ZF(C) is trying the capture this. This will become more explicit in the following section in which we will discuss a conception of sets that emphasises the open-ended and indefinitely extensible nature of sets.

1.2 Potentialism to the Rescue

As we have seen, modern set theory can be interpreted in terms of a hierarchy. Moreover, it has been discussed that there are some problems with ZF(C), such as that it has been said to be ad hoc since it artificially prevents paradoxes from arising without solving the problems underlying the paradoxes. A view that has endorsed this criticism is Potentialism, which argues that the set-theoretic hierarchy is inherently potential in nature and that this characteristic should be incorporated in the formal system of set theory. Intuitively, the iterative conception of sets is often discussed in terms of time; sets are formed over time. Potentialism proposes to "replace the language of time and activity with the more bloodless language of potentiality and actuality" (Parsons, 1977, 355). In this section, we will briefly discuss some of Potentialism's history, what Potentialism's problems with set theory as it currently stands are and what it proposes instead. This section will mostly focus on the philosophical side of their story, while the more formal details will be left for the next section.

Based on the paradoxes that arose from naive set theory, many researchers working on the foundations of mathematics have concluded that the universe of sets is not a set itself. The question that follows is *why* this is the case, the *why*-question for short (Soysal, 2017). Some state-of-the-art proposed solutions to the *why*-question and the other problems mentioned in the previous section are built on a conception of the hierarchy of sets as *potential* (see e.g. Linnebo (2010); Uzquiano (2003)).

We will give a clearer description of Potentialism throughout this section, but, roughly, Potentialism argues that no matter how many stages of the hierarchy of sets have been formed, there is always the possibility of a next stage; the hierarchy is open-ended and indefinitely extensible. This way, the hierarchy is never "complete". Since only "complete" totalities can form sets, the totality of sets cannot form a set. This potential nature of the hierarchy of sets fits the iterative conception of sets quite naturally since this incorporates the idea of there always being a next stage. However, while the iterative conception of sets is often used to motivate ZF(C), this potential nature is not represented in it.

> The apparent potential character of the hierarchy of sets contrasts starkly with the axiomatic set theory that the iterative conception is ordinarily used to motivate, namely Zermelo–Fraenkel (ZF) set theory. This theory is formulated in a language with no modal vocabulary. Moreover, the theory quantifies freely over 'all sets', thus apparently assuming that 'all sets' are simultaneously available as a legitimate range of quantification. So this theory appears to treat the hierarchy of sets as if it was an actual or completed hierarchy. (Linnebo, 2013, 205)

So, while the iterative hierarchy does show a potential nature in the sense that it is always possible to further extend it, this potential character is not mirrored in the language of ZF(C).

Before going into more detail on Potentialism in set theory and how it contrast with its opposing view, Actualism, we will first give a more precise explanation of what Potentialism more generally entails by discussing it from a historical perspective.

1.2.1 Potential and actual Infinity

The idea of potentiality concerns infinity more generally. From Aristotle on, until the nineteenth century many endorsed the idea that infinity is always potential rather than actual (Linnebo and Shapiro, 2017). Aristotle (*Physics*,3.6,20627-29) characterises infinity as that "there is always another and another to be taken. And the thing taken will always be finite, but always different". Put this way, infinity can be seen as an ever-growing finitude; the infinite lies in that this process of being able to take new finite things never ends rather than any object actually being infinite (Lear, 1979; Sorabji, 1983).

Following this potentialist view of infinity, the successor principle for the natural numbers is as follows: it will always be the case that, for any natural number, we *can* produce a successor. Or, in other words, it is necessary that, for any natural number, possibly there is a successor. This is opposed to actual infinity, which entails that for any natural number *there is* a successor.

In the nineteenth century, however, the views of many researchers on infinity changed, largely due to Cantor's work (Linnebo and Shapiro, 2017). With his work on the real numbers and the transfinite, Cantor came to distinguish between to types of multiplicities: inconsistent and consistent multiplicities. In a letter to Dedekind (28 July 1899), Cantor wrote:

> For on the one hand a multiplicity can be such that the assumption that *all* of its elements are together leads to contradictions, so that it is impossible to conceive of the multiplicity as a unity, as "one finished thing". Such infinities I call *absolutely infinite* or *inconsistent multiplicities* [...]
>
> When on the other hand the totality of elements of a multiplicity can be thought without contradiction as "being together", so that their collection into "*one* thing" is possible, I call it a *consistent multiplicity* or *set*. (Cantor and Zermelo, 1932)

So, we have consistent multiplicities, such that the set of such a multiplicity would not lead to contradictions, and we have inconsistent multiplicities, such that the set of such a multiplicity would lead to a contradiction. The consistent and inconsistent infinite multiplicities are, respectively, identified with actual and potential infinity (Cantor, 1886).

Although the set-theoretic paradoxes did prove to be problematic for Cantor's conception of set theory in general, the view of consistent infinite multiplicities as actual has stuck.

> Of course, the set-theoretic paradoxes discovered around the turn of the century caused alarm. But many mathematicians no doubt agreed with David Hilbert's conviction that a solution can be found and accordingly that "[n]o one shall drive us out of the paradise which Cantor created for us" (Hilbert, 1925, 191). Thankfully, in the course of the first half of the twentieth century, our place in Cantor's paradise was secured — not in the way Hilbert envisaged, but thanks to the now-standard axiomatization ZFC and the closely connected iterative conception of sets. (Linnebo and Shapiro, 2017, 1)

So, Cantor's views on actual infinity persisted, and much of how infinity is commonly seen to this day is still due to his work. However, as we will see now, this standard view is not shared by all researchers.

1.2.2 Modern Potentialism

Modern Potentialism, in the form that we are interested in here, is concerned with the foundation of mathematics. While Cantor's views changed the general outlook on infinity, Potentialism argues that, at least concerning sets, this was unwarranted. According to Linnebo (2013) the standard way of interpreting modern set theory is mistaken in how it solves the paradoxes since "the theory quantifies freely over 'all sets', thus apparently assuming that 'all sets' are simultaneously available as a legitimate range of quantification. So this theory appears to treat the hierarchy of sets as if it was an actual or completed hierarchy." (Linnebo, 2013, 205). The problem with the standard, actualist treatment of the hierarchy of sets is that it seems to have trouble answering the *why*-question; if all sets are simultaneously available, why could they not form a set?

Roughly, we can contrast the potential and actual way of looking at the hierarchy of sets by the following two set formation principles:

(1) Necessarily, for any objects xx, possibly there is their set $\{xx\}$; $\Box\forall xx\Diamond\exists y\mathrm{Set}(y, xx)$, i.e. possibly there is a set containing exactly these objects.

(2) For any objects xx, there is their set; $\{xx\}$, $\forall xx\exists y\mathrm{Set}(y, xx)$, i.e. there is a set containing exactly these objects.

For the potentialist, even though we have (1), it does not follow that we have (2); the entire hierarchy of sets that we can form exists potentially but not actually. The hierarchy as a whole is not a set according to this view since elements of sets have to actually and not merely potentially exist; the sets in the potential hierarchy are merely a plurality and will never all actually exist. Note that the potentiality used here can be interpreted in a variety of ways. Linnebo (2013, 207-208) states that it is not a metaphysical modality in the sense that it is usually being used after Kripke since sets exist by metaphysical necessity if they exist at all. The modality is more closely related to the before mentioned Aristotelean distinction between actual and potential infinity. This interpretation remains somewhat vague, but we will leave further discussion on this for section 1.4. In general, the modality is related to the extension of our ontology: "A claim is possible, in this sense, if it can be made to hold by a permissible extension of the mathematical ontology; and it is necessary if it holds under any permissible such extension." (Linnebo, 2013, 208).

To make the distinction between the actualist and potentialist view of sets clearer, let us compare their interpretations of the iterative conception of sets. As mentioned at the start of this section, the intuitive explanation of the iterative conception of sets often uses reference to time and activity. In an actualist conception of the iterative hierarchy, this is often taken to be merely metaphorical. Or stronger even, as Boolos (1989, 8) claims, "thoroughly unnecessary". However, in the potentialist conception of the iterative hierarchy this reference to time and activity is not done away with as merely metaphorical. Although, as seen in the earlier cited passage from Parsons (1977, 355), the language of time and activity is often replaced by more familiar modal notions.

Potentialism is argued to be a better alternative to an actualist interpretation of the hierarchy of sets since a potentialist view can avoid a problem concerning arbitrariness that comes up for the actualist. The actualist, seeing all the sets that can be described by the cumulative hierarchy as actually existing at once, is argued not to be able to explain why the objects that make up the totality of all sets cannot be a set itself. The problem is that they freely quantify over all of them and there is no essential difference between what the hierarchy is and what a set is (Linnebo, 2013, 205). In this sense, disallowing the existence of a set of all sets, would be to cut the hierarchy off at an arbitrary level. This problem does not arise for the potentialist since there is a definite difference in the nature of sets and of the hierarchy of sets. Sets and their elements are completed and actual, whereas the hierarchy as a whole will always remain potential since the process of extending the hierarchy can go on indefinitely. The actualist, however, is argued not able to make this distinction in any proper way, or as Studd puts it:

> I can see no hope for elaborating a non-modal, tenseless stage-theory - or for that matter, any other view - in order to meet this challenge, in a non-arbitrary and principled way. (Studd, 2013, 700)

Moreover, the other problem that was mentioned in section 1.1, *indefinite extensibility*, can in the potential view of the hierarchy of sets be more easily understood and is no longer an underlying problem in need of a solution. It can simply be seen as "a byproduct of the potential character of the set-theoretic universe." (Uzquiano, 2015, 149).

The potentialist's view of the conditions under which some things can form a set is related to Cantor's views on absolute, or inconsistent, and consistent multiplicities. As we have seen, when a multiplicity is such that "the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'" (Ewald 1996, 931) then it is of the first kind. However, when the multiplicity can be thought together and seen as finished, then it is of the second kind. Sets can only contain multiplicities of the second kind (Ewald 1996, 927-932). This relates to the potentialist's view since, in the same way, the hierarchy cannot be seen as a set precisely because it is not *one finished thing* and cannot be conceived of as such since it always remains potential.

Subsequently, Linnebo proposes two constraints on when some objects can form a set; not only do the objects need to be actual, they also need to form a "consistent multiplicity", following Cantor's distinction between two kinds multiplicities. Furthermore, Linnebo (2013, 214-215) uses two core ideas for the nature of sets, namely, that they are constituted by their elements and that there is a priority of the elements of a set to the set itself. The first means that the nature of any specific set is exhausted by its elements. While the second asserts that the elements of a set have to be available prior to the existence of the set. Together, these ideas give a conception of sets such that only if some things actually and consistently exist, they can form a set, and this set would be defined by exactly those things.

1.3 Potentialism Formally

The formal side of Potentialism has proven to be a very fruitful field (see e.g. Hamkins and Linnebo (2017)) to which we could dedicate more than a thesis by itself. However, due to the philosophical considerations that will be discussed in section 1.4, we will not discuss it in detail. Instead, in this section, we will only discuss Linnebo's (2013) proposal to formally implement Potentialism, some properties of the resulting system and some minimal results.

Formally, in the current literature, Potentialism is implemented by doing set theory using a standard modal logic, such that *possibly* in (1) is modelled as the modal diamond operator. So, the potential character of the hierarchy and quantifiers' "implicit modal character" (Linnebo 2010: 155) is made explicit: the universal and existential quantifiers are respectively interpreted as $\Box\forall$ and $\Diamond\exists$. $\Box\forall\varphi$ should be interpreted as that φ will be true for every set formed, no matter how far one would extend the hierarchy. $\Diamond\exists\varphi$ should be interpreted as that φ will be true for some set that can be formed by some extension of the hierarchy (Linnebo 2010: 155-156). Given the assumptions that Linnebo makes (2013: 213) the modalised quantifiers behave just like regular quantifiers. Stronger even, "modal set theory is compatible with ordinary nonmodal ZF set theory but looks at the same subject matter under 'a finer resolution'" (Linnebo, 2013, 214).

The modal first-order language that Linnebo (2013, 210) discusses consists of "classical **S4.2**, the usual introduction and elimination rules for identity and the first-order quantifiers, and the axiom $x \neq y \to \Box(x \neq y)$". The modal logic used is a directed partial order, meaning that the accessibility relation is reflexive, transitive, anti-symmetric and directed. This makes the system sound with respect to **S4.2** (Linnebo, 2013, 209). The use of this modal logic is motivated by the desired relation between the accessibility relation and the domain assignment: $wRw' \to D(w) \subseteq D(w')$ (Linnebo, 2013, 208). This principle means that if w' is accessible from w then w' is an expansion of the mathematical ontology of w. This also means that the worlds in the model represent the stages of the process of set formation, such that the domain of each world contains the sets formed thus far. Aside from this, Linnebo (2013, 209) also requires the accessibility relation to satisfy well-foundedness. These requirements together imply that the system proves the Converse Barcan Formula, $\Box\forall x\varphi(x) \to \forall x\Box\varphi(x)$ (Linnebo, 2013, 210).

Finally, there is one further requirement that Linnebo (2013, 209) discussed, maximality.

Maximality expresses the idea that at every stage, all sets that can be formed are indeed formed. Although this idea goes beyond the minimal conception of the process of set formation, according to Linnebo, it is a necessary requirement for justifying the power set axiom. Linnebo, therefore, discusses his formal system for Potentialism both with and without this requirement. Note that maximality would change the earlier mentioned accessibility relation from being just directed, to being linear (Linnebo, 2013, 209, fn7). A comparison between the different resulting systems from a mathematically motivated philosophical perspective can be found in Hamkins (2018, 32-35), in which an outlines is given of the different potentialist considerations with how they are expressed in the validities following from, among others, linear and directed frameworks.

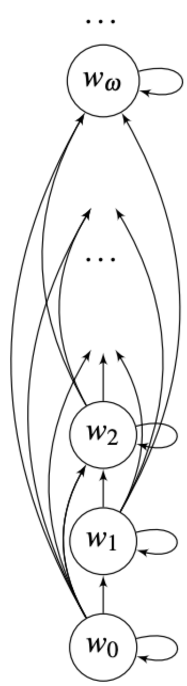

Figure 1.2: The potential hierarchy of sets

Moreover, Linnebo uses a plural logic in order to clearly show the relation between pluralities and sets. This means that, aside from our regular singular variables x, y that can be bound by the existential and universal quantifier, we also have plural variables xx, yy such that $\forall xx$... and $\exists yy$... should, respectively, be read as “given any things xx...” and “for some things yy...” (Linnebo, 2013, 210).

The plural logic is bound by the following few principles (Linnebo, 2013, 211-212):

$$u \prec xx \rightarrow \Box(u \prec xx) \qquad (\text{STB}^{+} \prec)$$

$$u \not\prec xx \rightarrow \Box(u \not\prec xx) \qquad (\text{STB}^{-} \prec)$$

$$\forall u(u \prec xx \to \Box\theta) \to \Box\forall u(u \prec xx \to \theta) \tag{INEXT-φ}$$

The first two principles state that being one of some objects and not being of some objects is stable across worlds. These two axioms ensure that when we move from one world to a next, an object that was part of the things xx will remain part of the things xx and an object that was not will still not be (Linnebo, 2013, 211). However, we want the plurality to be stable in a stronger way, we want to ensure that the objects that are among xx stay exactly the same across worlds. This is done by the third axiom schema, INEXT-φ, above (Linnebo, 2013, 211-212).

We can also assign stableness to formulas just in case the following two principles hold:

$$\varphi(u) \to \Box\varphi(u) \tag{STB$^+ - \varphi$}$$

$$\neg\varphi(u) \to \Box\neg\varphi(u) \tag{STB$^- - \varphi$}$$

The use of plural logic has two main contributions in the overall system: (i) it is a way of making arbitrary pluralities from the relevant domain available, and (ii) it provides a formalisation of the notion *extensional definiteness* (Linnebo, 2013, 212-213). The first is done by the unrestricted comprehension schema for pluralities:

$$\exists xx\forall u(u \prec xx \leftrightarrow \varphi(u)) \tag{P-Comp}$$

where xx is not free in $\varphi(u)$. Meaning that there are some things xx such that they are precisely those that satisfy φ.

Linnebo (2013, 212) characterises the notion *extensional definiteness* with the help of "the idiom of possible worlds" as follows: "a formula φ is extensionally definite at a world w just in case its extension remains the same at any later world $w' \geq w$." and formalises it as:

$$\exists xx\Box\forall u(u \prec xx \leftrightarrow \varphi(u)) \tag{ED-φ}$$

which entails the earlier seen principles ($\mathrm{STB}^+ - \varphi$), ($\mathrm{STB}^- - \varphi$) and ($\mathrm{INEXT} - \varphi$) (Linnebo, 2013, 212). These principles are used in the comparison of modal and nonmodal theories.

Aside from this, Linnebo (2013, 214-218) also formulates some principles about the nature of sets. As we discussed in section 1.2, Linnebo uses two core ideas about the nature of sets: (i) they are constituted by their elements, and (ii) there is a priority of the elements of a set to the set itself.

This first idea motivates the axiom of extensionality (Linnebo, 2013, 215):

$$x = y \leftrightarrow \forall u(u \in x \leftrightarrow u \in y) \tag{EXT}$$

While the second idea, together with the assumption that the accessibility relation between states is well-founded, permits the adoption of the foundation axiom (Linnebo, 2013, 215):

$$\forall x(\exists y(y \in x) \to \exists y(y \in x \land \forall z(z \in x \to z \notin y))) \tag{F}$$

Given this plural logic, Linnebo (Linnebo, 2013, 219) formulates the principle for set formation as

$$\Box\forall xx\Diamond\exists y\Box\forall u(u \in y \leftrightarrow u \prec xx) \tag{C}$$

This means that for the objects that are available at a specific point in the process of set formation, there is a later stage that has sets with these objects as elements (Linnebo, 2013, 218-219).

To be able to compare his modal version of set theory and standard non-modal ZF, Linnebo (2013, 213) uses the following definition.

