Defending Classes

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In this work we describe a variety of formal theories that try to capture some of our pre-theoretical notions of collection, while being careful not to fall prey to paradox. We also present some considerations that a satisfactory theory of collections must take into account, focusing in self-instantiation and unrestricted quantification. We then offer a historical account of Cantor's notion of set as a well asserting several differences between this notion and that of a Russellian class. We will also defend the use of classes in addition to sets in our theories of collections. Finally, we asses how the theories of collections surveyed fulfill the different requirements laid out for a theory to be considered satisfactory.

I do remember an apothecary [...] and in his needy shop a tortoise hung, an alligator stuff'd, and other skins of ill-shap'd fishes; and about his shelves a beggarly account of empty boxes, green earthen pots, bladders, and musty seed remnants of packthread, and old cakes of roses, were thinly scatter'd, to make up a show.

Shakespeare, *Romeo and Juliet*

Filter
VERB
1 [*with object*] (. . .)
1.1 Process or assess (items) in order to reject those that are unwanted.
'you'll be put through to a secretary whose job it is to filter calls'
'the brain has the ability to filter out information it considers non-essential'

Oxford English Dictionary

I try all things, I achieve what I can.

Herman Melville, *Moby-Dick, or, the Whale*

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Caminante, son tus huellas
el camino y nada más;
Caminante, no hay camino,
se hace camino al andar.
Al andar se hace el camino,
y al volver la vista atrás
se ve la senda que nunca
se ha de volver a pisar.
Caminante no hay camino
sino estelas en la mar.

Antonio Machado

A las aladas almas de las rosas
del almendro de nata te requiero,
que tenemos que hablar de
muchas cosas,
compañero del alma, compañero.

Miguel Hernández

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And so, after everyone has been duly acknowledge, *let's do some science!*

Contents

Chapter 1

An unexpected wake-up call

1.1 Introduction

In the first chapter of this work we will look at some pre-theoretic ideas we might have about collections and at one of the first formalisations of them in the work of Gottlob Frege. We will also learn how Frege's proposal fails to be satisfactory given its inconsistency and draw some lessons from this useful for the further study of formal theories of collections in the remainder of this work.

The chapter starts by looking at the ideas regarding objects, properties and groups of objects that we might have prior to a formalisation attempt and that we would like our theories to capture in §1.2, we also look at the logic of such description. In §1.3 we look at Frege's pioneer formalisation of such ideas. Putting particular emphasis on the principle known as Basic Law V. This will be seen to play a key role in deriving the contradictions that undermine his theory. Finally, we look at some potential diagnosis of the inconsistency which will be informative when discussing other theories that avoid such pitfall.

1.2 What we want to capture

The structure of the world furnishes many objects and these are very different. Some are big like the stars and some small like the electrons. Some are concrete like a pencil and others are abstract like the natural numbers. Another feature of the fabric of reality are properties, or more generally relations. As in the case of objects, there are many of these and of very different nature, from the property of being a whale to that of being a number or being identical to yourself. Although objects and properties seem like very different entities, they stand nevertheless in a close relationship. Indeed, we can associate to an object the properties which it instantiates and so for instance my pencil will be tied to the property of being sharp or of being in Amsterdam. Similarly to every property we can associate the objects of which this property is true, and so in this case Moby Dick will be one of the objects associated to the property of being a whale. Note that in the same way as we can quantify over objects, for instance by saying that some objects are big or that all of them are equal to themselves, we can also talk about quantifying over properties by saying that some property is instantiated by only one object or, again, that all properties are identical to themselves.[1]

Note also that there seems to be a very close relationship between predicates and some incomplete sentences of our languages, or open formulas in the logical jargon. In a sense, the property of being a whale seems to arise from the formula '() is a whale', and so for every open formula so there is a corresponding property. There also seems to be a close tie between how objects behave when their names are used to complete the gaps in open formulas and relations, namely that if 'Moby Dick' is used to complete the sentence '() is a whale' and this is true, then the object named by 'Moby Dick' does indeed instantiate the property of being a whale.[2]

We can also look at the objects associated to some property in a new light if we consider them not individually, but as a novel kind of object, a collection or aggregate. It also seems that when we talk about collections we have an easy way to determine whether two of these are the same. Namely, it suffices to look at whether their components are the same. Such criteria is of course nonsensical when dealing with objects which are not aggregates, what the mathematical jargon calls atoms or *urelemente*. And so Moby Dick is a part, or in a less mereological fashion, a member of the object associated in this way with the property of being a whale.[3] Note that these collections are peculiar objects since to each property there seems to correspond a unique and distinctive such object, that composed of those and only those objects that instantiate such property, and so in line with this thought, it seems that to know whether two properties are the same it suffices to know whether these associated collections coincide, and conversely to know if the collections are the same it suffices to know that their associated properties are instantiated by the same objects.

1.2.1 The logic of this picture

The logic of the picture we sketched above is that of the Second-order predicate calculus. This section aims to be a quick introduction to such a system and can be obviated by the familiar reader.[4] We work with a language $\mathcal{L}$ that includes the following simple terms:

- Object names: $a, b, \ldots$
- Variable names: $x, y, \ldots$

[1] These remarks are setting the stage for the use of second order predicate calculus, see next section for more on such a system.

[2] As we will see below this is encoded in the comprehension principle of second order logic.

[3] Note however that we needn't associate to each object which is a collection a property, since for each arbitrary array of objects we can associate a new object namely that collecting them. Here no mediating appeal to a property was made.

[4] Here we follow closely the presentation in (Zalta, 2019a, section 1). The reader seeking a deeper treatment of the topic is referred to (Shapiro, 1991, §II.3).

- n-place relation names: $P^n, Q^n, \ldots (n \geq 1)$
- n-place relation variables: $F^n, G^n, \ldots (n \geq 1)$

The object names and variables denote elements in a domain of objects, and n-place relation names and variables denote elements in a domain of n-place relations. Moreover, the domains of objects and relations are disjoint. For ease in notation, we will write $P, Q, \ldots$ instead of $P^2, Q^2, \ldots$ for n-place relation names or variables for $n \leq 2$.

The formulas of $\mathcal{L}$ are built up from the simple terms as follows:

- If Π is an n-place relation term and $v_1, \ldots, v_n$ are any object terms, then $\Pi v_1 \ldots v_n$ is an atomic formula.
- If v_1 and v_2 are any object terms, $v_1 = v_2$ is an atomic formula.
- If ϕ, ψ are formulas, then $\sim\phi$ and $\phi \supset \psi$ are molecular formulas.
- Where ϕ is any formula and α any variable, then $\forall\alpha\phi$ is a quantified formula.

Note that $\mathcal{L}$ is a second-order language since the last item in the above definition allows quantified formulas of form both $\forall x\phi$ or $\forall F\phi$. We now provide the usual definitions for the other connectives and quantifiers:

- $\phi \,\&\, \psi := \sim(\phi \supset \sim\psi)$
- $\phi \vee \psi := \sim\phi \supset \psi$
- $\phi \equiv \psi := (\phi \supset \psi) \,\&\, (\psi \supset \phi))$
- $\exists\alpha\phi := \sim\forall\alpha\sim\phi$

Let ϕ, ψ and χ be any formulas of $\mathcal{L}$, and let α be any variable and τ any term of the same type as α, then our logic has the following rules and axioms:

- Axioms of propositional logic, for instance:
  - $\phi \supset (\psi \supset \phi)$
  - $(\phi \supset (\psi \supset \chi)) \supset ((\phi \supset \psi) \supset (\phi \supset \chi))$
  - $(\sim\phi \supset \sim\psi) \supset ((\sim\phi \supset \psi) \supset \phi)$
- Universal instantiation: $\forall\alpha\phi \supset \phi^\tau_\alpha$, where ϕ^τ_α is the result of uniformly substituting τ for the free occurrences of α in ϕ, and τ is substitutable for α, in the sense that no variable free in τ becomes bounded by a quantifier in ϕ^τ_α. A consequence of this principle is existential introduction, this is $\phi^\tau_\alpha \supset \exists\alpha\phi$.
- Quantifier distribution: $\forall\alpha(\phi \supset \psi) \supset (\phi \supset \forall\alpha\psi)$, where α is a variable not free in ϕ.

- Laws of identity:
  - $x = x$
  - $x = y \supset (\phi \supset \phi')$, where ϕ' is the result of substituting one or more occurrences of y for x in ϕ.
- Modus ponens (MP): from ϕ and $\phi \supset \psi$, derive ψ.
- Rule of generalisation (GEN): from ϕ, derive $\forall\alpha\phi$.

Our calculus also includes comprehension principles that guarantees the existence of an n-place relation corresponding to every open formula with n free object variables:

Comprehension Principle for n-place relations

$$\exists G \forall x_1 \dots \forall x_n (G x_0 \dots x_n \equiv \phi), \tag{1.1}$$

where ϕ is any formula in which G doesn't occur free.

With regards to notation, if ϕ is a formula with free variables $x_1, \dots, x_n$, we will name by $[\lambda x_1 \dots x_n \phi]$ the corresponding n-place relation under the comprehension principle. In accordance, then, with this choice of notation, we have the following syntactic rule, expressing that a collection of objects $y_0, \dots, y_n$ will satisfy the property $[\lambda x_0 \dots x_n \phi]$ associated with a formula ϕ if and only if the result of substituting $y_0, \dots, y_n$ for $x_0, \dots, x_n$ in ϕ yields a true sentence, as is indeed required by comprehension.

λ-Conversion

$$\forall y_0 \dots \forall y_n ([\lambda x_0 \dots x_n \phi] y_0 \dots y_n \equiv \phi^{y_0,\dots,y_n}_{x_0,\dots,x_n}) \tag{1.2}$$

Note finally, that the system described above is consistent. Indeed, we can consider a domain of objects with one element, say o, and the domain of n-place relations containing two predicates P_n and Q_n for each n, such that Po is the case but Qo is not, this is a model for the theory. What we do require is that the predicates are closed under the connectives.

1.3 Frege's system

We can find an attempt to formalise the picture described in §1.2 regarding, objects, properties and collections, in Gottlob Frege's work at the turn of the XXth century.[5] For him, as we observed above there are two fundamental and distinct entities in the world, objects and functions.[6] Most interesting for us will be what Frege called concepts. These are a specific type of function, in particular, endomorphisms[7] taking objects to two special objects namely truth

[5]For the exposition of Frege's theory in this section we follow closely (Zalta, 2019a, §2)

[6]These functions correspond to what we have above called properties and relations and this distinction is not all that relevant for our purposes.

[7]This is, a map where domain and range coincide.

and falsity.[8] Consider again the property of being a whale. Then, thought as a function, this will take Moby Dick to the value truth, but Nemo to the value false. However, note that if we see the statement Wa to mean that a instantiates the property W[9], Frege would read this as saying that the image of a under function W is the object truth, and would take Wa to be a name for this special object.

For Frege, these concepts are unsaturated in the sense that, as mentioned in §1.2 there is a corresponding sentence which is incomplete. Indeed, its corresponding name is of the form '() is a whale' and has no place in the logical syntax unless we have an indicator of an argument, which would in a sense undermine its unsaturated character. Further, as opposed to objects, functions do not exist on their own but only refer when an object is subject to it, then they denote a concept, so while '() is a whale' doesn't refer the corresponding lambda term, i.e. the name of its corresponding property, $[\lambda x W x]$ does. Although note the indication of an argument in $[\lambda x W x]$. Incidentally, this attitude can be seen in Frege's famous dictum 'never try to define the meaning of a word in isolation, but only as it is used in the context of a proposition' (Frege, 1893, p. 116).[10]

Frege also observes the correspondence we noted before between open formulas and predicates when it comes to instantiation and satisfaction. In his writings, this takes the shape of a substitution rule that allows to replace any free concept name with an open formula.[11] Now, even though for Frege function names fail to refer unless an argument is indicated in the notation, we can indeed talk about what he calls the course-of-values of a function. This is an object that encodes the value associated to every object under a function, and so in some way or another the information that Moby Dick gets mapped under being a whale to truth but Nemo to false will be present in such object. Note this is what we would actually call the function in set-theoretic parlance, indeed, the ordered pairs $(MobyDick, T), (Nemo, F)$ belong to the set-theoretic function $Whale : Obj \to \{T, F\}$, similarly the pair $(Nemo, Marlin)$ is encoded in the function 'father of ()'.

1.3.1 Basic Law V or the original sin

In the case of concepts, Frege takes the course-of-values of such entities as only encoding information about which objects are mapped to truth under it, this he calls their extension. Even more economically we can just take as the extension the collection of such objects, and so the extension of being a whale will include in it Moby Dick. If f is a concept then Frege denotes the course of values of f as $\acute{\epsilon} f(\epsilon)$, with $\acute{\epsilon}$, when applied to a name for an object $f(\epsilon)$ a term-forming

[8]Indeed to further belabour the point of footnote 6 notice that when we think of concepts as relations and not functions, these will nevertheless be functional relations defined everywhere.

[9]Hereafter we use the logical notation introduced in §1.2.1, i.e. a is an object name and W is a 1-place relation, or property, name.

[10]See (Heck & May, 2013) for more on the notion of unsaturatedness in Frege.

