Polyhedral Completeness in Intermediate and Modal Logics

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis explores a newly-defined polyhedral semantics for intuitionistic and modal logics. Formulas are interpreted inside the Heyting algebra of open subpolyhedra of a polyhedron, and the modal algebra of arbitrary subpolyhedra with the topological interior operator. This semantics enjoys a Tarski-style completeness result: **IPC** and **S4.Grz** are complete with respect to the class of all polyhedra. In this thesis I explore the general phenomenon of completeness with respect to some class of polyhedra.

I present a criterion for the polyhedral completeness of a logic based on Alexandrov's *nerve* construction. I then use this criterion to exhibit an infinite class of polyhedrally-complete logics of each finite height, as well as demonstrating the polyhedral completeness of Scott's logic **SL**. Taking a different approach, I provide an axiomatisation for the logic of all convex polyhedra of each dimension n.

The main conceptual contribution of this thesis is the development of a combinatorial approach to the interaction between logic and geometry via polyhedral semantics.

Contents

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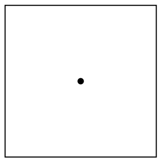
Chapter 0



Introduction

Let no one ignorant of geometry enter here

According to legend, inscribed above the door to Plato's Academy

Geometry and logic share a long friendship. Mathematical logic in the West probably first found its feet in connection with the emergence of geometry as an *a priori* discipline [KK62]. The Ancient Greeks inherited a collection of empirically-verified geometric observations from the Egyptians and Babylonians, and it was their great achievement to systematise the study and place it on a solid logical basis, culminating in Euclid's celebrated *Elements*.

It is only relatively recently, however, that a deeper connection between geometry and logic has started to emerge. Multiple lines of research have explored links with diverse areas, from type theory to model theory. In this thesis I contribute to a line of research which seeks to traverse another fibre of this connection, by relating intuitionistic and modal logic with algebras of polyhedra. In order to warm up, and to provide some context for the present investigation, let us take a brief tour of a selection of already-established logic-geometry links. This is by no means a comprehensive overview, but I hope that these examples serve to give a flavour of the situation as it stands today. Numerous other examples can be found in the Handbook of Spatial Logics [APB07], which provides an excellent survey of some of the recent developments in this area.

Perhaps the most direct link between logic and geometry can be found in Alfred Tarski's seminal work investigating the logical foundations of geometry, following David Hilbert's programme of rigorously founding Euclid's work [Hil50]. In [Tar59], Tarski shows how all of *elementary geometry* — "that part of Euclidean geometry which can be formulated and established without the help of any set-theoretical devices" — can be formalised in first-order logic using only the notions of *betweenness* and *equidistance* as non-logical concepts. He demonstrates that such a system is decidable, but not finitely axiomatisable. This work was further developed by Tarski and his followers, looking for instance at non-Euclidean geometries. More recently, fragments of elementary geometry have been formalised using modal logic, by interpreting the modal $\Box$ and $\Diamond$ in interesting ways. For instance, taking *lines in the plane* as the domain of valuation, one may consider the modalities $[\|]$ — interpreted as "holds in all parallel lines" — and $[\times]$ — interpreted as "holds in all intersecting lines" [BG02] (see also [BGKV07] for an overview of this kind of modal formalism).

Another fibre of the logic-geometry link is *topos theory*. Alexander Grothendieck invented toposes as a generalisation of topological spaces, to deal with the numerous situations occurring in mathematics which involve topology-like and continuity-like arguments, but where a genuine topological space is absent [Vic07]. Grothendieck's toposes play an important role in algebraic geometry, but topos theory also has close connections to logic via the more general notion of an *elementary topos*. The situation is summed up succinctly by Saunders MacLane and Ieke Moerdijk, who begin their prologue to [MM94] as follows.

> A startling aspect of topos theory is that it unifies two seemingly wholly distinct mathematical subjects: on the one hand, topology and algebraic geometry, and on the other hand, logic and set theory. Indeed, a topos can be considered both as a "generalized space" and as a "generalized universe of sets".

As a somewhat related example, consider the relation, discovered around the turn of the 21$^{\text{st}}$ century, between Martin-Löf type theory and homotopy theory. The former is a theory of types originally intended for use as a foundation for constructive mathematics. Its pertinent feature is that it accommodates two kinds of equality: the usual *definitional* equality, and a (new) *propositional* equality. Propositional equality between objects a and b amounts to the *identity type*, $\mathrm{Id}(a,b)$, being inhabited. Objects of type $\mathrm{Id}(a,b)$ can be thought of both as proofs of the equality of a and b, and as 'paths' from a to b. But for any two such p,q in $\mathrm{Id}(a,b)$, one may form *their* identity type $\mathrm{Id}(p,q)$. If p and q are thought of as paths from a to b, then the objects of $\mathrm{Id}(p,q)$ should be thought of as homotopies between p and q. And this scheme continues to higher levels. This culminates in an elegant relationship between type theory and (higher) homotopy theory, which allows the two-way transfer of ideas. Indeed, Steve Awodey begins his [Awo10] on the subject as follows.

> The purpose of this informal survey article is to introduce the reader to a new and surprising connection between Geometry, Algebra, and Logic, which has recently come to light in the form of an interpretation of the constructive type theory of Per Martin-Löf into homotopy theory, resulting in new examples of certain algebraic structures which are important in topology.

In a rather different vein, another branch of logic, namely *model theory*, is a source of numerous connections. The area of *geometric stability theory* seeks to classify the models of first-order theories in terms of general dimension-like properties stemming from notions of independence [Pil96]. The key examples of this are linear independence in vector spaces and algebraic independence in algebraically closed fields. Geometric stability theory has found applications in diophantine geometry, beginning with Ehud Hrushovski's proof of the Mordell-Lang conjecture [Hru96] (see also Anand Pillay's exposition [Pil97]).

From topological to polyhedral semantics. The genesis of many connections between logic and geometry was the discovery of topological semantics for intuitionistic and modal logic, as pioneered by Marshall Stone [Sto38], Tang Tsao-Chen [Tsa38], Alfred Tarski [Tar39] and John C. C. McKinsey [McK41]. This semantics is now well-known. In short, one starts with a topological space X, and interprets intuitionistic formulas inside the Heyting algebra of open sets of X, and modal formulas inside the modal algebra of subsets of X with $\Box$ interpreted as the topological interior operator. A celebrated result due to Tarski [Tar39] states that this provides a complete semantics for intuitionistic propositional logic (**IPC**) on the one hand, and the modal logic **S4** on the other. Moreover, one can even obtain completeness with respect to certain individual spaces. Specifically, McKinsey and Tarski showed [MT44] that for any separable metric space X without isolated points, if **IPC** $\nvdash \phi$, then ϕ has a countermodel based on X, and similarly with **S4** in place of **IPC**. Later, this result was refined still further by Helena Rasiowa and Roman Sikorski, who showed that one can do without the assumption of separability [RS63].

This result traces out an elegant interplay between topology and logic; however, it simultaneously establishes limits on the power of this kind of interpretation. Indeed, examples of separable metric spaces without isolated points are the n-dimensional Euclidean space $\mathbb{R}^n$ and the Cantor space 2^ω. What McKinsey and Tarski's result shows is that — topologically speaking — the logics of these spaces are the same, namely **IPC** or **S4**. The upshot is that topological semantics does not allow logic to capture much of the geometric content of a space.

A natural idea is that, if we want to remedy the situation and allow for the capture of more information about a space, then we need a more fine-grained algebra than the Heyting algebra of open sets, or the modal algebra of arbitrary subsets with the interior operator. This idea was developed by Marco Aiello, Johan van Benthem, Guram Bezhanishvili and Mai Gehrke. They consider the modal logic of *chequered* subsets of $\mathbb{R}^n$: finite unions of sets of the form $\prod_{i=1}^n C_i$, where each $C_i \subseteq \mathbb{R}$ is convex ([ABB03] and [BBG03]; see also [BB07]).

In this thesis, I pick up a line of research, initiated by [BMMP18] and further investigated in [Gab+18], which takes this algebra-refinement idea one step further. Since our aim is to be able to capture some of the geometric content of a space, it is natural to restrict attention to topological spaces and subsets which are *polyhedra* (of arbitrary dimension). It turns out that this works: after making this restriction, one finds oneself in an environment which is still logic-friendly. That is, the set $\mathrm{Sub}_o(P)$ of open subpolyhedra of P is a Heyting algebra under $\subseteq$ (and a similar result holds in the modal case). The main result of [BMMP18] is that more is true. A polyhedral analogue of Tarski's theorem holds: these polyhedral semantics are complete for **IPC** and **S4.Grz**. Furthermore, this approach delivers at least some of what we wanted: logic can capture the dimension of the polyhedron in which it is interpreted, via the bounded depth schema.

Two lines of approach. The Main Question driving the investigation in this thesis is the following. Which other geometric properties of polyhedra can be captured under these polyhedral semantics? Another way of putting this is: which logics are complete with respect to some class of polyhedra? In fact, these two dual expressions exemplify the two ways in which we will approach the problem. Starting with a class **C** of polyhedra (say, specified by a certain geometric property), one can ask: what is the logic of **C**? This is the approach taken in Chapter 4, where we consider the logic of the class of convex polyhedra in each dimension.

Going in the other direction, one starts with a logic $\mathscr{L}$, and asks whether it is the logic of some class of polyhedra. This is the theme of Chapter 3, in which we investigate a class of logics axiomatised by certain *Jankov-Fine formulas* — formulas which encode the intuition of 'forbidding configurations in Kripke frames'. The key piece of theory here, and what ultimately forms a bridge between the two approaches, is a connection between the fundamental notion of polyhedral *triangulation* and the construction of the *nerve* of a poset. This connection will furnish us in Chapter 2 with a criterion for the polyhedral completeness of a logic expressed purely in terms of finite posets. Using this criterion, the logic approach transforms into a combinatorial problem on finite posets.

Outline and main results. In Chapter 1, I remind the reader of the key logical machinery which will be active in this thesis, and cover the basic parts of polyhedral geometry which we will need. I then unite these two areas by showing that the set $\mathrm{Sub}_{\mathrm{o}}(P)$ of open subpolyhedra of a polyhedron P forms a locally-finite Heyting algebra, following [BMMP18].

This unison is deepened in Chapter 2. I define the nerve $\mathcal{N}(F)$ of a poset F, and give two ways in which it is used to relate logic with polyhedral geometry. (1) The nerve enables the geometric realisation of F: there is a polyhedron P which maps in an appropriate way onto F. This then yields the final piece in the proof, following [BMMP18], that polyhedral semantics is complete for **IPC**. (2) The nerve construction is closely related to the operation of *barycentric subdivision* on a triangulation. Exploiting this relation I present a proof — from joint work with Nick Bezhanishvili, David Gabelaia and Vincenzo Marra — of the *Nerve Criterion for polyhedral completeness*: a logic L is complete with respect to some class of polyhedra if and only if it is the logic of a class of finite frames closed under taking nerves. Viewing this result in terms of Kripke frames, we can say that "the logic of a polyhedron is the logic of the iterated nerves of any one of its triangulations". The criterion yields many negative results, showing in particular that there are continuum-many non-polyhedrally-complete logics with the finite model property.

At this point, the only logics known to be polyhedrally-complete are, speaking intuitionistically, **IPC** and the logic $\mathbf{BD}_n$ of bounded depth n, for each n. In Chapter 3, I expand the known domain of polyhedrally-complete logics, following the logical approach mentioned above. I consider logics defined using *starlike trees* as *forbidden configurations* — i.e. logics defined by the *Jankov-Fine formulas* of a collection of trees with a special property: trees which only branch at the root. Exploiting the Nerve Criterion, I prove that every such logic is polyhedrally-complete if and only if it has the finite model property. This yields an infinite class of polyhedrally-complete logics of each finite height, as well as one of infinite height: Scott's logic **SL**. As forbidden configurations, starlike trees turn out to have a clear geometric meaning, expressing connectedness properties of polyhedral spaces, and this provides one answer to the Main Question. One might wonder if a generalisation is possible to arbitrary trees, or even to a wider class of frames. As to the latter, some negative results are known; see Corollary 4.12. For the former, the situation is rather obscure, and it is not clear whether it is possible to account for the additional complexity introduced by allowing branching at higher points of the tree; see the discussion on 'general trees' in Section 3.4.

Chapter 4 takes the other approach, and considers a very geometrically-motivated question: what is the logic of the class of n-dimensional convex polyhedra? This turns out to be axiomatised by the Jankov-Fine formulas of three simple starlike trees, and Chapter 4 is devoted to a proof of this fact. For soundness, the proof follows using the results in Chapter 3 (however, a direct geometric proof is possible). As to completeness, the proof proceeds in two stages. The first stage is combinatorial, and involves showing that the logic axiomatised by these Jankov-Fine formulas is complete with respect to a certain class of finite posets called *saw-topped trees*. To conclude, I show that each of these finite posets can be realised geometrically in an n-dimensional convex polyhedron.

A triad of fields. This thesis develops the interplay of a triad of fields (Figure 1). Geometric methods are combined with techniques from the logical combinatorics of finite frames, as well as combinatorial geometry, in order to deepen the exciting new link recently established between logic and polyhedra. This area is still in its infancy, and there are many interesting open problems and directions for future research. The natural ultimate goal would be a full classification of all polyhedrally-complete logics, which

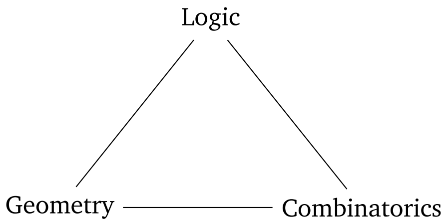
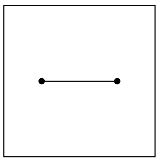


Figure 1: The triad of fields

would provide a comprehensive answer to the Main Question of this thesis. But other directions present themselves, such as questions of decidability, or the intriguing prospect of using logical methods to prove classical theorems in geometry. I briefly explore these ideas and others in the conclusion.

Chapter 1



Background and Set-up

In this chapter, I go over the needed background from logic and geometry, and set up the first link between them. Throughout, I will assume familiarity with basic topology and linear algebra. I will also assume knowledge of elementary notions from category theory, such as *functor*, *equivalence* and *duality*. For a clear introduction to these concepts I refer the reader to [Awo06], but this knowledge is not essential for most of the thesis.

1.1 Logical Machinery

The main kind of logic considered in this thesis is *intuitionistic logic*. In this section I remind the reader of the main definitions and results which we will need later on, and also set up some notation, primarily following [CZ97]. Another kind of logic, namely *modal logic*, is also important. However, as we shall see, results in the present setting transfer freely between intuitionistic logic and modal logic, and it suffices to consider only one of the two. I opt for the former, in line with [BMMP18].

Intermediate logics. We begin with a set $\mathfrak{Prop}$ of propositional variables, and generate the set $\mathfrak{Form}$ of formulas in the usual way, using the connectives $\bot$, $\wedge$, $\vee$ and $\to$. A *logic* $\mathscr{L}$ is a deductively-closed set of formulas. Write $\mathscr{L} \vdash \phi$ for $\phi \in \mathscr{L}$. The logic **IPC** is the standard intuitionistic propositional logic. An *intermediate logic* is a consistent logic extending **IPC**. Classical propositional logic, **CPC**, is the largest intermediate logic. I will usually use the term 'logic' as a shorthand for 'intermediate logic'. The actual gory syntax of the logics plays a rather ancillary role in this thesis, and we will mainly be concerned with its semantic aspects. I will outline here two standard types of structure in which intermediate logics are interpreted.

Posets as Kripke frames. A *Kripke frame* for intuitionistic logic is simply a poset $(F, \leqslant)$. For technical reasons, let us allow $\varnothing$ as a frame. The relation $\vDash$ is defined in the usual way. Given a class of frames $\mathbf{C}$, its *logic* is:

$$\mathrm{Logic}(\mathbf{C}) := \{\phi \in \mathfrak{Form} \mid \forall F \in \mathbf{C} \colon F \vDash \phi\}$$

Conversely, given a logic $\mathscr{L}$, define:

$$\mathrm{Frames}(\mathscr{L}) := \{F \text{ Kripke frame} \mid F \vDash \mathscr{L}\}$$
$$\mathrm{Frames}_{\mathrm{fin}}(\mathscr{L}) := \{F \text{ finite Kripke frame} \mid F \vDash \mathscr{L}\}$$

A logic $\mathscr{L}$ has the *finite model property* (f.m.p.) if it is the logic of a class of finite frames. Equivalently, if $\mathscr{L} = \text{Logic}(\text{Frames}_{\text{fin}}(\mathscr{L}))$.

Proposition 1.1. (1) **IPC** is the logic of the class of all frames.

(2) **IPC** has the finite model property, so that **IPC** is the logic of the class of all finite frames.

Proof. See [CZ97, Theorem 2.43, p. 45 and Theorem 2.57, p. 49]. □

The structure of Kripke frames. Let us carve out some additional vocabulary and notation. Fix posets F and G. A *subframe* of F is a subset $H \subseteq F$ regarded as a subposet. For any $x \in F$, its *upset*, *downset*, *strict upset* and *strict downset* are defined, respectively, as follows.

$$\begin{aligned}\uparrow(x) &:= \{y \in F \mid y \geqslant x\}\\ \downarrow(x) &:= \{y \in F \mid y \leqslant x\}\\ \Uparrow(x) &:= \{y \in F \mid y > x\}\\ \Downarrow(x) &:= \{y \in F \mid y < x\}\end{aligned}$$

For any set $S \subseteq F$, its *upset* and *downset* are defined, respectively, as follows.

$$\begin{aligned}\uparrow U &:= \bigcup_{x \in U} \uparrow(x)\\ \downarrow U &:= \bigcup_{x \in U} \downarrow(x)\end{aligned}$$

A subframe $U \subseteq F$ is *upwards-closed* or a *generated subframe* if $U = \uparrow U$. It is *downwards-closed* if $\downarrow U = U$. The *Alexandrov topology* on F is the set of its upwards-closed subsets. This constitutes a topology on F. In the sequel, we will freely switch between thinking of F as a poset and as a topological space. Note that the closed sets in this topology correspond to downwards-closed sets.

A *top element* of F is $t \in F$ such that $\mathsf{depth}(t) = 0$. The set of top elements in F is denoted by $\mathsf{Top}(F)$; let $\mathsf{Trunk}(F) := F \setminus \mathsf{Top}(F)$. The *top width* of F is $|\mathsf{Top}(F)|$. For any $x, y \in F$, say that x is an *immediate predecessor* of y and that y is an *immediate successor* of x if $x < y$ and there is no $z \in F$ such that $x < z < y$. Write $\text{Succ}(x)$ for the collection of immediate successors of x.

A *chain* in F is $X \subseteq F$ which as a subposet is linearly-ordered. The *length* of the chain X is $|X|$. The chain X is *strict* if there are no $x < y < z$ such that $x, z \in X$ but $y \notin X$. Take any subframe $H \subseteq F$. A chain $X \subseteq H$ is maximal (in H) if there is no chain $Y \subseteq H$ such that $X \subset Y$ (i.e. such that X is a proper subset of Y). The *height* of H is the element of $\mathbb{N} \cup \{\infty\}$ defined by:

$$\mathsf{height}(H) := \sup\{|X| - 1 \mid X \subseteq H \text{ is a chain}\}$$

For notational uniformity, say that this value is also the *depth* of H, $\mathsf{depth}(H)$. Let $\mathsf{height}(\varnothing) = \mathsf{depth}(\varnothing) = -1$. Note that these definitions apply when $H = F$. For any $x \in F$, define its *height* and *depth* as follows.

$$\begin{aligned}\mathsf{height}(x) &:= \mathsf{height}(\downarrow(x))\\ \mathsf{depth}(x) &:= \mathsf{depth}(\uparrow(x))\end{aligned}$$

The *height* of a logic $\mathscr{L}$ is the element of $\mathbb{N} \cup \{\infty\}$ given by:

$$\mathsf{height}(\mathscr{L}) := \sup\{\mathsf{height}(F) \mid F \in \text{Frames}(\mathscr{L})\}$$

A frame F has *uniform height* n if every top element has height n.

The poset F is *rooted* if it has a minimum element, which is called the *root*, and is usually denoted by $\bot$. Define:

$$\mathrm{Frames}_\bot(\mathscr{L}) := \{F \in \mathrm{Frames}(\mathscr{L}) \mid F \text{ is rooted}\}$$
$$\mathrm{Frames}_{\bot,\mathrm{fin}}(\mathscr{L}) := \{F \in \mathrm{Frames}_{\mathrm{fin}}(\mathscr{L}) \mid F \text{ is rooted}\}$$

PROPOSITION 1.2. $\mathrm{Frames}_\bot(\mathbf{CPC}) = \{\bullet\}$, the singleton of the 1-element poset.

Proof. Note that if $F \vDash p \vee \neg p$ and F is rooted then $F = \bullet$. $\square$

The *comparability relation* $\bowtie$ on F is defined:

$$x \bowtie y \quad \Longleftrightarrow \quad (x < y \text{ or } y < x)$$

Say that x and y are *comparable* if $x \bowtie y$. The *comparability graph* of F is the graph $(F, \bowtie)$. A *path* in F is a path in its comparability graph — in other words, a sequence $p = x_0 \cdots x_k$ of elements of F such that for each i we have $x_i \bowtie x_{i+1}$. Write $p\colon x_0 \rightsquigarrow x_k$. The path p is closed if $x_0 = x_k$. The poset F is *path-connected* if between any two points there is a path.

PROPOSITION 1.3. When F is finite, it is path-connected if and only if it is connected as a topological space.

Proof. See [BG11, Lemma 3.4]. $\square$

A *connected component* of F is a subframe $U \subseteq F$ which is connected as a topological subspace and is such that there is no connected V with $U \subset V$.

PROPOSITION 1.4. (1) The connected components partition F.

(2) Connected components are downwards-closed.

(3) When F is finite, each connected component is upwards-closed.

Proof. These are standard results in topology. See e.g. [Mun00, §25, p. 159]. $\square$

An *antichain* in F is a subset $Z \subseteq F$ in which no two elements are comparable (i.e. a so-called *independent set* in the comparability graph of F). The *width* $\mathrm{width}(F)$ of F is the cardinality of the largest antichain in F.

P-morphisms. A function $f : F \to G$ is a *p-morphism* if it satisfies the following two conditions.

$$\forall x, y \in F : (x \leqslant y \Rightarrow f(x) \leqslant f(y)) \tag{\textbf{forth}}$$
$$\forall x \in F : \forall z \in G : (f(x) \leqslant z \Rightarrow \exists y : (x \leqslant y \wedge f(y) = z)) \tag{\textbf{back}}$$

REMARK 1.5. The (**forth**) condition expresses that f is monotonic, and is equivalent to requiring that f be continuous. The (**back**) condition is equivalent to requiring that f be open.

An *up-reduction* from F to G is a surjective p-morphism f from an upwards-closed set $U \subseteq F$ to G. Write $f : F \circ\!\!\rightarrow G$.

PROPOSITION 1.6. If there is an up-reduction $F \circ\!\!\rightarrow G$ then $\mathrm{Logic}(F) \subseteq \mathrm{Logic}(G)$. In other words, if $G \nvDash \phi$ then $F \nvDash \phi$.

Proof. See [CZ97, Corollary 2.8, p. 30 and Corollary 2.17, p. 32]. □

Corollary 1.7. If **C** is any collection of frames and $\mathscr{L} = \text{Logic}(\mathbf{C})$, then:

$$\mathscr{L} = \text{Logic}(\text{Frames}_{\perp}(\mathscr{L}))$$

Proof. First, $\mathscr{L} \subseteq \text{Logic}(\text{Frames}_{\perp}(\mathscr{L}))$. Conversely, suppose $\mathscr{L} \nvdash \phi$. Then there exists $F \in \mathbf{C}$ such that $F \nvDash \phi$, hence there is $x \in F$ such that $x \nvDash \phi$ (for some valuation on F), meaning that $\uparrow(x) \nvDash \phi$. Now, $\uparrow(x)$ is upwards-closed in F, hence $\mathsf{id}_{\uparrow(x)}$ is an up-reduction $F \circ\!\!\rightarrow \uparrow(x)$. Then by Proposition 1.6, we get that $\uparrow(x) \vDash \mathscr{L}$, so that $\uparrow(x) \in \text{Frames}_{\perp}(\mathscr{L})$. □

Trees. A finite poset T is a *tree* if it has a root $\perp$, and every other $x \in T \setminus \{\perp\}$ has exactly one immediate predecessor. A *branch* in T is a maximal chain. Given any finite, rooted poset F, its *tree unravelling* $\mathscr{T}(F)$ is the set of its strict chains which contain the root. Define the function $\mathsf{last}\colon \mathscr{T}(F) \to F$ by:

$$X \mapsto \max(X)$$

Proposition 1.8. $\mathscr{T}(F)$ is a tree and last is a p-morphism.

Proof. See [CZ97, Theorem 2.19, p. 32]. □

Heyting algebras. A *Heyting algebra* is a set A equipped with operations $\wedge$, $\vee$ and $\to$ together with distinguished elements 0 and 1, such that $(A, \wedge, \vee, 0, 1)$ is a bounded lattice and $\to$, called the *Heyting implication*, satisfies:

$$c \leqslant a \to b \quad \Longleftrightarrow \quad c \wedge a \leqslant b$$

A map $h\colon A \to B$ between Heyting algebras is a *homomorphism* if it preserves $\wedge$, $\vee$, $\to$, 0 and 1. A *Heyting subalgebra* is a subset $B \subseteq A$ such that B is a Heyting algebra under $\wedge$, $\vee$, $\to$, 0 and 1. Given $S \subseteq A$, the Heyting subalgebra *generated* by S is defined:

$$\langle S \rangle := \bigcap \{B \subseteq A \mid B \text{ is a subalgebra and } S \subseteq B\}$$

It is straightforward to see that $\langle S \rangle$ is a subalgebra, and the smallest subalgebra containing S. The Heyting algebra A is *locally-finite* if for every finite $S \subseteq A$ the subalgebra $\langle S \rangle$ is also finite.

An *assignment* on A is a function $I\colon \mathfrak{Prop} \to A$. The *value* $[\![\phi]\!]_I$ of any formula ϕ under this assignment is defined inductively as follows.

$$\begin{aligned}
[\![\perp]\!]_I &= 0 \\
[\![\psi \wedge \chi]\!]_I &= [\![\psi]\!]_I \wedge [\![\chi]\!]_I \\
[\![\psi \vee \chi]\!]_I &= [\![\psi]\!]_I \vee [\![\chi]\!]_I \\
[\![\psi \to \chi]\!]_I &= [\![\psi]\!]_I \to [\![\chi]\!]_I
\end{aligned}$$

A formula ϕ is *valid* on A, notation $A \vDash \phi$, if $[\![\phi]\!]_I = 1$ for every assignment I. Extend the Logic($\mathbf{C}$) notation to classes of Heyting algebras. Let us record some basic facts about the interaction between logic and Heyting algebras.

Proposition 1.9. Let A and B be Heyting algebras with B a subalgebra of A. Then $\text{Logic}(A) \subseteq \text{Logic}(B)$.

Proof. See [CZ97, Proposition 7.59, p. 220]. □

Proposition 1.10. The logic of a Heyting algebra is the logic of its finitely-generated subalgebras. That is, for any Heyting algebra A, we have:

$$\mathrm{Logic}(A) = \mathrm{Logic}(B \mid B \text{ finitely-generated subalgebra of } A)$$

Proof. The left-to-right inclusion is by Proposition 1.9. For the right-to-left, assume that $A \nvDash \phi$ for some formula ϕ. Then there is an assignment I on A such that $[\![\phi]\!]_I \neq 1$. Let $p_1, \ldots, p_m$ be the propositional variables occurring in ϕ. Without loss of generality, we may assume that the domain of I is $\{p_1, \ldots, p_m\}$. Let $B := \langle I(p_1), \ldots, I(p_m)\rangle$. Then I is also an assignment on B, and $[\![\phi]\!]_I \in B$. Thus $B \nvDash \phi$. □

Co-Heyting algebras. Perhaps less well known than their cousins, co-Heyting algebras are structures dual with Heyting algebras. A *co-Heyting algebra* is a set C equipped with operations $\wedge$, $\vee$ and $\leftarrow$ together with distinguished elements 0 and 1, such that $(C, \wedge, \vee, 0, 1)$ is a bounded lattice, and $\leftarrow$, called the *co-Heyting implication*, satisfies:

$$a \leftarrow b \leqslant c \quad \Longleftrightarrow \quad a \leqslant b \vee c$$

Co-Heyting algebras are intimately related with Heyting algebras. In fact, every co-Heyting algebra can be regarded as a Heyting algebra in the following way. As lattices, co-Heyting and Heyting algebras can be seen as categories; then given any co-Heyting algebra A, its opposite category A^{op} is a Heyting algebra, and *vice versa*. This schema of dualities allows us to transfer definitions and results between Heyting and co-Heyting algebras.

For more information on co-Heyting algebras I refer the reader to [MT46, §1] and [Rau74], where they are called 'Brouwerian algebras'. I mention these dual algebras because, as we will see, the logical structure of a polyhedron is more immediately approached from the co-Heyting perspective. However, since we are interested in connections with intuitionistic logic, it is most natural to flip to Heyting algebras.

Topological semantics. In advance of our upcoming encounter with polyhedral semantics, let us see how, as mentioned in the introduction, we can interpret formulas inside a topological space X. The collection of open sets $\mathcal{O}(X)$ of X forms a Heyting algebra. We take $\varnothing$, X, $\cap$ and $\cup$ for 0, 1, $\wedge$ and $\vee$, respectively, and define the Heyting implication $\rightarrow$ by:

$$U \rightarrow V := \mathrm{Int}(U^{\complement} \cup V)$$

where Int is the topological interior operator, and $-^{\complement}$ is the complement operator.

Proposition 1.11. With these assignments, $\mathcal{O}(X)$ is a Heyting algebra.

Proof. See [CZ97, Proposition 8.31, p. 247]. □

This means that we can interpret formulas inside topological spaces. Write $X \vDash \phi$ for $\mathcal{O}(X) \vDash \phi$, and extend the other Heyting algebra notation to X. The completeness result mentioned in the introduction can now be written down explicitly.

Theorem 1.12 (McKinsey-Tarski Theorem)**.** Let X be any separable metrisable space without isolated points. Then $\mathbf{IPC} = \mathrm{Logic}(X)$.

Proof. The original proof is in [MT44]. Helena Rasiowa and Roman Sikorski proved this result without the separability requirement [RS63]. For a newer, more topological proof, see [BBLM18]. For some modern proofs of specific cases, see [BB07, §2.5, pp. 241–250]. □

The topological space X also comes with a co-Heyting algebra, namely its collection of closed sets $\mathscr{C}(X)$. Co-Heyting implication on $\mathscr{C}(X)$ is defined:

$$C \leftarrow D := \mathrm{Cl}(C \setminus D)$$

where Cl denotes the topological closure operator. Now, the present topological setting provides concrete realisation of the schema of dualities between Heyting and co-Heyting algebras. Indeed, the complement operator $-^{\mathsf{C}}$ gives an isomorphism $\mathscr{O}(X)^{\mathrm{op}} \cong \mathscr{C}(X)$.

Finite Esakia duality. The Alexandrov topology means that every Kripke frame F can be thought of as a topological space. The collection of open sets of this space then forms a Heyting algebra, as above. Denote this Heyting algebra by $\mathrm{Up}\, F$ — the algebra of upwards-closed sets in F — and let us call it the *dual Heyting algebra* of F.

Can this process be reversed? It turns out that if we want to associate a dual structure to each Heyting algebra, in general we need something richer than a Kripke frame. The *Esakia duality* establishes a duality between the category of Heyting algebras and the category of so-called *Esakia spaces*. (Note that this duality occurs on a different level to the schema of dualities between Heyting algebras and co-Heyting algebras, where the duality occurred when we considered those algebras *themselves* as categories.) We won't need the full result here, but it turns out that when one restricts to the finite case, what results is a duality between finite Heyting algebras and finite Kripke frames.

Let A be a Heyting algebra. A *filter* on A is a subset $W \subseteq A$ such that the following conditions hold.

(a) W is non-empty.

(b) W is upwards-closed.

(c) For every $a, b \in W$ we have $a \wedge b \in W$.

The filter W is *prime* if in addition it satisfies the following.

(d) W is proper.

(e) Whenever $a \vee b \in W$ we have $a \in W$ or $b \in W$.

The *spectrum* $\mathrm{Spec} A$ of A is the set of all prime filters in A. It forms a poset under $\subseteq$. When A is finite, call $\mathrm{Spec} A$ the *dual poset* of A; note that $\mathrm{Spec} A$ is again finite.

So, to every (finite) poset, we associate a (finite) Heyting algebra, and to every finite Heyting algebra, we associate a finite poset. In order to extend this to an equivalence of categories, we need to see how maps between structures are transformed. The natural maps in the category of Kripke frames are p-morphisms. Given a p-morphism $f : F \to G$, define:

$$\begin{aligned} \mathrm{Up}(f)\colon\ \mathrm{Up}\, G &\to \mathrm{Up}\, F \\ U &\mapsto f^{-1}[U] \end{aligned}$$

Going the other direction, given a homomorphism $h\colon A \to B$ between Heyting algebras, define:

$$\begin{aligned} \mathrm{Spec}(h)\colon\ \mathrm{Spec}\, B &\to \mathrm{Spec}\, A \\ W &\mapsto h^{-1}[W] \end{aligned}$$

THEOREM 1.13. Thus defined, Up and Spec form an equivalence of categories between the category of finite Kripke frames with p-morphisms and the category of finite Heyting algebras with homomorphisms.

Proof. See [DT66]. The original proof in Russian of the general Esakia duality is found in [Esa85]. An English translation is forthcoming [Esa19]. English proofs are also given in [CJ14] and [Mor05, §5]. In the finite case, we have isomorphisms $A \cong \operatorname{Up}\operatorname{Spec} A$ and $F \cong \operatorname{Spec}\operatorname{Up} F$ for any finite Heyting algebra A and finite poset F. The former is part of Brikhoff's Representation Theorem [Bir37]. Both isomorphisms may be found in [DP90, pp. 171-172]. □

Importantly, this duality is logic-preserving.