Definition 1.3.1. *We refer to the complex strings $\Box\forall$ and $\Diamond\exists$ as* modalised quantifiers*. When $\mathcal{L}$ is a nonmodal language of first or second order, let $\mathcal{L}^{\Diamond}$ be the modal language that results from adding the modal operators $\Box$ and $\Diamond$ to $\mathcal{L}$. Given a nonmodal formula φ of $\mathcal{L}$, its* potentialist translation *$\varphi^{\Diamond}$ is the fully modalised formula of $\mathcal{L}^{\Diamond}$ that results from replacing each ordinary quantifier in φ with the corresponding modalised quantifier.*

By adding some further assumptions to the system, all potentialist translations of the axioms of ZF can be derived. Moreover, the overall structure of the potential hierarchy is isomorphic to the cumulative hierarchy V_α and modal set theory and ZF are mutually interpretable. An important theorem about the bridge between modalised quantifiers and ordinary quantifiers is the following (Linnebo, 2013, 214):

Theorem 1.3.1. *Let $\vdash$ be the relation of classical deducibility in a language $\mathcal{L}$, although if $\mathcal{L}$ is a plural language, we mean classical deducibility without the use of any plural comprehension axioms. Let $\vdash^{\Diamond}$ be deducibility in $\mathcal{L}^{\Diamond}$ by $\vdash$, S4.2, and the stability axioms for $\mathcal{L}^{\Diamond}$. Let $\varphi_1, ..., \varphi_n$ and ψ be $\mathcal{L}$-formulas. Then we have:*

$$\varphi_1, ..., \varphi_n \vdash \psi \textit{ iff } \varphi_1^{\Diamond}, ..., \varphi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}$$

As Linnebo (2013, 214) explains, this theorem shows that modal set theory, as he has presented it, is compatible with Zermelo Fraenkel set theory since this implies that something is derivable in ZF if and only if its modalised version is derivable in Linnebo's system.

1.4 The Problems of Potentialism

In the previous sections we have looked at an effort to answer the question *why* the universe of sets is not a set itself, the *why*-question. In this section, we will discuss an argument against the potentialist account of set theory given by Soysal (2017). Soysal argues that Potentialism is not able to give an answer to the *why*-question beyond, what she calls, the *minimal explanation.* To understand the criticism, we will first discuss this minimal explanation and why some have found this explanation unsatisfying. Then, we will focus on the modalities involved in the potentialist account of the hierarchy of sets and why there is no interpretation of these that can push Potentialism beyond the minimal explanation.

1.4.1 The minimal explanation

As we have seen, having a set of all sets can be problematic. The minimal explanation for why there is no universal set refers to precisely these problems; the assumption of there being such a set contradicts some axioms of ZF(C) (Soysal, 2017, 2).

While it is true that the assumption of having a universal set contradicts some of ZF(C)'s axioms, this minimal explanation has left many unsatisfied. In her paper, (Soysal, 2017, 2-4) quotes some of the expressions of unsatisfaction. Studd (2013, 699), for instance, asks: "What is it about the world that allows some sets to form a set, whilst prohibiting others from doing the same?". We know that some sets cannot form a set since this would lead to contradictions but *why* is the case that in some instances these contradictions arise, whereas in others they does not? As Soysal (2017, 3) mentions, the minimal explanation is not outright rejected as an answer in general; the minimal explanation does form an answer to why the universal set is not a set in ZF(C). However, it fails to provide a more in-depth explanation on the nature of sets such that a universal set cannot exist. What the researchers who reject the minimal explanation seem to be asking for is an account of the nature of sets such that it motivates the axioms as well as the prohibition of a universal set.

Soysal (2017, 12-17), however, goes on to argue that Potentialism is not able to give such a deeper explanation either since there is no interpretation of the potentialist's modalities that can ground an answer to the *why*-question. She divides attitudes towards the modal notions in Potentialism in two: (i) those that leave the modality as unexplained by taking it as a mathematical primitive and (ii) those that take the modality as "somehow tracking the practice of set theorists to accept the existence of larger and larger cardinals" (Soysal, 2017, 12). Eventually, neither of these interpretations are able to give a solution to the *why*-question that is somehow "deeper" than the minimal explanation.

Before we go on to discuss the two attitudes towards the modal notions in Potentialism and their problems, recall that a formula φ is derivable in ZF if and only if $\varphi^{\Diamond}$ that results from replacing each quantifier in φ with its corresponding modal quantifier, can be proved in modal set theory (Linnebo 2013, 214; Studd 2013, 710). This means that there are no results that are provable in either ZF or the formulation of modal set theory we described that are not provable in the other. So, the complication of the language should in some other way pay off; otherwise, it is just unnecessary complication.

1.4.2 Modal notions unexplained

Views that leave the modality unexplained include, for instance, Parsons' view (1977; 1983), in which the modality is taken to be a mathematical modality, Studd (2013, 707) who leaves it open how the modality has to be interpreted (within certain bounds), and also Linnebo (2010, 158) when he states that we do not need to specify a notion of the modality; any interpretation that fits the idea that sets are potential relative to their members will do.

Soysal's argument against the forms of Potentialism that leave the modal notions uninterpreted consists of two steps. First, she states that the modalities of the potential view come at a cost (2017, 13-14); the modalities form a complication of the language of set theory and of our conception of set-theoretical practice. In order to explain set-theoretical practice, we need to explain how the set-theorist relates to the potential hierarchy of sets. Broadly there are two options: (i) seeing the set-theorist as working at a certain world or stage of the hierarchy, such that this stage contains all sets that are established by the best version of set theory so far, which Soysal (2017, 13) mentions with reference to Linnebo (2010, 159, fn. 21), and (ii) seeing the set-theorist as taking an external perspective on the hierarchy (Soysal, 2017, 13). The first option is problematic due to the unclarity of "being the best version of sets theory". Not only is it not clear what it means to be the best version of a theory, it is also unclear what scope being the best version takes; does this mean the best version of set theory that a specific set-theorist is aware of or the best version acroos the whole community? The second option is problematic as well; if one can take an external perspective to the hierarchy of sets and quantify over all possible sets, then why can we not form a (possible) set of all possible sets? It appears that the *why*-question simply resurfaces in a potential setting in this case and one would have to resort to a potentialist version of the minimal explanation (Soysal, 2017, 13-14).

The second step in Soysal's argument is to note that, since the potential and iterative hierarchies are isomorphic and the modal and non-modal version of set theory are mutually interpretable, we cannot turn to the set of true sentences that contain modalities to gain an understanding of the modality at work (Soysal, 2017, 14). Therefore, it seems that there is not really anything further added to set theory by modalising its quantifiers. And thus, no deeper explanation of the *why*-question can be given by Potentialism.

One thing to note is that this set of all potential sets referred to in the first step of Soysal's argument would always be merely potential, even if the modality is taken as a primitive. For a set to be actual is taken to mean that the set has been produced by following a procedure similar to that of the iterative hierarchy; iterating the "set of" procedure. Since, following this procedure, we cannot reach the set of all potential sets, the universal set has to be merely potential. This is the difference that Linnebo (2013) sketched between the nature of the hierarchy of sets and sets themselves, namely that the hierarchy is never complete and thereby potential in character and thus intrinsically different than a set; a set being completed and actual in character. However, as Soysal (2017, 14) mentions, this does not give us a more in-depth answer to the *why*-question if we do not know what it means for sets or the universe of sets to be potential.

So, even with this side note, it seems that the potential view of the hierarchy of sets without an interpretation of the modal notions involved is not able to give a satisfactory answer to our *why*-question that is deeper than an answer already available within ZF(C); the set of all sets simply does not occur in the hierarchy and its occurrence would be in contradiction with some of the axioms.

1.4.3 Modal notions track expansion of the universe

Different defenders of Potentialism that do not leave the modal notions unexplained use various interpretations of the modal notions such that they roughly track the universe of sets expanding. Two examples of such accounts can be found in Parsons (1983) and Linnebo and Rayo (2012).

Soysal (2017, 15) argues that, in general, interpretations of the modal notions are built on two considerations: (i) *specifiability*: potentialists claim that sets exist if and only if they are specified by a theory, and (ii) *expandability*: theories can always be expanded. From this, the potentialist deduces that, since sets exist if and only if they are specified by theories and theories can always be expanded, which means that it is never the case that all sets are specified by a certain theory, there is never completed totality of "all sets" which can form a set.

The second consideration, *expandability*, is clearly related to the open-ended and indefinitely extensible nature of sets in the Potentialist's view. It matches the idea that it is always possible to form further sets.

However, the first consideration, *specifiability*, is less clearly retraceable to any specific claim in Potentialism. The idea that sets exist if and only if they are specified by a theory is closer to a formalist view of mathematical objects. It seems too strong a formulation of an idea that Linnebo does support. For instance, Linnebo (2013, 207-208) states that the modality in the potentialist' account is tied to a process of building up larger and larger domains of mathematical objects. However, as earlier discussed in section 1.4, this modality is not metaphysical since mathematical objects exist of metaphysical necessity, if at all. As we will see in the next paragraph, it is instead linked to *recognising* additional ontology, which seems to differ from what Soysal calls *specifiability*.

We will get into more details on problems with Soysal's argument later this section and in section 4.3.1. Let us for now follow the rest of Soysal's argument first. To show how we might get from these two considerations to Potentialism, Soysal (2017, 15) quotes the following passage:

> [The potentialist] view makes it impossible to draw a clean separation between the question of how one might extend one's expressive resources and the question of how many sets exist. By increasing one's expressive resources in the right sort of way, one is led to recognize additional ontology. So in so far as one believes that the process of extending one's expressive resources is essentially open-ended - and therefore that there is no definite fact of the matter about what sets are. (Linnebo and Rayo, 2012, 292)

The problem for this move from *specifiability* and *expandability* to Potentialism, according to Soysal, is that the combination of *specifiability* and *expandability* is not incompatible with the existence of a universal set. The example that Soysal (2017, 16) mentions of a form of set theory that has both a universal set and to which expandability applies is NFU^+. NFU^+ is a form of set theory derived from Quine's system of axiomatic set theory called New Foundations (NF) (Quine, 1937). NFU^+, formulated by Jensen (1969), adds the existence of urelements and the infinity and choice axioms to NF (Holmes et al., 2012).

NFU^+ contains a universal set by way of an axiom. Moreover, *Expandability* applies to NFU^+ as well, since the axioms of NFU^+ allow for a never-ending expansion of the domain. Lastly, although this form of set theory does not explicitly have *specifiability* as a constraint, there is nothing, Soysal (2017, 16) argues, that contradicts it either. So, NFU^+ with a specifiability constraint would form a counterexample to the idea that *expandability* and *specifiability* together do not allow for the existence of a universal set. Therefore, these constrants do not directly lead to an answer to the *why*-question.

Moreover, Soysal (2017, 16) argues that if we do not count this as a counterexample because of some specific property of NFU^+, we have refrained from the general *why*-question and are left with a more specific form, asking why there is no universal set in a specific form of set theory. However, as we have already seen, the minimal explanation is equally able to give an answer to this question as Potentialism is. So, once again, Potentialism is redundant for giving an answer to the *why*-question and a merely "costly detour" (Soysal, 2017, 14).

Although this argument by Soysal does tell us something about the potentialist's view and the *why*-question, it does not seem to warrant the conclusion that Soysal draws from it. First, as we have already mentioned, the specifiability constraint does not seem to clearly match or capture the potentialist's view. However, it seems that the weaker claim that the potentialist makes instead, would not be in contradiction with NFU^+ either, so this does not counter Soysal's argument.

There is, however, a stronger counterargument against Soysal's critique. Note that the potentialist is not aiming to give an explanation of why there is no universal set without assuming anything else about the nature of sets. The potentialist's view, for instance, does not just use *expandability*, but more specifically expansion by iteration of the "set of" procedure. This way, we can still exclude NFU$^+$, since it assumes certain things and misses certain other ideas about sets that do not fit the potentialist's conception of sets.

What Soysal's argument then shows is merely that one cannot base the argument against a universal set purely on *expandability* and *specifiability* since these criteria are not in incompatible with the existence of a universal set. However, there is a stronger version of Soysal's argument that can be made which will be discussed in section 4.3.1, together with a more detailed discussion on what the potentialist conception of sets entails that does exclude NFU$^+$ as an alternative.

1.4.4 The problems of Potentialism

The problem signalled by Soysal's criticism is specifically directed at how the modalities involved in Potentialism should be interpreted, leading to the conclusion that a satisfactory interpretation seems impossible. If the modality is left unexplained, there is no way to give a further explanation of why the universe of sets is not a set itself than the minimal explanation does. On the other hand, if the modality is explained based on *expandability* and *specifiability* principles, the universal set cannot excluded.

Aside from the problems mentioned by Soysal, one could also argue that Potentialism as currently implemented still does not fully capture the potential nature of the hierarchy. This criticism towards Potentialism is related to the problems that were mentioned by Soysal concerning the interpretation of the modality, although it is more focussed on the purpose that the modality is supposed to fulfil.

Linnebo (2013, 207-208) mentions that the modality is not related to metaphysical existence since mathematical objects exist by metaphysical necessity if they exist. Rather, we should interpret the modality in Aristotelian terms; the modality has to do with the distinction between actual and potential infinity. However, this does not bring us much further since according to Aristotle, potential existence is to a certain extend undefinable (*Theta*, 6, 1048a37). In any case, the potentialist seems to want to express is that the hierarchy as a whole never actually exist, but comes about step-wise through the process of set formation. In Linnebo's interpretation (see e.g. Linnebo (2013); Linnebo and Rayo (2012)), one could understand this process of set formation as building ontologies in a step-wise manner.

Potentialism, as currently implemented, is not fully able to express the process of building ontologies or expanding the expressive resources since the modality cannot capture the set formation process. Within the current modal implementation of Potentialism, the relation between potential and actual existence is modelled, but not the process that is needed for going from being potential to being actual, the *dynamic* process of expansion. This does not mean, however, that Potentialism in general will be unable to provide a satisfactory solution to the *why*-question and cannot form a more justifiable conception of the foundations of mathematics. In the next section, we will briefly look at what options there are for solving these problems within Potentialism. For now, note that there are problems with the standard interpretations of the hierarchy of sets beyond the *why*-question that Potentialism did answer too, such as the issues with arbitrariness and indefinite extensibility. So, while Potentialism, as currently implemented, is not ideal, it is in certain aspects an improvement upon other interpretations of the concept of set.

Lastly, it seems that not only the interpretation of the modality is unclear, but more generally, the notion of existence is. Right now, it is not completely clear what it means that for the actualist the hierarchy as a whole exists at once, while for the potentialist this is not the case. There is at least one sense in which it exists for either interpretation, namely metaphysically (if at all). However, what further sense of existence of the hierarchy does the actualist commit to that the potentialist does not? The answer to this question would of course also answer how the modality in the potentialist account should be interpreted, however, stated like this, we can relate this question to more general work on whether there are different ways of existing and what it means to necessarily or possibly exist (see e.g. Gibson (1998)). We will come back to the notion of existence in section 4.2 in the context of the system proposed in the following chapters. However, for now, there is no precise way to define the difference between the potentialist and actualist depending on their notions of existence.

1.5 What's next?

In this chapter, we have seen that the foundations of mathematics as currently most often used and implemented is not in line with some intuitions one might have about mathematics and the existence of its objects. The worry for the most commonly used interpretation of set theory is that it is ad hoc. Potentialism tried to answer to this worry by more closely capturing the open-ended nature of the hierarchy of sets. However, Potentialism is not able to offer a deeper explanation of *why* there is no set of all sets. Moreover, it seems that it still does not fully capture the potential nature of the hierarchy.

The problem signalled by Soysal's criticism is directed explicitly at how the modalities involved in the account should be interpreted, leading to the conclusion that a satisfactory interpretation seems impossible. There is, however, an intuitive way to change these modalities based on the conception of sets that Potentialism is trying to capture. To counter Soysal's criticism that the *why*-problem simply moves from an actualist to a potentialist setting, we need a way to express that the whole model of possible worlds does not exist at once either and find a form of Potentialism that is not merely a "costly detour".

A natural way to do so, is by using a modality that can capture the *process* of set formation, instead of merely the relation between potentially and actually existing. This way, there is also a more precise sense in which sets could be said to potentially or actually exist. There are, however, no tools available for this in standard modal logic since its operators are all static. In order to represent a process like the process of set formation, and change to our ontology or the extension of the word set, we need a dynamic setting that allows us to model change.

Therefore, while Potentialism did offer a move from an actualist to a potential view of the hierarchy, it is still missing a move from a static to a dynamic view of the set-theoretical hierarchy. What this move to a dynamic setting means and entails, will be explained in the next chapters.

2

Dynamic Set Theory

In this chapter, we will propose a new implementation for Potentialism based on the problems that were discussed in the previous chapter. In section 1.5, it was mentioned that Potentialism still misses a move from a static to a dynamic setting and that to solve some of the earlier mentioned problems we need a way to express that the whole model of possible worlds does not exist at once. We will start by expanding the explanation on this by discussing the intuitions behind the system that will be proposed: dynamic set theory (DST). Then, in the following section, 2.2, we will formalise these intuitions by giving a new notion of set formation and proposing a semantics that fits the Potentialist views better. In order to do this, we will use a combination of plural logic, modal logic and dynamic semantics. After this, in section 2.3, we will discuss how to recover the axioms of ZF.

2.1 Intuitions

In order to explain the intuitions behind the system that will be proposed in this chapter, we will first highlight some details of the potentialist's philosophical considerations. First, consider again the quote that was given in section 1.4 from Linnebo and Rayo (2012, 292).

> [The potentialist] view makes it impossible to draw a clean separation between the question of how one might extend one's expressive resources and the question of how many sets exist. By increasing one's expressive resources in the right sort of way, one is led to recognize additional ontology. So insofar as one believes that the process of extending one's expressive resources is essentially open-ended - and therefore that there is no definite fact of the matter about what sets are. (Linnebo and Rayo, 2012, 292)

Although it is debatable whether this quote represents the view of the potentialist in general, or even Linnebo's own view, it might partially clarify the intuitions for the new system that will be proposed. In order to model the process of extending one's expressive resources and thereby recognising additional ontology, we need to be able to model a processes. In the modal implementation of Potentialism, as presented in section 1.3, it is merely possible to represent what sets are possible to form from one stage to another. However, it is unable to represent the process of actually forming those sets.

In order to represent this process, we need a few extra tools that are not available to us within a standard modal logic. Specifically, we need some operators to express actions that transform one model into a new one. The action operator should model the action of extending the model such that the outcome is a new model including the sets resulting from extending the old one.

A formal tool that fits this description quite closely is dynamic semantics. Dynamic semantics has primarily been used to model the growth of information over time. In natural language semantics, for instance, it is used to model a piece of discourse as potentially updating an information state with new information such that meaning is seen as the way the information state of the receiver would be changed by the new information (see e.g. Nouwen et al. (2016), Aloni et al. (1997),Groenendijk and Stokhof (1991),Groenendijk and Stokhof (1990)). In the case of natural language, dynamic semantics comes with a specific view on meaning and information.

> Dynamic semantics is a perspective on natural language semantics that emphasizes the growth of information in time. It is an approach to meaning representation where pieces of text or discourse are viewed as instructions to update an existing context with new information, the result of which is an updated context. In a slogan: meaning is context change potential. (Nouwen et al., 2016)

These ideas match the conception of sets that we are trying to capture since we want to emphasise the growth of information, the process of set formation, over time. Moreover, as we will discuss in detail in section 4.2.2, we will model not the growth of objects in our ontology, but the growth of objects recognised by our language. This way, an update to an existing context, namely recognising certain sets as existing, will result in a new context with further sets recognised.