[11]Put more precisely, for any free concept F we can replace it by an expression of form $[\lambda x \phi(x)]$, and then use the λ-conversion principle of our logic in the substituted instances. Note also that this principle is equivalent to the comprehension principle in our logic.

operator. Returning to the collection of things satisfying it associated to a property, its extension, one question we can ask is how can we determine when two properties have the same extension, or equivalently when two concepts have the same range-of-values? Regarding this, Frege has the following criterion:

> The course-of-values of the concept f is identical to the course-of-values of the concept g if and only if f and g agree on the value of every argument (i.e., if and only if for every object x, $f(x) = g(x)$). (Zalta, 2019b, §2.4.1)

And so to know whether two concepts have the same extension we just need to know that when looked at as a function, their value agrees in all the objects in their domain. The attentive reader will notice that this is exactly capturing what we said in §1.2 regarding collections associated to properties and their identity conditions. This is Frege's infamous Basic Law Five, or put formally and slightly modifying the notation from (Zalta, 2019a, §20):

$$\acute{\epsilon} f(\epsilon) = \acute{\epsilon} g(\epsilon) \leftrightarrow \forall x(f(x) = g(x)) \quad (\mathbf{BLV})^{12}$$

Note that Frege originally formulated the law with equality as opposed to logical equivalence since, as said above, for him true sentences are just names denoting the object truth. In line with our λ notation to name concepts and slightly simplifying that of courses-of-value, we will take $\epsilon[\lambda x \phi]$ to name the extension of this concept. So while the latter names a concept the other denotes a very different thing according to Frege, namely an object. In any case, here we do not take ϵ to be a term-forming operator but rather a function symbol from concepts to objects. Note that if F is a concept variable, ϵF ranges over extensions. Note too that above we characterised extensions as the collection of objects satisfying a given concept, hence we can also employ set-theoretic notation in a natural way and so instead of $\epsilon[\lambda x \phi]$ we could naturally write $\{x|\phi(x)\}$, or $\hat{x}(\phi(x))$. Here we are talking of the notion objects satisfying a concept and this collection being the extension, and so we see in operation here the notion of membership, Frege formalises this as follows:

$$x \in y^{13} := \exists G(y = \epsilon G \;\&\; Gx) \quad (\in)$$

And so x is a member of y if it is the extension of a concept under which x falls. Above we remarked that it was natural to think that to each property there correspond the collection of things satisfying it which in our acquired terminology can be rephrased as saying that to every concept there corresponds an extension. This is indeed a principle derivable in Frege's system[14]:

Proposition 0.1 (Existence of extensions). $\forall G \exists x(x = \epsilon G)$

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[12] Note that in making this statement we are enriching our logical language with function symbols, functional application and a term-forming operator.

[13] Or $x \cap y$ in Frege's original use.

[14] Here we take this to be *SOL* with identity (see §1.2.1) supplemented by ϵ and *BLV*

Proof.

1.	$x = x$	Axiom
2.	$\forall x(x = x)$	1, GEN
3.	$\epsilon F = \epsilon F$	2, Universal instantiation
4.	$\exists x(x = \epsilon F)$	3, Existential generalisation
5.	$\forall G \exists x(x = \epsilon G)$	4, GEN

Q.E.D.

Using the notation introduced above we can now give a simpler formulation of BLV for concepts:

Basic Law V (*Concepts*)
$$\epsilon F = \epsilon G \equiv \forall x(Fx \equiv Gx)^{15} \tag{1.3}$$

And so the extension of the concept F, or collection of Fs, is the same as that of Gs if and only if the same objects fall under both.

Another natural property regarding extensions and objects that we mentioned before is that an object should be a member of the extension of a concept if and only if this object satisfies the concept to which such extension corresponds. This encodes the thought that extensions are a very special kind of object. Namely, a collection of other objects. Precisely those objects instantiating the associated property, this is what underpins that to each concept there corresponds a unique extension. This property is indeed provable in Frege's system:

Proposition 0.2 (Law of extensions)**.** $\forall G \forall x(x \in \epsilon G \equiv Gx)$

Proof. Let c be an arbitrary object and F an arbitrary concept, we want to show that $c \in \epsilon F \equiv Fc$.

($\Rightarrow$) Suppose then that $c \in \epsilon F$, and so by def. of $\in$ we get that $\exists G(\epsilon G = \epsilon F \& Gc)$, let this concept be H, $\epsilon H = \epsilon F \& Hc$ so we have that but then by BLV we get that $\forall x(Hx \equiv Fx)$ since $\epsilon H = \epsilon F$, and so Hc implies that Fc.

($\Leftarrow$) Suppose now that Fc, then by the existence of extension and identity we have $\epsilon F = \epsilon F$ and so by conjunction introduction that $\epsilon F = \epsilon F \& Fc$ and by existential introduction that $\exists G(\epsilon F = \epsilon G \;\&\; Gc)$ and so by definition that $c \in \epsilon F$. Q.E.D.

Given the nature of extensions and the fact that these are characterised by the objects falling under the property they arise from, and as observed above these are exactly the objects that are members of the extension, one should only need to know the members of two extensions to determine their equality. This is the principle of extensionality and it is indeed derivable in Frege's system.

Proposition 0.3 (Principle of extensionality).

$$Extension(x)\ \&\ Extension(y) \supset (\forall z(z \in x \equiv z \in y) \supset x = y)$$

Proof. We have by definition that $\exists G(x = \epsilon G)$ and $\exists F(y = \epsilon F)$, since we also have that $\forall z(z \in x \equiv z \in y)$ and so that $\forall z(z \in \epsilon G \equiv z \in \epsilon F)$, then from the law of extensions it follows that $\forall w(Fw \equiv Gw)$ and so by BLV we have that $\epsilon F = \epsilon G$ and so that $x = y$. Q.E.D.

1.3.2 In contradiction

It turns out however, that Frege's system is inconsistent. This was communicated to him by Bertrand Russell, hence the name of the subsequent contradiction being known as Russell's paradox. Frege added two derivations of the contradiction to his *Grundgesetze*, we now look at these, the first is more direct, the second might be more familiar us modern readers.

The first way to derive a contradiction Frege offers relates to the extension associated to the concept *being an x that is the extension of some concept under which x doesn't fall under*. This concept exists by the comprehension principle. Using our notation we denote this by $[\lambda x \exists F(x = \epsilon F \& {\sim} Fx)]$. We also know by the extension principle that there is an extension associated to this concept, namely $\epsilon[\lambda x \exists F(x = \epsilon F \& {\sim} Fx)]$, but then we can show that this extension falls under its associated concept if and only if only it does not. Let us abbreviate $[\lambda x \exists F(x = \epsilon F \& {\sim} Fx)]$ by R, then

Proposition 0.4 (Russell's paradox, first derivation).

$$R(\epsilon R) \equiv {\sim} R(\epsilon R)$$

Proof. ($\Rightarrow$) Suppose that $R(\epsilon R)$ which is shorthand for $[\lambda x \exists F(x = \epsilon F \& {\sim} Fx)](\epsilon R)$, then by λ-conversion $\exists F(\epsilon R = \epsilon F \& {\sim} F(\epsilon R))$, take this concept to be H, then we have that $\epsilon R = \epsilon H$ and ${\sim} H(\epsilon R)$, then by BLV we conclude that ${\sim} R(\epsilon R)$.

($\Leftarrow$) Suppose that ${\sim} R(\epsilon R)$, by λ-conversion ${\sim}(\exists F(\epsilon R = \epsilon F \& {\sim} F(\epsilon R)))$ and so $\forall F {\sim}(\epsilon R = \epsilon F \& {\sim} F(\epsilon R)))$, this is $\forall F(\epsilon R = \epsilon F \supset F(\epsilon R)))$, instantiating with R, $\epsilon R = \epsilon R \supset R(\epsilon R))$, and since $\epsilon R = \epsilon R$ we conclude that $R(\epsilon R)$. Q.E.D.

The second derivation of the contradiction makes use of membership and so is more familiar to what today would appear as Russell's paradox in, say, a set-theory textbook. First note that in Frege's system one can derive the following result regarding the existence of an object, namely an extension, that has as members those and only those elements instantiating a given concept.

Proposition 0.5 (Naive comprehension for extensions).

$$\forall F \exists y \forall x(x \in y \equiv Fx)$$

Proof. By an instantiation of the law of extensions for F, $\forall x(x \in \epsilon F \equiv Fx)$, by existential weakening, $\exists y \forall x(x \in \epsilon F \equiv Fx)$, and so by universal generalisation $\forall F \exists y \forall x(x \in \epsilon F \equiv Fx)$. Q.E.D.

Note that we can now consider the predicate $[\lambda z z \notin x]$, abbreviated by R, and show the following

Proposition 0.6 (Russell's paradox, second derivation)**.**

$$\exists x(x \in x \equiv x \notin x)$$

Proof. Apply naive comprehension to R to get $\exists y \forall x(x \in y \equiv [\lambda z z \notin z]x)$, take this object to be r, then $\forall x(x \in r \equiv [\lambda z z \notin z]x)$ and instantiate with r, so $r \in r \equiv [\lambda z z \notin z]r$ apply $\lambda-$conversion to get $r \in r \equiv r \notin r$ and apply existential weakening. Q.E.D.

1.3.3 Drawing conclusions

We now return to the title of this section. I have tried to indicate why we might think of the system described above as formalising some innocent and fairly uncontroversial ideas about objects, concepts and their corresponding extensions. It turns out, however, that Frege's system is inconsistent. This is at least a wake up call in the sense that it forces us to reevaluate our formalisation of our ideas about collections, given that our first attempt at it has failed and so more care than one might have expected is required. It is also unexpected in that the principles used in the formalisation are the ones seemingly innocent of SOL and the equally seemingly sensible BLV.

When it comes to an attempted diagnosis of the root of the problem it seems there is a tension between the comprehension principle, ensuring that there is a concept corresponding to each open formula, the existence of extensions principle which ensures that to each concept there corresponds an extension, and BLV which rules the relationship between extensions and concepts. More explicitly the problem seems to be that this framework requires, on the one hand to be the same number of concepts and extensions, but on the other it requires more concepts than there are objects. Indeed, if we take, the seemingly innocuous view of concepts in which these are exhausted by the objects that instantiate them. Namely $F = G \equiv \forall x(Fx = Gx)$, a so-called extensional view. Then, by the principle of extensions, to each concept F there corresponds an extension ϵF. It is now convenient to break up BLV into its two different implications in order to evaluate the constraints it poses in this correspondence:

BLVa

$$\forall x(Fx \equiv Gx) \supset \epsilon F = \epsilon G \tag{1.4}$$

BLVb

$$\epsilon F = \epsilon G \supset \forall x(Fx \equiv Gx) \tag{1.5}$$

According to the contrapositive of $BLVa$, if two extensions differ there is some object instantiating one of them that doesn't instantiate the other, and so given our extensional view of concepts these are different, and thus no same concept will be assigned to two different extensions, and so this correspondence is functional. More problematic seems the condition imposed by $BLVb$, namely that this function be injective. Indeed, consider the contrapositive of this principle, it states that if there is some object that falls under one concept but not another, and so that under our extensional view this means that the concepts are different, then the extensions associated with these will in turn differ. Thus, there are cannot be more concepts than extensions.

Note however, that when we conjoin BLV with the comprehension principle of SOL we need there to be more concepts than objects, and so in particular than extensions. Recall that the comprehension principle for concepts states that for any open formula in the language, or in other words for any expressible condition on objects, there is a corresponding concept. Now, although it may seem that this is enough to demand more concepts than, at least $2^{|objects|}$, there are indeed non-standard models of the theory with the same number of concepts and objects, that take advantage of the fact that the language can only express countably infinite many conditions and so we can take a countable infinite amount of objects to attain this feature. The problematic element is thus the addition of BLV, and the indefinitely many new concepts that it generates by reference to the extensions it creates.

In the next chapter we will provide a survey of different approaches taken by several authors in order to attain an inconsistency-free theory of extensions, or collections, sets, classes, and so on. There are several avenues open, from revisions of the logic to the modification of the second-order comprehension principle putting some constraints on the formulas that have a corresponding extension. In any case, we can at this point note that there will be a trade-off between the apparent compelling nature the tenets of Frege's theory $(SOL + BLV)$ and its simplicity, and the freedom from paradox of the axiomatic theories that try to salvage Frege's attempt at formalising our ideas when it comes to the notion such as formula, property or collection of things satisfying it among others.

1.4 Summary

In this chapter we gave an account of what our informal ideas about collections might be as well as introducing their formalisation by Frege, we also highlighted the inconsistency of such a system and gave some reasons as to why this is the case which consistent theories should take on board.

We began in §1.2 by looking at the ideas about objects, properties and their associated collections of objects satisfying them that we might hold before embarking on a theoretical trip to formalise them. In §1.2.1 we looked in some detail at second order logic, the logical system that seems to encode some of the thoughts expressed at the beginning of the section regarding properties.

In §1.3 we looked at the system formalising the ideas in the previous section by Frege, in §1.3.1 we placed especial emphasis on Frege's Basic Law V, a principle giving the conditions for the equality of two extensions given the properties they arise from. Although noting the plausibility of such a principle in §1.3.2 we gave two derivations of a contradiction that use such law. Finally, in §1.3.3 we made some remarks as to the reasons for such contradictions appearing in Frege's system regarding concepts and their associated extensions. As we also mentioned these warnings will have to be taken on board by the theories of collections that aim to capture our ideas about these entities without falling prey to inconsistency as in the case of Frege.

Chapter 2

A taxonomy of theories of collections

2.1 Introduction

In §1 we saw how Frege's attempt at formalising what we could think of as our informal ideas about collections might failed due to inconsistency. In this chapter we will present some theories of collections that avoid the paradoxes that affected Frege's system while still trying to stay faithful, as much as possible at least given that the shadow of contradiction always lurks in the background, to the ideas we want to capture with such a formalisation. Although evidently not exhaustive, it is hoped that the systems here chosen serve as a good sample space of the different approaches that might be taken in this respect, these include typed theories, theories of just sets, sets and classes or just classes or theories using non-classical logic.

The chapter starts by looking in §2.2 at the pioneering theory of collections trying to preserve consistency after the discovery of the paradoxes, this is of course Russell's theory of types here problematic formulas such as the problematic $x \notin x$ will be banned from the syntax since they do not respect some type restrictions imposed in the language.