PROPOSITION 1.14. Let F be a frame and A be a finite Heyting algebra. Then:

$$\operatorname{Logic}(F) = \operatorname{Logic}(\operatorname{Up} F)$$
$$\operatorname{Logic}(A) = \operatorname{Logic}(\operatorname{Spec} A)$$

Proof. For the first equality, see [CZ97, Corollary 8.5, p. 238], noting that our Kripke frames are special cases of what are there called 'intuitionistic general frames'. The second equality follows from the first using the finite Esakia duality. □

COROLLARY 1.15. If $\mathbf{C}$ is a class of locally-finite Heyting algebras, then $\operatorname{Logic}(\mathbf{C})$ has the finite model property.

Proof. Take $A \in \mathbf{C}$. Since it is locally-finite, by Proposition 1.10 we have:

$$\operatorname{Logic}(A) = \operatorname{Logic}(\{B \mid B \text{ finite subalgebra of } A\})$$

Therefore, by Proposition 1.14, we get:

$$\begin{aligned}\operatorname{Logic}(\mathbf{C}) &= \bigcap_{A \in \mathbf{C}} \operatorname{Logic}(A) \\ &= \operatorname{Logic}(\operatorname{Spec} B \mid A \in \mathbf{C} \text{ and } B \text{ finite subalgebra of } A)\end{aligned}$$

Thus $\mathbf{C}$ has the finite model property. □

Jankov-Fine formulas as forbidden configurations. A very important class of formulas which will reoccur throughout the thesis is the class of Jankov-Fine formulas. These formulas allow logic to capture quite precisely the notion of up-reduction. To every finite rooted frame Q, we associate a formula $\chi(Q)$, the *Jankov-Fine* formula of Q (also called the *Jankov-De Jongh formula* of Q). The precise definition of $\chi(Q)$ is somewhat involved, but the exact details of this syntactical form are not relevant for our considerations. What matters to us is its notable semantic property.

THEOREM 1.16. For any frame F, we have that $F \vDash \chi(Q)$ if and only if F does not up-reduce to Q.

Proof. See [CZ97, §9.4, p. 310], for a treatment in which Jankov-Fine formulas are considered as specific instances of more general 'canonical formulas'. A more direct proof is found in [Bez06, §3.3, p. 56], which gives a complete definition of $\chi(Q)$. See also [BB09] for an algebraic version of this result. □

Jankov-Fine formulas formalise the intuition of 'forbidden configurations'. The formula $\chi(Q)$ 'forbids' the configuration Q from its frames. Later, we will use these formulas as definitional devices, but for now note the following handy corollary.

COROLLARY 1.17. Let $\mathscr{L} = \operatorname{Logic}(\mathbf{C})$ where $\mathbf{C}$ is a class of frames. Then:

$$\operatorname{Frames}_{\perp,\mathrm{fin}}(\mathscr{L}) = \{F \text{ finite rooted frame} \mid \exists G \in \mathbf{C} \colon G \circ\!\!\rightarrow F\}$$

Proof. First, if F is a finite rooted frame such that there is $G \in \mathbf{C}$ and an up-reduction $G \circ\!\!\rightarrow F$, then by Proposition 1.6 we have that $F \in \mathrm{Frames}_{\perp,\mathrm{fin}}(\mathscr{L})$. Conversely take F finite and rooted, and assume that there is no $G \in \mathbf{C}$ with $G \circ\!\!\rightarrow F$. Then by Theorem 1.16, $G \vDash \chi(F)$ for every $G \in \mathbf{C}$; whence $\mathscr{L} \vdash \chi(F)$. This means that $F \nvDash \mathscr{L}$, so that $F \notin \mathrm{Frames}_{\perp,\mathrm{fin}}(\mathscr{L})$. □

The logics of bounded depth. We now meet a well-known schema of logics which will pop up at various points throughout the thesis. For $n \in \mathbb{N}$, define the *axiom of bounded depth n* inductively as follows, where $p_0, p_1, \ldots$ is an infinite set of distinct propositional variables.

$$\mathbf{bd}_0 := p_0 \vee \neg p_0$$
$$\mathbf{bd}_{n+1} := (p_{n+1} \vee (p_{n+1} \to \mathbf{bd}_n))$$

Then let the *logic of bounded depth n* be $\mathbf{BD}_n := \mathbf{IPC} + \mathbf{bd}_n$. Note that $\mathbf{CPC} = \mathbf{BD}_0$.

Proposition 1.18. A frame F validates $\mathbf{BD}_n$ if and only if $\mathsf{height}(F) \leqslant n$.

Proof. See [CZ97, Proposition 2.38, p. 43]. □

Let Ch_k be the chain (linear order) on $k+1$ elements.

Proposition 1.19. A frame F validates $\mathbf{BD}_n$ if and only if there is no p-morphism $F \to \mathrm{Ch}_{n+1}$.

Proof. See [CZ97, Table 9.2, p. 291 and §9.1]. □

An important property of each logic $\mathbf{BD}_n$ is that any extension automatically has the f.m.p.

Theorem 1.20 (Segerberg's Theorem)**.** Let $\mathscr{L}$ be a logic extending $\mathbf{BD}_n$. Then $\mathscr{L}$ has the finite model property. Hence $\mathbf{BD}_n = \mathrm{Logic}(F \text{ frame} \mid \mathsf{height}(F) \leqslant n)$.

Proof. See [CZ97, Theorem 8.85, p. 272]. □

The modal story. Modal formulas are built up in the usual way from $\mathfrak{Prop}$ using the connectives $\perp$, $\Box$, $\wedge$, $\vee$ and $\to$. The following standard modal logics are relevant here.

- Propositional modal logic $\mathbf{K}$.
- $\mathbf{S4} := \mathbf{K} + (\Box p \to p) + (\Box p \to \Box\Box p)$.
- $\mathbf{S4.Grz} := \mathbf{S4} + \mathbf{grz}$, where $\mathbf{grz} := \Box(\Box(p \to \Box p) \to p) \to p$.

A *modal algebra* is a set M equipped with operations $\Box$, $\wedge$, $\vee$ together with distinguished elements 0 and 1 such that $(M, \wedge, \vee, 0, 1)$ is a Boolean algebra and $\Box$ satisfies $\Box 1 = 1$ and $\Box(a \wedge b) = (\Box a \wedge \Box b)$. An *assignment* is a function $I : \mathfrak{Prop} \to M$. The *value* $[\![\phi]\!]_I$ of any modal formula ϕ under I is computed inductively just as in the case of Heyting algebras, with:

$$[\![\psi \to \chi]\!]_I = \neg[\![\psi]\!]_I \vee [\![\chi]\!]_I$$

The formula ϕ is *valid* on M if $[\![\phi]\!]_I = 1$ for every assignment I on M. Let us extend our logical notation to modal algebras. A modal algebra M is an **S4**-*algebra* (also called *interior algebra* or *closure algebra*) if $M \vDash \mathbf{S4}$, and a *Grzegorczyk algebra* if $M \vDash \mathbf{S4.Grz}$.

Modal logic enjoys an intimate connection with intuitionistic logic. The *Gödel translation*, Tr, mapping intuitionistic formulas to modal formulas, is defined inductively as follows.

$$\begin{aligned}
\mathrm{Tr}(\bot) &= \bot \\
\mathrm{Tr}(p) &:= \Box p \\
\mathrm{Tr}(\psi \wedge \chi) &:= \mathrm{Tr}(\psi) \wedge \mathrm{Tr}(\chi) \\
\mathrm{Tr}(\psi \vee \chi) &:= \mathrm{Tr}(\psi) \vee \mathrm{Tr}(\chi) \\
\mathrm{Tr}(\psi \to \chi) &:= \Box(\mathrm{Tr}(\psi) \to \mathrm{Tr}(\chi))
\end{aligned}$$

PROPOSITION 1.21. Let M be a **S4**-algebra, and let $M_\Box$ be the subset of M consisting of all those a such that $\Box a = a$. Then $M_\Box$ is a Heyting algebra, with $\wedge$, $\vee$, 0 and 1 inherited from M, and $\to$ defined:

$$a \to b := \Box(\neg a \vee b)$$

Moreover, for any intuitionistic formula ϕ, we have:

$$M_\Box \vDash \phi \quad \Longleftrightarrow \quad M \vDash \mathrm{Tr}(\phi)$$

Proof. See [CZ97, Lemma 8.28 and Proposition 8.31, pp. 246–247]. □

Note the link with the topological semantics above. Indeed, when X is a topological space, the set $\mathscr{P}(X)$ is a **S4**-algebra with $\Box$ interpreted as the topological interior operator. We saw above that $\mathscr{O}(X)$ was a Heyting algebra, with the Heyting implication $\to$ defined as $U \to V := \mathrm{Int}(U^{\mathsf{C}} \cup V)$.

THEOREM 1.22. Let A be a Heyting algebra. Then there is a **S4**-algebra M such that $A \cong M_\Box$.

Proof. See [CZ97, Corollary 8.35, p. 249]. □

REMARK 1.23. This **S4**-algebra M is not in general unique up to isomorphism.

The pinnacle of the connection between modal and intuitionistic logic is the Blok-Esakia Theorem. Taking logics to be sets of formulas, we can view the class of intermediate logics as a lattice; denote this by Ext**IPC**. Similarly, denote the lattice of (normal) modal logics extending **S4**.**Grz** by NExt**Grz**.

THEOREM 1.24 (Blok-Esakia Theorem)**.** There is an isomorphism $\sigma : \mathrm{Ext}\mathbf{IPC} \to \mathrm{NExt}\mathbf{Grz}$ such that for any $\mathscr{L} \in \mathrm{Ext}\mathbf{IPC}$:

$$\mathscr{L} = \{\mathrm{Tr}(\phi) \mid \phi \in \sigma\mathscr{L}\}$$

Proof. This theorem was proved independently by Willem Blok [Blo76] and Leo Esakia [Esa76]. For alternative proofs, see [CZ97, §9.6] and [Jeřo09]. □

1.2 Polyhedra

In this section, I will describe those parts of polyhedral geometry which will be necessary for establishing and strengthening the link between logic and polyhedra. I will mainly be following [Sta67] and [Mun84], along with the exposition given in [BMMP18].

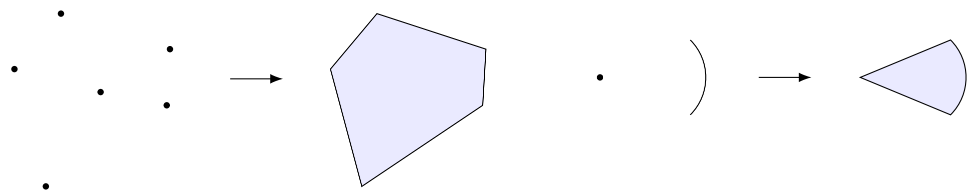


Figure 1.1: Some examples of the convex hull operation in $\mathbb{R}^2$

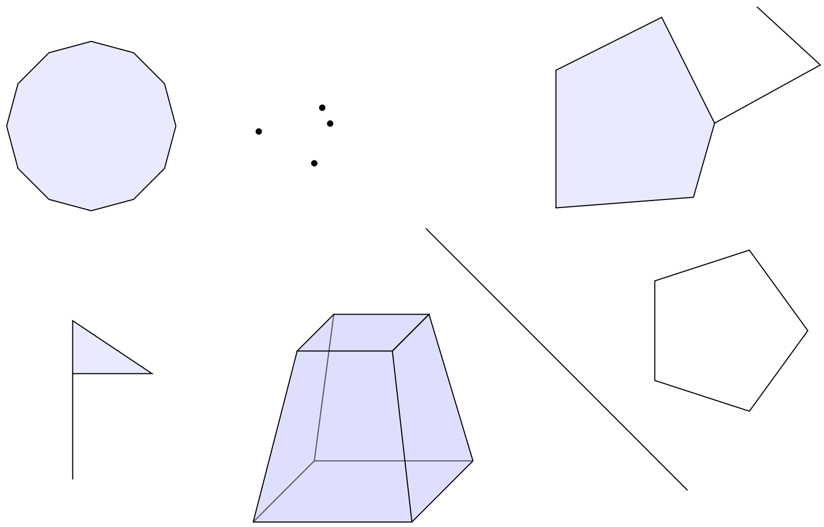


Figure 1.2: A medley of different polyhedra in $\mathbb{R}^3$. Each connected region is a polyhedron, as is the union of any number of those regions.

Polytopes and polyhedra. Every polyhedron considered here lives in some Euclidean space $\mathbb{R}^n$. Take $x_0, \ldots, x_d \in \mathbb{R}^n$. An *affine combination* of $x_0, \ldots, x_d$ is a point $r_0 x_0 + \cdots + r_d x_d$, specified by some $r_0, \ldots, r_d \in \mathbb{R}$ such that $r_0 + \cdots + r_d = 1$. A *convex combination* is an affine combination in which additionally each $r_i \geqslant 0$. Given a set $S \subseteq \mathbb{R}^n$, its *convex hull* $\operatorname{Conv} S$ is the collection of convex combinations of its elements. See Figure 1.1 for a couple of examples of the convex hull operation in $\mathbb{R}^2$. A subspace $S \subseteq \mathbb{R}^n$ is *convex* if $\operatorname{Conv} S = S$. A *polytope* is the convex hull of a finite set. A *polyhedron* in $\mathbb{R}^n$ is a set which can be expressed as the finite union of polytopes. To get an idea of the variety of subspaces which fall under the term 'polyhedron', see Figure 1.2. Note that every polyhedron is closed and bounded, hence compact.

Simplices. A set of points $x_0, \ldots, x_d$ is *affinely independent* if whenever:

$$r_0 x_0 + \cdots + r_d x_d = \mathbf{0} \quad \text{and} \quad r_0 + \cdots + r_d = 0$$

we must have that $r_0, \ldots, r_d = 0$. This is equivalent to saying that the vectors:

$$x_1 - x_0, \ldots, x_d - x_0$$

are linearly independent. A *d-simplex* is the convex hull σ of $d + 1$ affinely independent points $x_0, \ldots, x_d$, which we call its *vertices*. Write $\sigma = x_0 \cdots x_d$; its *dimension* is $\operatorname{Dim} \sigma :=$

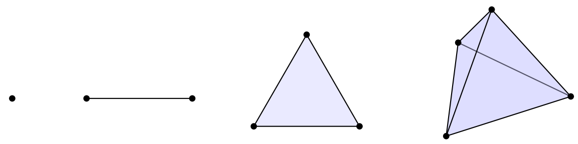


Figure 1.3: A quintessential 0-,1-,2- and 3-simplex

d. The idea is that a d-simplex is the simplest kind of polyhedron of dimension d. See Figure 1.3 for representations of some simplices of dimensions 0 to 3 (of course, not all simplices are so *regular*). In fact, as we shall soon see, every polyhedron can be decomposed into these basic building blocks.

PROPOSITION 1.25. Every simplex determines its vertex set: two simplices coincide if and only if they share the same vertex set.

Proof. See [Mau80, Proposition 2.3.3, p. 32]. □

A *face* of σ is the convex hull τ of some non-empty subset of $\{x_0, \ldots, x_d\}$ (note that τ is then a simplex too). Write $\tau \preccurlyeq \sigma$, and $\tau \prec \sigma$ if $\tau \neq \sigma$.

Since $x_0, \ldots, x_d$ are affinely independent, every point $x \in \sigma$ can be expressed uniquely as a convex combination $x = r_0 x_0 + \cdots + r_d x_d$. Call the tuple $(r_0, \ldots, r_d)$ the *barycentric coordinates* of x in σ. The *barycentre* $\widehat{\sigma}$ of σ is the special point whose barycentric coordinates are $(\frac{1}{d+1}, \ldots, \frac{1}{d+1})$. The *relative interior* of σ is defined:

$$\operatorname{Relint} \sigma := \{r_0 x_0 + \cdots + r_d x_d \in \sigma \mid r_0, \ldots, r_d > 0\}$$

The relative interior of σ is 'σ without its boundary' in the following sense. The *affine subspace spanned by* σ is the set of all affine combinations of $x_0, \ldots, x_d$. Then the relative interior of σ coincides with the topological interior of σ inside this affine subspace. Note that $\operatorname{Cl} \operatorname{Relint} \sigma = \sigma$, the closure being taken in the ambient space $\mathbb{R}^n$.

Triangulations. A *simplicial complex* in $\mathbb{R}^n$ is a finite set Σ of simplices satisfying the following conditions.

(a) Σ is $\prec$-downwards-closed: whenever $\sigma \in \Sigma$ and $\tau \prec \sigma$ we have $\tau \in \Sigma$.

(b) If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is either empty or a common face of σ and τ.

The *support* of Σ is the set $|\Sigma| := \bigcup \Sigma$. Note that by definition this set is automatically a polyhedron. We say that Σ is a *triangulation* of the polyhedron $|\Sigma|$. See Figure 1.4 for examples of triangulations of the polyhedra in Figure 1.2. Notice that Σ is a poset under $\prec$, called the *face poset*. Here we see the first suggestion of a connection with logic via Kripke frames. A *subcomplex* of Σ is subset which is itself a simplicial complex. Note that a subcomplex, as a poset, is precisely a downwards-closed set. Given $\sigma \in \Sigma$, its *open star* is defined:

$$\mathrm{o}(\sigma) := \bigcup \{\operatorname{Relint}(\tau) \mid \tau \in \Sigma \text{ and } \sigma \subseteq \tau\}$$

PROPOSITION 1.26. The relative interiors of the simplices in a simplicial complex Σ partition $|\Sigma|$. That is, for every $x \in |\Sigma|$, there is exactly one $\sigma \in \Sigma$ such that $x \in \operatorname{Relint} \sigma$.

Proof. See [Mau80, Proposition 2.3.6, p. 33]. □

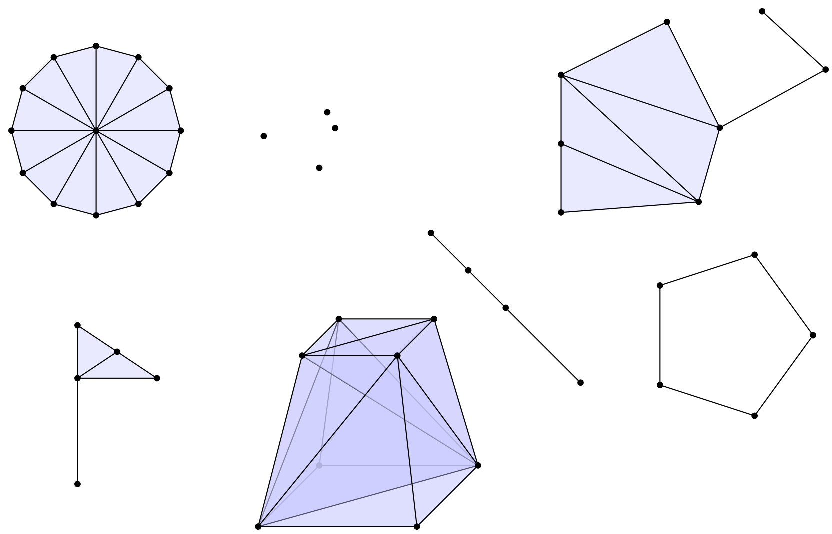


Figure 1.4: Triangulations of the medley of polyhedra given in Figure 1.2

In light of Proposition 1.26, for any $x \in |\Sigma|$ let us write σ^x for the unique $\sigma \in \Sigma$ such that $x \in \operatorname{Relint} \sigma$.

Proposition 1.27. Let Σ be a simplicial complex, take $\tau \in \Sigma$ and $x \in \operatorname{Relint} \tau$. Then no proper face $\sigma \prec \tau$ contains x. This means that σ^x is the inclusion-smallest simplex containing x.

Proof. See [BMMP18, Lemma 3.1]. □

The next result is a basic fact of polyhedral geometry, and will play a fundamental role in connecting it with logic throughout this thesis. For Σ a triangulation and S a subspace of the ambient Euclidean space $\mathbb{R}^n$, define:

$$\Sigma_S := \{\sigma \in \Sigma \mid \sigma \subseteq S\}$$

This, being a downwards-closed subset of Σ, is a subcomplex of Σ.

Lemma 1.28 (Triangulation Lemma)**.** Any polyhedron admits a triangulation which simultaneously triangulates each of any fixed finite set of subpolyhedra. That is, for a collection of polyhedra $P, Q_1, \ldots, Q_m$ such that each $Q_i \subseteq P$, there is a triangulation Σ of P such that Σ_{Q_i} triangulates Q_i for each i.

Proof. See [RS72, Theorem 2.11 and Addendum 2.12, p. 16]. □

A note on terminology. The term 'polyhedron' is ancient, and over the years it has acquired a variety of meanings. A remark on the present terminology is in order. I will partly be following [BMMP18, Remark 2.11]. In one very traditional usage (though still present in some fields today), 'polyhedron' is reserved for *convex* sets. Another possible restriction, in line with historical terminology, is that 'polyhedron' applies only to *three-dimensional solids*. As is standard in the field of piecewise-linear topology however, the usage in the present thesis is not subject to these restrictions (c.f. classic textbooks [Sta67; RS72]).

I should note however that the standard usage of 'polyhedron' is in fact more general than the present one. In PL topology, a 'polyhedron' is the union of a *locally-finite* simplicial complex. The latter is defined as a (possibly infinite) set Σ of simplices satisfying (a) and (b) in our definition of 'simplicial complex' above, subject to the condition that every point $x \in \bigcup \Sigma$ has an open neighbourhood which intersects only finitely-many simplices. Now, it is a standard fact that 'compact polyhedra' (in the more general sense) coincide with what we are referring to here as 'polyhedra' (see [RS72, Theorem 2.2, p. 12]). Hence we are effectively using the term 'polyhedron' as a shorthand for 'compact polyhedron'; such usage is common in the literature (see, e.g. [Mau80]).

The dimension of a polyhedron. I said above that the dimension of a d-simplex $\sigma = x_0 \cdots x_d$ is exactly d. Since the vertices $x_0, \ldots, x_d$ are affinely independent, this is the same as the linear-space dimension of the affine subspace spanned by σ. The *dimension* of simplicial complex Σ is:

$$\operatorname{Dim} \Sigma := \max\{\operatorname{Dim} \sigma \mid \sigma \in \Sigma\}$$

REMARK 1.29. Note that $\operatorname{Dim} \Sigma = \mathsf{height}(\Sigma)$ as a poset.

PROPOSITION 1.30. Let Σ, Δ be simplicial complexes. If $|\Sigma| = |\Delta|$ then $\operatorname{Dim} \Sigma = \operatorname{Dim} \Delta$.

Proof. See [Sta67, Proposition 1.6.12, p. 30]. □

With this in mind, we define the *dimension* $\operatorname{Dim} P$ of a polyhedron P to be the dimension of its triangulations. When $P = \varnothing$, let $\operatorname{Dim} P := -1$.

Barycentric subdivision. Triangulations allow us in some ways to approximate the structure of a polyhedron. The finer the triangulation, the better the approximation. Barycentric subdivisions afford us a systematic way of generating finer and finer triangulations, starting from a base. This process allows us to extract, in the limit, all the relevant information about a polyhedron, in a way made precise by the Nerve Criterion, which we will meet in Chapter 2.

Let Σ, Δ be simplicial complexes. Δ is a *subdivision* or *refinement* of Σ, notation $\Delta \lhd \Sigma$, if $|\Sigma| = |\Delta|$ and every simplex of Δ is contained in a simplex of Σ.

LEMMA 1.31. If $\Delta \lhd \Sigma$ then for every $\sigma \in \Sigma$ we have:

$$\sigma = \bigcup \{\tau \in \Delta \mid \tau \subseteq \sigma\}$$

Proof. Let $S := \{\tau \in \Delta \mid \tau \subseteq \sigma\}$. Clearly $\bigcup S \subseteq \sigma$. Conversely, for $x \in \sigma$, let $\tau^x \in \Delta$ be such that $x \in \operatorname{Relint} \tau^x$. Since Δ refines Σ, there is some $\rho \in \Sigma$ such that $\tau^x \subseteq \rho$; assume that ρ is inclusion-minimal with this property. It follows from [Spa66, §3, Lemma 3, p. 121] that $\operatorname{Relint} \tau^x \subseteq \operatorname{Relint} \rho$, meaning that $x \in \sigma \cap \operatorname{Relint} \rho$. By condition (b) on Σ, we have that $\sigma \cap \rho$ is face of ρ. But then by Proposition 1.27, $\rho \preccurlyeq \sigma$, since otherwise $\sigma \cap \rho$ would be a proper face of ρ containing $x \in \operatorname{Relint} \rho$. Therefore $\tau^x \subseteq \rho \subseteq \sigma$ so that $x \in \bigcup S$. □

The *barycentric subdivision* $\operatorname{Sd} \Sigma$ of Σ is particularly important kind of subdivision. The idea is that we put a new vertex at the barycentre of each simplex in Σ, then build up the rest of the simplicial complex around this. Spelling this in detail is somewhat involved, and the technical details will not be needed in this thesis. Hopefully the examples in Figure 1.5 should provide the intuition behind the construction, but for a full definition I refer the reader to [Mun84, §15, p. 83].

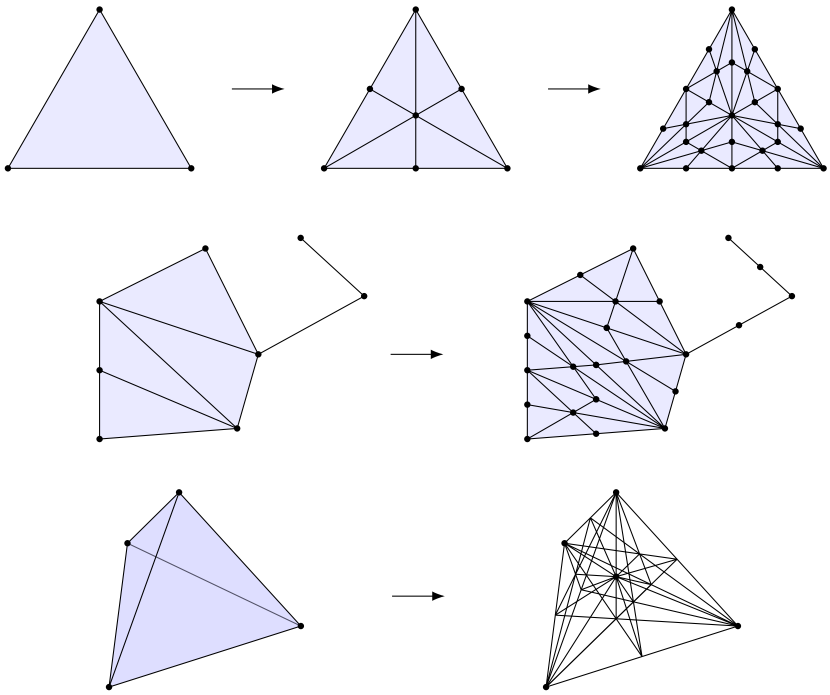


Figure 1.5: Examples of barycentric subdivision (the right-most tetrahedron is drawn without filled-in faces to aid clarity)

1.3 Logic and Polyhedra in Concert

Now that the background in logic and geometry has been established, we are in a position to connect the two. As indicated above, the initial contact is made between polyhedra and co-Heyting algebras. By exploiting duality, we then attain our connection with Heyting algebras and intuitionistic logic.

The co-Heyting algebra of sub-polyhedra. Fix throughout a polyhedron P. Let $\mathrm{Sub_c}P$ denote the set of subpolyhedra of P. We will see that $\mathrm{Sub_c}P$ is a co-Heyting algebra.

PROPOSITION 1.32. $\mathrm{Sub_c}P$ is a distributive lattice under $\cap$ and $\cup$.

Proof. I follow [BMMP18, Corollary 2.12]; see also [Mau80, Proposition 2.3.6, p. 33]. First note that $\varnothing$ and P are minimal and maximal elements in $\mathrm{Sub_c}P$. Also, by definition, the union of two polyhedra is again a polyhedron. So take $Q, R \in \mathrm{Sub_c}P$ and consider $Q \cap R$. By the Triangulation Lemma 1.28, there is a triangulation Σ of P such that Σ_Q and Σ_R triangulate Q and R respectively. Since Σ_Q and Σ_R are subcomplexes of Σ, so is $\Sigma_Q \cap \Sigma_R$. I will show that $|\Sigma_Q \cap \Sigma_R| = Q \cap R$, which will complete the proof. First, by definition $|\Sigma_Q \cap \Sigma_R| \subseteq Q \cap R$. Conversely, take $x \in Q \cap R$. Since $|\Sigma_Q| = Q$ and $|\Sigma_R| = R$, there are $\sigma_Q \in \Sigma_Q$ and $\sigma_R \in \Sigma_R$ such that $x \in \sigma_Q \cap \sigma_R$. Then, by condition (b) in the definition of simplicial complex, $\sigma_Q \cap \sigma_R$ must be a common face of σ_Q and σ_R. But then by condition (a), we must have $\sigma_Q \cap \sigma_R \in \Sigma_Q \cap \Sigma_R$, so that $x \in |\Sigma_Q \cap \Sigma_R|$. $\square$

THEOREM 1.33. $\mathrm{Sub_c}P$ is a co-Heyting algebra, and a subalgebra of $\mathscr{C}(P)$.

Proof. I follow [BMMP18, Lemma 3.1]; see also [Mau80, Proposition 2.3.7, p. 34]. Take Q, R subpolyhedra of P; I will show that $C := \mathrm{Cl}(Q \setminus R)$ is a polyhedron. First note that, by taking $R \cap Q$, we may assume that $R \subseteq Q$. Using the Triangulation Lemma 1.28, let Σ be a triangulation of Q such that Σ_R triangulates R. Define:

$$\Delta := \{\sigma \in \Sigma \mid \exists \tau \in \Sigma \setminus \Sigma_R : \sigma \preccurlyeq \tau\}$$

Note that Δ is a subtriangulation of Σ. I will show that $|\Delta| = C$, which will give us the result. For the left-to-right inclusion, take $\sigma \in \Delta$ and let $\tau \in \Sigma \setminus \Sigma_R$ be such that $\sigma \preccurlyeq \tau$. Since $\tau = \mathrm{Cl}\,\mathrm{Relint}\,\tau$, it suffices to show that $\mathrm{Relint}\,\tau \subseteq Q \setminus R$. So take $x \in \mathrm{Relint}\,\tau$, and note already that $x \in Q$. By Proposition 1.27, x is not contained in any proper face of τ. Hence, by condition (b) on simplicial complexes, for any $\rho \in \Sigma$, if $x \in \rho$ then $\tau \preccurlyeq \rho$. Therefore, by choice of τ and condition (a) on Σ_R, x is not contained in any simplex of Σ_R, whence $x \notin R$.

For the right-to-left inclusion, take $x \in C$. Since $C = \mathrm{Cl}(Q \setminus R)$ in $\mathbb{R}^n$, there is a sequence $(x_k)_{k \in \mathbb{N}}$ in $Q \setminus R$ which converges to x. For each x_k, the simplex σ^{x_k} lies in $\Sigma \setminus \Sigma_R$. Since the latter is finite, there is a simplex $\tau \in \Sigma \setminus \Sigma_R$ which contains infinitely-many x_k's. By restricting to these x_k's, we obtain a subsequence $(x_{k_i})_{i \in \mathbb{N}}$ which lies in τ and converges to x. But, τ is closed, whence $x \in \tau \subseteq |\Delta|$. □

Triangulation subalgebras and local-finiteness. Triangulations have an important algebraic correspondent, which will be used to show that $\mathrm{Sub_c}P$ is locally-finite. Take a triangulation Σ of P. Its elements are themselves polyhedra, and in fact subpolyhedra of P; therefore $\Sigma \subseteq \mathrm{Sub_c}P$. Let $\mathrm{P_c}(\Sigma)$ be the sublattice of $\mathscr{C}(P)$ generated by Σ. The following is Lemma 3.6 in [BMMP18].

Proposition 1.34. $\mathrm{P_c}(\Sigma)$ is a co-Heyting subalgebra of $\mathrm{Sub_c}P$.

Proof. Note that any non-empty $Q, R \in \mathrm{P_c}(\Sigma)$ are by definition triangulated by Σ_Q and Σ_R, respectively. But then it follows from the proof of Theorem 1.33 that $Q \leftarrow R = \mathrm{Cl}(Q \setminus R) = |\Delta| = \bigcup \Delta$, where:

$$\Delta := \{\sigma \in \Sigma \mid \exists \tau \in \Sigma \setminus \Sigma_R : \sigma \preccurlyeq \tau\}$$

Therefore $Q \leftarrow R \in \mathrm{P_c}(\Sigma)$. □

Call a subalgebra $A \subseteq \mathrm{Sub_c}P$ a *triangulation subalgebra* if $A = \mathrm{P_c}(\Sigma)$ for some Σ. The following two results are Lemma 3.2 and Corollary 3.7 in [BMMP18].

Proposition 1.35. Every finitely-generated subalgebra of $\mathrm{Sub_c}P$ is contained in some triangulation algebra.

Proof. Take $Q_1, \ldots, Q_m \in \mathrm{Sub_c}P$. Let Σ, by the Triangulation Lemma 1.28, be a triangulation of P which also triangulates $Q_1, \ldots, Q_m$. Note that the distributive lattice D generated by $Q_1, \ldots, Q_m$ is contained in $\mathrm{P_c}(\Sigma)$ by definition. Further, if $R, S \in D$ then $R \leftarrow S \in \mathrm{P_c}(\Sigma)$ just as in the proof of Proposition 1.34. Therefore the subalgebra generated by $Q_1, \ldots, Q_m$ is contained in $\mathrm{P_c}(\Sigma)$. □

Corollary 1.36. $\mathrm{Sub_c}P$ is locally-finite.

Proof. This follows from Proposition 1.35 since every triangulation subalgebra $\mathrm{P_c}(\Sigma)$ is finite. □

We also have the following properties of triangulation subalgebras, which will be useful in the sequel.

PROPOSITION 1.37. (1) Triangulation algebras determine their corresponding triangulations. That is, for any two triangulations Σ and Δ, if $\mathrm{P_c}(\Sigma) = \mathrm{P_c}(\Delta)$ then $\Sigma = \Delta$.

(2) If Σ and Δ are triangulations which are isomorphic as posets then $\mathrm{P_c}(\Sigma) \cong \mathrm{P_c}(\Delta)$.

(3) If Δ refines Σ, then $\mathrm{P_c}(\Sigma)$ is a subalgebra of $\mathrm{P_c}(\Delta)$.

