Intensional Kleene logics for vagueness

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written by

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Abstract

In this thesis we study the phenomenon of vagueness using multi-valued logics. First, we develop a collection of logics based on strong and weak Kleene truth-tables. All these logics are tolerant, they can express a change in degree for some property which is too small to be perceived as significant by a competent speaker. In classical logic this form of tolerance brings to the sorites paradox. The logics developed in the thesis can account for tolerance while retaining their consistency. These logics are also intensional, since they make use of non-rigid terms and intensional objects.

Two among the defined logics are chosen in order to study the notion of vagueness that they convey and how they can account for problematic subjects, such as vague objects and vague identity. Finally, we provide two interpretations of vagueness, an epistemic one based on Kantian epistemology, and one in a discursive setting using the notion of topic.

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Contents

Chapter 1

Introduction

Vagueness is a fact of our linguistic practice. Natural languages are manifestly able to express statements that competent speakers consider vague. This feature, despite being a flaw of our language from a Fregean perspective[1], at the same time reveals the remarkable inferential power of the logic (or logics) underlying our everyday use of language, which allows us to reason about uncertain facts. On a theoretical ground, accepting that vagueness is a structural feature of natural language implies that any attempt to comprehend natural language which eradicate the possibility of vagueness from it is destined to be partial, if not plainly mistaken.

As Hegel wrote, "[t]he forms of thought are first set out and stored in human *language* [...]. In everything that the human being has interiorized, in everything that in some way or other has become for him a representation, in whatever he has made his own, there has language penetrated, and everything that he transforms into language and expresses in it contains a category, whether concealed, mixed, or well defined." (G.W.F. Hegel, *Science of logic*, *Preface to the second edition*, translation by G. di Giovanni). Even though this Hegelian quotation is, in its original context, far from the contemporary sensibility in analytic philosophy, its message can be transposed, without twisting it, into a regulative maxim. We should not underestimate what might appear as a flaw of language, demoting it to an error that a regimentation of such language should get rid of. Instead we must account for odd but pervasive features of language, in spite of their apparent resistance to an adequate formalization within a classical framework.

This does not mean that any deviation from classical logic should be fideistically

[1] Cfr. Frege (1902), §56.

accepted as a consequence of a proper formalization, since this deviation itself requires a philosophical justification. This is especially true for phenomena like vagueness, which in some cases can imply unorthodox ontological thesis like the possible existence of vague objects.

In this thesis we will extensively study one of the many formal treatments of vagueness within analytic philosophy of language, namely 3-valued logics and some particular semantics based on this framework. Since this is a philosophical research, first of all we are going to define the meaning of our object of study, vagueness (§1.1). Then we will provide a short survey on the mainstream approaches in the field (§1.2).

1.1 Phenomena of vagueness

Vagueness is a phenomenon we have extensive experience of on a pre-theoretical level, since vast portions of our natural languages are to some extent vague[2]. In order to begin a philosophical enquiry on vagueness we pretend more precision though. First we restrict our notion of vagueness, distinguishing it from context-dependency, underspecificity, and ambiguity.

When I say that "athlete p is fast", I might be accused of making a vague use of the predicate "being fast", because p is fast for an average human being, but his performance is not outstanding compared to a leopard. The uncertainty of this sentence derives from a lack of background information, it is not clear what the term of comparison is. Once that extra information is provided, the sentence "athlete p is a fast human being" is not longer uncertain. This is a case of context-dependency: once the context is made explicit, the alleged vagueness is gone. On the contrary, what we will consider vagueness is intrinsic to our language, speakers cannot get rid of the vagueness of a predicate by an *ad hoc* stipulation without losing part of the meaning of the predicate.

Another lack of information that does not amount to vagueness is underspecificity or generality. "p is the person taller than 150cm" is not a vague description of p. There is a large number of individuals who satisfy this description, hence it is a very imprecise way for identifying a person. Nonetheless the sentence is not vague, the cut-off point at 150cm sharply distinguishes among those who could be p and those who are certainly not. The sentence is underspecific, it does not give us enough information for individuating the person p, although the scarce

[2]If not the totality of natural language, as Russell (1923) claims.

information it provides is completely non-vague.

Nor vagueness is ambiguity. An ambiguous word has two or more meanings. "Bank" can refer to the section of a river or to the institution where money is deposited. As the two previous cases, also ambiguity can be resolved with additional explicit information. Usually the preceding flow of a discourse is enough to disambiguate the terms involved in it. In a formal setting, ambiguity is removed by adjusting the syntax, e.g. introducing two terms bank_1 and bank_2, each of which denotes unambiguously.

Following Keefe (2000), we individuate three defining features of vagueness: *borderline cases*, *absence of sharp boundaries*, and *sorites paradox*.

We start with Peirce's definition of vagueness, which captures the fundamental property of vague predicates: when P is vague, there are cases in which we are not able to tell whether P holds or not.

> A proposition is vague when there are possible states of things concerning which it is intrinsically uncertain whether, had they been contemplated by the speaker, he would have regarded them as excluded or allowed by the proposition. (Peirce (1902))

In other words, P is vague when some of its instances are borderline cases of P-ness. In a borderline case the usual bivalent perspective is not capable to decide whether an object x satisfies P or not. Notoriously, classical logic cannot account for a similar situation. Let us consider a person p, whose height is the same as the average for someone of his gender and age. Is p tall? Since p is not above the average height, it does not seem the case that p is tall. But at the same time we would not say that p is not-tall, precisely because he is not below average. Formally, p does not fall either under the extension of the predicate T or under its complement, which is technically impossible in classical semantics. Under this interpretation Tp is a truth-value gap.

It is crucial to notice that in borderline cases no amount of further information can settle the problem of vagueness in a classical way. Peirce's definition continues as follows:

> By intrinsically uncertain we mean not uncertain in consequence of any ignorance of the interpreter, but because the speaker's habits of language were indeterminate. (Peirce (1902))

Vagueness is constitutive of natural language, the uncertainty in borderline cases is not a form of ignorance[3], which could otherwise be resolve if we were provided with the lacking information.

We can easily see now why context-dependency, underspecificity and ambiguity were not classified as vagueness. All three of them can be resolved once the lacking information is obtained, whether it is the knowledge of the context of evaluation, of further specifications, or of the intended sense of an ambiguous word. In particular, notice that borderline cases are not a form of context-dependency. Returning to the example of the average tall p, we can better define a class of individuals and claim that, in that context, p is indeed tall. In this case though we are more properly using the predicate "being tall in context c", rather than simply "being tall", so we are forcing an interpretation which is not faithful to the intended one.

Closely related to borderline cases is the absence of sharp boundaries. We have seen how the borderline zone constitutes a sort of penumbra[4], where classical categories cease to work properly. Using the standard set-theoretic interpretation, a vague predicate P divides the domain of the discourse into three extensions: the objects which are P, the objects which are not-P, and an intermediate area of the objects which are vaguely P. This third section is the penumbra. According to this description, there are sharp boundaries between the three extensions. Our experience though tells us otherwise. If we search for the exact demarcation point from one of the classical sections to the penumbra, we find a second penumbra which divides the two, a blurred zone where we cannot tell whether we are still in the classical extension or not. Moreover the borders of the second penumbra are themselves blurred, and so on if we search for a third penumbra. In other words vagueness itself seems to be vague[5], which in the literature is known as higher-order vagueness[6].

Everyone agrees that a newborn is a child. Similarly, someone who reaches the legal age, let us say 18 years, is definitely no longer a child. Now, let us assume that someone who is 12 years old can be considered a vague case of child. Between

[3] As we will see soon, epistemic theories of vagueness reduce it to a form of ignorance, but a special one, as claimed by the epistemicism of Williamson (1994). What we are denying here is that vagueness can be caused by contingent ignorance, we cannot settle a borderline case by using better measurement tools. Even accepting an epistemic reading, vagueness remains a structural feature of our cognitive apparatus, which cannot be eliminated in principle.

[4] Cfr. Russell (1923).

[5] Cfr. Sorensen (1985) and Hyde (1994).

[6] The notion of higher-order vagueness has been criticized or even rejected by some authors, e.g. Wright (2010).

12 and 18 years, when does the vague childhood end? And if we can individuate a second-order vague zone between vague-child and not-child, at which point in time does a person stop being a vague-child and become a vague-vague-child? This simple example shows how pervasive vagueness can be.

Higher-order vagueness is the reason why we cannot adopt a pragmatic solution and resolve vagueness simply renouncing to use the terms which fall inside the penumbra:

> Someone might seek to obtain precision in the use of words by saying that no word is to be applied in the penumbra, but unfortunately the penumbra is itself not accurately definable, and all the vaguenesses which apply to the primary use of words apply also when we try to fix a limit to their indubitable applicability. (Russell (1923))

Finally, there is the sorites paradox, the most characteristic paradox of vagueness since the antiquity[7]. In its simplest informal version, the sorites ($\sigma\omega\rho\acute{o}\varsigma$, heap) runs as follows. A single grain of sand is not a heap, and neither two grains of sand constitute a heap. Generalizing the reasoning, if we have something which is not a heap, adding a single grain of sand does not turn it into a heap. Unfortunately the consequence of these premises is that any arbitrary large number of grain of sands, even billions of them, will never be a heap.

More in general, in a sorites sequence we have a series of objects from d_0, which is definitely P, to the opposite d_f, which is definitely not-P. The other objects between these two extremes are ordered according to the gradation in P-ness. The higher the degree of P-ness, the closer an object d is to d_0, vice versa the lower its degree is the closer d is to d_f. As a last, crucial remark, the gradation of P between an object d_i at the adjacent d_{i+1} is so small that even if it is perceivable, a competent speaker would not claim that the step change the classical truth-value of the application of P. In other words, if Pd_i is true then Pd_{i+1} is not false, and if Pd_i is false then Pd_{i+1} is not true. Basic classical logic seems to force us to infer that if Pd_0 is true, then Pd_1 is also true, and so Pd_2 and Pd_i for every $i \geq 0$. But similarly, since Pd_f is false then Pd_{f-1} is false, and so on for each Pd_i for all $i \geq 0$. Every member of the sequence is both P and not-P. This apparently sound reasoning brings us to a conclusion which is both formally inconsistent and counterintuitive, since in a similar sequence the experience shows

[7]The history of vagueness in the western philosophical tradition can be found in Williamson (1994).

that the small changes between each step if taken singularly are insufficient for a change in truth-value, but when they are taken together they sum up to a decisive difference.

The source of the sorites paradox is the indistinguishability under the relevant property between each adjacent member in a sequence. As we have seen, in a sorites sequence if d_i is P then we would not claim that d_{i+1} is not-P. In this case d_i and d_{i+1} are P-indistinguishable, in the sense that there is no change in the classical value of the application of P to each of them. In the literature this indistinguishability is known as *tolerance*, the term introduced by Wright (1975), who defines it as "a degree of change too small to make any difference". In fact during one step in a sorites series we tolerate a change in the degree of some property without a corresponding change in truth-value.

Once we accept the possibility of vagueness, for a vague predicate P can be built a corresponding sorites paradox. What is much more controversial is the notion of vague identity, although a completely similar paradox can be provided for it. As in the case of the sorites, the antiquity already devised such a paradox, in the form of the paradox of Theseus' ship:

> The thirty-oared galley in which Theseus sailed with the youths and returned safely was preserved by the Athenians down to the time of Demetrius of Phalerum. At intervals, they removed the old timbers and replaced them with sound ones, so that the ship became a classic illustration for the philosophers of the disputed question of growth and change, some of them arguing that it remained the same, and others that it became a different vessel. (Plutarch, *Life of Theseus*, translation by I. Scott-Kilvert)

Although Theseus' ship is usually regarded as a puzzle of composition or of persistence over time, we are going to assimilate it to the phenomena of vagueness and, more precisely, to the category of tolerance. Hence we are proposing to interpret Theseus' ship as a case of vague identity. Notoriously this notion has been challenged by the arguments of Evans (1978) and Salmon (1982), who claim that vague identity is an inconsistent concept. Informally, let a and b be vaguely identical, then a has the property of being vaguely identical to b. At the same time b is definitely identical to itself, therefore b lacks the property of being vaguely identical to b. Hence a and b differ for some property. By Leibinz's law, which is at the foundation of our notion of identity, this implies that a and b are different, i.e. not vaguely identical but strictly non-identical. The Evans-Salmon argument has been largely influential but it has not settled the problem of vague identity once and for all[8]. Nonetheless it poses a challenge to the already suspicious notion of vague identity. One of the purposes of this thesis is to provide a formal account and a suitable philosophical interpretation for vague identity.

1.2 Logics for uncertainty

We have determined a definition of vagueness. Now we illustrate the most influential accounts of vagueness in the philosophical debate[9]: *degrees of truth*, *supervaluationism*, and *epistemic interpretations*.

Degree theories of vagueness extend the set of truth-values beyond the classical true and false, and assign to vague sentences non-classical values. We can can have finite and infinite-valued logics in this way. Among the finite-valued logics, 3-valued ones are the mainstream. The non-classical value $1/2$, intermediate between true and false, is read as "vague". The following problem is how to philosophically justify this new truth-value. Using strong Kleene truth-tables, which are widely taken as the semantics for 3-valued logics oriented to vagueness[10], $1/2$ finds a natural interpretation as "lacking sufficient classical information to be determined as true or false". For example, the semantic clause for disjunction is $V(\varphi \vee \psi) = max\{V(\varphi), V(\psi)\}$. We can easily see that it is enough for one of the disjunct to be true in order for the whole disjunction to be true, returning the same intended meaning of classical disjunction.

Deviating from classical logic, degree theories claim that vague sentences have a different, non-classical mode of truth. Extending this reasoning we arrive to infinite-valued logics. In an infinite-valued logic, sentences are assigned a truth-value in the real closed interval $[0, 1]$. The degree of truth of a sentence is a real number $r \in [0, 1]$. If 0 is completely false and 1 is completely true, any intermediate value r can be read as how close to true r is. The semantics for infinite truth-values is just an extension of strong Kleene truth-tables, e.g. $V(\varphi \vee \psi) = max\{V(\varphi), V(\psi)\}$, but now $V(\varphi), V(\psi) \in [0, 1]$. Infinite-valued logics avoid the problem in which their finite-valued counterpart incur, namely the presence of sharp boundaries, due to the density of real numbers. What no degree theory can

[8]For examples of the debate about the Evans-Salmon argument cfr. Lewis (1988), Parsons (2000), and Edgington (2000)

[9]We do not consider the non-solutions to vagueness, namely those theories which deny that vagueness can receive any adequate formal treatment, like Russell (1923) and Dummett (1975)

[10]E.g. Tye (1994).

account for (at least in this standard formulation), are what Fine (1975) called penumbral connections. Even if inside the penumbra of vagueness classical values do not apply to atomic sentences, it seems reasonable to require that tautologies are still true. φ can be $1/2$, nevertheless $\varphi \vee \neg\varphi$ must be true under any interpretation. This criticism is tied to the strong Kleene reading of $1/2$, in fact no matter how scarce our classical information is, the law of excluded middle is intuitively required to be valid.

This last remark underlies an interpretation that reads vagueness as a form of uncertainty which can be, in principle, settled if the context is completely specified. This is the idea behind supervaluationism. A vague predicate has a vast range of applications, and its vagueness is the result of the semantic indecision of the speaker. Each of these applications is a precisification, which formally corresponds to a completely classical model, in which every fact is bivalently decided. A sentence is vague because there is discordance in its value among the various precisifications, since there is no reason for a speaker to choose one over the others[11]. On the contrary, when all the precisifications agree on the value of a formula, the speaker is allowed to consider the formula having that value, since that choice would be confirmed by every refinement of the context at hand.

More formally, for a model $M = \langle D, V^3 \rangle$ with a (strong Kleene) 3-valued valuation V^3, let a precisification be a classical model $M_p = \langle D, V_p \rangle$ that agrees with M with respect to every classically evaluated atomic formula α, namely if $V^3(\alpha) = 1$ then $V_p(\alpha) = 1$, and if $V^3(\alpha) = 0$ then $V_p(\alpha) = 0$. By straightforward induction it follows that the classical truth-value of every formula φ of arbitrary complexity is preserved by the precisification. A supervaluation V^S based on M is a function such that $V^S(\varphi) = 1$ if for every precisification $V_p(\varphi) = 1$, $V^S(\varphi) = 0$ if for every precisification $V_p(\varphi) = 0$, and $V^S(\varphi) = 1/2$ otherwise. When $V^S(\varphi) = 1$ we say that φ is supertrue, if $V^S(\varphi) = 0$ φ is superfalse.

It follows immediately that all classical tautologies are supertrue, and classical contradictions are superfalse. In this way, the problem of penumbral connections is solved in a supervaluational setting, e.g. $\varphi \vee \neg\varphi$ is supertrue. Supervaluationism has further problems though. Supervaluations are not compositional, in the sense that the truth-value of subformulae do not functionally determine the value of the formulae containing them. E.g., let $V^3(\varphi) = V^3(\psi) = 1/2$. For the supervaluation V^S based on V^3 we have that $V^S(\varphi \vee \psi) = 1/2$ but $V^S(\varphi \vee \neg\varphi) = 1$. The function

[11] The concept of vagueness in supervaluationism resembles that of ambiguity, as explicitly claimed by Fine (1975). A similar reading is endorsed by Lewis (1982). Keefe (2000), pp.156-159, argues against this identification between vagueness and ambiguity.

associated to the connective $\vee$ behaves differently, although it receives $\langle 1/2, 1/2 \rangle$ as argument in both cases. Moreover, even though supervaluations preserve classical logic, once we add a definiteness operator (whose semantics is: φ is definitely true iff φ is supertrue) many classical inferences fails, like contraposition[12].

Both degree theories and supervaluationism propose, to some extent, a revision of classical semantics. On the contrary, epistemic theories of vagueness claim that vagueness is an epistemic phenomenon and that classical logic is to be preserved as the correct logic. Hence vagueness does not point at mistakes in the form of our reasoning but at our ignorance about what we are talking about. Influential epistemic theories of vagueness are Sorensen (1988) and Williamson (1994). Williamson's theory, named epistemicism, claims that predicates have sharply determined extensions and that their application is completely classical. In the world any object is either P or not-P, there are no further intermediate ways of being P. We consider some sentences to be vague because we do not know the boundaries of the predicates we use. This ignorance though is not an accident, a contingent lack of information that a better education may resolve. Vague predicates are such because they possess an intrinsic feature, a gap of uncertainty in their application or, in Williamson's words, a margin for error, within which competent speakers are not able to discern a significant change in the degree of the application of the predicate. This is a hard limitation, rooted in our cognitive structure, this is the form of ignorance that epistemic interpretations ascribe to vagueness.

On the one hand, the great advantage of epistemic theories is that classical logic is preserved in all its strength and elegance. On the other hand, these theories sound *prima facie* totally counterintuitive and they have to answer to many philosophical objections. How is it possible that our use of language is disjoint from the alleged true meaning of the predicates we employ? And in which sense does a predicate have an extension independently from the speakers of a language? These are just a couple of the objections that an epistemic theory has to answer.

1.3 Overview

The purpose of this thesis is to study logics for vagueness and, in particular, logics able to account for tolerance. In chapter 2 a survey on many 3-valued semantics is provided. We start with a modal extension of the 3-valued version of the tolerant logic first presented by Cobreros *et al.* (2012). This logic, **sK**, is

[12]A list of some of the invalid inferences is found in Williamson (1994), pp.149-153.

based on strong Kleene truth-tables. Moreover **sK** is an intensional logic, in the sense of Fitting (2004), based on a language which contains non-rigid designators. Extensions and variations of **sK** are provided. All of these logics share the property of being conservative extensions of classical logic, by virtue of the fact that the relation of logical consequence which defines each of these logics goes from true premises to non-false conclusions (either 1 but also $1/2$ is accepted). As proved by Cobreros *et al.* (2012), this entailment captures all classical logic. At the same time, special constraints on the semantics make tolerant inferences, like the single steps of sorites paradox, possible without inconsistent consequences. In chapter 2 we also devise semantics based on weak Kleene truth-tables and study their properties. Surprisingly, there are little differences between logics based on strong and weak Kleene tables once logical consequence is defined as above.

In chapter 3 we discuss some applications of two of the logics defined in the previous chapter: **SK**, based on strong Kleene tables, and **WK**, based on weak ones. **SK** is employed to provide an account for vague objects, tolerance and vague identity, moreover we give an interpretation based on Kantian epistemology of the notion of vagueness conveyed by this logic. Finally **WK** is used as the semantics for a logic for topics.

Chapter 2

Formal semantics

Our study of vagueness begins with the formal framework which we are going to apply to matters of philosophy of language in chapter 3. In the current chapter we provide a collection of 3-valued modal logics[1], showing their properties and relations.

The simplest logics from which we start are **sK** (§2.2) and **wK** (§2.3), based respectively on strong and weak Kleene truth-tables of Kleene (1952). These are 3-valued quantified modal logics provided with special predicates able to express tolerance, a relation that holds between terms indistinguishable under some respect. **sK** is not based on the well-known 3-valued logics $\mathbf{K}_3$ or **LP**, instead it is a modal extension of the 3-valued translation of the tolerant logic originated in Cobreros *et al.* (2012). This logic is characterized by a mixed entailment which goes from true premises to non-false conclusions (hence a non-classical conclusion is still sound).

The modal logic underlying **sK** and **wK** is the constant domain version of the minimal normal modal logic **K**. Extending this framework to stronger normal modal logics is an easy task, obtained imposing constraints on the accessibility relation or, with a syntactic approach, adding the respective axioms. We do not intend to follow this further research here though.

Another feature shared by all the logics introduced in this chapter is that they are intensional. The investigation of intensionality in order to formalize natural language dates back to Montague (1970) and Gallin (1975). Using in particular the machinery of Fitting (2004), besides individual variables we make use of non-rigid terms, intensions. The distinction between these two categories of terms will

[1] A survey on this wide topic is Garson (2001). Most of the framework developed here is based on Fitting & Mendelsohn (1998) and Garson (2013)

correspond to different ontological levels in chapter 3, but already in the current chapter we are going to show their distinct formal properties.

Extensions of **sK** and **wK** will be studied, obtained introducing new sentential operators (§2.4) and allowing for multiple denoting terms (§2.6). Moreover the proof theory for some of these logics will be provided in the form of tree-style systems (§2.5).

As a terminological note, when it is indifferent whether the fundamental semantics is based on strong or weak Kleene, we use the denomination **iK** logic.

2.1 Preliminary notions

Let $\mathscr{L}_{CL}$ be a first-order language containing countably infinitely many predicate letters P^n for each finite arity $n \in \mathbb{N}$, individual variables $x_1, x_2... \; y, z...$ and intensional variables $f_1, f_2... \; g, h...$. The logical symbols of the language are $\vee, \neg, =, \forall$ and $\Box$, moreover we use the symbol λ for predicate abstraction.

Form($\mathscr{L}_{CL}$) is the set of well-formed formulae of $\mathscr{L}$, which are defined as follows:

$$\varphi := P^n x_1, ... \; x_n \mid x_1 = x_2 \mid \neg\varphi \mid \varphi \vee \varphi \mid [\lambda x.\varphi](f) \mid \forall x \varphi \mid \Box\varphi$$

In quantified and abstract formulae, the bounded variable must be free for substitution under the scope of the quantifier or the λ operator. The notions of open and closed formulae are as usual.

For every monadic predicate P of the original language we extend $\mathscr{L}_{CL}$ to $\mathscr{L}$ adding a special dyadic predicate I_P, whose intuitive reading is "is indistinguishable with respect to P from"[2]. The vocabulary of $\mathscr{L}$ is obtained form that of $\mathscr{L}_{CL}$ together with the further clause that $I_P x_1 x_2$ is a well-formed formula.

The terms of $\mathscr{L}$ are individual variables and intensional variables. $\mathscr{L}$ is a typed language, where individual variables are of type O and intensional variables of type I. Each predicate of arity n is of type $\langle O_{(1)}, ...O_{(n)}\rangle$, both the identity symbol $=$ and each predictae I_P have type $\langle O, O\rangle$. Formulae of the form Pf are not allowed. Similarly, quantifiers and λ operator bound individual variables. A

[2]Indistinguishability predicates I_P are introduced to express tolerance. We build these only from monadic predicates of $\mathscr{L}_{CL}$ for two reasons. First, this choice makes the semantic clauses for I_P much easier. Moreover, the fundamental problem of tolerant reasoning is expressed without loss of generality by the transition in the gradation of some property, named by a monadic predicate.

more expressive language which allows for predicate of intensional or mixed type and quantification over intensions is not the aim of the current study.

A *tolerant (quantified) Kripke frame* is a structure $\mathscr{F} = \langle W, R, D, \sim \rangle$, where W is a non-empty set of points or worlds, R is a binary relation over W, and D is a non-empty set. Intuitively D is the domain of extensional objects, over which standard first-order quantifiers range. The last element $\sim$ is a collection of functions: for each monadic predicate P there is a function $\sim_P$: $W \mapsto D^2$. We write $\sim_{P,w}$ instead of $\sim_P (w)$ and call it the tolerance or indistinguishability relation for P at w[3]. We impose $\sim_{P,w}$ to be reflexive and symmetric for every P and w, but we do not require it to be transitive.

Since we are using a language with open formulae, we need to assign values to free variables. Let $Var(\mathscr{L})$ be the set of variables, both individual and intensional, of $\mathscr{L}$. We define an assignment as a function $a : W \times Var(\mathscr{L}) \mapsto D$ such that for each individual variable x, $a(w, x)$ is a constant function. In this way individual variables denote as proper names, according to the standard account of Kripke (1972). Although we do not restrict individual variables to proper names, we assume them to refer to their denotations directly, in the sense that they rigidly pick their denotations through all the worlds.

Whenever a frame $\mathscr{F}$ contains a tolerance relation $\sim$, we assume that the valuation V of any model based on $\mathscr{F}$ satisfies the following closeness constraint[4]:

Definition 2.1.0.1 (Closeness). For every $d_1, d_2 \in D$, if $d_1 \sim_{P,w} d_2$, $d_1 = a(w, x_1)$ and $d_2 = a(w, x_2)$, then $|V_{w,a}(Px_1) - V_{w,a}(Px_2)| < 1$.