Note, however, that the kind of dynamic semantic that will be implemented here is in some aspect quite different from the kind used to model natural language. For instance, a rule that is common in dynamic semantics for natural language is the non-distributivity of disjunction, meaning that the order of disjuncts matter (Aloni, 2016). This rule does not make sense in the context of set theory since, as we will see in the rest of this chapter, the dynamic component in set theory only relates to the quantifiers. So, while we will use the general framework of dynamic semantics, we will not adopt the rules that are commonly associated with it within specific uses of the framework.

Another field, that is more closely related to the kind of dynamic semantics that will be adopted here, is dynamic epistemic logic. Dynamic epistemic logic is a family of logics that results from adding dynamic operators for model transformation to a logical language (Baltag and Renne, 2016). This model transformation is most often interpreted as an action, for instance, an agent making a move in a game or communicating a message to someone. Aside from this dynamic component, dynamic epistemic logic models information (Baltag and Renne, 2016). In the system that we aim to obtain here, the information consists of truths and falsities of set theory, while the dynamic component lies in recognising new sets, extending one's expressive resources.

To explain the process of model transformation in more detail, consider the following example. Take two agents, a and b, and some proposition p. Agent a does not know whether p is true or not, which means that a cannot distinguish between p and $\neg p$. Agent b does know whether p. Moreover, p is actually true. This is represented in the following model:

Now, agent b publicly announces p, resulting in a updating their knowledge with p. This is represented in the following model:

In this case, the action, namely a updating their knowledge with p, results in removing one of the states from the model since the action means that it is no longer considered

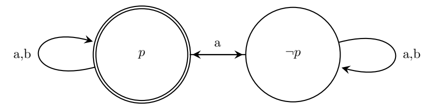

Figure 2.1: Dynamic Epistemic model before update

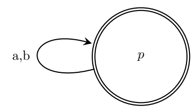

Figure 2.2: Dynamic Epistemic model after update

possible that $\neg p$.

The details of how this system works are not relevant here, with the exception of the update. Formally this is represented with the dynamic operator $!\varphi$. This operator transforms a model to remove all states such that $\neg\varphi$. Although we are indeed interested in model transformations, we will use an operator that has an opposite working, namely, adding new states to the model.

While dynamic epistemic logic differs from the dynamic semantics we are looking for here, since the transformation seen deletes rather than adds states to the model, it is more similar to our dynamic semantics than the dynamic component in natural language semantics, for instance, in how it deals with the influence of information on an agent's knowledge.

So, in general, dynamic semantics matches with potentialist' perspective on the hierarchy of sets since in the potentialist' view of the hierarchy of sets it is always possible to further extend the hierarchy, which can be seen as a form of growth of information. Dynamic semantics will make it possible to model this process of extending the hierarchy. This way of implementing the potentialist's view of the hierarchy of sets will result in a hierarchy that will look more like figure 2.3, instead of the static model for modal set theory in figure 1.2.

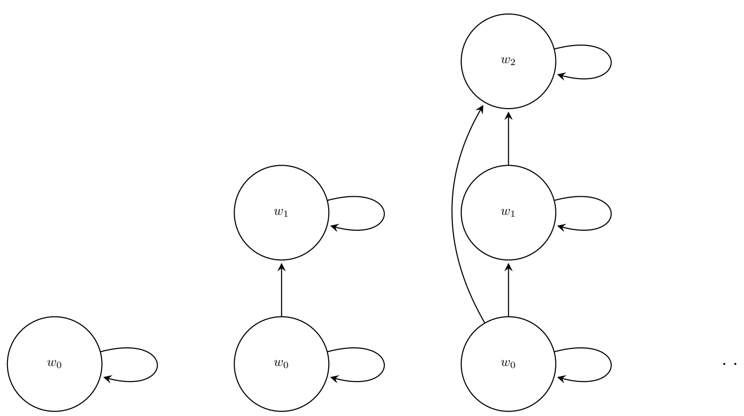

Figure 2.3: The dynamic hierarchy of sets

Going from left to right throughout these stages, and infinitely many stages further, is done by the update or extending operator that transforms the model such that we get a new state in which we have the newly formed sets. The update operators that we will use will be denoted with $\langle\ \rangle^n$ and $[\ ]^n$. Informally $\langle\ \rangle^n\varphi$ and $[\ ]^n\varphi$, respectively mean that there is a state in the model resulting from a finite string of updates to the original model such that φ and for any state in the model resulting from any finite string of updates to the model, φ. Finally, we will denote the model resulting from a finite string of updates n to a prior model $\mathcal{M}^m$ as $\mathcal{M}^{m+n}$. The semantics of these operators and the details of how this process functions will be worked out in more detail and more formally in the next section.

2.2 Dynamic Set Theory

In this section, we will formalise some of the intuitions mentioned in the previous section. Since the presentation of the formal side of Potentialism was largely focused on Linnebo (2013), we will prove analogue results in our dynamic setting. We will start with identifying the appropriate modal logic for our system, defining the plural logic used, outlining the semantics with a focus on our new dynamic operators, and finally looking at the notion of set existence in this setting. Let us first define models of dynamic set theory.

Definition 2.2.1 (Model of DST)**.** *Let $\mathcal{M} = \langle W, w_0, \leq, \mathcal{D}, \mathcal{I}\rangle$ be a model of dynamic set theory, such that*

- *W is a non-empty set of states.*
- *w_0 is the root of the model.*
- *R is a binary relation on $W \times W$.*
- *$\mathcal{D}$ is a function that assigns to each state a domain, such that $wRw' \to \mathcal{D}(w) \subseteq \mathcal{D}(w')$.*
- *$\mathcal{I}$ is a function that assigns for each atomic φ out of Φ a subset $\mathcal{I}(\varphi)$ of W, such that $\mathcal{I} : \Phi \to \mathcal{P}(W)$.*

Before looking into the properties of R, we will first motivate the use of an initial state w_0 and the requirement $wRw' \to \mathcal{D}(w) \subseteq \mathcal{D}(w')$ on the domain function.

Using a pointed model gives us the ability to, keep track of the updates done to the model thus far. Moreover, this state is present in any model of DST. So, instead of the point w_0 being used as the "actual state", as is usually done with pointed models, it is used as the initial state or the point of generation.

Our domain function requirement is based on the intuitions behind the expansion of the mathematical ontology recognised. A successor compared to the state it succeeds should contain all the mathematical ontology that was previously recognised and, additionally, the new mathematical ontology that became recognised following the update from which the successor state resulted.

Furthermore, note that the domain of the last state in any model is comprised of all the sets formed thus far. Or, more formally, for w_n such that there is no $w_{n+1} \neq w_n$ for which w_nRw_{n+1}, $\mathcal{D}(w_n)$ contains all sets formed thus far.

2.2.1 Identifying the appropriate modal logic

As we have seen in section 1.3, Linnebo (2013) uses modal logic **S4.2**. The modal logic we use will be the same. We will present its motivation in more detail here. Note that, the relations that come about due to an update to the model will also be constraint by this modal logic, meaning that any relation resulting from an update to a model satisfies the requirements of **S4.2** as well.

Given our requirement $wRw' \to \mathcal{D}(w) \subseteq \mathcal{D}(w')$, we know that the most natural accessibility relation is at least a *partial order*, so we have at least reflexivity, transitivity and anti-symmetry. This means that instead of R we will, following convention, denote our accessibility relation as $\leq$.

Aside from this, we also need to consider some constraints on the accessibility relations related to the order in which sets are formed and the amount of sets that are formed. Intuitively, we want that the order in which sets are formed to be irrelevant; that a set is not formed at a certain state does not exclude it from being formed in a successor state. This corresponds to our accessibility relation being *directed* or, formally, satisfying $\forall w \forall w' \exists w''(w \leq w'' \wedge w' \leq w'')$. Without this requirement, one could have branches in the model that will never connect anymore, meaning that some sets will not be formed in that branch of the model.

Aside from this, Linnebo (2013) also considers *maximality*. Intuitively, this principle amounts to the requirement that all sets that can be formed at a certain state will indeed be formed. This requirement would make our accessibility relation linear, instead of just directed. Since the maximality requirement has to do with the set formation principle as well, it will be discussed in more detail in section 2.2.5.

2.2.2 Plural logic

The use of a plural logic was motivated by the ability it provides to distinguish between pluralities and sets. We will use the same principles as Linnebo (2013) does, outlined in section 1.3. The first principle is a plural comprehension schema.

$$\exists xx \forall u(u \prec xx \leftrightarrow \varphi(u)) \tag{P-COMP}$$

where $\varphi(u)$ does not contain xx free, and a plurality can be empty.

In order to guarantee the necessary generality of the other principles we need to replace the standard modal operators used by Linnebo with dynamic operators. Although the principles as formulated by Linnebo already guarantee that within any model the axioms hold, it is useful to dynamically formulate since this makes it explicit that over any string of updates the principles hold as well. This gives us the following axioms:

$$u \prec xx \to [\ \ ]^n(u \prec xx) \tag{STB$^+ \prec$}$$

$$u \not\prec xx \to [\ \ ]^n(u \not\prec xx) \tag{STB$^- \prec$}$$

$$\forall u(u \prec xx \to [\ \ ]^n\theta) \to [\ \ ]^n\forall u(u \prec xx \to \theta) \tag{INEXT-φ}$$

Note, however, that the interpretation of the inextensibility schema for φ depends on the semantics used, which will be outlined in section 2.2.3. For now, recall that $[\ \ ]^n\varphi$ means that for any string of updates to the model, the resulting model satisfies φ. This means that $\text{STB}^+ \prec$ ensures that if something is one in a plurality, it will be also one in the plurality in any state in any model resulting from a finite string of updates to the original model. $\mathrm{STB}^- \prec$ ensures that if something is not one in a plurality, it will still not be one in the plurality in any state in any model resulting from a finite string of updates to the original model. Finally, INEXT-φ expresses that for any two states w, w' such that wRw', the objects that are one of a plurality at w' are among the objects that are one of the plurality at w.

These principles define how pluralities behave. Given these principles we can define the resulting system just as Linnebo (2013, 212) does:

Definition 2.2.2 (PFO & MPFO)**.** *Let* PFO *be the system that adds to standard first-order logic the standard introduction and elimination rules for the plural quantifiers and the comprehension schema (P-Comp). Let* MPFO *be the system that adds to PFO the modal logic* **S4.2***, the stability axioms (*$\mathrm{STB}^+ \prec$*), (*$\mathrm{STB}^- \prec$*), and all the instances of the inextensiblity schema (*$\mathrm{INEXT} - \prec$*).*

In the next section, we will define the semantics for plural quantifiers. Moreover, further principles relating pluralities to sets will be outlined in section 2.2.5.

2.2.3 Semantics

We will now consider the assignment of truth-values for any arbitrary formula of the language. We will outline both a classical and an intuitionistic version of the semantics, the first in line with Linnebo (2013) and Hamkins and Linnebo (2017), the second based on the motivations that Lear (1977) presents.

First, the classical semantics for Dynamic Set Theory:

Definition 2.2.3 (Semantics for DST)**.**

$w \vDash \varphi(x_1, ..., x_n)$ *iff if* φ *is atomic,* $\varphi(x_1, ..., x_n)$ *holds at* w

$w \vDash \neg\varphi$ *iff* $w \nvDash \varphi$

$w \vDash \varphi \wedge \psi$ *iff* $w \vDash \varphi$ *and* $w \vDash \psi$

$w \vDash \varphi \vee \psi$ *iff* $w \vDash \varphi$ *or* $w \vDash \psi$

$w \vDash \varphi \rightarrow \psi$ *iff if* $w \vDash \varphi$ *then* $w \vDash \psi$

$w \vDash \exists x\varphi x$ *iff there is a* $d \in \mathcal{D}(w)$ *such that* $w \vDash \varphi(d)$

$w \vDash \forall x\varphi x$ *iff for all* $d \in \mathcal{D}(w)$*,* $w \vDash \varphi(d)$

$w \vDash \exists xx\varphi xx$ *iff there is* xx *in* $\mathcal{D}(w)$ *such that for all* $u \prec xx$*,* $w \vDash \varphi(u)$

$w \vDash \forall xx\varphi xx$ *iff for all* xx *in* $\mathcal{D}(w)$ *and for all* $u \prec xx$*,* $w \vDash \varphi(u)$

$w \vDash \Diamond\varphi$ *iff there is a* w' *such that* wRw' *and* $w' \vDash \varphi$

$w \vDash \Box\varphi$ *iff for all* w' *such that* wRw'*,* $w' \vDash \varphi$

$\mathcal{M}^m, w \vDash \langle\ \rangle^n\varphi$ *iff there is a finite string, length* n*, of updates to* $\mathcal{M}$ *such that there is a state* w' *such that* $w \leq w'$ *in the resulting model* $\mathcal{M}^{m+n}$ *such that* $\mathcal{M}^{m+n}, w' \vDash \varphi$

$\mathcal{M}^m, w \vDash [\ ]^n\varphi$ *iff for any finite string, length* n*, of updates to* $\mathcal{M}$ *and for any state* w' *such that* $w \leq w'$ *in the resulting model* $\mathcal{M}^{m+n}$*,* $\mathcal{M}^{m+n}, w' \vDash \varphi$

While a large part of this is quite standard, there are a few things to note about the semantics. First, the plural quantifiers. The existentially quantified statement, asserts that there is a plurality in the domain of the relevant state such that all things that are a part of this plurality satisfy φ. The universally quantified statement, asserts that for any plurality this is the case.

Secondly, note that we use both static and dynamic modal opertors. The static modal operators range over the states accessible from the relevant state in the current model. The dynamic modal operators, on the other hand, range over all the states accessible from the current state in any model resulting from a finite number of updates to the current model.

We will call the intuitionistic version of our dynamic set theory, DST_i. In order to formulate the intuitionistic version of the semantics, we first look at the motivations for the semantics for modal set theory as given by Lear (1977), and then translate this to our dynamic setting.

In the paper "Sets and Semantics", Lear (1977) works out a formulation of set theory that can capture the idea that the extension of "set" can change over time.

> The assignment of values for arbitrary formulas of the language fits nicely with the intuition of the extension of "set" changing over time with one's improved understanding of the concept of set and the structure of the set-theoretic universe. The assignment $\mathcal{W}$ is two-valued in this sense: there are two values, 0,1, and over each $\mathcal{V}(t)$, $t \in \mathcal{T}$, every sentence φ receives a value. A difference from the standard semantics is that, if φ receives the value 0, one cannot infer that $\neg\varphi$ receives the value 1. (Lear, 1977, 93)

The semantics that will be presented here is based on the same intuitions and principles. However, instead of taking time as the basis, we will use states and updates to models to capture the intuition of the extension of the concept of set changing. We will now present the semantics and look into some of the specific details, both outlining how this differs from the classical semantics for DST and from the semantics as given by Lear (1977, 93-94).

Definition 2.2.4 (Semantics for DST_i)**.**

$w \vDash \varphi(x_1, ..., x_n)$ iff if φ is atomic, $\varphi(x_1, ..., x_n)$ holds at w

$w \vDash \neg\varphi$ iff for all w' such that $w \leq w'$, $w' \nvDash \varphi$

$w \vDash \varphi \wedge \psi$ iff $w \vDash \varphi$ and $w \vDash \psi$

$w \vDash \varphi \vee \psi$ iff $w \vDash \varphi$ or $w \vDash \psi$

$w \vDash \varphi \to \psi$ iff for all w' such that $w \leq w'$, if $w' \vDash \varphi$ then $w' \vDash \psi$

$w \vDash \exists x\varphi x$ iff there is a state w' such that $w \leq w'$ and $d \in \mathcal{D}(w')$ such that $w' \vDash \varphi(d)$

$w \vDash \forall x\varphi x$ iff for all w' such that $w \leq w'$ and $d \in \mathcal{D}(w')$, $w' \vDash \varphi(d)$

$w \vDash \exists xx\varphi xx$ iff there is a w' such that $w \leq w'$ and there is xx in $\mathcal{D}(w')$ such that for all $u \prec xx$, $w' \vDash \varphi(u)$

$w \vDash \forall xx\varphi xx$ iff for all w' such that $w \leq w'$ and for all xx in $\mathcal{D}(w')$ such that for all $u \prec xx$, $w' \vDash \varphi(u)$

$w \vDash \Diamond\varphi$ iff there is a w' such that $w \leq w'$ and $w' \vDash \varphi$

$w \vDash \Box\varphi$ iff for all w' such that $w \leq w'$, $w' \vDash \varphi$

$\mathcal{M}^m, w \vDash \langle\ \ \rangle^n\varphi$ iff there is a finite string, length n, of updates to $\mathcal{M}$ such that there is a state w' such that $w \leq w'$ in the resulting model $\mathcal{M}^{m+n}$ such that $\mathcal{M}^{m+n}, w' \vDash \varphi$

$\mathcal{M}^m, w \vDash [\ \ ]^n\varphi$ iff for any finite string, length n, of updates to $\mathcal{M}$ and for any state w' such that $w \leq w'$ in the resulting model $\mathcal{M}^{m+n}$, $\mathcal{M}^{m+n}, w' \vDash \varphi$

There are a few things to note about this semantics, starting with the negation clause. The interpretation of the $\neg\varphi$ is different than in a classical setting; it models the idea that no matter how the hierarchy of sets expands, if $\neg\varphi$ is true then there will be no set in any further state such that it satisfies φ. A similar interpretation holds for the implication. Instead of requiring that the current state satisfies "if φ then ψ", we require that for any accessible state this is the case.

There are some clauses in the semantics of DST_i that are not part of the semantics as given by Lear (1977), namely those for the plural quantifiers and the modalities. The plural quantifiers, just as the negation, behave intuitionistically since they are not bound to a specific state, but bound to the states accessible to it.

The other operators have the same semantics as in the classical setting. Note, that we can associate the interpretation of the combination of an update operator and a quantifier (either dynamic box with a universal quantifier or dynamic diamond with an existential quantifier), behaves in a similar way as what Lear (1977) describes for his quantifiers.

> [L]et us consider the interpretation of the universal quantifier first. The interpretation of the universal quantifier differs from the classical interpretation which simply ranges over a given domain. The interpretation of the universal quantifier offered here attempts to compensate for the fact that the extension of "set" will always be limited, while preserving the intuitions underlying one's assertion of a universal statement. The idea is that the sentence "$\forall x\varphi(x)$" receives value 1 at a given time if, no matter how the extension should enlarge in the future, $\varphi(\quad)$ will be true of every element c in the expanded universe. (Lear, 1977, 94)

So, the universal quantifier is supposed to in some way represent both the always incomplete extension of the concept of set and the standard intuitions on what asserting a universal statement amounts to. In our case, the expansion of the universe is not modelled over time but over updates of the model. The idea then is that the sentence $[\ \ ]^n\forall x\varphi(x)$ receives the value 1 at a state if, no matter how many updates to the model will follow, $\varphi(d)$ will be true of every $d \in \mathcal{D}(w')$ for every w' accessible from w in the expanded universe.