Proof. (1) It follows from conditions (a) and (b) on simplicial complexes that $\mathrm{P_c}(\Sigma)$ consists exactly of the unions of elements of Σ, and similarly for Δ. Assume $\mathrm{P_c}(\Sigma) = \mathrm{P_c}(\Delta)$ and take $\sigma \in \Sigma$. Then $\sigma \in \mathrm{P_c}(\Delta)$, so $\sigma = \bigcup S$ for some $S \subseteq \Delta$, and similarly each $\tau \in S$ is $\tau = \bigcup T_\tau$ for some $T_\tau \subseteq \Sigma$. Hence:

$$\sigma = \bigcup \bigcup_{\tau \in S} T_\tau$$

But then by condition (b) on Σ, every $\rho \in \bigcup_{\tau \in S} T_\tau$ must either be equal to σ or be a proper face of σ. Since $\operatorname{Relint} \sigma$ contains no proper face of σ, we must have $\sigma \in T_\tau$ for some $\tau \in S$. But then $\sigma \subseteq \tau \subseteq \sigma$, and so $\sigma \in \Delta$. Applying this argument also in the other direction, we get that $\Sigma = \Delta$.

(2) This is immediate from the definition of $\mathrm{P_c}$.

(3) By Lemma 1.31, every $\sigma \in \Sigma$ is the union of simplices in Δ. Whence $\Sigma \subseteq \mathrm{P_c}(\Delta)$. Therefore, by definition $\mathrm{P_c}(\Sigma) \subseteq \mathrm{P_c}(\Delta)$. □

The other side of the coin: the Heyting algebra $\mathrm{Sub_o}P$. Now that we have established the locally-finite co-Heyting algebra $\mathrm{Sub_c}P$, it is time to recast it as a Heyting algebra. To obtain a concrete representation, let us take inspiration from the topological case above, and apply the complement operator $-^{\mathsf{C}}$. Following [BMMP18], an *open subpolyhedron* of P is the set-theoretic complement of a (closed) subpolyhedron in P. Let $\mathrm{Sub_o}P$ be the set of open subpolyhedra in P. By the duals of Proposition 1.32, Theorem 1.33, and Corollary 1.36, this is a locally-finite Heyting algebra, and a subalgebra of $\mathscr{O}(P)$. Given any triangulation Σ of P, let $\mathrm{P_o}(\Sigma)$ be the Heyting subalgebra generated by the complements of the simplices in Σ — i.e. the dual to the co-Heyting subalgebra $\mathrm{P_c}(\Sigma)$.

Note that 'open subpolyhedra' are not 'polyhedra' in the sense used here. They are however 'polyhedra' in the more general sense of PL topology. Indeed, one might think to use this more general notion as an alternative way of constructing an algebra of open polyhedra. But in fact, the open PL-polyhedra are *too* general for our purposes: the open PL-polyhedra in $\mathbb{R}^n$ are exactly the arbitrary open sets of $\mathbb{R}^n$ (see [FP90, Corollary 3.2.22, p. 109]). Thus, the consideration here of open subpolyhedra, though not standard, paves the way to a new semantics for intuitionistic logic by reinstating the duality between open and closed sets.

DEFINITION 1.38 (Polyhedral completeness)**.** A logic $\mathscr{L}$ is *polyhedrally-complete* if it is the logic of a class of polyhedra.

This is the key definition. The quest of the present thesis is to investigate polyhedral completeness. Our results so far allow us to make a first, important observation.

PROPOSITION 1.39. If $\mathscr{L}$ is polyhedrally-complete, then it has the finite model property.

Proof. Assume that $\mathscr{L} = \mathrm{Logic}(\mathbf{C})$, where $\mathbf{C}$ is a class of polyhedra. Then by the dual of Corollary 1.36, it is the logic of a class of locally-finite Heyting algebras; whence by Corollary 1.15 it has the f.m.p. □

And the modal case. By Theorem 1.22, there is an **S4**-algebra M such that $M_\square = \mathrm{Sub}_\mathrm{o} P$. In fact, there is such an algebra with a rather natural form, as considered in [Gab+18]. An *open half-space* in $\mathbb{R}^n$ is the set of points $(x_1, \dots, x_n)$ satisfying, for some $a_1, \dots, a_n, b \in \mathbb{R}$:

$$a_1 x_1 + \cdots + a_n x_n < b$$

The corresponding *closed half-space* is the set of points satisfying:

$$a_1 x_1 + \cdots + a_n x_n \leqslant b$$

Define a *polytopal set* in $\mathbb{R}^n$ to be a subspace which is the intersection of finitely-many open and closed half-spaces. More compactly, we can say that a polytopal set is the solution set of a system of linear inequalities. A *polyhedral set* is then the union of finitely-many polytopal sets.

Given any polyhedral set P, let $\mathrm{Sub}P$ denote the collection of polyhedral subsets. This is a modal algebra when $\square$ is interpreted as the topological interior operator. Moreover, we have the following.

PROPOSITION 1.40. $\mathrm{Sub}P$ is a Grzegorczyk algebra.

Proof. This follows from [Fon18, Theorem 3.8.3, p. 105]. $\square$

PROPOSITION 1.41. When P is a polyhedron (in our sense), we have $(\mathrm{Sub}P)_\square = \mathrm{Sub}_\mathrm{o} P$.

Proof. See [Fon18, Theorem 3.5.1, p. 86]. $\square$

Using this modal algebra, one can proceed to investigate the modal logic of polyhedra. However, the Blok-Esakia Theorem tells us that this investigation is essentially the same as that of the intuitionistic logic of polyhedra. Indeed, the isomorphism σ allows us to move freely between the resulting logics. Since it is thus redundant to keep track of both kinds of logic, from now on I will focus solely on the intuitionistic side.

Posets dual to triangulation subalgebras. Fix a triangulation Σ of P. We investigate the triangulation subalgebra $\mathrm{P}_\mathrm{o}(\Sigma)$ a little more. First, recall the definition (page 19) of the open star $\mathrm{o}(\sigma)$ of a simplex $\sigma \in \Sigma$.

PROPOSITION 1.42. The open star of a simplex is an open subpolyhedron. That is, $\mathrm{o}(\sigma) \in \mathrm{P}_\mathrm{o}(\Sigma)$.

Proof. See [Mau80, Proposition 2.4.3, p. 43] and [BMMP18, p. 12]. $\square$

Define:

$$\begin{aligned} \gamma^\uparrow \colon \mathrm{Up}\,\Sigma &\to \mathrm{P}_\mathrm{o}(\Sigma) \\ U &\mapsto \bigcup_{\sigma \in U} \mathrm{Relint}\,\sigma \end{aligned}$$

To see that $U \in \mathrm{Up}\,\Sigma$ really lands in $\mathrm{P}_\mathrm{o}(\Sigma)$, note that:

$$\begin{aligned} \gamma^\uparrow(U) &= \bigcup_{\sigma \in U} \gamma^\uparrow(\uparrow(\sigma)) \\ &= \bigcup_{\sigma \in U} \bigcup \{\mathrm{Relint}(\tau) \mid \tau \in \Sigma \text{ and } \sigma \subseteq \tau\} \\ &= \bigcup_{\sigma \in U} \mathrm{o}(\sigma) \end{aligned}$$

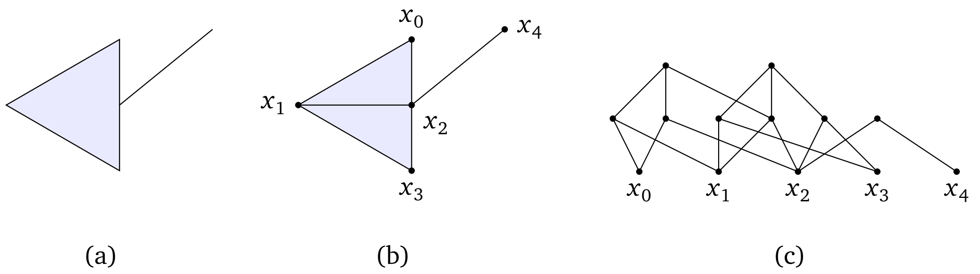


Figure 1.6: Computation of the face poset: (a) a 2-dimensional polyhedron, (b) a triangulation of this polyhedron, and (c) the face poset of this triangulation.

Since U is finite, by Proposition 1.42 we get that $\gamma^{\uparrow}(U) \in \mathrm{P_o}(\Sigma)$.

Proposition 1.43. $\gamma^{\uparrow}$ gives an isomorphism of Heyting algebras $\mathrm{Up}\,\Sigma \cong \mathrm{P_o}(\Sigma)$.

Proof. See [BMMP18, Lemma 4.3]. □

This proposition gives a concrete description of $\mathrm{P_o}(\Sigma)$ and means that Σ, as a face poset, is its dual. The characterisation is very handy, since the simplicial complex Σ tends to be much easier to visualise than the algebra $\mathrm{P_o}(\Sigma)$. See Figure 1.6 for an example of the computation of this poset. For any polyhedron P, we have the following chain of equalities.

$$\begin{aligned}
\mathrm{Logic}(P) &= \mathrm{Logic}(\mathrm{Sub_o}P) \\
&= \mathrm{Logic}(B \mid B \text{ finitely-generated subalgebra of } \mathrm{Sub_o}P) && \text{(Proposition 1.10)} \\
&= \mathrm{Logic}(\mathrm{P_o}(\Sigma) \mid \Sigma \text{ triangulation of } P) && \text{(Proposition 1.35)} \\
&= \mathrm{Logic}(\mathrm{Up}\,\Sigma \mid \Sigma \text{ triangulation of } P) && \text{(Proposition 1.43)} \\
&= \mathrm{Logic}(\Sigma \mid \Sigma \text{ triangulation of } P) && \text{(Proposition 1.14)}
\end{aligned}$$

This leads us to our first maxim.

Maxim I. The logic of a polyhedron is the logic of its triangulations.

Thus we obtain a purely combinatorial description of $\mathrm{Logic}(P)$ in terms finite objects: its triangulations.

Using the isomorphism in Proposition 1.43, we can also define the *dimension* of an open polyhedron $Q \in \mathrm{P_o}(\Sigma)$ to be $\mathrm{Dim}(Q) := \mathsf{height}(\downarrow(U))$, where $U = (\gamma^{\uparrow})^{-1}(Q)$.

Proposition 1.44. $\mathrm{Cl}\,Q$ is a polyhedron and $\mathrm{Dim}(Q) = \mathrm{Dim}(\mathrm{Cl}\,Q)$. Hence the dimension of Q is independent of the triangulation Σ.

Proof. We have, by Proposition 1.43, that:

$$Q = \gamma^{\uparrow}(U) = \bigcup_{\sigma \in U} \mathrm{Relint}\,\sigma$$

Hence, as U is finite:

$$\mathrm{Cl}\,Q = \bigcup_{\sigma \in U} \mathrm{Cl}\,\mathrm{Relint}\,\sigma = \bigcup_{\sigma \in U} \sigma = |\downarrow U|$$

noting that, since $\downarrow U$ is closed it is a subcomplex of Σ. But now:

$$\mathrm{Dim}(|\downarrow U|) = \mathsf{height}(\downarrow U) = \mathrm{Dim}(Q) \qquad \square$$

Polyhedral maps. Here I will present some basic interactions between logic and polyhedra at the level of morphisms. These results come from joint work with Nick Bezhanishvili, David Gabelaia and Vincenzo Marra. Let P be a polyhedron and F be a poset. A function $f : P \to F$ is a *polyhedral map* if the preimage of any open set in F is an open subpolyhedron of P. Note that such a function is continuous.

Proposition 1.45. Let $f : P \to F$ be a function from a polyhedron P to a finite poset F, and write $f^* := f^{-1}[-]\colon \mathscr{P}(F) \to \mathscr{P}(P)$ for the inverse image function.

(1) The function f is polyhedral if and only if f^* descends to a lattice homomorphism $f^*\colon \mathrm{Up}\,F \to \mathrm{Sub}_{\mathrm{o}}P$.

(2) The function f is polyhedral and open if and only if f^* descends to a homomorphism of Heyting algebras $f^*\colon \mathrm{Up}\,F \to \mathrm{Sub}_{\mathrm{o}}P$.

Proof. Clearly f^* is a homomorphism of Boolean algebras, so (1) follows from the definitions. As for (2), let us first assume that f is polyhedral and open, and take $U, V \in \mathrm{Up}\,F$ with the aim of showing that $f^*(U \to V) = f^*(U) \to f^*(V)$. The left-to-right inclusion follows from the fact that f^* is a lattice homomorphism. For the right-to-left, writing X^{C} for the complement of X, we have the following chain of inclusions.

$$\begin{aligned} f[f^*(U) \to f^*(V)] &= f\left[\mathrm{Int}\left(f^{-1}[U]^{\mathsf{C}} \cup f^{-1}[V]\right)\right] \\ &\subseteq \mathrm{Int}\left(f\left[f^{-1}[U]^{\mathsf{C}} \cup f^{-1}[V]\right]\right) && (f \text{ is open}) \\ &= \mathrm{Int}\left(f\left[f^{-1}[U^{\mathsf{C}} \cup V]\right]\right) \\ &\subseteq \mathrm{Int}(U^{\mathsf{C}} \cup V) \\ &= U \to V \end{aligned}$$

Applying $f^* = f^{-1}$ to both sides, we get that $f^*(U) \to f^*(V) \subseteq f^*(U \to V)$.

For the converse implication, assume that f^* is a Heyting algebra homomorphism. By (1), f is polyhedral, so take $W \subseteq F$ with the aim of showing that $f^{-1}[\mathrm{Int}\,W] = \mathrm{Int}(f^{-1}[W])$. First let $A := \mathrm{Int}((\uparrow W)^{\mathsf{C}} \cup W) \cup \mathrm{Int}(W^{\mathsf{C}})$ and $B := \mathrm{Int}\,W$. A routine calculation verifies that $A^{\mathsf{C}} \cup B = W$, and moreover that $A, B \in \mathrm{Up}\,F$. Then:

$$\begin{aligned} f^{-1}[\mathrm{Int}\,W] &= f^*[A \to B] \\ &= f^*[A] \to f^*[B] && (f^* \text{ is a homomorphism}) \\ &= \mathrm{Int}(f^*[A]^{\mathsf{C}} \cup f^*[B]) \\ &= \mathrm{Int}(f^*[A^{\mathsf{C}} \cup B]) \\ &= \mathrm{Int}(f^{-1}[W]) \end{aligned} \qquad \square$$

Let Σ be a simplicial complex and F be a poset. Given any function $f : \Sigma \to F$, define the map $\widehat{f} : |\Sigma| \to F$ by:

$$\widehat{f}(x) := f(\sigma^x)$$

Proposition 1.46. When $f : \Sigma \to F$ is a p-morphism, $\widehat{f} : |\Sigma| \to F$ is an open polyhedral map.

Proof. For any $U \in \operatorname{Up} F$, we have that:

$$\widehat{f}^{-1}[U] = \bigcup\{\operatorname{Relint}\sigma \mid \sigma \in \Sigma \text{ and } \sigma \in f^{-1}[U]\} = \gamma^{\uparrow}(f^{-1}[U])$$

Since f is monotonic, $f^{-1}[U]$ is upwards-closed in Σ, whence as above $\widehat{f}^{-1}[U]$ is an open sub-polyhedron of $|\Sigma|$. Now take an open set $W \subseteq |\Sigma|$, with the aim of showing that $\widehat{f}[W]$ is open. Define:

$$\Sigma \# W := \{\sigma \in \Sigma \mid \operatorname{Relint}(\sigma) \cap W \neq \varnothing\}$$

Then:

$$\widehat{f}[W] = \{f(\sigma^x) \mid x \in W\} = f[\Sigma \# W]$$

If $\sigma \in \Sigma \# W$ and $\sigma \preccurlyeq \tau$, then as $\sigma \subseteq \tau = \operatorname{Cl}\operatorname{Relint}\tau$ and W is open, we have $\tau \in \Sigma \# W$; i.e. $\Sigma \# W$ is upwards-closed. But now, f is open and so $\widehat{f}[W]$ is also upwards-closed. □

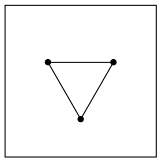
Another important class of maps is that of PL homeomorphisms. First, for any $X, Y \subseteq \mathbb{R}^n$, a function $X \to Y$ is an *affine map* if it is of the form $x \mapsto Mx + b$, where M is a linear transformation and $b \in \mathbb{R}^n$. Now let P, Q be polyhedra. A homeomorphism $f : P \to Q$ is *piecewise-linear* if there is a triangulation Σ of P such that for each $\sigma \in \Sigma$ the restriction $f|_{\sigma}$ is affine. Call such maps *PL homeomorphisms* for short.

Proposition 1.47. The inverse of a PL homeomorphism is a PL homeomorphism.

Proof. See [RS72, p. 6]. □

Proposition 1.48. Any PL homeomorphism $f : P \to Q$ between polyhedra, along with its inverse $g : Q \to P$, induce mutually inverse isomorphisms of Heyting algebras $f^* \colon \operatorname{Sub_o} Q \to \operatorname{Sub_o} P$ and $g^* \colon \operatorname{Sub_o} P \to \operatorname{Sub_o} Q$.

Proof. The inverse image of a subpolyhedron under a PL homeomorphism is again a subpolyhedron [RS72, Corollary 2.5, p. 13], meaning the inverse image of an open subpolyhedron is an open subpolyhedron. Furthermore, homeomorphisms are open maps. Hence $f^* \colon \mathscr{P}(Q) \to \mathscr{P}(P)$ and $g^* \colon \mathscr{P}(P) \to \mathscr{P}(Q)$ descend to functions as in the statement. These are mutually inverse isomorphisms of lattices by definition. The fact that they also preserve Heyting implication follows just as in the proof of Proposition 1.45. □



Chapter 2

Nerves and Triangulations

In this chapter, I introduce the notion of the nerve of a poset. This is a classical construction originally due to Pavel Alexandrov [Ale98], which has found numerous applications in geometry, topology and combinatorics (see e.g. [Bjö95]). The nerve has an important place in the present thesis, and it will be used to deepen the logic-geometry connection established in Chapter 1.

2.1 Nerves for Geometric Realisation

Definition 2.1 (Nerve of a poset)**.** The *nerve* $\mathcal{N}(F)$ of a poset F is the collection of non-empty finite chains in F ordered by inclusion.

See Figure 2.1 for an example of the nerve construction, and compare to Figure 1.6.

A key fact about the nerve and its relation to logic is that there is always a p-morphism $\mathcal{N}(F) \to F$.

Definition 2.2. Let $\max\colon \mathcal{N}(F) \to F$ be the map which sends a chain X to its maximal element.

Proposition 2.3. $\max\colon \mathcal{N}(F) \to F$ is a p-morphism.

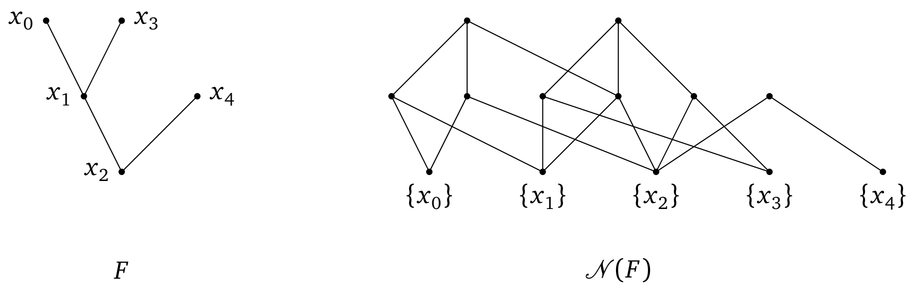


Figure 2.1: An example of the construction of the nerve of a poset

Proof. First, max is clearly monotonic. Second, for the **back** condition, take $X \in \mathcal{N}(F)$ and $y > \max X$. Since y is greater than the maximal element of X, it is greater than every element of X; whence $X \cup \{y\}$ is a chain. Then $X \subseteq X \cup \{y\}$ and $\max(X \cup \{y\}) = y$. $\square$

We will also need the following basic observation.

PROPOSITION 2.4. $\mathsf{height}(\mathcal{N}(F)) = \mathsf{height}(F)$.

Proof. Let $X = \{x_0, \ldots, x_k\}$ be a chain in F, then the following is a chain in $\mathcal{N}(F)$ of the same length.

$$\{\{x_0\}\} \subset \{\{x_0\}, \{x_0, x_1\}\} \subset \{\{x_0\}, \{x_0, x_1\}, \{x_0, x_1, x_2\}\} \subset \cdots \subset \{\{x_0\}, \ldots, X\}$$

Conversely, let $\mathscr{X}$ be a chain of length $k+1$ in $\mathcal{N}(F)$. Write $\mathscr{X} = \{X_0, \ldots, X_k\}$, such that $X_0 \subset \cdots \subset X_k$. Then, for each i we have $|X_{i+1}| > |X_i|$, so that X_k is a chain in F of length at least $k+1$. $\square$

Geometric Realisation. The first use to which we put the nerve is in generating realisations of finite posets as polyhedra, following [BMMP18]. For any Euclidean space $\mathbb{R}^n$ let $e_1, \ldots, e_n$ be the elements of its standard basis.

DEFINITION 2.5 (Induced simplicial complex)**.** Let F be a finite poset, and enumerate it as $F = \{x_1, \ldots, x_n\}$. The *simplicial complex induced by* F is the set of simplices:

$$\nabla F := \{\mathrm{Conv}\{e_{i_1}, \ldots, e_{i_k}\} \mid \{x_{i_1}, \ldots, x_{i_k}\} \in \mathcal{N}(F)\}$$

PROPOSITION 2.6. ∇F is a simplicial complex.

Proof. (a) Take $X = \{x_{i_1}, \ldots, x_{i_k}\} \in \mathcal{N}(F)$. Faces of $\mathrm{Conv}\{e_{i_1}, \ldots, e_{i_k}\}$ correspond to subsets $\{x_{j_1}, \ldots, x_{j_m}\} \subseteq X$. Any such subset is also a chain in F, meaning that we have $\mathrm{Conv}\{e_{j_1}, \ldots, e_{j_m}\} \in \nabla F$. (b) Clearly, the intersection of any two simplices in ∇F is either empty or a common face of both of them. $\square$

REMARK 2.7. Note that, as posets, $\nabla F \cong \mathcal{N}(F)$.

DEFINITION 2.8 (Geometric realisation)**.** The *geometric realisation* of F is the polyhedron $|\nabla F|$.

REMARK 2.9. Of course, the definition of ∇F and the resulting realisation $|\nabla F|$ depends on the enumeration of F. But the different possible realisations are, logically speaking, indistinguishable, and for our purposes we don't need to worry about this subtlety.

PROPOSITION 2.10. For any finite poset F, we have that $\mathrm{Logic}(|\nabla F|) \subseteq \mathrm{Logic}(F)$.

Proof. First, the p-morphism $\max\colon \mathcal{N}(F) \to F$ induces a p-morphism $\nabla F \to F$. This in turn, via Proposition 1.46, induces an open polyhedral map $|\nabla F| \to F$. Then, by Proposition 1.45 (2), we get a homomorphism $\mathrm{Up}\, F \to \mathrm{Sub_o}|\nabla F|$. Finally, Proposition 1.9 means that $\mathrm{Logic}(|\nabla F|) = \mathrm{Logic}(\mathrm{Sub_o}|\nabla F|) \subseteq \mathrm{Logic}(F)$. $\square$

Polyhedral completeness. Using this last piece, we are now in a position to prove the main result of [BMMP18], that **IPC** is the logic of the class of all polyhedra.

THEOREM 2.11. **IPC** $= \mathrm{Logic}(P \mid P$ is a polyhedron$)$.

Proof. Since every $\mathrm{Sub_o}P$ is a Heyting algebra, we get that:

$$\mathbf{IPC} \subseteq \mathrm{Logic}(P \mid P \text{ is a polyhedron})$$

For the converse inclusion, assume that $\mathbf{IPC} \nvdash \phi$. By Proposition 1.1, there is a finite poset F such that $F \nvDash \phi$. But then, by Proposition 2.10, we get that $|\nabla F| \nvDash \phi$. $\square$

2.2 The Nerve Criterion for Polyhedral Completeness

The second use of the nerve is essentially a strengthening of Maxim I. We will see that in order to understand the logic of a polyhedron, it suffices to take any triangulation and consider only its iterated nerves. The foundation of this result is the classical link between nerves and barycentric subdivision, but the complete proof will be somewhat involved, and we will need to import several results from polyhedral geometry. The proof is joint work with Nick Bezhanishvili, David Gabelaia and Vincenzo Marra.

The main theorem. The main theorem of this chapter is first stated algebraically. The all-important Nerve Criterion concerning polyhedral completeness will be extracted from this later.

DEFINITION 2.12 (kth derived subdivision). Let Σ be a simplicial complex. The *kth derived subdivision of* Σ, denoted by $\Sigma^{(k)}$, is the result of applying the barycentric subdivision operation k-times on Σ. I.e. $\Sigma^{(k)} = \mathrm{Sd}^k\,\Sigma$.

DEFINITION 2.13 (kth derived triangulation subalgebra). Let A be a triangulation subalgebra of $\mathrm{Sub_o}P$ for some polyhedron P. By the dual of Proposition 1.37 (1), there is a unique triangulation Σ of P such that $A = \mathrm{P_o}(\Sigma)$. For any $k \in \mathbb{N}$, let $A^{(k)} := \mathrm{P_o}(\Sigma^{(k)})$.

THEOREM 2.14. Let P be a polyhedron and let A be any triangulation subalgebra of $\mathrm{Sub_o}P$. For any finitely-generated subalgebra B of $\mathrm{Sub_o}P$, there is $k \in \mathbb{N}$ such that B is isomorphic to a subalgebra of $A^{(k)}$.

Let us see how to prove this theorem.

Rational polyhedra and unimodular triangulations. The intuition behind Theorem 2.14 is that any triangulation can be approximated from any other by taking iterated barycentric subdivisions. The difficulty one might face with spelling out such an intuition is dealing with the 'continuum nature' of $\mathbb{R}^n$. It might be imagined that, if we start with a triangulation Σ on irrational vertices and try to approximate it using the iterated barycentric subdivisions of a triangulation on rational vertices, the approximations would never quite capture all of Σ. The approach taken here is effectively to show that it suffices to restrict attention to the rational case. In order to make this idea precise, we need some definitions. I will mainly be following [Mun11].

A polytope in $\mathbb{R}^n$ is *rational* if it may be written as the convex hull of finitely many points in $\mathbb{Q}^n \subseteq \mathbb{R}^n$. A polyhedron in $\mathbb{R}^n$ is *rational* if it may be written as a union of a finite collection of rational polytopes. A simplicial complex Σ is *rational* if it consists of rational simplices. Note that when this is the case, $|\Sigma|$ is a rational polyhedron.

For any $x \in \mathbb{Q}^n \subseteq \mathbb{R}^n$, there is a unique way to write out x in coordinates as $x = (\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n})$ such that for each i, we have $p_i, q_i \in \mathbb{Z}$ coprime. The *denominator* of x is defined:

$$\mathrm{Den}(x) := \mathrm{lcm}\{q_1, \dots, q_n\}$$

Note that $\mathrm{Den}(x) = 1$ if and only if x has integer coordinates. Letting $q = \mathrm{Den}(x)$, the *homogeneous correspondent* of x is defined to be the integer vector:

$$\widetilde{x} := \left(\frac{qp_1}{q_1}, \ldots, \frac{qp_n}{q_n}, q \right)$$

A rational d-simplex $\sigma = x_0 \cdots x_d$ is *unimodular* if there is an $(n+1) \times (n+1)$ matrix with integer entries whose first d columns are $\widetilde{x_0}, \ldots, \widetilde{x_d}$, and whose determinant is ± 1. This is equivalent to requiring that the set $\{\widetilde{x_0}, \ldots, \widetilde{x_d}\}$ can be completed to a $\mathbb{Z}$*-module basis* of $\mathbb{Z}^{d+1}$. A simplicial complex is *unimodular* if each one of its simplices is unimodular.

Farey subdivisions. In order to obtain the main result concerning barycentric subdivisions, we go via another kind of subdivision which is more amenable to the rational case.

PROPOSITION 2.15. For any $x, y \in \mathbb{Q}^n$, there is a unique $m \in \mathbb{Q}^n$ such that $\widetilde{m} = \widetilde{x} + \widetilde{y}$, and this lies in the relative interior of the 1-simplex $\mathrm{Conv}\{x, y\}$.

Proof. Let $H_{n+1} \subseteq \mathbb{R}^{n+1}$ be the hyperplane specified by:

$$H_{n+1} := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 1\}$$

Identify $\mathbb{Q}^n$ with the set of rational points of H_{n+1}. Under this identification, $\widetilde{m} = \widetilde{x} + \widetilde{y}$ lies in the affine cone:

$$\{a\widetilde{x} + b\widetilde{y} \mid a, b > 0\}$$

A routine computation then proves the geometrically evident fact that m is the point of intersection of the line spanned in R^{n+1} by the vector $\tilde{m}$, with the hyperplane H_{n+1}; from which the result follows. $\square$

For $x, y \in \mathbb{Q}^n$, let this $m \in \mathbb{Q}^n$ be their *Farey mediant*. The Farey mediant behaves in a similar way to the barycentre of x and y.

Using the notion of Farey mediant, one can define the notion of a Farey subdivision. Just as in the case of barycentric subdivision, the precise formulation is somewhat involved, while the technical details are not so important for the present thesis. Thus, as before, I will present the idea, coupled with some diagrams, in order to give the essential intuition. For a complete definition, I refer the reader to [Mun11, §5.1, p. 55].

Let Σ_1, Σ_2 be rational simplicial complexes in $\mathbb{R}^n$. Then Σ_2 is an *elementary Farey subdivision* of Σ_1 if it is obtained from Σ_1 by subdividing exactly one of its 1-simplices $\mathrm{Conv}\{x, y\}$ through the introduction of the Farey mediant m of x and y as the single new vertex of Σ_2. If Σ_2 can be obtained from Σ_1 through finitely many successive elementary Farey subdivisions, then we say Σ_2 is a *Farey subdivision* of Σ_1. See Figure 2.2 for examples of this operation.

To relate Farey subdivisions with barycentric subdivisions, note that one may define an *elementary barycentric subdivision* analogously to the Farey case, by taking a single 1-simplex and adding a new vertex at its barycentre. The following technical lemma will be useful below; its proof uses the details of the full definition of elementary Farey and barycentric subdivision.

LEMMA 2.16. Let Σ, Δ be simplicial complexes with Σ rational, assume that $\gamma \colon \Sigma \to \Delta$ is an isomorphism of Σ and Δ as posets, and take a 1-simplex $\sigma \in \Sigma$. Then the elementary Farey subdivision of Σ along σ and the elementary barycentric subdivision of Δ along $\gamma(\sigma)$ are isomorphic as posets.

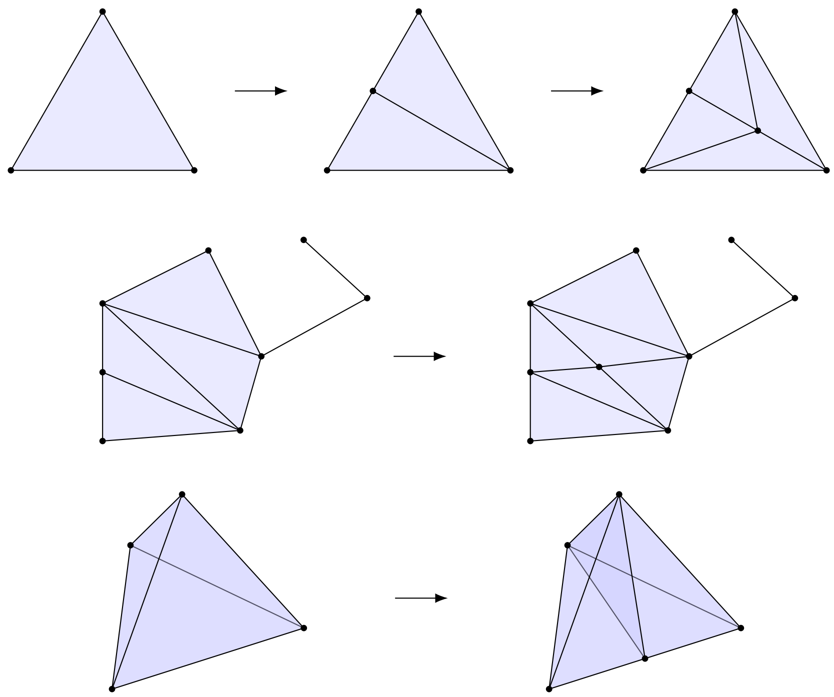


Figure 2.2: Examples of elementary Farey subdivisions

Proof. Indeed, at the level of posets, elementary Farey subdivision and elementary barycentric subdivision are the same operation: we take a 1-simplex and add a new vertex somewhere in its interior, then construct the rest of the complex around this. For more details see [Ale30, §III]. □

The following is a fundamental fact of rational polyhedral geometry, and captures the idea of 'rational approximation'.

Lemma 2.17 (The De Concini-Procesi Lemma)**.** Let P be a rational polyhedron, and let Σ be a unimodular triangulation of P. There exists a sequence $(\Sigma_i)_{i\in\mathbb{N}}$ of unimodular triangulations of P with $\Sigma_0 = \Sigma$ such that:

(a) For each $i \in \mathbb{N}$, Σ_{i+1} is an elementary Farey subdivision of Σ_i, and

(b) For any rational polyhedron $Q \subseteq P$, there is $i \in \mathbb{N}$ such that Σ_i triangulates Q.

Proof. See [Mun11, Theorem 5.3, p. 57]. □

From $\mathbb{R}$ to $\mathbb{Q}$. We will now see how to relate general polyhedra to rational polyhedra, and general simplicial complexes to unimodular simplicial complexes.

Lemma 2.18. Let P be a polyhedron, and let Σ be a triangulation of P. There exist an integer $n \in \mathbb{N}$, a rational polyhedron $Q \subseteq \mathbb{R}^n$, and a unimodular triangulation Δ of Q such that P and Q are PL-homeomorphic via a map that induces an isomorphism of Σ and Δ as posets.