This constraint gives to $\sim$ its intended interpretation of indistinguishability relation. We allow two things to be P-indistinguishable only if their gap with respect to P is not too wide. Since we will assign to vague statements the numerical value $1/2$, closeness amounts to the fact that there is no classical change in truth-value from d_1 to d_2 with respect to their P-ness. If we assume that a change is the passage from truth to falsity or vice versa with respect to some quality, closeness formalizes that phenomenon of tolerant reasoning in which we are unable to detect a significant (i.e. classical) change in a gradation, generating that state of

[3]Cobreros *et al.* (2012) work in a non-modal setting, with a single indifference relation $\sim$ for each model. The same could be done in a modal setting, obtaining a simpler semantics. This complication seems nonetheless unavoidable: the possibility that predicates may change extension across worlds justifies the fact that two objects which might have been indistinguishable at some world could be distinguishable at another one.

[4]Cfr. Cobreros *et al.* (2015) and Smith (2008), pp.145-158.

uncertainty which is peculiar of sorites paradox.

2.2 Strong Kleene models for vagueness

A *strong Kleene model* (sK-model) is a structure $\langle \mathscr{F}, V\rangle$, where $\mathscr{F} = \langle W, R, D, \sim\rangle$ is a tolerant Kripke frame. V is valuation function which assigns to each predicate P^n and world $w \in W$ a function $V_w(P) : D^n \mapsto \{1, 1/2, 0\}$ and that satisfies the closeness constraint of definition 2.1.0.1.

Given an assignment a and a world $w \in W$, we define the valuation $V_{w,a}$ recursively as follows:

- $V_{w,a}(Px_1...x_n) = V_w(P)(a(w, x_1), ...a(w, x_n))$
- $V_{w,a}(I_P xy) = 1$ if $a(w, x) \sim_{P,w} a(w, y)$; $V_{w,a}(I_P xy) = 0$ otherwise.
- $V_{w,a}(x_1 = x_2) = 1$ if $a(w, x_1) = a(w, x_2)$, $V_{w,a}(x_1 = x_2) = 0$ otherwise.
- $V_{w,a}(\neg\varphi) = 1 - V_{w,a}(\varphi)$
- $V_{w,a}(\varphi \vee \psi) = max\{V_{w,a}(\varphi), V_{w,a}(\psi)\}$
- $V_{w,a}(\forall x\varphi) = min\{V_{w,a'}(\varphi) \mid a'$ is a x-variant of $a\}$
- $V_{w,a}(\Box\varphi) = min\{V_{v,a}(\varphi) \mid v \in R[w]\}$
- $V_{w,a}([\lambda x.\varphi](f)) = V_{w,a'}(\varphi)$ where a' is like a except that $a'(w, x) = a(w, f)$

It is easy to check that the above semantic clauses for $\neg$ and $\vee$ correspond exactly to strong Kleene truth-tables:

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

	$\neg$
1	0
1/2	1/2
0	1

For the remaining logical connectives the middle truth-value 1/2 behaves accordingly to the usual interpretation, namely as lacking sufficient classical information to be determined as either 1 or 0.

Conjunction is defined as $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$, implication as $\varphi \rightarrow \psi := \neg\varphi \vee \psi$, the existential quatifier as $\exists x\varphi := \neg\forall x\neg\varphi$, and the possibility operator as $\Diamond\varphi := \neg\Box\neg\varphi$. These definitions allow us to obtain the expected semantic clauses:

- $V_{w,a}(\varphi \wedge \psi) = V_{w,a}(\neg(\neg\varphi \vee \neg\psi)) = 1 - V_{w,a}(\neg\varphi \vee \neg\psi) = 1 - max\{V_{w,a}(\neg\varphi), V_{w,a}(\neg\psi)\} = 1 - max\{1 - V_{w,a}(\varphi), 1 - V_{w,a}(\psi)\} = 1 - (1 - min\{V_{w,a}(\varphi), V_{w,a}(\psi)\}) = min\{V_{w,a}(\varphi), V_{w,a}(\psi)\}$

- $V_{w,a}(\exists x\varphi) = V_{w,a}(\neg\forall\neg\varphi) = 1 - V_{w,a}(\forall\neg\varphi) = 1 - min\{V_{w,a'}(\neg\varphi) \mid a'$ is a x-variant of $a\} = 1 - (1 - max\{V_{w,a'}(\varphi) \mid a'$ is a x-variant of $a\} = max\{V_{w,a'}(\varphi) \mid a'$ is a x-variant of $a\}$

- $V_{w,a}(\Diamond\varphi) = V_{w,a}(\neg\Box\neg\varphi) = 1 - V_{w,a}(\Box\neg\varphi) = 1 - min\{V_{v,a}(\varphi) \mid v \in R[w]\} = 1 - (1 - max\{V_{v,a}(\varphi) \mid v \in R[w]\}) = max\{V_{v,a}(\varphi) \mid v \in R[w]\}$

With the above semantics we obtain the logics $\mathbf{K}_3^m$ and $\mathbf{LP}^m$, which are a modal extensions for a richer language of the well-known 3-valued logics $\mathbf{K}_3$, the standard strong Kleene logic as introduced by Kleene (1952), pp.332-340, and **LP**, the logic of paradox developed by Priest (1979).

Definition 2.2.0.1 ($\mathbf{K}_3^m$)**.** A formula φ is $\mathbf{K}_3^m$-satisfiable iff there are a sK-model $M = \langle W, R, D, \sim, V\rangle$, a world $w \in W$ and an assignment a such that $V_{w,a}(\varphi) = 1$. A formula φ is $\mathbf{K}_3^m$-valid iff $V_{w,a}(\varphi) = 1$ for every world w of every sK-model. A formula φ is a $\mathbf{K}_3^m$-consequence of a set of formulae Γ, $\Gamma \vDash^{K_3^m} \varphi$, iff for every sK-model, if $V_{w,a}[\Gamma] = \{1\}$[5] then $V_{w,a}(\varphi) = 1$. $\mathbf{K}_3^m$ is the logic $\langle \mathscr{L}, \vDash^{K_3^m}\rangle$.

Definition 2.2.0.2 ($\mathbf{LP}^m$)**.** A formula φ is $\mathbf{LP}^m$-satisfiable iff there are a sK-model $M = \langle W, R, D, \sim, V\rangle$, a world $w \in W$ and an assignment a such that $V_{w,a}(\varphi) \neq 0$. A formula φ is $\mathbf{LP}^m$-valid iff $V_{w,a}(\varphi) \neq 0$ for every world w of every sK-model. A formula φ is a $\mathbf{LP}^m$-consequence of a set of formulae Γ, $\Gamma \vDash^{LP^m} \varphi$, iff for every wK-model, if $V_{w,a}[\Gamma] \subseteq \{1, 1/2\}$ then $V_{w,a}(\varphi) \neq 0$. $\mathbf{LP}^m$ is the logic $\langle \mathscr{L}, \vDash^{LP^m}\rangle$.

$\mathbf{K}_3^m$ and $\mathbf{LP}^m$ extend their non-modal counterparts and share many of their peculiarities. E.g. it is easy to check that in $\mathbf{K}_3^m$ the law of excluded middle is not valid and that in $\mathbf{LP}^m$ *modus ponens* is unsound.

Both these logics have to sacrifice some classical validities. What we want though is to devise a logic able to account for vagueness and at the same time we want to retain as much classical reasoning as possible. As Smith (2008), pp.221-223, and Cobreros *et al.* (2012) proved, we can preserve the whole classical logic

[5] We use the square brackets notation to denote the image of a set through a function. Given a function $f : A \mapsto B$ and $S \subseteq A$, we define $f[S] := \bigcup_{s \in S} \{f(s)\}$.

CL[6], employing an entailment which goes from true premises to non-false conclusions.

Definition 2.2.0.3 (sK). A formula φ is a **sK**-consequence of a set of formulae Γ, $\Gamma \vDash^{sK} \varphi$, iff for every sK-model $M = \langle W, R, D, \sim, V\rangle$, if $V_{w,a}[\Gamma] = \{1\}$ then $V_{w,a}(\varphi) \neq 0$. A formula is **sK**-valid iff it is $\mathbf{LP}^m$-valid. **sK** is the logic $\langle \mathscr{L}, \vDash^{sK}\rangle$.

Theorem 2.2.1. *For every* $\Gamma \cup \{\varphi\} \subseteq Form(\mathscr{L}_{CL})$, $\Gamma \vDash^{sK} \varphi$ *iff* $\Gamma \vDash^{CL} \varphi$.

Proof. ($\Rightarrow$) Since the set of classical valuations is a subset of the set of strong Kleene valuations, it follows immediately that if $\Gamma \vDash^{sK} \varphi$ then $\Gamma \vDash^{CL} \varphi$.

($\Leftarrow$) Let $\Gamma \nvDash^{sK} \varphi$, hence for some sK-model $M = \langle W, R, D, \sim, V\rangle$ we have $V_{w,a}[\Gamma] = \{1\}$ and $V_{w,a}(\varphi) = 0$. Let $M' = \langle W, R, D, V'\rangle$ be a classical model with the same frame as M (except for $\sim$), and with V' defined as follows: if $V_{w,a}(Px_1...x_n) = 1$ then $V'_{w,a}(Px_1...x_n) = 1$, and if $V_{w,a}(Px_1...x_n) = 0$ then $V'_{w,a}(Px_1...x_n) = 0$. It is provable by straightforward induction on the complexity of φ that if $V_{w,a}(\varphi) = 1$ then $V'_{w,a}(\varphi) = 1$, and if $V_{w,a}(\varphi) = 0$ then $V'_{w,a}(\varphi) = 0$. It follows that $V'_{w,a}[\Gamma] = \{1\}$ and $V'_{w,a}(\varphi) = 0$, hence $\Gamma \nvDash^{CL} \varphi$. □

Theorem 2.2.1 proves that **sK** coincides with classical logic when we consider the classical fragment $\mathscr{L}_{CL}$ of the language. Therefore **sK** is a conservative extension of classical logic, while $\mathbf{K}_3^m$ and $\mathbf{LP}^m$ are its proper sublogics.

It could be objected that while the entailment relation validates all classical tautologies, the original meaning of classical validity is betrayed by **sK**. A classical tautology is a formula which is evaluated true in every model, while a **sK**-validity is only non-false in every model. On the one hand this is a shortcoming common to multi-valued logics, their inability to account for penumbral connections[7]. In fact, consider the =-free fragment of $\mathscr{L}$ and evaluate every atomic formula $1/2$: by induction it follows that every formula is $1/2$ as well, including classical tautologies. On the other hand, if we intend a tautology as something which can never be falsified[8], then **sK**-validities retain the main feature of classical tautologies.

While **sK** preserves classical logic, its metatheory is not completely classical. A remarkable classical property which the entailment in **sK** lacks is transitivity,

[6]By classical logic here we intend the logic characterized by a standard logical consequence relation over all quantified constant domain Kripke frames for a language with intensional variables. An example of such logic, which differs slightly from more orthodox expositions of quantified normal modal logic **K**, can be provided by the intensional logics of Fitting & Mendelsohn (1998) and Fitting (2004).

[7]Cfr. Fine (1975).

[8]What is called quasi-tautology in Bergmann (2008), pp.84-85.

in fact the relation $\vDash^{sK}$ is not transitive[9]. Let $\varphi \vDash^{sK} \psi$ and $\psi \vDash^{sK} \chi$, and let M be a model such that $V_{w,a}(\varphi) = 1$, hence $V_{w,a}(\psi) \geq 1/2$. The second entailment states that if $V_{w,a}(\psi) = 1$ then $V_{w,a}(\chi) \geq 1/2$, it requires a strictly true premise ψ, which is not guaranteed solely by the first entailment and $V_{w,a}(\varphi) = 1$. It follows that these two entailments are not sufficient to claim their concatenation $\varphi \vDash^{sK} \chi$.

This feature will play a crucial role in stopping the concatenation of *modus ponens* in sorites sequences, therefore accounting for tolerant reasoning, as we will see in chapter 3, §3.3. We can already notice that sK-models are a suitable framework for a degree theory of vagueness, precisely because of the ability of dealing with tolerance. The tolerance principle $\forall x, y((Px \wedge I_P xy) \rightarrow Py)$ is in fact **sK**-valid, as it is the entailment $Px \wedge I_P xy \vDash^{sK} Py$, thanks to the closeness constraint.

sK mantains some characteristic properties of constant domain modal logics[10]. The Barcan schema $\forall x \Box \varphi \rightarrow \Box \forall x \varphi$ and its converse $\Box \forall x \varphi \rightarrow \forall x \Box \varphi$ are in fact both $\mathbf{LP}^m$-valid, hence **sK**-valid too. Identity retains many of its properties, first among them its necessity, in fact $\forall x, y(x = y \rightarrow \Box x = y)$ is **sK**-valid. The validity of this formula follows immediately from the definition of assignment, which is not sensitive to the world of evaluation in the case of individual variables. This is not the case for intensional variables and in fact $[\lambda x.x = y](f) \rightarrow \Box[\lambda x.x = y](f)$ is not **sK**-valid, a counterexample is provided by a model such that $W = \{w, v\}, R = \{\langle w, v\rangle\}, a(w, f) = d_1, a(v, f) = d_2$ and $a(w, y) = a(v, y) = d_1$. This result is a direct consequence of the fact that intensional variables are non-rigid designators. In this way **sK** can account for contingent identity statements.

A similar asymmetry holds for Leibniz's law too: $\forall x, y(x = y \wedge \varphi(x) \rightarrow \varphi[y/x])$ is **sK**-valid, but when φ is a formula where the main connective is a modal operator and with a *de dicto* reading of the consequent we can obtain a formula $[\lambda y.(x = y \wedge \Diamond\psi(x))](f) \rightarrow \Diamond[\lambda y.\psi[y/x]](f)$ which is not **sK**-valid. Consider a model such that $W = \{w, v\}, R = \{\langle w, v\rangle\}, V_{w,a}(Px) = 1$ for an assignment such that $a(w, x) = d_1 = a(w, f)$, and let $V_{v,a}(Px) = 1$ but $V_v(P)(d_2) = 0$ and $a(v, f) = d_2$. It follows that $V_{w,a}(\lambda y.(x = y \wedge \Diamond Px)](f)) = 1$ while $V_{w,a}(\Diamond[\lambda y.Py](f)) = 0$. These differences between individual and intensional variables will be philosophically justified in chapter 3.

Before moving on, we return to a topic briefly mentioned, quantification over intensional variables. In order to introduce intensional quantification a domain

[9]Cfr. Cobreros *et al.* (2012) and Cobreros *et al.* (2017).
[10]Cfr. Hughes & Cresswell (1996).

of intensions D_I is added to a sK-model. For simplicity we rename the original (extensional) domain D as D_O. The members of D_I are intensions, functions $f : W \mapsto D$, and each intensional variable is interpreted into the intension with the same name as itself, namely $a(f) = f \in D_I$. These semantic clauses should be added in order to evaluate intensional quantified formulae:

- $V_{w,a}(\forall f \varphi(f)) = min\{V_{w,a}(\varphi[g/f]) \mid g \in D_I\}$
- $V_{w,a}(\exists f \varphi(f)) = max\{V_{w,a}(\varphi[g/f]) \mid g \in D_I\}$

We are using a substitutional reading of the quatifiers in this case, since we can safely assume intensions to be, at least, linguistic objects, hence their set D_I has at most countable cardinality.

We will not make use of intensional quantification because in sK-models these quantifiers would be a needless domain restriction. In fact in $\mathscr{L}$ predicates accept only individual variables as arguments and these variables are interpreted in extensional objects. Even when λ abstraction is employed, the value of the substituted intension is just its extension at the considered world, namely a member of D_O. Moving to a richer language that introduces predicates of type $\langle T_1, ... T_n\rangle$, with each $T_i \in \{O, I\}$, would make intensional quantifiers a meaningful addition.

2.3 Weak Kleene models for vagueness

A *weak Kleene model* (wK-model) is a structure $\langle \mathscr{F}, V\rangle$, where $\mathscr{F} = \langle W, R, D, \sim\rangle$ is a tolerant Kripke frame. V is a valuation function which assigns to each predicate P^n and world $w \in W$ a function $V_w(P) : D^n \mapsto \{1, 1/2, 0\}$ and that satisifes the closeness constraint (definition 2.1.0.1).

Given an assignment a and a world $w \in W$, we define the valuation $V_{w,a}$ recursively as follows:

- $V_{w,a}(Px_1...x_n) = V_w(P)(a(w, x_1), ... a(w, x_n))$
- $V_{w,a}(I_P xy) = 1$ if $a(w, x) \sim_{P,w} a(w, y)$; $V_{w,a}(I_P xy) = 0$ otherwise.
- $V_{w,a}(x_1 = x_2) = 1$ if $a(w, x_1) = a(w, x_2)$, $V_{w,a}(x_1 = x_2) = 0$ otherwise.
- $V_{w,a}(\neg\varphi) = 1 - V_{w,a}(\varphi)$ if $V_{w,a}(\varphi) \in \{1, 0\}$, $V_{w,a}(\neg\varphi) = 1/2$ otherwise.
- $V_{w,a}(\varphi \vee \psi) = max\{V_{w,a}(\varphi), V_{w,a}(\psi)\}$ if $V_{w,a}(\varphi) \in \{1, 0\}$ and $V_{w,a}(\psi) \in \{1, 0\}$, $V_{w,a}(\varphi \vee \psi) = 1/2$ otherwise.

- $V_{w,a}(\forall x\varphi) = min\{V_{w,a'}(\varphi) \mid a' \text{ is a } x\text{-variant of } a\}$ if $\{V_{w,a'}(\varphi) \mid a' \text{ is a } x\text{-variant of } a\} \subseteq \{1, 0\}$, $V_{w,a}(\forall\varphi) = 1/2$ otherwise.

- $V_{w,a}(\Box\varphi) = min\{V_{v,a}(\varphi) \mid v \in R[w]\}$ if $\{V_{v,a'}(\varphi) \mid v \in R[w]\} \subseteq \{1, 0\}$, $V_{w,a}(\Box\varphi) = 1/2$ otherwise.

- $V_{w,a}([\lambda x.\varphi](f)) = V_{w,a'}(\varphi)$ where a' is like a except that $a'(w, x) = a(w, f)$

It is easy to check that the above semantic clauses for $\neg$ and $\vee$ correspond exactly to weak Kleene truth-tables:

$\vee$	1	1/2	0
1	1	1/2	1
1/2	1/2	1/2	1/2
0	1	1/2	0

	$\neg$
1	0
1/2	1/2
0	1

All the other connectives are defined as it was done for strong Kleene logics. The connectives behave accordingly to the infectious character of the non-classical value 1/2 in weak Kleene logics[11]. 1/2 is intended as a nonsensical or meaningless value, not necessarily ascribable to ungrammaticality but also to other sources, such as categorical mistake. In the propositional setting the nonsensical value 1/2 propagates by compositionality through the whole formula, infecting it[12]. This property cannot be straightforwardly extended to the quantified modal case, as we are going to see.

Let $Sub(\varphi)$ be the set of subformulae of φ and $Atom(\varphi)$ the set of atomic formulae contained in φ. The following restricted formulation of the contamination principle holds:

Fact 2.3.0.1 (**Contamination principle**). *For every formula φ of the $\{\neg, \vee\}$-fragment of $\mathscr{L}$, $V_{w,a}(\varphi) = 1/2$ iff $V_{w,a}(\psi) = 1/2$ for some $\psi \in Sub(\varphi)$ iff $V_{w,a}(\psi) = 1/2$ for some $\psi \in Atom(\varphi)$.*

It is now clearer why nonsensicality is a suitable interpretation for 1/2. When the component of a complex formula is nonsensical, we consider the containing formula meaningless as well. An intuitive example is provided by our natural understanding of grammaticality: once one fragment of a sentence is syntactically incorrect, competent speakers usually perceive the whole sentence as ungrammatical.

[11] Cfr. Ferguson (2015) and Szmuc (2016).

[12] Cfr. Ciuni & Carrara (2016)

Moving to the full language $\mathscr{L}$, fact 2.3.0.1 loses its validity. Consider a formula of the form $\forall x\varphi$: we might have $V_{w,a}(\forall x\varphi) = 1/2$ without having $V_{w,a}(\varphi) = 1/2$, e.g. for a model such that $V_{w,a}(Px) = 1$ and $V_{w,a}(Py) = 1/2$, hence $V_{w,a}(\forall xPx) = 1/2$ but $V_{w,a}(Px) \neq 1/2$. Similar reasonings hold for formulae of the form $\Box\varphi$ and $[\lambda x.\varphi](f)$, for which the contamination principle generally fails.

Despite this limiting result, the intended interpretation of $1/2$ as nonsensical is preserved. For a quantified formula $\forall x\varphi(x)$ to be nonsensical, it is a sufficient condition that for some element d of the domain the instance $\varphi(x)$ is nonsensical when x is assigned d as value (depending on the cardinality of the domain we are not guaranteed that for a fixed assignment any instance is actually evaluated as $1/2$). Similarly $\Box\varphi(x)$ is nonsensical at w iff at some R-successor of w $\varphi(x)$ is evaluated as nonsensical.

The provided semantics allows us to define $\mathbf{B}_3^m$ and $\mathbf{PWK}^m$, which are modal extensions of the logics of nonsense $\mathbf{B}_3$, introduced by Bochvar (1938), and **PWK**, developed independently by Halldén (1949) and Prior (1957).

Definition 2.3.0.1 ($\mathbf{B}_3^m$)**.** A formula φ is $\mathbf{B}_3^m$-satisfiable iff there are a wK-model $M = \langle W, R, D, \sim, V\rangle$, a world $w \in W$ and an assignment a such that $V_{w,a}(\varphi) = 1$. A formula φ is $\mathbf{B}_3^m$-valid iff $V_{w,a}(\varphi) = 1$ for every world w of every wK-model. A formula φ is a $\mathbf{B}_3^m$-consequence of a set of formulae Γ, $\Gamma \vDash^{B_3^m} \varphi$, iff for every wK-model, if $V_{w,a}[\Gamma] = \{1\}$ then $V_{w,a}(\varphi) = 1$. $\mathbf{B}_3^m$ is the logic $\langle \mathscr{L}, \vDash^{B_3^m}\rangle$.

Definition 2.3.0.2 ($\mathbf{PWK}^m$)**.** A formula φ is $\mathbf{PWK}^m$-satisfiable iff there are a wK-model $M = \langle W, R, D, \sim, V\rangle$, a world $w \in W$ and an assignment a such that $V_{w,a}(\varphi) \neq 0$. A formula φ is $\mathbf{PWK}^m$-valid iff $V_{w,a}(\varphi) \neq 0$ for every world w of every wK-model. A formula φ is a $\mathbf{PWK}^m$-consequence of a set of formulae Γ, $\Gamma \vDash^{PWK^m} \varphi$, iff for every wK-model, if $V_{w,a}[\Gamma] \subseteq \{1, 1/2\}$ then $V_{w,a}(\varphi) \neq 0$. $\mathbf{PWK}^m$ is the logic $\langle \mathscr{L}, \vDash^{PWK^m}\rangle$.

$\mathbf{B}_3^m$ and $\mathbf{PWK}^m$ share with their non-modal versions some noticeable failures of classical principles. $\mathbf{B}_3^m$ is non-adjunctive, in the sense that $\varphi \nvDash^{B_3^m} \varphi \vee \psi$, and $\mathbf{PWK}^m$ is non-simplifying, since $\varphi \wedge \psi \nvDash^{PWK^m} \varphi$[13]. These peculiarities make the meaning of $\vee$ in $\mathbf{B}_3^m$ and $\wedge$ in $\mathbf{PWK}^m$ hard to interpret. Nevertheless, for the restricted language $\mathscr{L}_{CL}$ we can reobtain classical logic using a mixed logical consequence relation.

Definition 2.3.0.3 (**wK**)**.** A formula φ is **wK**-consequence of a set of formulae Γ, $\Gamma \vDash^{wK} \varphi$, iff for every wK-model $M = \langle W, R, D, \sim, V\rangle$, if $V_{w,a}[\Gamma] = \{1\}$ then

[13]Cfr. Ciuni & Carrara (2019a).

$V_{w,a}(\varphi) \neq 0$. A formula is **wK**-valid iff it is **PWK**^{m}-valid. **wK** is the logic $\langle \mathscr{L}, \vDash^{wK} \rangle$.

Theorem 2.3.1. *For every* $\Gamma \cup \{\varphi\} \subseteq Form(\mathscr{L}_{CL})$, $\Gamma \vDash^{wK} \varphi$ *iff* $\Gamma \vDash^{CL} \varphi$.

Proof. Similar to theorem 2.2.1. □

In **wK** the properties mentioned for **sK** still hold: both the Barcan schemas are valid, identity is necessary in the form $\forall x, y(x = y \to \Box x = y)$ but this is not the case for intensional variables ($\nvDash^{wK} [\lambda x.x = y](f) \to \Box[\lambda x.x = y](f)$), and similarly Leibniz's law $\forall x, y(x = y \land \varphi(x) \to \varphi[y/x])$ is valid although some of its abstract instances are not ($\nvDash^{wK} [\lambda y.(x = y \land \Diamond Px)](f) \to \Diamond[\lambda y.Py](f)$).

Like **sK**, **wK** is a conservative extension of classical logic, even if the entailment is not classical, more precisely $\vDash^{wK}$ is non-transitive like $\vDash^{sK}$. This feature allows **wK** to account for tolerant inferences while maintaining consistency, for the same reasoning which holds for **sK**.

As a last remark, we notice that all the models introduced have constant domains, hence we are using a possibilist reading of quantification. What exists at some world exists everywhere. In order to move towards a more flexible system we could employ a standard procedure of free logics and add an existence predicate E to the language[14]. Instead of a fixed domain D now the model contains a function $D^E : W \to D$ and quantifiers at a world now ranges over the domain $D^E(w)$ of the world of evaluation w. We add the constraint that for every $w \in W, D^E(w) \neq \emptyset$ and the clause $V_{w,a}(E(t)) = 1$ iff $t \in D^E(w)$, 0 otherwise. The model domain over which the denotation of every term at each world ranges is $D = \bigcup_{w \in W} D^E(w)$.

2.4 Correspondences and extensions

We have introduced two logics which both expand classical logic. We are still left with the question about the relation between these two systems. We are going to prove that **sK** and **wK**, despite the different truth-tables at their bases, individuate the same logic.