Next we will look at Quine's New Foundations in §2.3, although still the type theoretic flavour will be present when dealing with paradoxical properties, in this theory variables are not typed and so we find entities such as the universal collection, not existing in Russell's type theory.

After that in §2.4 we examine the most popular theory of collections today among mathematicians, that of Zermelo and supplemented by the axioms of choice and regularity, we will see how here the paradoxes are avoided by restricting comprehension to already existing sets, and so the Russell contradiction informs us that there is no universal set in this theory, thus we will see that this theory encodes a doctrine of limitation of size, namely that some things are too big to constitute sets.

After that in §2.5 we look at the standard way to introduce classes into the picture of the theory of collections this is the system of von Neumann and incorporates to Zermelo's system proper classes which are objects that are not members of other objects, this adds expressive power to the theory. Here, the paradoxes are then taken to establish that some collections will fail to be sets, they will be only classes.

In §2.6 we study the theory of Ackermann, in which unlike in that of von Neumann one can not define the set predicate, this is because Ackermann takes seriously the idea that sets are generated through an open ended process that never concludes and so at no given point we can fix for good the extension of the set predicate, here we will also discuss how this thoughts can be formalised through Kripke semantics for set theory, we will also see how unlike in Zermelo's theory limitation of size principles will not play a main role here.

We then move to §2.7 where we look at a more recent theory of classes that of Schindler, here we will have a class corresponding to every formula including the paradoxical ones. Moreover, we will also have classical logic as our system in this theory, however, although bivalence will hold, Schindler draws our attention to the notion of range of significance. This informs us of which classes makes sense to predicate of some property and which does not, hence the paradox here is telling us that the Russell class is not in its range of significance of itself, Schindler goes on to read off that the contradictions are due to some sort of circularity as in the case just highlighted. We also look at a consistency proof of the theory.

Finally, in §2.8 we look at Maddy's theory of classes as in the case of Schindler's here we will have a class for every property, the difference is that now we drop classical logic and adopt a paracomplete logic, Maddy then shows how using an analogous solution as that produced by Kripke to the paradoxes of truth one can construct a hierarchy of models that thinks that the paradoxical Russell statement is neither true nor false but undecided. We also look into an axiomatisation of the theory that turns out to be incomplete and unsound.

2.2 Russell: a different type of language

Chronologically, it seems fitting that since it was Russell who communicated Frege the presence of an inconsistency in his system we start our survey of alternative formulations of theories of collections with his proposal.[1]

Russell notes that as mentioned in our second derivation of the inconsistency in §1.3.2, we the concept that lands us in trouble is that of being a member of itself. For him, the trouble lies in that formulas such as $x \in x$ or $x \notin x$ are problematic, in the sense that they do not determine a concept, or the that the concept they determine fails to have an extension. He takes issue with the naive

[1]Here we follow the textbook treatment in (Mendelson, 1997, pp. 289-293), for the original from Russell consult (Russell, 1908).

comprehension principle for extensions explained in §1.3.2. Note, however, that these are not the only formulas that one must be careful about, but also more generally any of the form $x_0 \in x_1$ & $x_1 \in x_2$ & $\ldots$ & $x_{n-1} \in x_n$ & $x_n \in x_0$ for $n \geq 1$ ought to be put in quarantine.

In order to achieve get rid of statements embodying these forms of circularity, Russell proposes us to think of the universe of objects as being stratified into countable infinitely many different domains called types. We start at the bottom of this hierarchy with some collection of objects which are not collections themselves and so have no members[2], these objects are of type 0. Then we have the objects which are indeed collections and have as elements objects with no members, i.e. objects of type 0, these are the objects of type 1. And so in general, for any $n > 1$, objects of type n will be collections whose members are of type $n - 1$.

Note that in the same way as in SOL we distinguish the variables ranging over objects and those ranging over relations, in the language of the theory of types, which is FOL with the membership relation $\in$, we distinguish each variable according to the type of the objects it ranges over. And so x^n will range over objects of type n and, importantly, all variables are typed. The atomic formulas then will look like this $x^n \in y^n + 1$, and the rest of the formulas are built as usual with connectives and quantifiers. Note then that Russell's aim of blocking formulas such as $x \in x$ is achieved, since both arguments in the relation share the type which is not allowed by the syntax.

However, one should be careful when defining the equality, since this should be relativised to each type. In particular, we will take two objects of the same type to be equal if and only if they belong to exactly the same objects of the subsequent type:

Typed definition of equality

$$x^n = y^n := \forall z^{n+1}(x^n \in z^{n+1} \equiv y^n \in z^{n+1}) \tag{2.1}$$

Moreover, we have the usual axiom of extensionality, but again relative to our typed framework:

Typed axiom of extensionality

$$\forall y^{n+1} \forall z^{n+1}(\forall x^n(x^n \in y^{n+1} \equiv x^n \in y^{n+1}) \supset y^{n+1} = z^{n+1}) \tag{2.2}$$

We see that the axiom above is not informative enough for members of type 0, and so the preceding definition of equality is also required. Now we turn to the comprehension axiom schema. This guarantees that for each formula there is an associated collection of elements satisfying it, or in a more Fregean parlance, that there is an extension associated to each concept associated to a

[2]These are called atoms or urelements in the set theoretic jargon.

formula. The caveat is of course that attention should be paid to the types, and so what we get is that for each open formula, there is an associated collection of objects of a given type that satisfy the formula, this collection will be in turn of the next higher type:

Typed schema of comprehension

$$\exists y^{n+1} \forall x^n (x^n \in y^{n+1} \equiv \phi(x^n)) \tag{2.3}$$

For any ϕ in the language in which y^{n+1} is not free.

Moreover, by extensionality, this collection is unique and we can denote it by $\{x^n | \phi(x^n)\}$ using set-builder notation.

Now, note that when in this typed framework we define the usual set-theoretic constructions we will get a distinct copy of each object corresponding to each of the types. So, for instance, the comprehension schema provides an empty set for each non-zero type by applying it to $x^n \neq x^n$, say $\emptyset^{n+1}$, and so there is no such thing as *the* empty set. Similar remarks apply to the natural numbers. This fact makes it cumbersome for mathematicians to work in type theory, more worryingly for Russell, one cannot prove the existence of an infinite set in this framework and this must be introduced as an axiom. This is a concern for his logicist project. To formulate the axiom we will need the usual Kuratowski definition of ordered pair $(x^n, y^n) := \{\{x^n\}, \{x^n, y^n\}\}$, where $\{x^n, y^n\} = \{u^n | u^n = x^n \vee u^n = y^n\}$. Note then that (x^n, y^n) has type $n+2$, thus a binary relation on a set A, being a set of ordered pairs will have type $2 + type(A)$, and so a relation on the universe of sets of type 0, $V^1 := u^0 | u^0 = u^0$ will have type 3. Now we are ready to formulate the axiom:

Typed axiom of infinity

$$\exists x^3((\exists u^0 \exists v^0((u^0, v^0) \in x^3)) \;\&\; \forall u^0 \forall v^0 \forall w^0((u^0, u^0) \notin x^3 \;\&\; ((u^0, v^0) \in x^3 \;\&\; (v^0, w^0) \in x^3 \supset (u^0, w^0) \in x^3) \;\&\; ((u^0, v^0) \in x^3 \supset \exists z^0((v^0, z^0) \in x^3)))) \tag{2.4}$$

The axiom above states that there is a non-empty, irreflexive, transitive binary relation on V^1, x^3, then we can conclude that there are infinitely many objects of type 0.

The theory with the three axioms above in the language we described is known as Russell's simple theory of types, or ST. Russell presented a different theory of types in his monumental work with Alfred North Whitehead *Principia Mathematica*, the ramified theory of types. However, his concern when producing such a system were not the paradoxes arising from Frege's framework but the impredicativity of the definitions of set allowed by ST.[3] Although ubiquitous in mathematics, some authors such as Russell, Whithehead or Poincaré

[3] Recall that we say a set impredicative if it is defined with a formula in which we can find a quantified variable that ranges over a collection that contains the defined set

doubted the legitimacy of such definitions. To avoid them, the theory of types is supplemented by adding a hierarchy of orders for each type higher than 0, this theory has a further axiom, that of reducibility which allows us to collapse the hierarchy of orders to its lowest level in order that the mathematical challenges of the approach be overcome. Moreover, ST has been expanded with other axioms, for instance those of Peano Arithmetic at type 0 are used by Gödel to prove incompleteness, and also by expanding the hierarchy of types to higher infinities.

2.3 Quine: a new type of foundation

In the previous section we mentioned that although solving the paradoxes,ST is a cumbersome theory for doing mathematics. One of the philosophers that tried to simplify the theory of types will still retaining a similar solutions to the paradoxes tenets was Willard Van Orman Quine. Quine's type theory is called New Foundations,[4] NF, crucially, this theory uses a single type of variable and the predicate symbol $\in$. The definition of equality is the same as for ST, as well as the extensionality axiom, although now there is no need to be careful with types:

NF definition of equality

$$x = y := \forall z(x \in z \equiv y \in z) \tag{2.5}$$

NF axiom of extensionality

$$\forall y \forall z(\forall x(x \in y \equiv x \in y) \supset y = z) \tag{2.6}$$

Now, in order to write an axiom of comprehension that avoids paradoxical collections, in a theory like NF where we cannot make use of the resources that the typed variables provide us in ST, we will use the notion of a formula being stratified.

Definition 1 (*Stratified formula*)**.** We say that a formula ϕ in our language is stratified if one can assign natural numbers to the variables of ϕ so that:

1. All occurrences of the same free variable are assigned the same number.

2. All occurrences of a variable bound by the same quantifier are assigned the same number.

3. Any subformula of the form $x \in y$ of ϕ, is such that the number assigned to y is the successor of that assigned to x.

With this definition in place we can now formulate the comprehension axiom as follows:

[4]Here we follow the textbook treatment in (Mendelson, 1997, pp. 293-297), for the original from Quine see (Quine, 1937).

NF schema of comprehension

$$\exists y \forall x(x \in y \equiv \phi(x)) \tag{2.7}$$

For any ϕ in the language which is stratified.

Note that even if the axiomatisation of the theory is infinite as presented here it can be shown that one can give a finite axiomatisation of NF.[5] It is now obvious that the paradox of Russell does not arise, since the formula $x \in x$ is not stratified and so there is no extension corresponding to it given the comprehension principle just stated.

Regarding the mathematical power of the system, in the same way as in Russell's theory of types, on can construct here in a fashion way the usual set-theoretic objects. The difference however is that here we will not have as before a copy of each object in every type, and so there is for instance a unique empty set $\emptyset$ and a unique universal set V as opposed to one for each type. Now by considering V we can see a distinguishing feature of NF with respect to ST, namely that some collections are self-membered:

Proposition 0.7. $V \in V$

Proof. By def of V we have by comprehension that $\forall x(x \in V \equiv x = x)$, but since $V = V$, then $V \in V$. Q.E.D.

The proposition above is not the only difference between NF and the more usual theories of sets. In fact, in NF the proof of Cantor's theorem that $\mathbb{P}(A) > A$ does not go through since this uses a set arising from a non-stratified formula. As a result, this undermines the proof of Cantor's result to the effect that there is no biggest cardinal. If NF proved $\mathbb{P}(A) > A$, then since $\mathbb{P}(V) = V$, a contradiction would arise. Indeed, in NF $f : u \mapsto \{u\}$ is not a stratified formula. A theory with such a property is called not strongly Cantorian. What we can prove in NF, is the peculiar property that $S(A) := \{x | \exists u(u \in A \,\&\, x = \{u\})\} < V$, and so that $V > S(V)$, so V has the property that it is not equinumerous with the set of singletons of its elements. Regarding the usual axioms, NF can disprove the axiom of choice and prove the axiom of infinity, which is taken for some mathematicians as too strong of a result, casting a shadow of inconsistency over the system. However there is a recent paper claiming its consistency.[6] This system is also attractive as a basis for a foundation of category theory.[7]

Another system proposed by Quine is ML, this has a language with capitalised variables, $X, Y, Z, \ldots$, called class variables, in this theory we distinguish some special kind of class, called the sets. Sets are those classes that have other classes as members, i.e. $S(X) := \exists Y(X \in Y)$, we introduce lower case variables to range over sets. Equality, is defined as follows:

[5] The first such formalisation is that in (Hailperin, 1944).
[6] See (Holmes, 2015).
[7] See (Feferman, 1974)

ML definition of equality
$$X = Y := \forall Z(Z \in X \equiv Z \in Y) \tag{2.8}$$

We then have the following axiom of equality:

ML axiom of equality
$$\forall X \forall Y \forall Z((X = Y \,\&\, Z \in X) \supset Y \in Z) \tag{2.9}$$

We also have unrestricted comprehension for classes:

ML schema of class comprehension
$$\exists Y \forall x(x \in Y \equiv \phi(x)) \tag{2.10}$$
For any ϕ in the language.

And a stratified comprehension principle for sets:

NF schema of comprehension
$$\forall y_1 \dots y_n \exists z \forall x(x \in z \equiv \phi(x, y_1, \dots, y_n)) \tag{2.11}$$
For any ϕ in the language which is stratified and has free variables $x, y_1, \dots, y_n$ and whose quantifiers range over sets.

ML is a conservative extension of NF, its advantages include ease and power of expression, as well as better behaved natural numbers and the ability to prove the full generality of induction.

2.4 Zermelo: the mathematician's theory of choice

Leaving theories of a more type theoretical flavour aside, we now look at the first and the one which has become the most popular theory from which to carry out mathematics, Ernst Zermelo's axiomatic theory of sets, Z[8]. The only objects in this theory will be sets, the language of the theory is *FOL* with a membership predicate $\in$. The definition of equality is the same as in ML:

Z definition of equality
$$x = y := \forall z(z \in x \equiv z \in y) \tag{2.12}$$

The list of axioms are then as follows:[9] [10]

[8] Here we follow the brief exposition in (Mendelson, 1997, pp. 288-9), for a more detailed treatment the reader is referred to (Jech, 2003), for the original paper from Zermelo one might consult (Zermelo, 1908).