The combination of the dynamic diamond and existential quantifier is built on similar intuitions, and gets, thereby, just as the plural existential quantifier, an intuitionistic character. Asserting an existentially quantified sentence means that under some expansion of the universe the sentence is true. More importantly, if we deny an existentially quantified sentence, this means that there is no expansion of the extension of set such that this sentence will be true.

There are different motivations involved in choosing either version of the semantics. If we want to stay true to the system of Potentialism as worked out by Linnebo (2013), we should use the classical semantics. This is not only because the semantics used by Linnebo are classical, but also because without a classical semantics we will not be able to recover all the results that he gives. As we will see, we can not motivate all the axioms of ZF using an intuitionistic version of the semantics. However, on the other hand, there is a more philosophical case to be made for the intuitionistic version of the semantics. As Lear (1977) states, the use of intuitionistic semantics fits well with the idea that the extension of "set" changes over time with gained understanding of the concept and hierarchy of sets throughout further steps in the structure. We will discuss the difference between the two in more detail in section 2.4 from a formal perspective and in section 4.2.3 from a philosophical perspective.

Given either system, we now also know that our models satisfy the dynamic version of the Converse Barcan Formula.

Theorem 2.2.1 (Converse Barcan Formula)**.** *Any model of $DST_{(i)}$ satisfies* $[\ ]^n \forall x \varphi(x) \to \forall x [\ ]^n \varphi(x)$.

Proof. Let $\mathcal{M}^m, w$ be arbitrary such that $\mathcal{M}^m, w \vDash [\ ]^n \forall x \varphi(x)$. Then, for all $\mathcal{M}^{m+n}$ such that $\mathcal{M}^m [\ ]^n \mathcal{M}^{m+n}$ and for any w' in $\mathcal{M}^{m+n}$ such that $w \leq w'$, $\mathcal{M}^{m+n}, w' \vDash \forall x \varphi(x)$. This means that $\mathcal{M}^{m+n}, w' \vDash \varphi(d)$ for all $d \in \mathcal{D}(w')$. Then, since from $w \leq w'$ it follows that $\mathcal{D}(w) \subseteq \mathcal{D}(w')$, we know that $\mathcal{M}^{m+n}, w' \vDash \varphi(d)$ for all $d \in \mathcal{D}(w)$. This means that $\mathcal{M}^m, w \vDash [\ ]^n \varphi(d)$ for all $d \in \mathcal{D}(w)$. Therefore, $\mathcal{M}^m, w \vDash \forall x [\ ]^n \varphi(x)$. Since $\mathcal{M}, w$ were arbitrary, we have $[\ ]^n \forall x \varphi(x) \to \forall x [\ ]^n \varphi(x)$. $\square$

2.2.4 The dynamics of DST

We now have a system with two update operators, $\langle\ \rangle^n$ and $[\ ]^n$. The idea is to, whenever there is a quantifier in nonmodal set theory, add these update operators to them such that $\forall$ becomes $[\ ]^n \forall$ and $\exists$ becomes $\langle\ \rangle^n \exists$. By combining the quantifiers, ranging over finite models, with update operators, the intuitive meaning of the existential and universal quantifier are preserved. Universally quantified statements still range over all sets, whether these sets are part of the current model or not. Since the dynamic operators are only used in combination with the quantifiers, in the case of assertions that are about a specific stage of the hierarchy, and thus a specific model, our language is static. It is only when we aim to make assertions about what exists and what universally is true that we use the dynamic features of the language.

We have defined the dynamic operators in such a way the model keeps track of the numbers of updates that have been done by the index it receives. We can use the index of a model to identify the model with the level in the hierarchy in ZF(C) and Linnebo's system. The nth update to the model can be identified with (a part of) the nth stage in the iterative hierarchy and (one of) the nth states in the potential hierarchy. Whether it is identified with a part of a stage or the entire stage, and equally one of the states or all of the states of the potential hierarchy, at that level depends on the adoption of the maximality principle. In case the model is linear, we can exactly identify them with each other, while, in case the model is directed we can only partially identify them with each other.

Note that, due to the semantics of the update operators, the difference between the classical and intuitionistic system is not as big as it would otherwise be. In the case we are interested in quantified statements, the results will be the same, due to the semantics of the update operators. It is only within states and models that there is a difference between the results of the classical and intuitionistic semantics of DST. The implications of this will be further discussed in section 4.2.3. Note that, constructivist considerations do influence the recovery of the axioms of ZF and the translation results between dynamically modal and nonmodal set theory. We will discuss these different results in section 2.4 and chapter 3.

2.2.5 Set existence and the nature of sets

Now that we have the general outline of our formal environment, we can finally look into the process of set formation in our system. In order to do this, we need to define the relation between pluralities and sets and how we get from the former to the latter.

Linnebo (2013, 219) only uses the following principle (C) for the existence of sets:

$$\Box\forall xx\Diamond\exists y\Box\forall u(u \in y \leftrightarrow u \prec xx) \tag{C}$$

This means that, necessarily, given any objects xx it is possible for them to form a set. In order to translate this principle to our setting, we need to replace the modal operators with our dynamic update operators. This translation is not completely obvious, however. First, note that xx has to be from the domain before updating, otherwise the principle will cause versions of paradoxes such as Russell's paradox. Moreover, it is not evident that we want to replace all modal operators in (C) with dynamic update operators.

Let us consider (C) step by step. In the setting of DST_i, the first box is made unnecessary by the semantics for the universal plural operator. We have $w \vDash \forall xx\varphi xx$ if and only if for all w' such that wRw' and for all xx in $\mathcal{D}(w')$ such that for all $u \prec xx$, $w' \vDash \varphi(u)$, meaning that we already include all the pluralities in any domain of the accessible states. In the setting of DST, however, the box is necessary to make sure we range over all pluralities in the model in states accessible from the current.

The diamond operator in (C) is supposed to model the possible formation of sets. In the setting of $\mathrm{DST}_{(i)}$, we replace this diamond operator with an update operator, such that we transform our original model in such a way that the resulting model satisfies the remaining part of the formula, $\exists y\Box\forall u(u \in y \leftrightarrow u \prec xx)$.

Then, finally, the second box is used to express that the members of y are, everywhere, exactly those things that are a part of plurality xx. This means that, in the setting of $\mathrm{DST}_{(i)}$, we need to replace this box operator with an update operator as well. Otherwise, $\forall u(u \in y \leftrightarrow u \prec xx)$ is not guaranteed in further updates to the model. This gives us:

$$\forall xx\langle\ \rangle^n\exists y[\ ]^n\forall u(u \in y \leftrightarrow u \prec xx) \tag{DST_i-C}$$

$$\Box\forall xx\langle\ \rangle^n\exists y[\ ]^n\forall u(u \in y \leftrightarrow u \prec xx) \tag{DST-C}$$

Before we go on to recover the axioms of ZF, we first need a few more principles on the nature of sets and the relation between conditions and sets. As discussed at the end of section 1.2, Linnebo uses two core ideas on the nature of sets: (i) sets are extensionally defined, and (ii) there is a priority of the elements of a set to the set itself. These ideas vindicate the adoption of a few principles that are important for the recovery of the axioms of ZF.

First, the idea that sets are extensionally defined, already motivates the axiom of extensionality:

$$[\ ]^n\forall x[\ ]^n\forall y([\ ]^n\forall z(z \in x \leftrightarrow z \in y) \to x = y) \tag{$\mathrm{DST}_{(i)}$ EXT}$$

Note that in DST all the universal quantified should be supplemented with update operators, to give us the necessary generality.

The second core idea, the priority of the elements of sets over their sets, is expressed by the adoption of the foundation axiom:

$$[\ ]^n\forall x(\langle\ \rangle^n\exists y(y \in x) \rightarrow \langle\ \rangle^n\exists y(y \in x \wedge [\ ]^n\forall z(z \in x \rightarrow z \notin y))) \tag{DST F}$$

Linnebo (2013, 216-217) justifies the adoption of foundation by this priority principle and the assumption that the accessibility relation between possible worlds is well-founded. Note, however, that the adoption of Foundation is not acceptable in our intuitionistic setting since it leads to instances of the excluded middle, which is constructively not permissible (Crosilla, 2019). We will discuss an intuitionistically acceptable alternative, with similar desired consequences, in section 2.3.2.

Before we go on to consider the recovery of axioms of ZF, we will finally come back to a principle mentioned in section 1.3, the *maximality principle*. This is the principle that expresses that at each state, all sets that can be formed from the pluralities of the preceding world are, in fact, formed (Linnebo, 2013, 209). This principle is fairly natural from the perspective of the cumulative hierarchy, since this way, the states correspond directly to stages in the hierarchy (Menzel, ming, 17). Linnebo flags the use of this principle since it is not part of a minimal conception of the process of set formation. However, it is necessary for the recovery of the powerset axiom of ZF, since this requires the existential definiteness of subsets, which we will discuss at the start of the following section.

2.3 Recovering the Axioms of ZF

In order to recover the axioms Empty Set, Separation, Pairing, Union and Power of ZF, we need a few further principles that correspond to Linnebo's (2013) (ED-∈) and (ED-⊆).

The first principle,

$$\exists yy\Box\forall u(u \prec yy \leftrightarrow u \in y) \tag{ED-$\in$}$$

expresses the idea that being an element of a particular set is extensionally definite, meaning that at every state in which the set exists, it has the same elements. (Linnebo, 2013, 215) In DST and DST$_i$, this is expressed by

$$\exists yy[\ ]^n\forall u(u \prec yy \leftrightarrow u \in y) \tag{DST$_{(i)}$ ED-$\in$}$$

here the combination of the update operator and the universal quantifier expresses what $\Box\forall x$ does in Linnebo's case.

The second principle,

$$\exists yy\Box\forall u(u \prec yy \leftrightarrow u \subseteq y) \tag{ED-$\subseteq$}$$

expresses the idea that being a subset of a particular set is extensionally definite, meaning that at every state in which the set exists, it has the same subsets. This principle depends on whether one excepts the maximality principle since without it one can easily construct counterexamples to it (Linnebo, 2013, 217-218).

> Consider a set a and some subset $b \subseteq a$ that is present at some later world. When a was formed, all of its elements must already have been available. So *a fortiori* all the elements of b must have been available. When a was formed, we therefore had the *ability* to form b. But was this ability *exercised*? According to

the principle of Maximality - which says that we always form all the sets that we are capable of forming - the answer is *yes* and (ED - $\subseteq$) will thus hold. But as we have seen, without Maximality, it is easy to construct a counterexample to (ED - $\subseteq$). (Linnebo, 2013, 218)

In DST and DST$_i$, (ED - $\subseteq$) is expressed by

$$\exists yy[\ \ ]^n \forall u(u \prec yy \leftrightarrow u \subseteq y) \qquad \text{(DST}_{(i)} \text{ ED-}\subseteq)$$

Again, the combination of the update operator and the universal quantifier expresses what $\Box\forall x$ does in Linnebo's case.

We will now prove a lemma that guarantees the stableness of the formulas in our languages. First, however, we need a definition corresponding to definition 1.3.1 for our dynamic modal language and formulas.

Definition 2.3.1. *We refer to the complex strings* $[\ \ ]^n\forall$ *and* $\langle\ \ \rangle^n\exists$ *as* dynamically modalised quantifiers. *When* $\mathcal{L}$ *is a nonmodal language of first or second order, let* $\mathcal{L}^{\langle\ \ \rangle^n}$ *be the modal language that results from adding the modal operators* $[\ \ ]^n$ *and* $\langle\ \ \rangle^n$ *to* $\mathcal{L}$*. Given a nonmodal formula* φ *of* $\mathcal{L}$*, its* potentialist translation $\varphi^{\langle\ \ \rangle^n}$ *is the fully modalised formula of* $\mathcal{L}^{\langle\ \ \rangle^n}$ *that results from replacing each ordinary quantifier in* φ *with the corresponding modalised quantifier.*

Let $\mathcal{L}_{\in}$ *be the language of ordinary nonmodal set theory, and* $\mathcal{L}_{P\in}$ *be the corresponding plural language. Then,* $\mathcal{L}_{\in}^{\langle\ \ \rangle^n}$ *and* $\mathcal{L}_{P\in}^{\langle\ \ \rangle^n}$ *are the dynamic modal languages that result from adding the dynamic operators to each nonmodal language respectively.*

The following lemma serves to ensure that the modalised version of a formula receives the same evaluation in any state of the hierarchy.

Lemma 2.3.1. *Let* φ *be a fully modalised well-formed formula of modal language* $\mathcal{L}^{\langle\ \ \rangle^n}$*. Then **S4.2** and the stability axioms for* $\mathcal{L}^{\langle\ \ \rangle^n}$ *prove that* φ*,* $\langle\ \ \rangle^n\varphi$ *and* $[\ \ ]^n\varphi$ *are all equivalent and thus, in particular, that* φ *is stable.*

Proof. Since we work in an extension of modal logic **T**, it suffices to prove $\langle\ \ \rangle^n\varphi \rightarrow [\ \ ]^n\varphi$. We will use a proof by induction over the complexity of φ.

i. φ is atomic. Then we know that $\langle\ \ \rangle^n\varphi \rightarrow [\ \ ]^n\varphi$ by our stability axioms.

   Induction hypotheses: let ψ, ψ_1 and ψ_2 be such that $\langle\ \ \rangle^n\psi \rightarrow [\ \ ]^n\psi$, $\langle\ \ \rangle^n\psi_1 \rightarrow [\ \ ]^n\psi_1$ and $\langle\ \ \rangle^n\psi_2 \rightarrow [\ \ ]^n\psi_2$.

ii. $\varphi = \neg\psi$. Then $\langle\ \ \rangle^n\varphi \rightarrow [\ \ ]^n\varphi$ follows from the induction hypothesis.

iii. $\varphi = \psi_1 \wedge \psi_2$. Then $\langle\ \ \rangle^n\varphi$ implies $\langle\ \ \rangle^n\psi_1 \wedge \langle\ \ \rangle^n\psi_2$, which by the induction hypothesis implies $[\ \ ]^n\psi_1 \wedge [\ \ ]^n\psi_2$. This then implies $[\ \ ]^n\varphi$.

iv. $\varphi = \psi_1 \vee \psi_2$. Then $\langle\ \ \rangle^n\varphi$ implies $\langle\ \ \rangle^n\psi_1 \vee \langle\ \ \rangle^n\psi_2$, which by the induction hypothesis implies $[\ \ ]^n\psi_1 \vee [\ \ ]^n\psi_2$. This then implies $[\ \ ]^n\varphi$.

v. $\varphi = \psi_1 \rightarrow \psi_2$. Then $\langle\ \ \rangle^n\varphi$ implies $\langle\ \ \rangle^n\psi_1 \rightarrow \langle\ \ \rangle^n\psi_2$, which by the induction hypothesis implies $[\ \ ]^n\psi_1 \rightarrow [\ \ ]^n\psi_2$. This then implies $[\ \ ]^n\varphi$.

vi. $\varphi = \exists x\psi$. Since we work in an extension of **S4**, we have $\langle\ \rangle^n\varphi \to \varphi$ for φ of this form. Therefore, it suffices to show that $\varphi \to [\ ]^n\varphi$. By the induction hypothesis we know that φ is equivalent to $\langle\ \rangle^n\exists x[\ ]^n\psi$. By the converse Barcan formula, we then have $\forall x[\ ]^n\exists y(x = y)$. Thereby, $\exists x[\ ]^n\psi \to [\ ]^n\exists x\psi$. Therefore, $\langle\ \rangle^n\exists x[\ ]^n\psi$ implies $\langle\ \rangle^n[\ ]^n\exists x\psi$. By (G) it then follows that $[\ ]^n\langle\ \rangle^n\exists x\psi$, which is $[\ ]^n\varphi$.

vii. $\varphi = \forall x\psi$. Since we work in an extension of **S4**, we have $\langle\ \rangle^n\varphi \to \varphi$ for φ of this form. Therefore, it suffices to show that $\varphi \to [\ ]^n\varphi$. By the induction hypothesis we know that φ is equivalent to $\langle\ \rangle^n\forall x[\ ]^n\psi$. By the converse Barcan formula, we then have $\forall x[\ ]^n\exists y(x = y)$. Thereby and the properties of **S4.2**, $\forall x[\ ]^n\psi \to [\ ]^n\forall x\psi$. Therefore, $\langle\ \rangle^n\forall x[\ ]^n\psi$ implies $\langle\ \rangle^n[\ ]^n\forall x\psi$. By (G) it then follows that $[\ ]^n\langle\ \rangle^n\forall x\psi$, which is $[\ ]^n\varphi$.

$\square$

In order to recover the axioms infinity and replacement, we need principles that correspond to Linnebo's (ED-Repl) and ($\Diamond$-Refl) in combination with a lemma to ensure that we cannot derive Russell's paradox by the latter principle (Linnebo, 2013, 221-223).

$$\Box\forall u\Diamond\exists v\Box\forall v'(\psi^{\Diamond}(u, v') \leftrightarrow v = v') \to \Box\forall xx\Diamond\exists yy(\forall u \prec xx)(\exists v \prec yy)\psi^{\Diamond}(u, v) \quad \text{(ED-Repl)}$$

In the case of classical DST we have to exchange the box and diamond operators in the antecedent by their corresponding dynamic operators, to secure the same level of generality. However, in the consequent we do need the box and diamond operators, since we otherwise do not range over all the relevant pluralities in the model. This gives us the following:

$$[\ ]^n\forall u\langle\ \rangle^n\exists v[\ ]^n\forall v'(\psi^{\Diamond}(u, v') \leftrightarrow v = v') \to \Box\forall xx\Diamond\exists yy(\forall u \prec xx)(\exists v \prec yy)\psi^{\langle\ \rangle^n}(u, v) \quad \text{(DST ED-Repl)}$$

In the case of intuitionistic DST we can again exchange the box and diamond operators in the antecedent for their corresponding dynamic operators. Furthermore, we can remove the box and diamond in the consequent, since the plural quantifiers already range over the entire model. This gives us the following:

$$[\ ]^n\forall u\langle\ \rangle^n\exists v[\ ]^n\forall v'(\psi^{\Diamond}(u, v') \leftrightarrow v = v') \to \forall xx\exists yy(\forall u \prec xx)(\exists v \prec yy)\psi^{\langle\ \rangle^n}(u, v) \quad \text{(DST}_i\text{ ED-Repl)}$$

The last principle to consider is the following reflection principle:

$$\varphi^{\Diamond} \to \Diamond\varphi \quad (\Diamond\text{-Refl})$$

used to express that "truths about the potential hierarchy of sets are 'reflected' in truths about individual possible worlds." (Linnebo, 2013, 222). In our case we need to replace the regular modal operators with dynamic operators, giving us:

$$\varphi^{\langle\ \rangle^n} \to \langle\ \rangle^n\varphi \quad (\langle\ \rangle^n\text{-Refl})$$

As Linnebo (2013, 222-223) explains, this principle is not entirely unproblematic; it seems that applying principle (C) to this reflection principle allows us to derive the claim that possibly every plurality forms a set allowing us to derive the same reasoning as, for instance, in Russell's paradox. There is a way to circumvent this, however. Our plural quantifiers do not range over all pluralities whatsoever but over those pluralities that can form sets. Thereby, there is a one-to-one correspondence between the pluralities and sets. To formally ensure this, we need the following lemma, which can be proved by Linnebo's (2013, 223) suggested method.