Lemma 2.4.0.1. *Let* $M = \langle \mathscr{F}, V \rangle$ *be a wK-model. A strong refinement of* M *is a sK-model* $M^s = \langle \mathscr{F}, V^s \rangle$ *such that for every atomic formula* α *of* $\mathscr{L}$, *if* $V_{w,a}(\alpha) = 1$ *then* $V^s_{w,a}(\alpha) = 1$ *and if* $V_{w,a}(\alpha) = 0$ *then* $V^s_{w,a}(\alpha) = 0$. *It holds that*

[14] Cfr. Bencivenga (1986).

for every $\varphi \in Form(\mathscr{L})$*, if* $V_{w,a}(\varphi) = 1$ *then* $V^s_{w,a}(\varphi) = 1$ *and if* $V_{w,a}(\varphi) = 0$ *then* $V^s_{w,a}(\varphi) = 0$.

Proof. The proof proceeds by induction on the complexity of φ.

(φ is atomic) When we are not considering predicates of the form I_P or $=$, the property holds by definition. For atomic formulae containing I_P or $=$ the property holds since M and M^s share the same frame $\mathscr{F}$.

Let us assume that (i.h.) the property holds for every formula of complexity lower than φ.

($\varphi := \neg\psi$) If $V_{w,a}(\neg\psi) = 1$ then $V_{w,a}(\psi) = 0$, by (i.h.) $V^s_{w,a}(\psi) = 0$, hence $V^s_{w,a}(\neg\psi) = 1$. Similarly for $V_{w,a}(\neg\psi) = 0$.

($\varphi := \psi \vee \chi$) If $V_{w,a}(\psi \vee \chi) = 1$ then $V_{w,a}(\psi) = 1$ or $V_{w,a}(\chi) = 1$. By (i.h.) this implies $V^s_{w,a}(\psi) = 1$ or $V^s_{w,a}(\chi) = 1$, hence $V^s_{w,a}(\psi \vee \chi) = 1$. Similarly for $V_{w,a}(\psi \vee \chi) = 0$.

($\varphi := \forall x\psi$) If $V_{w,a}(\forall x\psi) = 1$ then $V_{w,a'}(\psi) = 1$ for every x-variant a' of a, then by (i.h.) $V^s_{w,a'}(\psi) = 1$ for every x-variant a' of a, so $V^s_{w,a}(\forall x\psi) = 1$. Similarly for $V_{w,a}(\forall x\psi) = 0$.

($\varphi := \Box\psi$) If $V_{w,a}(\Box\psi) = 1$ then $V_{v,a}(\psi) = 1$ for every $v \in R[w]$, then by (i.h.) $V^s_{v,a}(\psi) = 1$ for every $v \in R[w]$, so $V^s_{w,a}(\Box\psi) = 1$. Similarly for $V_{w,a}(\Box\psi) = 0$.

($\varphi := [\lambda x.\psi](f)$) If $V_{w,a}([\lambda x.\psi](f)) = 1$ then $V_{w,a'}(\psi) = 1$ for some a' such that $a'(w,x) = a(w,f)$, then by (i.h.) $V^s_{w,a'}(\psi) = 1$ for some a' such that $a'(w,x) = a(w,f)$, so $V^s_{w,a}([\lambda x.\psi](f)) = 1$. Similarly for $V_{w,a}([\lambda x.\psi](f)) = 0$.

□

Lemma 2.4.0.2. *Let* $M = \langle \mathscr{F}, V\rangle$ *be a sK-model. A weak refinement of* M *is a wK-model* $M^w = \langle \mathscr{F}, V^w\rangle$ *such that for every atomic formula* α *of* $\mathscr{L}$*, if* $V_{w,a}(\alpha) = 1$ *then* $V^w_{w,a}(\alpha) = 1$ *and if* $V_{w,a}(\alpha) = 0$ *then* $V^w_{w,a}(\alpha) = 0$*. It holds that for every* $\varphi \in Form(\mathscr{L})$*, if* $V_{w,a}(\varphi) = 1$ *then* $V^w_{w,a}(\varphi) = 1$ *and if* $V_{w,a}(\varphi) = 0$ *then* $V^w_{w,a}(\varphi) = 0$.

Proof. Contrary to the proof of lemma 2.4.0.1, the current one requires an intermediate step. Let $M^c = \langle \mathscr{F}, V^c\rangle$ be a sK-model such that if $V_{w,a}(\alpha) \geq 1/2$ then

$V^c_{w,a}(\alpha) = 1$ and if $V_{w,a}(\alpha) = 0$ then $V^c_{w,a}(\alpha) = 0$. We can prove by straightforward induction on the complexity of φ that if $V_{w,a}(\varphi) = 1$ then $V^c_{w,a}(\varphi) = 1$ and if $V_{w,a}(\varphi) = 0$ then $V^c_{w,a}(\varphi) = 0$. Moreover by construction all the atoms of M^c have a classical value, hence by semantics every formula receives a classical valuation in M^c. Let now M^w be a weak refinement of M^c.

We can prove by induction on the complexity of φ that if $V^c_{w,a}(\varphi) = 1$ then $V^w_{w,a}(\varphi) = 1$ and if $V^c_{w,a}(\varphi) = 0$ then $V^w_{w,a}(\varphi) = 0$. The induction is immediate since strong and weak Kleene valuations agree when receive as input only classical values.

By transitivity we conclude that M^w is also a refinement of M and that if $V_{w,a}(\varphi) = 1$ then $V^w_{w,a}(\varphi) = 1$, and if $V_{w,a}(\varphi) = 0$ then $V^w_{w,a}(\varphi) = 0$. □

Theorem 2.4.1. ***sK*** $=$ ***wK***

Proof. (**sK** $\subseteq$ **wK**) Let $\Gamma \not\models^{wK} \varphi$, so there is a wK-model $M = \langle \mathscr{F}, V \rangle$ such that $V_{w,a}[\Gamma] = \{1\}$ and $V_{w,a}(\varphi) = 0$. By lemma 2.4.0.1 there is a sK-model $M^s = \langle \mathscr{F}, V^s \rangle$ such that $V^s_{w,a}[\Gamma] = \{1\}$ and $V^s_{w,a}(\varphi) = 0$, so $\Gamma \not\models^{sK} \varphi$.

(**wK** $\subseteq$ **sK**) Let $\Gamma \not\models^{sK} \varphi$, so there is a sK-model $M = \langle \mathscr{F}, V \rangle$ such that $V_{w,a}[\Gamma] = \{1\}$ and $V_{w,a}(\varphi) = 0$. By lemma 2.4.0.2 there is a wK-model $M^w = \langle \mathscr{F}, V^w \rangle$ such that $V^w_{w,a}[\Gamma] = \{1\}$ and $V^w_{w,a}(\varphi) = 0$, so $\Gamma \not\models^{wK} \varphi$. □

Now that we have obtained the equivalence between **sK** and **wK**, the next step is to enrich the object language. With this move we will be able to introduce two new and non-equivalent logics.

Let $\mathscr{L}_E$ be $\mathscr{L}$ extended with the sentential operators Δ_d, Δ_c, which in the following will be called external operators[15]. The vocabulary of $\mathscr{L}_E$ contains all the formulae of $\mathscr{L}$ plus formulae of the form $\Delta_d\varphi$ and $\Delta_c\varphi$. Δ_d is a determinacy operator, which returns 1 if its argument is 1 and 0 otherwise, Δ_c is a classicality or meaningfulness operator, which returns 1 if its argument has a classical truth-value and 0 otherwise.

An *external strong Kleene model* (sK_E-model) is a structure $\langle \mathscr{F}, V \rangle$, where $\mathscr{F}$ is a tolerant Kripke frame and V is a valuation identical to that of an sK-model, with the addition of the recursive clauses:

- $V_{w,a}(\Delta_d\varphi) = 1$ if $V_{w,a}(\varphi) = 1$, $V_{w,a}(\Delta_d\varphi) = 0$ otherwise.

[15]In Bochvar (1938) there is in fact a distinction between an internal and an external system. Cfr. Malinowski (2007), pp.24-25, and Bergmann (2008), pp.80-84.

- $V_{w,a}(\Delta_c\varphi) = 1$ if $V_{w,a}(\varphi) \in \{1, 0\}$, $V_{w,a}(\Delta_c\varphi) = 0$ otherwise.

An *external weak Kleene model* (wK_E-model) is defined similarly, starting with a wK-model and adding to the valuation the above clauses.

Employing again a mixed entailment, we define the following extensions of **sK** and **wK**.

Definition 2.4.1.1 ($\mathbf{sK}_E$ and $\mathbf{wK}_E$). A formula φ is a sK_E-consequence of a set of formulae Γ, $\Gamma \vDash^{sK_E} \varphi$, iff for every sK_E-model $M = \langle \mathscr{F}, V \rangle$, if $V_{w,a}[\Gamma] = \{1\}$ then $V_{w,a}(\varphi) \neq 0$. $\mathbf{sK}_E$ is the logic $\langle \mathscr{L}_E, \vDash^{sK_E} \rangle$. The logic $\mathbf{wK}_E$ is defined similarly with respect to wK_E-models.

When it does not matter which logic we are considering we will write $\mathbf{iK}_E$ and iK_E-model. $\mathbf{sK}_E$ and $\mathbf{wK}_E$ are obviously conservative extensions of, respectively, **sK** and **wK**, therefore they preserve classical logic too.

With the semantics clauses for external operators, their previous denominations become now perspicuous. Δ_d is a determinacy operator in the sense that marks as true only what is definitely true, i.e. classically true. This operator is originally used, in a non-modal setting and with different notation, by Bochvar (1938), and allows within the logic $\mathbf{B}_3$ to recapture all classical logic[16]. The classicality or normality[17] operator Δ_c tell us instead when a formula has a classical truth-value or not. In a strong Kleene setting, Δ_c indicates whether there is enough information to settle the classical truth-value of a sentence, in weak Kleene instead Δ_c states whether a sentence is meaningful (or grammatical) or not. In this latter sense a similar operator is employed by Halldén (1949)[18].

From the external operators we can define their intensional variants, $\Delta_d^{\uparrow} := \Box\Delta_d$ and $\Delta_d^{\uparrow} := \Box\Delta_c$[19]. These derivative operators add an intensional component

[16] A proof of this fact is in Bergmann (2008), pp.82-84.

[17] Cfr. Ciuni & Carrara (2019b).

[18] Cfr. Williamson (1994), pp.103-105.

[19] A more fine-grained semantics can be developed, taking $\Delta_d^{\uparrow}$ and $\Delta_c^{\uparrow}$ as primitive and associating to each of them an accessibility relation. An extended Kripke frame is a structure $\mathscr{F}^+ = \langle W, R_\Box, R_d, R_c, D, \sim \rangle$. $\mathscr{F}^+$ is like a tolerant Kripke frame, with the following additions: $R_\Box$ is the accessibility relation for $\Box$, R_d that for $\Delta_d^{\uparrow}$, and R_c that for $\Delta_c^{\uparrow}$. The semantic clause for $\Box$ is as usual, those of the remaining operators are:

- $V_{w,a}(\Delta_d^{\uparrow}\varphi) = min\{V_{v,a}(\Delta_d\varphi) \mid v \in R_d[w]\}$
- $V_{w,a}(\Delta_c^{\uparrow}\varphi) = min\{V_{v,a}(\Delta_c\varphi) \mid v \in R_c[w]\}$

It follows immediately that when $R_\Box = R_d = R_c$ we have $V_{w,a}(\Delta_d^{\uparrow}) = V_{w,a}(\Box\Delta_d)$ and $V_{w,a}(\Delta_c^{\uparrow}) =$

to the previous ones. $\Delta_d^\uparrow$ expresses intensional determinacy, stating whether something is definitely true across all the accessible worlds, while $\Delta_c^\uparrow$ states whether something is classically determined at every accessible world, so on the converse it allows us to express when a sentence is intensionally undetermined or vague, i.e. when it is vague at some accessible world.

Fact 2.4.1.1.

- *When M is a sK_E-model, $V_{w,a}(\Delta_d\varphi) = V_{w,a}(\Delta_c\varphi \wedge \varphi)$ for every $w \in W$, hence $\Delta_d\varphi$ can be defined in $\boldsymbol{sK}_E$ as $\Delta_d\varphi := \Delta_c\varphi \wedge \varphi$. This is not the case for $\boldsymbol{wK}_E$ (just consider a model where $V_{w,a}(\varphi) = 1/2$).*
- *A unary operator $\circ$ is called normal in a logic $\boldsymbol{L}$ when $\circ(\varphi \to \psi) \to (\circ\varphi \to \circ\psi)$ is $\boldsymbol{L}$-valid. Δ_d is normal in both $\boldsymbol{sK}_E$ and $\boldsymbol{wK}_E$. Δ_c is normal in $\boldsymbol{wK}_E$ but not in $\boldsymbol{sK}_E$.*

The introduction of the external operators marks the divergence between logics based on strong and weak Kleene semantics. The logics $\mathbf{sK}_E$ and $\mathbf{wK}_E$ are indeed different, as we show.

Theorem 2.4.2. *Neither $\boldsymbol{sK}_E \subseteq \boldsymbol{wK}_E$ nor $\boldsymbol{wK}_E \subseteq \boldsymbol{sK}_E$.*

Proof. ($\mathbf{sK}_E \nsubseteq \mathbf{wK}_E$) Consider the formula $\Delta_c(\Delta_d\varphi \leftrightarrow (\Delta_c\varphi \wedge \varphi))$. As stated above, $\Delta_d\varphi$ and $\Delta_c\varphi \wedge \varphi$ always receive the same strong evaluation, hence $V_{w,a}(\Delta_d\varphi \leftrightarrow (\Delta_c\varphi \wedge \varphi)) = 1$ for every sK_E-model, and so $V_{w,a}(\Delta_c(\Delta_d\varphi \leftrightarrow (\Delta_c\varphi\wedge\varphi))) = 1$, the formula is $\mathbf{sK}_E$-valid. On the contrary, while $\Delta_d\varphi \leftrightarrow (\Delta_c\varphi\wedge\varphi)$ is also $\mathbf{wK}_E$-valid, this is not the case for $\Delta_c(\Delta_d\varphi \leftrightarrow (\Delta_c\varphi \wedge \varphi))$, in fact consider a model where $V_{w,a}(\varphi) = 1/2$, then $V_{w,a}(\Delta_d\varphi \leftrightarrow (\Delta_c\varphi \wedge \varphi))) = 1/2$, which implies $V_{w,a}(\Delta_c(\Delta_d\varphi \leftrightarrow (\Delta_c\varphi \wedge \varphi))) = 0$.

($\mathbf{wK}_E \nsubseteq \mathbf{sK}_E$) $\neg\Delta_c\varphi \vDash^{wK_E} \varphi \wedge \psi$, in fact for any wK_E-model when the antecedent is true then $V_{w,a}(\varphi) = 1/2$, which by the contamination principle implies $V_{w,a}(\varphi \wedge \psi) = 1/2$. This doesn't hold in general for $\mathbf{sK}_E$, in fact when $V_{a,w}(\psi) = 0$ then $V_{w,a}(\varphi \wedge \psi) = 0$, hence $\neg\Delta_c\varphi \nvDash^{sK_E} \varphi \wedge \psi$. □

$V_{w,a}(\Box\Delta_c)$.

In this way obtain the multi-modal logics $\mathbf{sK}_E^+$ and $\mathbf{wK}_E^+$. The philosophical question left open is how to justify the different accessibility relations. E.g. if $R_\Box$ is assumed to stand for metaphysical possibility, then it seems safe to assume that $R_c \subseteq R_\Box$, while under an epistemic interpretation of $R_\Box$ we could have cases in which $v \in R_c[w]$ but $v \notin R_\Box[w]$, because the implicit agent at w may consider meaningfulness a linguistic feature which should be evaluated even outside the scope of epistemic possibility. This is an example of the problems that need to be answered if we want to endorse this semantics.

2.5 Proof systems

Many logics have been introduced in the previous paragraphs. Here we are going to provide the proof theory in the form of refutation trees for the strongest systems, $\mathbf{sK}_E$ and $\mathbf{wK}_E$. The completeness of these proof systems follows the technique of Priest (2008).

2.5.1 sK_E-trees

A node for a sK_E-tree is a string either of the form wRv or of the form $w : \varphi, i$, where w is the prefix, $\varphi \in Form(\mathscr{L}_E)$, and $i \in \{s, t\}$ is the label. s stands for strictly true and it corresponds to the formula being evaluated exactly as 1, t means tolerantly true and it corresponds to the formula being at least $1/2$.

A sK_E-tree is a poset of nodes with a minimal element. The definition of branch is as usual. Each node is obtained by the application of one of the following rules to previous nodes on the same branch. When a node is labelled as i it is indifferent which value i takes, but in the same rule all the occurrences of i must take the same value.

$$(\sim\text{P})\ \frac{\begin{array}{c} w : Px, s \\ w : I_P xy, s \end{array}}{w : Py, t} \qquad (\sim\text{N})\ \frac{\begin{array}{c} w : \neg Px, s \\ w : I_P xy, s \end{array}}{w : \neg Py, t}$$

$$(\sim\text{r})\ \frac{}{w : I_P xx, s} \qquad (\sim\text{s})\ \frac{w : I_P xy, s}{w : I_P yx, s}$$

$$(=\text{P})\ \frac{\begin{array}{c} w : x = y, s \\ w : Px_1 ... x ... x_n, i \end{array}}{w : Px_1 ... y ... x_n, i} \qquad (=\text{N})\ \frac{\begin{array}{c} w : x = y, s \\ w : \neg Px_1 ... x ... x_n, i \end{array}}{w : \neg Px_1 ... y ... x_n, i}$$

($P(x)$ is any atomic formula) ($P(x)$ is any atomic formula)

$$(=\text{r})\ \frac{}{w : x = x, s}$$

$$(\vee\text{T})\ \frac{w : \varphi \vee \psi, i}{w : \varphi, i \qquad w : \psi, i} \qquad (\vee\text{F})\ \frac{w : \neg(\varphi \vee \psi), i}{\begin{array}{c} w : \neg\varphi, i \\ w : \neg\psi, i \end{array}}$$

$$(\forall T)\ \frac{w : \forall x\varphi, i}{w : \varphi[y/x], i} \quad \text{(y already appears on the branch)} \qquad (\forall F)\ \frac{w : \neg\forall x\varphi, i}{w : \neg\varphi[c/x], i} \quad \text{(c is new on the branch)}$$

$$(\Box T)\ \begin{array}{c} w : \Box\varphi, i \\ \underline{wRv} \\ v : \varphi, i \end{array} \qquad (\Box F)\ \frac{w : \neg\Box\varphi, i}{\begin{array}{c} wRv \\ v : \neg\varphi, i \end{array}} \quad \text{(v is new on the branch)}$$

$$(\lambda T)\ \frac{w : [\lambda x.\varphi](f), i}{w : \varphi[c_{f,w}/x], i} \quad \text{($c_{f,w}$ is new on the branch)} \qquad (\lambda F)\ \frac{w : \neg[\lambda x.\varphi](f), i}{w : \neg\varphi[c_{f,w}/x], i} \quad \text{($c_{f,w}$ is new on the branch)}$$

$$(\Delta_d T)\ \frac{w : \Delta_d\varphi, s}{w : \varphi, s} \qquad (\Delta_d F)\ \frac{w : \neg\Delta_d\varphi, s}{w : \neg\varphi, t}$$

$$(\Delta_c T)\ \frac{w : \Delta_c\varphi, s}{w : \varphi, s \quad\quad w : \neg\varphi, s} \qquad (\Delta_c F)\ \frac{w : \neg\Delta_c\varphi, s}{\begin{array}{c} w : \varphi, t \\ w : \neg\varphi, t \end{array}}$$

For every formula φ of the form $x{=}y$, $\neg x{=}y$, $I_P xy$, $\neg I_P xy$, $\Delta_d\psi$, $\neg\Delta_d\psi$, $\Delta_c\psi$, $\neg\Delta_c\psi$, we have the following rule:

$$(t \Rightarrow s)\ \frac{w : \varphi, t}{w : \varphi, s}$$

A branch is closed (marked by $\perp$) when it contains one of the following nodes:

$$\begin{array}{c} w : \varphi, s \\ w : \neg\varphi, s \\ \perp \end{array} \qquad \begin{array}{c} w : \varphi, s \\ w : \neg\varphi, t \\ \perp \end{array} \qquad \begin{array}{c} w : \neg\varphi, s \\ w : \varphi, t \\ \perp \end{array}$$

A branch is open if it is not closed, it is complete if at every node has been applied the corresponding rule. A tree is complete if all its branches are complete, it is closed if all its branches are closed, and it is open if it has at least one open branch.

Definition 2.5.0.1. $\Gamma \vdash^{sK_E} \varphi$ iff every complete open sK_E-tree starting with the nodes $w : \gamma, s$ for each $\gamma \in \Gamma$ and $w, \neg\varphi, s$ is closed.

Theorem 2.5.1. *sK_E-trees are sound with respect to $\boldsymbol{sK}_E$.*

The proof is a routinely exercise, it is sufficient to check that a valuation which makes every node of a branch b true still makes true the branch b' which is obtained from b applying any rule.

We prove the completeness of sK_E-trees by the contrapositive, showing that from an open branch of a complete tree we can build a model for the formulae occurring on the branch, in particular for the formulae of the root.

Let us preliminary define the relation $\simeq \subseteq Var(\mathscr{L}_E)^2$ such that given an open branch b of a complete sK_E-tree, $x \simeq y$ iff $w : x = y, s \in b$. In order to account for terms of the form $c_{f,w}$, which are introduced by rules (λT) and (λF), we add countably infinitely many of them to $Var(\mathscr{L}_E)$. The rule (=r) explicitly states the reflexivity of =, while symmetry and transitivity are easily derivable with the rule (=P), hence = is an equivalence relation, and so is $\simeq$. Let $[x]_\simeq := \{y \mid x \simeq y\}$.

Definition 2.5.1.1. Let b be an open branch of a complete sK_E-tree. The sK_E-model $M^b = \langle W, R, D, \sim, V\rangle$ and assignment a induced by b are defined as follows:

- $W = \{w \mid w$ is the prefix of some node of $b\}$
- $R = \{\langle w, v\rangle \mid wRv \in b\}$
- $D = \{[x]_\simeq \mid x$ is free in some node of $b\}$
- $a(w, x) = [x]_\simeq$ for x individual variable of any form; $a(w, f) = [c_{f,w}]_\simeq$ for f intensional variable.
- $\sim_{P,w} = \{\langle [x]_\simeq, [y]_\simeq\rangle \mid w : I_P xy, s \in b\}$
- V is such that for every atomic formula α not containing any predicate I_P or =, if $w : \alpha, s \in b$ then $V_{w,a}(\alpha) = 1$, if $w : \neg\alpha, s \in b$ then $V_{w,a}(\alpha) = 0$, if $w : \alpha, t \in b$ then $V_{w,a}(\alpha) \geq 1/2$, if $w : \neg\alpha, t \in b$ then $V_{w,a}(\alpha) \leq 1/2$.

The model M^b is indeed a sK_E-model, in fact $\sim_{P,w}$ is a reflexive and symmetric relation, due to ($\sim$r) and ($\sim$s), and V satisfies the closeness property: by construction $\langle [x]_\simeq, [y]_\simeq\rangle \in \sim_{P,w}$ iff $w : I_P xy, s \in b$, and if $w : Px, s \in b$ then by ($\sim$P) $w : Py, t \in b$, hence $V_{w,a}(Px) = 1$ and $V_{w,a}(Py) \neq 0$. Similarly for $w : \neg Px, s$, using ($\sim$N).

Lemma 2.5.1.1. *The model M^b and assignment a induced by b is such that if $w : \varphi, s \in b$ then $V_{w,a}(\varphi) = 1$, if $w : \neg\varphi, s \in b$ then $V_{w,a}(\neg\varphi) = 0$, if $w : \varphi, t \in b$ then $V_{w,a}(\varphi) \geq 1/2$, if $w : \neg\varphi, t \in b$ then $V_{w,a}(\neg\varphi) \leq 1/2$.*

Proof. The proof proceeds by induction on the complexity of φ.

- (φ is atomic or a negated atomic formula) If φ does not contain any predicate I_P or $=$, then the property holds by construction of V.

  Let $\varphi := I_Pxy$: if $w : I_Pxy, s \in b$ then $\langle [x]_\simeq, [y]_\simeq \rangle \in \sim_{P,w}$, hence $V_{w,a}(I_Pxy) = 1$; if $w : I_Pxy, t \in b$ then by $(t \Rightarrow s)$ also $w : I_Pxy, s \in b$, so $V_{w,a}(I_Pxy) \geq 1/2$; if $w : \neg I_Pxy, s \in b$ then, since b is complete, $w : I_Pxy, s \notin b$, hence $\langle [x]_\simeq, [y]_\simeq \rangle \notin \sim_{P,w}$ and $V_{w,a}(I_Pxy) = 0$; if $w : \neg I_Pxy, t \in b$ then by $(t \Rightarrow s)$ also $w : \neg I_Pxy, s \in b$, so $V_{w,a}(I_Pxy) \leq 1/2$.

  Let $\varphi := x = y$: if $w : x = y, s \in b$ then $x \simeq y$, hence $a(w, x) = [x]_\simeq = [y]_\simeq = a(w, y)$, so $V_{w,a}(x = y) = 1$; if $w : x = y, t \in b$ then by $(t \Rightarrow s)$ also $w : x = y, s \in b$ and $V_{w,a}(x = y) \geq 1/2$; if $w : \neg x = y, s \in b$ then $w : x \simeq y, s \notin b$, so $[x]_\simeq \neq [y]_\simeq$ and $V_{w,a}(x = y) = 0$; if $w : \neg x = y, t \in b$ then by $(t \Rightarrow s)$ also $w : \neg x = y, s \in b$ and $V_{w,a}(x = y) \leq 1/2$.

Let us assume that (i.h.) the property holds for every formula of complexity lower than φ.

- ($\varphi := \psi \vee \chi$ or $\varphi := \neg(\psi \vee \chi)$) If $w : \psi \vee \chi, s \in b$ then by ($\vee$T) either $w : \psi, s \in b$ or $w : \chi, s \in b$, then by (i.h.) $V_{w,a}(\psi) = 1$ or $V_{w,a}(\chi) = 1$, hence $V_{w,a}(\psi \vee \chi) = 1$. If $w : \neg(\psi \vee \chi), s \in b$ then by ($\vee$F) both $w : \neg\psi, s \in b$ and $w : \neg\chi, s \in b$, then by (i.h.) $V_{w,a}(\psi) = 0$ and $V_{w,a}(\chi) = 0$, hence $V_{w,a}(\psi \vee \chi) = 0$. The remaining cases are similar.