Proof. This is a standard argument. Fix a bijection β from the vertices of Σ to the standard basis of $\mathbb{R}^n$, where n is the number of vertices in Σ. Take a simplex $\sigma = x_0 \cdots x_d$

in Σ. Note that the points $\beta(x_0), \ldots, \beta(x_d)$ are affinely independent; let $\alpha(\sigma)$ be the d-simplex spanned by their convex hull: $\alpha(\sigma) := \mathrm{Conv}\{\beta(x_0), \ldots, \beta(x_d)\}$. Since the vertices of $\alpha(\sigma)$ are standard basis elements, $\alpha(\sigma)$ is a unimodular simplex by definition. Let $f_\sigma \colon \sigma \to \alpha(\sigma)$ be the linear map determined by $f_\sigma(x_i) = \beta(x_i)$ for each i, and let $g_\sigma \colon \alpha(\sigma) \to \sigma$ be its inverse, determined by $g_\sigma(\beta(x_i)) = x_i$.

Now, let $Q := \bigcup_{\sigma \in \Sigma} \alpha(\sigma)$. For any simplices $\sigma \preccurlyeq \tau$, the map f_σ agrees with f_τ on σ. Hence we may glue these maps together to form a map $f : P \to Q$, i.e. $f(x) = f_\sigma(x)$, where σ is any simplex of Σ containing x. Similarly, we may glue together the maps g_σ for $\sigma \in \Sigma$ to form an inverse to f. By definition f is a PL homeomorphism. Finally, note that $\Delta := \{\alpha(\sigma) \mid \sigma \in \Sigma\}$ is a triangulation of Q, and that f induces the poset isomorphism $\sigma \mapsto \alpha(\sigma)$ between Σ and Δ. □

LEMMA 2.19. Let Σ be a unimodular triangulation of the rational polyhedron P, and suppose Σ' is a Farey subdivision of Σ. There is a triangulation Δ of P which is isomorphic as a poset to Σ', and $k \in \mathbb{N}$ such that $\Sigma^{(k)}$ refines Δ.

Proof. The proof works by replacing each elementary Farey subdivision by an elementary barycentric subdivision. We induct on the number $m \in \mathbb{N}^{>0}$ of elementary Farey subdivisions needed to obtain Σ' from Σ. If $m = 1$, let $\mathrm{Conv}\{x, y\}$ be the 1-simplex of Σ being subdivided through its Farey mediant. Then the first barycentric subdivision $\Sigma^{(1)}$ of Σ refines the elementary barycentric subdivision Σ^* of Σ along $\mathrm{Conv}\{x, y\}$. By Lemma 2.16, Σ^* and Σ' are isomorphic.

For the induction step, suppose $m > 1$, and write $(\Sigma_i)_{i=0}^m$ for the finite sequence of triangulations connecting $\Sigma = \Sigma_0$ to $\Sigma' = \Sigma_m$ through elementary Farey subdivisions. By the induction hypothesis, there is $k \in \mathbb{N}$ such that $\Sigma^{(k)}$ refines a triangulation Δ isomorphic to Σ_{m-1}; let us fix one such isomorphism γ. Let $\mathrm{Conv}\{x, y\}$ be the 1-simplex of Σ_{m-1} that must be subdivided through its Farey mediant in order to obtain Σ_m. Let further σ be the simplex of Δ that corresponds to $\mathrm{Conv}\{x, y\}$ through the isomorphism γ. Since the 1-simplices are exactly the height-1 elements of Δ, we get that σ is a 1-simplex. Then $\Sigma^{(k+1)}$ refines Δ^*, the latter denoting the elementary barycentric subdivision of Δ along σ. But Δ is isomorphic to Σ_{m-1}, and therefore by Lemma 2.16, Δ^* is isomorphic to Σ_m. □

LEMMA 2.20 (Beynon's Lemma)**.** Let P be a rational polyhedron, and let Σ be a triangulation of P. There exists a rational triangulation of P which is isomorphic as a poset to Σ.

Proof. This is the main result of [Bey77]. □

Putting it all together. It is time to combine all our ingredients and prove the main theorem of the chapter.

Proof of Theorem 2.14. Let Σ be the triangulation of P such that $A = \mathrm{P_o}(\Sigma)$. Using Lemma 2.18, Proposition 1.37 (2) and Proposition 1.48 we may assume without loss of generality that P is rational and Σ is unimodular. By Proposition 1.35, there is a triangulation Δ of P such that B is isomorphic to a subalgebra of $\mathrm{P_o}(\Delta)$. By Beynon's Lemma 2.20 and Proposition 1.37 (2), we may assume that Δ is rational (and hence each member of B is, too). By the De Concini-Procesi Lemma 2.17, there is a Farey subdivision Σ' of Σ that refines Δ. Therefore by Proposition 1.37 (3), B is isomorphic to a subalgebra of $\mathrm{P_o}(\Sigma')$. By Lemma 2.19, there is $k \in \mathbb{N}$ such that $\Sigma^{(k)}$ refines Σ' up to isomorphism. Hence by Proposition 1.37 (3) again, $A^{(k)}$ contains a subalgebra isomorphic to $\mathrm{P_o}(\Sigma')$, and therefore also a subalgebra isomorphic to B. □

Bringing nerves back onto the stage. Now that we have proved the main theorem as stated, it will be fruitful to examine it from another perspective. The reader may have noticed that the statement of Theorem 2.14 doesn't mention nerves. The reason that these latter constructions are relevant here is the following.

PROPOSITION 2.21. Let Σ be a simplicial complex. The barycentric subdivision of Σ is isomorphic as a poset to the nerve of Σ:

$$\mathrm{Sd}\,\Sigma \cong \mathcal{N}(\Sigma)$$

Proof. Let me give an intuitive proof as to why this is the case. For more detail, I refer the reader to [Mau80, Proposition 2.5.10, p. 51] and [RW12, §3].

In our informal definition, the construction of the barycentric subdivision of a simplicial complex Σ involved putting a new vertex at the barycentre of each simplex of Σ, and constructing the rest of $\mathrm{Sd}\,\Sigma$ around this. Let us consider in a little more detail what this involves. For each simplex $\sigma \in \Sigma$, we have a new 0-simplex, which we will label $\{\sigma\}$. The first step in 'building up the rest of $\mathrm{Sd}\,\Sigma$' would be to add in some 1-simplices. A little reflection and diagram staring (consider again Figure 1.5) indicates that we should put a 1-simplex between $\{\sigma\}$ and $\{\tau\}$ exactly when $\sigma \prec \tau$ or $\tau \prec \sigma$, i.e. when $\{\sigma,\tau\}$ is a chain in Σ. Let us label such a new 1-simplex $\{\sigma,\tau\}$. The next stage would be to add in some 2-simplices. Some further reflection and diagram staring should indicate that we should add a 2-simplex connecting σ, τ and ρ exactly when $\{\sigma,\tau,\rho\}$ is a chain in Σ. Label such a 2-simplex by $\{\sigma,\tau,\rho\}$. Continuing in this fashion, we eventually arrive at an isomorphism $\mathrm{Sd}\,\Sigma \cong \mathcal{N}(\Sigma)$. □

COROLLARY 2.22. For P a polyhedron and Σ a triangulation of P we have:

$$\mathrm{Logic}(P) = \mathrm{Logic}(\mathcal{N}^k(\Sigma) \mid k \in \mathbb{N})$$

Proof. Indeed:

$$\begin{aligned}
\mathrm{Logic}(P) &= \mathrm{Logic}(\mathrm{Sub}_{\mathrm{o}}P) \\
&= \mathrm{Logic}(A \mid A \text{ finitely-generated subalgebra of } \mathrm{Sub}_{\mathrm{o}}P) && \text{(Proposition 1.35)} \\
&= \mathrm{Logic}(\mathrm{P}_{\mathrm{o}}(\Sigma^{(k)}) \mid k \in \mathbb{N}) && \text{(Theorem 2.14)} \\
&= \mathrm{Logic}(\Sigma^{(k)} \mid k \in \mathbb{N}) && \text{(as above)} \\
&= \mathrm{Logic}(\mathcal{N}^k(\Sigma) \mid k \in \mathbb{N}) && \text{(Proposition 2.21)}
\end{aligned}$$

□

This leads us to our second maxim, as well as the much-anticipated Nerve Criterion.

MAXIM II. The logic of a polyhedron is the logic of the iterated nerves of any one of its triangulations.

COROLLARY 2.23 (The Nerve Criterion)**.** A logic $\mathscr{L}$ is polyhedrally-complete if and only if it is the logic of a class of finite frames closed under the nerve construction $\mathcal{N}$.

Proof. Assume that $\mathscr{L}$ is the logic of a class $\mathbf{C}$ of polyhedra. For each $P \in \mathbf{C}$ fix a triangulation Σ_P, and let:

$$\mathbf{C}^* := \{\mathcal{N}^k(\Sigma_P) \mid P \in \mathbf{C} \text{ and } k \in \mathbb{N}\}$$

Then:

$$\begin{aligned}
\mathrm{Logic}(\mathbf{C}^*) &= \bigcap_{P\in\mathbf{C}} \mathrm{Logic}(\mathscr{N}^k(\Sigma_P) \mid k\in\mathbb{N}) \\
&= \bigcap_{p\in\mathbf{C}} \mathrm{Logic}(P) && \text{(Corollary 2.22)} \\
&= \mathrm{Logic}(\mathbf{C}) = \mathscr{L}
\end{aligned}$$

Conversely, assume that $\mathscr{L} = \mathrm{Logic}(\mathbf{D})$, where $\mathbf{D}$ is a class of finite frames closed under $\mathscr{N}$. Let:

$$\mathbf{D}_* := \{|\nabla(F)| : F \in \mathbf{D}\}$$

I will show that $\mathscr{L} = \mathrm{Logic}(\mathbf{D}_*)$. First suppose that $\mathscr{L} \nvdash \phi$, so that $F \nvDash \phi$ for some $F \in \mathbf{D}$. Then by Proposition 2.10 we have that $|\nabla(F)| \nvDash \phi$, so that $\mathrm{Logic}(\mathbf{D}_*) \nvdash \phi$. Conversely, suppose that $\mathrm{Logic}(\mathbf{D}_*) \nvdash \phi$, so that $|\nabla(F)| \nvDash \phi$ for some $F \in \mathbf{D}$. By definition $\nabla(F)$ is a triangulation of $|\nabla(F)|$, hence by Corollary 2.22 there is $k \in \mathbb{N}$ such that $\nabla(F)^{(k)} \nvDash \phi$. But $\nabla(F) \cong \mathscr{N}(F)$ by definition, and so by Proposition 2.21 we get $\mathscr{N}^{k+1}(F) \cong \nabla(F)^{(k)}$. Thus, as $\mathbf{D}$ is closed under $\mathscr{N}$, we get that $\mathscr{L} \nvdash \phi$. □

Some first consequences of the Nerve Criterion. Let us see how the Nerve Criterion can be applied to provide some negative answers to the question *which logics are polyhedrally-complete*? We begin with a definition which encompasses a wide class of logics (see [BB17]).

DEFINITION 2.24 (Stable logic). A logic $\mathscr{L}$ is *stable* if $\mathrm{Frames}_\perp(\mathscr{L})$ is closed under monotone images.

PROPOSITION 2.25. The following well-known logics are all stable (for more information on these logics see [CZ97, Table 4.1, p. 112]).

(i) The logic of weak excluded middle, $\mathbf{KC} = \mathbf{IPC} + (\neg p \vee \neg\neg p)$.

(ii) Gödel-Dummett logic, $\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p)$.

(iii) $\mathbf{LC}_n = \mathbf{LC} + \mathbf{BD}_n$.

(iv) The logic of bounded width n, $\mathbf{BW}_n = \mathbf{IPC} + \bigvee_{i=0}^n (p_i \to \bigvee_{j\neq i} p_j)$.

(v) The logic of bounded top width n, defined:

$$\mathbf{BTW}_n := \bigwedge_{0\leqslant i<j\leqslant n} \neg(\neg p_i \wedge \neg p_j) \to \bigvee_{i=0}^{n} (\neg p_i \to \bigvee_{j\neq i} \neg p_j)$$

(vi) The logic of bounded cardinality n, defined:

$$\mathbf{BC}_n := p_0 \vee (p_0 \to p_1) \vee ((p_0 \wedge p_1) \to p_2) \vee \cdots \vee ((p_0 \wedge \cdots \wedge p_{n-1}) \to p_n)$$

Proof. See [BB17, Theorem 7.3]. □

In fact:

THEOREM 2.26. There are continuum-many stable logics.

Proof. See [BB17, Theorem 6.13]. □

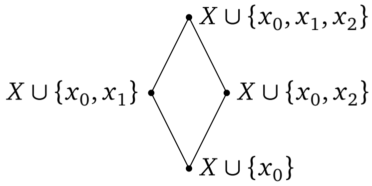
THEOREM 2.27. Every stable logic has the finite model property.

Proof. See [BB17, Theorem 6.8]. □

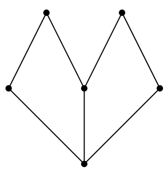
Hence, stable logics are good candidates for polyhedrally-complete logics (c.f. Proposition 1.39). However:

THEOREM 2.28. If $\mathscr{L}$ is a stable logic other than **IPC**, and Frames($\mathscr{L}$) contains a frame of height at least 2, then $\mathscr{L}$ is not polyhedrally-complete.

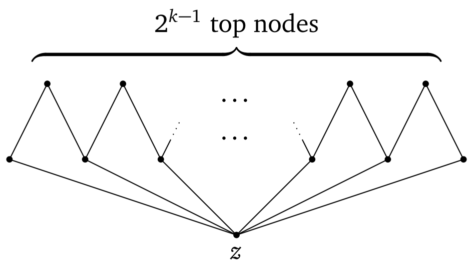
Proof. Suppose not. Since $\mathscr{L}$ is polyhedrally-complete, by the Nerve Criterion 2.23, there is a class **C** of finite frames closed under $\mathcal{N}$ such that $\mathscr{L} = \text{Logic}(\mathbf{C})$. Since Frames($\mathscr{L}$) contains a frame of height at least 2, by Proposition 1.18 we must have $\mathscr{L} \nvdash \mathbf{BD}_1$. Since $\mathscr{L} = \text{Logic}(\mathbf{C})$, there is therefore $F \in \mathbf{C}$ such that $\mathsf{height}(F) \geqslant 2$. This means there are $x_0, x_1, x_2 \in F$ with $x_0 < x_1 < x_2$. Without loss of generality, we may assume that x_2 is a top element and that x_1 is an immediate predecessor of x_2 and x_0 an immediate predecessor of x_1. Now, by assumption $\mathcal{N}^k(F) \in \mathbf{C}$ for every $k \in \mathbb{N}$. Let us examine the structure of these frames a little. Note that $\{x_0, x_1, x_2\}$ is a chain. Let X be a maximal chain in $\Downarrow(x_0)$. We have the following relations occurring in $\mathcal{N}(F)$.



Moreover, by assumptions on x_0, x_1, x_2 and X, we have that $X \cup \{x_0, x_1, x_2\}$ is a top element of $\mathcal{N}(F)$, with $X \cup \{x_0, x_1\}$ and $X \cup \{x_0, x_2\}$ immediate predecessors, and $X \cup \{x_0\}$ an immediate predecessor of those. So, we may apply this argument once more, to obtain the following structure sitting at the top of $\mathcal{N}^2(F)$.



Iterating, we see that at the top of $\mathcal{N}^k(F)$ we have the following structure.



Let z be the base element of this structure, as indicated. Now, take $k \in \mathbb{N}$ and let $\{t_1, \ldots, t_m\}$ be the top nodes of $\mathcal{N}^k(F)$ produced by this construction, where $m = 2^{k-1}$. By Proposition 1.6, $\uparrow(z) \in \text{Frames}_\perp(\mathscr{L})$. Let G be any arbitrary poset with up to m

elements $\{y_1, \ldots, y_m\}$ (possibly with duplicates) plus a root $\bot$. Define $f : \uparrow(z) \to G$ as follows.

$$x \mapsto \begin{cases} y_i & \text{if } x = t_i, \\ \bot & \text{otherwise.} \end{cases}$$

Then f is monotonic. Since $\mathscr{L}$ is stable, this means that $G \in \mathrm{Frames}_\bot(\mathscr{L})$. Thus (since, by Proposition 1.1 and Corollary 1.7, **IPC** is the logic of finite rooted frames) we get that $\mathscr{L} = \mathbf{IPC}$. $\lightning$ $\square$

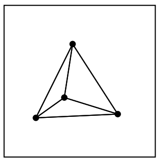
We can get one positive result relatively cheaply however.

Theorem 2.29. $\mathbf{BD}_n$ is polyhedrally-complete for each $n \in \mathbb{N}$. In fact, $\mathbf{BD}_n$ is the logic of the class of polyhedra of dimension at most n.

Proof. By Proposition 2.4, for any poset F we have $\mathsf{height}(F) = \mathsf{height}(\mathscr{N}(F))$. Since by Theorem 1.20, $\mathbf{BD}_n$ is the logic of frames of height at most n, it is polyhedrally-complete by the Nerve Criterion. The fact that $\mathbf{BD}_n$ is the logic of the class of polyhedra of dimension at most n follows from the proof of Theorem 2.11, and the fact that the dimension of a simplicial complex is the same as its height as a poset. $\square$

Remark 2.30. Note that the first part of this result follows from the second and so can be obtained without the use of the Nerve Criterion, as is done in [BMMP18]. I include it only in order to give a first indication of the utility of the Criterion, and to warm up before examining its more involved applications in the next chapter.

Chapter 3



Starlike Polyhedral Completeness

Up to this point, the following is a complete list of the logics known to be polyhedrally-complete.

$$\mathbf{IPC}, \mathbf{BD}_0, \mathbf{BD}_1, \mathbf{BD}_2, \ldots$$

In other words, we know of one polyhedrally-complete logic of each finite height. On the other hand, Theorem 2.28 provides a continuum of polyhedrally incomplete logics which have the finite model property, including many well-known examples. What does the rest of the landscape look like? In this chapter, I exploit the Nerve Criterion to fill out parts of this map. I will show that on a certain infinite class of intermediate logics — called 'starlike logics' — polyhedral completeness coincides with the finite model property. This will yield an infinite class of polyhedrally-complete logics of each finite height. Furthermore, it will show us that Scott's logic, defined (see [CZ97, p. 40]):

$$\mathbf{SL} := \mathbf{IPC} + ((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$$

is polyhedrally-complete.

3.1 The Logical Approach

Exploiting the Nerve Criterion, the proofs in this chapter involve combinatorial manipulations of frames (following the 'logic approach' mentioned in the introduction). In this section, I collect a miscellany of definitions and tools which play an important part in the techniques of this chapter.

Nerve-validation. It will be convenient to reformulate the Nerve Criterion in terms of a new validity concept.

Definition 3.1 (Nerve-validation). Let F be a poset and ϕ be a formula. *F nerve-validates ϕ*, notation $F \vDash_{\mathcal{N}} \phi$, if for every $k \in \mathbb{N}$ we have $\mathcal{N}^k(F) \vDash \phi$.

Remark 3.2. Note that, since we have a p-morphism $\mathcal{N}(G) \to G$ for every G, by Proposition 1.6 this is equivalent to requiring that $\mathcal{N}^k(F) \vDash \phi$ for infinitely-many $k \in \mathbb{N}$.

Lemma 3.3. *A logic $\mathscr{L}$ is polyhedrally-complete if and only if it has the finite model property and every rooted finite frame of $\mathscr{L}$ is the up-reduction of a poset which nerve-validates $\mathscr{L}$.*

Proof. Assume that $\mathscr{L}$ is polyhedrally-complete. Then by the Nerve Criterion 2.23 it is the logic of a class $\mathbf{C}$ of finite frames which is closed under $\mathscr{N}$, and so in particular has the f.m.p. Then by Corollary 1.17, every finite rooted frame F of $\mathscr{L}$ is the up-reduction of some $F' \in \mathbf{C}$. Since $\mathbf{C} \subseteq \mathrm{Frames}(\mathscr{L})$ and is closed under $\mathscr{N}$, such an F' nerve-validates $\mathscr{L}$.

Conversely, let $\mathbf{C}$ be the class of all finite rooted frames which nerve-validate $\mathscr{L}$. Note that $\mathbf{C}$ is closed under $\mathscr{N}$. Further, clearly $\mathscr{L} \subseteq \mathrm{Logic}(\mathbf{C})$. To see the reverse inclusion, suppose that $\mathscr{L} \nvdash \phi$. Since $\mathscr{L}$ has the f.m.p., there is $F \in \mathrm{Frames}_{\bot,\mathrm{fin}}(\mathscr{L})$ such that $F \nvDash \phi$. By assumption, F is the up-reduction of $F' \in \mathbf{C}$. Then by Proposition 1.6, $F' \nvDash \phi$, meaning that $\mathrm{Logic}(\mathbf{C}) \nvdash \phi$. □

Pointed up-reductions. The following tiny technical lemma, showing that we can always assume that up-reductions are of a particular form, will be valuable to us in the sequel, since it will simplify the treatment of certain forbidden configurations.

Definition 3.4 (Pointed up-reductions). Let F and Q be finite posets, and let Q have root $\bot$. An up-reduction $f : F \to Q$ is *pointed* with *apex* $x \in F$ if $\mathrm{dom}(f) = \uparrow(x)$ and $f^{-1}\{\bot\} = \{x\}$.

Lemma 3.5. If there is an up-reduction $F \circ\!\!\rightarrow Q$ then there is a pointed up-reduction $F \circ\!\!\rightarrow Q$.

Proof. Take $f : F \circ\!\!\rightarrow Q$, and choose $x \in f^{-1}\{\bot\}$ maximal. Then $f|_{\uparrow(x)}$ is still a p-morphism, and is moreover a pointed up-reduction $F \circ\!\!\rightarrow Q$. □

Corollary 3.6. Let F, Q be finite posets, with Q rooted. Then $F \vDash \chi(Q)$ if and only if there is no pointed up-reduction $F \circ\!\!\rightarrow Q$.

P-congruences. An alternative way of viewing a p-morphism $f : F \to G$ is as a kind of congruence relation on F (see [CZ97, p. 262]). This way of thinking will enable a convenient method of constructing p-morphisms.

Definition 3.7 (P-congruence). A *p-congruence* on a frame F is an equivalence relation $\sim$ such that whenever $x \leqslant y$ we have $[x] \subseteq \downarrow[y]$.

Definition 3.8 (Quotient frame). Let $\sim$ be a p-congruence on F. The *quotient frame* $F/\sim$ has as elements the equivalence classes of $\sim$, and its relation is given by:

$$[x] \leqslant [y] \quad \Leftrightarrow \quad [x] \subseteq \downarrow[y]$$

The *quotient map* is $q : F \to F/\sim$, given by $x \mapsto [x]$.

Proposition 3.9. The quotient map is a p-morphism.

Proof. See [CZ97, Theorem 8.68(i), p. 263]. □

Theorem 3.10 (First Isomorphism Theorem). Let $f : F \to G$ be a surjective p-morphism. Then relation $\sim$ on F defined by:

$$x \sim y \quad \Leftrightarrow \quad f(x) = f(y)$$

is a p-congruence, and moreover $F/\sim \cong G$ via the map $[x] \mapsto f(x)$.

Proof. See [CZ97, Theorem 8.68(ii), p. 263]. □

PROPOSITION 3.11. Let F be a frame and $\mathscr{W}$ be a set of pair-wise disjoint subsets of $\mathsf{Top}(F)$. The relation $\sim_{\mathscr{W}}$, defined as follows, is a p-congruence.

$$x \sim_{\mathscr{W}} y \quad \Leftrightarrow \quad x = y \text{ or } \exists W \in \mathscr{W} : x, y \in W$$

Proof. This is immediate from the definition. □

DEFINITION 3.12. Define $F/\mathscr{W} := F/\sim_{\mathscr{W}}$. Relabel the element $[x] \in F/\mathscr{W}$ as x whenever $x \in F \setminus \bigcup \mathscr{W}$. Let $q_{\mathscr{W}}$ be the quotient map on $\sim_{\mathscr{W}}$.

3.2 Starlike Polyhedral Completeness

Starlike trees. Let us meet the main actors of this chapter.

DEFINITION 3.13 (Starlike trees). A tree T is a *starlike tree* if every $x \in T \setminus \{\bot\}$ has at most one immediate successor.

The terminology 'starlike' comes from graph theory [WS79]. If we were to place the root of a starlike tree at the centre of a diagram and arrange its branches radially outward, it would look like a star. It will be useful to carve out some notation with which we can conveniently point to each starlike tree (up to isomorphism). Note that a starlike tree is determined by the multiset of its branch heights. The following notation is inspired by that used in the theory of multisets.

DEFINITION 3.14. Let $n_1, \dots, n_k, m_1, \dots, m_k \in \mathbb{N}^{>0}$, with $n_1, \dots, n_k$ distinct. Then $T = \langle n_1^{m_1} \cdots n_k^{m_k} \rangle$ is the starlike tree such that if we remove the root $\bot$ we are left with exactly, for each i, m_i chains of length n_i. Let $\langle \epsilon \rangle = \bullet$, the singleton poset. Call $\alpha = n_1^{m_1} \cdots n_k^{m_k}$ (or ϵ) the *signature* of T. We will always assume that $n_1 > n_2 > \cdots > n_k$.

In other words, $T = \langle n_1^{m_1} \cdots n_k^{m_k} \rangle$ is composed of, for each i, m_i branches of length $n_i + 1$. See Figure 3.1 for some examples of starlike trees together with their signatures. I will sometimes write 1^0 for ϵ.

DEFINITION 3.15. Let $\alpha = n_1^{m_1} \cdots n_k^{m_k}$ be a signature. The *length* of α is defined as $|\alpha| := m_1 + \cdots + m_k$. Let $|\epsilon| := 0$. For $j \leqslant |\alpha|$, the *jth height*, $\alpha(j)$, is n_i, where:

$$m_1 + \cdots + m_{i-1} \leqslant j < m_1 + \cdots + m_i$$

DEFINITION 3.16. Let α and β be signatures. Say that $\alpha \leqslant \beta$ if $|\alpha| \leqslant |\beta|$ and for every $j \leqslant |\alpha|$ we have $\alpha(j) \leqslant \beta(j)$.

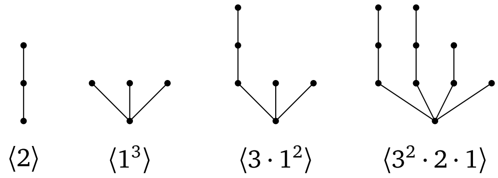


Figure 3.1: Some examples of starlike trees

Visually, this means that if we represent $\alpha = n_1^{m_1} \cdots n_k^{m_k}$ on a grid as a block n_1-tall and m_1-wide, followed by a block n_2-tall and m_2-wide, and so on, and similarly for β, that β covers α. Considering the examples in Figure 3.1, we have the following relations:

$$1^3 < 3 \cdot 1^2 < 3^2 \cdot 2 \cdot 1, \quad 2 < 3 \cdot 1^2$$

Remark 3.17. When $\alpha = n_1^{m_1} \cdots n_k^{m_k}$ and β are signatures, we have $\alpha \leqslant \beta$ if and only if $|\alpha| \leqslant |\beta|$ and for every $i \leqslant k$, we have:

$$\beta(m_1 + \cdots + m_i) \geqslant n_i$$

Proposition 3.18. If $\alpha \leqslant \beta$ then there is a p-morphism $\langle\beta\rangle \to \langle\alpha\rangle$.

Proof. Let us first fix labellings on $\langle\alpha\rangle$ and $\langle\beta\rangle$. Label the root of $\langle\alpha\rangle$ with $\perp$. We may arrange the branches of $\langle\alpha\rangle$ in a sequence so that the jth branch has height $\alpha(j)$. Let us label the non-root elements of the jth branch in ascending order as $a(j,1), \dots, a(j,\alpha(j))$, and similarly for $\langle\beta\rangle$, with $b(j,i)$ for $j \leqslant |\beta|$ and $i \leqslant \beta(j)$.

Now, define $f : \langle\beta\rangle \to \langle\alpha\rangle$ as follows. Note, for $j \leqslant |\alpha|$, we have $\alpha(j) \leqslant \beta(j)$. For $i \leqslant \beta(j)$ let:

$$f(b(j,i)) := a(j, \min(i, \alpha(j)))$$

For $j > |\alpha|$ and $i \leqslant \beta(j)$, let:

$$f(b(j,i)) := a(1, \alpha(1))$$

A routine calculation shows that f is a p-morphism. □

Remark 3.19. Note that the starlike tree $\langle k\rangle$ is the chain on $k+1$ elements, Ch_k. I will use this former notation for chains from now on.

Definition 3.20 (k-fork)**.** For $k \in \mathbb{N}^{>0}$, the *k-fork* is the starlike tree $\langle 1^k\rangle$.

Starlike trees as forbidden configurations. We are interested in Jankov-Fine formulas $\chi(T)$, where T is a starlike tree. On both posets and polyhedra, such formulas will turn out to express a class of connectedness properties. Let us first see some new terminology.

Definition 3.21. Let F be a finite poset. Define $\mathfrak{C}(F)$ to be the set of connected components of F. The *connectedness type* $\mathrm{c}(F)$ of F is the signature $n_1^{m_1} \cdots n_k^{m_k}$ such that $\mathfrak{C}(F)$ contains for each i exactly m_i sets of height $n_i - 1$, and nothing else. Let $\mathrm{c}(\varnothing) := \epsilon$.

Remark 3.22. Note that when F is connected, $\mathrm{c}(F) = n+1$, where $n = \mathsf{height}(F)$.

Definition 3.23 (α-partition)**.** Let $\alpha > \epsilon$ be a signature. An *α-partition* of F is a partition:

$$F = C_1 \sqcup \cdots \sqcup C_{|\alpha|}$$

into open sets such that C_j has height at least $\alpha(j) - 1$. For notational uniformity, say that F has an ϵ-partition if $F = \varnothing$.

Remark 3.24. So an α-partition is an open partition in which the number and heights of the connected components are specified by α.

Lemma 3.25. A finite poset F has an α-partition if and only if $\alpha \leqslant \mathrm{c}(F)$.

Proof. Let $\beta := \mathrm{c}(F)$, and write $\alpha = n_1^{m_1} \cdots n_k^{m_k}$. We may assume $\beta > \epsilon$. Then we can partition F into its connected components:

$$F = \hat{C}_1 \sqcup \cdots \sqcup \hat{C}_{|\beta|}$$

such that $\hat{C}_j$ has height $\beta(j)-1$. Take $\alpha \leqslant \beta$. We construct an α-partition $(C_j \mid j \leqslant |\alpha|)$ in blocks. First, since $\alpha \leqslant \beta$, we have that $\beta(m_1) \geqslant n_1$. This means that each of $\hat{C}_1, \ldots, \hat{C}_{m_1}$ has height at least n_1. Let $C_1, \ldots, C_{m_1}$ be these components $\hat{C}_1, \ldots, \hat{C}_{m_1}$. Next, we have that $\beta(m_1 + m_2) \geqslant n_2$, meaning that each of $\hat{C}_{m_1+1}, \ldots, \hat{C}_{m_1+m_2}$ has height at least n_2. Let $C_{m_1+1}, \ldots, C_{m_1+m_2}$ be these components. Continue constructing $(C_j \mid j \leqslant |\alpha|)$ in this fashion. Note that we don't run out, since $|\alpha| \leqslant |\beta|$. Finally, take the remaining $|\beta|-|\alpha|$ components and add them to C_1.

Conversely, assume that $(C_j \mid j \leqslant |\alpha|)$ is an α-partition of F. First note that since this is an open partition, we must have that $|\alpha| \leqslant |\mathfrak{C}(F)| = |\beta|$. Now consider C_1. Let:

$$\Gamma := \{l \leqslant |\beta| : \hat{C}_l \subseteq C_1\}$$

Since C_1 is open and closed, for each $\hat{C}_l$, either $\hat{C}_l \subseteq C_1$ or $\hat{C}_l \cap C_1 = \varnothing$. Hence:

$$C_1 = \bigcup_{l \in \Gamma} \hat{C}_l$$

Because each $\hat{C}_l$ is upwards- and downwards-closed, this means that:

$$\mathsf{height}(C_1) = \max\left\{\mathsf{height}(\hat{C}_l) \;\middle|\; l \in \Gamma\right\}$$

Therefore, as $\beta(1)$ is maximal in $\{\beta(j) \mid j \leqslant |\beta|\}$, we get that $\alpha(1) \leqslant \beta(1)$.

Applying this argument inductively on $F \setminus C_1$, we get that $\alpha \leqslant \beta = \mathrm{c}(F)$. $\square$

COROLLARY 3.26. When F is connected, F has an α-partition if and only if $\alpha = k$, where $k \leqslant \mathsf{height}(F)+1$.

DEFINITION 3.27 (α-connectedness)**.** Let F be a poset and α be a signature. F is *α-connected* if there is no $x \in F$ such that there is an α-partition of $\Uparrow(x)$.

REMARK 3.28. By Lemma 3.25, this is equivalent to requiring that $\alpha \nleqslant \mathrm{c}(\Uparrow(x))$ for each $x \in F$.

We can now express the meaning of $\chi(\langle\alpha\rangle)$ on frames.

PROPOSITION 3.29. For F a finite poset and α any signature, $F \vDash \chi(\langle\alpha\rangle)$ if and only if F is α-connected.

Proof. First label the elements of $\langle\alpha\rangle$ as in the proof of Proposition 3.18. Assume that $F \nvDash \chi(\langle\alpha\rangle)$. Then by Corollary 3.6 there is a pointed up-reduction $f : F \to \langle\alpha\rangle$ with apex x. This means that $f^{-1}[\langle\alpha\rangle \setminus \{\bot\}] = \Uparrow(x)$. For each $j \leqslant |\alpha|$, let:

$$C_j := f^{-1}\{a(j,1), \ldots, a(j,\alpha(j))\}$$

Since $\{a(j,1), \ldots, a(j,\alpha(j))\}$ is upwards-closed, so is C_j. Note that the C_j's are disjoint. Hence $(C_j \mid j \leqslant k)$ is an open partition of $\Uparrow(x)$. Now, pick $x_1 \in f^{-1}\{a(j,1)\}$. Since f is a p-morphism, there is $x_2 \in f^{-1}\{a(j,2)\}$ with $x_1 < x_2$. Continuing in this fashion, we find a chain of length $\alpha(j)$ in C_j, whence $\mathsf{height}(C_j) \geqslant \alpha(j)-1$. But then $(C_j \mid j \leqslant k)$ is an α-partition of $\Uparrow(x)$, meaning that F is not α-connected.