Lemma 2.3.2. *Let φ be a fully dynamically modalised sentence of $\mathcal{L}_{P\in}^{\langle\ \rangle^n}$. Let φ' be the result of replacing every plural variable uu_i of φ with a singular variable x_i (assumed not already to occur in φ') and replacing every occurrence of "$\prec$" with an occurrence of "$\in$". Then NS$^-$ + (C) proves $\varphi \leftrightarrow \varphi'$.*

Proof. To prove this claim, one can prove a more general claim where φ may contain free variables. Assume the free plural variables in φ are $uu_1, ..., uu_n$. Then the claim is that the mentioned theory proves

$$x_1 = \{uu_1\} \wedge ... \wedge x_n = \{uu_n\} \rightarrow (\varphi \leftrightarrow \varphi')$$

where x = {uu} abbreviates $\forall v(v \in x \rightarrow v \prec uu)$.

This can be proved by an induction on the number of quantifiers in φ. □

With these tools at hand, we will now go through the axioms.

2.3.1 Extensionality

As we have already seen, the axiom of extensionality is directly motivated by the idea that sets are extensionally defined. Moreover, the interpretation of the universal quantifier as combined with an update operator gives us the necessary generality. Therefore, for both DST and DST$_i$ we have:

$$[\ \ ]^n \forall x [\ \ ]^n \forall y ([\ \ ]^n \forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)) \tag{DST$_{(i)}$ EXT}$$

2.3.2 Foundation

In the case of classical DST, the axiom of foundation is motivated by the priority of the elements of sets over sets, in combination with the assumption that the accessibility relation between states is well-ordered. Therefore, for DST we have:

$$[\ \ ]^n \forall x (x \neq \emptyset \rightarrow \langle\ \ \rangle^n \exists y \in x (y \cap x = \emptyset)) \tag{DST F}$$

In the case of intuitionistic DST, we need an alternative since it has been shown that the axiom of foundation in combination with empty set, extensionality, separation and unordered pairs implies the law of the excluded middle, making the logic classic again (see e.g. Myhill (1973); Crosilla (2019); Incurvati (2008) for proofs and explanation).

Instead, the intuitionistic version of ZF proposed by Myhill (1973) uses the principle $\forall x(\forall y \in x.\varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x)$, which is the principle of transfinite induction for $\in$. Again, the interpretation of the universal quantifier with an update operator gives us the necessary generality. Therefore, for DST$_i$, we have:

$$[\ \ ]^n \forall x([\ \ ]^n \forall y \in x.\varphi(y) \to \varphi(x)) \to [\ \ ]^n \forall x \varphi(x) \qquad \text{(DST}_i \text{ IND)}$$

Note, however, that we need $\varphi(x)$ and $\varphi(y)$ to be stable here. This can be guaranteed by lemma 2.3.1.

2.3.3 Empty Set

To prove that $\mathrm{DST}_{(i)}$ proves versions of the empty set axiom, note that $u \neq u$ is extensionally definite. Then, applying our versions of (C) to the pluralities defined by $u \neq u$, gives us the dynamic version of the empty set axiom:

$$\langle\ \ \rangle^n \exists x[\ \ ]^n \forall u(u \notin x) \qquad \text{(DST}_{(i)} \text{ EMPT)}$$

2.3.4 Separation

To prove that $\mathrm{DST}_{(i)}$ proves versions of the separation axiom schema, consider $u \in a \wedge \varphi(u)$, where $\varphi(u)$ is stable. This can be guaranteed by lemma 2.3.1. Since we have extensional definiteness of membership, we know that $u \in a \wedge \varphi(u)$, where $\varphi(u)$ is stable, is extensionally definite. Then, applying our versions of (C) to the pluralities defined by $u \in a \wedge \varphi(u)$, where $\varphi(u)$ is stable, gives us the dynamic version of the separation axiom schema:

$$\langle\ \ \rangle^n \exists x[\ \ ]^n \forall u(u \in x \leftrightarrow u \in a \wedge \varphi(u)) \qquad \text{(DST}_{(i)} \text{ SEP)}$$

2.3.5 Pairing

To prove that $\mathrm{DST}_{(i)}$ proves versions of the pairing axiom, note that $u = a \vee u = b$ is extensionally definite. Then, applying our versions of (C) to the pluralities defined by $u = a \vee u = b$, gives us the dynamic version of the pairing axiom:

$$\langle\ \ \rangle^n \exists x[\ \ ]^n \forall u(u \in x \leftrightarrow u = a \vee u = b) \qquad \text{(DST}_{(i)} \text{ PAIR)}$$

2.3.6 Union

To prove that $\mathrm{DST}_{(i)}$ proves versions of the union axiom, consider $\exists x(u \in x \wedge x \in a)$. Since we have extensional definiteness of membership, it follows that $\exists x(u \in x \wedge x \in a)$ is extensionally definite. Then, applying our versions of (C) to the pluralities defined by $\exists x(u \in x \wedge x \in a)$, gives us the dynamic version of the union axiom:

$$\langle\ \ \rangle^n \exists x[\ \ ]^n \forall u(u \in x \leftrightarrow \langle\ \ \rangle^n \exists v(u \in v \wedge v \in a)) \qquad \text{(DST}_{(i)} \text{ UN)}$$

2.3.7 Powerset

To prove that $\mathrm{DST}_{(i)}$ proves versions of the powerset axiom, consider $u \subseteq a$. If we assume the maximality principle, discussed earlier, we have extensional definiteness of subsets. In this case it follows that $u \subseteq a$ is extensionally definite. Then, applying our versions of (C) to the pluralities defined by $u \subseteq a$, gives us the dynamic version of the powerset axiom:

$$\langle\ \ \rangle^n \exists x[\ \ ]^n \forall u(u \in x \leftrightarrow u \subseteq a) \qquad \text{(DST}_{(i)} \text{ POW)}$$

Note, however, that the powerset axiom has been argued to be too strong from an intuitionistic perspective.

> Power set seems especially nonconstructive and impredicative compared with the other axioms: it does not involve, as the others do, putting together or taking apart sets that one has already constructed but rather selecting out of the totality of all sets those that stand in the relation of inclusion to a given set. (Myhill, 1975, 351)

So, the powerset axiom has us range over all objects fitting a certain condition, instead of merely considering those sets that have been constructed. This is not really a problem of the powerset axiom since it is only with the axioms of separation, specifying the subsets of a set, that the powerset axiom has this strength. (see e.g. (Incurvati, 2008, 90) and (Lear, 1977, 99)) Myhill replaces the powerset axiom with one of its consequences, called the exponentiation axiom.

In our case, however, we will stick to the regular powerset axiom because the considerations for choosing intuitionistic semantics in our case differ from the broader constructivist's views. We will discuss these considerations in more detail in section 4.2.3.

2.3.8 Replacement

To prove that $\mathrm{DST}_{(i)}$ proves versions of the replacement axiom schema, consider a set x and assume that $\psi^{\langle\ \rangle^n}(u,v)$ is functional, or formally $[\ ]^n\forall u\langle\ \rangle^n\exists v[\ ]^n\forall v'(\psi^{\langle\ \rangle^n}(u,v')\leftrightarrow v=v')$. By lemma 2.3.1 it suffices to show that there may be a set y such that $[\ ]^n\forall u\in x(\exists v\in y)\psi^{\langle\ \rangle^n}(u,v)$. Let xx be the elements of x and apply the existential definiteness of replacement to derive $\langle\ \rangle^n\exists yy(\forall u\prec xx)(\exists v\prec yy)\psi^{\langle\ \rangle^n}(u,v)$. Then, applying the DST version of (C), we know that yy possibly form a set. Then, it follows that $\langle\ \rangle^n\exists y(\forall u\in x)(\exists v\in y)\psi^{\langle\ \rangle^n}(u,v)$, as desired. Therefore, we have:

$$[\ ]^n\forall u\langle\ \rangle^n\exists! v\psi(u,v)\rightarrow[\ ]^n\forall x\langle\ \rangle^n\exists y(\forall u\in x)(\exists v\in y)\psi^{\langle\ \rangle^n}(u,v) \qquad (\mathrm{DST}_{(i)}\ \mathrm{REP})$$

2.3.9 Infinity

To prove that $\mathrm{DST}_{(i)}$ proves versions of the infinity axiom, note that $\mathrm{DST}_{(i)}$ proves $\langle\ \rangle^n\exists x(x=\emptyset)$ and $[\ ]^n\forall u\langle\ \rangle^n\exists v(v=\{u\})$ by the earlier mentioned axioms. Applying ($\langle\ \rangle^n$-Refl) to the conjunction of the two formulas gives us

$$\langle\ \rangle^n(\exists x(x=\emptyset)\wedge\forall u\exists v(v=\{u\}) \qquad \text{(i)}$$

By some of the rules and axioms available in **S4** we can turn a proof of $p\rightarrow\Diamond q$ into a proof of $\Diamond p\rightarrow\Diamond q$. To prove this we need to show that if $\mathcal{M},w\vDash p\rightarrow\Diamond q$ then $\mathcal{M}\vDash\Diamond p\rightarrow\Diamond q$. Assume that $\forall w$, if $\mathcal{M},w\vDash p$ then $\mathcal{M},w\vDash\Diamond q$. There are two cases to consider:

i. Assume that $\mathcal{M},w\nvDash p$. Suppose that $\mathcal{M},w\vDash\Diamond p$. Then there is some w' such that wRw' and $\mathcal{M},w'\vDash p$. Since $\forall w$, if $\mathcal{M},w\vDash p$ then $\mathcal{M},w\vDash\Diamond q$, then $\mathcal{M},w'\vDash\Diamond q$. This means that there is some w'' such that $w'Rw''$ and $\mathcal{M},w''\vDash q$. By transitivity it follows that $\mathcal{M},w\vDash\Diamond q$. Therefore, if $\mathcal{M},w\vDash\Diamond p$ then $\mathcal{M},w\vDash\Diamond q$.

ii. Assume that $\mathcal{M},w\vDash p$. Then, since $\forall w$, if $\mathcal{M},w\vDash p$ then $\mathcal{M},w\vDash\Diamond q$, $\mathcal{M},w\vDash\Diamond q$. Therefore $\mathcal{M},w\vDash\Diamond p$ then $\mathcal{M},w\vDash q$.

Therefore, any proof of $p \to \Diamond q$ can be turned into a proof of $\Diamond p \to \Diamond q$.

This also means that we can turn a proof of $p \to \langle\ \rangle^n q$ into a proof of $\langle\ \rangle^n p \to \langle\ \rangle^n q$. Therefore, in order to prove that (i) implies the $\text{DST}_{(i)}$ version of the infinity axiom, we can ignore the first dynamic operator in (i).

Assume that $\exists x(x = \emptyset) \land \forall u \exists v(v = \{u\})$. By applying (P-Comp) to a tautology we get a plurality xx, which is the plurality of all objects at the relevant state, such that

$$\emptyset \prec xx \land \forall u(u \prec xx \to \{u\} \prec xx) \tag{ii}$$

which we can strengthen to $[\ ]^n$(ii): $[\ ]^n(\emptyset \prec xx \land \forall u(u \prec xx \to \{u\} \prec xx))$.

Now, by the $\text{DST}_{(i)}$ versions of (C) we know that it is possible for xx to form a set, such that

$$[\ ]^n \forall u(u \in x \leftrightarrow u \prec xx) \tag{iii}$$

Together $[\ ]^n$(ii) and (iii) imply

$$\emptyset \in x \land [\ ]^n \forall u(u \in x \to \{u\} \in x)$$

Therefore,

$$\langle\ \rangle^n \exists x(\emptyset \in x \land [\ ]^n \forall u(u \in x \to \{u\} \in x)) \tag{$\text{DST}_{(i)}$ INF}$$

2.4 DST and DST_i

As we have seen, in some cases, the results between DST and DST_i differ not just due to the different formulations of (C), but also due to the more general difference in semantics. Therefore, we will now briefly discuss the differences between the two resulting systems.

As, we saw the differences in the semantics lie in the interpretation of the negation, implication, and the quantifiers, both singular and plural. Let us first consider negation, implication and the plural quantifiers. As we saw, when they are not combined with any update operator, they range over, respectively for classical and intuitionistic semantics, either the current state or all the states accessible from the current state in the current model. Note, however, that when they are combined with an update operator, the results of the classical semantics and intuitionistic semantics will be the same. This is because of the interpretation of the update operators. Then, since we defined to interpret any singular quantified statement of the nonmodal language as $\langle\ \rangle^n \exists$ and $[\ ]^n \forall$ respectively for the existential and universal quantifier, the results of the classical semantics and intuitionistic semantics concerning singular quantified statements will be the same.

Therefore, whenever we evaluate statements ranging over models, the outcomes of DST and DST_i are the same, whereas if we evaluate statements within a certain model there are differences between the results. This result will discussed from a more philosophical point of view in section 4.2.3.

The only difference between the axioms of ZF that we can recover in DST and DST_i is the axiom of foundation. In DST we can motivate adopting the axiom of foundation by the priority of the elements of sets over sets in combination with the assumption that the accessibility relation between states is well-ordered. In intuitionistic versions of set theory, the axiom of foundation is not permissible, since it implies instances of the law of the excluded middle. Therefore the principle of transfinite induction is adopted instead. Whether we should favour DST or DST$_i$ based on this consideration will be further discussed in section 4.2.3 as well. Note, however, that the axiom schema of separation is usually weakened in intuitionistic versions of set theory as well. However, here we argued that since we are not using intuitionistic logic for the usual constructivist reasons, we do not have to weaker separation. To what extend this argument applies elsewhere as well will also be considered in section 4.2.3.

In general, it is easy to see that the differences between DST and DST$_i$ are not as big as, for instance ZF and IZF, due to the interpretation of the dynamic modalities. This is because the dynamic modalities, as already mentioned in section 2.2.3, fit the same character as Lear (1977, 94) described for the interpretation of the quantifiers.

3

Properties of Dynamic Set Theory

In this chapter, we will work towards translation results between both classical and intuitionistic DST, and classical and intuitionitic ZF. As we have seen in last chapter, we were able to recover all axioms of ZF in DST. However, for DST_i we had to use s weaker version of the axioms. Before we can see to what extend these systems are interpretable in each other, we need to compare their deduction systems. So, in section 3.1 we will look at some theorems concerning the deduction relation in both DST and DST_i compared with classical and intuitionistic ZF. With these results at hand, we will first discuss the easiest case, DST and ZF. Then, we will look at DST_i and intuitionistic ZF. And, finally, DST_i and ZF.

3.1 Deducibility

Before proving results on the interpretableness of versions of DST into ZF, we need to compare the deducibility relation of the language of nonmodal set theory and DST. The theorems that will be proved correspond to theorem 1.3.1 in section 1.3. Together with lemma 2.3.1, this theorem shows that our dynamically modalised quantifiers behave in the same way as ordinary quantifiers do. As we will see in the last chapter, this is an important factor philosophically since this means that set-theorist can make use of standard quantifiers without contradicting talk about the dynamic potential hierarchy of sets. Aside from this, it is also essential to proving that corresponding nonmodal and modal versions of set theory are mutually interpretable.

Theorem 3.1.1. *Let $\vdash$ be the relation of classical deducibility in a language $\mathcal{L}$, although if $\mathcal{L}$ is a plural language, we mean classical deduction without the use of any plural comprehension axioms. Let $\vdash^{\langle\ \rangle^n}$ be deducibility in $\mathcal{L}^{\langle\ \rangle^n}$ by $\vdash$, **S4.2**, and the stability axioms for $\mathcal{L}^{\langle\ \rangle^n}$. Let $\varphi_1, ..., \varphi_n$ and ψ be $\mathcal{L}$-formulas. Then we have:*

$$\varphi_1, ..., \varphi_n \vdash \psi \text{ iff } \varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} \psi^{\langle\ \rangle^n}$$

Proof. We will use a proof by induction on proofs.

($\Rightarrow$) Since the only hard cases are the quantifiers we will only look at the introduction and elimination rules for the universal and existential quantifier.

i. *Universal elimination:* Assume that $\varphi_1, ..., \varphi_n \vdash \forall x\psi$. Conclude by the universal elimination rule that $\varphi_1, ..., \varphi_n \vdash \psi(t)$ for some t not free in $\varphi_{i\in\{1,...,n\}}$. By the induction hypothesis it follows that $\varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} [\ ]^n \forall x \psi^{\langle\ \rangle^n}$. Then, trivially, also $\varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} \psi^{\langle\ \rangle^n}(t)$ for some t not free in $\varphi_{i\in\{1,...,n\}}^{\langle\ \rangle^n}$.

ii. *Universal introduction:* Assume that $\varphi_1, ..., \varphi_n \vdash \psi(t)$ for some t not free in $\varphi_{i\in\{1,...,n\}}$. Conclude by the universal introduction rule that $\varphi_1, ..., \varphi_n \vdash \forall x\psi$. By the induction hypothesis we have $\varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} \psi^{\langle\ \rangle^n}(t)$ for some t not free in $\varphi_{i\in\{1,...,n\}}^{\langle\ \rangle^n}$. Then by the universal introduction rule it follows that $\varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} \forall x\psi^{\langle\ \rangle^n}$. By using the deduction theorem, necessitation rule, distribution axiom and principles available in our modal logic we get $[\ ]^n\varphi_1^{\langle\ \rangle^n}, ..., [\ ]^n\varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} [\ ]^n \forall x\psi^{\langle\ \rangle^n}$. By lemma 2.3.1 it follows that $\varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} [\ ]^n \forall x\psi^{\langle\ \rangle^n}$.

iii. *Existential elimination:* Assume that $\varphi_1, ..., \varphi_n \vdash \exists x\psi$. Conclude by the existential elimination rule that $\varphi_1, ..., \varphi_n \vdash \psi(t)$ for some t not free in $\varphi_{i\in\{1,...,n\}}$. By the induction hypothesis it follows that $\varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} \langle\ \rangle^n \exists x\psi^{\langle\ \rangle^n}$. Then, trivially, also $\varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} \psi^{\langle\ \rangle^n}(t)$ for some t not free in $\varphi_{i\in\{1,...,n\}}^{\langle\ \rangle^n}$.

iv. *Existential introduction:* Assume that $\varphi_1, ..., \varphi_n \vdash \psi(t)$ for some t not free in $\varphi_{i\in\{1,...,n\}}$. Conclude by the existential introduction rule that $\varphi_1, ..., \varphi_n \vdash \exists x\psi$. By the induction hypothesis we have $\varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} \psi^{\langle\ \rangle^n}(t)$ for some t not free in $\varphi_{i\in\{1,...,n\}}^{\langle\ \rangle^n}$. Then by the existential introduction rule it follows that $\varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} \exists x\psi^{\langle\ \rangle^n}$. By using the properties of $S4$ this gives us $\langle\ \rangle^n\varphi_1^{\langle\ \rangle^n}, ..., \langle\ \rangle^n\varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} \langle\ \rangle^n \exists x\psi^{\langle\ \rangle^n}$. By lemma 2.3.1 it follows that $\varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} \langle\ \rangle^n \exists x\psi^{\langle\ \rangle^n}$.