[9] With $u \subset x := \forall y(y \in u \supset y \in x)$

[10] With $u \cup x$ the set resulting from applying union to the set resulting from applying ordered pairs to u and x.

Substitutivity of equality
$x = y \supset (x \in z \equiv y \in z)$
Pairing
$\forall x \forall y \exists z \forall u (u \in z \equiv u = x \vee u = y)$
Emptyset
$\exists x \forall y (y \notin x)$
Union
$\exists y \forall u (u \in y \equiv \exists v (u \in v \,\&\, v \in x))$
Power set
$\forall x \exists y \forall u (u \in y \equiv u \subseteq x)$
Separation
$\forall x \exists y \forall u (u \in y \equiv (u \in x \,\&\, \phi(x)))$
For any ϕ in the language, with y not free in ϕ.
Infinity
$\exists x (\emptyset \in x \,\&\, \forall z (z \in x \supset z \cup \{z\} \in x))$

Note here that Russell's paradox is blocked by the fact that comprehension is restricted to subsets of some given set. So given the nature of the Russell property this informs us that the universal set does not exist in this theory.

The intention of the theory is, from an emptyset and an infinite set, build using the axioms above all the sets required to equip mathematicians with the tools required to carry out their investigations, what is called the iterative, or von Neumann, hierarchy of sets. It turns out, however, that Z is too weak of a theory, even though we can show that for all n, the ordinal $\omega + n$ exists, we can not show that the set of all such ordinals, and so its least upper bound, $\omega + \omega$ does exist. Abraham Frankel proposed a way out of this difficulty[11] and this was recast by Skolem in the language of Z[12], this is the axiom schema of replacement:[13]

Replacement
$$Fun(\phi) \supset \forall w \exists z \forall v (v \in z \equiv \exists u (u \in w \,\&\, \phi(u, v))) \tag{2.13}$$
For any $\phi(x, y)$ in the language.

This schema tells us that if we have a function whose domain is a set, also its range will be a set. This theory is known as Zermelo-Frankel set theory, ZF, and it is usually understood as containing also an axiom of regularity banning cycles of membership in sets:[14]

[11] See (Fraenkel, 1922)
[12] See (Skolem, 1922).
[13] With $Fun(\phi) := \forall x \forall u \forall v (\phi(x, u) \,\&\, \phi(x, v) \supset u = v)$.
[14] With $y \cap x$ the set resulting from applying comprehension to $y \cup x$ under $z \in x \,\&\, z \in y$.

Regularity

$$x \neq \emptyset \supset \exists y(y \in x \,\&\, y \cap x = \emptyset \tag{2.14}$$

If one also adds the axiom of choice one gets the theory known as ZFC which is the standard foundation for mathematics today. The axiom of choice states that for any set there is a function, the choice function for the set, such that for any non-empty subset of this set the image of the subset under the function is a member of the subset, this statement is equivalent to saying that for any set of pairwise disjoint non-empty sets there is a set, the choice set, containing one member from each of the sets in the original sets and only these, or formally:

Choice

$$\forall u(u \in x \supset u \neq \emptyset \,\&\, \forall v(v \in x \,\&\, v \neq u \supset v \cap u = \emptyset)) \supset \\ \exists y \forall u(u \in x \supset \exists! w(w \in u \cap y)) \tag{2.15}$$

Briefly returning to ST, we can associate to it a first-order theory ST* with a predicate for membership $\in$ and predicates T_n corresponding to each type, such that $\forall x^n \phi(x^n)$ will be translated to $\forall x(T_n(x) \supset \phi(x))$ and if $x^{k_1}, \ldots, x^{k_l}$ are free variables of ϕ, we prefix to the result $T_{k_i}(x_1) \,\&\, , \ldots, \,\&\, T_{k_l}(x_l) \supset$ and changing each x_i^k to x_i, the axioms of this new theory are the translations of those of ST, then ZFC is stronger than ST, in the sense that it can prove the consistency of ST*.

2.5 von Neumann: the classical theory of classes

The standard way to enlarge the ontology of collections in an axiomatic theory with the inclusion of classes is, by supplementing the sets present in ZFC following the theory laid out by von Neumann and revised and simplified by mathematicians like Bernays and Gödel, from them it is that this theory receives its name, NBG.[15] The language of NBG is a FOL with equality and the membership predicate $\in$. Note that as we did previously here we use capitalised variables to range over classes, then we also have two predicate symbols, C and S to express when a collection is a class or a set, we then define when a collection is a proper class, meaning that it is a class but not a set:

NBG definition of proper class

$$PC(X) := \sim S(X) \tag{2.16}$$

We then use lower case to denote variables ranging over sets, $\forall x \phi(x)$ abbreviates $\forall X(S(X) \supset \phi(X))$ for instance. We now look at some axioms dealing

[15]Here we follow the exposition in (Mendelson, 1997, pp. 225-87), for a more primary source one can consult the following monograph by Gödel (Gödel, 1940).

with the relation between sets, classes and proper classes:

Everything is a class
$\forall X C(X)$
Elements of classes are sets
$\forall X \forall Y (X \in Y \supset S(X))$

We also have the usual axiom of extensionality, stating that there is nothing more to classes than the sets it contains:

Extensionality
$\forall X \forall Y \forall u((u \in X \equiv u \in Y) \supset X = Y)$

We then have class existence axioms, some of them analogous to those of ZFC:

$\in$-reduction
$\exists X \forall u \forall v((u, v) \in X \equiv u \in v)$
Set pairing
$\forall x \forall y \exists z \forall u(u \in z \equiv u = x \vee u = y)$
Emptyset
$\exists x \forall y(y \notin x)$
Set Union
$\exists y \forall u(u \in y \equiv \exists v(u \in v \,\&\, v \in x))$
Set Power set
$\forall x \exists y \forall u(u \in y \equiv u \subseteq x)$
Domain
$\forall X \exists Z \forall u(u \in Z \equiv \exists v((u, v) \in X))$
Infinity
$\exists x(\emptyset \in x \,\&\, \forall z(z \in x \supset z \cup \{z\} \in x))$
Class intersection
$\forall X \forall Y \exists Z \forall u(u \in Z \equiv u \in X \,\&\, u \in Y)$
Class complement
$\forall X \exists Z \forall u(u \in Z \equiv u \notin X)$
Ordered tuples
(1.)$\forall X \exists Z \forall u \forall v((u, v) \in Z \equiv u \in X)$
(2.)$\forall X \exists Z \forall u \forall v \forall w((u, v, w) \in Z \equiv (v, w, u) \in X)$
(3.)$\forall X \exists Z \forall u \forall v \forall w((u, v, w) \in Z \equiv (u, w, v) \in X)$
Replacement
$\forall x \forall F(Fun(F) \supset \exists z \forall v(v \in z \equiv \exists u(u \in x \,\&\, (u, v) \in F))$
Regularity
$\forall X(X \neq \emptyset \supset \exists y(y \in X \,\&\, y \cap X = \emptyset))$
Choice
$\forall x \exists X(Fun(X) \,\&\, \forall y(y \subseteq x \,\&\, y \neq \emptyset) \supset \exists z((y, z) \in F \,\&\, z \in y))$

Note that in contrast with ZFC, NBG is a finitely axiomatisable theory, since the distinction between sets and classes allow us to formulate replacement with a single axiom, for instance. Now, given our interest in avoiding paradoxes and assessing at what cost different axiomatic theories do this, we are keen on finding out to which extent is comprehension restricted in this theory. It turns out that one can prove that there is a class corresponding to any formula in which only set variables are bound by quantifiers. This we call, for these purposes, a predicative formula.

Class existence theorem
$\exists Z \forall X_1 \ldots X_n((x_1, \ldots, x_n) \in Z \equiv \phi(x_1, \ldots, x_n, Y_1, \ldots, Y_n))$
For any predicative $\phi(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ in the language.

Proof. The proof proceeds by induction on the number, say k, of connectives and quantifiers. Note that we can assume formulas do not contain a subformula of the form $Y_i \in W$ since this can be replaced by $\exists(x = Y_i \& x \in W)$. Further, we can also assume no formula of form $X \in X$ occurs. We now give a brief sketch of the proof. The base case, when $k = 0$ has three different subcases, when $\phi := x_i, \phi := x_i \in x_j$ and $\phi := x_i \in Y_i$, the result then follows by applying the axioms of $\in$-reduction, intersection, complement, domain and ordered tuples and domain. We also have three cases in the induction step $\phi := \sim\psi, \phi := \psi \supset \chi$ and $\phi := \forall x \psi$, which can be proven again by using the axioms mentioned before. Q.E.D.

Note that instead of a finite axiomatisation of the theory we could have chosen to substitute the axioms used in the proof of the class existence theorem by an axiom stating this result. Now, we have by the class existence theorem, there is a class $Y := \{x | x \notin x\}$ s.t. $\forall x(x \in Y \equiv x \notin x)$, i.e. $\forall X(S(X) \supset (X \in Y \equiv X \notin X)$. Assume then that $S(Y)$, so $Y \in Y \equiv Y \notin Y$ so $Y \in Y \,\&\, Y \notin Y$. So this contradiction informs us that the Russell collection is not a set, but a proper class. One can similarly prove that the universal class, the complement of the emptyset, is a proper class.

Similarly as with ML and NF, NBG is a conservative extension of ZFC and equiconsistent with it. Its advantages over it include ease and power of expression, or finite axiomatisation, they also differ with respect to the existence of certain models.

We now look at the theory MK, which is an strengthening of NBG with a more powerful class existence axiom schema, its name is due to mathematicians Morse and Kelley.[16]. The theory appeared in an appendix to a general topology book by Kelley and independently in writings of Morse, Mostowski and Quine. Formally, we replace the axioms of $\in$-reduction, intersection, complement, domain and ordered tuples by the following schema:

[16]Here we follow (Mendelson, 1997, p. 287), for Kelley's formulation one should consult (Kelley, 1955)

MK comprehension schema
$$\exists Y \forall x(x \in Y \equiv \phi(x)) \quad (2.17)$$
For any ϕ in the language, with Y not free in ϕ.

So note that here we have comprehension for any predicate and not only predicative formulas, MK is also stronger than NBG in the sense that it can show the consistency of the latter theory and also all its axioms, hence it is a proper extension, in fact it is as strong as NBG plus an inaccessible cardinal.

2.6 Ackermann: Imagination and existence

In this section we look at the theory of sets and classes developed by Wilhelm Ackermann[17]. This theory sits in stark contrast with the theories of ZFC and its extension to the universe of classes NBG. There the paradoxes where avoided by the axioms encoding some doctrine of limitation of size, namely that comprehension can only be applied within sets or to predicative formulas in which case some times the associated object is a proper class and not a set. As we will see, in A, the name of Ackermann's theory, there are only very lax limitation of size principles, namely that subsets and members of sets are sets, and a seemingly strong comprehension principle.

The language of A is a FOL with equality and an binary predicate for membership, $\in$ as well as a predicate S denoting being a set. The key difference between A and other theories of classes is that here we will see that being a set is not definable in terms of membership. We will use capitalised variables to range over classes and lower case variables to range through sets via the usual abbreviations. As familiar by naow, we have that a defining feature of the classes is that they are exhausted by their members:

Extensionality
$$\forall X \forall Y \forall Z((Z \in X \equiv Z \in Y) \supset X = Y)$$

We then have the limitation of size axioms mentioned above:

Heridity
$$\forall Y \forall x(Y \in x \supset S(Y))$$
Subsets
$$\forall X \forall x(Y \subseteq x \supset S(Y))$$

And we can also add the usual regularity axiom for sets:

[17] For the original source consult (Ackermann, 1956), here we follow (Goodstein, Fraenkel, & Bar-Hill, 1958, pp. 148-53)

Regularity
$x \neq \emptyset \supset \exists y(y \in x \,\&\, y \cap x = \emptyset$

Note that, interestingly, what we do not have is that members of classes are sets, which was the way we defined the set-predicate in NBG and ML. We now look at the comprehension schemas of A, first there is an unrestricted comprehension for classes but such that this class arising from comprehension will have, as we saw in NBG and was used to block the paradoxes, sets as members:

A class comprehension
$\forall x(\phi(x) \supset S(x)) \supset \exists Y \forall u(u \in Y \equiv \,\&\, \phi(u)))$
For any ϕ in the language.

Observe that as in NBG, the argument behind Russell's paradox shows that the Russell class is not a set. Also, like in NBG here the class of all sets V is not a set, this is an application of the subset axiom once we know that the Russell class is not a set.

We now look at the distinctive axiom of A, comprehension for sets, this states that if the only classes satisfying some property are sets, then there is a set of exactly these sets:

A set comprehension
$\forall x_1 \ldots x_n(\forall Y \phi(Y, x_1 \ldots, x_n) \supset S(Y) \exists z \forall Y(Y \in z \equiv \phi(Y, x_1 \ldots, x_n)))$
For any ϕ in the language not involving S and whose all parameters are sets.

As noted above, this axiom does not have, as that of ZFC or those of subset and heredity of A, a flavour of limitation of size. Before looking at Ackermann's motivation for such schema, we look into the technical limitations of the principle. These are two, first we require that the set predicate does not appear in ϕ. Suppose then this was not the case, then we could apply the schema to $S(X) \,\&\, \phi(X)$, so that for any $\phi(X)$ we would conclude that $\{x|\phi(x)\}$ is a set, and so in particular $\{x|x \notin x\}$ is a set and so this restriction is required for the consistency of A. Now we look into the restriction barring proper class parameters from appearing in ϕ. If this was not the case, we could have chosen $X \in Y$ as our $\phi(X, Y)$ and so we would have that whenever Y is a class only containing sets, $\{x|x \in Y\}$ is a set, but then by class comprehension we can substitute prarameter Y by any class $\{x|\phi(x)\}$ to obtain that any such class is a set, which then produces Russell's paradox.