- ($\varphi := \forall x\psi$ or $\varphi := \neg\forall x\psi$) If $w : \forall x\psi, s \in b$ then by ($\forall$T) $w : \psi[y/x], s \in b$ for each y appearing in b, so by (i.h.) $V_{w,a}(\psi[y/x]) = 1$ and since by construction $D = \{[y]_\simeq \mid y \text{ is free in } b\}$ then $V_{w,a}(\forall x\psi) = 1$. If $w : \neg\forall x\psi, s \in b$ then by ($\forall$F) $w : \neg\psi[c/x], s \in b$, so by (i.h.) $V_{w,a}(\psi[c/x]) = 0$ and $V_{w,a}(\forall x\psi) = 0$. The remaining cases are similar.

- ($\varphi := \Box\psi$ or $\varphi := \neg\Box\psi$) If $w : \Box\psi, s \in b$ then by ($\Box$T) $v : \psi, s \in b$ for every v such that $wRv \in b$, then by (i.h.) $V_{v,a}(\psi) = 1$ for all $v \in R[w]$, since by construction $W = \{w \mid w \text{ is the prefix of some node of } b\}$ and $R = \{\langle w, v\rangle \mid wRv \in b\}$, therefore $V_{w,a}(\Box\psi) = 1$. If $w : \neg\Box\psi, s \in b$ then by ($\Box$F) $wRv \in b$ and $v : \psi, s \in b$, hence by (i.h.) $V_{v,a}(\psi) = 0$ and $V_{w,a}(\Box\psi) = 0$. The remaining cases are similar.

- ($\varphi := [\lambda x.\psi](f)$ or $\varphi := \neg[\lambda x.\psi](f)$) If $w : [\lambda x.\psi](f), s \in b$ then by (λT) $w : \psi[c_{f,w}/x], s \in b$, then by (i.h.) $V_{w,a}(\psi[c_{f,w}/x]) = 1$. Now by construction

$a(w,f) = [c_{f,w}]_{\simeq}$ and $a(w,c_{f,w}) = [c_{f,w}]_{\simeq}$, so an assignment a' which differs from a at most for $a(w,f) = a(w,c_{f,w})$ is a itself, therefore $V_{w,a}([\lambda x.\psi](f)) = 1$. The remaining cases are similar.

- ($\varphi := \Delta_d\psi$ or $\varphi := \neg\Delta_d\psi$) If $w : \Delta_d\psi, s \in b$ then by (Δ_dT) $w : \psi, s \in b$, then by (i.h.) $V_{w,a}(\psi) = 1$, therefore $V_{w,a}(\Delta_d\psi) = 1$. If $w : \neg\Delta_d\psi, s \in b$ then by (Δ_dF) $w : \neg\psi, t \in b$, then by (i.h.) $V_{w,a}(\psi) \leq 1/2$, therefore $V_{w,a}(\Delta_d\psi) = 0$. The cases labelled by t are subsumed by ($t \Rightarrow s$) under the ones for s.

- ($\varphi := \Delta_c\psi$ or $\varphi := \neg\Delta_c\psi$) If $w : \Delta_c\psi, s \in b$ then by (Δ_cT) either $w : \psi, s \in b$ or $w : \neg\psi, s \in b$, then by (i.h.) either $V_{w,a}(\psi) = 1$ or $V_{w,a}(\psi) = 0$, therefore $V_{w,a}(\Delta_c\psi) = 1$. If $w : \neg\Delta_c\psi, s \in b$ then by (Δ_cF) $w : \psi, t \in b$ and $w : \neg\psi, t \in b$, then by (i.h.) $V_{w,a}(\psi) \geq 1/2$ and $V_{w,a}(\psi) \leq 1/2$, which amounts to $V_{w,a}(\psi) = 1/2$, therefore $V_{w,a}(\Delta_c\psi) = 0$. The cases labelled by t are subsumed by ($t \Rightarrow s$) under the ones for s.

□

Theorem 2.5.2. *sK_E-trees are complete with respect to $\boldsymbol{sK}_E$.*

Proof. Let $\Gamma \not\models^{sK_E} \varphi$, so there exists a complete open sK_E-tree starting with $w : \gamma, s$ for each $\gamma \in \Gamma$ and $w, \neg\varphi, s$. Let b be an open branch of the tree. By lemma 2.5.1.1, the model $M^b = \langle \mathscr{F}, V \rangle$ and assignment a induced by b are such that $V_{w,a}[\Gamma] = \{1\}$ and $V_{w,a}(\varphi) = 0$, therefore $\Gamma \not\models^{sK_E} \varphi$. □

2.5.2 $\boldsymbol{wK}_E$-trees

A node for a wK_E-tree has the same form of node for a sK_E-tree, with the exception that the label is $i \in \{s, n\}$. n stands for nonsensical and it corresponds to the non-classical truth-value $1/2$.

The nodes of a wK_E-tree are obtained by the application of one of the following rules to previous nodes on the same branch.

$$(\sim\text{P})\ \frac{\begin{array}{c} w : Px, s \\ w : I_P xy, s \end{array}}{w : Py, s \qquad w : Py, n} \qquad (\sim\text{N})\ \frac{\begin{array}{c} w : \neg Px, s \\ w : I_P xy, s \end{array}}{w : \neg Py, s \qquad w : \neg Py, n}$$

$$(\sim\text{r})\ \frac{}{w : I_P xx, s} \qquad (\sim\text{s})\ \frac{w : I_P xy, s}{w : I_P yx, s}$$

$$(=\mathrm{P})\ \frac{\begin{array}{c} w : x = y, s \\ w : Px_1...x...x_n, i \end{array}}{w : Px_1...y...x_n, i}$$
($P(x)$ is an atomic formula)

$$(=\mathrm{N})\ \frac{\begin{array}{c} w : x = y, s \\ w : \neg Px_1...x...x_n, i \end{array}}{w : \neg Px_1...y...x_n, i}$$
($P(x)$ is an atomic formula)

$$(=\mathrm{r})\ \frac{}{w : x = x, s} \qquad (\neg n)\ \frac{w : \neg\varphi, n}{w : \varphi, n}$$

$$(\vee\mathrm{T})\ \frac{w : \varphi \vee \psi, s}{\begin{array}{cc} w : \varphi, s & w : \psi, s \\ w : \Delta_c\psi, s & w : \Delta_c\varphi, s \end{array}} \qquad (\vee\mathrm{F})\ \frac{w : \neg(\varphi \vee \psi), s}{\begin{array}{c} w : \neg\varphi, s \\ w : \neg\psi, s \end{array}}$$

$$(\vee n)\ \frac{w : \varphi \vee \psi, n}{w : \varphi, n \qquad w : \psi, n}$$

$$(\forall\mathrm{T})\ \frac{w : \forall x\varphi, s}{w : \varphi[y/x], s}$$
(y already appears on the branch)

$$(\forall\mathrm{F})\ \frac{w : \neg\forall x\varphi, s}{\begin{array}{c} w : \neg\varphi[c/x], s \\ w : \forall x\Delta_c\varphi, s \end{array}}$$
(c is new on the branch)

$$(\forall n)\ \frac{w : \forall x\varphi, n}{w : \varphi[c/x], n}$$
(c is new on the branch)

$$(\Box\mathrm{T})\ \frac{\begin{array}{c} w : \Box\varphi, s \\ wRv \end{array}}{v : \varphi, s}$$

$$(\Box\mathrm{F})\ \frac{\begin{array}{c} w : \neg\Box\varphi, s \\ wRu \end{array}}{\begin{array}{c} wRv \\ v : \neg\varphi, s \\ u : \Delta_c\varphi, s \end{array}}$$
(v is new on the branch)

$$(\Box n)\ \frac{w : \Box\varphi, n}{\begin{array}{c} wRv \\ v : \varphi, n \end{array}}$$
(v is new on the branch)

$$(\lambda\mathrm{T})\ \frac{w : [\lambda x.\varphi](f), i}{w : \varphi[c_f/x], i}$$
(c_f is new on the branch)

$$(\lambda\mathrm{F})\ \frac{w : \neg[\lambda x.\varphi](f), s}{w : \neg\varphi[c_f/x], s}$$
(c_f is new on the branch)

$$(\Delta_d\mathrm{T})\ \frac{w : \Delta_d\varphi, s}{w : \varphi, s} \qquad (\Delta_d\mathrm{F})\ \frac{w : \neg\Delta_d\varphi, s}{w : \neg\varphi, s \qquad w : \varphi, n}$$

$$(\Delta_c\mathrm{T})\ \frac{w : \Delta_c\varphi, s}{w : \varphi, s \qquad w : \neg\varphi, s} \qquad (\Delta_c\mathrm{F})\ \frac{w : \neg\Delta_c\varphi, s}{w : \varphi, n}$$

A branch is closed (marked by $\bot$) when it contains the following nodes:

$$\begin{array}{ccc} w:\varphi,s & w:\varphi,s & w:\neg\varphi,s \\ w:\neg\varphi,s & w:\varphi,n & w:\varphi,n \\ \bot & \bot & \bot \end{array}$$

A branch is closed also when it contains the following node for φ of the form $x{=}y$, $I_P xy$, $\Delta_d\psi, \Delta_c\psi$:

$$\begin{array}{c} w:\varphi,n \\ \bot \end{array}$$

Definition 2.5.2.1. $\Gamma \vdash^{wK_E} \varphi$ iff every complete open wK_E-tree starting with the nodes $w:\gamma, s$ for each $\gamma \in \Gamma$ and $w, \neg\varphi, s$ is closed.

Theorem 2.5.3. *wK_E-trees are sound with respect to $\boldsymbol{wK}_E$.*

As before, the proof is a routinely exercise.

Definition 2.5.3.1. Let b be an open branch of a complete wK_E-tree. The wK_E-model $M^b = \langle W, R, D, \sim, V\rangle$ and assignment a induced by b is defined as in definition 2.5.1.1, with the following exception:

- V is such that for every atomic formula A not containing any predicate I_P or $=$, if $w : A, s \in b$ then $V_{w,a}(A) = 1$, if $w : \neg A, s \in b$ then $V_{w,a}(A) = 0$, if $w : A, n \in b$ then $V_{w,a}(A) = 1/2$.

The model M^b is a wK_E-model for the same reasoning as before, namely the rules force $\sim_{P,w}$ to be a reflexive and symmetric relation and V satisfies the closeness property.

Lemma 2.5.3.1. *The model M^b and assignment a induced by b is such that if $w : \varphi, s \in b$ then $V_{w,a}(\varphi) = 1$, if $w : \neg\varphi, s \in b$ then $V_{w,a}(\neg\varphi) = 0$, if $w : A, n \in b$ then $V_{w,a}(A) = 1/2$.*

Proof. The proof proceeds by induction on the complexity of φ. Many cases are similar to those in lemma 2.5.1.1. Moreover all the cases for $w : \neg\varphi, n \in b$ are reduced by $(\neg n)$ to $w : \varphi, n \in b$ for arbitrary φ, and the cases $w : \varphi, n \in b$ for φ of the form $x{=}y, I_P xy$, $\Delta_d\psi, \Delta_c\psi$ should not be considered, since the branch would be closed by the occurrence of that node. The only interesting cases to be considered are the following:

- ($\varphi := \neg\forall x\psi$) From $w : \neg\forall x\psi, s \in b$ follows $w : \neg\psi[c/x], s \in b$ and $w : \forall x\Delta_c\psi, s \in b$ by ($\forall$F), hence by ($\forall$T) and (Δ_cT) either $w : \psi[y/x], s \in b$ or $w : \neg\psi[y/x], s \in b$ for each y appearing free in b, so by (i.h.) $V_{w,a}(\psi[c/x]) = 0$, and $V_{w,a}(\psi[y/x]) = 1$ or $V_{w,a}(\psi[y/x]) = 0$ for each $[y]_\simeq \in D$, which amounts to $V_{w,a}(\forall x\psi) = 0$.
- ($\varphi := \neg\Box\psi$) From $w : \Box\psi, s \in b$ follows $wRv \in b$ and $v : \psi, s \in b$ by ($\Box$F), moreover it imples that for every u such that $wRu \in b$ we have $u : \Delta_c\psi, s \in b$, hence by ($\Delta_c$T) either $u : \psi, s \in b$ or $u : \neg\psi, s \in b$. By (i.h.) $V_{v,a}(\psi) = 0$ and for all $u \in R[w]$ either $V_{v,a}(\psi) = 1$ or $V_{v,a}(\psi) = 0$, since by construction $W = \{w \mid w \text{ is the prefix of some node of } b\}$ and $R = \{\langle w, v\rangle \mid wRv \in b\}$, therefore $V_{w,a}(\Box\psi) = 0$.

□

Theorem 2.5.4. *wK_E-trees are complete with respect to* $\boldsymbol{wK}_E$.

Proof. Let $\Gamma \nvDash^{wK_E} \varphi$, so there exists a complete open wK_E-tree starting with $w : \gamma, s$ for each $\gamma \in \Gamma$ and $w, \neg\varphi, s$. Let b be an open branch of the tree. By lemma 2.5.3.1, the model $M^b = \langle \mathscr{F}, V\rangle$ and assignment a induced by b is such that $V_{w,a}[\Gamma] = \{1\}$ and $V_{w,a}(\varphi) = 0$, hence $\Gamma \nvDash^{wK_E} \varphi$. □

2.6 Vague identity

The previous logics cannot account for vague identity statements. The object language $\mathscr{L}_E$ can express that something is vaguely identical to something else with $\neg\Delta_c x = y$, but this formula is unsatisfiable in any of the previous logics. The semantic clauses only allows for sharp distinctions between true and false identity. Vague identity is explosive in those systems, in fact $\neg\Delta_c x = y \vDash^{iK_E} \psi$ for arbitrary ψ.

In the following we are going to introduce $\mathbf{iK}^{3v}$, extensions of **iK** logics which thanks to multiple denoting assignment functions are able to deal with vague identity. After, we will move to $\mathbf{iK}^{2v}$, a deviation from the previous systems which employs a modified language and are able to work with vague identity too. Finally we will recapitulate the relations between all the logics introduced, ordering their hierarchy.

Multiple denoting assignments[20] characterize a noticeable difference between

[20]A thorough study of multiple denoting terms is provided, in the setting of the logic of intentionality, by Priest (2016), ch.8.

individual and intensional variables. The following semantics make use of assignments that associate to intensional variables subsets of objects of the domain. On the previous account, an intension differs from extensional objects (i.e. members of the domain) only because it is a sum of the objects that it picks throughout the possible worlds. Introducing a multiple denoting assignment for intensions, the denotation of an intensional variable at a world becomes a potentially new object, something that cannot be found inside the domain as a single entity, but a collection of extensional objects. This distinction underlies interesting ontological differences, which will be extensively treated in chapter 3.

2.6.1 iK^{3v} and extensions

A sK^{3v}-model is a structure $\langle \mathscr{F}, V \rangle$ built exactly as a sK-model. The change here regards assignments.

Definition 2.6.0.1 (General and admissible assignments). A *general assignment* is a relation $g \subseteq W \times \mathit{Var}(\mathscr{L}) \times D$. Given a model M and an assignment g, we denote the set of $d \in D$ which are denotations of a variable k at w as $g[w,k] := \{d \in D \mid \langle w,k,d \rangle \in g\}$. A general assignment is *admissible* when it is functional with respect to individual variables, i.e. such that there is exactly one $d \in D$ such that $d \in a[w,x]$ for each individual variable x.

For a general assignment g, the strong valuation V is extended recursively as follows:

- $V_{w,g}(Px_1...x_n) = 1$ if $V_w(P)(d_1,...d_n) = 1$ for each $d_1 \in g[w,x_1],...d_n \in g[w,x_n]$; $V_{w,g}(Px_1...x_n) = 0$ if $V_w(P)(d_1,...d_n) = 0$ for each $d_1 \in g[w,x_1],...d_n \in g[w,x_n]$; $V_{w,g}(Px_1...x_n) = 1/2$ otherwise.
- $V_{w,g}(x = y) = 1$ if $g[w,x] = g[w,y]$; $V_{w,g}(x = y) = 0$ if $g[w,x] \cap g[w,y] = \emptyset$; $V_{w,g}(x = y) = 1/2$ otherwise.
- $V_{w,g}(I_P xy) = 1$ if $d \sim_{P,w} d'$ for each $d \in g[w,x_1], d' \in g'[w,y_1]$; $V_{w,g}(I_P xy) = 0$ if $d \nsim_{P,w} d'$ for each $d \in g[w,x_1], d' \in g'[w,y_1]$; $V_{w,g}(I_P xy) = 1/2$ otherwise.
- $V_{w,g}([\lambda x.\varphi](f)) = V_{w,g'}(\varphi)$ where g' is a *general* assignment like g except that $g'[w,x] = g[w,f]$.

It is easy to see that sK^{3v}-models are generalizations of sK-models, where the latter are special cases of sK^{3v}-models where the assignments are considered functional with respect to every type of variable. We call these assignments *standard*[21].

A remarkable feature of the above semantics is that now we can satisfy vague identity statements. The clause for identity read that relation in set-theoretic terms, interpreting the truth-value of identity sentences as the degree of overlap between the denotations of the two variables[22]. Complete coincidence amounts to true identity, disjointness to false identity, while partial overlap is vague identity. This criterion can be applied also to the semantics of the previous sections, although without multiple denoting terms this whole machinery becomes redundant, since cases of overlap are impossible to obtain in those semantics. In this way we have an easy extensional reading of vague identity, although this notion might remain obscure. We will investigate it in chapter 3, §3.3..

This framework can be immediately extended to the language $\mathscr{L}_E$, obtaining sK^{3v} from sK^{3v}_E, adding the semantic clauses for external operators.

Definition 2.6.0.2 ($\mathbf{sK}^{3v}$ and $\mathbf{sK}^{3v}_E$)**.** A formula φ is $\mathbf{sK}^{3v}$-satisfiable iff there are a sK^{3v}-model $M = \langle W, R, D, \sim, V\rangle$, a world $w \in W$ and an admissible assignment a such that $V_{w,a}(\varphi) \neq 0$. A formula φ is $\mathbf{sK}^{3v}$-valid iff $V_{w,a}(\varphi) \neq 0$ for every world w of every sK^{3v}-model. A formula φ is a $\mathbf{sK}^{3v}$-consequence of a set of formulae Γ, $\Gamma \vDash^{sK^{3v}} \varphi$, iff for every sK^{3v}-model and *admissible* assignment a, if $V_{w,a}[\Gamma] = \{1\}$ then $V_{w,a}(\varphi) \neq 0$. $\mathbf{sK}^{3v}$ is the logic $\langle \mathscr{L}, \vDash^{sK^{3v}}\rangle$. The logic $\mathbf{sK}^{3v}_E$=$\langle \mathscr{L}_E, \vDash^{sK^{3v}_E}\rangle$ is defined similarly, with respect to sK^{3v}_E-models.

The key move here is that the logics $\mathbf{sK}^{3v}$ and $\mathbf{sK}^{3v}_E$ are defined using admissible assignments. General assignments come into play when the semantic clause for abstract formulae $[\lambda x.\varphi(x)](f)$ is used. In that case the clause may require for an individual variable to be assigned multiple denotations, which cannot happen with admissible assignments only. General assignments are then just a technical tool necessary in order to perform the recursion of V over the whole language, but admissible assignments are those which define the logics we are interested in.

This strategy can be applied to wK- and wK_E-models, obtaining wK^{3v}- and wK^{3v}_E-models. The framework is exactly as above.

[21] Technically speaking, general and admissible assignments are formally different from standard assignments, since the former are relations and the latter functions. With a little abuse of formalism we will ignore this complication.

[22] A similar criterion is employed in Parsons (2000).

Definition 2.6.0.3 ($\mathbf{wK}^{3v}$ and $\mathbf{wK}^{3v}_E$). A formula φ is $\mathbf{wK}^{3v}$-satisfiable iff there are a wK^{3v}-model $M = \langle W, R, D, \sim, V\rangle$, a world $w \in W$ and an admissible assignment a such that $V_{w,a}(\varphi) \neq 0$. A formula φ is $\mathbf{wK}^{3v}$-valid iff $V_{w,a}(\varphi) \neq 0$ for every world w of every wK^{3v}-model. A formula φ is a $\mathbf{wK}^{3v}$-consequence of a set of formulae Γ, $\Gamma \vDash^{wK^{3v}} \varphi$, iff for every wK^{3v}-model and *admissible* assignment a, if $V_{w,a}[\Gamma] = \{1\}$ then $V_{w,a}(\varphi) \neq 0$. $\mathbf{wK}^{3v}$ is the logic $\langle \mathscr{L}, \vDash^{wK^{3v}}\rangle$. The logic $\mathbf{wK}^{3v}_E$=$\langle \mathscr{L}_E, \vDash^{wK^{3v}_E}\rangle$ is defined similarly, with respect to wK^{3v}_E-models.

In the following we are going to study the relations between the logics just defined.

Lemma 2.6.0.1. *Let $M = \langle \mathscr{F}, V\rangle$ be an iK^{3v}-model. Given an admissible assignment a, let h_a be a choice function such that for each intensional variable f and world w it picks out exactly one element from $a[w, f]$, and such that that if $a[w, f] = a[w, k]$ for any intensional variable k then $h_a(w, f) = h_a(w, k)$. Let a' be an admissible assignment such that a' agrees with a with respect to individual variables and $a'[w, f] = \{h_a(w, f)\}$ for every intensional variable f. It holds that if $V_{w,a}([\lambda x.\varphi](f)) = 1$ then $V_{w,a'}([\lambda x.\varphi](f)) = 1$, and if $V_{w,a}([\lambda x.\varphi](f)) = 0$ then if $V_{w,a'}([\lambda x.\varphi](f)) = 0$.*

Proof. The proof consists in an induction on the complexity of φ. The only interesting cases are the atomic ones for predicates not of the form I_P.

($\varphi := Px_1...y...x_n$) If $V_{w,a}([\lambda y.Px_1...y...x_n](f)) = 1$ then $V_{w,g}(Px_1...y...x_n) = 1$, where g is a general assignment which differs from a at most for $g[w, y] = a[w, f]$. Now if $V_{w,g}(Px_1...y...x_n) = 1$ then $V_{w,g}(P)(d_1, ...d_y, ...d_n) = 1$ for each $d_1 \in g[w, x_1], ...d_y \in g[w, y], ...d_n \in g[w, x_n]$. Let a' be the assginment defined above and obtained from a with the choice fuction h_a. Since h_a picks exactly one $d_f \in a[w, f]$ and since for any of these d_f we have $V_{a,w}(Px_1...y...x_n) = 1$, for the assignment g' which differs from a' at most for $g'[w, y] = a'[w, f]$ we have $V_{g',w}(Px_1...y...x_n) = 1$. We conclude that $V_{w,a'}([\lambda y.Px_1...y...x_n](f)) = 1$. The same reasoning holds for $V_{w,a}([\lambda y.Px_1...y...x_n](f)) = 0$.

($\varphi := x = y$) If $V_{w,a}([\lambda x.x = y](f)) = 1$ then $V_{w,g}(x = y) = 1$ for g that differs from a at most for $g[w, x] = a[w, f]$. By construction of a' we have that $a'[w, f] = \{h_a(w, f)\} = g[w, x]$, hence for g' that differs from a' at most for $g'[w, x] = a'[w, f]$ we have $V_{w,g'}(x = y) = 1$ and $V_{w,a'}([\lambda x.x = y)(f) = 1$. A similar reasoning holds for $V_{w,a}([\lambda x.x = y](f)) = 0$. Notice that even if the proof is straightforward when the right-side term of identity is an individual variable, which has a single denotation under an admissible assignment, in the case of $[\lambda x, y.x = y](f, f')$ the property still holds, since by the construction rules of a' we are guaranteed that the same element is picked by h_a both for f and f', in fact if $a[w, f] = a[w, f']$ then $h_a(w, f) = h_a(w, f')$.

□

An important feature of the assignment built from h_a is that it is functional with respect to each type of variable, therefore it can be turned into a standard assignment with trivial modifications.

Corollary 2.6.0.1. *Let $M = \langle \mathscr{F}, V\rangle$ be an iK^{3v}-model and a an admissible assignment. There exists an iK-model (of the same kind of M, namely with $i = s$ or $i = w$) $M' = \langle \mathscr{F}, V'\rangle$ and a standard assingment a' such that if $V_{w,a}([\lambda x.\varphi](f)) = 1$ then $V'_{w,a'}([\lambda x.\varphi](f)) = 1$ and if $V_{w,a}([\lambda x.\varphi](f)) = 0$ then if $V'_{w,a'}([\lambda x.\varphi](f)) = 0$.*

Proof. Take $V' = V$. The assignment a' is the function obtained from the admissibile assignment built from a with the choice function h_a as described in lemma 2.6.0.1. □

Theorem 2.6.1. ***sK** = **sK**^{3v} = **wK** = **wK**^{3v}*

Proof.

- ($\mathbf{sK} = \mathbf{sK}^{3v}$)

  ($\mathbf{sK} \subseteq \mathbf{sK}^{3v}$) Let $\Gamma \not\models^{sK^{3v}} \varphi$, then for some sK^{3v}-model $M = \langle \mathscr{F}, V\rangle$, world w and admissible assignment a we have $V_{w,a}[\Gamma] = \{1\}$ and $V_{w,a}(\varphi) = 0$. Let us build a sK-model $M' = \langle \mathscr{F}, V'\rangle$ such that $V(P)^{-1}(1) \subseteq V'(P)^{-1}(1)$ and $V(P)^{-1}(0) \subseteq V'(P)^{-1}(0)$[23], and let a' be a standard assignment obtained from a by corollary 2.6.0.1. It is provable by straightforward induction on the complexity of φ that if $V_{w,a}(\varphi) = 1$ then $V'_{w,a'}(\varphi) = 1$ and if $V_{w,a}(\varphi) = 0$ then $V'_{w,a'}(\varphi) = 0$. The only non-trivial case is the one for $\varphi := [\lambda x.\psi](f)$, which is guaranteed by corollary 2.6.0.1. Hence $\Gamma \not\models^{sK} \varphi$.