($\Leftarrow$) Consider an operation $\varphi \mapsto \varphi^-$ taking every formula φ to a formula φ^- in which all modal operators of φ are deleted. This also applies to our axioms and inference rules such that all axioms of our dynamic modal theory are mapped to a theorem of the nonmodal theory that corresponds to it and every inference rule of our dynamic modal theory is mapped to a legitimate inference of the corresponding nonmodal theory. It follows then that if $\varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash^{\langle\ \rangle^n} \psi^{\langle\ \rangle^n}$ then $\varphi_1, ..., \varphi_n \vdash \psi$, since $(\varphi^{\langle\ \rangle^n})^- = \varphi$. □

Note, however, that this proof is only available to us when using classical DST. To prove a similar result for intuitionistic DST, we would have to let $\vdash$ be the relation of intuitionistic deducibility (see e.g. Bezhanishvili et al. (2006) for an outline of the deduction system). This means that we also have the following theorem, by similar reasoning.

Theorem 3.1.2. *Let $\vdash_i$ be the relation of intuitionistic deducibility in a language $\mathcal{L}$, although if $\mathcal{L}$ is a plural language, we mean intuitionistic deduction without the use of any plural comprehension axioms. Let $\vdash_i^{\langle\ \rangle^n}$ be deducibility in $\mathcal{L}^{\langle\ \rangle^n}$ by $\vdash_i$, **S4.2**, and the stability axioms for $\mathcal{L}^{\langle\ \rangle^n}$. Let $\varphi_1, ..., \varphi_n$ and ψ be $\mathcal{L}$-formulas. Then we have:*

$$\varphi_1, ..., \varphi_n \vdash_i \psi \textit{ iff } \varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash_i^{\langle\ \rangle^n} \psi^{\langle\ \rangle^n}$$

We do not have this theorem between classical deduction and intuitionistic dynamic deduction, since intuitionistic logic is weaker than classical logic. This means that we can infer less in intuitionistic logic than in classical logic. However, since there are no inferences in intuitionistic logic that cannot be made under classical logic, we do have one side of this theorem.

Theorem 3.1.3. *Let $\vdash$ be the relation of classical deducibility in a language $\mathcal{L}$, although if $\mathcal{L}$ is a plural language, we mean classical deduction without the use of any plural comprehension axioms. Let $\vdash_i^{\langle\ \rangle^n}$ be deducibility in $\mathcal{L}^{\langle\ \rangle^n}$ by $\vdash_i$, **S4.2**, and the stability axioms for $\mathcal{L}^{\langle\ \rangle^n}$. Let $\varphi_1, ..., \varphi_n$ and ψ be $\mathcal{L}$-formulas. Then we have:*

$$\varphi_1, ..., \varphi_n \vdash \psi \text{ if } \varphi_1^{\langle\ \rangle^n}, ..., \varphi_n^{\langle\ \rangle^n} \vdash_i^{\langle\ \rangle^n} \psi^{\langle\ \rangle^n}$$

3.2 Interpretability

Let us first define the systems DST and DST_i according to the principles added for recovering the axioms.

Definition 3.2.1 (DST)**.** *Let DST denote the $\mathcal{L}_{P\in}^{\langle\ \rangle^n}$-theory that adds the axioms (DST C), (DST EXT), (DST ED-$\in$), (DST F), (DST ED-$\subseteq$), (DST ED-Repl) and ($\langle\ \rangle^n$-Refl) to MFPO.*

Definition 3.2.2 (DST_i)**.** *Let DST denote the $\mathcal{L}_{P\in}^{\langle\ \rangle^n}$-theory that adds the axioms (C), (DST_i EXT), (DST_i ED-$\in$), (DST_i F), (DST_i ED-$\subseteq$), (DST_i ED-Repl) and ($\langle\ \rangle^n$-Refl) to MFPO_i.*

3.2.1 DST and ZF

Theorem 3.2.1. *The dynamic set theory DST is interpretable in nonmodal set theory ZF and is therefore consistent provided that ZF is.*

Proof. First note that DST proves all the dynamic translations of the axioms of ZF and that, by theorem 3.1.1, something is deducible in ZF if and only if its dynamic translation is deducible in DST.

In order to proof that dynamic set theory DST is interpretable in nonmodal theory ZF we need to define a translation from $\mathcal{L}_{P\in}^{\langle\ \rangle^n}$ to $\mathcal{L}_{\in}$, so that we can verify that the axioms of DST map into truths of ZF and the logical relations will be preserved. To this end, we will recursively define $\varphi \mapsto [\varphi]^{V_0}$ as follows:

$$\begin{aligned}
[u \prec xx]^{V_\alpha} &= u \in xx \\
[\forall x \varphi]^{V_\alpha} &= (\forall x \in V_\alpha)[\varphi]^{V_\alpha} \\
[\forall xx \varphi]^{V_\alpha} &= (\forall xx \in V_{\alpha+1)}[\varphi]^{V_\alpha} \\
[[\ \ ]^n \varphi]^{V_\alpha} &= (\forall \beta \geq \alpha)[\varphi]^{V_\beta}
\end{aligned}$$

Note that, in line one and three we use plural variables on the right-hand side, however by lemma 2.3.2 we know can replace these with singular variables. Therefore, this can be seen as notational shorthand.

It is easy to see how the axioms of **S4.2** ($[\ \ ]^n(\varphi \to \psi) \to ([\ \ ]^n\varphi \to [\ \ ]^n\psi), [\ \ ]^n\varphi \to \varphi, [\ \ ]^n\varphi \to [\ \ ]^n[\ \ ]^n\varphi$) are mapped to truths of first-order logic.

Let us first go through the axioms related to the plural logic. First, (DST P-COMP), which translates as formulas of the form:

$$(\forall \alpha)(\exists xx \in V_{\alpha+1})(\forall u \in V_\alpha)(u \in xx \leftrightarrow [\varphi]^{V_\alpha})$$

which are theorems of ZF.

(DST STB$^+$ $\prec$), (DST STB$^-$ $\prec$) and (DST INEXT-$\prec$) are, respectively mapped into formulas of the form:

$$(\forall\alpha)(u \in xx) \to (\forall\beta \geq \alpha)(u \in xx)$$

$$(\forall\alpha)(u \notin xx) \to (\forall\beta \geq \alpha)(u \notin xx)$$

$$(\forall\alpha)(\forall u \in V_\alpha)((u \in xx) \to (\forall\beta \geq \alpha)[\theta]^{V_\beta}) \to (\forall\beta \geq \alpha)(\forall u \in V_\beta)[u \in xx \to \theta]^{V_\beta}$$

For the first two it easy to see that these map to theorems of ZF. The third also maps to theorems of ZF by the increasing domains of each stage of the hierarchy.

Our set existence principle (DST C) translates into:

$$(\forall\alpha)(\forall xx \in V_{\alpha+1})(\exists\beta \geq \alpha)(\exists y \in V_\beta)(\forall\gamma \geq \beta)(\forall u \in V_\gamma)(u \in y \leftrightarrow u \in xx)$$

This principle is trivially true in ZF.

Then, for axioms (DST EXT) and (DST F) it is easy to see that they map into truths of first-order logic. So, we have (DST ED-$\in$) (DST ED-$\subseteq$), (DST ED-Repl) and ($\langle\ \rangle^n$-Refl) left. These, respectively translate into the following formulas:

$$(\forall\alpha)(\exists xx \in V_{\alpha+1})(\forall\beta \geq \alpha)(\forall u \in V_\beta)(u \in yy \leftrightarrow u \in y)$$

$$(\forall\alpha)(\exists xx \in V_{\alpha+1})(\forall\beta \geq \alpha)(\forall u \in V_\beta)(u \subseteq yy \leftrightarrow u \in y)$$

$$\begin{aligned}(\forall\alpha)(\forall\beta \geq \alpha)(\forall u \in V_\beta)(\exists\gamma \geq \beta)(\exists v \in V_\gamma)(\forall\delta \geq \gamma)(\forall v' \in V_\delta)([\psi^{\langle\ \rangle^n}(u,v') \leftrightarrow v = v']^{V_\delta} \\ \to (\forall xx \in V_{\delta+1})(\exists yy \in V_{\delta+1})(\forall u \in xx)(\forall v \in yy)[\psi^{\langle\ \rangle^n}(u,v)]^{V_\delta}\end{aligned}$$

$$(\forall\alpha)(\exists\beta \geq \alpha)(\varphi \to [\varphi]^{V_\beta})$$

For the first two, it is again easy to see how these map into theorems of ZF. The third translation is simply a disguised version of the ordinary replacement schema. Lastly, our translation of $\langle\ \rangle^n$-Reflection is the ordinary reflection principle in disguise, since for any φ that does not contain dynamic operators and is singular, we can replace $[\varphi]^{V_\beta}$ with φ^{V_β}, which is the relativisation of φ to V_β

What remains to be shown is that deducibility in **S4.2**, by necessitation, is also a licensed step in ZF. To this end we need to prove that if we can prove that $[\varphi]^{V_0}$ then we can prove that $\forall\alpha[\varphi]^{V_\alpha}$. This can be done by an induction of proofs, analogue to Linnebo (2013, 224-225).

Since, by theorem 3.1.1, we now that every step licensed by an inference rule of DST is also licensed by an inference rule of ZF, all the dynamic translations of the axioms of ZF are proved by DST and all the translations of axioms of DST are valid theorems of ZF, it follows that dynamic set theory DST is interpretable in nonmodal set theory ZF and is therefore consistent provided that ZF is. □

3.2.2 $\mathbf{DST}_i$ and $\mathbf{ZF}_i$

Theorem 3.2.2. *The dynamic set theory DST_i is interpretable in nonmodal set theory ZF_i and is therefore consistent provided that ZF_i is.*

Proof. This proof would follow the same structure as the last, but use theorem 3.1.2 instead of theorem 3.1.1. Therefore, the dynamic set theory DST_i is interpretable in nonmodal set theory ZF_i and is therefore consistent provided that ZF_i is. ☐

3.2.3 $\mathbf{DST}_i$ and ZF

Theorem 3.2.3. *The dynamic set theory DST_i is interpretable in nonmodal set theory ZF and is therefore consistent provided that ZF is.*

Proof. Since we only need the step of the inference rules of DST_i to be licensed by steps of inference rules of ZF, and not the other way around, this proof can use theorem 3.1.3 instead of theorem 3.1.1. Therefore, the dynamic set theory DST_i is interpretable in nonmodal set theory ZF and is therefore consistent provided that ZF_i is. ☐

4

Philosophy of Dynamic Set Theory

In the last two chapters, we have focused on the formal properties of Dynamic Set Theory by laying out the semantics of the system, recovering the axioms of ZF and proving translation results between DST and ZF. However, DST is primarily motivated by philosophical ideas and problems with other forms of set theory and conceptions of sets. Therefore, in this last chapter, we will go back to the philosophy behind set theory in general, Potentialism and our novel approach to these, DST. We will start, in section 4.1, by discussing whether DST is still in line with the original intuitions and motivations for Potentialism. Then, in section 4.2, we will consider the concept of set that fits with the formal system of DST. Although this is of course similar to the idea endorsed by Potentialism as discussed in section 1.2, there are some important differences. These differences will be highlighted by looking at another interpretation of Potentialism formulated by Uzquiano (2015). In section 4.3, we will go back to one of the problems for Potentialism discussed in section 1.4, namely the *why*-question. We will both consider whether DST is able to give a more satisfactory account of *why* the universe of sets is not a set itself and whether such an account is in general even possible.

4.1 Principles of Potentialism

In this section, we will discuss whether DST is still in line with Potentialism by looking at whether DST is still able to solve the problems for Actualism and ZF(C) that Potentialism solved. First, we will consider in what way DST captures the open-endedness of the hierarchy of sets and the indefinitely extensible nature of sets. Then, we will consider the problems with ad hocness and arbitrariness that ZF(C) has been argued to have due to the lack of an independent motivation for the axioms from the conception of sets. Although we will not consider the conception of set that is used in DST in full extent yet, we will consider whether DST is able to avoid these problems with ad hocness and arbitrariness.

4.1.1 Open-endedness and indefinite extensibility

The open-endedness of the concept of set refers to the phenomenon that whenever we define a totality of sets, we can always define a larger totality by applying the “set of” operation. This way, the concept of set is indefinitely extensible; since “for any definite characterisation of it, there is a natural extension of this characterisation, which yields a more inclusive concept” (Dummett, 1963, 195). This feature of sets is also used in, for instance, deriving Russell’s paradox.

As we discussed in section 1.1, Dummett argued that the indefinitely extensible nature of sets is the real problem underlying the set-theoretic paradoxes. Therefore, prohibiting the paradoxes from arising itself is not enough. According to Dummett (see e.g. Dummett (1963, 1991, 1994b,a)) we should adopt an intuitionistic logic in set theory, since, due to indefinite extensibility and there not being anything else that determines the truth-value of mathematical statements, there are statements with no determinate truth-value.

In Potentialism as it is currently implemented, open-endedness and indefinite extensibility can be seen as a byproduct of the potential nature of the hierarchy of sets. Formally, this is captured by the modal operators employed. Since it is always possible, from every stage of the hierarchy, and thereby every state in the model, to go to a next stage with new sets, the hierarchy is open-endedness, and we can never formulate a definite *actual* characterisation of sets, since there is a natural extension of this characterisation. Note, however, that we could still form a *potential* characterisation of sets, since the entire structure of possible states is there, although not actually, potentially.

In Dynamic Set Theory, open-endedness and indefinite extensibility can, again, be seen as a byproduct of the potential nature of the hierarchy of sets. In our case, this is formally captured by our update operators. It is always possible, from every stage in the hierarchy, to update the current model to a new model containing new sets, making the hierarchy open-ended. This way, it is also never possible to formulate a definite characterisation of all potential sets since we can always extend to a more inclusive model. Whether we can still form a dynamic potential characterisation of all sets in this case is harder to say since we never have the entire structure of possible states. One could still argue that we have a characterisation through all possible updates to models, but since we defined updates to be of a finite string length there would still never be a complete overview of the hierarchy.

4.1.2 Avoiding arbitrariness and ad hocness

One of the further problems for ZF(C) was that it could be seen as ad hoc. The axioms were chosen in such a way that the paradoxes of naive set theory are avoided, but without a real motivation from the concept of set itself. Moreover, the actualist interpretation of the hierarchy of sets has been said to cut the hierarchy off at an arbitrary level. We will now discuss to what extent this ad hocness and arbitrariness is avoided in DST.

Let us first briefly look into the ad hocness problem. The problem is that the choice of axioms in ZF would be guided to prohibit the set-theoretic paradoxes from arising, but lack an independent or further justification. This argument is not supported by all; as we already saw in section 1.1, Boolos (1971) stated that ZF is independently motivated by the iterative conception of sets. Now, there are two closely related questions at play to see whether ZF(C) is indeed ad hoc or not: (i) does ZF(C) (exactly) capture the iterative conception of sets, and (ii) is the iterative conception itself ad hoc.

Concerning the second question one may note that the iterative conception of sets is assumed in this thesis. There are, however, both arguments against the use of this conception of sets and other proposals for conceptions of sets instead of the iterative conception. Quine (1951, 50-51), argued that the only truly intuitive conception of sets is captured by the naive comprehension schema, but, as we have seen, this has led to paradoxes such as Russell's paradoxes. Other conceptions of sets related to ZF(C) include, for instance, the limitation of size conception (see e.g. Hallett (1986)). A discussion of this will be left for section 4.3.3. For now, we will assume that the iterative conception does at least as well as other, non-contradictory, conceptions of sets.

Whether ZF(C) (exactly) captures the iterative conception of sets is disputed as well. On the one hand, there are those who argue that this is indeed the case. Boolos (1971), for instance, outlines axiom by axiom how the iterative conception and ZF relate. On the other hand, there are those that argue that ZF(C) is not able to fully capture the iterative conception. An example is Menzel (1986, 46-47), who argues that the axioms of ZF(C) are not just ad hoc but are in conflict with its conceptual foundations.

> [T]he axioms in a certain sense embody the iterative conception of set: the natural models of ZFC (with or without urelements) are cumulative hierarchies. Yet, in order to avoid the BF paradox the axioms rule out the possibility a set whose existence seems to *follow* from the iterative conception. Thus, despite the fact that it embodied the iterative conception, it cannot capture the iterative conception in its full generality. (Menzel, 1986, 47)

So, ZFC is to a certain extent in line with the iterative conception. However, it cannot capture it entirely by its efforts to avoid the Burali-Forti paradox.

In general, aside from its relation to the iterative conception, the ad hocness argument is focused on how the naive comprehension schema is restricted and how the other axioms are chosen to recover some results from naive set theory. For instance, Hellman and Cook (2018, 53), argue that ZF(C)'s resolution of Burali-Forti paradox, although it does the job mathematically, is unsatisfactory from a philosophical perspective: "In short, the 'resolution' offered by first-order ZFC is a paradigm of the ad hoc.".

What the consequences of these forms of ad hocness are, will be discussed in section 4.3.3. What we can say is that, from the potentialist point of view, ZF(C) at least does not incorporate the inherent modal character of the iterative conception of sets. However, interpreted this way, the critique would be the same as what was discussed in the previous subsection, namely that ZF(C) does not fully capture the open-ended and indefinitely extensible nature of sets. So, if the iterative conception is inherently modal in character as the potentialist interprets it, the potentialist version of the ZF(C) axioms would do at least slightly better. This argument, however, is not convincing to someone who is not already drawn to Potentialism since they might deny there is such an "inherent modal character".