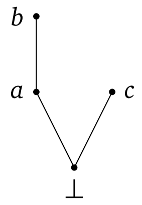


Figure 3.2: Scott's tree

Conversely, assume that F is not α-connected, so that there is $x \in F$ and an α-partition $(C_j \mid j \leqslant k)$ of $\Uparrow(x)$. For each C_j, we have, by definition, that $\mathsf{height}(C_j) \geqslant \alpha(j) - 1$. Hence by Proposition 1.19 there is a p-morphism $f_j \colon C_j \to \langle \alpha(j) - 1 \rangle$. Define $f : \uparrow(x) \to \langle \alpha \rangle$ as follows.

$$y \mapsto \begin{cases} \bot & \text{if } y = x, \\ f_j(y) & \text{if } y \in C_j \end{cases}$$

Then f is a p-morphism, so an up-reduction $F \circ\!\!\rightarrow \langle \alpha \rangle$. ☐

Remark 3.30. Note in particular it follows that $\mathbf{BD}_n = \mathbf{IPC} + \chi(\langle n+1 \rangle)$. This is just Proposition 1.19 of course.

Starlike logics. We are now in a position to define the principle class of logics that will be investigated in this chapter.

Definition 3.31. Let $\mathscr{S} := \{\alpha \text{ signature} \mid \alpha \neq 1^2\}$.

Definition 3.32 (Starlike logics)**.** Take $\Lambda \subseteq \mathscr{S}$ (possibly infinite). The *starlike logic* $\mathbf{SFL}(\Lambda)$ based on Λ is the logic axiomatised by $\mathbf{IPC}$ plus $\chi(\langle \alpha \rangle)$ for each $\alpha \in \Lambda$. Write $\mathbf{SFL}(\alpha_1, \ldots, \alpha_k)$ for $\mathbf{SFL}(\{\alpha_1, \ldots, \alpha_k\})$.

As made precise by Proposition 3.29, on frames starlike logics express connectedness properties. Let us meet a well-known infinite-height starlike logic.

Proposition 3.33. $\mathbf{SL} = \mathbf{SFL}(2 \cdot 1)$. So Scott's logic is a starlike logic.

Proof. See [CZ97, §9 and Table 9.7, p. 317]. ☐

This is our first sighting of the starlike tree $\langle 2 \cdot 1 \rangle$, which will reappear at various points throughout the rest of the thesis. Let us give it a name: *Scott's tree*. Observe its image and labelling in Figure 3.2.

The main theorem and its fruits. Here is the main theorem of the chapter.

Theorem 3.34. A starlike logic is polyhedrally-complete if and only if it has the finite model property.

Most of the rest of the chapter will be devoted to a proof of this theorem. But first let us see some of its nice consequences for the programme of classifying the polyhedrally-complete logics.

Corollary 3.35. For each $n \in \mathbb{N}$, there are infinitely-many polyhedrally-complete logics of height n.

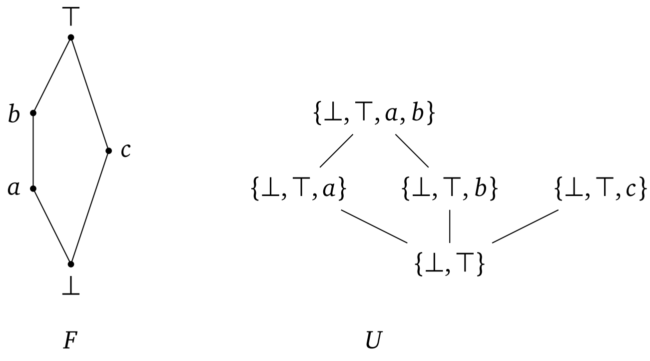


Figure 3.3: $\mathrm{Frames}_{\mathrm{fin}}(\mathbf{SL})$ is not closed under $\mathscr{N}$

Proof. Take $\Lambda \subseteq \mathscr{S} \setminus \{n+1\}$. Then by Remark 3.30:

$$\mathbf{SFL}(\{n+1\} \cup \Lambda) = \mathbf{BD}_n + \mathbf{SFL}(\Lambda)$$

By Segerberg's Theorem 1.20, $\mathbf{SFL}(\{n+1\} \cup \Lambda)$ has the f.m.p. Then by the Main Theorem 3.34, $\mathbf{SFL}(\{n+1\} \cup \Lambda)$ is polyhedrally-complete.

To see that this covers infinitely-many logics of height n, consider $\mathbf{SFL}(n+1, 1^k)$ for each $k > 0$. Note that the starlike tree $\langle n \cdot 1^i \rangle$ is a frame of $\mathbf{SFL}(n+1, 1^k)$ for $i < k-1$, but not for any $i \geqslant k-1$. Therefore, $(\mathrm{Frames}(\mathbf{SFL}(n+1, 1^k)) \mid k \in \mathbb{N}^{>0})$ are all distinct. □

Corollary 3.36. Scott's Logic **SL** is polyhedrally-complete.

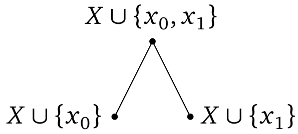
Proof. This follows from Proposition 3.33, the Main Theorem 3.34 and the fact that Scott's logic has the f.m.p. (see [CZ97, Example 11.50, p. 405]). □

Remark 3.37. It follows from Corollary 3.36 that we cannot simplify the Nerve Criterion to "a logic is polyhedrally-complete if and only if its class of finite frames is closed under the nerve $\mathscr{N}$". Indeed, consider the frame F given in Figure 3.3. By Proposition 3.33 and Proposition 3.29, F is a finite frame of **SL**. But the frame U in Figure 3.3 occurs as a generated subframe of $\mathscr{N}(F)$, which maps p-morphically onto Scott's tree $\langle 2 \cdot 1 \rangle$. Hence $\mathscr{N}(F) \notin \mathrm{Frames}_{\mathrm{fin}}(\mathbf{SL})$.

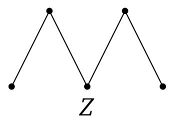
The difork case. The reader will have noticed that the difork $\langle 1^2 \rangle$ is omitted from the definition of a starlike logic, and consequently from the Main Theorem 3.34. In fact, polyhedral semantics is quite fond of this tree: when we take it as a forbidden configuration, the resulting landscape of polyhedrally-complete logics is as sparse as possible.

Proposition 3.38. Let $\mathscr{L}$ be a polyhedrally-complete logic containing $\mathbf{SFL}(1^2)$. Then $\mathscr{L} = \mathbf{CPC}$, the maximum logic.

Proof. Suppose for a contradiction that $\mathscr{L}$ is a polyhedrally-complete logic containing $\mathbf{SFL}(1^2)$ other than **CPC**. By the Nerve Criterion 2.23, $\mathscr{L} = \mathrm{Logic}(\mathbf{C})$ where $\mathbf{C}$ is a class of finite posets closed under $\mathscr{N}$. Since $\mathscr{L} \neq \mathbf{CPC}$, by Proposition 1.2 there must be $F \in \mathbf{C}$ with $\mathrm{height}(F) \geqslant 1$. This means that F has a chain $x_0 < x_1$. As in the proof of Theorem 2.28, we may assume that x_1 is a top element of F and that x_0 is an immediate predecessor of x_1. Take X a maximal chain in $\Downarrow(x_0)$. Then, as in that proof, we obtain the following structure lying at the top of $\mathscr{N}(F)$.



Applying the nerve once more, we obtain the following structure at the top of $\mathscr{N}^2(F)$.



Since **C** is closed under $\mathscr{N}$, we get that $\mathscr{N}^2(F) \in \text{Frames}(\mathscr{L})$. But $\Uparrow(Z)$ maps p-morphically onto $\langle 1^2 \rangle$, contradicting that $\mathscr{L} \vdash \chi(\langle 1^2 \rangle)$. ↯ □

3.3 The Proof of the Starlike Completeness Theorem

The proof uses Lemma 3.3. Given a finite rooted frame of a starlike logic **SFL**(Λ), we will find a frame F' which nerve-validates **SFL**(Λ) and which maps p-morphically onto F. To do so we proceed in three steps.

(1) We examine what it means for a frame to nerve-validate $\chi(\langle \alpha \rangle)$.

(2) We see that we may assume that F is *graded*: this is a useful structural property of posets.

(3) Using this additional structure, the final frame F' and the p-morphism $F' \to F$ are constructed.

Nerve-validation. Let us first see what it means for F to nerve-validate $\chi(\langle \alpha \rangle)$.

Definition 3.39. Let F be a poset and $x < y$ in F. The *diamond* and *strict diamond* of x and y are defined, respectively:

$$\begin{aligned} \updownarrow(x, y) &:= \uparrow(x) \cap \downarrow(y) \\ \Updownarrow(x, y) &:= \updownarrow(x, y) \setminus \{x, y\} \end{aligned}$$

Definition 3.40. A poset F is *α-diamond-connected* if there are no $x < y$ in F such that there is an α-partition of $\Updownarrow(x, y)$. The poset F is *α-nerve-connected* if it is α-connected and α-diamond-connected.

With a slight conceptual change, α-connectedness and α-diamond-connectedness can be harmonised.

Definition 3.41. For any poset F, we take a new element ∞, and let $\check{F} := F \cup \{\infty\}$, where ∞ lies above every element of F.

Then F is α-nerve-connected if and only if there are no $x < y$ in $\check{F}$ for which there is an α-partition of $\Updownarrow(x, y)$.

Theorem 3.42. *Let F be a finite poset and take $\alpha \in \mathscr{S}$. Then $F \vDash_{\mathscr{N}} \chi(\langle \alpha \rangle)$ if and only if F is α-nerve-connected.*

Proof. Assume that F is not α-nerve-connected with the aim of showing $F \not\vDash_{\mathscr{N}} \chi(\langle\alpha\rangle)$. Choose $x < y$ in $\check{F}$ such that $\Updownarrow(x, y)$ has an α-partition. That is, there is an open partition $(C_j \mid j \leqslant |\alpha|)$ of $\Updownarrow(x, y)$ such that $\mathsf{height}(C_j) = \alpha(j)$. Choose a chain $X \subseteq F$ which is maximal with respect to (i) $x, y \in X$ (ignoring the case $y = \infty$), and (ii) $X \cap \Updownarrow(x, y) = \varnothing$. I will show that $\Uparrow(X)^{\mathscr{N}(F)}$ has an α-partition. Note that by maximality of X, elements $Y \in \Uparrow(X)^{\mathscr{N}(F)}$ are determined by their intersection $Y \cap \Updownarrow(x, y)$. For $j \leqslant |\alpha|$, let:

$$\widehat{C}_j := \{Y \in \Uparrow(X)^{\mathscr{N}(F)} \mid Y \cap C_j \neq \varnothing\}$$

Take $j, l \leqslant |\alpha|$ distinct. Since (by Proposition 1.4) both C_j and C_l are upwards- and downwards-closed in $\Updownarrow(x, y)$, there is no chain $Y \in \Uparrow(X)^{\mathscr{N}(F)}$ such that $Y \cap C_j \neq \varnothing$ and $Y \cap C_l \neq \varnothing$. This means that:

(1) $\widehat{C}_j$ and $\widehat{C}_l$ are disjoint.

(2) For any $Y \in \Uparrow(X)^{\mathscr{N}(F)}$ we have $Y \in \widehat{C}_j$ if and only if $Y \cap \Updownarrow(x, y) \subseteq C_j$. Hence each $\widehat{C}_j$ is upwards- and downwards-closed in $\Uparrow(X)^{\mathscr{N}(F)}$.

Furthermore, since $(C_j \mid j \leqslant |\alpha|)$ covers $\Updownarrow(x, y)$, we get that $(\widehat{C}_j \mid j \leqslant |\alpha|)$ covers $\Uparrow(X)^{\mathscr{N}(F)}$. Finally, any maximal chain in $\widehat{C}_j$ is a sequence of chains $Y_0 \subset \cdots \subset Y_l$ such that $|Y_{i+1} \setminus Y_i| = 1$; this then corresponds to a maximal chain in C_j. Therefore:

$$\mathsf{height}(\widehat{C}_j) = \mathsf{height}(C_j)$$

Ergo $(\widehat{C}_j \mid j \leqslant |\alpha|)$ is an α-partition of $\Uparrow(X)^{\mathscr{N}(F)}$, meaning that $\mathscr{N}(F)$ is not α-connected. Then, by Proposition 3.29, $\mathscr{N}(F) \not\vDash \chi(\langle\alpha\rangle)$, hence by definition $F \not\vDash_{\mathscr{N}} \chi(\langle\alpha\rangle)$.

For the converse direction, I will show that if F is α-nerve-connected, then so is $\mathscr{N}(F)$, which will give the result by induction (note that α-nerve-connectedness is stronger than α-connectedness, and hence by Proposition 3.29 if $\mathscr{N}^k(F)$ is α-nerve-connected then $\mathscr{N}^k(F) \vDash \chi(\langle\alpha\rangle)$). So assume that F is α-nerve-connected. I will first prove α-connectedness. Take $X \in \mathscr{N}(F)$ with the aim of showing that $\Uparrow(X)^{\mathscr{N}(F)}$ has no α-partition.

Firstly, assume that X has more than one 'gap'; that is, there are distinct $w_1, w_2 \in F \setminus X$ such that $X \cup \{w_1\}$ and $X \cup \{w_2\}$ are still chains, but such that there exists $z \in X$ with $w_1 < z < w_2$. For $i \in \{1, 2\}$, let $u_i \in X \cap \Downarrow(w_i)$ be greatest and $v_i \in X \cap \Uparrow(w_i)$ be least. See Figure 3.4 for a representation of the situation. Take $Y, Z \in \Uparrow(X)^{\mathscr{N}(F)}$; I will give a path $Y \rightsquigarrow Z$. First, for $i \in \{1, 2\}$, by adding in w_i if necessary, we may assume that $Y \cap \Updownarrow(u_i, v_i) \neq \varnothing$, and similarly for Z. We then have the following path in $\Uparrow(X)^{\mathscr{N}(F)}$ (note that we need to be careful that none of the elements on the path equal X):

$$\begin{tikzcd}[column sep=tiny]
Y \arrow[dr, dash] & & (Y \setminus \Updownarrow(u_1, v_1)) \cup \{w_1\} \arrow[dl, dash] \arrow[dr, dash] & & (Z \setminus \Updownarrow(u_1, v_1)) \cup \{w_1\} \arrow[dl, dash] \arrow[dr, dash] & & Z \arrow[dl, dash] \\
& Y \setminus \Updownarrow(u_1, v_1) & & X \cup \{w_1\} & & Z \setminus \Updownarrow(u_1, v_1) &
\end{tikzcd}$$

Hence, $\Uparrow(X)^{\mathscr{N}(F)}$ is path-connected, so by Proposition 1.3, it is connected. Therefore, by Corollary 3.26, it suffices to show that $\mathsf{height}(\Uparrow(X)^{\mathscr{N}(F)}) < \mathsf{height}(F)$. But this follows from Proposition 2.4.

Hence we may assume that X has exactly one gap (when X has no gaps, $\Uparrow(X)^{\mathscr{N}(F)} = \varnothing$). This means that there are $x, y \in X$ with $x < y$ such that $X \cap \Updownarrow(x, y) = \varnothing$ and X is maximal outside of $\Updownarrow(x, y)$. As before then, elements $Y \in \Uparrow(X)^{\mathscr{N}(F)}$ are determined by their intersection $Y \cap \Updownarrow(x, y)$. Suppose that $\Uparrow(X)^{\mathscr{N}(F)}$ has an α-partition $(\widehat{C}_j \mid j \leqslant |\alpha|)$. For each $j \leqslant |\alpha|$, let:

$$C_j := \bigcup \widehat{C}_j \cap \Updownarrow(x, y)$$

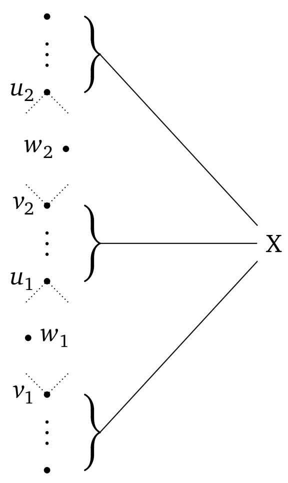


Figure 3.4: The set-up when X has more than one gap

Note that $\bigcup_{j\leqslant|\alpha|} C_j = \Updownarrow(x,y)$. For each $j \leqslant |\alpha|$, since $\widehat{C}_j$ is downwards-closed (by Proposition 1.4), we have that, for $z \in \Updownarrow(x,y)$:

$$z \in C_j \quad\Longleftrightarrow\quad \exists Y \in \widehat{C}_j : z \in Y \quad\Longleftrightarrow\quad X \cup \{z\} \in \widehat{C}_j$$

This means in particular that the C_j's are pairwise disjoint. Further, if $z \in C_j$ and $w \in \Updownarrow(x,y)$ with $w < z$, then $X \cup \{w,z\}$ is a chain, and so as $\widehat{C}_j$ is upwards-closed, we have $X \cup \{w,z\} \in \widehat{C}_j$, meaning that $w \in C_j$; similarly when $w > z$. Whence each C_j is upwards- and downwards-closed. Finally, as above, maximal chains in $\widehat{C}_j$ correspond to maximal chains in C_j of the same length, whence:

$$\mathsf{height}(\widehat{C}_j) = \mathsf{height}(C_j)$$

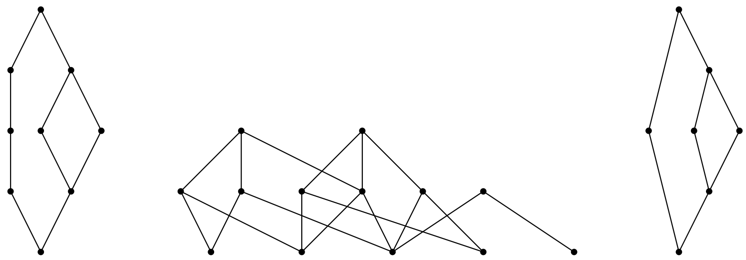
But then $(C_j \mid j \leqslant |\alpha|)$ is an α-partition of $\Updownarrow(x,y)$, contradicting the fact that F is α-nerve-connected. $\lightning$

This shows that $\mathscr{N}(F)$ is α-connected. What about α-diamond-connectedness? In fact we can show this without using any assumptions on F. Take $X, Y \in \mathscr{N}(F)$ with $X \subset Y$. I will show that $\Updownarrow(X,Y)^{\mathscr{N}(F)}$ has no α-partition. We may assume that $|Y \setminus X| \geqslant 2$, otherwise $\Updownarrow(X,Y)^{\mathscr{N}(F)} = \varnothing$. Note that this means in particular that $\alpha > 1$, since F is α-connected. If $|Y \setminus X| = 2$, then $\Updownarrow(X,Y)^{\mathscr{N}(F)}$ is the antichain on two elements, which, since $\alpha \neq 1^2$ by assumption, has no α-partition. So assume that $|Y \setminus X| \geqslant 3$; I will show that in fact $\Updownarrow(X,Y)^{\mathscr{N}(F)}$ is connected. Take distinct $Z, W \in \Updownarrow(X,Y)^{\mathscr{N}(F)}$. Choose $z \in Z \setminus X$ and $w \in W \setminus X$. Since $|Y \setminus X| \geqslant 3$, we have that $X \cup \{z,w\} \in \Updownarrow(X,Y)^{\mathscr{N}(F)}$. Hence the following is a path in $\Updownarrow(X,Y)^{\mathscr{N}(F)}$:

$$\begin{tikzcd}[column sep=small]
Z \arrow[dr, no head] & & X \cup \{z,w\} \arrow[dl, no head] \arrow[dr, no head] & & W \arrow[dl, no head] \\
& X \cup \{z\} & & X \cup \{w\} &
\end{tikzcd}$$

Therefore, $\Updownarrow(X,Y)^{\mathscr{N}(F)}$ is connected. Finally, note that:

$$\mathsf{height}(\Updownarrow(X,Y)^{\mathscr{N}(F)}) \leqslant \mathsf{height}(\mathscr{N}(F)) = \mathsf{height}(F) \qquad \square$$



(a) (b) (c)

Figure 3.5: Graded posets: (a) and (b) are examples of graded posets, while (c) is a non-example.

REMARK 3.43. Note that the proof shows an interesting property of the formulas $\chi(\langle\alpha\rangle)$: we have $F \vDash_{\mathscr{N}} \chi(\langle\alpha\rangle)$ if and only if $\mathscr{N}(F) \vDash \chi(\langle\alpha\rangle)$. This is not true in general. For example, formulas expressing bounded width can take many iterations of the nerve construction to become falsified.

Graded posets. The next step is to show that we can put $F \in \mathrm{Frames}_{\perp,\mathrm{fin}}(\mathbf{SFL}(\Lambda))$ into a special form. The following definition comes from combinatorics (see e.g. [Sta97, p. 99]).

DEFINITION 3.44 (Graded poset). A *rank function* on a poset F is a map $\rho : F \to \mathbb{N}$ such that:

(i) whenever x is minimal in F, we have $\rho(x) = 0$,

(ii) whenever y is the immediate successor of x, we have $\rho(y) = \rho(x) + 1$.

If F is non-empty and has a rank function, then it is *graded*.

The notion of gradedness has a strong visual connection. When a poset is graded, we can draw it out in well-defined layers such that any element's immediate successors lie entirely in the next layer up. See Figure 3.5 for some examples and non-examples of graded posets.

PROPOSITION 3.45. Let F be a finite poset.

(1) F is graded if and only if for every $x \in F$, all maximal chains in $\downarrow(x)$ have the same length.

(2) When F is graded, $\rho(x) = \mathsf{height}(x)$ for every $x \in F$, and $\mathsf{height}(F) = \max \rho[F]$.

(3) Rank functions, when they exist, are unique.

Proof. (1) See [Sta97, p. 99]. Assume that F is graded, and take X a maximal chain in $\downarrow(x)$ for some $x \in F$. Let $k = \rho(x)$. I will show that $|X| = k + 1$. Since X is a chain, the ranks of each of its elements are distinct. Since X is maximal, $x \in X$. Suppose for a contradiction that there is $j < k$ such that there is no $x \in X$ of rank j. We may assume that j is minimal with this property. We can't have $j = 0$, since otherwise X wouldn't contain any minimal element, so wouldn't be a maximal chain. Hence, there is $y \in X$ with $\rho(y) = j - 1$. Let z be next in X after y. Then y

has an immediate successor w such that $w \leqslant z$. By definition, $\rho(w) = j$, so $w \notin X$. But $X \cup \{w\}$ is a chain, contradicting the maximality of X. $\lightning$ Therefore, $|X| = k + 1$.

Conversely, define $\rho : F \to \mathbb{N}$ by:

$$x \mapsto \mathsf{height}(x)$$

Let us check that ρ is a rank function. (i) Clearly, when x is minimal, $\rho(x) = 0$. (ii) Suppose for a contradiction that there are $x, y \in F$, with y an immediate successor of x, such that $\rho(y) \neq \rho(x) + 1$. First, by definition, $\rho(y) > \rho(x)$, so we must have $\rho(y) > \rho(x) + 1$. Choose maximal chains $X \subseteq {\downarrow}(x)$, $Y \subseteq {\downarrow}(y)$. Note that by assumption:

$$|Y| > |X| + 1$$

But now, since y is an immediate successor of x, both $X \cup \{y\}$ and Y are maximal chains in ${\downarrow}(y)$ of different heights. $\lightning$

(2) This follows from the proof of (1).

(3) This follows from (3). □

Corollary 3.46. (1) Every tree is graded.

(2) For any finite poset F, its nerve $\mathscr{N}(F)$ is graded, with rank function given by $\rho(X) = |X| - 1$.

Proof. For (2), note that for any $X \in \mathscr{N}(F)$ we have $\mathsf{height}(X) = |X| - 1$. □

Gradification in the presence of Scott's tree. The task now is, given a finite rooted frame F of $\mathbf{SFL}(\Lambda)$, to find a finite graded rooted frame F' of $\mathbf{SFL}(\Lambda)$ and a p-morphism $f : F' \to F$. We will do this using two different methods, depending on whether or not we have Scott's tree $\langle 2 \cdot 1 \rangle$ present. Let us first consider the case $2 \cdot 1 \in \Lambda$. The following lemmas show us that this case is not too complicated.

Lemma 3.47. Take $\Lambda \subseteq \mathscr{S}$ such that $2 \cdot 1 \in \Lambda$ but $n \notin \Lambda$ for any $n \in \mathbb{N}$.

(1) If there is no $k \in \mathbb{N}^{>0}$ such that $1^k \in \Lambda$, then $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(2 \cdot 1)$.

(2) Otherwise, let $k \in \mathbb{N}^{>0}$ be minimal such that $1^k \in \Lambda$. Then $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(2 \cdot 1, 1^k)$.

Proof. (1) Take $\alpha \in \Lambda$. Then by assumption $\alpha(1) \geqslant 2$, hence, as $\alpha \neq 2$, we have $2 \cdot 1 \leqslant \alpha$. Then by Proposition 3.18 there is a p-morphism $\langle \alpha \rangle \to \langle 2 \cdot 1 \rangle$. Hence by the semantic meaning of Jankov-Fine formulas, Theorem 1.16, we have that $\chi(\langle \alpha \rangle) \to \chi(\langle 2 \cdot 1 \rangle)$ is valid. This means that $\mathbf{SFL}(\Lambda) \subseteq \mathbf{SFL}(2 \cdot 1)$. The converse direction is immediate.

(2) Take $\alpha \in \Lambda$. If $\alpha(1) \geqslant 2$ then by the proof of (1) we have $\chi(\langle \alpha \rangle) \to \chi(\langle 2 \cdot 1 \rangle)$. So assume that $\alpha(1) < 2$. Since $\alpha \neq \epsilon$, we have $\alpha(1) = 1$, meaning that $\alpha = 1^l$ for some $l \in \mathbb{N}^{>0}$. By assumption $k \leqslant l$. But then $1^k \leqslant \alpha$, giving that $\chi(\langle \alpha \rangle) \to \chi(\langle 1^k \rangle)$ as before. □

Corollary 3.48. Take $\Lambda \subseteq \mathscr{S}$ such that $2 \cdot 1 \in \Lambda$ and there is $n \in \mathbb{N}$ with $n \in \Lambda$; assume that n is the minimal such natural number.

(1) If there is no $k \in \mathbb{N}^{>0}$ such that $1^k \in \Lambda$, then $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(n, 2 \cdot 1)$.

(2) Otherwise, let $k \in \mathbb{N}^{>0}$ be minimal with $1^k \in \Lambda$. Then $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(n, 2 \cdot 1, 1^k)$.

Proof. This follows from Lemma 3.47 and the fact that when $n_1 < n_2$ we have $\chi(\langle n_1 \rangle) \to \chi(\langle n_2 \rangle)$. $\square$

Using this, the 'meaning' of $\mathbf{SFL}(\Lambda)$ can be expressed relatively simply. Note that this meaning is expressed in terms of the depth of elements $x \in F$. Up until this point we have mainly been concerned with the height of elements.

Lemma 3.49. Take $\Lambda \subseteq \mathscr{S}$ such that $2 \cdot 1 \in \Lambda$, and let F be a finite poset. Let $n \in \mathbb{N}$ be minimal such that $n \in \Lambda$, or ∞ if no such signature is present. Similarly, let $k \in \mathbb{N}^{>0}$ be minimal with $1^k \in \Lambda$, or ∞. Then $F \vDash \mathbf{SFL}(\Lambda)$ if and only if the following three conditions are satisfied for every $x \in F$.

(i) We have $\mathsf{height}(F) \leqslant n - 1$.

(ii) Whenever $\mathsf{depth}(x) = 1$, we have $|{\Uparrow}(x)| < k$.

(iii) Whenever $\mathsf{depth}(x) > 1$, the set ${\Uparrow}(x)$ is connected.

Proof. By Corollary 3.48 and the fact that $F \vDash \chi(\langle n \rangle)$ if and only if $\mathsf{height}(F) \leqslant n-1$, it suffices to treat the case $n = \infty$. Now by Lemma 3.47, $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(2 \cdot 1, 1^k)$ when $k < \infty$, and $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(2 \cdot 1)$ otherwise.

Assume that $F \vDash \mathbf{SFL}(\Lambda)$. (ii) In the case $k < \infty$, take $x \in F$ with $\mathsf{depth}(x) = 1$. Note that ${\Uparrow}(x)$ is an antichain, so $(\{y\} \mid y \in {\Uparrow}(x))$ is an open partition of ${\Uparrow}(x)$. Since $x \vDash \chi(\langle 1^k \rangle)$, by Lemma 3.25 and Proposition 3.29 we must have $|{\Uparrow}(x)| < k$. (iii) Now take $x \in F$ with $\mathsf{depth}(x) > 1$, and suppose for a contradiction that ${\Uparrow}(x)$ is disconnected. Then we can partition ${\Uparrow}(x)$ into disjoint upwards-closed sets U, V. Since $\mathsf{depth}(x) > 1$, one of U and V (say U) must have height at least 1. But then (U, V) is a $(2 \cdot 1)$-partition of ${\Uparrow}(x)$, contradicting that $F \vDash \chi(\langle 2 \cdot 1 \rangle)$ by Proposition 3.29. $\lightning$

Conversely, assume that $F \nvDash \mathbf{SFL}(\Lambda)$. I will show that one of (ii) and (iii) is violated. If $F \nvDash \chi(\langle 2 \cdot 1 \rangle)$, then by Proposition 3.29 there is $x \in F$ and a $(2 \cdot 1)$-partition (U, V) of ${\Uparrow}(x)$. But then $\mathsf{height}(U) \geqslant 1$, meaning that $\mathsf{depth}(x) > 1$, and furthermore ${\Uparrow}(x)$ is disconnected, violating (iii). So let us assume that $k < \infty$, that $F \vDash \chi(\langle 2 \cdot 1 \rangle)$ but that $F \nvDash \chi(\langle 1^k \rangle)$. Again, we get $x \in F$ and a 1^k-partition $(C_1, \ldots, C_k)$ of ${\Uparrow}(x)$. We must have that $\mathsf{height}(C_1) = 0$, otherwise $(C_1, C_2 \cup \cdots \cup C_k)$ is a $(2 \cdot 1)$-partition of ${\Uparrow}(x)$. Similarly $\mathsf{height}(C_i) = 0$ for every $i \leqslant k$. This means that $\mathsf{depth}(x) = 1$, and that $|{\Uparrow}(x)| \geqslant k$, violating (ii). $\square$

Theorem 3.50. Let $\Lambda \subseteq \mathscr{S}$ be such that $2 \cdot 1 \in \Lambda$. Let F be a finite rooted poset such that $F \vDash \mathbf{SFL}(\Lambda)$. Then there is a finite graded rooted poset F' and a p-morphism $f : F' \to F$ such that $F' \vDash \mathbf{SFL}(\Lambda)$.

This is the the 'gradification' theorem. Let me outline the construction before coming to the full proof.

- We first split F up into its tree unravelling $\mathscr{T}(F)$.
- We then lengthen branches so that the tree has a uniform height.
- Lastly, we join top nodes of this tree in order to recover any α-connectedness that we lost.

See Figure 3.6 for an example of this process.

Proof. Let $n := \mathsf{height}(F)$. We may assume $\epsilon \notin \Lambda$. If $2 \in \Lambda$, then by Remark 3.30, $n \leqslant 1$, so F is already graded. So assume that $2 \notin \Lambda$.

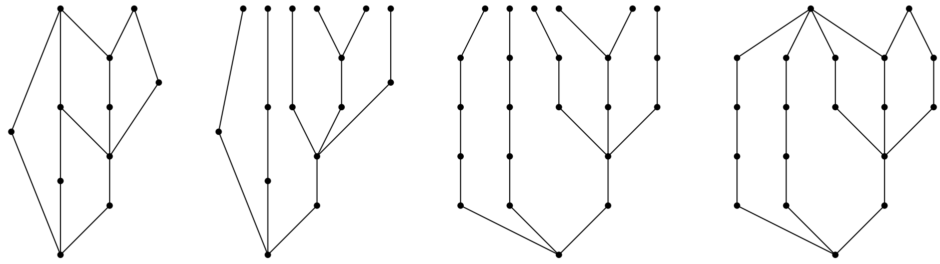


Figure 3.6: An example of gradification in the presence of Scott's tree

Start with the tree unravelling $T = \mathscr{T}(F)$ of F. Form a new tree T_0 by replacing each top node $t \in \mathsf{Top}(T)$ with a chain of new elements $t^*(0), \ldots, t^*(m_t)$, where $m_t = n - \mathsf{height}(t)$. The relations between these new elements and the rest of T is as follows:

$$t^*(0) < \cdots < t^*(m_t),$$
$$x < t^*(0) \quad \Longleftrightarrow \quad x < t \qquad \forall x \in T$$

Note that in T_0 all branches have the same length $n+1$. Define the p-morphism $g \colon T_0 \to T$ by:

$$x \mapsto \begin{cases} x & \text{if } x \in \mathsf{Trunk}(T), \\ \mathsf{last}(t) & \text{if } x = t^*(i) \text{ for some } t \in \mathsf{Top}(T) \text{ and } i \leqslant m_t \end{cases}$$

Form F' from T_0 by identifying, for top nodes $t, s \in \mathsf{Top}(T)$, the elements $t^*(m_t)$ and $s^*(m_s)$ whenever $\mathsf{last}(t) = \mathsf{last}(s)$. That is, let $F' := T_0/\mathscr{W}$, where:

$$\mathscr{W} := \{\{t^*(m_t) \mid \mathsf{last}(t) = u\} \mid u \in \mathsf{Top}(F)\}$$

Note that we have a p-morphism $f = \mathsf{last} \circ g \circ q_{\mathscr{W}} \colon F' \to F$. Furthermore, F is clearly finite and rooted. As to gradedness, take $x \in F'$ with the aim of showing that all maximal chains in $\downarrow(x)$ are of the same length, utilising Proposition 3.45. If $x \in \mathsf{Trunk}(F')$, then $\downarrow(x)^{F'}$ is a linear order. So assume that $x \in \mathsf{Top}(F')$. Then any maximal chain X in $\downarrow(x)$ corresponds to a branch of T_0, and therefore has length $n+1$.