We can now formalise what we expressed at the beginning of the section regarding the impossibility, as opposed to the other theories of classes we mentioned before, of defining the set predicate by any condition which uses only set parameters and doesn't contain the predicate itself. Indeed, suppose $\psi(X)$ was such a formula and equivalent to $S(X)$, then take $\psi(X)$ & $X \notin X$ as $\phi(X)$ in the schema of set comprehension we get that the $\{x|x \notin x\}$ is a set, and so this assumption is incompatible with the consistency of A.

We will now look into the motivations of Ackermann for the set comprehension axiom. For Ackermann, even though set theory deals both with sets and proper classes, the former are the true subject of the theory. However, Ackermann takes what is usually considered to be a *façon de parler* to explain the iterative conception of set embodied in theories such as ZFC, namely that sets are generated from the emptyset through the operations of union and powerset, in a cumulative process seriously. Hence we can now understand why in his view the extension of the predicate being a set cannot any given point be thought as well-defined and so we cannot be certain of whether some class X will end up being a set. This is precisely because this process of set formation has not yet concluded at any given time for iteration can always be continued.[18] Thus, a predicate can only be considered as well-defined in the sense just explained if it doesn't use the set-predicate or class parameters, as required by the set comprehension schema. Note that it seems that the conditions imposed by this schema seem insufficient since although barring class parameters, it does allow statements that quantify over all classes and it is not clear why the totality of classes is less ill-defined than their individual instances.[19] Be that as it may, the following quote from Dana Scott summarises the situation:

> A remarkably simple axiomatization of a system of set theory is presented which the reviewer feels deserves serious consideration. The system is formalized in an applied first-order calculus with identity using a binary predicate e (membership) and a singulary predicate M (being a set). It is quite essential for the consistency of the system that M is not definable in terms of E. The axiom of extensionality is assumed so that all the individuals can be considered as collections of individuals, but it is easily proved from the axioms that there are collections that are not sets and even contain non-sets. The necessity for the existence of such improper collections in the theory makes comparison with the standard systems of set theory somewhat difficult.(Zalta, 2019b, p. 10)

As Scott points out, the different principles present in each theory are motivated by different views about the nature of sets and classes. Thus the thought that a comparison of A with a more traditional theory such as ZF will not be easy bussiness. However, it turns out that A is a conservative extension of ZF. So any satement aboutZF provable in A was already provable inZF, and all theorems ofZF are provable in A. Hence, ZF is equiconsistent with A. Related to these points, we here show the following more modest result:

Theorem 1. *All theorems of* Z *are provable in* A.

Proof. We proceed by establishing that the axioms of Z are theorems of A.

[18]Note that even though this way of expressing the structure of the set-theoretic universe in terms of iteration of some operations over time is quite informal this can be indeed formalised through semantics for modal logic or intuitionistic logic, as we will see later in this section.

[19]Note that here the objector seems to be assuming the so-called All in One principle, more on this in §3.3.3.

- Extensionality: This follows directly from A extensionality.
- Pairing: Given sets a and b, consider the expression with these as paramenters $X = a \vee X = b$, this is only satisfied by sets and is allowed in the set comprehension schema, and so there is a set containing only a and b.
- Union: Proven similarly to pairing using the expression $\exists Y(X \in Y \& Y \in b)$.
- Power set: Proven similarly to pairing using the expression $X \subseteq b$.
- Separation: Let b be a set and $\phi(x)$ any open formula, then by class comprehension, the following class exists $\{x|x \in b \,\&\, \phi(x)\}$ which is a set by the subset axiom.
- Empty set: Apply set comprehension to the formula $X \neq X$.
- Infinity: Consider the following expression:

$$\forall Y((\emptyset \in Y \,\&\, \forall Z(Z \in Y \supset Z \cup \{Z\} \in Y)) \supset X \in Y)$$

  I.e. X is contained in every inductive class. Note now that V, the class of all sets, is such a class, since it contains the empty set and by the axioms of union and ordered pairs, if b is a set so is $b \cup \{b\}$ and so every class satisfying the formula above will be a member of V and hence a set, and so we can apply this formula to the set comprehension schema, to obtain a set:

$$\omega := \{x|\forall Y((\emptyset \in Y \,\&\, \forall Z(Z \in Y \supset Z \cup \{Z\} \in Y)) \supset x \in Y)\}$$

  Then clearly $\emptyset \in \omega$, also suppose that $b \in \omega$, then b is contained in every inductive class and so $b \cup \{b\}$ will also be contained in such a class and so will also be a member of ω, thus ω is an inductive set, as the axiom of infinity requires.
- Regularity: This follows directly from A regularity.

Q.E.D.

Recall from the quote from Scott that there are some classes that contain non-sets, however the axioms do not mention these entities directly. Nevertheless, we can prove their existence, even if establishing their presence through a proof by contradiction might make be not very informative when assessing their place in the theory.

Proposition 1.1. $\exists X \exists Y(Y \in X \,\&\, {\sim}S(Y))$

Proof. Suppose for the sake of contradiction that no such a class existed, then no class X would be a member of any class B, unless X was a set, however by pairing any set is a member of some class, then the property of being a set would be equivalent to being a member of a class, i.e. $S(X) \equiv \exists Y(X \in Y)$, but we showed above that this property is not definable in A in this way, a contradiction.

Q.E.D.

This theory has also been proposed as a basis for category theory.[20] We now look at another theory that can be understood as sharing the same main themes as Ackermann's proposal, this theory was formalised by Reinhardt after some notes from Shoenfield.[21] The core principle of the theory is the following:

P If ϕ is a property of stages and one can imagine a situation in which all the stages having ϕ have been built up, then there exists a stage s beyond all the stages which have ϕ.

This principle takes, as we pointed out before that also Ackermann did, the idea that sets are build up in some cumulative and continuous process seriously. Note that there is in this principle a distinction between stages that have actually been built or exist and stages and imagining a situation in which these stages have been built, the principle tells us that that possibility or imagination is enough to guarantee the existence of a stage beyond this stages with the property. We could illustrate the principle as follows, suppose that we are at stage 0 of the set theoretic hierarchy, and we consider the property of being a finite stage, then we could imagine a situation in which all the finite stages have been built, indeed this is the situation in which the set-building procedure has reached V_ω, then there exists a stage beyond all these, for instance the one we name $V_{\omega+1}$. The reference to imagination can be understood here in more or less constructive terms, namely that this act of imagination brings the stage into existence, or in a more platonic fashion, that the imaginative act allows us to access epistemically the realm of mathematical objects.

More formally, the language of Reinhardt's theory, called S after Shoenfield, is FOL with equality and the membership relation. It will also have a predicate S, to indicate something is a set, as well as a constant V, the imaginary set of all sets. Here the imaginable sets are those in the range of quantification and the existing ones are the ones falling under the predicate. Also note that the existing properties in S are modelled as formulas in which all of its parameters are existing sets. A stage x being built up in situation y, will be written as $x \in y$ as it will be the fact that y is beyond x. x exists and is a set will be written as $x \in V$. As usual we adopt the axiom of extensionality:

Extensionality

$$\forall x \forall y \forall z((z \in x \equiv z \in y) \supset x = y)$$

We also have the same limitation of size principles as in A:

Heridity

$$\forall y \forall x(y \in V \ \& \ x \in y \supset x \in V)$$

Subsets

$$\forall x \forall y(y \in V \ \& \ x \subseteq y \supset x \in V)$$

We now have a comprehension schema for imagined entities which takes the form of Zermelo's separation axiom, and so we see a classical treatment of these imagined entities:

[20] See for instance (Muller, 2001).

[21] Here we follow the presentation in (Reinhardt, 1974b).

Imagined comprehension
$\forall x \exists y \forall u(u \in y \equiv (u \in x \,\&\, \phi(u)))$
For any ϕ in the language, with y not free in ϕ.

Finally we get to the axiom codifying principle S:

S schema
$y_1, \ldots, y_n \in V \,\&\, \exists x \forall u(\phi(y_1, \ldots, y_n, u) \supset u \in x) \supset \exists s(s \in V \,\&\, \forall u(\phi(y_1, \ldots, y_n, u) \supset u \in s))$
For any ϕ in the language, with x not free in ϕ.

Indeed, note that this is an adequate formalisation since principle S tells us that if we can imagine some situation x, i.e. in our language $\exists x$, such that for any stage u with property P, here coded by $\phi(y_1, \ldots, y_n, u)$ with the parameters existing, this u has been built in x, then there is a stage s such that this exists, i.e. $s \in V$, and for any aforementioned u, s is beyond it, i.e. $u \in s$. One could now add the usual axioms of regularity and choice.

Note that the distinction in Reinhardt's theory is between entities imagined and existing, but that this distinction could also be understood in the distinction that Ackermann makes between sets and classes. Then the idea of an existing property could be understood as a property of classes that do not depend on their existence on V, Reinhardt calls such properties independent. Then S can be rephrased as saying that if P is an independent property of classes such that for some class x, $P \subseteq x$, then there is a set u, such that $P \subseteq u$. Comparing A and S , one can immediately notice that the principle S is stronger than Ackermann's set comprehension, since the latter is obtained from the former by existential weakening on V, and so while A requires everything satisfying the property to be a set, for S it is enough that this entity is a class. The other axioms of A follow immediately, from the class comprehension schema once we take V as the class we are separating from.

Finally in this section we turn our attention to a theory of Powell,[22] this theory takes seriously the idea of classes as closely tied to properties by including in the language a predication symbol $\ni$, we also have the universe of sets V as a constant. Here classes are defined as collections of sets and so if we take capitalised variables to range over classes, we get that $\forall X \phi(X) := \forall x(\forall y(y \in x \supset y \in V) \supset \phi(x))$, note that we assume that classes only occur to the left of predication, then we can read $X \ni x$ as x instantiates X, or more similarly to set-membership talk,

[22]Here we follow (Reinhardt, 1974b, §6), for the original presentation see (Jech & Powell, 1971)

that x belongs to the class X. We then have the following axioms:

Heredity
$x \in y \in V \supset x \in V$
Separation
$\forall x \exists y \forall u (u \in y \equiv (u \in x \,\&\, \phi(x)))$
For any ϕ in the language, with y not free in ϕ.
Predication
$\forall x \in V (Y \ni x \equiv x \in Y)$
Extensionality
$\forall x \in V((x \in Y \equiv x \in Z) \supset Y = Z)$
P Comprehension
$\forall x_1 \dots x_n \in V \exists Q \forall u (Q \ni u \equiv \phi(Z_1, \dots, Z_n, x_1 \dots x_n, u))$
For any ϕ in the language, which doesn't include V and where the P_i only occur on the left of predication ϕ.

To these axioms we can then add regularity and choice. In this theory it is not only the case that one can derive the axioms of ZFC, but also large cardinal axioms.

2.6.1 Intuitionistic semantics for set theory

The aim of this section is to offer a possible formalisation of what is taken as the underlying narrative regarding the structure and construction of the set theoretic universe for theories explored in this section such as A and S.[23] The story here is that this structure is built in stages according to some allowed set-theoretic operations performed in previously existing sets in this way expanding the domain of the quantifiers ranging over sets at some given stage. One of the features of this formalisation is that we need to be able to have a global picture of the development of set theory. This means that we need to know how the different stages relate to each other in terms of their order of appearance in the hierarchy of sets. A very useful construction when studying relational structures are Kripke frames:

Definition 2 (*(Kripke) Frame*)**.** A frame is a pair (W, R), where W is a non-empty set and R is a reflexive and transitive relation.

For our purposes we take the set to be the collection of stages in the set-theoretic hierarchy and the relation the order of the different stages in the hierarchy.

Definition 3 (*(Kripke) Model*)**.** A Kripke model is a tuple (W, R, f, g) which consists of a frame (W, R) together with:

[23] Here we follow closely the formalisation in (Lear, 1977), further relevant references are (Incurvati, 2008, §3) and (Paseau, 2001, §II)

- a function f that maps each element of T to the domain of a classical model of set theory, s.t. $wRw' \Rightarrow f(w) \subseteq f(w')$, and
- a function g that takes a relation and a stage and gives a relation with domain the given stage t, s.t. $tRt' \Rightarrow g(R,t) \subseteq g(R,t')$.

These seem reasonable constraints to place since usually the only relation we deal with is membership. This expresses the thought that once it is decided that a set belongs to another set, this will not cease to be the case even if we enrich the set theoretic universe with novel entities. Moreover f will inform us of the sets conforming the universe at each stage. For instance, under the usual von Neumann construction of the hierarchy: $f(0) = \emptyset$ and $f(1) = \{\emptyset\}$, note that since the hierarchy is understood to grow and not to shrink in the process of development. After all, we are just enriching our notion of set, we take that if some stage follows another the sets already present in the previous stage will not get lost in the new one, hence the condition on f.