The other problem was that the actualist interpretation of the hierarchy of sets was argued to cut the hierarchy of at an arbitrary level. As we have seen in section 1.2, from the actualist perspective, all sets in the hierarchy actually exist at once. This led to the problem why the objects that make up the totality of all sets cannot be a set itself according to this view since there is no essential difference in the nature of a set and the hierarchy of sets. Disallowing the existence of a set of all sets would the hierarchy of at an arbitrary level.

This problem does not arise for the potentialist since for the potentialist there *is* a definite distinction between sets and the hierarchy; the first being actual in nature, while the second will always remain merely potential. In order for a set to *actually* exist, its elements have to be actual, opposed to merely potential. Since it is always possible to form further sets, a part of the hierarchy will always remain potential. Therefore the hierarchy as a whole cannot form a set.

This idea is still present in our dynamic version of Potentialism; we merely changed what it means for something to potentially exist. However, note that, to see how well DST fares in both aspects more precisely we need to understand the concept of set in DST and need to have a more well-defined notion of (potential) existence and an interpretation of the dynamic modalities. Therefore, we will come back to this in section 4.2.

4.2 The Concept of Set

In this section, we will look at the concept of set in DST. As we have seen, the philosophical principles of Potentialism are largely recovered in DST. However, there are important differences between Potentialism as currently implemented and DST. To show this, we will first look at the linguistic interpretation of Potentialism as used by Uzquiano (2015). Then, we will propose an interpretation of the dynamic modalities and show some similarities between Uzquiano's interpretation and DST and compare Uzquiano's, Linnebo's and DST's proposal with each other.

4.2.1 The linguistic turn

In the paper "Varieties of Indefinite Extensibility", Uzquiano (2015) advances a different version of Potentialism based on a linguistic model of indefinite extensibility. The broad idea is to "[conceive] of indefinite extensibility as a feature of the set-theoretic vocabulary and not the concepts they are supposed to express." (Uzquiano, 2015, 150). So, the concept of set itself is not indefinitely extensible, rather indefinite extensibility is a feature of the language we use to talk about sets.

Specifically, Uzquiano combines two ideas by Williamson (1998) and Gödel (1947). Williamson argues that whatever we assign the word "set" to, we can always find a more inclusive meaning.

> For given any reasonable assignment of meaning to the word "set" we can assign it a more inclusive meaning while feeling that we are going on in the same way, and make correlative changes to the words in an iterative account of sets, to preserve it too. The inconsistency is not in any one meaning we assign the iterative account; it is in the attempt to combine all the different meanings that we could reasonably assign it into a single super-meaning. (Williamson, 1998, 20)

This indeed moves indefinite extensibility to our use of language instead of playing on the conceptual level.

Uzquiano combines this with the iterative conception of sets as described by Gödel.

> The concept of set, however, according to which a set is anything obtainable from the integers (or some other well-defined objects) by iterated application of the operation "set of" and not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever; that is, the perfectly "naive" and uncritical working with this concept of set has so far proved self-consistent. (Gödel, 1947, 180)

This means that sets are those things that arise from iteratively applying the "set of" operation to the objects already in place.

Combined, these two lead to the idea that through iteratively applying the "set of" operation, we will keep on gaining more inclusive meanings of set.

To express these ideas, Uzquiano (2015) uses a different language with different principles than the language and principles we presented in section 1.3 from Linnebo (2013). Uzquiano (2015, 150-151) uses two primitive predicates to express the relation between pluralities and sets: α for "available for collection" and $\equiv$ for "is a set of". The most important principles are the following three:

$$\forall xx(\forall x(x \prec xx \to \alpha(x)) \to \exists x : x \equiv xx) \tag{Collection}$$

$$\forall xx \forall yy \forall x \forall y(x \equiv xx \wedge y \equiv yy \leftrightarrow (x = y \leftrightarrow \forall z(z \prec xx \leftrightarrow z \prec yy))) \tag{Extensionality}$$

$$\forall x(\mathrm{Set}(x) \to \Diamond\alpha(x)) \tag{Availability$^\Diamond$}$$

The first principle, collection, expresses that for all objects, if they are available for collection, then they have been collected into a set. The second principle, extensionality, expresses that for all objects, there is at most one set. From these two principles, it follows that it cannot be the case that all sets are available for collection (Uzquiano, 2015, 151). Instead, Uzquiano (2015, 152) uses a modal version of this idea, expressed by the third principle, Availability$^\Diamond$. The diamond operator in this principle expresses this process of reinterpretation of the language. This principle expresses that for any set, there is a reinterpretation of the predicate α on which it is true that $\alpha(x)$.

The modal logic that Uzquiano (2015) uses is largely in line with that of Linnebo (2013) and DST. However, there is one crucial difference that is related to the difference in interpretation of the modalities. While Linnebo (2013) uses expanding domains, Uzquiano (2015) works in a constant domain setting validating for both single and plural quantifiers both the Barcan and converse Barcan formula.

$$\Box\forall x\varphi \to \forall x\Box\varphi \qquad \Box\forall xx\varphi \to \forall xx\Box\varphi \tag{CBF}$$

$$\forall x\Box\varphi \to \Box\forall x\varphi \qquad \forall xx\Box\varphi \to \Box\forall xx\varphi \tag{BF}$$

The motivation Uzquiano gives is as follows:

> The validity of singular and plural versions of (CBF) and (BF) illustrates the fact that the phenomenon of indefinite extensibility is concerned not with ontology but rather with language. (Uzquiano, 2015, 154)

So, the validity of both (CBF) and (BF) expresses that while the ontology and thereby the concept of set remains the same, our language does not.

While there are many further interesting aspects to the version of Potentialism presented by Uzquiano (2015), we will not discuss this version of Potentialism further since similar lines of criticism that were discussed in section 1.4 apply here. Instead, in the next part, we will compare the interpretation of our dynamic version of set theory with both Linnebo (2013) and Uzquiano (2015).

4.2.2 Dynamic modalities

We have now seen two versions of Potentialism. Linnebo (2013) uses modal notions to distinguish between potential and actual existence, with different possible ways to interpret the modalities. Uzquiano (2015) uses modal notions to track the process of reinterpreting set-theoretic language. To give an interpretation of our dynamic modalities, we will use a combination of both accounts of Potentialism.

As we have seen in the last few chapters, our formal account of dynamic Potentialism is largely in line with Linnebo's. The divergence lies primarily in the replacement of the modal operators with dynamic modal operators. Let us first recall our general motivations for this to move to a dynamic setting. In Potentialism as modally implemented in the system presented by Linnebo (2013) it is easy to see that the hierarchy as a whole does not *actually* exist. However, we needed something further to fully capture the potential nature of sets and the hierarchy of sets; we needed a way to express that the whole model of possible worlds does not exist as once in a potential sense either. Moreover, we needed to find a form of Potentialism that is not merely a "costly detour", as Soysal (2017) called it. In order to do this, we need a way to model the *process* of expanding the expressive resources, not just the relation between sets and the hierarchy.

In the system that we outlined in the previous two chapters, we are still able to show this relation between sets and the hierarchy of sets; sets being actual in nature and the hierarchy always remaining potential. This way, as we saw in section 4.1, all the philosophical motivations for Potentialism are still reflected. Aside from this, DST also models the *process* of the hierarchy expanding by the application of the update operator. However, without an interpretation of what this process of expansion means, DST would merely be a more complicated way of modelling what was already modelled in other versions of Potentialism. This would make DST an even more costly detour. Especially because, just like in the modal version of Potentialism from Linnebo (2013), our dynamically modalised quantifiers behave in the same way as ordinary quantifiers do, as was shown by theorem 3.1.1.

The proposal is to interpret the process of expansion and the dynamic operators in a way that is similar to the interpretation offered by Uzquiano (2015). The broad idea is to take the update operators not to refer to an expansion of what sets exist in an ontological sense. Instead, we take the update operators to refer to the process of expanding what, according to our language, exists.

This makes the meaning of the expression "expanding expressive resources and thereby recognising further ontology", used earlier to characterise the potentialist's views, clearer. We are not modelling the expansion of the ontology; DST does not show how objects that were previously non-existent (in some sense) are brought into existence. We are modelling the expansion of what part of the ontology is *recognised* by our expressive resources. By applying the "set of" operation we transform our old model into a new one that represents a more inclusive meaning of the word "set".

There are a few important differences between DST and the system proposed by Uzquiano (2015) as well. Consider again one of the formal properties of Uzquiano's system we discussed in section 4.2.1; the validity of both the Barcan and the converse Barcan formula resulting in stable domains. This property of the model is due to the distinction that Uzquiano (2015) makes between the ontology and what part of it is recognised in the model. In DST, however, we do not model this relation; we do not distinguish between the mathematical ontology and what our language specifies, in our formal account.

Since we are not modelling this distinction between what ontologically exists and what our language is ranging over, but merely the process of reinterpretation by itself, this also makes DST to some extent metaphysically neutral. DST contradicts neither the Platonist nor the anti-realist since it does not make claims about whether the objects we refer to exist or not and in what sense they might. We refer to objects that are specified by our language, which are the objects that set-theorist make use of in their set-theoretic practice. However, we make no claims about the objects that are not specified by the language and about the objects specified by our language, we only claim that they are part of our expressive resources. From this perspective it also makes sense that we do not use stable domains, as Uzquiano (2015) does. The ontology recognised by our language does increase, therefore the domain should only satisfy the converse Barcan formula and not the Barcan formula.

Although there is nothing in the system presented by Linnebo (2013) that explicitly counters this linguistic interpretation of Potentialism, DST does make this more explicit and is a more fitting representation. One could argue that the move from one state to another in the structure presented by Linnebo (2013) models the reinterpretation of the language. However, what it does not show is how this process of reinterpretation works. Moreover, the structure being there in its entirety in a potential way, in Linnebo's (2013) account of Potentialism, does bring up questions concerning how we should view this structure. These questions do not arise in the case of DST since the structure does not exist in its entirety in any sense; the number of iterations of the update operation is always finite.

Moreover, DST does not endorse one of the possible interpretations of the modality that Linnebo mentions: "A claim is possible, in this sense, if it can be made to hold by a permissible extension of the mathematical ontology; and it is necessary if it holds under any permissible such extension." (Linnebo, 2013, 208). Instead of trying to model the permissible extensions of the mathematical ontology, we are modelling the permissible extensions of the language about our mathematical ontology.

This means that are update operators $\langle\ \rangle^n\varphi$ and $[\ ]^n\varphi$ should be, respectively, interpreted as "there is a permissible extension of the part of the mathematical ontology our set-theoretic language ranges over such that φ" and "under all permissible extensions of the part of the mathematical ontology our set-theoretic language ranges over, φ". What is a permissible extension is determined by the axioms and our principle (C), which states that every plurality which is available in the current set-theoretic language can under an extension of the language form a set.

4.2.3 Classical and Intuitionistic DST

In section 2.4, we already made a comparison between the classical and intuitionistic version of DST based on its formal properties. Here, we will comsider which of the two systems matches better with the philosophical considerations for Potentialism.

In general, the reason why one might adopt intuitionistic semantics in potentialist set theory can be found in the argument we earlier mentioned by Lear (1977). Lear (1977, 93) argued that since the extension of "set" changes over time when one gains an improved understanding of the concept of set and, thereby, of the set-theoretic universe, that some statements cannot simply be evaluated over one's current understanding of the concept of set and the set-theoretic universe. This motivation for adopting intuitionistic semantics matches with the interpretation of the dynamic modalities since there is never a reinterpretation of our language such that our expressive resources are all sets. However, if we evaluate a statement that makes a claim about all sets or the existence of a specific set, we cannot simply evaluate it over our current expressive resources since this evaluation would not be conclusive and does not match our intuitions on what it means to assert such a statement.

As we noted in section 2.4, whenever we evaluate statements ranging over the updates of models, the outcomes of DST and DST_i are the same, whereas if we evaluate statements within a certain model there are differences between the outcomes. Thus, we are only interested in statements evaluated within a certain model. We will now consider which way of evaluating statements fits best with the philosophical considerations for Potentialism. To this end we only have to consider statements of the form $\neg\varphi$ and $\varphi \to \psi$, such that φ and ψ do not contain any singular quantifiers. This is because in both DST and DST_i the quantifiers would be supplemented with their corresponding dynamic operator, which means the statement would not be evaluated within a model anymore and the other logical connectives have the same interpretation in intuitionistic and classical DST.

Concerning statements of the form $\neg\varphi$, Lear (1977, 93) argues that one cannot infer that $\neg\varphi$ receives value 1 if φ receives value 0. However, in the case that φ does not contain any quantifiers we can simply evaluate the statement at the current state. This is because formulas of this form only range over specific objects in the current domain and by our stableness axioms it follows that for all accessible states from the current state, the valuation it receives will remain the same. Therefore, from a potentialist point of view, evaluating $\neg\varphi$ at a state and not over states or updates is acceptable. The same argument applies to statements of the form $\varphi \to \psi$. It is only when we evaluate statements that make a claim about the existence of something such that φ or about the universality of φ, that evaluating at a single state is insufficient. In these cases, there is no difference between DST and DST_i since statements of these forms are in the same way evaluated across transformations of models.

One might argue that based on these considerations we should already favour DST over DST_i from a potentialist point of view. Let us still consider the axioms, specifically, the combination of the powerset axiom and the separation axiom schema and the foundation axioms.

In intuitionistic versions of set theory, the powerset axiom is usually weakened since, in combination with the separation axiom schema, it ranges over objects that do not constructively exist. In our case, we are not using intuitionistic logic for the usual constructivist reasons since we are not really constructing sets; we are modelling the expansion of the expressive resources of our language. For this reason, objects that are not constructively acceptable can still be a part of the expressive resources of our language. What is constructively problematic is that the combination of the powerset axiom and the separation axiom schema allows us to form a set out of all the subsets of a specific set. The problem is that, from a constructivist perspective, the subsets of a specific set have not necessarily been constructed. However, by the assumption of the maximality principle, which states that all the sets can be formed at a given state are formed, we have formed all the subsets of a specific set. Therefore, keeping the separation axiom schema as is, is not problematic in our case.

In the intuitionistic version of DST, we exchanged the foundation axiom for the transfinite induction schema since the foundation axiom implies the law of the excluded middle. From a potentialist point of view this is not required either. Whenever the statement does not contain singular quantifiers, the law of excluded middle is acceptable from a potentialist point of view in any case since, as we saw, statements of this form can be evaluated at the current state. Whenever the statement does contain singular quantifiers, these will be combined with their corresponding dynamic modalities so that they will be evaluated across the updates of models. In this case, instances of the excluded middle are not problematic either. Therefore, from a purely potentialist point of view, the foundation axiom is acceptable and can even be motivated from the assumption of the priority of the elements of sets to sets and the well-foundedness of the accessibility relation.

Thus, since the addition of dynamic modalities to the quantifiers already capture every part of intuitionistic logic that one would need from a purely potentialist point of view, we can conclude that the classical potentialist should adopt classical DST.

4.3 The *Why*-Question

In this section, we will discuss the *why*-question, why is the universe of sets not a set itself, as posed by Soysal (2017), in light of the interpretation of the dynamic modalities of DST. The problem for Potentialism with a modal implementation was the interpretation of the modalities involved. So we will start by discussing how the interpretation of the dynamic modalities involved in DST relate to the *why*-question. Then we will discuss whether, with this interpretation at hand, we are able to give a more in-depth explanation of the *why*-question than a version of the minimal explanation. Lastly, we will discuss the *why*-question in a broader light to see what is required to answer this question, whether this is achievable and what constraints on possible answers to the question might be.

4.3.1 Answering the *why*-question

In section 1.4, we discussed the arguments that Soysal (2017) gave against Potentialism based on their ability to answer the *why*-question. The argument was directed against the modalities involved in Potentialism. As Soysal (2017, 12) noted, there are two general ways to deal with these modalities, either they are left unexplained and taken as a sort of primitive, or they are interpreted in some way that relates to tracking how the universe of sets expands. As is clear from the previous section, Dynamic Set Theory falls in the second category since we use an interpretation of the dynamic modalities as a reinterpretation of the language. We will now discuss to what extent this interpretation is able to give a "deeper" explanation of why the universe of sets is not a set itself.

Let us first recap the argument against using the modal notions as tracking the universe of sets extending. Soysal (2017, 15) assumes that interpretations of the modal notions are generally built on two considerations: (i) *specifiability*: the claim that sets exist if and only if they are specified by a theory, and (ii) *expandability*: the claim that theories can always be expanded. The idea is then that these two claims together would rule out the possibility of a set of all sets. However, as Soysal (2017) argues, this is not guaranteed since there are examples of forms of set theory that can incorporate both claims but also have a universal set.

It is clear that *expandability* is a part of Dynamic Set Theory. The dynamic operators are used to model the process of expansion. The other claim, *specifiability*, does not seem to match with DST since, as we explained in section 4.2.2, DST is not committed to ontological claims about the existence of sets that are not specified by the language.

We can rephrase *specifiability* as follows: sets are part of our expressive resources if and only if they are specified by a theory. In this case, it seems that Soysal's argument can still apply, since this version of *specifiability* does not contradict any claim in NFU^+ either. Therefore, these two claims do not guarantee that a universal set cannot exist either. This leads to Soysal's conclusion that the modalities are redundant and a merely "costly detour" (2017, 14) again. Since DST is, just as the current standard implementation of Potentialism in set theory, designed in such a way to guarantee precisely those results that ZF gives us, there is no formal benefit to using DST either. Moreover, philosophically *expandability* and *specifiability* do not provide a more in-depth explanation of why there is no universal set than the minimal explanation does.

So, Dynamic Set Theory and its interpretation of the dynamic modalities do not do much better against the criticism raised by Soysal (2017) if we reformulate *specifiability*. However, as was already foreshadowed in section 1.4, the argument given by Soysal can be countered in other ways. We will now work out the considerations mentioned in that section in more detail.

First, let us consider the two claims, *specifiability* and *expandability*. There are a few considerations to take into account to see whether the argument given by Soysal (2017) works: (i) what do these two claims mean exactly, (ii) do they fit with the conception of set of Potentialism, (iii) do they encompass the conception of set of Potentialism.

The meaning of *expandability* is quite clear. For, *specifiability*, however, it is slightly less obvious what Soysal (2017) means exactly. In order to understand this claim, we need to understand what is meant with existence. Here, just like with the interpretation of the modalities, it cannot refer to their metaphysical being since if they exist, they exist of metaphysical necessity. Instead, it should be in line with what the potentialist means with actual existence. In DST, however, we replaced talk of potential and actual existence with talk of potential and actual existence according to our language.