  ($\mathbf{sK}^{3v} \subseteq \mathbf{sK}$) A standard assignment is a special case of general assignment, hence every sK-model is already a sK^{3v}-model, therefore $\Gamma \not\models^{sK} \varphi$ implies $\Gamma \not\models^{sK^{3v}} \varphi$.

[23]For each predicate P^n, $V(P)^{-1}(v)$ is the set of all $\langle d_1, ...d_n\rangle \in D^n$ such that $V(P)(d_1, ...d_n) = v$.

- ($\mathbf{wK} = \mathbf{wK}^{3v}$)

  ($\mathbf{wK} \subseteq \mathbf{wK}^{3v}$) As in the case for $\mathbf{sK} \subseteq \mathbf{sK}^{3v}$, using corollary 2.6.0.1.

  ($\mathbf{wK}^{3v} \subseteq \mathbf{wK}$) A standard assignment is a special case of general assignment, hence every wK-model is already a wK^{3v}-model, therefore $\Gamma \not\models^{wK} \varphi$ implies $\Gamma \not\models^{wK^{3v}} \varphi$.

- ($\mathbf{sK} = \mathbf{wK}^{3v}$ and $\mathbf{wK} = \mathbf{sK}^{3v}$) By theorem 2.4.1 we have $\mathbf{sK} = \mathbf{wK}$, which together with the two previous points concludes the proof.

□

Theorem 2.6.2. $\boldsymbol{sK}^{3v} \subset \boldsymbol{sK}_E^{3v} \subset \boldsymbol{sK}_E$

Proof. That $\mathbf{sK}^{3v} \subset \mathbf{sK}_E^{3v}$ is trivial, while $\mathbf{sK}_E^{3v} \subseteq \mathbf{sK}_E$ holds since sK_E-models are just special sK_E^{3v}-models. Now consider the formula $[\lambda x, y.\neg\Delta_c(x = y)](f, g)$: this formula is not $\mathbf{sK}_E$-satisfiable (that logic does not allow for vague identity), hence $[\lambda x, y.\neg\Delta_c(x = y)](f, g) \models^{sK_E} \psi$ for any ψ. On the contrary $[\lambda x, y.\neg\Delta_c(x = y)](f, g)$ is indeed $\mathbf{sK}_E^{3v}$-satisfiable, therefore $[\lambda x, y.\neg\Delta_c(x = y)](f, g) \not\models^{sK_E^{3v}} \psi$. Hence $\mathbf{sK}_E^{3v} \not\supseteq \mathbf{sK}_E$. □

Theorem 2.6.3. $\boldsymbol{wK}^{3v} \subset \boldsymbol{wK}_E^{3v} \subset \boldsymbol{wK}_E$

Proof. By the same reasoning and the same formula used in theorem 2.6.2. □

Theorem 2.6.4. $\boldsymbol{wK}_E^{3v} \not\subseteq \boldsymbol{sK}_E$, $\boldsymbol{sK}_E^{3v} \not\subseteq \boldsymbol{wK}_E$, $\boldsymbol{wK}_E^{3v} \not\subseteq \boldsymbol{sK}_E^{3v}$, $\boldsymbol{sK}_E^{3v} \not\subseteq \boldsymbol{wK}_E^{3v}$

Proof.

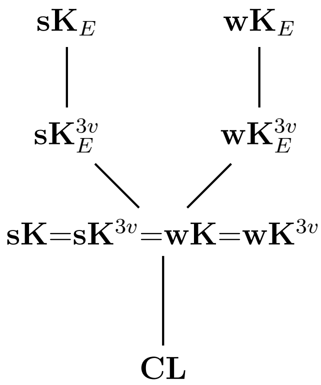
- ($\mathbf{wK}_E^{3v} \not\subseteq \mathbf{sK}_E$) Consider one of the formulae used in theorem 2.4.2: $\neg\Delta_c\varphi \models^{wK_E} \varphi \wedge \psi$, while it doesn't hold in $\mathbf{sK}_E$.

- ($\mathbf{sK}_E^{3v} \not\subseteq \mathbf{wK}_E$) Again using the other formula of theorem 2.4.2: $\Delta_c(\Delta_d\varphi \leftrightarrow (\Delta_c\varphi \wedge \varphi))$ is $\mathbf{sK}_E^{3v}$-valid but not $\mathbf{wK}_E$-valid.

- ($\mathbf{wK}_E^{3v} \not\subseteq \mathbf{sK}_E^{3v}$ and $\mathbf{sK}_E^{3v} \not\subseteq \mathbf{wK}_E^{3v}$) By the two points above and $\mathbf{sK}_E^{3v} \subset \mathbf{sK}_E$ (theorem 2.6.2) and $\mathbf{wK}_E^{3v} \subset \mathbf{wK}_E$ (theorem 2.6.3).

□

The results of theorems 2.4.1, 2.4.2 and theorems from 2.6.1 to 2.6.4 are summed up in the following Hasse diagram:



2.6.2 iK^{2v} and extensions

The semantics defined in the previous paragraph introduce a further distinction between individual and intensional variables. Under admissible assignments, which are the kind of assignments relevant to characterize the logics we are interested in, intensional variables can denote multiple objects. Nevertheless, despite the formalism, the two types of variables behave very much alike. Now we provide new semantic structures where the difference between the two variables plays a greater role.

We are going to introduce cK-models, structures similar to the previous ones but in which the basic valuation function is classical. It will follow that, under admissible assignments, formulae not containing the λ operator always receive classical truth-values. Vagueness is confined to the level of intensions in this way. This makes cK-models a suitable framework for a theory of vagueness which wants to claim at the same time that there is no form of ontic vagueness and that nonetheless vagueness is a structural feature of natural language.

A preliminary step involves a modification in the language. Starting from $\mathscr{L}_{CL}$ we extend the language with predicates of the form I_P as before, but now I_P is of type $\langle I, I\rangle$, hence formulae of the form $I_P fg$ are well-formed[24]. Intuitively these are still read as indistinguishability predicates, which now relate intensional instead of individual variables. The new language is denoted as $\mathscr{L}'$, and its extension with the external operators is $\mathscr{L}'_E$.

[24]We denominate these indistinguishability predicates with the same nomenclature I_P with a little abuse of notation, despite they are entities of different type.

Before going on, we introduce the set D_I of all intensional variables of the language and we assume it to be infinitely denumerable (hence we do not count as intension every function from a set of worlds W to an arbitrarily large set of objects D). The domain D_I will play a role in following semantic clauses, yet we assume it to be fixed by the language and be the same for every model. For this reason we will not include it explicitly in the structure of a model.

A cK-model is a structure $\langle \mathscr{F}, C \rangle$, where $\mathscr{F} = \langle W, R, D, \sim \rangle$ is a tolerant Kripke frame with the exception that $\sim$ is a collection of functions such that for each monadic predicate P there is a function $\sim_P$: $W \mapsto D_I^2$. C is a classical valuation, i.e. a mapping which assigns to every predicate P^n a function $C(P^n) : W \times D^n \mapsto \{1, 0\}$. As for iK^{3v}-models, assignments are now relations $g \subseteq W \times Var(\mathscr{L}) \times D$ and are divided into general and admissible per definition 2.6.0.1.

For a cK-model and a general assignment g, the strong 3-valued valuation $V^{C,s}$ based on C is defined recursively as follows:

- $V^{C,s}_{w,g}(Px_1...x_n) = 1$ if $C_w(P)(d_1, ...d_n) = 1$ for each $d_1 \in g[w, x_1], ...d_n \in g[w, x_n]$; $V^{C,s}_{w,g}(Px_1...x_n) = 0$ if $C_w(P)(d_1, ...d_n) = 0$ for each $d_1 \in g[w, x_1], ...d_n \in g[w, x_n]$; $V^{C,s}_{w,g}(Px_1...x_n) = 1/2$ otherwise.

- $V^{C,s}_{w,g}(I_P f_1 f_2) = 1$ if $f_1 \sim_{P,w} f_2$; $V^{C,s}_{w,g}(I_P f_1 f_2) = 0$ otherwise.

The other cases are just as in the valuation of a sK^{3v}-model. Similarly we define the weak 3-valued valuation $V^{C,w}$ based on C, which uses the same atomic clauses above and the clauses for a wK^{3v}-model in the other cases. When it does not matter whether V is based on strong or weak Kleene, we just write $V^{C,i}$.

The first atomic clause states that a formula Px is vague iff P holds for some of its denotations but not for others. In fact $V^{C,i}_{w,a}(Px) = 1/2$ iff $\exists d, d' \in D$ such that $C_w(P)(d) = 1$ and $C_w(P)(d') = 0$. Since C is a classical valuation and for no $d \in D$ it can be the case that P both holds and not holds, then a necessary condition for the vagueness of Px is that x denotes multiple objects[25].

[25]Under admissible assignments only intensional variables have multiple denotations. The semantic clause for atomic formulae draws an analogy with supervaluations. For an intensional variable f, $V^{C,i}_{w,a}([\lambda x.Px](f)) = 1$ (or 0) amounts to $C_w(P)(d) = 1$ (or 0) for each of its denotations $d \in a[w, f]$, that is to say that $[\lambda x.Px](f)$ is true (respectively false) iff all of its precisifications are true (false). In this way truth for intensions is similar to supertruth in supervaluationism, and falsity to superfalsity (cfr. Van Fraassen (1966), Fine (1975)). This account is similar to the second option of Priest (2016), p.163, for a semantics for multiple denoting terms, where the condition of mixture of truth-values among the denotations of a term, which $V^{C,i}$ evaluates

The same atomic clause contains the main feature which makes these latter semantics different from the ones presented in the previous paragraphs. Under admissible assignments, in a cK-model all the formulae containing only individual variables receive classical truth-values.

Fact 2.6.4.1. *Let M be a cK-model and a and admissible assignment. For every formula φ not containing intensional variables and $w \in W$, $V^{C,i}_{w,a} \in \{1, 0\}$.*

Proof. The proof is an induction on the complexity of φ. The base case follows from the semantic clause for atomic formulae and the fact that a is an admissible assignment, therefore it assigns to each individual variable x a singleton from D. The inductive step is straightforward, since abstract formulae, containing intensional variables, are not taken into account. □

This fact tells us that two formulae Px and Py either differ in truth-value with one being 1 and the other 0, or they do not differ at all. It cannot be the case that their difference is $1/2$. This makes any indistinguishability relation over the denotations of individual variables totally trivial. For this reason we have changed the type of I_P predicates and the definition of $\sim$ as a relation over D_I instead of over D.

We need to rephrase the definition of closeness accordingly to these modifications:

Definition 2.6.4.1 (Closeness for $\mathscr{L}'$). If $f_1 \sim_{P,w} f_2$ then $|V^{C,i}_{w,a}([\lambda x.P(x)](f_1)) - V^{C,i}_{w,a}([\lambda x.P(x)](f_2))| < 1$.

Unravelling the definition we obtain that if $f_1 \sim_{P,w} f_2$ then $|C_w(P)(d) - C_w(P)(d')| < 1$ for each $d \in a[f, w], d' \in a[f', w]$. Now intensions are the objects which can be indistinguishable under some respect, hence only at the level of D_I vagueness can arise. This formal difference can be interpreted in ontological terms. The members of D are the objects which compose the world, they are sharply determined, therefore classical under every respect. On the contrary, when we move to D_I we are at a representational level. Intensions can be thought of as linguistic entities, which can be undetermined due to the limits of our representational system. This interpretation will be properly developed in chapter 3.

as $1/2$, is interpreted by Priest as a truth-value gap. If we change the semantics in such a way that $[\lambda x.Px](f)$ is true (respectively false) iff at least one of its precisifications is true (false), we obtain an account which resembles subvaluations (cfr. Hyde (1997)) and Priest's first option (p.163).

Adding the external operators Δ_d and Δ_c to $\mathscr{L}'$ we obtain the language $\mathscr{L}'_E$, which with the relative semantic clauses provide cK_E-models. We can now define the logics characterized by mixed entailment based on cK-models.

Definition 2.6.4.2 ($\mathbf{sK}^{2v}$ and $\mathbf{sK}^{2v}_E$)**.** A formula φ is $\mathbf{sK}^{2v}$-satisfiable iff there are a cK-model $M = \langle W, R, D, \sim, C\rangle$, a world $w \in W$ and an admissible assignment a such that $V^{C,s}_{w,a}(\varphi) \neq 0$. A formula φ is $\mathbf{sK}^{2v}$-valid iff $V^{C,s}_{w,a}(\varphi) \neq 0$ for every world w of every cK-model. A formula φ is a $\mathbf{sK}^{2v}$-consequence of a set of formulae Γ, $\Gamma \vDash^{sK^{2v}} \varphi$, iff for every cK-model and *admissible* assignment a, if $V^{C,s}_{w,a}[\Gamma] = \{1\}$ then $V^{C,s}_{w,a}(\varphi) \neq 0$. $\mathbf{sK}^{2v}$ is the logic $\langle \mathscr{L}', \vDash^{sK^{2v}}\rangle$.

The logic $\mathbf{sK}^{2v}_E$=$\langle \mathscr{L}'_E, \vDash^{sK^{2v}_E}\rangle$ is defined similarly, with respect to cK_E-models.

Definition 2.6.4.3 ($\mathbf{wK}^{2v}$ and $\mathbf{wK}^{2v}_E$)**.** A formula φ is $\mathbf{wK}^{2v}$-satisfiable iff there are a cK-model $M = \langle W, R, D, \sim, C\rangle$, a world $w \in W$ and an admissible assignment a such that $V^{C,w}_{w,a}(\varphi) \neq 0$. A formula φ is $\mathbf{wK}^{2v}$-valid iff $V^{C,w}_{w,a}(\varphi) \neq 0$ for every world w of every cK-model. A formula φ is a $\mathbf{wK}^{2v}$-consequence of a set of formulae Γ, $\Gamma \vDash^{wK^{2v}} \varphi$, iff for every cK-model and *admissible* assignment a, if $V^{C,w}_{w,a}[\Gamma] = \{1\}$ then $V^{C,w}_{w,a}(\varphi) \neq 0$. $\mathbf{wK}^{2v}$ is the logic $\langle \mathscr{L}', \vDash^{wK^{2v}}\rangle$.

The logic $\mathbf{wK}^{2v}_E$=$\langle \mathscr{L}'_E, \vDash^{wK^{2v}_E}\rangle$ is defined similarly, with respect to cK_E-models.

As in $\mathbf{iK}^{3v}$ logics, even in the above definitions the use of general assignments serves the technical purpose of allowing the recursion for abstract formulae, while admissible assignments are the criterion for the validity of a formula.

Now we move to the relations between the systems just introduced and those from the previous sections.

Theorem 2.6.5. *For every* $\Gamma \cup \{\varphi\} \subseteq$ *Form*$(\mathscr{L}_{CL})$*,* $\Gamma \vDash^{sK^{2v}} \varphi$ *iff* $\Gamma \vDash^{CL} \varphi$ *iff* $\Gamma \vDash^{wK^{2v}} \varphi$

Proof.

- ($\Gamma \vDash^{sK^{2v}} \varphi$ iff $\Gamma \vDash^{CL} \varphi$) The left-to-right direction is immediate, since classical models are a special case of cK-models, where the assignment is functional with respect to every variable. For the other direction, let $\Gamma \nvDash^{sK^{2v}} \varphi$, hence for some cK-model $V^{C,s}_{w,a}[\Gamma] = \{1\}$ and $V^{C,s}_{w,a}(\varphi) = 0$. The reasoning of lemma 2.6.0.1 can be repeated, obtaining a similar lemma for cK-models. Hence we can restrict the assignment a to a standard assignment a' with a choice function h_a that preserves all the classical values for atomic formulae under a. It is easy to prove by induction that this extends to arbitrary φ. The only interesting case is the one for $\varphi := [\lambda x.\psi](f)$, which is guaranteed by the adapted verion of the mentioned lemma. The cK-countermodel can be immediately transformed into a classical model such that $M' = \langle \mathscr{F}, V' \rangle$ such that $V'_{w,a'}[\Gamma] = \{1\}$ and $V'_{w,a'}(\varphi) = 0$, hence $\Gamma \not\models^{CL} \varphi$.

- ($\Gamma \models^{wK^{2v}} \varphi$ iff $\Gamma \models^{CL} \varphi$) Again the left-to-right direction follows from the fact that classical models are special cK-models. As in the previous point, with an adaptation of lemma 2.6.0.1 we can build from $\Gamma \not\models^{sK^{2v}} \varphi$ a classical countermodel.

$\square$

The equality with classical logic for the restricted vocabulary implies the following corollary.

Corollary 2.6.5.1. ***CL*** $\subset$ ***sK***^{2v} *and* ***CL*** $\subset$ ***wK***^{2v}

As it was the case for the previous logics without external operators, $\mathbf{sK}^{2v}$ and $\mathbf{wK}^{2v}$ are equivalent, while the introduction of external operators marks the difference between these logics.

Theorem 2.6.6. ***sK***^{2v} $=$ ***wK***^{2v}

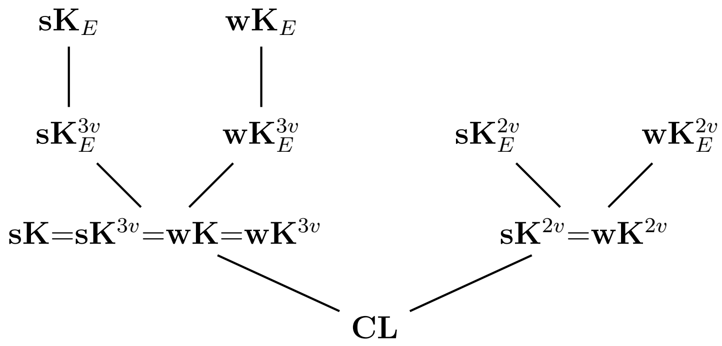
Proof. ($\mathbf{wK}^{2v} \subseteq \mathbf{sK}^{2v}$) Let $M = \langle \mathscr{F}, C \rangle$ be a cK-model and a an admissible assignment such that for some formulae Γ, φ we have $V^{C,s}_{w,a}[\Gamma] = \{1\}$ and $V^{C,s}_{w,a}(\varphi) = 0$. We can repeat the induction in lemma 2.4.0.2 and show that the weak valuation $V^{C,w}$ based on C preserves the classical truth-values assigned by $V^{C,s}$. The induction is straightforward. Therefore $V^{C,w}_{w,a}[\Gamma] = \{1\}$ and $V^{C,w}_{w,a}(\varphi) = 0$.

($\mathbf{sK}^{2v} \subseteq \mathbf{wK}^{2v}$) For the other direction, let the cK-model $M = \langle \mathscr{F}, C \rangle$ be such that $V^{C,w}_{w,a}[\Gamma] = \{1\}$ and $V^{C,w}_{w,a}(\varphi) = 0$. Let a' be the assignment built from a and a choice function h_a per lemma 2.6.0.1. It is easy to show that if $V^{C,w}_{w,a}(\varphi) = 1$ then $V^{C,w}_{w,a'}(\varphi) = 1$ and if $V^{C,w}_{w,a}(\varphi) = 0$ then $V^{C,w}_{w,a'}(\varphi) = 0$. Under a' every intensional variable has a unique denotation, and since the valuation C is classical it follows that all the atomic formulae and the atomic formulae prefixed by the λ operator receives a classical truth-value, therefore by the semantic clauses for $V^{C,w}$ every formula of arbitrary complexity has a classical value. Since weak and strong valuations agree when receive as imput only classical values, it follows that $V^{C,s}_{w,a'}[\Gamma] = \{1\}$ and $V^{C,s}_{w,a'}(\varphi) = 0$. $\square$

Theorem 2.6.7. $\boldsymbol{sK}^{2v} \subset \boldsymbol{sK}_E^{2v}$, $\boldsymbol{wK}^{2v} \subset \boldsymbol{wK}_E^{2v}$, $\boldsymbol{sK}_E^{2v} \not\subseteq \boldsymbol{wK}_E^{2v}$, $\boldsymbol{wK}_E^{2v} \not\subseteq \boldsymbol{sK}_E^{2v}$

Proof. The first two inclusions are obvious. For $\mathbf{sK}_E^{2v} \not\subseteq \mathbf{wK}_E^{2v}$: $\Delta_c(\Delta_d\varphi \leftrightarrow (\Delta_c\varphi \wedge \varphi))$ is $\mathbf{sK}_E^{2v}$-valid without being $\mathbf{wK}_E^{2v}$-valid. For $\mathbf{wK}_E^{2v} \not\subseteq \mathbf{sK}_E^{2v}$: $\neg\Delta_c\varphi \vDash^{wK_E^{2v}} \varphi \wedge \psi$ but it doesn't hold in $\mathbf{sK}_E^{2v}$. □

Theorems from 2.6.5.1 to 2.6.7 allow us to expand the previous Hasse diagram, completing the hierarchy of the logics introduced during this chapter:



Chapter 3

Philosophical remarks

In the previous chapter a number of 3-valued systems based on strong and weak Kleene truth-tables have been introduced and their main formal properties studied. A comprehension of their mathematical behaviour is still not enough for evaluating whether they are adequate for modelling vagueness and what interpretation of vagueness is provided by those models. It is the aim of this chapter to consider two of those systems and explore the relative advantages and shortcomings when applied to languages that talk about vague facts (we will return later to the question whether these are speaking merely about vague application of predicates or directly of vague objects).

The first system we are going to study in detail is $\mathbf{sK}_E^{2v} = \langle \mathscr{L}'_E, \vDash^{sK_E^{2v}} \rangle$ (definition 2.6.4.2), which is obtained from a model fully classical with respect to individual variables, while vagueness arises only at the level of intensional variables. This neat separation between the two types of variables will be object of in depth consideration. Since in the following $\mathbf{sK}_E^{2v}$ will be the strong Kleene system which will receive the longest consideration, in order to make the reading easier we denominate it **SK** (for *S*trong *K*leene), and similarly we will talk about *SK*-models. We are going to consider **SK** as the foundation for a theory of vague objects (§3.1), how to interpret the domains of quantification (§3.2), and how this logic accounts for vague identity (§3.3).

On the second part of this chapter we are going to move the focus over a system based on weak Kleene tables, namely $\mathbf{wK}_E^{3v} = \langle \mathscr{L}_E, \vDash^{sK_E^{3v}} \rangle$ (definition 2.6.0.3), which would be renamed **WK** (for *W*eak *K*leene), being the only weak Kleene logic that is going to play a large role in the following. Weak Kleene semantics has received barely any consideration when compared to its strong counterpart. Despite this lack of interest towards a seemingly eccentric logic, we are going to provide an interpretation that can, in a larger discursive setting, deal with the phenomenon of vagueness too (§3.4).

3.1 Towards an account of vagueness *simpliciter*

A crucial question that any study of vagueness, however formal, should address if it claims any speculative depth is: "how much vagueness does the system account for?". That is the same as to show how deep the vagueness expressed by the language runs into the universe of discourse. On a propositional level, the answer is as immediate as unsatisfactory: states of affairs, or whatever propositional letters stand for, can be indeterminate. A philosophical investigation cannot go beyond the level of state of affairs in this case, unable to refine the underlying theory of vagueness enough to consider how indeterminacy can affect objects, their properties and relations. Certainly a formal study of vagueness can stop at the propositional level and attain worthy results, though that enquiry would hardly delve deep enough to reach the ontology underlying the mathematical system, because of lack of accuracy in the formal machinery employed, namely propositional calculus. Adopting first-order logic provides a more fine-grained level of detail, we can express the vagueness of an object under some respect. What is the nature of this vagueness is a question we are going to address later, although the fact that things might be indeterminate for some property has a strong intuitive appeal and does not require a pre-theoretical justification. There is another more fundamental possibility which is open at first-order level, but that is much more controversial, the possibility of vague objects, vague *per se*, not under some respect.

The starting question here boils down to the question: "what are vague objects?". This is not an innocent question, since it implicitly requires us to provide a justification of the very possibility of vague, fuzzy or indeterminate objects, which has been deemed by most of the contemporary philosophical debate from extravagant things that could exist only in the realm of pure formalism to something utterly irreconcilable with any sound ontology[1]. Here we mean by vague object (without any further specification) or vagueness *simpliciter* what is intended when in everyday language a speaker asserts that something is vague. This is not the case when it is asserted that some x is vague under some respect P, which falls under the category of P-vagueness and amounts to $V_{w,a}(Px) = 1/2$ at the context

[1] Among others, important authors like Dummett (1975) and Lewis (1993) argued against the existence of vague objects.

of evaluation w. Whether P-vagueness is a special case of vagueness *simpliciter* is not a trivial question and we are going to return to it later.

In order to start our enquiry into vague objects, a common ground must be established about the term "object". Clarifying the semantic content of that widely polysemic word is required to begin a philosophical discourse immune to objections relevant to different definitions of the term. By "object" here and in the following we intend everything which is the reference of a variable. The advantage of this definition is to include in the category of objects everything we can talk about, with the possible downside of a resulting inflated ontology. This inflation can be positively reconsidered in the light of the difference between being and existence which can be made explicit in the formal language with the introduction of an existence predicate E, but which has been implicitly assumed since the introduction of the constant domain framework. In this way many of the objects that are accepted by the given definition are *possibilia*, therefore what we consider the real space is not filled with dubious entities which can hardly have any empirical instance. This extremely wide collection of objects, despite being an offence for those "who have a taste for desert landscapes"[2], does not seem a blatant violation of Ockham's razor, since it provides a functional ontology in the context at hand.

Among the multitude of admissible objects just defined, only one part receives an explicit representation in the formal models for **SK**: those objects which are the denotation of an individual variable, namely the members of the domain. A whole different set of objects seems to be left aside, the objects which are referred to by intensional variables. These objects are taken into account by SK-models, but only implicitly. In fact the domain of intensions is considered fixed for the language $\mathscr{L}'_E$. There has been no need to introduce such a domain D_I within the structure of SK-models since $\mathscr{L}'_E$ lacks a quantifier ranging over intensions. That quantifier, given also the lack of predicates of intensional type, would have been rather useless, able to express statements regarding a restriction of the domain of extensional objects D[3]. What the objects that intensions stand for will be answered later. For now it is clear that they are ontologically and modally different from extensional objects. In

[2]Quine (1948).