Let us now use Lemma 3.49 to verify that our construction preserves α-connectedness for $\alpha \in \Lambda$ and complete the proof. Let $k \in \mathbb{N}^{>0}$ be minimal such that $1^k \in \Lambda$, or ∞ if no such signature is present. For $u \in \mathsf{Top}(F)$ let $\widehat{u}$ be the equivalence class of those elements $t^*(m_t)$ such that $\mathsf{last}(t) = u$. Note that by construction, for $x \in \mathsf{Trunk}(T)$ and $u \in \mathsf{Top}(F)$:

$$x < \widehat{u} \quad \Longleftrightarrow \quad \mathsf{last}(x) < u \tag{$\star$}$$

We need to check the three conditions of Lemma 3.49.

(i) Note that $\mathsf{height}(F') = \mathsf{height}(F)$.

(ii) For any $x \in F'$ with $\mathsf{depth}(x) = 1$, either $x \in \mathsf{Trunk}(T)$ or $x = t^*(n_t - 1)$ for some top node $t \in T$. In the former case, the fact that $|{\Uparrow}(x)| \leqslant k$ follows from ($\star$) and the fact that $|{\Uparrow}(\mathsf{last}(x))^F| \leqslant k$. In the latter case we have ${\Uparrow}(x) = \left\{\widehat{\mathsf{last}(t)}\right\}$.

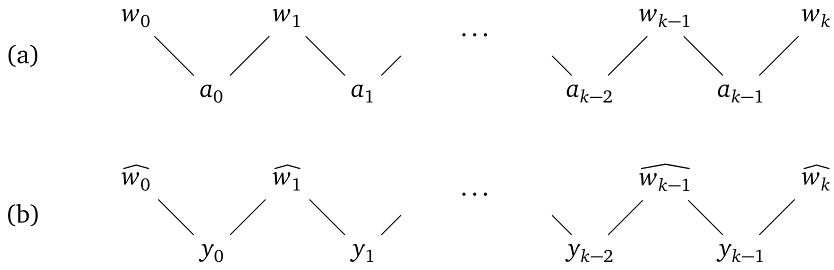


Figure 3.7: The form of the paths in $\Uparrow(\mathsf{last}(x))^F$ and $\Uparrow(x)^{F'}$

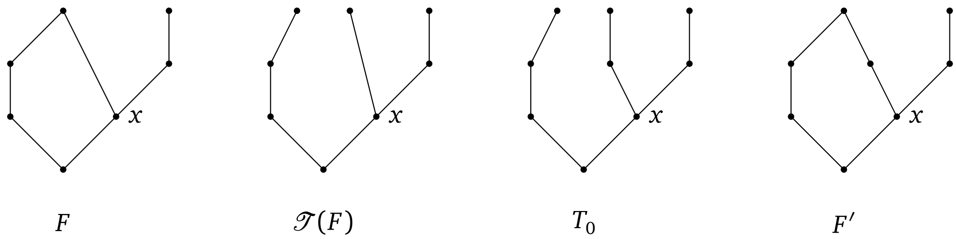


Figure 3.8: The technique in the proof of Theorem 3.50 doesn't work in general

(iii) Similarly, for any $x \in F'$ with $\mathsf{depth}(x) > 1$, either $x \in \mathsf{Trunk}(T)$ or $x = t^*(r)$ for some top node $t \in T$ and $r < n_t - 1$. In the latter case, $\Uparrow(x)$ is a chain, so connected. For the former case, it suffices to show that any two top elements $\widehat{u}, \widehat{v} \in \Uparrow(x)$ are connected by a path in $\Uparrow(x)$. Note that $\mathsf{depth}(\mathsf{last}(x))^F > 1$. Now, since $F \vDash \chi(\langle 2 \cdot 1 \rangle)$, by Lemma 3.49 there is a path $u \rightsquigarrow v$ in $\Uparrow(\mathsf{last}(x))^F$. We may assume that this path is of form given in Figure 3.7 (a), where $w_0, \ldots, w_k$ are top nodes in F. Using ($\star$), this path then translates into a path $\widehat{u} \rightsquigarrow \widehat{v}$ as in Figure 3.7 (b), where $y_i \in \mathsf{last}^{-1}\{a_i\} \cap \Uparrow(x)$ for each i. □

Remark 3.51. In Chapter 4, it will be useful to note that the proof of Theorem 3.50 produces frames of a particular form: when we remove $\mathsf{Top}(F')$ from F', what is left is a tree of uniform height $n - 1$, where $n = \mathsf{height}(F')$.

Gradification without Scott's tree. Now that the situation $2 \cdot 1 \in \Lambda$ has been dealt with, let us turn to the case $2 \cdot 1 \notin \Lambda$. Unfortunately, the proof of Theorem 3.50 crucially relied on the fact that the original frame F was $(2 \cdot 1)$-connected. Consider for instance the frame F given in Figure 3.8, which at x is not $(2 \cdot 1)$-connected. If we apply the construction to F, we end up with a frame F' in which x sits below two connected components of height 1, that is, $\mathrm{c}(\Uparrow(x)^{F'}) = 2^2$. Hence F' is not 2^2-connected, while F is. Taking $2 \cdot 1$ away from Λ is a double-edged sword however, since it allows for more complex constructions in F'.

The following reusable lemma will come in handy a couple of times.

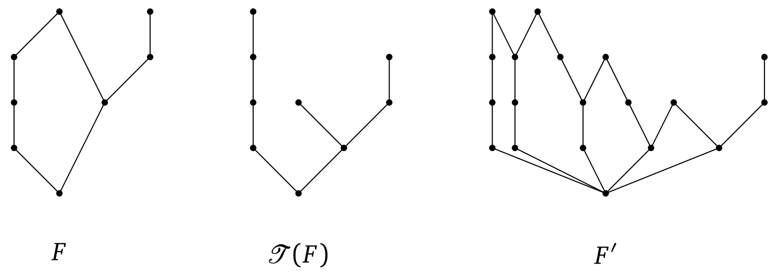


Figure 3.9: An example of gradification in the absence of Scott's tree.

LEMMA 3.52. Let $f : F' \to F$ be a surjective p-morphism between finite posets, and take $x \in F'$. Assume that for any $y, z \in \mathrm{Succ}(x)$ there is a path $y \rightsquigarrow z$ in $\Uparrow(x)$ whenever there is a path $f(y) \rightsquigarrow f(z)$ in $\Uparrow(f(x))$. Then:

$$\mathfrak{C}(\Uparrow(x)) = \{f^{-1}[C] \mid C \in \mathfrak{C}(\Uparrow(f(x)))\}$$

In particular, if $\mathsf{height}(f^{-1}[C]) = \mathsf{height}(C)$ for any $C \in \mathfrak{C}(\Uparrow(f(x)))$ then:

$$\mathrm{c}(\Uparrow(x)) = \mathrm{c}(\Uparrow(f(x))$$

Proof. Note that, by Proposition 1.4, since f is a p-morphism and F and F' are finite, $\{f^{-1}[C] \mid C \in \mathfrak{C}(\Uparrow(f(x)))\}$ is a partition of $\Uparrow(x)$ into upwards- and downwards-closed sets. So it suffices to show that $f^{-1}[C]$ is connected for every $C \in \mathfrak{C}(\Uparrow(f(x)))$. Take $y_0, z_0 \in f^{-1}[C]$. Since $f^{-1}[C]$ is downwards-closed in $\Uparrow(x)$, there are $y, z \in \mathrm{Succ}(x) \cap f^{-1}[C]$ such that $y \leqslant y_0$ and $z \leqslant z_0$. Then $f(y), f(z) \in C$, so by assumption there is a path $f(y) \rightsquigarrow f(z)$ in $\Uparrow(f(x))$. But then by assumption there is a path $y \rightsquigarrow z$ in $\Uparrow(x)$, which lies in $f^{-1}[C]$ since the latter is upwards- and downwards-closed. □

THEOREM 3.53. Let $\Lambda \subseteq \mathscr{S}$ be such that $2 \cdot 1 \notin \Lambda$. Let F be a finite rooted poset such that $F \vDash \mathbf{SFL}(\Lambda)$. Then there is a finite graded rooted poset F' and a p-morphism $f : F' \to F$ such that $F' \vDash \mathbf{SFL}(\Lambda)$.

The construction works in two steps as follows (see Figure 3.9 for an example).

- Again, we start by splitting F up into its tree unravelling $\mathscr{T}(F)$.
- Then, in order to connect the frame back up again while ensuring that it remains graded, we construct 'zigzag roller-coasters' connecting top nodes of different heights.

Proof of Theorem 3.53. As in the proof of Theorem 3.50, we may assume that $\epsilon, 1, 2 \notin \Lambda$.

Start with $T = \mathscr{T}(F)$. For every two distinct $p, q \in \mathsf{Top}(T)$ such that $\mathsf{last}(p) = \mathsf{last}(q) = t$, we will build a 'roller-coaster' structure $Z(p,q)$, which will furnish a bridge between p and q. Every such structure $Z(p,q)$ is independent, so that they can all be added to T at the same time. First note that by Corollary 3.46, T is graded; let $\rho : T \to \mathbb{N}$ be its rank function.

Now, take distinct $p, q \in \mathsf{Top}(T)$ such that $\mathsf{last}(p) = \mathsf{last}(q) = t$. Let $l := \rho(q) - \rho(p)$. By swapping p and q, we may assume that $l \geqslant 0$. I will join p and q with a zigzagging path, which consists of lower points $a_0, \ldots, a_l$, upper points $b_0, \ldots, b_{l-1}$ and intermediate points $c_0, \ldots, c_{l-1}$. The relations between these points are as follows (see Figure 3.10).

$$a_i < c_i < b_i, \qquad a_{i+1} < b_i$$

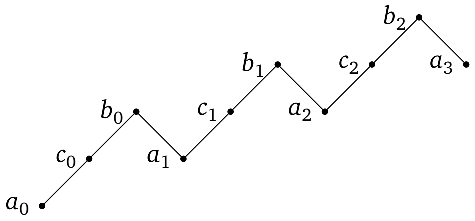


Figure 3.10: The relations between the zigzag points in case $l = 3$.

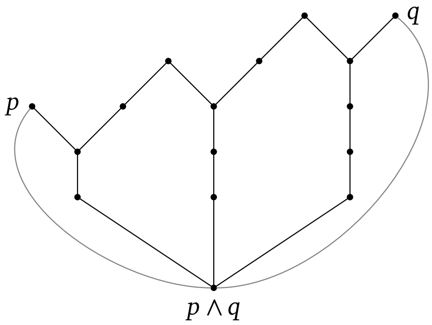


Figure 3.11: The zigzag path and the ladder structure in place.

Consider $p \wedge q$ (i.e. the intersection of p and q, regarded as strict chains containing the root), and let $k := \rho(p) - \rho(p \wedge q) - 1$. Note that $k \geqslant 0$ since p and q are incomparable. Moreover, $k \geqslant 1$. Indeed, suppose for a contradiction that $k = 0$, so that p is an immediate successor of $p \wedge q$. Then $\mathsf{last}(p)$ is an immediate successor of $\mathsf{last}(p \wedge q)$. But $\mathsf{last}(q) = \mathsf{last}(p)$, so we have, as strict chains:

$$p = (p \wedge q) \cup \{\mathsf{last}(p)\} = (p \wedge q) \cup \{\mathsf{last}(q)\} = q$$

contradicting that p and q are distinct. $\lightning$

To ensure that the new poset F' is still graded, we need to dangle some scaffolding down from the zigzag path to $p \wedge q$. Below each lower point a_i we will dangle a chain of $k + i - 1$ points $d(i, 1), \ldots, d(i, k + i - 1)$. The relations are as follows:

$$d(i, 1) < d(i, 2) < \cdots < d(i, k + i - 1) < a_i$$

Finally, let $\mathrm{Z}(p, q)$ denote the whole structure of the zigzag path plus the dangling scaffolding. Attach $\mathrm{Z}(p, q)$ to T by adding the following relations and closing under transitivity (see Figure 3.11).

$$a_0 < p, \qquad a_l < q, \qquad \forall i \colon p \wedge q < d(i, 1)$$

Let F' be the result of adding $\mathrm{Z}(p, q)$ to T for every pair p, q, and define the function $f : F' \to F$ by:

$$f(x) := \begin{cases} \mathsf{last}(x) & \text{if } x \in T \\ \mathsf{last}(p) & \text{if } x \in Z(p, q) \text{ for some } p, q \end{cases}$$

First, let us see that f is a p-morphism. The **forth** condition follows from the fact that last is monotonic, and that:

- if $x \leqslant y$ with $x \in T$ and $y \in Z(p,q)$, then by construction $x \leqslant p \wedge q$, meaning that $f(x) = \mathsf{last}(x) \leqslant \mathsf{last}(p \wedge q) \leqslant \mathsf{last}(p) = f(y)$, and
- if $x \leqslant y$ with $x \in Z(p,q)$ and $y \in T$, then by construction $y \in \{p,q\}$, so that $f(x) = \mathsf{last}(p) = f(y)$.

The **back** condition follows from the fact that last is open, and that each $Z(p,q)$ maps to a top node.

Second, for any pair p,q, we can extend the rank function ρ to the new structure Z(p,q) as follows (as indicated by the heights of the nodes in Figure 3.11):

$$\begin{aligned} \rho(a_i) &= \rho(p) + i - 1 \\ \rho(b_i) &= \rho(p) + i + 1 \\ \rho(c_i) &= \rho(p) + i \\ \rho(d(i,j)) &= \rho(p \wedge q) + j \end{aligned}$$

To see that, thus extended, ρ is still a rank function, it suffices to check that the newly-ranked $Z(p,q)$ fits into T as a ranked structure. That is, we need to check the following equations.

$$\begin{aligned} \rho(p) &= \rho(a_0) + 1 \\ \rho(q) &= \rho(a_l) + 1 \\ \rho(d(i,1)) &= \rho(p \wedge q) + 1 \end{aligned}$$

But these follow by definition. In this way we see that F' is graded.

Finally, it remains to be shown that $F \vDash \mathbf{SFL}(\Lambda)$. So take $x \in F$. First, whenever $x \in Z(p,q)$ for some p,q, by construction $\Uparrow(x)$ is α-connected for every signature other than ϵ, 1^2, $2 \cdot 1$ and k where $k \geqslant \mathsf{height}(F) + 1$. Hence we may assume that $x \in T$. Let us use Lemma 3.52. Take $y,z \in \mathrm{Succ}(x)$ such that there is a path $f(y) \rightsquigarrow f(z)$ in $\Uparrow(\mathsf{last}(x))$, with the aim of finding a path $y \rightsquigarrow z$ in $\Uparrow(x)$.

Assume that $y \in Z(p,q)$ for some p,q. Then since $y \in \mathrm{Succ}(x)$ and $x \in T$, by construction $x = p \wedge q$. All of $Z(p,q)$ is connected in $\Uparrow(x)$, hence there is a path $y \rightsquigarrow p$. Let $p' \in T$ be the immediate successor of x which lies below p (this exists since T is a tree). Then we have a path $y \rightsquigarrow p'$ in $\Uparrow(x)$. Therefore, we may assume that $y \in T$, and similarly that $z \in T$.

So, we have a path $\mathsf{last}(y) \rightsquigarrow \mathsf{last}(z)$. We now proceed in a similar fashion to the proof of Theorem 3.50. We may assume that the path $\mathsf{last}(y) \rightsquigarrow \mathsf{last}(z)$ has the form in Figure 3.12 (a), where $t_0, \ldots, t_k$ are top nodes in F. Let $u_0 := y$ and $u_k := z$. For each $i \in \{1, \ldots, k-1\}$, choose $u_i \in \mathsf{last}^{-1}\{a_i\}$. For $i \in \{0, \ldots, k-1\}$, take $p_i, q_i \in \mathsf{last}^{-1}\{t_i\}$ such that $u_i \leqslant p_i$ and $u_{i+1} \leqslant q_i$. For each such i, since $\mathsf{last}(p_i) = \mathsf{last}(q_i)$, there is a path $p_i \rightsquigarrow q_i$ which lies in $Z(p_i, q_i)$, and hence lies in $\Uparrow(x)$. Compose all these paths as in Figure 3.12 to form a path $y \rightsquigarrow z$ in $\Uparrow(x)$ as required.

It now remains to show that if $C \in \mathfrak{C}(\Uparrow(\mathsf{last}(x)))$, then $\mathsf{height}(f^{-1}[C]) = \mathsf{height}(C)$. First, since f is a p-morphism, $\mathsf{height}(f^{-1}[C]) \geqslant \mathsf{height}(C)$. Conversely, let $X \subseteq f^{-1}[C]$ be a maximal chain. Assume X intersects with some $Z(p,q)$. Then we can replace the part $X \cap (Z(p,q) \cup \{p,q\})$ with the unique maximal chain in $\Uparrow(p \wedge q)^T$ containing q (this exists since T is a tree). Then by construction this does not decrease the length of X nor does it move X outside of $f^{-1}[C]$ (since the latter is upwards- and downwards-closed). Therefore, we may assume that $X \subseteq T$, so X corresponds to a chain $\mathsf{last}[X]$ of the same length in C.

Therefore, by Lemma 3.52 we get that $\mathrm{c}(\Uparrow(x)) = \mathrm{c}(\Uparrow(\mathsf{last}(x))$. Applying Lemma 3.25, we have that $\Uparrow(x)$ has an α-partition if and only if $\Uparrow(\mathsf{last}(x))$ has an α-partition. $\square$

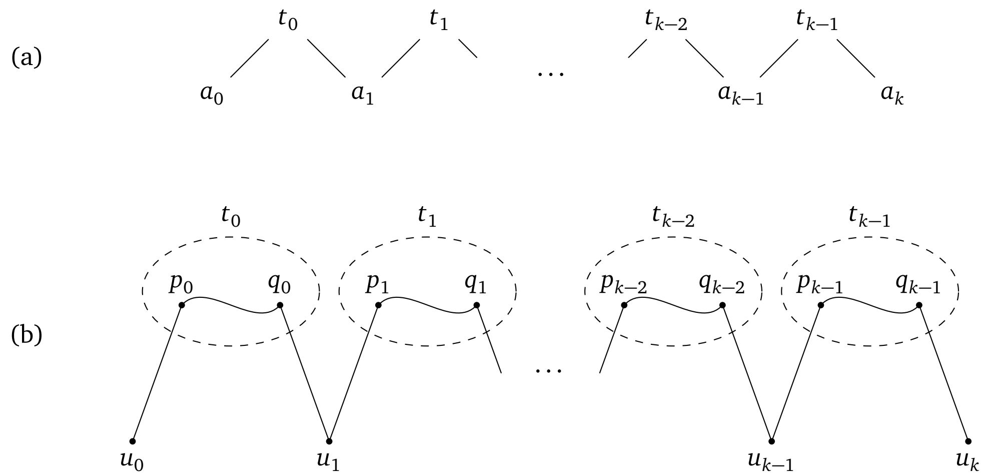


Figure 3.12: The form of the paths in $\Uparrow(\mathsf{last}(x))$ and $\Uparrow(x)$

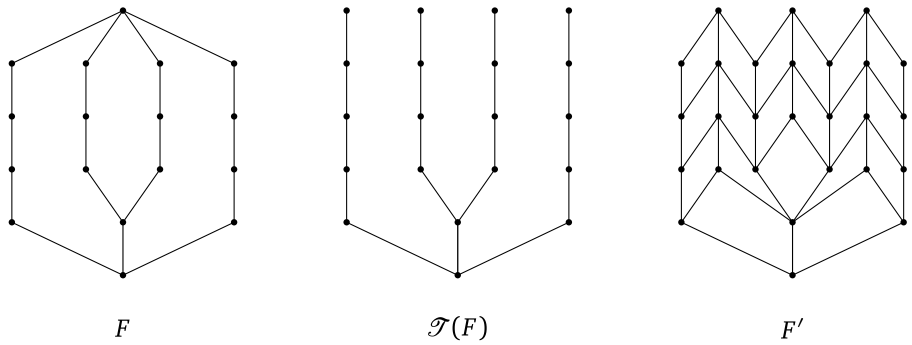


Figure 3.13: An example of nervification, using the graded structure of F

Nervification. We now find ourselves, having suitably prepared F, in a position to make use of its additional graded structure. The general method of the final construction, in which we transform F into a frame which nerve-validates $\mathbf{SFL}(\Lambda)$, is the same as in Theorem 3.50 and Theorem 3.53. We begin with the tree unravelling $\mathscr{T}(F)$, perform some alterations, then rejoin top nodes. A key difference here is that we won't rejoin every top node to every other top node whose 'last' value is the same. Instead, we line up all the top nodes mapping to the same element and link each top node to at most two other top nodes: its neighbours. See Figure 3.13 for an example of the construction.

Definition 3.54. Let T be a finite tree. Then for each $x \in T$, we have that $\downarrow(x)$ is a chain. For $k \leqslant \mathsf{height}(x)$, let $x^{(k)}$ be the element of this chain which has height k. Let $x^{(-k)}$ be the element which has height $\mathsf{height}(x) - k$.

Definition 3.55. For $n \in \mathbb{N}$, let $\mathscr{S}_n := \mathscr{S} \setminus \{1^k \mid k < n\}$.

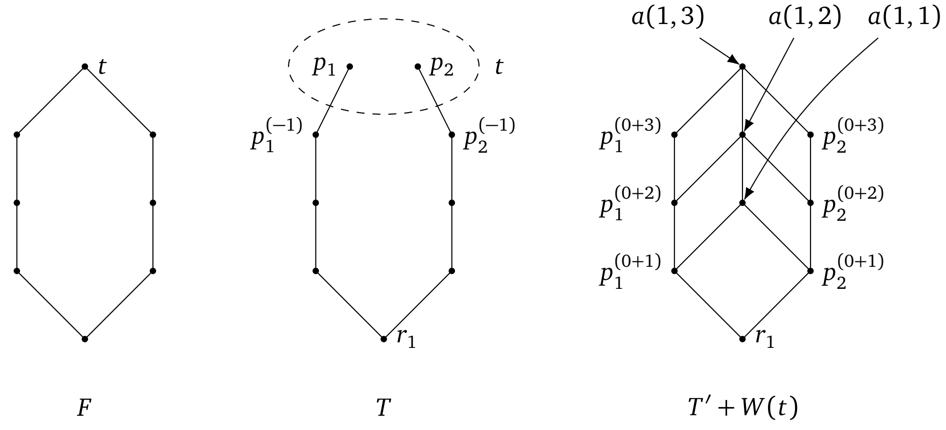


Figure 3.14: The chevron structure in a case with two branches.

Theorem 3.56. Take $\Lambda \subseteq \mathscr{S}$ and let F be a finite graded rooted poset of height n such that $F \vDash \mathbf{SFL}(\Lambda)$. Then there is a poset F' and a p-morphism $f : F' \to F$ such that $F \vDash \mathbf{SFL}(\Lambda)$ and such that F is α-diamond-connected for every $\alpha \in \mathscr{S}_n$.

Proof of Theorem 3.56. We may assume that $\epsilon, 1 \notin \Lambda$. Further, if $2 \in \Lambda$, then $\mathsf{height}(F) = 1$, so F is already α-diamond-connected for every $\alpha \in \mathscr{S}_n$. Hence we may assume that $2 \notin \Lambda$.

Once more, start with $T = \mathscr{T}(F)$. Chop off the top nodes: let $T' := \mathsf{Trunk}(T)$. For each $t \in \mathsf{Top}(F)$, we will add a new structure $W(t)$, which lies only above elements of T'. Let $\rho : F \to \mathbb{N}$ be the rank function on F. Note that $\rho \circ \mathsf{last} \colon T \to \mathbb{N}$ is the rank function on T.

Take $t \in \mathsf{Top}(F)$. Enumerate $\mathsf{last}^{-1}\{t\} = \{p_1, \ldots, p_m\}$. For each $i \leqslant m-1$, define:

$$r_i := p_i \wedge p_{i+1}$$
$$l_i := \rho(\mathsf{last}(r_i))$$
$$k_i := \rho(t) - \rho(\mathsf{last}(r_i)) - 1$$

Note that $k_i \geqslant 1$ just as in the proof of Theorem 3.53. Since F is graded and T is a tree, we have that:

$$|\Updownarrow(r_i, p_i)^T| = |\Updownarrow(r_i, p_{i+1})^T| = k_i$$

In other words, $p_i^{(l_i)} = p_{i+1}^{(l_i)} = r_i$. We will construct a 'chevron' structure which joins $p_i^{(-1)}$ to $p_{i+1}^{(-1)}$. For each $i \leqslant m-1$, take new elements $a(i,1), \ldots, a(i,k_i)$, and add them to T' using the following relations.

$$a(i,1) < \cdots < a(i,k_i), \qquad \forall j \leqslant k_i : p_i^{(l+j)}, p_{i+1}^{(l+j)} < a(i,j)$$

Let $W(t)$ be this new structure (i.e. the chain $\{a(i,1) < \cdots < a(i,k_i)\}$ in place). See Figure 3.14 and Figure 3.15 for examples of this process of adding chevrons.

The process of adding $W(t)$ is independent for each $t \in \mathsf{Top}(F)$. Let F' be the result of adding every $W(t)$ to T'. Define $f : F' \to F$ by:

$$f(x) := \begin{cases} \mathsf{last}(x) & \text{if } x \in T' \\ t & \text{if } x \in W(t) \text{ for some } t \in \mathsf{Top}(F) \end{cases}$$

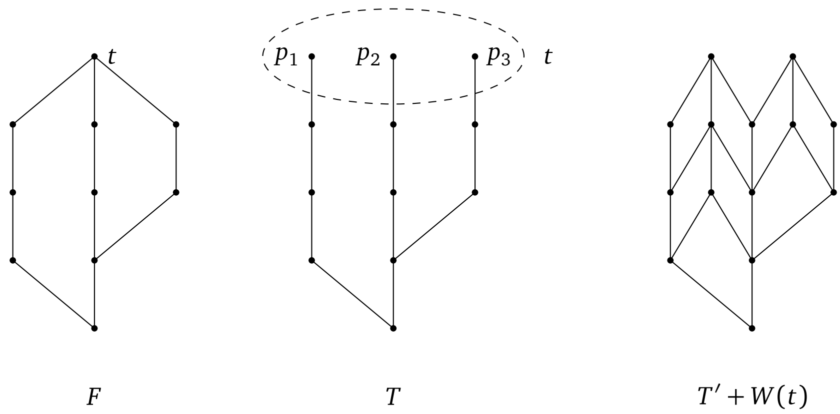


Figure 3.15: The chevron structure in a more complex case involving three branches.

Since we have made sure that each $W(t)$ contains, for each $p_i \in \mathsf{last}^{-1}\{t\}$, a node above $p_i^{(-1)}$ which maps to t, and that all of the new structure maps to a top node, f is a p-morphism.

Let us see that $F' \vDash \mathbf{SFL}(\Lambda)$. Take $x \in F'$. If $x \in W(t)$ for some t, then $\Uparrow(x)$ is either empty or a chain, hence $\Uparrow(x) \vDash \mathbf{SFL}(\Lambda)$. So we assume that $x \in T'$. The verification is now very similar to that in Theorem 3.53, making use of Lemma 3.52. Take $y, z \in \mathsf{Succ}(x)$ such that there is a path $f(y) \rightsquigarrow f(z)$ in $\Uparrow(\mathsf{last}(x))$. As in the proof of Theorem 3.53, by construction of $W(t)$ we may assume that $y, z \in T'$. Just as in that proof, we can construct a path $y \rightsquigarrow z$ from the path $f(y) \rightsquigarrow f(z)$, using the fact that whenever $t \in \Uparrow(\mathsf{last}(x)) \cap \mathsf{Top}(F)$, any $w, v \in f^{-1}\{t\}$ are connected by a path in $\Uparrow(x)^{F'}$ (this is how we constructed F'). It is straightforward then to check that if $C \in \mathfrak{C}(\Uparrow(\mathsf{last}(x)))$ we have $\mathsf{height}(f^{-1}[C]) = \mathsf{height}(C)$, giving that:

$$\mathsf{c}(\Uparrow(x)) = \mathsf{c}(\Uparrow(\mathsf{last}(x)))$$

To complete the proof, let us see that F' is α-diamond-connected for every $\alpha \in \mathscr{S}_n$. Take $x, y \in F'$ with $x < y$ and consider $\Updownarrow(x, y)$. There are several cases.

(a) Case $y \in T'$. We have that $\Updownarrow(x, y)^{F'} = \Updownarrow(x, y)^{T'}$, which is linearly-ordered since T' is a tree; hence it is connected and of height at most $n - 2$.

Hence $y = a(i, j)$ for $a(i, j) \in W(t)$ a new element. Let p_i, p_{i+1}, r_i, l_i be as above.

(b) Case $x \in W(t)$. Note that by construction $\Updownarrow(x, y)$ is linearly-ordered.

(c) Case $x = p_i^{(l+e)}$ for some e. If we have $\mathsf{height}(\Updownarrow(x, y)) = 1$, then $e = i - 1$ and $\Updownarrow(x, y)$ is the antichain on two elements, which is α-connected. Otherwise, by construction, $a(i, j - 1) \in \Updownarrow(x, y)$ which is connected to everything.

(d) Case $x = p_{i+1}^{(l+e)}$ for some e. This is symmetric.

(e) Case $x = r_i$. Again, if $\mathsf{height}(\Updownarrow(x, y))) = 1$ then $j = 1$ and $\Updownarrow(x, y)$ is the antichain on two elements, otherwise $a(i, 1) \in \Updownarrow(x, y)$ which is connected to everything.

(f) Otherwise, $x < r_i$ (since T' is a tree). Here $r_i \in \Updownarrow(x, y)$ which is connected to everything. □

Putting it all together. After a fair bit of labour, we now have all the ingredients we need for our proof. Let us put them together.

Proof of Theorem 3.34. By Lemma 3.3, we need to show that every finite rooted frame of **SFL**(Λ) is the up-reduction of one which nerve-validates **SFL**(Λ); in fact this up-reduction is just a p-morphism. So take such a frame F. We may assume that F is graded: when we have $2{\cdot}1 \in \Lambda$, apply Theorem 3.50, otherwise apply Theorem 3.53. Then by Theorem 3.56, there is a frame F' and a p-morphism $f : F' \to F$ such that F is α-nerve-connected for every $\alpha \in \Lambda$ (note that by Remark 3.30 we must have $\Lambda \subseteq \mathscr{S}_n$ where $n = \mathsf{height}(F)$). Then, by Theorem 3.42, F nerve-validates **SFL**(Λ), which completes the proof. $\square$

3.4 Epilogue

What does it all mean? Geometric interpretation. For a while now, we have been absorbed in some very combinatorial considerations, and the geometric part of the theory has been rather left on the sidelines. Theorem 3.34 furnishes us with a number of polyhedrally-complete logics, but the reader may very well wonder, for which classes of polyhedra are these logics complete? It is now time to answer this question.

Definition 3.57. Let α be a signature. A polyhedron P is *α-connected* if for every open subpolyhedron Q of P there is no partition $(B, C_1, \ldots, C_{|\alpha|})$ of Q such that $C_1, \ldots, C_{|\alpha|}$ are open subpolyhedra of P with $\mathrm{Dim}(C_j) \geqslant \mathrm{Dim}(B) + \alpha(j) - 1$, and $B \subseteq \mathrm{Cl}(C_1) \cap \cdots \cap \mathrm{Cl}(C_{|\alpha|})$.

For example, a polyhedron P is 1^3-connected if there is no partition (B, C_1, C_2, C_3) of an open subpolyhedron Q of P with C_1, C_2, C_3 open subpolyhedra and $B \subseteq \mathrm{Cl}(C_1) \cap \mathrm{Cl}(C_2) \cap \mathrm{Cl}(C_3)$.

Proposition 3.58. $P \vDash \chi(\langle \alpha \rangle)$ if and only if P is α-connected.

Proof. By Maxim I, $P \nvDash \chi(\langle \alpha \rangle)$ if and only if there is some triangulation Σ of P such that $\Sigma \nvDash \chi(\langle \alpha \rangle)$. Assume this is the case. Then by Proposition 3.29 there is $\sigma \in \Sigma$ and an α-partition $(C_j^* \mid j \leqslant |\alpha|)$ of $\Uparrow(\sigma)$. Let $Q := \mathrm{o}(\sigma)$, the open star of σ. By Proposition 1.42, Q is an open subpolyhedron. Now, by Proposition 1.43, the open sets C_j^* correspond to open subpolyhedra of P contained in Q via the isomorphism $\gamma^{\uparrow}$. Let $C_j := \gamma^{\uparrow}(C_j^*)$, for each $j \leqslant |\alpha|$. Finally let $B := \sigma$. Note that by definition $(B, C_1, \ldots, C_{|\alpha|})$ is a partition of Q, and that:

$$\begin{aligned} \mathrm{Dim}(C_j) &= \mathsf{height}(\downarrow C_j^*) && \text{(by definition)} \\ &= \mathsf{height}(C_j^*) + \mathsf{height}(\sigma) && (\Sigma \text{ graded, } C_j^* \subseteq \Uparrow(\sigma) \text{ downwards-closed}) \\ &= \mathsf{height}(C_j^*) + \mathrm{Dim}(\sigma) \\ &\geqslant \alpha(j) - 1 + \mathrm{Dim}(B) \end{aligned}$$

Furthermore, for any $\tau \in \Uparrow(\sigma)$ we have $\sigma \prec \tau$, so that $\sigma \subseteq \tau = \mathrm{Cl}\,\mathrm{Relint}\,\tau$. Whence $B \subseteq \mathrm{Cl}(C_1) \cap \cdots \cap \mathrm{Cl}(C_{|\alpha|})$.

Conversely, assume that $Q \in \mathrm{Sub_o}(P)$ and that $(B, C_1, \ldots, C_{|\alpha|})$ is a partition of Q as in the definition. Note that B, being the complement of a union of open subpolyhedra, is a polyhedron (by Proposition 1.32). Using the Triangulation Lemma 1.28, let Σ be a triangulation of P such that:

$$|\Sigma_{Q^{\mathsf{C}}}| = Q^{\mathsf{C}}, \quad |\Sigma_B| = B, \quad \forall j \leqslant |\alpha| : |\Sigma_{C_j^{\mathsf{C}}}| = C_j^{\mathsf{C}}$$

For each $j \leqslant |\alpha|$, let $C'_j := \Sigma \setminus \Sigma_{C_j^c}$. Note that this is upwards-closed, and that by Proposition 1.26, $C_j = \gamma^{\uparrow}(C'_j)$.