Note that we are not demanding for R to be a linear order, this is because in this way we can accommodate the possibility that at some point from some interpretation of the concept of set it develops two distinct and incompatible set theoretic universes, for instance one in which $V = L$ and another with measurable cardinals.[24]

A key feature of this system is that the semantics of the sentences in the language are different from those of classical logic. Indeed, these we will take into account, not just the actual status of the concept of set at the current stage but also at all subsequent stages in the future. The truth of the formulas is defined inductively as follows:

Definition 4 (*Intuitionistic semantics*)**.**

- $g(\phi(c_1, \ldots, c_n), t) = 1$ iff $c_1, \ldots, c_n \in g(\phi, t)$, for ϕ an atomic formula.
- $g(\phi \,\&\, \psi, t) = 1$ iff $g(\phi, t) = 1$ and $g(\psi, t) = 1$.
- $g(\phi \vee \psi, t) = 1$ iff $g(\phi, t) = 1$ or $g(\psi, t) = 1$.
- $g(\exists\phi(x), t) = 1$ iff for some $c \in f(t), g(\phi(c), t) = 1$.
- $g(\sim\phi, t) = 1$ iff for all $t'Rt$, $g(\phi, t') = 0$.
- $g(\phi \supset \psi, t) = 1$ iff for all $t'Rt$, if $g(\phi, t') = 1$, then $g(\psi, t') = 1$.
- $g(\forall x\phi(x), t) = 1$ iff for all $t'Rt$, for all $c \in f(t), g(\phi(c), t') = 1$.

As we just mentioned, note that the semantics for negated, conditional and universally quantified sentences are different from the classical semantics in the sense that we will look not only at the state in the system where the formula is evaluated but also in the subsequent stages in the frame. Consider, as an illustration the following frame:

[24] See again (Incurvati, 2008) and (Paseau, 2001) for further discussion regarding the condition of linearity on R.

$$
\begin{array}{c}
V_\kappa \\
| \\
\vdots \\
| \\
V_\omega \\
| \\
\vdots \\
| \\
V_0
\end{array}
$$

Where we understand the domains of the of each stage as the usual steps of the von Neumann hierarchy so that we have represented a model of ZFC with κ an inaccessible cardinal. We can thus see the differences with the usual semantics if we consider the sentence *All sets are finite*, at stage V_ω. Indeed, at this stage there are no infinite sets, however that is not enough to make the sentence true since in the subsequent world, $V_{\omega+1}$, there is ω, and so the new semantics tell us that this sentence will be false even in V_ω. According to the intuitionistic outlook, when we utter a universally quantified statement we are making a stronger assertion than the classical interpretation, not only saying that this fact is true in all the sets of a given stage but that it will remain true however we expand our notion of set, i.e. will be true in all subsequent stages in the model.

Similarly, for the negation consider the sentence *There is no infinite set* at V_ω, note that although the sentence *There is an infinite set* is false at this stage, its aforementioned negation is not since again, in the subsequent world, $V_{\omega+1}$, there is ω, and so we need to look forward to learn about the truth of negation under these semantics. Note that the law of excluded middle is not observed here always, when it is regarding a given formula we say that this has a fixed value. In fact, note that if we know that the negation of a sentence is true this is enough to know that the the sentence's truth-value is fixed at falsity. So we see again that the interpretation of the negation is stronger than the classical negation in which one looks only at the state in which the formula is evaluated. Now we need the stronger condition that no matter how set theory shall develop in the future, the negated sentence will never be true, which is definitely not the case with our sentence regarding infinite collections.

Consider know the sentence *If there are infinitely many sets, there is an infinite set*, and consider any V_α, such that $\alpha \in [0, \omega)$, classically this sentence would be trivially true in any such stage since the antecedent of the conditional is false there. However, the intuitionistic semantics render it false since any of this stages have a successor, V_ω, in which there are infinitely many sets but there is no infinite set, this is enough to render the sentence false in the finite stages too. So again, we see how this is a stronger interpretation of the connective since true conditionals due to trivialities like the above are avoided, so the hope here is that conditionals will come out true only for more significant mathematical reasons, namely that the existence of infinitely many sets would be incompatible with the lack of infinite sets in future extensions of the hierarchy. Nevertheless, the sentence under consideration will be true in $V_{\omega+1}$ and successive stages. In fact one can prove the following:

Theorem 2 (Monotonicity)**.** *If* (W, R, f, g) *is a Kripke model and* $w, w' \in W$ *are two worlds such that* wRw'*, then* $g(\phi, w) = 1$ *implies* $g(\phi, w') = 1$*.*

Proof. By induction on complexity of formula.

Base case: Suppose that $\phi(c_1, \ldots, c_n)$ is an atomic formula and that $g(\phi(c_1, \ldots, c_n), w) = 1$, then by definition of the semantics this means that $c_1, \ldots, c_n \in g(\phi, t)$, but since wRw', then $g(\phi, w) \subseteq g(\phi, w')$ by definition of g, and so $c_1, \ldots, c_n \in g(\phi, w')$ hence $g(\phi(c_1, \ldots, c_n), w') = 1$ as required.

Induction hypothesis: If (W, R, f, g) is a Kripke model and $w, w' \in W$ are two worlds such that wRw', then $g(\psi, w) = 1$ implies $g(\psi, w') = 1$, with ψ of lower complexity than ϕ.

Induction cases:

Suppose that ϕ is of the form $\sim\psi$, then $g(\sim\psi, w) = 1$ entails by definition of the semantics that $g(\psi, w') = 0$, and suppose that there was a w'' s.t. $w'Rw''$ with $g(\psi, w'') = 1$ then by transitivity we would have that wRw'' and so by hypothesis $g(\psi, w) = 0$, a contradiction so indeed $g(\sim\psi, w') = 1$ since all successors of w' falsify ψ.

Suppose that ϕ is of the form $\chi \,\&\, \psi$, then $g(\phi, w) = 1$ entails by definition of the semantics that $g(\chi, w) = 1$ and $g(\psi, w) = 1$ and by i.h. if wRw' $g(\chi, w') = 1$ and $g(\psi, w') = 1$ this guarantees $g(\phi, w') = 1$ as required.

Suppose that ϕ is of the form $\chi \vee \psi$, then $g(\phi, w) = 1$ entails by definition of the semantics that either $g(\chi, w) = 1$ or $g(\psi, w) = 1$ and by i.h. if wRw' either $g(\chi, w') = 1$ or $g(\psi, w') = 1$ this guarantees $g(\phi, w') = 1$ as required.

Suppose that ϕ is of the form $\chi \supset \psi$, then $g(\phi, w) = 1$, then consider any w'' s.t. $w'Rw''$ with $g(\chi, w'') = 1$ then by transitivity we would have that wRw'' and so by definition of the semantics of implication $g(\psi, w'') = 1$, so indeed $g(\phi, w') = 1$ as required.

Suppose that ϕ is of the form $\exists x\chi$, then $g(\phi, w) = 1$, by definition of the semantics this means that there is a $c \in f(w), g(\phi(c), w) = 1$. But since wRw', then $f(\phi, w) \subseteq f(\phi, w')$ by definition of f, and so there is a $c \in f(w')$, $g(\phi(c), w') = 1$ since $g(\phi(c), w) = 1$ and we apply the i.h. and so $g(\phi, w') = 1$ as required.

Suppose that ϕ is of the form $\forall x\psi$, then $g(\phi, w) = 1$, then consider any w'' s.t. $w'Rw''$ then by transitivity we would have that wRw'' and so by definition of the semantics of universal quantification $g(\psi, w'') = 1$, so indeed $g(\phi, w') = 1$ as required. Q.E.D.

So we see that what we observed above with our conditional sentence is not a coincidence, but that whenever a sentence is decided true by a model it remains so in the subsequent stages of the model. Note also that classical semantics do not necessarily make the monotonicity theorem true, indeed, consider the sentence *There is a biggest ordinal* in our previous frame. This will be true at successor stages of the hierarchy, false at the limit stage collecting these and true again at the subsequent stage and so on, hence the truth value of the sentence will flip indefinitely often.

Note that on a more metatheoretical level, although using intuitionistic semantics, these are apt for either a platonistic or intutionistic interpretation, in the platonistic view we will see the different stages of the frame as witnessing the set-theoretic universe growing over time, in the sense that the extension of the predicate *being a set* develops but always withing an independently well-defined universe of objects, the intuitionistic interpretation will actually see the new sets being, as opposed to the platonist's view, constructed or brought into existence over time.

Another point of departure will come in interpreting the semantics, throughout this section we have assumed the platonistic interpretation classifying sentences as true and false. In particular we noted that when a sentence is false this can be coupled with its negation being true, In which case the sentence has a fixed value, or its negation being false, in which case we can interpret this as showing that the extension of the set-predicate is not yet developed enough to settle the truth or falsity of the sentence for good. However, under the intuitionisic approach what we described as true sentences will be taken to be assertible and those false as not assertible at the given evaluation stage, in particular when we have that neither a sentence nor its negation is assertible as with *If there are infinitely many sets, there is an infinite set.* The intuitionist will maintain it becomes at $V_{\omega+1}$ assertible for before we lacked the relevant amount of the set-theoretic universe constructed in order to be able to utter the statement.

Finally note, that even though here we have used intuitionistic semantics one, might think a better formalisation of the construction of the hierarchy of set through stages could have been obtained by using the tools of modal logic. Very roughly, in this case we could interpret universally quantified sentences as boxed universals formulas, i.e. necessary and so true in all accessible states from a given point in the Kripke frame and existential ones as diamond existentials, i.e. such they must true in some accessible world. One might also want to go a step further in the modal path and claim that one, not only interprets the quantifiers modally in set theory, but also do set theory with a modal language.[25]

2.7 Schindler: the singularity of paradoxes

In this consider a more recent theory of classes due to Thomas Schindler.[26] We consider *FOL* $\mathfrak{L}$ with $=$ and $\in$ as nonlogical symbols and we add a relation xRy. This new relation symbol denotes that x is in the range of significance of y.

[25] See for instance (Parsons, 1983).
[26] Here we follow closely the exposition from Schindler in (Schindler, 2019)

$\mathfrak{L}$ also contains a class term $\{x|\phi\}$ corresponding to predicate ϕ s.t. $x \in FV(\phi)$, note in particular that ϕ might contain $\in, R$, other class terms or other free variables as parameters, and so we see that in this system we have a term for every class, including the paradoxical ones. By $total(x)$ we mean that x has no singularities, i.e. $\forall z(zRx)$. The idea behind the ranges of significance is that although in this system classical logic is used, and so bivalence holds. In particular for any class it either belongs or does not to any other class, in some cases, specially the ones leading to paradox, it makes no sense to predicate the property from which the class arises to some given object, this will be transcribed in the system by saying that the object does not lie in the range of significance of the class. In a sense we could say it is a category mistake to predicate the property of the class, in the same way that we might find something wrong with the sentence stating that the Higgs boson smells nice, we might agree this has a truth value although doubting the felicity of such statement. More concretely, the following are some conceptual axioms describing the general relation of a class and its range of significance.

Axiom 1 (*Class comprehension*).

$$\forall x(xR\{u|\phi\} \supset (x \in \{u|\phi\} \equiv \phi(x/u)))$$

where ϕ is any formula and x is free for u in ϕ.

And so comprehension holds provided that the object in question is in the range of significance of the class considered. And so even though the object will either belong or fail to do so with respect to the class, if the object is a singularity of the class, this fact prevents us from reaching any conclusion regarding this from the satisfaction of the property associated to the formula, or vice versa. Consider the Russell class, $r := \{u|u \notin u\}$:

Proposition 2.1. $\sim rRr$

Proof. By 1 $rRr \supset (r \in r \equiv r \notin r)$, so we conclude that $\sim rRr$. Q.E.D.

So as commented above, the paradox is transformed into the statement that r does not lie in its own range of significance. The following axiom ensures that no class contains objects alien to their range of significance.

Axiom 2 (*Singularity*).

$$\forall x(\sim xRy \supset x \notin y)$$

We can now show that it is not possible that $\{u|\phi\}$ contains some objects that are not ϕ's.

Proposition 2.2 (*No overspill theorem*).

$$\forall x(x \in \{u|\phi\} \supset \phi(x/u))$$

Proof. Suppose $x \in \{u|\phi\}$, then by contraposition of 2 $xR\{u|\phi\}$ so by 1 $x \in \{u|\phi\} \equiv \phi(x/u)$, thus $\phi(x/u)$. Q.E.D.

We now have our usual axiom of extensionality although relativised to ranges of significance to highlight that even if having the same members two classes differing in range of significance will be distinct.

Axiom 3 (*Extensionality*).

$$\forall x \forall y (R(x) = R(y) \,\&\, \forall z (z \in x \equiv z \in y) \supset x = y)^{27}$$

The next axioms deal with the relationship between the range of significance of a predicate and its logical form, ensuring they are well behaved under logical operations.

Axiom 4 (*Closure under connectives*).

$$\forall x (\sim xR\{u|\phi\} \supset xR\{u|\sim\phi\})$$

$$\forall x (xR\{u|\phi\} \,\&\, xR\{u|\psi\} \supset xR\{u|\phi \odot \psi\}), \text{ with } \odot = \{\,\&\,, \vee, \supset\}$$

Axiom 5 (*Closure under atomic predicates*).

$$\forall x \forall y (xRy \supset xR\{u|u \in y\})$$

$$\forall x \forall y (xRy \supset xR\{u|u = y\})$$

Axiom 6 (*Self-identity*).

$$total(\{u|u = u\})$$

This axiom ensures that not all predicates have an empty range of significance, at least the universal class has this property. $\{u|u = u\}$, the universal class will be denoted by V. Now we can show that the empty class will also be total:

Corollary 2.1. The empty class $\emptyset := \{u|u \neq u\}$ is total.

Proof. By 6 and 5. Q.E.D.

Corollary 2.2. $\forall x (x \in V)$

Proof. By 6 and 1. Q.E.D.

We can also show that the theory distinguishes between different collections that are usually taken to be proper classes by other theories.

Proposition 2.3. $V \neq r$

Proof. Immediate since these classes have different ranges of significance. Alternatively, since $V \in V$ by 2.2, if $V = r$, then $r \in r$, which by 2 means rRr contradicting 2.1. Q.E.D.

[27] $R(x) = R(y) := \forall z (zRx \equiv zRy)$.

The following axiom expresses the diagnosis from Schindler that the paradoxes are to be blamed on some form of circularity within the problematic classes, this we trivially see in the case of r.