[3]Lets extend the language with a universal quantifier ranging over intensions. The formula $\forall f[\lambda x.\varphi(x)](f)$ is true just in case for all $f \in D_I$ the substitution of the assignment of f for that of x makes $\varphi(x)$ true, but the assignment of any $f \in D_I$ falls entirely inside D, hence this quantification is merely a universal statement over a restricted domain. Introducing primitive predicates which take intensions among their arguments would make the new quantifier to increase the expressive power of the language.

fact intensional objects[4] have a sort of emergent nature, resulting from collections (even singletons) of extensional objects, although they are not plainly reducible to those because of their potentially non-rigid modal profile, which contrasts the modal rigidity of extensional objects. For these reasons, their linguistic origin and the lack of any direct correspondence in the explicit domain, we will identify intensions (intended as symbols) with their denotations (the intensional objects) and talk indifferently about one or the other.

Moving to the study of vague objects, the many semantics introduced in the previous chapter provides a complete account of P-vague objects for any predicate P[5] What we need for now is a formal definition of vagueness *simpliciter*. Some (not pair-wise exclusive) answers are the following, where d is an object:

(1.a) d is vague iff it is P-vague for some predicate P.

(1.b) d is vague iff it is P-vague for all predicates P.

(2.a) d is vague iff it is P-vague for some/all predicates $P \in S \subset Pred_{\mathscr{L}'_E}$.

(2.b) d is vague iff it is E-vague.

(3) d is vague iff it is the result of a multiple denotation, i.e. $d \subseteq D$.

We will eventually come to argue in favour of hypothesis (3). First we explain why the previous ones result unsatisfactory.

Prima facie hypothesis (1.a) seems precisely the definition of vagueness: something is vague if, under some respect, it is undecided whether a property applies or

[4]Someone might raise the objection that intensional objects are not objects at all, since they do not exist in space and concreteness should be a minimal requirement for objects. It is a matter of definition what we consider an object and in order to evaluate a definition we should consider the strength of its consequences rather than attack the definition itself. Without further arguments this objection is to be considered irrelevant, based on a physicalist prejudice. Even accepting the objection, some intensional objects can accommodate this physicalist reductionism. Let us consider Mount Everest. According to the common use of the term, Everest is an intensional object: it denotes many different but close sharply delimited regions of space, among which no agreement among the speakers can solve the problem of deciding which region is the correct one, not due to incomprehension but for a structural feature of our linguistic practice. Now, all these regions have the property of being situated in space, therefore, according to **SK** semantics, also Everest satisfies the property of being spatially located. Obviously not all intensions can satisfy this criterion of concreteness, nonetheless we have shown that with a selection of the admissible intensions we can account for a reductionist interpretation.

[5]Without loss of generality, we take into account only monadic predicates.

not to that object. According to (1.a), vagueness is a sort of *infectious* property: Socrates might be completely defined under almost every respect (he is certainly a man, mortal, Athenian, philosopher, etc.) but until we are able to decide whether he is tall or short we are forced to claim the vagueness of Socrates. What is counterintuitive here is the identical relevance that every property holds towards the determinacy of an object and how fragile this determinacy results. According to (1.a) every respect is equally entitled to settle the question of the vagueness of something in the case of a positive answer, this is the reason why we have used the term "infectious". In fact as in weak Kleene systems the non-classical value, i.e. vagueness (or nonsensicality, depending on the interpretation), spreads from a formula through the containing superformulae, similarly the vagueness under some respect in enough to imply in force of (1.a) the vagueness of the object. In conclusion the critical drawback of (1.a) is that the ontological status of anything is constantly jeopardized, since any object, no matter how tangible it is, might fall into the dubious category of vagueness just by failing to be classically determined for an arbitrary predicate which we can easily build as odd and *ad hoc* we like.

By the exactly opposite reasoning we arrive at hypothesis (1.b): since stating that something is vague is a strong ontological claim and it seems to be an oddity rather than the norm in a rigorous ontology, any claim of ontological vagueness needs a strong support to be asserted. More precisely it needs the strongest possible support. From a formal perspective this amounts to define vagueness *simpliciter* as vagueness under every respect. A vague object is then something totally indeterminate. The challenge now become how to conceive complete indeterminacy. Claiming that formally such a model is easy to build is not an answer to our question, since what is at stake here is the conceptual soundness of a completely indeterminate object, or, more radically, we cannot even assume without justification that its representation is conceivable. According to the standard reading of strong Kleene systems, vagueness is intended as lack of sufficient classical information: the P-vagueness of something amounts to its potentiality, *ceteris paribus*, to be either P or $\neg P$. A term which describes this state of complete absence of determination and full potentiality is "nothing". It seems that stripping something from every property in the search for a vague object we are left with nothing at all, or, in other words, only the pure logical space can satisfy this very peculiar condition of absolute indeterminacy. Even insisting on the reification of this state of indeterminacy, we have to face the new objection that by extensionality all the vague objects collapse into one and the same, the only vague object. This is the same drawback suffered by some naive theories of non-existence, e.g. theories of descriptions which claim that every description denotes but which, at the same time, do not want to inflate the ontology with a multitude of possible objects and are then forced to make all the void descriptions converging to the same empty object. But it is quite counterintuitive that we refer to the same object when we speak about the current king of France and about the planet between Mercury and the Sun. Without delving deeper into this extremely strong reading of vagueness, we think that the above arguments are a sufficient proof of the difficulty of following a similar path.

Moving forward, hypothesis (2.a) is a mediation between the previous two, providing a more cautious account for vagueness: from (1.a) we follow the intuition that some properties might compromise the determinacy of an object, but as the critique of the same hypothesis has shown a finer criterion should be employed when evaluating the contribute that every property provide to the global ontological status of the studied object. This implies that we should individuate a set S of predicates that are essential for the vagueness of something. S contains all and only the properties that constitute a determinate object, which would likely be much less than the totality of the predicates of the language[6]. In other words S provides an intensional definition of the property "be determinate". It follows that for an object failing to satisfy any[7] of the properties listed in S amounts to its vagueness. The next point is to pick out the members of S from all the predicates, which is not a trivial task at all. Does "being spatio-temporally located" constitute a fundamental quality for determinacy? What about "be discrete"? The list of uncertain predicates can be much longer, but this is already enough to understand the difficulty of a similar task.

Among the many qualities that might constitute the determinacy of an object, one draws more attention upon itself than any other: the existence predicate E. The employment of an existence predicate does not require the full machinery of free logics, even in the constant domain framework of **SK** we treat E as any other predicate, which individuates at every world all and only the existing objects. The semantic clauses for quantifiers remain unchanged, i.e. they are interpreted as outer quantifiers[8], hence the worlds of a model become inhabited by *possibilia.*

[6]This seems intuitively sound, since properties like e.g. "be a liberal" can easily receive vague instantiations in the everyday world without determine the vagueness of vaguely liberal people, who instead appear undeniably non-vague. Moreover putting $S = Pred(\mathscr{L}'_E)$ would lead back to the already dismissed hypothesis (1.a).

[7]Changing "any" with "every" is just a formalism that does not affect the line of reasoning.

[8]This might be considered a lack of expressive power and a serious misunderstanding of the meaning of quantifiers, but it is easy to define an inner quantifier, and consequently its dual, as

This is not a great change after all, it was already implicit in a system which makes use of intensional entities the possibility to deal with non-existent and even merely possible objects. This line of reasoning brings to hypothesis (2.b), a special case of (2.a): here the fundamental property for the determinacy of something is its existence. This choice might be accused of arbitrariness, but it can be replied that existence is not merely another predicate among others, as the history of philosophy shows clearly. Existence and the logic governing it has given rise to much theoretical inquiry, with respective paralogisms and fallacies, and the history of the ontological argument[9] is the most notable witness of such a troubled enterprise.

Returning to the topic at hand, vagueness can be assimilated to E-vagueness, a vague object would become an object whose existence is indeterminate. This solution might seem to have eluded the question, bringing a new one in the dispute: how can vague existence be possible? On a standard account, many-valued logic, even if provided with an existence predicate E, cannot express a vague instance of this predicate, in fact E is specially tied to the model, delimiting a subset of the domain, the existing objects, everything outside is simply non-existent (the extension of $\neg E$), there is no third way, as in standard set theory the relation of set membership is strictly bivalent. Moving to modal systems does not solve the problem, since the multitude of worlds behave all classically. What allows in **SK** to obtain vague existence are the intensional objects: the intensional variables have an extension which is a subset of the domain and from the mixture of values among their denotations results the possibility of non-classical values. In other words, as an intension f is P-vague (at w) when $V_{w,a}(Px_1) = 1$ and $V_{w,a}(Px_2) = 0$ for $a[w, x_1], a[w, x_2] \subseteq a[w, f]$ (we recall that in **SK** assignments are sets, per definition 2.6.0.1), in a similar fashion f can be E-vague. In this sense vagueness for any predicate and in particular E-vagueness can be read as an emergent property, since, in the context of **SK**, it is not already present at the bivalent (hence fully classical) level of the domain of extensional objects, taken as the fundamental on-

$\forall^E x\varphi := \forall x(Ex \to \varphi)$. We are going to use only outer quantifiers for formal simplicity.

[9]On this point, it is worth noting that Kant's well-known objection that existence is not a predicate is consistent with free logics and, more in general, with logics for languages containing a primitive existence predicate. In fact Kant's concern in the *Transcendental dialectic* is to block any inference that could conclude the existence of something by pure deduction from its properties. In this sense existence is not a predicate, i.e. it is not derivable from the definition of something. Moving to modern predicate logic, the feature of primitive predicates is exactly that they are, in fact, primitive, nothing but the model itself can inform of their truth or falsity, and E is no exception. In this way **SK** agrees with Kant's objection.

tological basis of the universe of discourse, but appears only at the higher level of intensions. Having moved E-vagueness to a non-fundamental but derived level not only provides a formal explanation but lowers the otherwise controversial charge that a similar claim, the possibility of vague existence, brings. Hypothesis (2.b) seems promising, yet it might be too restrictive. A couple of examples will show some plausibly intuitive failures of this criterion. When I hear the music produced by a very bizarre instrument which neither I heard before nor plays by any recognizable harmonic rule, I would easily say that such music is vague, not because of some alleged unclear existence (which is undeniable, intending music as the sum of physical waves that my brain elaborates as music) but because it is undefined with respect to what determines music. On the other hand let us imagine an allegedly fictional character, e.g. a detective lived during the Victorian era, described with the greatest care, such that every detail of this character has been clarified by the writer (we should not be concerned by the fact that a similar book would be infinitely long). Moreover the character and his story are perfectly consistent with the physical and historical reality, such that the verisimilitude is so convincing that we might actually doubt whether this is instead an historical depiction and not a fictional work. This character is vague according to our definition, although it may be argued that the uncertain existence of this object does not seem enough to deprive it of its determinacy. These examples show that E-vagueness is not as transparent as we hoped for when taken as vagueness *simpliciter*, nonetheless this notion has been useful and the arguments against it are not as cogent as the ones against hypotheses (1.a)-(2.a). We will call an E-vague object a *strictly vague* object.

This leads us to the final hypothesis. Despite being formally different from the previous ones, (3) is very close to (2.b) in its resulting effects. For this association between them, we will call a vague object according to (3) a *loosely vague* object. (3) defines as vague all the objects which are the result of a multiple denotion. This solution takes all the advantages of (2.b) getting rid of the strict condition on vagueness imposed by that hypothesis. On the one hand, the phenomenon of multiple denotations can only occur at the level of intensions, more precisely every form of vagueness is confined to that level. As before the ontological basis consisting in the elements of the domain is safe from any kind of vagueness, which could contaminate its theoretical plausibility and expose the whole theory to the burden of explaining how the ultimate blocks of the underlying ontology, what should be considered the real objects, can be indeterminate under any respect, that is much easier to explain at the level of linguistic entities such as intensions.

In this sense **SK** provides a cautious ontology, it draws a sharp distinction between two types of entities. On the one hand there are extensional objects, which are classically bivalent and in this way safe from any accuse of unintelligibility that can be moved against the non-classical truth-values of many-valued semantics. On the other hand we have intensions, whose nature is linguistic, they can be interpreted as descriptions, abstractions, or more generally as non-rigid designators, and whose non-classical behaviour can be more easily accepted thanks to their derivative origin, as something which cannot be given without the underlying domain. The emergent quality of vagueness is then preserved by (3), which on the other hand gets rid of the strict criterion imposed by E, whose role was too prominent in (2.b). Instead, a criterion which is different but still in some sense directly ontological is employed (on the contrary hypotheses (1.a)-(2.a) depend on a more linguistic criterion), namely the vagueness of an object is determined by the way in which that object is denoted[10]. On a more set-theoretic reading, determinate objects are identified with singletons of elements of the domain, vague objects are subsets with more than one object.

The definition of vague objects could be more precise, since these objects are denoted by intensions. Taking subsets of the domain as entities *per se* implies that these objects are bounded to each possible world, hence they proliferate with the number of worlds. But the main problem is that these alleged objects would not have any modal profile, neither a rigid one. We overcome this problem taking as objects not the extensions of intensions at each world, but their sums, the intensions themselves. Although they appear suspicious *prima facie*, intensional objects (which, as stated above, we identify with intensions) are something we can consistently conceive. As an example, the top card of a deck is something we can think about, which can be used to develop a good strategy taking in consideration the different values it may take, we can even touch its actual extension, namely the card that right now is physically situated on top of the deck. It is important to notice though that while we ponder about a move in the game which depends on the value of the still unknown top card, the object we are referring to is not the

[10]Reducing vagueness to indeterminacy of reference is the strategy of Stalnaker (1988) during his discussion of vague identity. This view is similar but not identical to our hypothesis (3). According to (3) indeterminate reference is not the source of vagueness but the source of its possibility. The vagueness ascribed by (3) to multiple denoting objects, which we called vagueness *simpliciter*, is more fundamental than vagueness under some respect, it is the possibility of being vague. Multiple denoting objects in **SK** might be vaguely identical, but if their extensions coincide they are truthfully identical in **SK**. Therefore in our setting referential indeterminacy is not the same as vagueness.

actual card (its extension at the actual world) but the intensional object, which ranges over an epistemic space of possible worlds corresponding to different states of the game consistent with the player's current information. This simple example shows how intensional objects are in some cases rather familiar things and rejecting them from our ontology should not be considered an obliged choice. Another more incisive example comes from ethics: when we consider other people and especially when it comes to provide a moral judgement, we are eminently interested in a person as an intensional entity, what matters is his modal profile (which is at least interpreted as temporal), and this is evident in the occidental conception of virtue as *habitus* (disposition), using the Scholastic formulation.

Now that we have a better understanding of intensional objects, we should consider which among them are to be considered vague. Structurally, intensions can be multiple denoting at none, some or all worlds. In order to have a loose criterion of vagueness (but not too loose, as in hypothesis (1.a)) we define as vague all those objects which are possibly multiple denoting[11], i.e. d is vague whenever $|a[w, d]| > 1$ for some $w \in W$. We will also call these *loosely vague* objects. These remarks bring us to a substitution of hypothesis (3) with:

(3′) d is vague iff it is multiple denoting at some $w \in W$.

In the following (3′) will be the intended definition of vagueness *simpliciter*. This loose vagueness is a more basic level than specific vagueness (i.e. P-vagueness): loose vagueness is the condition for the possibility of any specific vagueness, it is the prerequisite for an object to be possibly, but not necessarily, vague under any respect. Moreover it follows by definition that strictly vague objects are a special case of loosely vague ones, in fact for a strictly vague d we have $V_{w,a}(Ed) = 1/2$ for some w, which by the semantics of **SK** is possible only for an intension d such that $k \in C(E)$ and $k' \notin C(E)$ for some $k, k' \in a[w, d]$ (where C is the classical valuation on which the model is based), hence d is multiple denoting at w, which by (3′) amounts to its loose vagueness.

It should be noticed that in spite of their name, these vague objects just defined according to (3′) avoid any kind of fuzzy borders and undetermined boundaries, and because of this even these objects respect Quine's maxim "no entity without

[11] It might be replied that objects do not denote, terms do. This is not the case for intensions, which, by the given definition, are objects in all respects, and since we are identifying the intension with its corresponding intensional object then we obtain an entity that indeed denotes, although what it denotes is not itself in its entirety but, at each world, an intension denotes its modal slice there.

identity". In fact, despite the fact that we lack a symbol for intensional identity, identity among extensions of intensions at a worlds can be defined in term of identity between sets and this has to be checked through the whole set of worlds W in order to state the identity of two intensions. The resulting identity is an equivalence relation, so it satisfies the minimal requirements for an equality relation[12]. Together with the other provided arguments, this should suffice to at least accept the possibility of treating intensional objects as proper entities and vague objects as a special case of intensional ones[13].

3.2 A quasi-Kantian interpretation of vagueness in SK

We have provided a definition of vague objects which at least clarifies an otherwise obscure concept. In **SK** both vagueness under some respect (i.e. P-vagueness for any predicate P) and vagueness *simpliciter* can be formally represented, as we have already explained. What we still lack is an enquiry on the nature of the vagueness conveyed by **SK**. This amounts to provide an interpretation of the domains over which SK-models ranges, both the explicit one D of extensional objects and the implicit one D_I of intensions fixed by $\mathscr{L}'_E$, and what is the intended meaning of the fact that some objects are vague under some respect.

Many different interpretations have been developed by the literature: vagueness can be a deficiency in our representational system (Russell (1923)), a lack of meaning (Fine (1975)), a structural epistemic problem (Williamson (1994)), just to quote some of the most notorious positions. Evaluating whether these interpretations are correct for the semantics of **SK** is not totally straightforward, due to the presence of two types of entities in SK-models, extensional and intensional objects. In particular Williamson's epistemicism is interestingly close to the implicit

[12]Though it might not be the smallest equivalence relation, as in the case where two intensions share the same extensions at each world. In this case the intensions would be synonymous, although the identity relation just defined would not be able to discern between the two. Synonymity is in fact an hyperintensional property that we cannot capture in the current setting.

[13]This characterization of vague objects is situated completely inside the metatheory, it cannot be stated with the operators Δ_c or $\Delta_c^{\uparrow}$. An operator able to express vagueness *simpliciter* should range over the whole set of worlds regardless of the accessibility relation. A similar operator would exceed the expressive power of the considered language $\mathscr{L}'_E$, in the same way that a global necessity operator (ranging over the whole set of worlds) cannot be defined in terms of $\Box$ or of any component of the basic modal language, as proved e.g. in Blackburn *et al.* (2001), pp.54-55.

interpretation of vagueness provided by **SK**.

According to epistemicism, there is no vagueness at all in the world, vagueness is an entirely epistemic phenomenon. Williamson is a firm defender of classical logic and one of the least uncontroversial appeals of epistemicism is exactly its full endorsement of classical reasoning. The logic of vagueness may appear twisted, leading to gap or gluts of values, or even plainly contradictory as the conclusion of sorites arguments might suggest, but epistemicism claims that these troubles are born from a deep incomprehension of the phenomenon of vagueness, we have mistaken our lack of knowledge for a fact subsisting in the world. Things are not vague, it is us who are not able to meaningfully (i.e. classically) apply our categories to things in those cases in which the margin for error[14] is too wide, preventing us from discerning positive from negative cases and hence making knowledge impossible. The strong intuition of epistemicism is that the impossibility that this theory points at as the source of vagueness is not a practical limit of our language, which could be resolved with a Fregean regimentation of language. The limitation ascribed by epistemicism is a theoretical one, that no practical solution can overcome. In this way Williamson's epistemic reading of vagueness separates our ability to describe the world through our categories and the way in which the world actually is, saving at the same time vagueness as an essential feature of natural language and classical logic as the true logic of the world.

This theory is far from uncontroversial, as the philosophical debate has witnessed since the publication of Williamson (1994)[15]. We are not planning to defend epistemicism, though we will provide a new epistemic reading of vagueness. Nevertheless that theory is a useful starting point which, in Williamson's formulation, can be easily adapted to **SK** framework.

First, both **SK** and the logic underlying epistemicism are classical; in the latter case it is properly classical logic, in the former it is a more complex semantics which constitutes a conservative extension of classical logic. Technically speaking, the entailment relation in **SK** is not transitive, which is a striking difference from classical logic, although it can be easily proved, adapting theorem 2.2.1, that restricting $\mathscr{L}'_E$ to its classical fragment $\mathscr{L}_{CL}$ then **SK** coincides with classical logic. When facing phenomena that seems to imply a deviation from classical reason-

[14]Cfr. Williamson (1994), pp.230-234.

[15]Williamson (1994) was preceded by the papers Williamson (1992a) and Williamson (1992b), neither Williamson was the first to endorse an epistemic reading of vagueness. The second name mostly associated to the epistemic view of vagueness and precursor of Williamson is Sorensen (1988).

ing, a common strategy is to develop logical systems which to some extent differ from classical logic or constitute weaker sublogics in which paradoxical conclusions cannot be inferred any longer. On the other hand, classical logic, although not immune from critiques, in its simplicity and, for the most part, intuitive appeal, it is to be considered a canon for formal semantics, and even if we do not take classicality as a virtue, it is certainly not a flaw by itself.

Beyond this formal feature shared by **SK** and epistemicism, SK-models can be interpreted according to Williamson's reading of vagueness as structural ignorance. As already illustrated, the type distinction between individual and intensional variables in $\mathscr{L}'_E$ corresponds to the ontological difference between extensional and intensional objects. Assuming the epistemic reading, we reinterpret the two levels, extensional and intensional, as, respectively, the way in which the world is and the way in which we believe that the world is. The domain D is fully classical, since the valuation function C at the base of the model interprets each predicate classically, leaving no space for intermediate truth-values. In this way the domain is a faithful image of the world, according to Williamson's defence of classical logic as the true logic, the world as it is independently from our knowledge of it (or, to be more precise, regardless of our possibility of effectively knowing it). Inside the object language $\mathscr{L}'_E$ individual variables are employed to represent this world. On a different level there are intensional variables, which constitute our way to speak about the world, with the limits imposed not only by our linguistic practice but by our own cognitive apparatus.

As Williamson individuates the origin of vagueness within his epistemology, ascribing its source to the margin for error, we can similarly adapt the notion of margin for error to multiple denotations. Restricting our consideration to moandic predicates, the clauses of **SK** allow non-classicality, interpreted as vagueness, to appear only in the occasion of multiple denotations. In particular P-vaguess (at a fixed context w) for the intension f is the result of a mixture of both classical truth-values among the members of the domain which falls under the extension of f. The denotation of f can be intended not merely as a set of objects among which we are unable to pick the proper reference of f due to indecision or ambiguity, but as the set whose members are indistinguishable, to the best of our epistemic possibilities, when it comes to identify what f is. It follows that the properties of f are determined by the distribution of those same properties among its denotations.

We have in this way reinterpreted multiple denotations as a form of ignorance and the evaluation of atomic formulae follows a strategy similar to supervaluations: instead of a set of all the classical refinements of a model, we have the set of all the denotations of f, among which, as among the classical refinements, we have no reason to choose one over another, and when P holds for all these denotations despite our inability to distinguish among them we can conclude, by an extended reasoning by cases, that P holds for f, and similarly when P is false for every denotation. This is an evident parallel with the notions of super-truth and super-falsity, while vagueness amounts to the occurrence both of satisfying and of non-satisfying denotations with respect to P, as in the supervaluational setting happens when the refinement of a model does not guarantee that a formula holds, namely when both satisfying models and countermodels can result from the process of refinement.

This shows how **SK** can accommodate an epistemic reading of vagueness, specifically that prescribed by Williamson's epistemicism. The definition introduced previously of loosely vague objects, which identifies vagueness with the possibility of having multiple denotations, fits well with the current setting. What was the source of vagueness before, multiple denotations, can now be reinterpreted as margin for error, where the margin is no longer to be considered a gradation under some respect which is small enough to make two things indistinguishable. Instead the new notion of margin comes before any classification, appearing already at a more basilar level which precedes any predication, the level, we could say, of reference. What happens in Williamson's margin for error is that given an object d, which by itself is bivalently determined in every respect, for us it may be vague whenever for some property P our faculties are not able to discern d from d' for their P-ness, even though d is P and d' is not. On the contrary in **SK** the possibility of vagueness is already given when d is presented at its most fundamental ontological level, that of its denotation. Here the space of multiple denotations is the same as a margin for error, moreover we have a simple criterion for the presence of a similar margin, which does not revolves around the pragmatics of the use of every single predicate: when the set of denotations is not a singleton, then we have a margin for error. We conclude that the notion of loose vagueness is a more fundamental notion of vagueness, which by itself does not imply P-vagueness for any particular P, but it is the prerequisite for any specific vagueness[16].

[16] A constraint can be added to models in order to ensure that multiple denotations at w implies P-vagueness for at least one P, namely requiring all couples of distinct members of the domain to be different under some respect, that is to say that we are considering models that satisfy the principle of identity of indiscernibles: $\forall x \forall y(\forall P(Px \leftrightarrow Py) \rightarrow x = y)$ (contrary to Leibniz's law, which can be assumed as a metaphysical principle and expressed formally as a first-order schema, the identity of indiscernible can be expressed only as a metatheorem of a first-order theory, although we should not be bothered here by this technicality). Before accepting this

Despite its classicality and the efforts of Williamson and other followers of this epistemic view of vagueness, epistemicism remains still too controversial and counterintuitive to achieve a leading role in the literature about vagueness. Epistemicism challenges our usual understanding of knowledge and even the argument of the margin for error has to face serious consequences, as it seems to imply that a complete knowledge of the words we use transcends the use we make of it. The main criticism that in the current context makes epistemicism an untenable position does not revolve around the solution it provides for the phenomenon of vagueness. Instead a more fundamental difficulty can be found at the level of the philosophy of language that is validated from the assumption of a similar theory of vagueness.