Now, choose $\sigma \in \Sigma_B$ of maximum dimension $\mathrm{Dim}(B)$. For each $j \leqslant |\alpha|$, let $C^*_j := C'_j \cap \uparrow(\sigma)$. We have that:

$$\sigma \subseteq |\Sigma_B| = B \subseteq \operatorname{Cl} C_j = \operatorname{Cl} \gamma^{\uparrow}(C'_j) = \bigcup_{\tau \in C'_j} \operatorname{Cl} \operatorname{Relint} \tau = |{\downarrow} C'_j|$$

as in the proof of Proposition 1.44. This means that $\sigma \in \downarrow C'_j$, so that C^*_j is not empty (since C'_j is upwards-closed). Furthermore, by maximality $B \cap \Uparrow(\sigma) = \varnothing$, whence $(C^*_j \mid j \leqslant |\alpha|)$ is an open partition of $\Uparrow(\sigma)$. In particular, by Proposition 1.4, each C^*_j is downwards-closed in $\Uparrow(\sigma)$. Then, using also that Σ is graded:

$$\begin{aligned}
\mathsf{height}(C^*_j) &= \mathsf{height}(\downarrow C^*_j) - \mathsf{height}(\sigma) \\
&= \mathsf{height}(\downarrow C'_j) - \mathsf{height}(\sigma) \\
&= \mathrm{Dim}(C_j) - \mathsf{height}(\sigma) && \text{(by definition)} \\
&= \mathrm{Dim}(C_j) - \mathrm{Dim}(B) \\
&\geqslant \alpha(j) - 1
\end{aligned}$$

Thus $(C^*_j \mid j \leqslant |\alpha|)$ is an α-partition of $\Uparrow(\sigma)$, meaning that, by Proposition 3.29, $\Sigma \nvDash \chi(\langle\alpha\rangle)$, so that $P \nvDash \chi(\langle\alpha\rangle)$. □

Corollary 3.59. Let $\Lambda \subseteq \mathscr{S}$. If $\mathbf{SFL}(\Lambda)$ has the finite model property then:

$$\mathbf{SFL}(\Lambda) = \mathrm{Logic}(P \text{ polyhedron} \mid \forall \alpha \in \Lambda \colon P \text{ is } \alpha\text{-connected})$$

Proof. By Theorem 3.34, $\mathbf{SFL}(\Lambda) = \mathrm{Logic}(\mathbf{C})$ for some class $\mathbf{C}$ of polyhedra. By Proposition 3.58, we have:

$$\mathbf{C} \subseteq \{P \text{ polyhedron} \mid \forall \alpha \in \Lambda \colon P \text{ is } \alpha\text{-connected}\}$$

and moreover if P is α-connected for every $\alpha \in \Lambda$ then $P \vDash \mathbf{SFL}(\Lambda)$. So the result follows. □

General trees. A natural question to ask is, given the result in Theorem 3.34 on starlike trees, what happens when we consider general trees? The situation turns out to be rather more complex, and it is not clear how to proceed. Indeed, following Lemma 3.3, the first part of determining whether $\chi(T)$ is polyhedrally-complete is to examine what it means to have $F \vDash_{\mathscr{N}} \chi(T)$, for F a finite frame. And the first steps in understanding *this* situation, is to see what it means to have $F \vDash \chi(T)$ and $\mathscr{N}(F) \vDash \chi(T)$. But when T has some branching above the root, this classification becomes quite hard.

The key point is that Lemma 3.5 allows us to assume that any up-reduction to a starlike tree is pointed, meaning that the only 'branching in the p-morphism' occurs at the root, so that a p-morphism to $\langle\alpha\rangle$ is the same as an α-partition. When we consider up-reductions to a general tree T however, we no longer have this luxury. Of course, we can always assume that such an up-reduction is pointed, and one might hope for the possibility of a recursive argument ultimately reducing to the starlike case. Unfortunately, the structure of a p-morphism to T doesn't appear to be 'local' in this sense. Consider for example, the posets F and T in Figure 3.16. Note that $\uparrow(x_i)$ for each i maps p-morphically onto the 3-fork $\langle 1^3\rangle$, however there is no up-reduction $F \circ\!\!\rightarrow T$. So, the investigation of the polyhedral completeness of the Jankov-Fine formulas of starlike trees will require some new techniques.

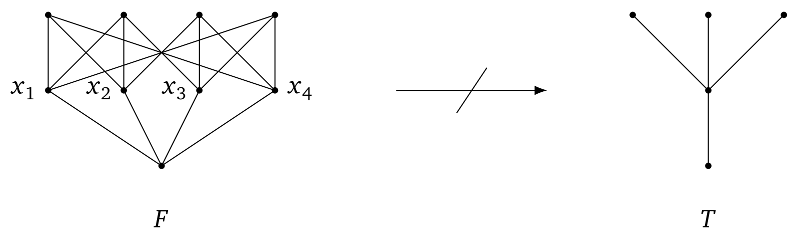
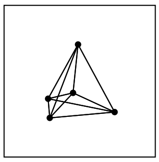


Figure 3.16: An example of the ‘non-locality’ of p-morphisms to general trees

Chapter 4



The Logic of Convex Polyhedra

It is now time to approach the Main Question from an alternative direction. Coming from the geometrical side one might ask: given a class **C** of polyhedra, what is the logic of **C**? In this chapter, I will provide an answer to this question when **C** is a particularly natural class of polyhedra: the class $\mathbf{CP}_n$ of convex polyhedra of dimension at most n. In particular, I will show that the logic of $\mathbf{CP}_n$ is axiomatised by the Jankov-Fine formulas of three simple starlike trees as follows (see Figure 4.1).

$$\mathbf{SFL}(n+1, 2\cdot 1, 1^3) = \mathbf{IPC} + \chi(\langle n+1\rangle) + \chi(\langle 2\cdot 1\rangle) + \chi(\langle 1^3\rangle)$$

The importance of convex polyhedra in polyhedral geometry is mirrored on the logical side: the logic of $\mathbf{CP}_n$ is the largest polyhedrally-complete logic of height n. The problem of finding an axiomatisation for the class of all convex polyhedra remains open.

4.1 The Logic $\mathbf{PL}_n$

The logic of the simplex. As a prelude to considering the logic of all convex polyhedra of dimension at most n, it is prudent to examine the humble simplex once more, from the perspective of the tight connection between logic and polyhedra uncovered in the preceding chapters. Of course, for each d there are many d-simplices. It is handy to single out one representative from this myriad. Let $e_0, \ldots, e_d$ be the standard basis vectors of $\mathbb{R}^{d+1}$. The *standard d-simplex* is $\Delta_d := \operatorname{Conv}\{e_0, \ldots, e_d\}$. We shall investigate $\operatorname{Logic}(\Delta_d)$. Using the notation from Chapter 3, recall that $\langle k\rangle$ is the chain on $k+1$ elements — as a poset it has height k.

Proposition 4.1. The logic of Δ_d is the logic of $\{\langle d\rangle, \mathcal{N}(\langle d\rangle), \mathcal{N}^2(\langle d\rangle), \ldots\}$.

Proof. Label the elements of $\langle d\rangle$ in ascending order as $a_0, \ldots, a_d$. Note that the following is a triangulation of Δ_d:

$$\Sigma := \{\operatorname{Conv} S \mid S \subseteq \{e_0, \ldots, e_d\}\}$$

Then $\Sigma \cong \mathcal{N}(\langle d\rangle)$ as posets via the map:

$$\operatorname{Conv} S \mapsto \{a_i \mid e_i \in S\}$$

Or, less explicitly, note that $\Sigma \cong \mathscr{P}(\{0,\dots,d\}) \cong \mathcal{N}(\langle d \rangle)$. Hence by Corollary 2.22:

$$\mathrm{Logic}(\Delta_d) = \mathrm{Logic}(\mathcal{N}(\langle d \rangle), \mathcal{N}^2(\langle d \rangle), \dots)$$

Moreover, since we have a p-morphism $\max\colon \mathcal{N}(\langle d \rangle) \to \langle d \rangle$ (Proposition 2.3), we can include $\langle d \rangle$ for free. $\square$

Remark 4.2. Note that the same proof works for any d-simplex, showing that there is only one 'logic of the d-simplex'. However, this fact follows from the more general result Corollary 4.5 below.

Corollary 4.3. When $d_1 < d_2$, we have $\mathrm{Logic}(\Delta_{d_2}) \subseteq \mathrm{Logic}(\Delta_{d_1})$.

Proof. I will show by induction that for each $k \in \mathbb{N}$ there is a chain $X = \{x_1, \dots, x_m\} \subseteq \mathcal{N}^k(\langle d_2 \rangle)$ of length $m = d_2 - d_1$ such that $\mathcal{N}^k(\langle d_1 \rangle) \cong \Uparrow(x_m)$. The result in particular follows from this by Proposition 4.1 and Proposition 1.6.

The base case $k = 0$ is immediate. So assume that we have $X \subseteq \mathcal{N}^k(\langle d_2 \rangle)$ as stated. For $i \leqslant m$, define $X_i := \{x_1, \dots, x_i\}$, and let $\mathscr{X} := \{X_1, \dots, X_m\} \subseteq \mathcal{N}^{k+1}(\langle d_2 \rangle)$. Note that $\mathscr{X}$ is a chain of length m. I will show that $\mathcal{N}^{k+1}(\langle d_1 \rangle) \cong \Uparrow(X_m)$. Define $f : \mathcal{N}(\Uparrow(x_m)) \to \Uparrow(X_m)$ by:

$$Y \mapsto X_m \cup Y$$

This is well-defined since if $Y \in \mathcal{N}(\Uparrow(x_m))$ then Y lies above each element of $X_m = X$. It is clearly order-preserving. Define $g\colon \Uparrow(X_m) \to \mathcal{N}(\Uparrow(x_m))$ by:

$$Z \mapsto Z \setminus X_m$$

To see that this is well-defined, note that:

$$\begin{aligned} \mathsf{height}(\Uparrow(x_m)) &= \mathsf{height}(\mathcal{N}^k(\langle d_1 \rangle)) && \text{(induction hypothesis)} \\ &= \mathsf{height}(\langle d_1 \rangle) && \text{(Proposition 2.4)} \\ &= d_1 \end{aligned}$$

and furthermore $\mathsf{height}(\mathcal{N}^k(\langle d_2 \rangle)) = d_2$. Hence X_m, having length $m = d_2 - d_1$, is a maximal chain in $\Downarrow(x_m)$. This means that for any chain $Z \subseteq \mathcal{N}^k(\langle d_2 \rangle))$ containing X_m we must have:

$$Z \setminus X_m = Z \cap \Uparrow(x_m) \subseteq \Uparrow(x_m)$$

Furthermore g is also order-preserving, and f and g are mutual inverses. Therefore:

$$\mathcal{N}^{k+1}(\langle d_1 \rangle) = \mathcal{N}(\mathcal{N}^k(\langle d_1 \rangle)) \cong \mathcal{N}(\Uparrow(x_m)) \cong \Uparrow(X_m) \qquad \square$$

The logic of convex polyhedra. Recall that a polyhedron P is convex if it is equal to its convex hull: $P = \mathrm{Conv}\, P$. Let $\mathbf{CP}_n$ be the class of convex polyhedra of dimension at most n. We are interested in $\mathrm{Logic}(\mathbf{CP}_n)$. The first key step in understanding this logic is the following observation, which makes the task significantly more manageable. Recall that Δ_n is the standard n-simplex.

Proposition 4.4. Every n-dimensional convex polyhedron is PL-homeomorphic to Δ_n.

Proof. See [RS72, Corollary 2.20, p. 21]. There it is shown that *n-cells* — which correspond to our n-dimensional convex polyhedra — are *n-balls* — meaning that they are PL-homeomorphic to the n-dimensional cube $[0,1]^n$. Since Δ_n is a convex polyhedron, the result follows. $\square$

COROLLARY 4.5. Let P and Q be n-dimensional convex polyhedra. Then $\mathrm{Logic}(P) = \mathrm{Logic}(Q)$.

Proof. By Proposition 4.4, P and Q are PL-homeomorphic. Then by Proposition 1.48, we have that $\mathrm{Sub}_{\mathrm{o}}P \cong \mathrm{Sub}_{\mathrm{o}}Q$, whence $\mathrm{Logic}(P) = \mathrm{Logic}(Q)$. □

COROLLARY 4.6. The logic of convex polyhedra of dimension at most n is the logic of the n-simplex. That is:

$$\mathrm{Logic}(\mathbf{CP}_n) = \mathrm{Logic}(\Delta_n)$$

Proof. Indeed:

$$\begin{aligned} \mathrm{Logic}(\mathbf{CP}_n) &= \mathrm{Logic}(\Delta_0, \cdots, \Delta_n) && \text{(Proposition 4.4)} \\ &= \mathrm{Logic}(\Delta_n) && \text{(Corollary 4.3)} \end{aligned}$$

□

REMARK 4.7. Corollary 4.6 in particular justifies calling $\mathrm{Logic}(\mathbf{CP}_n)$ 'the logic of polyhedra of dimension n' (as opposed to 'the logic of dimension *at most* n').

The largest logic. Convex polyhedra are rather special. One of the ways in which this specialness manifests itself logically is that $\mathrm{Logic}(\mathbf{CP}_n)$ is the largest polyhedrally-complete logic of height n. The result itself is not needed for the main theorem of this chapter, and I only present a proof sketch.

PROPOSITION 4.8. The logic of convex polyhedra of dimension n is the largest height-n polyhedrally-complete logic. That is, if $\mathscr{L}$ is a polyhedrally-complete logic of height n, then $\mathscr{L} \subseteq \mathrm{Logic}(\mathbf{CP}_n)$.

Sketch Proof. Since $\mathscr{L}$ is the logic of a class of polyhedra, by Maxim I it is the logic of the triangulations of those polyhedra. Since $\mathscr{L}$ has height n, there is such a triangulation Σ of a polyhedron P in this class with $\mathrm{Dim}(\Sigma) = n$. Then Σ contains an n-simplex σ. This means that Δ_n embeds into P via a map which is a PL-homeomorphism onto its image. By a slightly more general result than Proposition 1.48, this entails that:

$$\mathrm{Logic}(P) \subseteq \mathrm{Logic}(\Delta_n) = \mathrm{Logic}(\mathbf{CP}_n)$$

from which the result follows. □

The logic $\mathbf{PL}_n$. We now meet the candidate axiomatisation of $\mathrm{Logic}(\mathbf{CP}_n)$.

DEFINITION 4.9 ($\mathbf{PL}_n$). For $n \in \mathbb{N}$, let $\mathbf{PL}_n := \mathbf{BD}_n + \chi(\langle 2 \cdot 1 \rangle) + \chi(\langle 1^3 \rangle)$.

So (using Remark 3.30) we have that:

$$\mathbf{PL}_n = \mathbf{SFL}(n+1, 2 \cdot 1, 1^3) = \mathbf{IPC} + \chi(\langle n+1 \rangle) + \chi(\langle 2 \cdot 1 \rangle) + \chi(\langle 1^3 \rangle)$$

That is, $\mathbf{PL}_n$ is axiomatised by forbidding the $(n+1)$-chain, Scott's tree, and the 3-fork. See Figure 4.1 for representations of these starlike trees.

The aim of this chapter is to prove the following.

THEOREM 4.10. $\mathbf{PL}_n = \mathrm{Logic}(\mathbf{CP}_n)$.

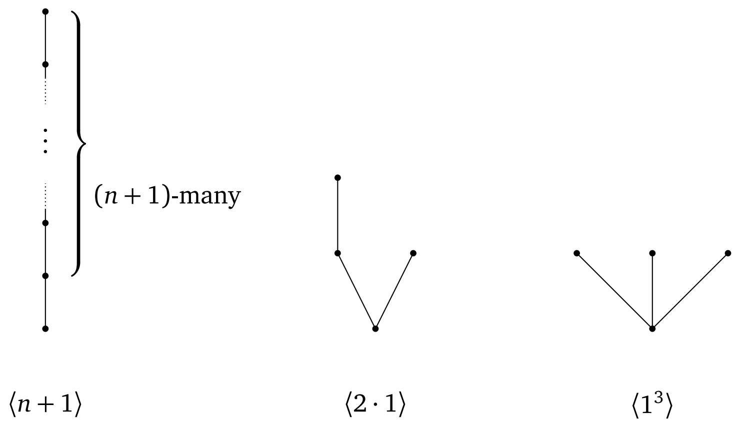


Figure 4.1: The three starlike trees whose Jankov-Fine formulas axiomatise $\mathbf{PL}_n$

By Theorem 3.42, and Segerberg's Theorem 1.20, we already know that $\mathbf{PL}_n$ is the logic of *some* class of polyhedra. The task is now to show that $\mathbf{PL}_n$ is the logic of the n-simplex, so that we can make use of Corollary 4.6.

REMARK 4.11. Scott's tree and the 3-fork are important starlike trees because they are the minimal non-chain non-difork starlike trees. That is, for any signature $\alpha \in \mathscr{S}$ which is not (k) for some $k \in \mathbb{N}$, we have that either $2\cdot 1 \leqslant \alpha$ or $1^2 \leqslant \alpha$. The fact that they play a principle role in axiomatising convexity is perhaps evidence for the importance of starlike trees in polyhedral semantics.

Theorem 4.10 has the following important corollary which is relevant to the general programme of the classification of polyhedrally-complete logics.

COROLLARY 4.12. Let $\mathscr{L}$ be a polyhedrally-complete logic of finite height n which is axiomatised by **IPC** plus $\chi(Q)$ for each Q in a collection $\mathscr{Q}$ of finite rooted frames. Then every $Q \in \mathscr{Q}$ up-reduces to $\langle n+1\rangle$, $\langle 2\cdot 1\rangle$ or $\langle 1^3\rangle$.

Proof. Suppose not, meaning that there is Q which doesn't up-reduce to $\langle n+1\rangle$, $\langle 2\cdot 1\rangle$ or $\langle 1^3\rangle$. By the characterisation of Jankov-Fine formulas (Theorem 1.16) this means that:

$$Q \vDash \chi(\langle n+1\rangle) + \chi(\langle 2\cdot 1\rangle) + \chi(\langle 1^3\rangle)$$

Hence $Q \vDash \mathbf{PL}_n$. But, of course, $Q \nvDash \chi(Q)$, meaning that $\mathbf{PL}_n \nvdash \chi(Q)$; hence:

$$\mathscr{L} \nsubseteq \mathbf{PL}_n$$

But by Theorem 4.10 and Proposition 4.8 $\mathbf{PL}_n$ is the largest polyhedrally-complete logic of height n. ↯ □

4.2 $\mathbf{PL}_n$ is the Logic of n-Dimensional Convex Polyhedra

In order to prove Theorem 4.10, we need to prove both containments — i.e. soundness and completeness.

Soundness. The first thing to prove is that $\mathbf{PL}_n$ is valid on all convex polyhedra of dimension n. By Corollary 4.6, it suffices to check that $\Delta_n \vDash \mathbf{PL}_n$. The proof is short, making use of the technology developed in the preceding chapters.

Theorem 4.13 (Soundness). $\mathbf{PL}_n$ is sound with respect to $\mathbf{CP}_n$.

Proof. I will show that $\Delta_n \vDash \mathbf{PL}_n$. By Proposition 4.1, it suffices to show that $\mathbf{PL}_n$ is valid on:

$$\mathbf{C} := \{\langle n\rangle, \mathscr{N}(\langle n\rangle), \mathscr{N}^2(\langle n\rangle), \ldots\}$$

By Proposition 2.4, every frame in $\mathbf{C}$ has height n. Hence, by Proposition 1.18, $\mathbf{BD}_n$ is valid on $\mathbf{C}$. Further, note that $\langle n\rangle$ is $(2\cdot 1)$-nerve-connected and 1^3-nerve connected, whence by Theorem 3.42:

$$\langle n\rangle \vDash_{\mathscr{N}} \chi(\langle 2\cdot 1\rangle) + \chi(\langle 1^3\rangle)$$

i.e. $\chi(\langle 2\cdot 1\rangle) + \chi(\langle 1^3\rangle)$ is valid on $\mathbf{C}$. □

Remark 4.14. In addition to the combinatorial proof given below, a more direct geometric proof can be given, making use of parts of classical dimension theory. The proofs that $\chi(\langle 1^3\rangle)$ and $\chi(\langle 2\cdot 1\rangle)$ are valid on the simplex were communicated to me in private correspondence by Vincenzo Marra and David Gabelaia.

Completeness. The proof that $\mathbf{PL}_n$ is complete with respect to n-dimensional convex polyhedra is a method, given a finite rooted frame F of $\mathbf{PL}_n$, of constructing a convex polyhedron P and an open polyhedral map $P \to F$. This yields the result by Proposition 1.45. The construction proceeds in two steps. The first is combinatorial in nature, and involves showing that F can be assumed to be of a particular form, called a 'saw-topped tree'. The second is more geometrical, and involves the construction of the polyhedron P using the structure of the saw-topped tree.

Theorem 4.15 (Completeness). $\mathbf{PL}_n$ is complete with respect to $\mathbf{CP}_n$.

The meaning of $\mathbf{PL}_n$ on frames. First, it will be convenient to spell out what it means, structurally, for a frame to satisfy $\mathbf{PL}_n$.

Lemma 4.16. Let F be a poset. Then $F \vDash \mathbf{PL}_n$ if and only if the following are satisfied.

(i) F has height at most n.

(ii) Whenever $\mathsf{depth}(x) = 1$, we have $|{\Uparrow}(x)| \leqslant 2$.

(iii) Whenever $\mathsf{depth}(x) > 1$, the set ${\Uparrow}(x)$ is connected.

Proof. This follows immediately from Lemma 3.49. □

Saw-topped trees. The type of poset that will be defined here has the important property of being *planar*: its Hasse diagram can be drawn in the plane with no overlapping lines. The notion of a planar poset has been studied somewhat in the literature (see [BLS99, §6.8, p. 101] for a survey), but we won't use any external results here.

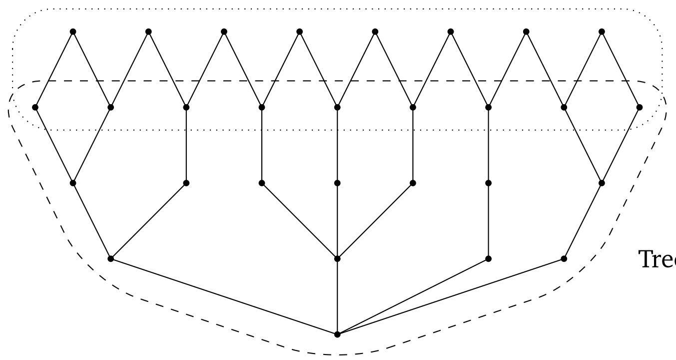


Figure 4.2: An example saw-topped tree

DEFINITION 4.17 (Plane ordering). Let T be a finite tree of uniform height. A linear ordering $\prec$ on $\mathsf{Top}(T)$ (or equivalently an enumeration $t_0, \ldots, t_{k-1}$ of $\mathsf{Top}(T)$) is a *plane ordering* (c.f. [Sta97, p. 294]) if when we arrange $\mathsf{Top}(T)$ in that order horizontally on the plane, we can draw the Hasse diagram of the rest of the tree below it so that no lines cross. Formally, for every $x \in T$ we have that $\uparrow(x) \cap \mathsf{Top}(T)$ is an interval with respect to $\prec$.

DEFINITION 4.18 (Saw-topped tree). Let T be a tree of uniform height, and let $\prec$ be a plane ordering on $\mathsf{Top}(T)$ with corresponding enumeration $t_0, \ldots, t_{k-1}$. The *saw-topped tree* based on $(T, \prec)$ consists of T plus new elements $s_0, \ldots, s_{k-2}$ with relations, for each i:

$$t_i, t_{i+1} < s_i$$

See Figure 4.2 for an example of a saw-topped tree.

LEMMA 4.19. Let F be a saw-topped tree of height n. Then $F \vDash \mathbf{PL}_n$.

Proof. Let F be based on $(T, \prec)$. Let us verify the conditions of Lemma 4.16. Conditions (i) and (ii) are immediate. As for (iii), take $x \in F$ with $\mathsf{depth}(x) > 1$. By construction, $x \in T$. Since $\prec$ is a plane ordering, we have that $\uparrow(x) \cap \mathsf{Top}(T)$ is an interval with respect to $\prec$. Therefore, the top two layers of $\Uparrow(x)$ are connected by the saw structure. $\square$

It will be convenient to work with saw-topped trees which are sufficiently uniformly wide.

DEFINITION 4.20 (Uniform width). A graded rooted poset F is *of width uniformly at least m*, for $m \in \mathbb{N}$, if $|\rho_k| \geqslant m$ for every $k \in \{1, \ldots, \mathsf{height}(F)\}$.

LEMMA 4.21. Let F be a saw-topped tree and $m \in \mathbb{N}$. Then F is the p-morphic image of a saw-topped tree such that $\mathsf{height}(F') = \mathsf{height}(F)$ and F' is of width uniformly at least m.

Proof. Let $s_0, \ldots, s_{k-2}$ be as in Definition 4.18. We can add a new chain to the right of F, as in Figure 4.3, mapping every new element to s_{k-2}. Applying this operation repeatedly yields the result. $\square$

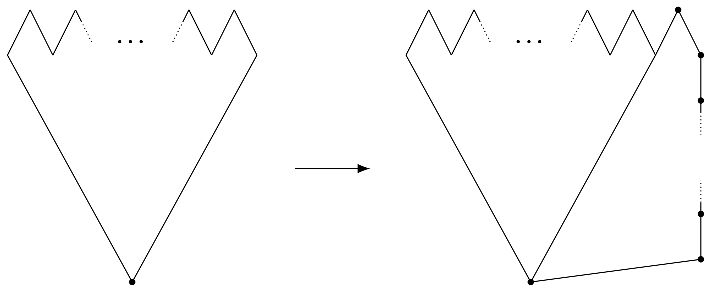


Figure 4.3: The widening of a saw-topped tree

The combinatorial step: preparation. In this step, we see that every rooted frame F of $\mathbf{PL}_n$ is the p-morphic image of a saw-topped tree. This saw-topped tree is constructed in two stages. In this first one, the frame F is prepared a little before we come to an inductive argument in the second stage.

Definition 4.22 (kth layer)**.** Let ρ be a rank function on F. The *kth layer* of F is the set $\rho_k := \rho^{-1}\{k\}$. Let F have height n. Define $\rho_S := \rho_{n-1} \cup \rho_n$.

We will make use of the following technical lemma.

Lemma 4.23. Let F be a finite rooted poset of height n with $F \vDash \chi(\langle 2 \cdot 1 \rangle)$, and which is graded with rank function ρ. Then ρ_S is connected in F.

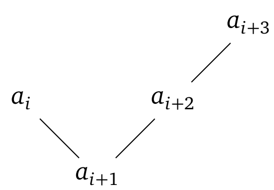
Proof. We may assume that $n \geqslant 2$. Take t, s top nodes with $\rho(s) = n$. Since $\bot \vDash \chi(\langle 2 \cdot 1 \rangle)$, by Proposition 3.29 there is a path $p \colon t \rightsquigarrow s$ in $\Uparrow(\bot)$. Let $p = a_0 \cdots a_k$. We may assume:

(i) that p steps in rank: $|\rho(a_{i+1}) - \rho(a_i)| = 1$,

(ii) that p is height-maximal: if $i < j < k$ and $a_j < a_i, a_k$, then there is no path $a_i \rightsquigarrow a_k$ in $\Uparrow(a_j)$.

Suppose for a contradiction that there is j such that $\rho(a_j) < n-1$. Since t, s is are top nodes, we must have $\rho(a_1) = \rho(a_0) - 1$ and $\rho(a_{k-1}) = n-1$. Hence by (i), there must be a sequence $a_i, a_{i+1}, a_{i+2}, a_{i+3}$ such that:

$$\rho(a_{i+1}) = \rho(a_i) - 1, \quad \rho(a_{i+2}) = \rho(a_i), \quad \rho(a_{i+3}) = \rho(a_i) + 1,$$



But now, by (ii), there is no path $a_i \rightsquigarrow a_{i+3}$ in $\Uparrow(a_{i+1})$. Therefore $a_{i+1} \nvDash \chi(\langle 2 \cdot 1 \rangle)$, a contradiction. ↯ This shows that there is a path $t \rightsquigarrow s$ in ρ_S (and also that F has uniform height n). $\square$

Definition 4.24 (P-path)**.** Let F be a graded poset with height n. A path $p = x_0 \cdots x_k$ in F is a *p-path*, if it satisfies the following properties.

(a) p has no immediate repetitions: $x_i x_{i+1}$ with $x_i = x_{i+1}$.

(b) For any $i \in \{1, \ldots, k-1\}$ such that $x_i \in \rho_{n-1}$, we have that $\Uparrow(x_i) = \{x_{i-1}, x_{i+1}\}$.

Lemma 4.25 (Preparation)**.** Take $n \geqslant 2$ and let F be a rooted poset such that $F \vDash \mathbf{PL}_n$. There is a rooted poset F' and a p-morphism $f : F' \to F$ with the following properties.

(I) $F' \vDash \mathbf{PL}_n$.

(II) F' is graded, with rank function ρ.

(III) F' has uniform height n.

(IV) For every $x \in \rho_{n-1}$ we have $|\Uparrow(x)| = 2$.

(V) Every pair $x, y \in \rho_S$ is connected by a p-path.

Proof. First, we may assume that F has height n. Otherwise, we may add a chain of $n - \mathsf{height}(F)$ elements below the root of F. The resulting frame still validates $\mathbf{PL}_n$ by Lemma 4.16, and we can map it p-morphically back to the original frame by collapsing the new chain to the root.

Second, by the proof of Theorem 3.50, following Remark 3.51, we may assume that F' is such that $F' \setminus \mathsf{Top}(F')$ is a tree of uniform height $n-1$. Hence F' is graded and of uniform height n. Therefore, we have (I), (II) and (III).

To get (IV), we need to perform the small modification of adding some extra top nodes to F. Obtain F' from F and a p-morphism $f' : F' \to F$ as follows. Note that every $x \in \rho_{n-1}$ has $|\Uparrow(x)| \leqslant 2$ by Lemma 4.16. So take $x \in \rho_{n-1}$ such that $|\Uparrow(x)| < 2$. First, since F has uniform height n, we must have $|\Uparrow(x)| \geqslant 1$; let $\Uparrow(x) = \{y\}$. Now add a new immediate successor y^* of x to F and define $f'(y^*) := y$. Adding in all of these elements to F, and completing f' by letting $f'|_F = \mathrm{id}$, we obtain F' and f'. Note that this operation preserves (I), (II) and (III) (we extend ρ to a rank function on F'), and secures (IV).

Finally, to verify condition (V), note that ρ_S is connected by Lemma 4.23. We can then transform any path $p = x_0 \cdots x_k$ in ρ_S into a p-path p' with the same end-points using the following two steps.

- Remove all immediate repetitions.
- For any $i \in \{1, \ldots, k-1\}$ such that $x_i \in \rho_{n-1}$ and $\Uparrow(x_i) \neq \{x_{i-1}, x_{i+1}\}$, there must be a unique $y \in \Uparrow(x_i) \setminus \{x_{i-1}, x_{i+1}\}$. Replace $x_{i-1} x_i x_{i+1}$ by $x_{i-1} x_i y x_i x_{i+1}$. □

Combinatorial step: inductive construction. With the frame suitably prepared, we are now in a position to inductively construct the saw-topped tree.

Theorem 4.26. $\mathbf{PL}_n$ is the logic of the class of saw-topped trees of height n and of width uniformly at least 2, for $n \geqslant 2$.

Proof. In light of Lemma 4.19 and Lemma 4.21, it suffices to show that any frame F of $\mathbf{PL}_n$ is the p-morphic image of a saw-topped tree of the same height. We may assume that F satisfies the properties in Lemma 4.25. The proof now proceeds by induction on n. For the base case $n = 2$, note that $\rho_0 = \{\bot\}$ (since F is rooted). Using (V) of Lemma 4.25 for F, let $p = x_0 \cdots x_k$ be a closed p-path in ρ_S (i.e. $x_0 = x_k$) which visits all of ρ_S and

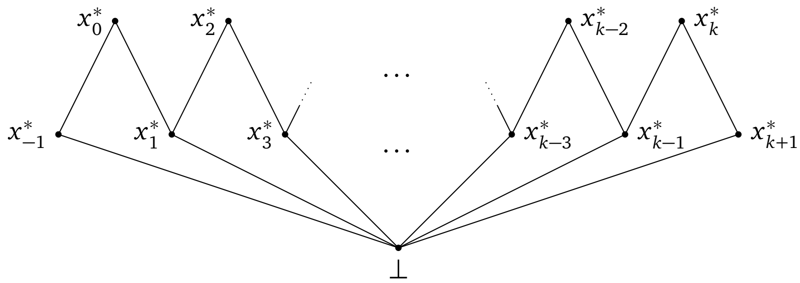


Figure 4.4: The relations in F' when $n = 2$

whose end-points lie in ρ_n. Construct the saw-topped tree F' by taking $\perp$ together with new elements $x_{-1}^*, x_0^*, \ldots, x_k^*, x_{k-1}^*$ with the following relations (see Figure 4.4).

$$\perp < x_i^* \qquad \forall i,$$
$$x_{-1}^* < x_0^*, \quad x_{k+1}^* < x_k^*,$$
$$x_i^* < x_j^* \quad \Longleftrightarrow \quad x_i < x_j \qquad \forall i, j \in \{0, \ldots, k\}$$

Then define the map $f : F' \to F$ by:

$$\perp \mapsto \perp,$$
$$x_{-1}^* \mapsto x_0,$$
$$x_{k+1}^* \mapsto x_k,$$
$$x_i^* \mapsto x_i \qquad \forall i \in \{0, \ldots, k\}$$

That f is a p-morphism amounts to the fact that p is a p-path.