Depending, then, on how we interpret *specifiability* either both claims or only *expandability* fit our dynamic operators. As we saw, the way *specifiability* is defined by Soysal (2017, 15), does not seem to fit DST. However, our reformulated version "sets are part of our expressive resources if and only if they are specified by a theory" does.

As we saw in section 1.4, the argument given by Soysal (2017) does, as of now (with either version of *specifiability*), not mean that Potentialism is not able to give a more in-depth answer to the *why*-question. What we can conclude is that *expandability* and *specifiability* together are not able to provide such an answer. In order to conclude that Potentialism in general is not able to give such an answer, Potentialism has to be built on only these two claims.

However, these two claims do not entirely encompass the nature of sets as endorsed by Potentialism. There are further constraints on what sets are like, which do rule out other forms of set theory such as NFU^+. One of these further constraints can be that the expansion has to be through iteration of the "set of" operation.

As was already mentioned in section 1.4, there is a stronger version of Soysal's argument that can be made. About NFU^+ one could complain that it does not fit the iterative conception of sets. However, Forster (2008) argues, the iterative conception does not rule out a universal set either. We will now briefly discuss Forster's proposal and how it could be used as an argument, similar to Soysal's, against Potentialism.

As Forster (2008, 97) explains, the cumulative hierarchy is usually taken to directly follow from the iterative conception of sets, it is often even used synonymously. However, the cumulative hierarchy is merely one specific way of working out the iterative conception of sets. In the cumulative hierarchy we, in Forster's words,

> lasso collections of sets, and then - before throwing them back into the herd of sets whence we plucked them - we perform some magic on the lasso contents (otherwise we would not get a new set). The magic is performed with the aid of a *wand*. (Forster, 2008, 98)

This means that gathering some objects together is not enough for forming a set; it would still be merely a plurality. We need to enforce some specific operation to actually make this plurality the contents of a set. After this, the plurality will still be available for future set forming, so it needs to be thrown back into the herd.

What Forster emphasises in his paper is the importance of this wand. If it where not for this wand, the amount of sets before and after lassoing would be the same. Moreover, the wand of the cumulative hierarchy has a very specific way of working but it seems that there are other possible wands with different ways of working that might be applied as well. According to Forster, the iterative conception of sets does not specify that the wand, or set-constructor, has to be exactly like this or that there can only be one constructor. (Forster, 2008, 99)

To illustrate this, Forster (2008, 100) outlines the case in which we have two constructors, where the first is like the one used in the cumulative hierarchy, constructing a set of out the things that we lassoed, while the second results in the complement of the set from the first. The first stage of construction then results in two things: the empty set, as normally, and its complement V, the universe of sets. The second stage results from applying both our constructors to the objects we now have, giving us: $\{\emptyset\}$, $V \setminus \{\emptyset\}$, $\{V\}$, $V \setminus \{V\}$, $\{\emptyset, V\}$ and $V \setminus \{\emptyset, V\}$.

Forster (2008) argues that both these constructors and even the use of two constructors conform to the iterative conception of sets. If this is indeed the case, it means that even when we specify that the expansion has to happen conform the iterative conception of sets, we are not able to rule out the existence of a universal set. In order for Potentialism to give a deeper answer to the *why*-question, it has to be further specified what the right conception of sets is. Of course, the most immediate response of the potentialist to this argument would be that construction via the second wand is not a legitimate way of set construction since the sets created by this wand do not consist of things that *actually* exist. From the interpretation of the dynamic modalities, the criticism against this wand would be that its use does not consist of a permissible extension of our language since the sets it creates do not consist of objects that do not belong to the expressive resources of the language before extending the language.

To counter this argument more generally, we need to not only explain why this second wand is not a legitimate way of constructing sets but also why the only legitimate way of constructing sets is via the first wand. This would not only form a way to counter Forster's argument, but could also form a more in-depth answer to the *why*-question. If we understand the wand in terms of reinterpretation of our language, such that the wand step-wise assigns a more inclusive meaning to the word "set", it is easy to see why the first wand is correct the set constructor. A set of all sets would never occur through this process since there is never an all-inclusive meaning of sets. However, it is only when one already beliefs that set construction is through iteration of the "set of" procedure that this argument is convincing. It seems, as we will further discuss in section 4.3.3, that it is not possible to form a completely satisfactory explanation of set construction that answers the *why*-question. However, as we will clarify in both section 4.3.2 and 4.3.3, there are further reasons to still favour dynamic set theory, with its linguistic interpretation, over other accounts of set theory and the concept of set.

There is a further problem that Soysal (2017, 13-14) posed for Potentialism, although this problem was more specifically directed towards the accounts of Potentialism that leave the modality uninterpreted. This problem relates to how we should see the set-theorist at work; are they working at a certain stage of the hierarchy, such that this stage contains all sets that are established by the best version of set theory so far, or are they taking an outside perspective on the hierarchy.

Due to our linguistic interpretation of the update operators, we can say that whenever a set-theorist is considering what sets can be formed from a specific stage of the hierarchy, they are actually considering how the set-theoretic language can be permissibly expanded. Since we are not modelling the *creation* of new sets the problem seems to be less invasive here. We can understand the set-theorist at work, when they are not creating sets but studying validities, as considering what is true under any or some premissible expansion of the language. The objects the language range over will, at least metaphysically, exist in any case (if at all).

4.3.2 An independent motivation

As we have seen, in general, the dynamic view of Potentialism will not completely solve the problems that Soysal posed for Potentialism since a completely satisfactory, in-depth answer to the *why*-question is still not attainable. However, there is a way in which we can still justify the dynamic version of Potentialism. Philosophically, the interpretation of the dynamic operators used is closer to Uzquiano's view of the modalities than Linnebo's, since it interprets the dynamic modalities as modelling reinterpretations of the language. In DST this interpretation is more evident than in Uzquiano (2015) since DST models the *process* of reinterpretation by its use of dynamic semantics.

As we saw in section 2.1, this way of modelling information processes has already been used in other fields. We will now discuss the relation of DST to linguistic research and other applications of dynamic semantics. This provides a further motivation to the use of dynamic modalities in DST. We will argue that, due to its use in modelling other information processes, our modality is a more natural variant than that of Linnebo (2013), that not just applies to mathematical information processes but is motivated independently of the foundations of mathematics.

The iterative conception of sets, whether we view it from a potentialist's or actualist's perspective, can be seen as defining a process; it is only when a set is the result of lassoing, waving our wand, and releasing the contents of the lasso back into the herd, that it can be a part of the hierarchy of sets. In order to gain an understanding of our hierarchy of sets, we need to understand, both in a formal and a philosophical sense, what this process is like.

Formally, a way to model process, and more specifically information processes, is by using dynamic semantics. Without the use of dynamic operators, one can model what a certain information state, a description of what a certain system is like given certain conditions, is like at a particular moment in time. However, one cannot model how such an information state might change or develop over time. This means that if one wants to model information *processes*, that dynamic operators are necessary.

Consider, for instance, the logic for modelling agents' knowledge mentioned in section 2.1. Without its dynamic elements, we could model what an agent knows, doubts and beliefs under some conditions, but we cannot model how an agent might revise their beliefs or what an agent might come to know given new input. In order to model this, we need to be able to update a model with the new input giving in a new model showing the situation resulting from this new input. In the same way, without using dynamic operators in our set theory, we can model what sets exist or might exist under certain conditions but not how they come about.

This gives us an independent motivation for using dynamic operators in set theory to model the process of expanding the expressive resources of set theory; their use has been proven fruitful in their application to other information processes. These applications also lead us to a natural, not purely mathematically motivated, interpretation of the dynamic operators.

As we saw in section 2.1, dynamic semantics used in natural language semantics comes with a specific perspective on language:

> Dynamic semantics is a perspective on natural language semantics that emphasizes

> the growth of information in time. It is an approach to meaning representation where pieces of text or discourse are viewed as instructions to update an existing context with new information, the result of which is an updated context. In a slogan: meaning is context change potential. (Nouwen et al., 2016)

With our linguistic interpretation of the dynamic operators, it is easy to see how a version of this view might apply to our case. The information that grows can be matched with the expanding expressive resources, while the existential and universal operators together with their dynamic operators are instructions to update the existing context, the existing model, with new information, new sets. This way, the manner in which we use the universal and existential operators can be seen as context change potential.

Aside from this, although this argument is less strong, modelling the process of set formation instead of just the model of what sets exist and may exist, can also be motivated by the way we intuitively think about infinity. In the introduction, we described that our intuitive way of understanding infinity is as a never-ending process. This means that when we consider the language describing infinity, it is natural to take this characteristic into account.

4.3.3 Questioning the *why*-question

Based on the criticism that Soysal gives to Potentialism, we might conclude that giving a satisfactory, in-depth answer to the *why*-question might not be fully possible. However, the conclusion that the minimal answer is at least as good as any, is too big a jump to conclusion. In this final part of the thesis we will briefly discuss why giving an in-depth answer to the *why*-question might not be possible, while the minimal answer still remains too minimal. We will argue that, while any account of set theory as the foundation of mathematics might to some extent be ad hoc, theories can be ad hoc in a less strong sense than brute forcing the prohibition of contradictions. We will try to arrive at some criteria for a satisfactory account and see how well our version of Potentialism fits these criteria.

Let us start with why giving an in-depth answer to the *why*-question might not be possible. An in-depth answer to the *why*-question would consist of an explanation of what sets are such that the existence of a universal set would be in contradiction with it. Moreover, this explanation has to be in line with our general mathematical intuitions and the role it has to fulfil as a foundation for mathematics.

As we saw, the iterative conception of sets is not able to fulfil these requirements, since following Forster's argument we can use a construction principle of sets in line with the iterative conception of sets that still allows for the existence of a universal set. Although we can give a more specific description of our conception of sets that counters Forster's argument, this explanation of the conception of sets is still not completely comprehensive in its answer to the *why*-question. Other conceptions of sets, such as the mentioned limitation of size conception do not seem to do much better.

The limitation of size conception has its ground in Cantor's distinction between absolute infinity and consistent infinity. Recall that only the second can be a set; only infinities that can be consistently thought together can be sets. The idea is that all contradictory collections, collections such that the assumption of them being a set would make it possible to derive a contradiction, are too big (Hallett, 1986, 165-179). From this idea one could derive that a collection of objects cannot form a set if the collection is in one-to-one correspondence with all sets, since this would lead to paradoxes. However, this conception does not seem to be less ad hoc than the iterative conception is and is even less explanatory concerning the nature of sets. As Boolos (1989, 7) stated "[The] limitation of size [...] is not a natural view, for one would come to entertain it only after one's preconceptions had been sophisticated by knowledge of the set-theoretic antinomies, including Russell's paradox, but those of Cantor and Burali-Forti as well.". One needs to know about the paradoxes that might arise to know about the limitations of size, but this would be after the fact.

Subsequently, one might conclude that a completely satisfactory answer to the *why*-question is out of reach. Consider now the minimal answer. The minimal answer to why the universe of sets is not a set itself is that the assumption of a universal set is in contradiction with some of the axioms of ZF(C). This answer tells us nothing about the nature of sets, nor does it motivate why we should use the axioms of ZF(C) instead of some other system that does include a universal set such as NFU^+ where its assumption is not contradictory.

However, that there is only so much we can do, which is what the impossibility of finding a completely satisfactory answer tells us, should never be taken to mean that there is nothing we can do or that we might as well do anything, which is what the minimal answer suggests. By answering that the reason that there is no universal set is that its assumption would be contradictory we give no motivation for using ZF(C) over other systems built on different conceptions of sets that can still recover similar mathematical results. In order to distinguish the accounts that do give such motivations in a satisfactory way from the ones present a version of the minimal answer or lead to unsatisfactory results, we will outline a few criteria.

The first criterion follows from the earlier mentioned criticism on ZF(C) by Menzel (1986). Menzel (1986, 47) argued that the way ZF(C) avoids the Burali-Forti paradox is not in line with the iterative conception, while the iterative conception is often used to motivate ZF(C). From this line of argument, we can distil the criterion that the choice of axioms should be in line with the choice of conception of sets. This is a very natural assumption since if we use a conception of sets to motivate our choice of axioms, these should at least be in line with each other.

From the argument against the limitations of size conception we can derive the criterion that the nature of sets should not be purely defined based on what it cannot consistently be. Without this we would not be able to know whether something can be a set or not without first supposing it is and seeing whether a contradiction arises. This is also problematic because this does not tell us anything about what sets are in a positive sense. The only conception of sets that this is based on, is that sets are objects that do not cause contradiction. However, to form a complete conception of sets we need to know more about its nature.

Finally, one might agree with the earlier mentioned problem by Quine (1951), that the only truly intuitive conception of sets is captured by the naive comprehension schema. This is problematic because, as we have seen, this conception of sets leads to paradoxes such as Russell's paradoxes. However, based on this consideration we might want to adopt a principle that has us keep as much of the sets resulting from the naive comprehension principle as is consistent. This is not as simple as it sounds since, as Incurvati and Murzi (2017) show, taking any maximally consistent set of instances of the naive comprehension schema does not give us a recursively axiomatisable collection of sets. Therefore, we should restrict the instances we use of the naive comprehension schema according to some "*non-naive* conceptions of sets and truth." (Incurvati and Murzi, 2017, 14). Therefore, based on this we can formulate the criterion that the way the naive comprehension axiom is restricted should be motivated from the conception of sets used and not just from consistency.

Although these criteria do not uniquely single out our conception of sets and our formal system, DST does at least satisfy the last two criteria. Whether the first principle, the choice of axioms being in line with the choice of conception of sets, applies would have to be further investigated. It is easy to see that the nature of sets is not purely defined based on what it cannot be. Moreover, the principles we use to restrict the naive comprehension schema are, at least to some extend, *non-naive*; it is independently motivated by the interpretation of information processes in research outside of mathematics. Therefore, the interpretation of the dynamic modalities of DST allows for an answer to the *why*-question that is more in-depth and better motivated than the minimal answer.

Conclusion

In this thesis, we developed a version of set theory that is able to capture the dynamic nature of sets inherent to a potentialist view of the iterative conception of sets by a dynamic implementation of Potentialism in set theory. The two main objectives of this thesis were to formulate a form of set theory that is truly dynamic and to offer a solution to the question *why* the universe of sets is not a set itself, the *why*-question. These objectives were achieved by a combination of formal and philosophical methods.

First, the philosophical need for a dynamic version of set theory was motivated and it was explained why other conceptions of sets have not been successful in providing a satisfactory answer to the *why*-question. Based on these considerations a dynamic version of set theory was formulated and it was proved that this system can recover the results of ZF. Finally, a philosophical interpretation of this formal system was offered and it was evaluated to what extent this interpretation is able to meet the challenges of providing an answer to the *why*-question and of formulating a justifiable conception of sets.

The first chapter served to meet this first step, providing a motivation for the need of a dynamic version of set theory. This was done by outlining the development of ZF(C) set theory, the iterative conception of sets and two opposing ways to interpret this conception of sets. It was shown that in order for Potentialism to better capture the iterative conception of sets, its static modal implementation is not sufficient. Not only is the interpretation of the modality involved problematic, but it also allows the problem Potentialism identified for Actualism to resurface in their setting. Thus, the two main problems for Potentialism at this stage are that it is not able to provide an interpretation of the modality that leads to an in-depth answer to the *why*-question and that the entire model of possible states with potential sets is still there at once.

In the second chapter, a dynamic version of set theory was developed. First, the intuitions on why dynamic semantics would fit the potentialist views were outlined. It was argued that, since we model the growth of information, we can apply methods that are used in modelling the growth of other kinds of information as well. It was also motivated on what level the dynamic component should figure, namely on statements that assert existence or universality since those are the statements that make claims about all sets. Based on these considerations, the model, language and semantics of dynamic set theory were worked out. The section on the semantics features both an intuitionistic and classical version since there have been arguments to adopt intuitionistic semantics to capture the potentialist conception of sets. Then, further principles were added to the system to capture the relation between sets and pluralities and to recover the axioms of ZF. It was shown that all the axioms of ZF can be recovered from the classical version of dynamic set theory, while in the intuitionistic version of dynamic set theory the foundation axiom has to be replaced by the transfinite induction principle. This chapter ended on the conclusion that the intuitionistic and classical semantics only lead to different results on the level of individual states, due to the semantics of the dynamic operators.

In the third chapter, it was shown that a formula is classically deducible in classical dynamic set theory if and only if it is classically deducible in ZF. It was also concluded that a formula is intuitionistically deducible in intuitionistic dynamic set theory if and only if it is intuitionistically deducible in ZF. Additionally, it was concluded that if a formula is intuitionistically deducible in intuitionistic dynamic set theory then it is classically deducible in ZF. The first two claims showed that the dynamic operators combined with their corresponding quantifier in dynamic set theory behave just like the normal quantifiers in ZF. Finally it was proved that dynamic set theory is interpretable in ZF. This served to show that dynamic set theory is indeed not trying to change the results that nonmodal set theory proves, but is merely trying to reformulate set theory in such a way that it fits and captures the conception of sets of Potentialism.

The subject of the final chapter was the philosophy of dynamic set theory. Its first aim was to show that dynamic set theory is still able to capture the open-endedness and indefinitely extensible nature of sets, and to see whether it was still able to solve the issues of ZF(C) and Actualism that Potentialism already provided a solution to. This aim was achieved since dynamic set theory still has the features that Potentialism used to solve this problems. The second aim was to develop an interpretation of the dynamic modalities in dynamic set theory. Instead of using the modalities to model the growth of the mathematical ontology, the modalities were interpreted as modelling the growth of our expressive resources. This means that our system is metaphysically innocent since it makes no claims about the existence or non-existence of specific sets. What it models instead is how we can step-wise assign more inclusive meanings to the word "set". Based on this interpretation it was also argued that from a purely potentialist point of view one should favour the classical version of dynamic set theory over the intuitionistic one. The final aim was to reconsider the *why*-question based on the interpretation of the modalities. It was argued that dynamic set theory is better able to give an answer to this question, partly because the problems concerning the interpretation of potential and actual existence were avoided by the linguistic interpretation of the modalities, and partly because there is an independent motivation for the dynamic modalities. Since dynamic semantics has been successfully applied in other fields to model the growth of information, there is an independent motivation to use it to model the growth of the expressive resources of set theory as well. Moreover, it provides a more natural interpretation of the modality that is not purely mathematical. Finally, the *why*-question was brought into question and it was argued that while a completely satisfactory answer to it might not be possible, there are still constraints on what an answer to this question and a conception of sets should be like. While it was clear to see that the conception of sets offered in this thesis fits two of these constraints, further work is needed to study how well dynamic set theory fits with the other.

Therefore, in this thesis a dynamic version of set theory was developed that allows for a more justifiable conception of sets and provides a more in-depth solution to the *why*-question.

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