The dramatic flaw of epistemicism is the separation that it operates between language and its use. On the one hand there is the world, which is composed of objects whose properties and relations determine, using a Wittgensteinian definition, facts. All these facts are bivalently decided, they holds or not, *tertium non datur* both according to Williamson's metaphysics, who assumes classical logic as the true logic, and to **SK**, whose atomic valuation function C is indeed classical and evaluates how the extensional objects, i.e. the elements of the world, behave. On the other hand there are the speakers, who talk about the world, using intensional notions which implies a trivalent logic where vague states of affairs can happen. Formally classical logic is just a special fragment of **SK**, thanks to strong Kleene's truth-tables, nonetheless there is a significant conceptual difference between the two systems. In a classical setting it is not simply that vagueness may happen but, as a matter of fact, it can never actually be the case, in a classical setting the very possibility of vagueness is unacceptable. From this picture appear two sides of language. One side leans towards the world, which describes it correctly and returns a faithful picture of it. On the other end there are us, the speakers from whom the linguistic process originated, but the picture that we produce on this side is more obscure, so imprecise that it leaves some grey areas where facts cannot be established according to classical values and remain indeterminate. Conciliating these two sides is the challenge that epistemicism must overcome if we want to legitimately assume it as the interpretation of vagueness in **SK**. We could argue that this opacity which happens on the 3-valued side results from a mismatch between our language and the world, the categories that we employ when talking might represent the world up to a margin of precision, they are something we impose on

constraint it is of crucial importance that the object language be powerful enough to express all the properties of an object that we consider exhaustive of its definition.

the world and we cannot pretend to obtain a perfect correspondence.

This answer is itself highly problematic. First, the alleged mismatch seems to be situated not between our language and the world, but inside the same language, between its opposed ends, the one facing the world and the one originating from the speaker. In this case it seems inappropriate to use the same term "language" to refer to this structure, since there are actually two objects, which corresponds to the two ends described. The previous answer then becomes just a way to postpone the question about the relation between language and world to a new level, one inside language, although now we are required to explain how these two sides of language exist and relate, which opens the original question anew. Even accepting that the mismatch happens between language and world, occurring during the interpretation that the language operates of outer experience, we are forced to accept also a strictly instrumental view of language. Language becomes merely a tool which translates whatever is outside us, the world, into something which we can manipulate with thought, talk about and, in conclusion, experiencing. This reading implies that there are three main, distinct figures in play here: the world, the experiencing subject and the tool that allows the latter to access the former, language, which is a third object, different from both the world of experience and the speaker. This naive view of language brings with itself a radical realist stance, which can certainly be defended with an appropriate philosophical argumentation. A similar task is way far from trivial, having to explain how language can both describe the world and be accessible for us while at the same time being something entirely different from both, and we are not going to attempt a similar enterprise.

Going further with our criticism, let us assume that all the previous objections were answered, hence we accept that the epistemicist view of language is actually able to explain how we can talk about the world is spite of the apparent mismatch happening somewhere during the linguistic act. The next objection is the incongruence between what we consider language and its use[17]. Language seems to have two different applications, which in turn underlie two different logical systems. On the one hand we have an abstract interpretation of language, which describes the world according to classical logic (in **SK** this corresponds to the fragment of $\mathscr{L}'_E$ which does not contain abstract formulae and intensional variables). On the other hand we have language as we use it (the whole $\mathscr{L}'_E$, especially the formulae involving intensions), which gives rise to vague sentences and formally results in a 3-valued logic. Again we are faced with two unacceptable choices: either we admit the difference and split the language into two objects, language for the world and

[17]Cfr. Wittgenstein (1953), §43.

language for us, falling under the previous objections, or we insist that language is one and the same, although there is a distinction between language and its use, and we are then required to explain what is language once disjoint from its use. Being unable to answer this latter question we consider this way aporetic and we will not pursue it.

This list of problems, even if it does not demolish epistemicism, it makes the application of such a theory to interpret **SK** not immediate and requiring a thorough justification. These objections concern mainly the interpretation of **SK** in an epistemicist setting, not epistemicism *per se*, it might be replied. This is true to a certain extent, although any epistemic thory of vagueness should be able to answer to all the objections raised above. Anyway what we are searching for is an interpretation of vagueness in **SK**, so this preliminary *pars destruens* was necessary to reject an epistemicist reading of vagueness, especially since we are going to propose an interpretation which itself is situated in the epistemic field.

The theory of vagueness that we are going to provide takes its inspiration from the main foundation of the great debate that was German classical philosophy, Kant's *Critique of Pure Reason.* Considering Kantian epistemology[18], let us remember the distinction between appearances and noumena. Appearances are what appear for us, what becomes an object thanks to the interaction between sensibility and understanding, appearances are the objects which compose the world of experience, whose objectivity is guaranteed by the transcendental structure of humanity, while without a transcendental subject, as things in themselves, there are no appearances at all. Noumena are those entities which reason needs to postulate in order to explain experience, what can justify the source of intuitions although no intuition corresponds to a noumenon, which properly is beyond any possibility of experience for us. That of noumenon is an incredibly rich and problematic notion and we will not even try to provide an adequate exegesis of Kant's conception of it, let us take it at face value.

As we saw during the criticism of epistemicism, the peculiar feature of **SK** of having two distinct levels, one of extensional and one of intensional objects, allows for a cautious ontology where vagueness only emerges at the non-basilar level of intensions while the fundamental domain of extensional objects remains classical. But at the same time this very feature requires a philosophical justification that explains the hiatus between the bivalent side and the trivalent one. In other words,

[18]In the following terms like appearance (*Erscheinung*), experience (*Erfahrung*), intuition (*Anschauung*), concept (*Begriff*), sensibility (*Sinnlichkeit*), understanding (*Verstand*) and reason (*Vernunft*) will be used in the technical sense in which they are employed by Kant.

the gap between extensions and intensions requires more than a formal treatment in order to be conceptually acceptable. How can the two sides of language coexist under a unifying logic? Using elements from Kantian philosophy, in a first approximation we can assimilate the difference between bivalent and trivalent level with that between noumena and appearances, therefore in the intended interpretation the elements of D_I are appearances, those of D noumena. Noumena can be seen as the source of our intuitions (even if only in a regulative and not objective sense), hence they stand for the things out in the world, although this is a world beyond any possibility of experience. Following Williamson's assumption that facts in the world behave accordingly to classical logic, we require the logic governing the domain of noumena to be classical, and indeed **SK** is fully classical until we do not make use of terms whose reference lie outside of D, thanks to the classical valuation C. Once we move to the full language, we are taking into consideration D_I too, which amounts to the level of appearances. In spite of the classicality of the entailment in **SK**, once intensions enter the scene we obtain a properly trivalent setting. The cause of indeterminacy can be ascribed to the epistemic process that leads from the material provided by noumena through sensitivity and understanding, ending in the representation[19]. As in general we are not guaranteed that the application of some criterion to an object results in an accurate depiction of its properties, in the same way we can impute the source of vagueness to some form of incongruence between the cognitive apparatus that we apply (intuitions and concepts) and the objects on which it is applied, the noumenic input. Returning to the theory of a mismatch that we have explored in the case of an epistemicist reading of **SK**, the main objection which compromised that interpretation, namely the apparent discontinuity that cut language into two parts, is now dissolved, since everything happens within the synthetic process of reason and the gap between D and D_I is explained in terms of Kantian epistemology. In other words, since both noumena and appearances are representations (the former, technically speaking, representations to which nothing correspond in the intuition, the latter are the result of the unifying process of the categories), they are both within the range of our language, assuming that the faculty of language is the same as the represen-

[19]Despite the misleading word, this is not a representational theory of vagueness. Representation (*Vorstellung*) in Kant does not reduce to a merely subjective thought, the very possibility of objectivity is given within the representational dimension. In a transcendental philosophy objects are something for us only thanks to the representation that we obtain through the application of categories, so in its proper meaning representation is something objective and universal (within humanity).

tational power of our transcendental structure. On the contrary vagueness is the result of the cognitive process, in fact only at the level of intensions, i.e. appearances, trivalence occurs. There is no gap here, since appearances are something other than noumena, in some sense they are derivative entities, they are the result of the synthetic activity of the mind, and this derivative nature is reflected in the semantics.

Despite these arguments, even without pretending to be Kantians this interpretation is untenable. Noumena are not objects in any sense of the term. Reason requires to think noumena, although nowhere in our experience we would ever be able to find them, not because they are so far away from us that any practical attempt to reach a noumenon would be constantly frustrated, but because there is nothing to be reached, there are no noumena outside appearances, properly there are no noumena at all outside pure thought. To put it briefly, since all the objects are given to us in the experience and, by definition, noumena reside outside it, noumena are no objects. Adopting Kripke's famous image there will never be a telescope powerful enough to let us see noumena[20], since we are actually searching something which is nowhere (inside the only world which is possible for us, the world of experience allowed by our transcendental structure), we are searching for nothing. Despite this warnings and any possible critique of our faculty of reason, we will always be tempted to objectify noumena, and this is not because of some vicious habit that centuries of metaphysics have instilled in our culture. As Kant remarkably shows in the *Transcendental dialectic* this tendency is rooted in the dialectical behaviour of our own reason. The constant tension to the reification of what corresponds to no object at all is a side effect of the unifying process in which reason consists, which on the one side drives us towards new discoveries as we group apparently different phenomena under common features, on the other side this same pulsion is the source of all the antinomies and paralogisms from which metaphysics suffers.

On the light of this deeper analysis, we can see how the formal system of **SK** badly adapts to a strictly Kantian interpretation of the elements of the domain D as noumena. The problem here is that a single language, $\mathscr{L}'_E$, is used to describe both extensional and intensional variables, or, to be more precise, the same set of predicates $Pred(\mathscr{L}'_E)$ applies to all the objects of a given SK-model (remember that the domain D_I of intensional objects is fixed by $\mathscr{L}'_E$ and implicitly assumed in every model). This indiscriminate use of predicates is exactly what the whole Kantian *Critique* forbids and it is the source of all unsolvable metaphysical dilemmas.

[20]Cfr. Kripke (1972), p.44.

Here predicates can be faithfully identified with Kantian concepts (*Begiffe*) of understanding, synthetic functions of our thought which brings under unity multiple representations. We are not claiming that any predicate stands for a category in the technical Kantian sense, as a pure concept, but it seems safe to claim that a predicate, or more precisely the property it stands for, is a unifying function, hence a concept. Now the main limitative result of the whole *Transcendental doctrine of elements* is that concepts can be applied only to intuitions, which themselves can be *a posteriori* or even *a priori*, as in the case of mathematics, only as far as our sensibility goes there is experience for us, outside of that there is, strictly speaking, no world at all, there is nothing for us. Every attempt to apply concepts to entities to which nothing corresponds in intuition, which is the very definition of noumenon, is to be considered spurious, an erroneous extension of the functions of understanding destined to generate paralogisms, or, using a modern terminology, this is precisely a categorical mistake.

Returning to the formal framework, if in a Kantian interpretation of **SK** the members of D stand for noumena, every formula of the form Px is to be considered a metaphysical nonsense, since predicates, as concepts, cannot be applied to things which cannot be given in any intuition. The type distinction between variables in $\mathscr{L}'_E$ and the fact that primitive predicates (not those of the form I_P) take only arguments of extensional type is not be regarded as a solution to this difficulty. As a matter of fact, despite formulae Px and $[\lambda x.Px](f)$ are syntactically different, λ abstraction practically allows to surpass type distinction and the semantic clause for abstract formulae in **SK** effectively provides meaning to the application of a predicate to intensional variables. The problem runs deeper and compromise the whole formalism, in fact the very classicality of D falls under this criticism, because if for no member $d \in D$ can meaningfully hold that $d \in C(P)$, neither it is meaningful to claim that this implies that $d \in D \backslash C(P)$, as classical negation requires, because again a negated property is a concept, hence we cannot predicate it of a noumenon. But this is not the end of our troubles, since even a 3-valued strong Kleene logic cannot account for the value to give to the elements of D, since in that case the intermediate value $1/2$ stands for the insufficiency of classical information to determine any classical value. On the contrary, when it comes to noumena there will never be enough information to settle the matter of their description, because there has never been an object, something that we are able to describe. Moreover since the semantics of the intensional level is determined by the extensional domain the informative content of every model would collapse, since every formula would be evaluated as $1/2$ (and when it comes to external operators

Δ_d and Δ_c despite these return only classical values nonetheless formulae would be equally evaluated everywhere). We obtain the same result if we introduce a fourth truth-value v_4 which means nonsensical and which results from the application of any predicate to a member of D, obtaining a 4-valued logic, in which the value v_4 can either be governed by weak Kleene truth-tables or be completely inert, not determining the value of intensional variables. In the former case we obtain an infectious value and this again implies, given the homogeneous distribution of v_4 for every property applied to any member of D, that by semantics the totality of $\mathscr{L}'_E$ would be evaluated as v_4, dissolving any informative content. In the latter case we need a new valuation function which assigns values to abstract formulae, restoring the recursion of the semantic clauses over the whole language, in both cases the semantics of **SK** falls apart. Another strategy is to repeat the original type distinction, adding a third type of objects other than extensional and intensional. Let us rename D as D_1 and add a new domain D_0 (with a relative set of individual variables of new type O_0), which is the set of noumena, while the member of D_1 becomes something within the possibility of our experience but somehow linked to noumena and able to describe them, a sort of reflection of D_0 to which we have a linguistic (hence conceptual) access. Now of D_0 we cannot speak meaningfully in any way, either avoiding formulae which refer to it or using the new value v_4 whenever this reference happens, but within D_1 (and the formulae which refer to elements inside it) we are able to speak about D_0, as a sort of Wittgenstein's ladder[21] that allows us to speak about what is properly unspeakable. Many questions are left open here, first of all what is the semantics linking all these levels, especially since at D_0 everything is equally evaluated v_4 under every respect, let alone the interpretation of D_1. This hypothesis is barely sketched and excessively contorted, but it shows how an attempt to reintroduce the interpretation of the elements of D as noumena requires a complete rebuilding of the whole semantics, which we will not pursue here. In conclusion, the notion of noumenon seems to elude every immediate formal treatment, if not every such treatment at all, and we should leave it to the original Kantian conception of noumenon as *Grenzbegriff* (Kant (1787), A255, B310-311), limiting concept, something "therefore only of negative use. But it is nevertheless not invented arbitrarily, but is rather connected with the limitation of sensibility, yet without being able to posit anything

[21]Wittgenstein (1921), prop.6.54: "My propositions are elucidatory in this way: he who understands me finally recognizes them as senseless, when he has climbed out through them, on them, over them. (He must so to speak throw away the ladder, after he has climbed up on it.) He must surmount these propositions; then he sees the world rightly."

positive outside of the domain of the latter."

It is because of these problems[22], which are not only a matter of history of philosophy, that we abandon the project of a Kantian interpretation for **SK**, but still we are going to use some of the insights that Kant's epistemology provided, moving to a *quasi-Kantian interpretation.* We retain the identification of the domain of intensions D_I with the level of appearances, what we need to change is the intended interpretation of D, since we cannot use noumena, for the many objections illustrated above. Nevertheless the reading of vagueness as a mismatch between our concepts and the interpretation that they operate of the world is a convincing hypothesis, so we are going to preserve as much Kantian epistemology as we can without falling into the aporia of the notion of noumenon. About D, we can intend it as a source of appearances (which is reflected formally by the functional dependency of the evaluation V of abstract formulae from the behaviour of the base mapping C relative to D). Not their ultimate source though, which lies beyond the boundary of the possibility of our experience, but a source located within the limits of our language[23], such that any application of predicates to elements of D is guaranteed to be legitimate.

This interpretation is called quasi-Kantian because we will not try to force the hermeneutics of Kant's works in order to find a suitable place for this sort of third realm D between noumena and appearances but still within the range of experience. The objects in D can be considered as elements of intuition, some kind of proto-representations, but as we said it might be counterproductive to search

[22]There is one last, more subtle objection, raised by Kant's doctrine itself. The alleged mismatch between D and D_I, which we have identified with the source of vagueness, translates, in Kantian epistemology, into an incongruence between sensibility and understanding. As an emergent feature, vagueness is to be intended here as a sort of mistake or limitation in the synthetic action of understanding while interpreting the material provided by sensibility, which gives rise to areas of indeterminacy. If we want to be faithful to Kant, this is openly in contrast with the result of the *Transcendental deduction* (A84-130, B116-169), which demonstrates the correctness of the application of the categories of understanding (and, by extension, of the concepts too) to the intuitions of sensibility. The ultimate impossibility for the above interpretation is then that in order to be defended it has to attack the crucial reasoning of the *Critique*, renouncing to be Kantian at all and dismissing its very foundations.

[23]This quasi-Kantian interpretation can also be considered, with the appropriate caveats, Wittgensteinian too, since it endorses the well-known Wittgenstein (1921), prop.5.6: "*The limits of my language* mean the limits of my world". In fact since language extends as far as concepts do, whatever is outside the world of experience is outside language as well (in the same way as concepts cannot be applied outside intuition), and similarly whatever is the world for us is such because of the objectifying action of concepts, so it is something we can talk about.

for an exact collocation for this new interpretation of D in Kant's works. The members of D are to be intended as the simplest units that can be given inside our cognitive process, before the complete synthesis that generates appearances but already within the possibility of our experience[24]. It is necessary to assume these elements as simple, since if experience is to be a synthetic process we cannot pretend it to start from something complex. The simplicity of the elements of D is also the reason of their classicality with respect to every predicate. Since they are as simple as possible for us to conceive, they also behaves in the simplest logical form under any predication, namely bivalently. For any P either it holds or not for a given $d \in D$, if this were not the case we would not be guaranteed that a synthetic process might even begin.

What happens when we move to D_I is that intensions operate a unification: as concepts are unifying functions of the understanding, so are intensions in this case. But intensions work on a more basilar ontological level, they unify a multitude of fundamental elements (members of D) into a single representation, the intensional object which has those elements as its extension[25]. Now that a new object is created, the intension, we can consider the result of the application of predicates to it and here, thanks to multiple denotation, vagueness can arise. This vagueness can be intended again as a mismatch between appearances and the elements of D, but since D is assumed to be within the limit of our language we should justify this discontinuity. Instead we can read the emergent quality of vagueness as a legitimate result of the synthetic operation of understanding, that completes the meaning of predicates, which in the world of appearances find their rightful application. That is to say that since, according to Kant, we can properly speak of objects for appearances then we should not consider vagueness as a deficiency of the representational system, but as a feature that cannot be observed at the basilar level of D, whose elements lack the complexity of those in D_I and this way they do not express the full range of meaning of predicates, bounded to a bivalent reading. Only in D_I the full extent of the meaning of predicates emerges, with the possibility of indeterminacy. As we can see the whole contrast between a bivalent and trivalent level is translated in this quasi-Kantian interpretation as the difference between D and D_I as steps of increasing complexity in the synthetic

[24]This is plainly contradictory in the orthodox Kant's system, there is no experience before the synthesis of the understanding, but, again, the fact that the interpretation is quasi-Kantian grants us some freedom.

[25]To be more precise we should say that the intensional object is the result of the sum of these unifications throughout the whole set of worlds W.

process of experience. In other words, returning to the tentative interpretation of the members of D as elements of intuition, we can reverse the standard reading of vagueness as indeterminacy and consider the classicality of D as the inability of sensitivity to provide a full reading of the empiric world. Sensitivity with intuitions cuts sharp distinctions in the application of predicates, it cannot account for all those cases of non-classicality that natural language show us as natural. On the contrary the understanding with its concepts fills the gaps left by the rough reading of sensitivity, which introduces a trivalent setting that is not to be taken as a more imprecise interpretation of experience but as a more genuine conception of the full range of application of predicates.

In conclusion, it could be replied that this quasi-Kantian interpretation is not a new epistemic theory of vagueness, but merely epistemicism in disguise. We should distinguish two sides, the formal and the philosophical, on both of them we argue that this is not the case and the current interpretation has some degree of originality. The formal side seems the weakest one, since the logic at the base of epistemicism is classical and the proposed logic for the quasi-Kantian interpretation is **SK**, a conservative extension of classical logic. There are some distinctions to be made though: the metatheories are different (in **SK** the entailment is notably non-transitive), in **SK** intensional variables are introduced which allows for the possibility the non-classical truth-value $1/2$, **SK** accounts for vague identity which instead is openly denied in epistemicism. These are just some of the major differences, but what is more relevant are the philosophical backgrounds that justify the two theories. Epistemicism is motivated by the need to save classical logic and at the same time explain vagueness in such a classical setting, resulting in the eradication of vagueness from the world as it gets moved to the epistemic dimension. On the contrary our interpretation is completely located inside Kantian epistemology, it is based on a transcendental argument that individuates vagueness in the cognitive process of construction of appearances by application of concepts. Moreover, and this is the most remarkable difference, while epistemicism gets rid of vagueness as something real, in the quasi-Kantian view since vagueness exists only at the level of appearances but, in a Kantian setting, the world is exactly the set of appearances, hence vagueness exists properly only in the world, the world which is given by our transcendental structure. By this reasoning we see how this interpretation is, in some sense, not only epistemic but ontological too, as ontology cannot be sharply detached from epistemology in Kant's philosophy, especially when it comes to the notion of appearance, which is at the same time the only thing that can be an object for us and the only one that we can objectively know, both in virtue of our unique transcendental structure. By the same reasoning the quasi-Kantian interpretation cannot be reduced to a from of representationalism, where vagueness is a feature that results from a deficiency in the representational system when applied to an object to be represented. Certainly Kantian philosophy considers appearances as representations, but these are objective representations, or, as we said before, the appearance is not to be thought as something which results from the interpretation of a thing in itself, the noumenon, because the latter is merely a limiting concept, without any correspondence in the world.

3.3 Tolerant reasoning and identity

One characterizing feature of vagueness is, besides the possibility of borderline cases, *tolerance*. As Wright (1975) defines it, tolerance is "a degree of change too small to make any difference", and by tolerant property we mean one such that it is affected by tolerant instances in its application. In other words a property is tolerant when there are cases in which we are unable to detect a change in its application after a small enough change in its degree. On the formal side, tolerant behaviour for a predicate P is described by the relation $\sim_P$ in the model (what we will call P-indistinguishability) and by the relative predicate I_P in the language.

We can read the whole phenomenon of tolerance as our inability in some cases to discern a change in the classical truth-value of a predication despite a change, although small, in the degree of the relative predicate. This reading is exactly the one endorsed by the closeness constraint (definition 2.6.4.1)[26], which forces the valuations of every two intensions for which $f \sim_{P,w} g$ to be such that $|V_w([\lambda x.Px](f)) - V_w([\lambda x.Px](g))| < 1$. This amounts to the fact that speakers are unable to classically discern the change in P-ness between f and g.

This introduction may suggest that tolerance is an epistemic phenomenon, caused by the cognitive impossibility of an agent to perceive a small enough change in some quality, not far from Williamson's theory of degree for error. The system **SK** validates this reading with the difference between D and D_I, in fact if we want to pursue a cautious interpretation which denies tolerance as a feature of the world **SK** allows us to confine tolerance to intensions, which can be intended as the properly linguistic level. Whatever the source of tolerance is, this topic is not

[26]Remind that in **SK** the tolerance relation $\sim$ ranges over the domain of intensions D_I, not over D, since D is completely classical and introducing a tolerance relation for its members would provide no information at all, because all the changes in the degree of a predicate for some elements of D are from true to false or vice versa.

our aim, it would suffice to notice that tolerant reasoning is an undeniable fact, at least as long as sorites sequences never cease to amaze the philosophical debate, in fact tolerance can be considered the root of the sorites paradox.

This well-known problem brings us to infer from the fact that a single grain of sand is not a heap and that adding a single grain does not turn something which is not an heap into a heap, to the conclusion that a pile of one thousand grains of sand is a heap. In this and the similar cases the reasoning has the form (UTP) $Px_0, \forall i(Px_i \rightarrow Px_{i+1}) \vDash Px_n$ for any arbitrary $n \in \mathbb{N}$. The inductive clause $\forall i(Px_i \rightarrow Px_{i+1})$ represents our inability to discern among small enough changes in the degree of a property in the case of a sequence of elements ordered according to their P-ness. That classical logic by itself cannot account for the sorites paradox is widely recognized, since for any predicate P that can be applied to a sorites sequence we inconsistently conclude that for every member of that sequence it holds $Px \wedge \neg Px$. What the sorites argument does is to begin an unstoppable induction[27] that forces us to accept an unwanted extension of the use of a predicate.

Returning to tolerance, each instance $Px_i \rightarrow Px_{i+1}$ of the universal inductive clause is in fact a tolerant implication, which, read as material conditional, states that it cannot be the case that Px_i is true while Px_{i+1} is false. The universal generalization of this tolerance principle is the problematic component of the sorites argument, hence tolerance is ultimately the engine of the unacceptable induction which leads to the inconsistency of the sorites. Within $\mathscr{L}'_E$ tolerance can be directly expressed thanks to the indistinguishability predicates of the form I_P, which in **SK** are given their intended meaning by the tolerance relation $\sim$ together with the closeness constraint. On the contrary in $\mathscr{L}'_E$ we cannot express the inductive clause of the sorites inference as a universal formula quantifying over indexes, but we have the tolerance principle as schema for any monadic predicate P:

(TP) $[\lambda x.Px](f), I_P fg \vDash [\lambda x.Px](g)$

All the logics defined in chapter 2 are tolerant precisely because they make (TP) valid.

As extensively illustrated in Cobreros *et al.* (2012), the mixed entailment relation from true to non-false (in their words, from strict to tolerant truth), is the key property for having a logic that validates (TP) and at the same time is consistent,

[27] Cfr. Boolos (1991).

thanks to the non-transitivity of entailment[28] which cannot take merely non-false premises. More in detail, let $V_{w,a}([\lambda x.Px](f)) = 1$ and $V_{w,a}(I_P fg) = 1$, the latter holds iff $f \sim_{P,w} g$. By the closeness constraint this implies $V_{w,a}([\lambda x.Px](g)) > 0$, therefore (TP) holds.

If we extend the language of **SK** in such a way that quantification over indexes is possible, the tolerance principle in its universally quantified form (UTP) is translated in terms of tolerance as $[\lambda x.Px](f_0), \forall i(I_P f_i f_{i+1}) \vDash [\lambda x.Px](f_n)$, which is not SK-valid. In fact this it is the result of the concatenation of successive instances of (TP) over the sequences of x_i, which is exactly what the mixed entailment forbids.

The difference in the validity of (TP) contrasted to the unsoundness of (UTP) in **SK** reflects the natural approach to tolerance. In the specific case tolerance does not constitute a problem, when we are giving a starting point x_i clearly determined as P we can without hesitation state the P-ness of the immediate successor x_{i+1}. Given the distribution of those points over a scale of gradual change of their P-ness, the change between each couple of points is too small to make any significant difference. The paradox emerges only at the general level, when we consider the result of many iteration of the previous step. Here the small modifications in the degree of P was accumulated and what singularly was an insignificant shift has now summed up to a decisive change. The semantics of **SK** returns this reading, validating (TP), the specific case, but not (UTP), the general one.