Now for the induction step $n > 2$. Let $\mathrm{Succ}(\perp) = \{a_0, \ldots, a_{m-1}\}$ be the immediate successors of $\perp$. By induction hypothesis, for each $a_i \in \mathrm{Succ}(\perp)$ there is a height-$(n-1)$ saw-topped tree F_i' and a p-morphism $f_i \colon F_i' \to \uparrow(a_i)$. For each F_i', its tree part T_i comes with plane ordering $\prec_i$; let s_i and t_i be the least and greatest elements in this ordering, respectively. Since $|\Uparrow(s_i)^{T_i}|, |\Uparrow(t_i)^{T_i}| = 1$, then by (IV) for F and the fact that f_i is a p-morphism, we must have $f_i(s_i), f_i(t_i) \in \rho_n$. For $i < m-1$, let $p_i = x_{i,0} \cdots x_{i,k_i}$ be a p-path in ρ_S from $f_i(t_i)$ to $f_{i+1}(s_{i+1})$ (using (V) for F).

Now, form F' by taking the following ingredients and combining them as in Figure 4.5.

- Each saw-topped tree F_i'.
- For each $i < m-1$, new elements $x_{i,0}^* \cdots x_{i,k_i}^*$ corresponding to $x_{i,0} \cdots x_{i,k_i}$.
- A chain of length $n-2$ (a rope ladder) to hang below each $x_{i,j}^*$, with j odd.

The result is evidently a saw-topped tree. Finally, construct the p-morphism $f : F' \to F$ as follows.

(a) Inside each saw-topped tree F_i', let f act as f_i.

(b) For each $x_{i,j}^*$, let $f(x_{i,j}^*) := x_{i,j}$.

(c) For each $x_{i,j}^*$ with j odd, send the rope ladder hanging below $x_{i,j}^*$ to $x_{i,j}$.

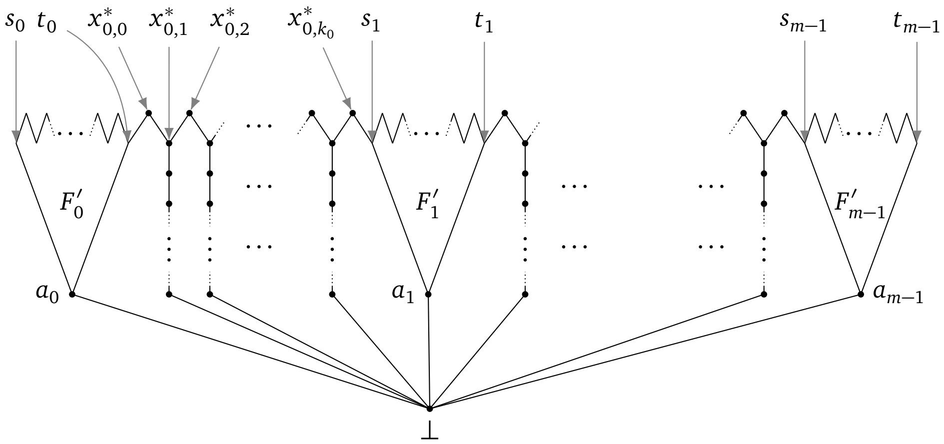


Figure 4.5: Construction of F' from $F'_0, \dots, F'_{m-1}$ and the paths $p_0, \dots p_{m-2}$.

To see that f is a p-morphism, the two important points are that $\Uparrow(f(s_i))^F$ and $\Uparrow(f(t_i))^F$ are empty for each i, and that for any $x^*_{i,j}$ with j odd, since p_i is a p-path:

$$\Uparrow\left(f(x^*_{i,j})\right)^F = \left\{x_{i,j-1}, x_{i,j+1}\right\} = \left\{f(x^*_{i,j-1}), f(x^*_{i,j+1})\right\} \qquad \square$$

Convex geometric realisation: intuition. In the second step of the proof, a saw-topped tree is 'realised' as a convex polyhedron P, in the sense that there is an open polyhedral map $P \to F$. So as to give some visual intuition for how the construction works, I will first work through a series of instructive examples, before coming to the full proof.

It will be convenient to work with 'tetrahedral-prism–based pyramids', which are examples of convex polyhedra. Let $e_0, \dots, e_n$ be the elements of the standard basis for $\mathbb{R}^{n+1}$. An n-dimensional tetrahedral-prism is the higher-dimensional prism which is made from two translated $(n-1)$-simplices.

Definition 4.27 (Standard n-dimensional tetrahedral-prism)**.** Let the *standard n-dimensional tetrahedral-prism* Φ_n be the point $\{e_0\}$ when $n = 0$, and for $n > 0$ the convex hull of the set:

$$\{e_i \mid i \leqslant n-1\} \cup \{e_i + e_n \mid i \leqslant n-1\}$$

The first five standard tetrahedral-prisms are the point, the line segment, the square, the triangular prism and the 4-dimensional tetrahedral prism. See Figure 4.6 for representations of these; note that the four-dimensional Φ_4 is represented as the wireframe of its projection into three-dimensional space.

An n-dimensional tetrahedral-prism–based pyramid is formed by taking an $(n-1)$-dimensional tetrahedral-prism as a base and adding an apex.

Definition 4.28 (Standard n-dimensional tetrahedral-prism–based pyramid)**.** Let the *standard n-dimensional tetrahedral-prism–based pyramid* Ψ_n be the point $\{e_0\}$ when $n = 0$, and for $n > 0$ the convex hull of $\Phi_{n-1} \cup \{e_n\}$.

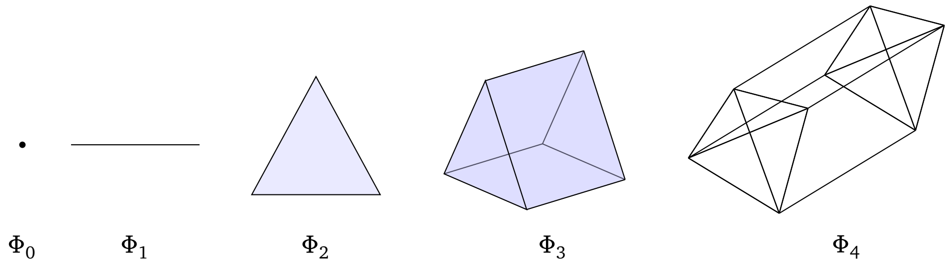


Figure 4.6: Representations of the first five standard tetrahedral-prisms

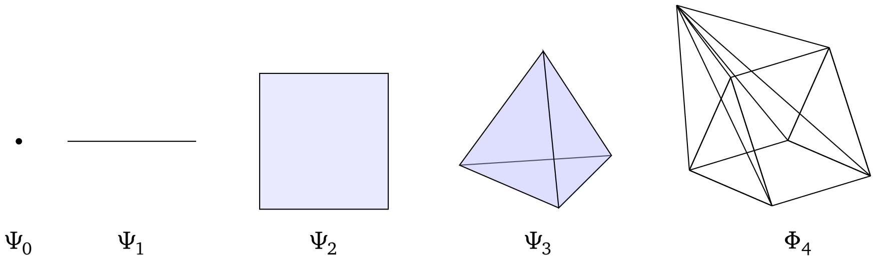


Figure 4.7: Representations of the first five standard tetrahedral-prism–based pyramids

For $n \geqslant 2$, we have that $\Psi_n = \operatorname{Conv} S$, where:

$$S = \{e_i \mid i \leqslant n-2\} \cup \{e_i + e_{n-1} \mid i \leqslant n-2\} \cup \{e_n\}$$

Note that each Ψ_n is convex and of dimension n. The first five standard tetrahedral-prism–based pyramids are the point, the line segment, the triangle, the tetrahedron and the triangular-prism–based pyramid. See Figure 4.7 for representations of these. Again, the four-dimensional Ψ_4 is represented as the wireframe of its projection into three-dimensional space.

Let us now begin with a height-2 example of the construction of the convex realisation of a saw-topped tree. At height 2, saw-topped trees all look very similar: they consist of a root below a saw structure. Consider Figure 4.8. The poset in the middle is an example of a height-2 saw-topped tree; the 2-dimensional tetrahedral-prism–based pyramid on the left is mapped onto this poset, where the preimages of the elements of the poset are given on the right. It is straightforward to verify that this is an open polyhedral mapping onto the saw-topped tree.

Our next two examples have height 3. Consider in Figure 4.9 an example of the simplest case: in which the ‘tree part’ has branching only at the root (i.e. it is a starlike tree). Note that by projecting downwards onto the base, it is possible to represent the preimages in 2-dimensions; see Figure 4.10. Using this projection method, Figure 4.11 illustrates how we can deal with a more complex tree structure.

Finally, let us consider a relatively complex example of height 4. See Figure 4.12. For simplicity and clarity, I omit the face-shading, and only label certain key points. Notice how the structure of the tree wraps around the tetrahedral prism.

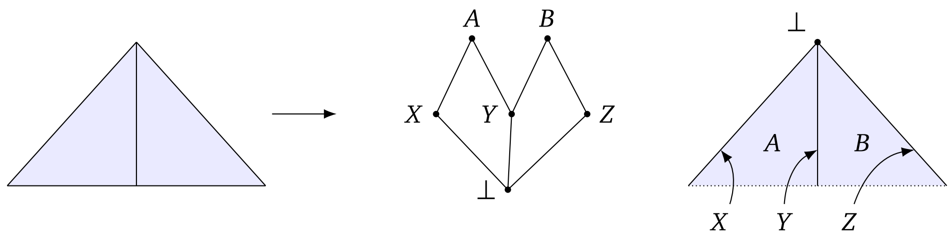


Figure 4.8: A height-2 example of convex geometric realisation

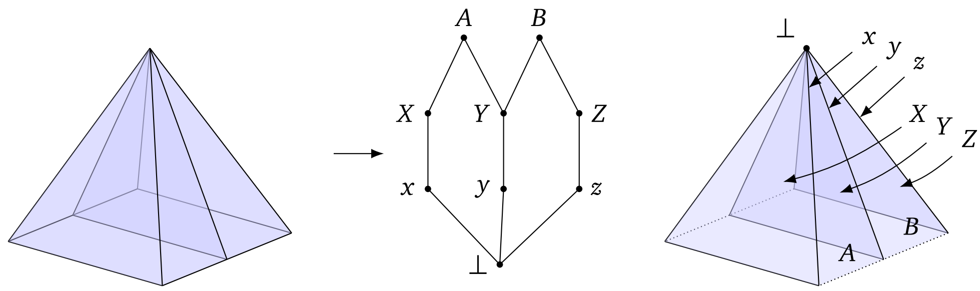


Figure 4.9: A simple height-3 example of convex geometric realisation

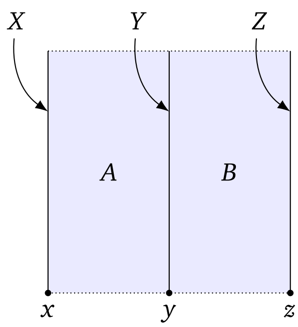


Figure 4.10: The projection onto the base of the simple height-3 example of convex geometric realisation in Figure 4.9

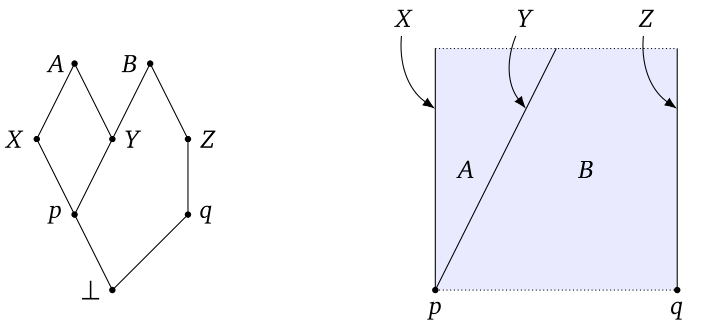


Figure 4.11: The projection onto the base of a more complex height-3 example of convex geometric realisation

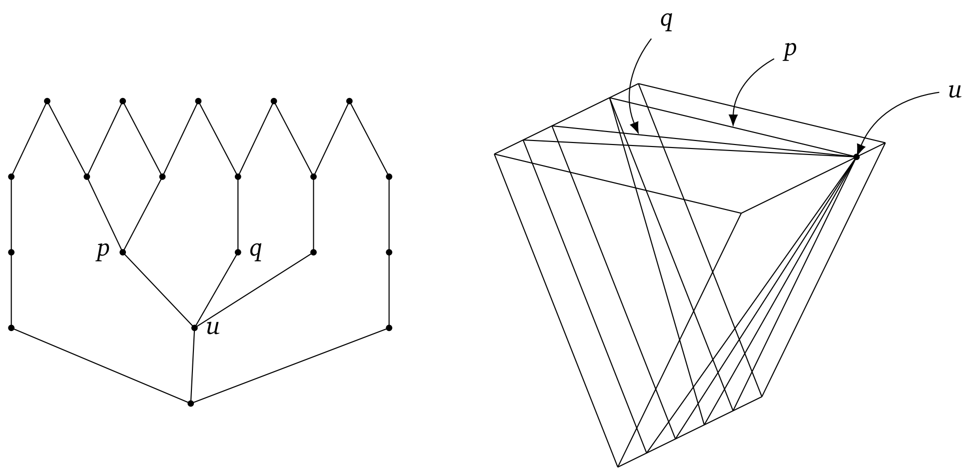
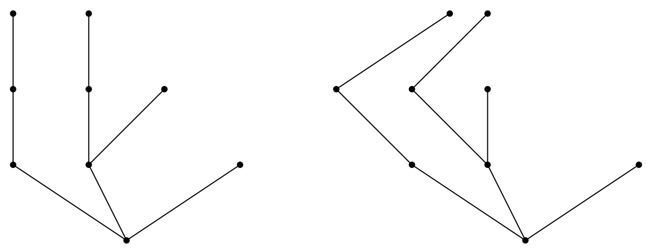


Figure 4.12: The projection onto the base of a height-4 example of convex geometric realisation



(a) (b)

Figure 4.13: Two plane drawings of the same tree. The drawing in (a) is vertically-distinguishing while the one in (b) is not.

Convex geometric realisation: full proof. With these examples in mind, let us turn now to the formal proof of completeness. In order to spell out the construction, it will be necessary to make somewhat more precise the notion of "drawing a tree in the plane", and in particular to see that such drawings can be assumed to be of a certain form.

Definition 4.29 (Plane drawing)**.** Let F be a poset and $d\colon F \to \mathbb{R}^2$ be an injection, such that $d = (d_1, d_2)$. Draw an edge xy between $f(x)$ and $f(y)$ whenever $y \in \mathrm{Succ}(x)$. Then d is a *plane drawing* (see [TM77]) of F if the following conditions hold.

(a) Whenever $x < y$ we have $d_2(x) < d_2(y)$.

(b) Two distinct edges x_1y_1 and x_2, y_2 only ever intersect at their end-points.

When we draw the vertices $f(x)$ and edges xy in the plane then we have a drawing of the Hasse diagram in which no edges overlap.

Definition 4.30 (Width of a drawing)**.** Let F be a finite poset. The *width* of a plane drawing d of F is:

$$\mathsf{width}(d) := \max\{d_1(x) \mid x \in F\} - \min\{d_1(x) \mid x \in F\}$$

Definition 4.31 (Vertically-distinguishing drawing)**.** Let d be a plane drawing of F. Then d is *vertically-distinguishing* if for any $x \in F$ and $y, z \in \mathrm{Succ}(x)$ either:

$$\forall y_0 \in \uparrow(y) \colon \forall z_0 \in \uparrow(z) \colon d_1(y_0) < d_1(z_0)$$

or:

$$\forall y_0 \in \uparrow(y) \colon \forall z_0 \in \uparrow(z) \colon d_1(y_0) > d_1(z_0)$$

See Figure 4.13 for an example and non-example of a vertically-distinguishing plane drawing of a tree.

LEMMA 4.32. Let $\prec$ be a plane ordering on a finite tree T of uniform height, and let ρ be the rank function on T. Then T has a plane drawing d satisfying the following conditions.

(i) T is vertically-distinguishing.

(ii) For every $x \in T$ we have $d_2(x) = \rho(x)$.

(iii) The top nodes in the drawing are ordered left-to-right as per $\prec$.

Proof. We prove this by induction on the height of T. The base case is immediate. So assume that $n = \mathsf{height}(T) > 0$. By induction hypothesis, for every $x \in \mathrm{Succ}(\perp)$ there is a plane drawing d^x of $\uparrow(x)$ satisfying the conditions. By translating, we may assume that $\min\{d_2^x(y) \mid y \in \uparrow(x)\} = 0$. Enumerate $\mathrm{Succ}(\perp) = \{x_1, \ldots, x_k\}$ according to $\prec$. That is, for each $i, j \leqslant k$ with $i < j$ ensure that:

$$\forall t_i \in \uparrow(x_i) \cap \mathsf{Top}(T) \colon \forall t_j \in \uparrow(x_j) \cap \mathsf{Top}(T) \colon t_i \prec t_j$$

(this is possible since $\uparrow(x) \cap \mathsf{Top}(T)$ is an interval for each $x \in \mathrm{Succ}(\perp)$). Let us define $d \colon T \to \mathbb{R}^2$. Let $d(\perp) := (0, 0)$. Then for $y \in T \setminus \{\perp\}$, there is a unique $x_i \in \mathrm{Succ}(\perp)$ such that $y \in \uparrow(x_i)$. Define:

$$d(y) := \left(d_1^{x_i}(y) + \sum_{j=1}^{i-1} \mathsf{width}(d^{x_j}) + i, \quad d^{x_i}(y) + 1 \right)$$

In other words, we place $\perp$ at the origin, then line up the drawings $d^{x_1}, \ldots, d^{x_k}$ of its successors side-by-side, shifted up by 1. It is straightforward to check that d is a plane drawing satisfying the conditions. $\square$

We will also need the following technical criterion which will help to build up a simplicial complex.

LEMMA 4.33. Let F be a poset and take any function $\alpha \colon F \to \mathbb{R}^n$. The collection:

$$\{\mathrm{Conv}\, \alpha[X] \mid X \in \mathcal{N}(F)\}$$

forms a simplicial complex if and only if $\mathrm{Conv}\, \alpha[X]$ and $\mathrm{Conv}\, \alpha[Y]$ are disjoint for any distinct $X, Y \in \mathcal{N}(F)$.

Proof. This follows from [Men99, Theorem 2], noting that the nerve $\mathcal{N}(F)$ is in particular an abstract simplicial complex, as defined there, with vertex set $\{\{x\} \mid x \in F\}$. $\square$

Proof of Theorem 4.15. The case $n = 0$ is immediate. For $n = 1$ note $\mathbf{PL}_1 = \mathrm{Logic}(\langle 1^2 \rangle)$ and that by Proposition 4.1 and Corollary 1.7:

$$\mathrm{Logic}(\Delta_1) = \mathrm{Logic}(\langle 1 \rangle, \mathcal{N}(\langle 1 \rangle), \cdots) = \mathrm{Logic}(\langle 1^2 \rangle)$$

Hence we may assume that $n \geqslant 2$.

By Theorem 4.26, and Proposition 1.45, it suffices to take a saw-topped tree F of height n and of width uniformly at least 2, and construct an open polyhedral map $f : \Psi_n \to F$. We do so by cutting up Ψ_n using a number of 'thin slices'. Assume that F is the saw-topped tree based on $(T, \prec)$, and let ρ be the rank function on F.

Let d be a plane drawing of T satisfying the conditions in Lemma 4.32. By scaling and translating, we may assume that $\mathsf{width}(d) = 1$ and that:

$$\min\{d_1(x) \mid x \in F\} = 0$$

Furthermore, since T is of width uniformly at least 2, by shifting the first and last points in each layer, we may assume that for each $k \in \{1, \dots, n-1\}$ there are $u_k, v_k \in \rho_k$ such that $d_1(u_k) = 0$ and $d_1(v_k) = 1$.

We will 'wrap d around the prism'. We do so by associating to each $x \in T \setminus \{\perp\}$ a point on the boundary of the $(n-1)$-dimensional tetrahedral-prism Φ_{n-1}, such that the elements of the kth layer ρ_k, for $k > 0$, appear in order along the edge $\operatorname{Conv}\{e_{k-1}, e_{k-1} + e_{n-1}\}$. Specifically, for each $x \in T \setminus \{\perp\}$, define:

$$\alpha(x) := e_{\rho(x)-1} + d_1(x) e_{n-1}$$

We also put the root of T at the apex of the tetrahedral-prism–based pyramid: $\alpha(\perp) := e_n$. Note that $\alpha\colon T \to \Psi_n$.

With these points singled out, we can now define the 'thin slices' which partition Ψ_n into regions. In Figure 4.8 for example, these are the lines labelled X, Y and Z. These thin slices are constructed using the same technique as was used for general geometric realisation (Definition 2.8), building up a simplicial complex using the nerve $\mathcal{N}(T)$. To each $X \in \mathcal{N}(T)$, associate the $(|X|-1)$-simplex:

$$\sigma(X) := \operatorname{Conv} \alpha[X]$$

Now let:

$$\Sigma := \{\sigma(X) \mid X \in \mathcal{N}(T)\}$$

To see that Σ is a simplicial complex, we can use Lemma 4.33. Take disjoint $X, Y \in \mathcal{N}(T)$ with the aim of showing that $\sigma(X) \cap \sigma(Y) = \varnothing$. Note that it suffices to assume that X is maximal in T and that Y is maximal in $T \setminus X$. Hence there is $w \in X$ and $x, y \in \operatorname{Succ}(w)$ such that:

$$X \cap \Uparrow(w) \subseteq \uparrow(x), \quad Y \subseteq \uparrow(y)$$

Let $m := \mathsf{height}(w)$. Now, d is vertically-distinguishing. Without loss of generality, assume that:

$$\forall x_0 \in \uparrow(x)\colon \forall y_0 \in \uparrow(y)\colon d_1(x_0) < d_1(y_0)$$

By construction this means that for any $(a_1, \dots, a_n) \in \sigma(X \cap \uparrow(x))$ and $(b_1, \dots, b_n) \in \sigma(Y)$ we have:

$$a_{n-1} < b_{n-1} \tag{$*$}$$

Now, suppose for a contradiction that there is:

$$c = (c_1, \dots, c_n) \in \sigma(X) \cap \sigma(Y)$$

Then, as $c \in \sigma(Y)$ and each point in $\alpha[Y]$ has zero in its first m coordinates, we must have that $c_i = 0$ for $i \leqslant m$, and moreover that $c_n = 0$. But this means that $c \in \sigma(X \cap \uparrow(x)) \cap \sigma(Y)$, contradicting $(*)$. $\lightning$

Hence Σ is a simplicial complex. Its realisation $|\Sigma|$ consists of l-many $(n-1)$-simplices with various faces identified (according to the structure of T); in particular they all meet at the apex e_n of the pyramid Ψ_n. In more detail, to each $t \in \mathsf{Top}(T)$, we can associate the $(n-1)$-simplex $\tau(t) := \sigma(\downarrow(t))$; then:

$$|\Sigma| = \bigcup_{t \in \mathsf{Top}(T)} \tau(t), \qquad e_n \in \bigcap_{t \in \mathsf{Top}(T)} \tau(t)$$

Now, take $s \in \rho_n$ (i.e. a top point of the saw). Let s have immediate predecessors t_1 and t_2. Let $\tau(s)$ be the convex polyhedron bounded by $\tau(t_1)$ and $\tau(t_2)$ inside Ψ_n (i.e. $\tau(s) := \operatorname{Conv}(\tau(t_1) \cup \tau(t_2))$). Note that:

$$\Psi_n = \bigcup_{s \in \rho_n} \tau(s)$$

We are now ready to define $f : \Psi_n \to F$. We first define the preimages of elements of T by upwards-induction on the tree structure.

- Let $f^{-1}\{\bot\} := \sigma(\downarrow(\bot)) = \{e_n\}$.
- For $x \in T$ with immediate predecessor y, let $f^{-1}\{x\} := \sigma(\downarrow(x)) \setminus \sigma(\downarrow(y))$.

Then, for $s \in \rho_n$ with immediate predecessors t_1 and t_2, let:

$$f^{-1}\{s\} := \tau(s) \setminus (\tau(t_1) \cup \tau(t_2))$$

Note that $f^{-1}\{s\} = \mathrm{Int}^{\Psi_n}(\tau(s))$ since $\tau(t_1) \cup \tau(t_2)$ is the boundary of $\tau(s)$.

Let us see that f so-defined is an open polyhedral map. First note that $f^{-1}\{s_i\}$ is an open subpolyhedron of Ψ_n. Second, for any $x \in T$ with immediate predecessor y we have that:

$$f^{-1}[\uparrow(x)] = \mathrm{Int}^{\Psi_n}\left(\bigcup\{\tau(s) \mid s \in \uparrow(x) \cap \rho_n\}\right) \setminus \sigma(\downarrow(y))$$

This is an open subpolyhedron, since $\sigma(\downarrow(y))$ is closed. Hence the preimage of an open set in F is an open subpolyhedron of Ψ_n. As to openness of f, it suffices to show that if $y < x$ in F, then $f^{-1}\{y\}$ lies in the boundary of $f^{-1}\{x\}$. But this is immediate from the construction of f. $\square$

Conclusion. We can now prove Theorem 4.10 by combining the soundness and completeness directions.

Proof of Theorem 4.10. By Theorem 4.13, we have that:

$$\mathbf{PL}_n \subseteq \mathrm{Logic}(\mathbf{CP}_n)$$

Conversely, by Theorem 4.15:

$$\mathrm{Logic}(\mathbf{CP}_n) \subseteq \mathbf{PL}_n$$ $\square$

Chapter 5

Conclusion

The Heyting algebra $\mathrm{Sub}_0 P$ opens up a rich connection between logic and polyhedral geometry, which is given life by the sustained import of geometrical ideas. The Triangulation Lemma is a potent ingredient of this connection, and is embedded in the techniques of this thesis. Together with the link between triangulations and nerves, its first yield is a Tarski-like completeness proof for **IPC** and **S4.Grz** in the class of all polyhedra. These ideas are developed further, culminating in the powerful Nerve Criterion, a product of the unison of logic with non-trivial arguments from rational polyhedral geometry.

It is at this point that the 'Combinatorics' leg of the triad of fields is extended (see Figure 5.1). Geometry has long enjoyed a deep relationship with combinatorics, and combinatorial methods are important in the logic of Kripke frames. In this thesis, I develop an approach which brings these two connections together. The Nerve Criterion is exploited to chart out a class of polyhedrally-complete logics axiomatised by the Jankov-Fine formulas of *starlike trees*. The proof that a starlike logic is polyhedrally-complete if and only if it has the finite model property, utilises a number of combinatorial techniques on finite posets. Such logics have a clear geometric meaning and play an important part in polyhedral semantics. Indeed, the largest starlike logic of height n is shown to coincide with the logic of convex polyhedra of dimension n. The proof of this fact blends combinatorial and geometric ideas, and serves as a fitting culmination of the various strands of the present thesis.

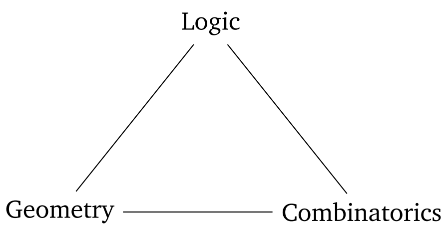


Figure 5.1: The triad of fields

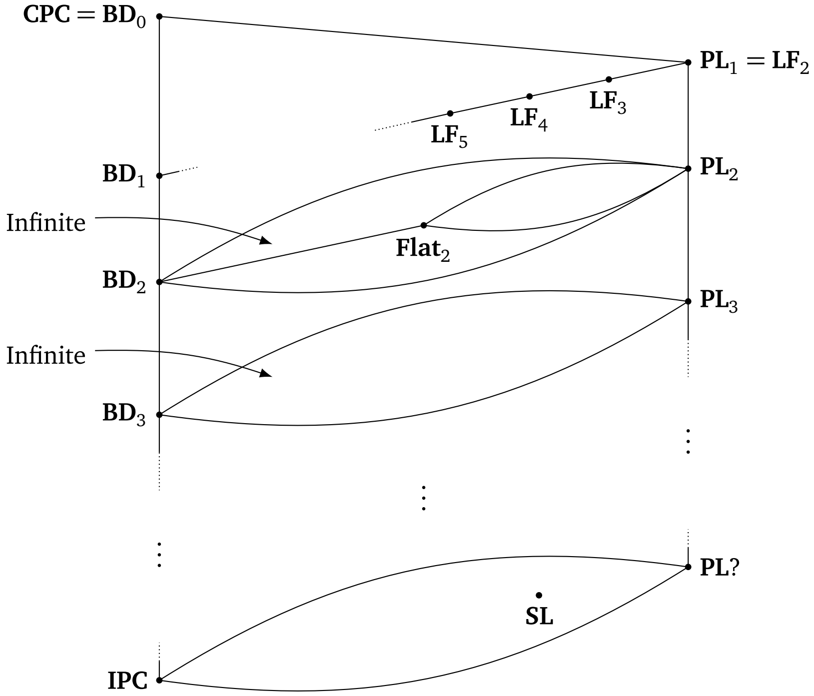


Figure 5.2: The landscape of polyhedrally-complete logics

Open Problems. Polyhedral semantics for logic is a very young area, and there are many open problems and directions for future research.

One ultimate goal would be a complete classification of polyhedrally-complete logics. The results in this thesis take several steps towards such a classification, and chart out key features of the landscape. For each finite height n, there is a smallest polyhedrally-complete logic: $\mathbf{BD}_n$ (see Theorem 3.34), a largest: $\mathbf{PL}_n$ (see Theorem 4.10 and Proposition 4.8), and infinitely-many polyhedrally-complete logics in between (Corollary 3.35). Beyond these results, David Gabelaia, Mamuka Jibladze, Evgeny Kuznetsov, and Levan Uridia recently investigated the lower-level structure of this landscape in more detail [GJKU19]. First, they show that every height-1 logic is polyhedrally-complete: these are $\mathbf{BD}_1$ plus the logic $\mathbf{LF}_k$ of the k-fork $\langle 1^k \rangle$ for each $k \geqslant 2$. Note that $\mathbf{LF}_k = \mathbf{SFL}(2, 1^{k+1})$. Second, turning to the height-2 case, they focus on logics of 'flat polygons': 2-dimensional polyhedra which can be embedded in the plane $\mathbb{R}^2$. Any such logic turns out to be axiomatised by a *subframe formula* (see [CZ97, p. 313]) plus the Jankov-Fine formulas of certain starlike trees; moreover, there is a smallest such logic: $\mathbf{Flat}_2$. See Figure 5.2 for a representation of the known landscape of polyhedrally-complete logics.

The two approaches listed in the introduction (logical and geometric) give rise to two different directions from which to tackle the classification problem. First, following on from Theorem 3.34, one way of obtaining some answers would be to determine which infinite-height starlike logics have the finite model property. Then going beyond the results in Chapter 3, it is natural to wonder (as already indicated) if polyhedral-completeness extends to logics axiomatised by the Jankov-Fine formulas of general trees, or even of arbitrary finite rooted posets. Corollary 4.12 provides a limit to this kind of axiomatisation, but beyond this little is known.

Taking the geometrical approach, the first question generated by Chapter 4 is: *what is the logic of all convex polyhedra?* The candidate is of course the following logic.

$$\mathbf{PL} := \mathbf{SFL}(2 \cdot 1, 1^3) = \mathbf{IPC} + \chi(\langle 2 \cdot 1 \rangle) + \chi(\langle 1^3 \rangle)$$

Note that if this logic has the finite model property, then by Theorem 4.10 it is indeed the logic of all convex polyhedra. Beyond convexity, the logic of other natural classes of polyhedra can by studied (such as the higher-dimensional versions of flat polygons mentioned above). Note that Proposition 1.48 reduces this to the study of PL-homeomorphism classes of polyhedra.

This last observation leads naturally to another question. Taking the modal perspective, what is the natural notion of bisimulation for polyhedra? Such a direction is in keeping with Felix Klein's Erlangen programme [Kle72] — with its emphasis on doing geometry by studying its *morphisms* — and may well help to provide bounds on the expressive power of the polyhedral semantics considered here.

In this thesis, we have examined a link between logic and geometry. One criterion for the success of such a link, beyond its naturalness, is its readiness to act as a conduit between the two fields, so that results from one area can be profitably transferred to the other. Here are two somewhat speculative ideas in this line — one in each direction. (1) After combinatorially proving the soundness of $\mathbf{PL}_n$ with respect to $\mathbf{CP}_n$ (Theorem 4.13), I remarked that the same result can be proved geometrically, using classical dimension theory. Then we have two proofs of the same thing coming from different areas, so one might hope for the possibility of supplying alternative logical-combinatorial proofs of (polyhedral) parts of dimension theory. (2) Medvedev's logic of finite problems (**ML**) is a well-known intermediate logic (see [CZ97, p. 53]), which appears to have a polyhedral character. If it can be placed within the framework of this thesis — or a similar framework — then this may open the door to the application of methods from geometric decidability theory to the question of the decidability of **ML**, which is a long-standing open problem.

On the subject of decidability, the notion of nerve-validity, introduced in Definition 3.1, may also have some bearing. After proving Theorem 3.42, I remarked on an interesting property of the formulas $\chi(\langle \alpha \rangle)$: a frame F nerve-validates $\chi(\langle \alpha \rangle)$ if and only if $\mathscr{N}(F) \vDash \chi(\langle \alpha \rangle)$. This raises an intriguing question: is it the case that, for every formula ϕ, there is $k \in \mathbb{N}$ such that F nerve-validates ϕ if and only if $\mathscr{N}^k(F) \vDash \phi$? This would entail, via Corollary 2.22, the decidability of the logic of any finite collection of polyhedra.

Lastly, the possibility of moving to a richer logic is always available to us. One motivation for this is that with the present semantics, logic cannot capture any of the homology of the polyhedron in which it is interpreted. This is because formula satisfaction is always local in a polyhedron (this fact is not so pronounced in the present thesis, where satisfaction at points of a polyhedron is eschewed in favour of the more abstract notion of triangulation). Homology seems a rather natural aspect for a logic to express; indeed, its axiomatic method is a well-developed line of research (see [Hat02, §2.3, p. 160]). Perhaps the addition of a universal modality will enable this expression.

Wrapping Up. Another fibre of the logic-geometry connection has been traced. I hope that this exciting new area of research will continue to develop and blossom into a beautiful interplay of ideas between these two fields.

List of Figures

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