Axiom 7 (*Circularity*).

$$\forall x(\exists y{\sim}xRy \supset \exists z{\sim}zRx)$$

So if x is a singularity it itself has a singularity.

Note that 7 is equivalent to $\forall x(\forall zzRx \supset \forall yxRy)$ and so total classes are in the range of significance of any class, and so total classes are never singularities. And so as we see in the subsequent theorem these are well behaved objects with respect to comprehension.

Theorem 3. *For every predicate ϕ, $\exists y\forall x(total(x) \supset (x \in y \equiv \phi(x/u)))$ where x is free for u in ϕ.*

Proof. By 7 $\forall x(total(x) \supset xR\{u|\phi\})$ so by 1 $\forall x(total(x) \supset (x \in \{u|\phi\} \equiv \phi)(x/u)))$ so $\exists y\forall x(total(x) \supset (x \in y \equiv \phi(x/u)))$ by existential weakening.

Q.E.D.

So 3 allows us to collect total classes at will.

2.7.1 Consistency proof and urelements

We now turn to a consistency proof, we will work within ZFCA[28] Let U be a countably infinite set of urelements. Further, let t and p be two urelements not in U. Put $U_T := \{(u,t)|u \in U\}$. The objects in U_T will be used to model membership between classes of equal rank. Let $V_\omega[U_T]$ be the smallest set X such that (α) $U_T \subseteq X$ and (β) whenever $x_1, \ldots x_n \in X$ then $(\{x_1, \ldots, x_n\}, t) \in X$. The collection of total classes T consists of all x such that $1.x \in V_\omega[U_T]$ or $2.\exists y_1, \ldots, y_n \in V_\omega[U_T]$ such that $x = (V_\omega[U_T] \setminus \{y_1, \ldots, y_n\}, t)$ Note that objects satisfying (1) have finite rank and those satisfying (2) have a first component that is cofinite in $V_\omega[U_T]$. The collection of proper (non-total) classes, P, consists of all x such that (3) $x = (y, p)$ for some $y \subseteq V_\omega[U_T]$. The domain of the model consists of T and P; $D := T \cup P$. Consider $x := (a, i) \in D$ with $i \in \{t, p\}$, we say that a is the set-component and i the index of x. If the set component of x is an urelement we call x itself an urelement. Let $\sigma_1((a, i)) = a$, we write $x \in_1 y$ if $x \in \sigma_1(y)$. Observe that $x \in_1 y$ only obtains if x has finite rank. Note also that P contains a 'copy' of each finite and cofinite set.

We now specify R as follows (i) if $x \in T$, then yRx if and only if $y \in D$, (ii) if $x \in P$, then yRx if and only if $y \in T$.

[28] ZFC with urelements. These is the usual system although some axioms are added to deal with the atoms. These include one stating that urelements have no elements or that there is something with no elements which is a set, extensionality is of course restricted to sets.

Proposition 3.1 (*Circularity*).

$$\forall x \in D(\exists y \in D{\sim}xRy \supset \exists z \in D{\sim}zRx)$$

Proof. Let $x \in D$ and assume there is some y such that ${\sim}xRy$. By definition, $x \in P$, but if $x \in P$ x is $R-$related to all and only the things in T, so $\exists z \in P$ ${\sim}zRx$ since P is non-empty. Q.E.D.

Put $D^* := T \setminus V_\omega[U_T]$, i.e. the objects with first component cofinite in $V_\omega[U_T]$. Let f be a bijection from D^* to U_T. We now define the relation to interpret identity between classes. $\forall x, y$ $x \equiv y$ if and only if (i) $x = y$, or (ii) $f(x) = y$, or (iii) $x = f(y)$. Note that this is an equivalence relation and that cases (ii) and (iii) only occur when both classes are total and one of them is in D^*.

We now define the relation to interpret class membership: For all $x, y \in D$:

1. $x, y \in P \Rightarrow {\sim}xEy$
2. $x \in P$ and $y \in T \setminus (D^* \cup U_T) \Rightarrow {\sim}xEy$
3. $x \in P$ and $y \in D^* \Rightarrow xEy$ and $xEf(y)$
4. If $x \in T$, then xEy if and only if $x \in_1 y \vee f(x) \in_1 y \vee x \in_1 f^{-1}(y) \vee f(x) \in_1 f^{-1}(y)$

So we can see that $x \in x$ if and only if $f(x) \in_1 x$, this is, if and only if the set component of x contains the urelement associated with x. Also note that the model will be minimal in the sense that only total classes will be self-membered.

Proposition 3.2 (*Singularity*).

$$\forall x \in D({\sim}xRy \supset {\sim}xEy)$$

Proof. Since ${\sim}xRy$, $y, x \in P$, so by definition of E, ${\sim}xEy$. Q.E.D.

Proposition 3.3 (*Extensionality*).

$$\forall x, y \in D\,(R(x) = R(y)\,\&\,\forall z \in D\,(z \in x \equiv z \in y) \supset x \sim y)$$

Proof. Either $x, y \in T$ or $x, y \in P$. We proceed by cases. Suppose first that $x, y \in U_T$. It suffices to show $\forall z(z \in_1 x \equiv z \in_1 y)$. Let $z \in_1 x$, then z has finite rank, so z is total, so zEx and so zEy, but since $y \notin U_T$, either $z \in_1 y$ or $f(z) \in_1 y$, but since $z \in V_\omega[U_T]$ then it follows that $z \in_1 y$. The other direction is similar. If $x \in U_T$ but $y \notin U_T$ one can proceed similarly to show $x = f(y)$ (or if $y \in U_T$ but $x \notin U_T$ one can proceed similarly to show $y = f(x)$). If both x and y are urelements we can show $f^{-1}(x) = f^{-1}(y)$. Q.E.D.

Proposition 3.4. $\forall x, y, z \in D\,(x \sim y\,\&\,xEz \supset yEz)$

Proof. We proceed by cases, if $x = y$ this is trivial. Suppose then without loss of generality that $x = f(y)$, then $x \in U_T$ so x is total, then xEz implies $x \in_1 z$ or $f(x) \in_1 z$ or $x \in_1 f^{-1}(z)$ or $f(x) \in_1 f^{-1}(z)$, i.e. $x \in_1 z$ or $x \in_1 f^{-1}(z)$ since $f(x)$ is not defined, now both $f(y) \in_1 z$ or $f(y) \in f^{-1}(z)$ implies yEz. Q.E.D.

This proposition tells us that objects that, in our model, are treated as equal are E-contained in the same objects.

Let $\mathcal{L}$ be the language of our class theory, to distinguish the membership symbol of $\mathcal{L}$ from that of ZFCA, we denote the former by ϵ. We denote by ϕ^* the relativisation of an $\mathcal{L}$-formula not containing any class-abstract, i.e. $(x = y)^* = x \sim y$, $(x \epsilon y)^* = xEy$ and $(\forall x \psi)^* = \forall x \in D\, \psi^*$.

Proposition 3.5. *If ϕ is an $\mathcal{L}$-formula, then $\forall x, y \in D\, (x \sim y \,\&\, \phi^*(x) \supset \phi^*(y))$*

Proof. We proceed by induction on the complexity of ϕ^*:

If $\phi^*(x) := x \sim s$, the result follows from $\sim$ being an equivalence relation.

If $\phi^*(x) := xEs$, then yEs follows from 3.5.

If $\phi^*(x) := sEx$, suppose to avoid triviality that, without loss of generality, $y = f(x)$, so $y \in U_T$ and $x, y \in T$, with $x \in D^*$. If $s \in P$, then the result follows by definition of E. So suppose $s \in T$, sEx implies $s \in_1 x$ or $f(s) \in_1 x$ or $s \in_1 f^{-1}(x)$ or $f(s) \in_1 f^{-1}(x)$, i.e. $s \in_1 x$ or $f(s) \in_1 x$ since $x \in D^*$, but as $f^{-1}(y) = x$ we have that $s \in_1 f^{-1}(y)$ or $f(s) \in_1 f^{-1}(y)$ and so in any case sEy.

If $\phi^*(x)$ is of the form xRs or sRx the result follows by definition of R.

The other clauses follow by induction. Q.E.D.

We now interpret the class abstracts, for simplicity we take $\sim$ and $\&$ as the only logical connectives of $\mathcal{L}$. First we capture the set of class abstracts that our theory proves to be total, which we will call Π-terms and are defined by the following simultaneous recursion:

1. $u = u \in \Pi$ and $\{u | u = u\} \in \Pi$
2. If $s \in \Pi$, then $u = s \in \Pi$ and $\{u | u = s\} \in \Pi$
3. If $s \in \Pi$, then $u \epsilon s \in \Pi$ and $\{u | u \epsilon s\} \in \Pi$
4. If $\phi \in \Pi$, then $\sim\phi \in \Pi$ and $\{u | \sim\phi\} \in \Pi$
5. If $\phi, \psi \in \Pi$, then $\phi \,\&\, \psi \in \Pi$ and $\{u | \phi \,\&\, \psi\} \in \Pi$
6. Nothing else is a Π-term.

We now define an interpretation $(-)^+$ from the set of Π-terms into D, in fact into T.

1. $(\{u|u = u\})^+ = (V_\omega[U_T], t)$
2. $(\{u|u = s\})^+ = (\{s^+\}, t)$
3. $(\{u|u\epsilon s\})^+ = s^+$
4. $(\{u|\sim\phi\})^+ = (V_\omega[U_T] \setminus \sigma_1((\{u|\phi\})^+), t)$
5. $(\{u|\phi \,\&\, \psi\})^+ = (\sigma_1((\{u|\phi\})^+) \cap \sigma_1((\{u|\psi\})^+), t)$

We now extend $(-)^+$ to an interpretation that maps all class abstracts of $\mathcal{L}$ into D. In order to cope with parameters we add to $\mathcal{L}$ a constant for each object in D not in the range of $(-)^+$. If a is such an object we write $\overline{a}$ for the corresponding constant and put $a = (\overline{a})^+$. Let $\phi(u, y_1, \ldots, y_n)$ be an $\mathcal{L}$-formula containing no class abstracts or individual constants. Furthermore, let $s_1, \ldots, s_n$ be a sequence of class terms or individual constants (of lower complexity than ϕ) and assume $s_1^+, \ldots, s_n^+ \in D$ are already defined. We now define $(\{u|\phi(u, s_1, \ldots, s_n)\})^+$ as above if $\{u|\phi(u, s_1, \ldots, s_n)\} \in \Pi$, otherwise, this will be $(\{u \in V_\omega[U_T] | \phi^*(u, s_1^+, \ldots, s_n^+)\}, p)$, which is well-defined since the set component of this pair is a subset of $V_\omega[U_T]$.

Proposition 3.6 (*Comprehension*)**.** *Let $\phi(u, y_1, \ldots, y_n) \in \mathcal{L}$ and $s_1, \ldots, s_n$ be a sequence of class abstracts or individual constants. Let $dR(\{u|\phi(u, s_1, \ldots, s_n)\})^+$. Then $dE(\{u|\phi(u, s_1, \ldots, s_n)\})^+ \equiv \phi^*(u, s_1^+, \ldots, s_n^+)$.*

Proof. Put $c := \{u|\phi(u, s_1, \ldots, s_n)\}$. Assume that $c \notin \Pi$, and so $c^+ = (\{u \in V_\omega[U_T]|\phi^*(u, s_1^+, \ldots, s_n^+)\}, p)$. Suppose then that dEc^+, since $c^+ \in P$, then $d \in T$ by definition of E, and so $d \in_1 c^+$ or $f(d) \in_1 c^+$ or $d \in_1 f^{-1}(c^+)$ or $f(d) \in_1 f^{-1}(c^+)$, i.e. since $c^+ \in P$ $f(d) \in_1 c^+$ or $d \in_1 c^+$. Now, if $d \in_1 c^+$, by definition of c^+, $\phi^*(u, s_1^+, \ldots, s_n^+)$. If $f(d) \in_1 c^+$, then $\phi^*(f(d), s_1^+, \ldots, s_n^+)$, since $d \sim f(d)$, 3.5 implies $\phi^*(d, s_1^+, \ldots, s_n^+)$. Assume now that $\phi^*(d, s_1^+, \ldots, s_n^+)$ and without loss of generality that $d \in V_\omega[U_T]$ (if this is not the case we can just work with $f(d)$ and use 3.5). By definition of c^+ $d \in_1 c^+$ and so dEc^+.

Assume now that $c \in \Pi$, thus $c^+ \in T$. We proceed by induction on the complexity of Π-terms. We show here two cases:

Assume $\phi := u = u$ so $\phi^* = u \sim u$, this holds for any object in D. Moreover, $c^+ = (V_\omega[U_T], t)$. Since $\sigma_1((c^*)^+)$ is infinite, dEc^+ holds for any $d \in P$ by definition of E. Moreover it is easy to see that dEc^+ for any $d \in T$.

We want to show now that $\phi := sE(\{u|\phi \,\&\, \psi\})^+$ if and only if $\phi^*(x) \,\&\, \psi^*(x)$. Now, $\{u|\phi\}$ and $\{u|\psi\}$ must be Π-terms. Take $x \in P$, (the case where $x \in T$ is proven similarly). Assume $xE(\{u|\phi \,\&\, \psi\})^+$, then $(\{u|\phi \,\&\, \psi\})^+$ must be co-finite and so $(\{u|\phi\})^+$ and $(\{u|\psi\})^+$ are co-finite so by definition of E, $xE(\{u|\phi\})^+$ and $xE(\{u|\psi\})^+$, so by inductive hypothesis $\phi^*(x) \,\&\, \psi^*(x)$. The other direction is similar. Q.E.D.