In the light of tolerance we can interpret the problematic application of vagueness to identity. If the intuitive implausibility were not enough, the Evans-Salmon argument is a constant threat to any study of vague identity. This argument, formulated by Evans (1978) and Salmon (1982), pp.243-45, is a refutation of the consistency of the notion of vague identity and it proceeds as a *reductio ad absurdum*[29].

The proof as formalized by Evans (1978) cannot be directly transposed in **SK**. For the sake of argumentation and without going into excessive technicalities, let us move to an untyped language $\mathscr{L}_E^u$ where predicates can indifferently take individual and intensional variables as arguments. In particular this holds for identity. $\mathscr{L}_E^u$ still contains an abstraction operator for predicate formation. Let $\nabla\varphi := \neg\Delta_c\varphi$.

[28]The lack of transitivity in tolerant logics is discussed in Cobreros *et al.* (2017) and Barrio *et al.* (2015).

[29]According to Lewis (1988), what Evans proved is not that there cannot be vague identity statements, which seems intuitively false, but that under an ontic reading of vagueness two objects cannot be vaguely identical. Therefore the result of Evan's argument is not that we should refuse vague identity, but that in order to retain it we have to abandon the ontic reading in favour of a semantic or representational interpretation of vagueness.

Evans' argument runs as follows[30]:

(E1) $\nabla a = b$

(E2) $[\lambda x.\nabla x = b](a)$

(E3) $\neg\nabla b = b$

(E4) $[\neg\lambda x.\nabla x = b](b)$

(E5) $a \neq b$

Let us start assuming to be indeterminate whether a and b are identical (E1), therefore a has the property of being vaguely identical to b (E2). On the contrary, b is definitely identical to itself (E3), hence b has the property of not being vaguely identical to b (E4). Since b has a property which a lacks, by the contrapositive of Leibniz's law we conclude that a is not identical to b (E5), contradiction.

About the Evans-Salmon argument there exists an extensive literature, here we are interested only in the impact that the argument has on **SK**. What we first notice is that the proof actually works in **SK**. As long as a and b are intensional variables (E1) can be assumed, otherwise it cannot be provided with any satisfying model (as we remember in **SK** only intensions can be vaguely identical). (E3) is a tautology, while (E2) and (E4) are just abstractions[31], legitimate in force of respectively (E1) and (E3). Finally, Leibniz's law is valid for individual variables in **SK**, but it is also valid for intensions as long as these are not substituted inside modal contexts. Now $\nabla := \neg\Delta_c$ does not open such a modal context (its

[30]Cobreros *et al.* (2014) refute Evans' argument using a second-order extension of their tolerant logic **st**. Thanks to the mixed entailment of **st**, which is defined as in **SK**, the premises of the argument are valid but their concatenation is not. Moreover they does not refute the argument stopping the abstraction steps, from (E1) to (E2) and from (E3) to (E4), which can be done with non-rigid terms. On the contrary they assume the interpretation of Lewis (1988) and take terms to be rigid, therefore adopting an ontic reading of vagueness.

[31]The steps (E2) and (E4) might be less straightforward than they seem. Following a remark from Parsons (2000), pp.50-52, we can argue that abstraction is a merely syntactic tool which does not guarantee that the resulting complex predicate stands for a genuine property. This is crucial for the application of Leibniz's law in step (E5), since if $[\lambda x.\nabla x = b]$ does not stand for a property then Evans' proof stops at (E4), in fact we would have no property for which a and b differs, therefore the contrapositive of Leibniz's law could not be applied. That every abstract stands for a property can be challenged by plainly contradictory predicates $[\lambda x.Px \wedge \neg Px]$ or by predicates which has no satisfying interpretation in accordance with their intended meaning, like the alleged property of being a round square $[\lambda x.Sx \wedge Rx]$.

scope is evaluated at the world of utterance), and since contraposition is a valid matatheorem of **SK** we conclude (E5).

Despite **SK** semantics allows to go throughout the demonstration to its conclusion, what we have really proved is $\nabla a = b \vDash^{SK} a \neq b$, but the mixed entailments is just telling that from the true premise $\nabla a = b$ it logically follows that $V(a \neq b) > 0$, which amounts to $V(a = b) < 1$, and in fact by the premise we already knew that precisely $V(a = b) = 1/2 < 1$.

This shows that vague identity is a completely legitimate and formally sound notion in **SK**. An interesting feature of this relation of vague identity is that it does not require the related object to be vague. To be more specific, the two objects must be intensions such that their denotations partially overlap (or at least one of them must be a multiple denoting intension), but they can be not P-vague for any predicate. Let us return to the typed language $\mathscr{L}'_E$. Let the model $M = \langle W, R, D, \sim, C \rangle$ and the assignment a be such that $a[w, f] = \{d_1, d_2\}, a[w, g] = \{d_2, d_3\}$ and C be such that $C_{w,a}(\alpha(\overline{d_1})) = C_{w,a}(\alpha(\overline{d_2})) = C_{w,a}(\alpha(\overline{d_3}))$ for every atomic formula α, with $\overline{d_i}$ name for d_i. Since C is a classical valuation, by semantics of **SK** it follows that $V_{w,a}([\lambda x.\alpha(x)](f)) \in \{1, 0\}$ and $V_{w,a}([\lambda x.\alpha(x)](g)) \in \{1, 0\}$. With respect to every predicate f and g have classical truth-values, by a reasoning similar to 2.6.4.1, so they are not vague under any determination. Nonetheless $V_{w,a}([\lambda x, y.x = y](f, g)) = 1/2$, so they are vaguely identical without being vague objects themselves. Using the definition of §3.1, f and g are vague objects in the most general sense of the term, as multiple denoting objects, but this is still a much weaker form of vagueness than any P-vagueness.

This non-classical reading of identity is successful in providing a semantics for indeterminate identity statements, but, as Williamson (2002) points out, this is not enough to be certain that what we have characterized is indeed faithful to the intended meaning of the identity relation. On the one hand it could be argued that vague identity as partial overlap is inequality in disguise, even in the case of multiple denoting objects when there is no complete correspondence we should claim that an identity statement is false, as we would in a classical bivalent logic.

On the other hand there are a few remarks to considers. First, it is prejudicial to claim that the only truthful reading of identity is the bivalent one. It is indeed intuitively difficult to conceive examples of vague identity, anyway the *prima facie* implausibility is not by itself an acceptable argument to reject the possibility of more fine-grained forms of identity, especially when we move from the world to language, where intensions have their legitimate place. Even if we want to preserve the classicality of the world, we can conciliate this with the assumption of a different behaviour of identity when it comes to linguistic entities, as we do in **SK**.

One way to justify this difference is tolerance. The whole set of denotations for a given intension is certainly a set with precise boundaries, but within the interpretation, especially under an epistemic reading, we cannot take for granted that a speaker knows exactly all the objects which constitute the denotation of a given intensional term he is using. The boundaries of an intension can be blurred, in fact the semantic content of a term whose referent cannot be ostensively pointed at is defined by the shared linguistic practice that ties the speaking community together, but no single speaker can arrogates to himself the exhaustive knowledge of that content, otherwise the whole phenomenon of vagueness would disappear. Returning to the geographical example of Mount Everest as an intensional term denoting multiple regions of space, it is uncontroversial that the region close to the peak is part of its denotation, but the further we move down from there the more dubious it gets to define whether we are still within that denotation.

If we accept that the very denotation of a term could be beyond our complete knowledge, when we compare the correspondence between two such terms the problem only amplifies. The two intensions might coincide in part, leaving a region of indeterminate overlap still undecided and, in the case of vagueness, undecidable. This denotational penumbra is not conceptually much different from the penumbra generated by predicate vagueness during a fine-grained gradation for some property.

Once we interpret vague identity as tolerant identity, paradoxes of identification such as Theseus' ship can be accounted for in **SK**[32]. These puzzles are characterized by a gradual transition in the composition of one object, and while each step is intuitively insufficient to justify a modification in the identity of the object, what results from the sum of these steps is something definitely different from the starting object. As in the case of sorites paradox, it sounds arbitrary to individuate a cut-off point at which the changing object ceases to be identifiable with the original one. Differently from sorites paradox, the notion of gradual change in identity is more controversial than that of change in the degree of a property and the respective semantics is less intuitive in the former case.

Considering the example of Theseus' ship, let d_i be the ship in which $i\%$ of the starting material has been substituted. d_0 is therefore the original ship of Theseus,

[32]Priest (2010) deals with this problem defining a non-transitive identity in a second-order variant of his logic **LP**. Cobreros *et al.* (2014) show that it is possible to account for the paradox using a tolerant notion of identity which preserves transitivity.

while d_{100} is definitely a new ship. Once an agent speaks about the ship during the reconstruction process, he refers to the ship at time t using the intensional variable s_t. Each of these intensions has a denotation of the form $a[w, s_t] = \{d_i \mid j \leq i \leq k\}$, with the interval $[j, k] \subseteq [0, 100]$. This amounts to the fact that at each time t there is a threshold of tolerance within which the speaker cannot distinguish the state of the ship, or that when it comes to its identification the state of the ship is irrelevant for the speaker until it ranges between d_j and d_k.

Let us assume that the intended speaker undoubtedly considers s_{t_0} the true Theseus' ship and $s_{t_{100}}$ a completely new construction. Consider a model where $a[w, s_{t_0}] = \{d_0, d_1\}, a[w, s_{t_{100}}] = \{d_{99}, d_{100}\}$ and for each $1 \leq i \leq 99, a[w, s_{t_i}] = \{d_{i-1}, d_i, d_{i+1}\}$. In this model for each k, $[\lambda x, y.x = y](s_{t_k}, s_{t_{k+1}})$ and $[\lambda x, y.x = y](s_{t_k}, s_{t_{k+2}})$ are non-false but tolerantly true, i.e. evaluated $1/2$, since the respective denotations partially overlap. This generates a penumbra which makes impossible for the speaker to claim whether the identity between the two states of the ship considered at two different times subsists. At the same time, $[\lambda x, y.x = y](s_{t_k}, s_{t_{k+n}})$ is false for $n > 2$, for the speaker the gap between s_{t_k} and $s_{t_{k+n}}$ is wide enough to justify the determinate difference between the two states of the ship. By this reasoning we obtain that while we cannot tell that the ship is definitely different from Theseus' one once only a few of the original pieces have been removed (i.e. $V_{w,a}([\lambda x, y.x \neq y](s_{t_0}, s_{t_1})) \neq 1$), the final result is certainly not the ship of Theseus anymore ($V_{w,a}([\lambda x, y.x \neq y](s_{t_1}, s_{t_{100}})) = 1$). This solution applies to similar paradoxes and it subsumes under tolerance an otherwise heterogeneous set of phenomena which range from sorites sequences to puzzles of composition.

3.4 A logic for topics

We turn our attention to the logic **WK**, based on weak Kleene truth-tables. As we remember from chapter 2, the peculiarity of weak Kleene semantics is that the non-classical value $1/2$ is infectious, in the sense that if a subformula is evaluated as $1/2$ then every superformula containing it is evaluated $1/2$ as well[33].

This rather bizarre semantic behaviour poses the challenge of providing a satisfying interpretation of $1/2$ in weak Kleene. Clearly we cannot adopt the intended meaning of $1/2$ of strong Kleene systems as lack of sufficient classical information. In the literature the early works of Bochvar (1938) and Halldén (1949) provided

[33]This is not strictly correct in a language containing an abstraction operation like $\mathscr{L}_E$, as noticed in fact 2.3.0.1.

the interpretation that has become the standard reading of the non-classical value of weak Kleene: $1/2$ is intended as nonsensical and the logics based on weak Kleene semantics are logics of nonsense. According to this interpretation a sentence is evaluated as $1/2$ whenever it is ungrammatical, i.e. when that piece of language is unintelligible for a competent speaker. Once an utterance is nonsensical, however we extend it the original ungrammatically infects the new sentence. "oyz#8w" is ungrammatical in English and every sentence containing it is ungrammatical as well, like "oyz#8w and Socrates is a philosopher", no matter what truth-functions are employed to compose the sentence[34]. This intuition corresponds to the infectious property of $1/2$ in weak Kleene semantics.

Further interpretations have been proposed, like those of Fitting (1994) and Szmuc (2019), nevertheless the overall effort of the research in the field of weak Kleene logics remains relatively modest. Recently Beall (2016) has sketched an interpretation of weak Kleene logic in terms of topics, where 1 corresponds to true and on-topic, 0 to false and on-topic, and $1/2$ to off-topic. In this way a weak Kleene model is to be intended as a discourse in which some topics are debated.

In this paragraph we are going to elaborate Beall's proposal, providing a full formal framework for a logic for topics. First we need a definition of topic, discussion and some of the axioms governing these terms:

(T1) A *simple topic* is a question whether a property holds for some individuals.

(T2) A sentence addresses a topic and how that topic is settled is expressed by the sentence's truth or falsity.

(T3) The *complex topic* addressed by a composite sentence is the sum of the topics addressed by its simpler components.

(T4) A *discussion* is a collection of topics, the discussed topics.

(T5) In a proper rational discussion all the discussed topics should be sharply determined.

These axioms will be taken as foundations for a formal theory of topics. (T1) is an informal but precise definition of topic, which can be made formal with the following:

[34]It can be replied that "oyz#8w and not oyz#8w" is always false and "oyz#8w or not oyz#8w" always true. This claim is much less indisputable in the case of weak Kleene logic compared to strong Kleene. Whether the claim holds or not, we have already acknowledged in chapter 2 that all the logics defined in this study are unable to account for penumbral connections, an unavoidable feature of most multi-valued logics.

Definition 3.4.0.1 (Topic). Given a WK-model $M = \langle W, R, D, \sim, V\rangle$, a topic or subject in M is a couple $\langle P^n, \langle d_1, ...d_n\rangle\rangle$ with $d_1, ...d_n \in D$.

A topic is a couple composed by a predicate and a n-tuple of elements about which the predicate may hold. In this way the topic can be interpreted as a still undetermined relation between a predicate and its argument. This definition satisfies (T1), in fact this undetermined relation can be read as the unanswered question whether the n-tuple of elements $\langle d_1, ...d_n\rangle$ is P or not[35].

The truth-value of a proposition settles the question about the topic addressed by that same proposition, this is what (T2) says: if the proposition is true then the relation expressed by the topic holds, if it is false it does not hold. It is immediate to define what the topic addressed by an atomic formula is. For a model M, world w and assignment a the topic addressed by $Px_1...x_n$ at w is $\tau_{w,a}(Px_1...x_n) = \{\langle P, \langle a(x_1, w)...a(x_n, w)\rangle\rangle\}$, where τ is the topic-function. In order to make the notation easier, in this paragraph we use the notation $a(w, x)$ with round brackets to denominate the single denotation of the individual variable x according to the admissible assignment a, namely $a(w, x) := d \in D$ such that $d \in a[w, x]$.

Moving to complex formulae (T3) tells us how to compute their addressed topics, following the intuition that a composite proposition talks about what its simpler components talk about. (T3) is therefore a sort of law of compositionality for topics. Using this insight, we can extend τ to the whole language. Let $T_{\mathscr{L}_E,M} = \{\langle P^n, \langle d_1, ...d_n\rangle \mid P^n \in \mathit{Pred}(\mathscr{L}_E)$ and $d_1, ...d_n \in D\}$ be the set of all the topics in M expressible by $\mathscr{L}_E$. Given a model M, world w and assignment a the function $\tau : \mathit{Form}(\mathscr{L}_E) \mapsto \mathscr{P}(T_{\mathscr{L}_E,M})$ is defined recursively as follows:

- $\tau_{w,a}(Px_1...x_n) = \{\langle P, \langle a(x_1, w)...a(x_n, w)\rangle\rangle\}$
- $\tau_{w,a}(x = y) = \{\langle =, \langle a(x, w), a(y, w)\rangle\rangle\}$
- $\tau_{w,a}(\neg\varphi) = \tau_{w,a}(\varphi)$

[35] According to (T1) topics are hybrid entities, composed by a linguistic term, the predicate, and a list of objects, the n-tuple $\langle d_1, ...d_n\rangle$. It could be objected that instead of the symbol P it would be better to take the property which the predicate stands for, namely $V(P)$. This choice does not change the substance of this exposition, nonetheless we prefer to stick to the use of predicates since topics are linguistic entities and by cardinality reasoning we are not guaranteed that every property (i.e. subset of D^n) is expressible by $\mathscr{L}_E$. If we intend topics as questions, per (T1), we want at least to be able to utter it, which amounts to its expressibility within the object language.

- $\tau_{w,a}(\varphi \vee \psi) = \tau_{w,a}(\varphi) \cup \tau_{w,a}(\psi)$
- $\tau_{w,a}(\forall x\varphi) = \bigcup\{\tau_{w,b}(\varphi) \mid b \text{ is a } x\text{-variant of } a\}$
- $\tau_{w,a}(\Box\varphi) = \bigcup\{\tau_{v,a}(\varphi) \mid v \in R[w]\}$
- $\tau_{w,a}([\lambda x.\varphi](f)) = \tau_{w,b}(\varphi)$ where b differs from a at most for $b(x,w) = a(f,w)$.

Considering a world w, some of the topics of M are settled at w, namely the topics $\tau_{w,a}(\varphi)$ for all φ such that $V_{w,a}(\varphi) \in \{1, 0\}$. In other words, the topics of all the classically evaluated formulae at w are settled. **WK** is a 3-valued logic though, so we are still left with a set of formulae which have no classical truth-value at w. Here Beall's insight comes into play: when a formula is evaluated as $1/2$ that formula is off-topic.

This last information allows us to naturally interpret the worlds of a model as discussions, where some topics are discussed and other not. To be more precise, the set of worlds W can be read as a collection of stages of a discussion[36], at each stage some topics have been discussed, therefore their truth-values have been decided with some argumentation. This is what we require per (T5), in a rational discussion (opposed to e.g. a merely rhetorical speech) the topics at hand should receive a classical value (what we meant by sharply determined in (T5)). This idealized notion of discussion intends a world w as a concluded discussion, where all the topics considered at w have been finally settled and in which no unsolvable problem is ever considered or, which is the same, in which once a topic is found undecidable it is discarded as off-topic.

We can now appreciate how τ returns a satisfying reading of the infectious property of $1/2$ in weak Kleene semantics. As we recall, in that system it is assumed that the nonsensicality of a part of a sentence is enough to compromise the whole sentence, similarly when we move to a discursive setting if something is off-topic then the employment of that piece of information in more complex argument makes that argument off-topic too. Let $\mathbb{T}(w) = \{\tau_{w,a}(\varphi) \mid V_{w,a}(\varphi) \in \{1, 0\}\}$ be the set

[36]We left open the problem of which constraint put on the accessibility relation R in order to model a discussion. In the current setting, R is taken as a general alethic accessibility. An interesting reading would be to take W as a set of progressive stages of a discussion, therefore giving to R a temporal reading. This choice resembles the reading of Kripke semantics for intuitionistic logic and in fact a persistence clause should be introduced in the semantics: when the classical value of φ is established at w then φ retains that same value at every successor of w. Persistence amounts to say that a topic is settled iff a (true) proof or refutation of the formula which expresses that topic is provided.

of on-topic subjects at w. We have an immediate extensional visualization of the infectious behaviour of off-topic subjects, in fact a formula φ is on-topic at w iff $\tau_{w,a}(\varphi) \subseteq \bigcup \mathbb{T}(w)$. If φ is off-topic then for some $t \in \tau_{w,a}(\varphi)$ we have $t \notin \bigcup \mathbb{T}(w)$, therefore for every ψ such that $\tau_{w,a}(\varphi) \subseteq \tau_{w,a}(\psi)$ it follows that $\tau_{w,a}(\psi) \nsubseteq \bigcup \mathbb{T}(w)$, which means that ψ is off-topic too. This is exactly the infectious property.

The topic-function τ is not sufficient as a substitute to the valuation V in a semantics for **WK**, wince τ is not able to capture the truth-value of formulae. Nonetheless τ is sensitive to differences which does not emerge at the semantic level, e.g. it distinguishes between tautologies as they may address different topics, which could be useful in order to implement some form of relevance in our logic. τ can also be used to characterize the topics in a model M, using it as a surjective mapping to the join-semilattice $\langle Fin(T_{\mathscr{L}_E,M})\backslash\{\emptyset\}, \subseteq\rangle$, where $Fin(T_{\mathscr{L}_E,M})$ is the set of all finite subsets of $T_{\mathscr{L}_E,M}$[37]. A finer-grained characterization can be provided building a function which maps each world w to its semilattice of topics $\langle \mathbb{T}(w), \subseteq\rangle$, a substructure of $\langle Fin(T_{\mathscr{L}_E,M})\backslash\{\emptyset\}, \subseteq\rangle$. This is just a suggestion and we are not going to follow the algebraic approach further.

Moving to **WK** properly, this logic is determined by a mixed entailment relation, which goes from true premises to non-false conclusions. As previously proved, this peculiar entailment characterizes a conservative extension of classical logic. Nevertheless the metatheory is partially non-classical, notably for the non-transitivity of logical consequence. This is a sufficient reason to ask for an interpretation of the structure of reasoning conveyed by **WK**.

The question can be rephrased as: what does it mean to allow for off-topic conclusions? A discussion, in the narrow sense of (T5), is a dialogue whose aim is to determine the truth or falsity of some matter at hand, the on-topic subjects. In classical reasoning, an argument is a concatenation of truth propositions, starting from axioms or already established facts and extended applying sound inferential rules. Concatenation is not a viable option is **WK** due to the lack of transitivity, but it is straightforward to prove that entailment is still transitive when only true

[37] $\langle Fin(T_{\mathscr{L}_E,M})\backslash\{\emptyset\}, \subseteq\rangle$ is only a join-semilattice since it does not contain the empty set as its bottom element, which would be the meet of the topics of any two atomic sentences, therefore it is not closed under meet. In the current interpretation of weak Kleene the non-classical value is not taken as nonsensical but as off-topic, therefore meaningful, while admitting the empty set in the semilattice amounts to recognize a meaningless topic, the null topic $\emptyset$. This is one of the reasons why we are not considering as a characterizing structure the Boolean algebra $\langle \mathscr{P}(T_{\mathscr{L}_E,M}), \subseteq\rangle$, the other reason is that our definition of topic does not allow for infinite topics, in fact these could be expressed only by infinitely long sentences, which are particularly inadmissible in a discursive setting.

conclusions are employed as new premises. Let us call this entailment from true to true strict entailment and define an *argument* in **WK** as a concatenation of strict entailments.

On a less idealized perspective, discussing can be a problematic procedure, in which mistakes are made and fallacies occur. Interpreting the accessibility relation R in a temporal sense, the change of truth-value for a formula between two world w, v such that Rwv can be intended as a form of belief revision, therefore the fact that a topic is settled only at some worlds is a form of defeasible knowledge. What the mixed entailment underlies is something different. Also within a discussion (i.e. in a single world w) inappropriate inferences may happen, one example is rhetoric argumentation. It is here that argumentation brings us outside the scope of the discussion, namely off-topic.

When this happens it is not evident whether we should accuse the speaker of using unsound reasoning, because technically nothing false has been stated (since, with some idealization, in **WK** what is off-topic is evaluated $1/2$, therefore, on an epistemic reading, the participants of the discussion have no knowledge of its classical truth-value). What is certainly plausible is that a line of reasoning which goes off-topic should be stopped, and this is exactly what the non-transitivity of entailment in **WK** does: given an argument which reaches an off-topic conclusion, that argument cannot be extended further, since the off-topic conclusion is not suitable to be the premise of a further entailment.

The notion of logical consequence within the discursive interpretation of **WK** can be read as a form of fallible reasoning, fallible not in the sense of unsound but instead in the sense of potentially leading to irrelevant information, namely in the case of off-topic conclusions. In other terms, a single entailment is a potential ending segment for an argument. Per the previous definition, arguments are concatenations of strict entailments, but if we extend one argument with a normal (mixed) entailment we must also conclude the argument, since the conclusion is guaranteed only to be non-false, which is not enough to start a new entailment, that conclusion could be off-topic and unsuitable for a premise. Despite the fact that it is an irrelevant contribution to the discussion, the off-topic conclusion is not removed from the conversation as if it were a false piece of information reached by unsound reasoning. Instead it is accepted (in the sense that the semantics allows to infer it) but, in some sense, left aside from being used again as new premise.

Within this setting vagueness becomes a special mode of off-topic. Following (T5), a necessary condition for a subject to be on-topic is that it must be sharply determined, in the sense the the discussion can eventually validate or refute it.

In the formal system this amounts to the fact t is on-topic at w iff the formula which addresses t receives a classical truth-value at that world. Now vagueness is characterized by the possibility of borderline cases, which by definition are those situations where classical bivalence fails. Whatever the nature of vagueness is, in a discursive setting a subject is vague not because of a form of laziness among the speakers who are not putting enough effort into resolving the issue. Instead vagueness is something intrinsic to the speakers' linguistic practice, which can be overcome with an agreement among the participants to the discussion or with the introduction of new background information. Either way this formally amounts to move to a world where the topic receives a classical value. In this sense vagueness is a form of irreducible off-topic, resolving vagueness is not something which can be done within a single discussion w (since being evaluated $1/2$ a vague statement cannot be discussed at w according to (T5)), instead it is a cross-world matter which lies outside the power of discussions as they are formalized in **WK**.

3.5 Conclusion

We have studied some of the philosophical implications of the theory of vagueness conveyed by the formalism of chapter 2. On the part of strong Kleene logics, **SK** is a suitable logic for an epistemic interpretation of vagueness, where a major role is played by the ontological difference between extensional and intensional objects. Vague objects, as objects which can potentially being vague under some respect, have found a place in the ontology underlying **SK**. Likewise, a justification for the notion of vague identity has been provided in terms of tolerance. Intensions, with the possibility of assigning them multiple denotations, have played a key role in both these subjects. The topic of intensionality can be greatly expanded, bringing the research outside the scope of philosophy of language. A closely related line of further study is that of the modal logic which better model vagueness. In fact we have restricted our consideration to logics which are variations of the minimal normal modal logic **K**, though it can be discussed if some constraints should be put on the accessibility relation when we consider vagueness. The unusual approach of weak Kleene semantics has been considered too, resulting in a logic for topics. What we have provided is a sketch of a formal theory of topics, which can be investigated outside the particular problem of vagueness, leading to a more ample and autonomous research.

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