Proposition 3.7 (*Negation*)**.** *Let $\forall x \in D\ (xR(\{u|\phi\})^+ \supset xR(\{u|\sim\phi\})^+)$*

Proof. If $\{u|\phi\} \in \Pi$ so is $\{u|{\sim}\phi\}$, and so $(\{u|\phi\})^+$ and $(\{u|{\sim}\phi\})^+$ are in T and have the same range of significance. If $\{u|\phi\} \notin \Pi$ neither is $\{u|{\sim}\phi\}$ and so $(\{u|\phi\})^+, (\{u|{\sim}\phi\})^+ \in P$ and so have the same range of significance. (So we have actually proven a biconditional). Q.E.D.

The other axioms of connectives are proven similarly as are those regarding $\in$ and $=$.

Proposition 3.8 (*Self-identity*)**.** *Let* $\forall x \in D\, xR(\{u|u = u\})^+$

Proof. Since $(\{u|u = u\})^+ = (v_\omega[U_T]), t)$ it is in T so any object in D is in its range. Q.E.D.

This concludes the consistency proof. Before concluding the section we look at how we might supplement this theory of classes with other entities. We now consider a language that contains additional predicates applying to people, numbers, sets, et. al., given that the motivation of the theory of classes is its application to a given domain of course the theme throughout this chapter being mathematics. Let us introduce a distinguished predicate, U for urelement, applying to these objects and introduce the following axiom stating that every urelement is in the range of significance of any class.

Axiom 8 (*Urelements*)**.** $\forall x(Ux \supset \forall xRy)$

Now, if T is a first order theory without $\in, R, U$ or classes (if T is the language of set theory we can work with two copies of $\in$). Let T^U be the relativisation of this theory to U. If T contains schemata, we extend these to allow $\in, R, U$ and class terms. We can show (by axioms 8 and 1) in T^U conjoined with our theory of classes that

$$\exists y \forall x(Ux \supset (x \in y \equiv \phi))$$

so this theory interprets the second-order version of T. We can add new axioms of this flavour to interpret higher-order versions of T, embedding the type hierarchy over T into our theory, this is taken to be a minimal adequacy result given the aim of Schindler of providing a type free theory of collections that improves on the what its typed counterparts have to offer.

2.8 Maddy: a paracomplete solution

To motivate the approach of her theory of classes to solving the paradoxes of set theory Penelope Maddy draws our attention to similar situations in the theory of truth. In particular, the problems surrounding such statements as *Everything I have ever said is false.* If it turns out that everything I have ever said apart from this statement is false, then the assumption that this statement has a truth value leads to paradox. Here we seem to have a statement without a truth value, where in the case of the Russell class we had a property without an extension. Saul Kripke has shown how the truth paradoxes can be solved by allowing truth value gaps as specified by a certain construction.[29] What Maddy proposes is that we adapt this solution to the case of logical classes by allowing gaps in the membership relation, this is that we abandon classical logic in favour of a paracomplete sytem. This is, for any property, assign an extension and an antiextension, but allow some things to fall in between. [30] Thus, Maddy proposes to adopt indeterminate membership as a key difference between classes and sets, and to use an imitation of Kripke's construction to show when these indeterminate membership relations occur.

More concretely, we consider a first order language $\mathfrak{L}$ with $=$ and $\in$ as nonlogical symbols and we add a term-forming operator $\widehat{\ }$, to form terms such as $\hat{x}(x = x)$ and $\hat{x}(x \in \emptyset)$.[31] To gain expressive power we will also include a constant $\overline{V}$ to stand for the class of all sets, and a constant $\overline{a}$ for each set a.

Definition 5 (*Terms and formulas of* $\mathfrak{L}$)**.**

1. All constants and variables are terms.
2. If t and t' are terms, then $t = t'$ and $t \in t'$ are formulas.
3. If ϕ and ψ are formulas, and x is a variable, then $\sim\phi, \phi \,\&\, \psi$ and $\forall x\phi$ are formulas.
4. If ϕ is a formula, and x is among the free variables of ϕ, then $\widehat{x}\phi$ is a term.

T is the collection of all terms, it is the union of S, the collection of all set constants, C, the collection of all terms of the form $\widehat{x}\phi$, and $\{\overline{V}\}$. C^* is the collection of closed terms in C, similarly, T^* is the collection of closed terms in T.

The intended model for this language contains all sets, $\overline{a}$ standing for a, here $\overline{V}$ will be a class with extension all sets and antiextension all classes. The variable part of the interpretations of $\mathfrak{L}$ is the extension and antiextension of the elements of C^*.

Definition 6 ($\mathfrak{L}-$ *structure*)**.** $\mathfrak{C} = \{(t, t^+_{\mathfrak{C}}, t^-_{\mathfrak{C}}) : t \in C^*\}$ is an $\mathfrak{L}$-structure iff $\forall t \in C^*$, $t^+_{\mathfrak{C}} \subseteq T^*$ and $t^-_{\mathfrak{C}} \subseteq T^*$ and $t^+_{\mathfrak{C}} \cap t^-_{\mathfrak{C}} = \emptyset$.

Note that it needn't be the case that $t^+_{\mathfrak{C}} \cup t^-_{\mathfrak{C}} = T^*$, and so we can have membership gaps, as mentioned above. The idea is that $t^+_{\mathfrak{C}}$ and $t^-_{\mathfrak{C}}$; represent the extension and antiextension respectively of the class term t.

Given a sentence τ we have three possibilities $\mathfrak{C} \models \tau$ ($\mathfrak{C}$ thinks τ is true), $\mathfrak{C} \not\models \tau$ ($\mathfrak{C}$ thinks τ is false) and $\mathfrak{C} \models^? \tau$ ($\mathfrak{C}$ does not have an opinion about τ).

Definition 7 (*Semantics for atomic sentences*)**.**

[29]See (Kripke, 1975).

[30]Here we follow closely the exposition of (Maddy, 2000), the other relevant work is (Maddy, 1983).

[31]Note that in the rest of this work we will adopt this notation to denote classes and leave the set-builder notation to talk about sets.

- τ is of the form $t \in t'$ for $t, t' \in T^*$, then:
  - $\mathfrak{C} \vDash \tau$ iff
    1. t is $\overline{a}$, t' is $\overline{b}$, and $a \in b$, or
    2. $t \in S$ and t' is $\overline{V}$, or
    3. $t' \in C^*$ and $t \in (t')^+_{\mathfrak{C}}$
  - $\mathfrak{C} \nvDash \tau$ iff
    1. t is $\overline{a}$, t' is $\overline{b}$, and $a \notin b$, or
    2. t is $\overline{V}$ and $t' \in S \cup \{\overline{V}\}$, or
    3. $t \in C^*$ and $t' \in S \cup \{\overline{V}$, or
    4. $t' \in C^*$ and $t \in (t')^-_{\mathfrak{C}}$
- If τ is of the form $t = t'$ for $t, t' \in T^*$, then $\mathfrak{C} \vDash \tau$ iff t and t' are the same term, and $\mathfrak{C} \nvDash \tau$ iff t and t' are different terms.

For complex sentences truth and falsity is defined via the strong Kleene rules.

Definition 8 (*Semantics for complex sentences*)**.** For sentences σ and τ,

1. $\mathfrak{C} \vDash \sim\sigma$ iff $\mathfrak{C} \nvDash \sigma$; $\mathfrak{C} \nvDash \sim\sigma$ iff $\mathfrak{C} \vDash \sigma$.
2. $\mathfrak{C} \vDash \sigma \,\&\, \tau$ iff $\mathfrak{C} \vDash \sigma$ and $\mathfrak{C} \vDash \tau$; $\mathfrak{C} \nvDash \sigma \,\&\, \tau$ iff $\mathfrak{C} \nvDash \sigma$ or $\mathfrak{C} \nvDash \tau$.
3. $\mathfrak{C} \vDash \forall x\phi$ iff for all $t \in T^*, \mathfrak{C} \vDash \phi(t/x)$; $\mathfrak{C} \nvDash \forall x\phi$ iff for some $t \in T^*, \mathfrak{C} \nvDash \phi(t/x)$.

One can then define $\vee, \supset, \equiv$ and $\exists$ from these in the usual way

Definition 9 ($\sqsubseteq$)**.** If $\mathfrak{C}$ and $\mathfrak{C}'$ are two $\mathfrak{L}$−structures, then $\mathfrak{C} \sqsubseteq \mathfrak{C}'$ iff for all $t \in C^*, t^+_{\mathfrak{C}} \subseteq t^+_{\mathfrak{C}'}$ and $t^-_{\mathfrak{C}} \subseteq t^-_{\mathfrak{C}'}$

.

Proposition 3.9 (Monotonicity)**.** *If $\mathfrak{C} \sqsubseteq \mathfrak{C}'$, then for any sentence σ, if $\mathfrak{C} \vDash \sigma$, then $\mathfrak{C}' \vDash \sigma$, and if $\mathfrak{C} \nvDash \sigma$, then $\mathfrak{C}' \nvDash \sigma$.*[32]

This tells us that once a sentence is decided, adding more elements to the extensions and antiextensions of classes does not disturb this fact.

With this machinery in place we construct the following sequence of $\mathfrak{L}$−structures:

$$\mathfrak{C}_0 = \{(\widehat{x}\phi, \widehat{x}\phi^+_0, \widehat{x}\phi^-_0) : \widehat{x}\phi \in C^*\} \text{ where } \widehat{x}\phi^+_0 = \widehat{x}\phi^-_0 = \emptyset$$

$$\mathfrak{C}_{\alpha+1} = \{(\widehat{x}\phi, \widehat{x}\phi^+_{\alpha+1}, \widehat{x}\phi^-_{\alpha+1}) : \widehat{x}\phi \in C^*\} \text{ where } \begin{cases} \widehat{x}\phi^+_{\alpha+1} = \{t \in T^* : \mathfrak{C}_\alpha \vDash \phi(t/x)\} \\ \widehat{x}\phi^-_{\alpha+1} = \{t \in T^* : \mathfrak{C}_\alpha \nvDash \phi(t/x)\} \end{cases}$$

For λ a limit ordinal,

[32] Here and in the rest of the section we omit the proofs of the results for the sake of brevity, the interested reader can find these in (Maddy, 2000), or otherwise should be able to provide them themselves without too much difficulty given an adequate supply of patience.

$$\mathfrak{C}_\lambda = \{(\widehat{x}\phi, \widehat{x}\phi^+_\lambda, \widehat{x}\phi^-_\lambda) : \widehat{x}\phi \in C^*\} \text{ where } \begin{cases} \widehat{x}\phi^+_\lambda = \bigcup_{\alpha<\lambda} \widehat{x}\phi^+_\alpha \\ \widehat{x}\phi^-_\lambda = \bigcup_{\alpha<\lambda} \widehat{x}\phi^-_\alpha \end{cases}$$

Define an $\mathfrak{L}-$structure U (for universe):

$$U = \{(\widehat{x}\phi, \widehat{x}\phi^+, \widehat{x}\phi^-) : \widehat{x}\phi \in C^*\} \text{ where } \begin{cases} \widehat{x}\phi^+ = \bigcup_{\alpha\in Ord} \widehat{x}\phi^+_\alpha \\ \widehat{x}\phi^- = \bigcup_{\alpha\in Ord} \widehat{x}\phi^-_\alpha \end{cases}$$

We are interested in U. By monotonicity, whatever becomes true or false at one of the $\mathfrak{C}_\alpha$, remains true or false in U. For example, $\mathfrak{C}_0 \vDash \overline{\emptyset} \in \overline{\{\emptyset\}}$ so $\overline{\emptyset} \in \hat{x}(x \in \overline{\{\emptyset\}})^+_1$ and so $\mathfrak{C}_1 \vDash \overline{\emptyset} \in \hat{x}(x \in \overline{\{\emptyset\}})$ thus $U \vDash \overline{\emptyset} \in \hat{x}(x \in \overline{\{\emptyset\}})$.

Note that since $\hat{z}(z = t \vee z = t')$ is not the same symbol as $\hat{z}(z = t' \vee z = t)$ it is easily established that:

Definition 10 (*Ordered tuple*)**.** For $t, t' \in T^*$, '(t,t')' is $\widehat{z}(z = t \vee z = t')$.

Proposition 3.10 (Equality of ordered tuples)**.** *For* $t, t' \in T^*, U \vDash ((t,t') = (u,u'))$ *iff* $U \vDash (t = u \,\&\, t' = u')$.

Notice also that these ordered classes are total, and so $U \nvDash ((t,t') = (u,u'))$ iff $U \vDash (t = u \& t' = u')$. We can define ordered n-tuples as usual: $(t,t',t'') = ((t,t'),t'')$. Now, if $x_0, \ldots, x_n$ are among the free variables of ϕ then $\hat{x}_0, \ldots, \hat{x}_n$ abbreviates $\hat{z}(\exists x_0, \ldots, \exists x_n(z = (x_0, \ldots, x_n) \,\&\, \phi))$, with z, the first variable not appearing in ϕ.

Continuing with the last example, recall that $\mathfrak{C}_0 \vDash \overline{\emptyset} \in \overline{\{\emptyset\}}$ and so $\mathfrak{C}_0 \vDash \exists x \exists y(\overline{\emptyset} \in \overline{\{\emptyset\}}) = (x,y) \& x \in y$, which means $(\overline{\emptyset}, \overline{\{\emptyset\}}) \in \hat{z}(\exists x \exists y(z = (x,y) \& x \in y))^+_1$ and so $\mathfrak{C}_1 \vDash (\overline{\emptyset}, \overline{\{\emptyset\}}) \in \hat{x}\hat{y}(x \in y)$ and by monotonicity, U agrees.

Now we can prove what Maddy considers one of the great advantages of her system, namely that the class of infinite collections is self-membered.

Theorem 4. $U \vDash \widehat{x}(x \text{ is infinite}) \in \widehat{x}(x \text{ is infinite})$

Finally, and as anounced above, the gaps in the membership relation allow for the Russell paradox to be sidestepped:

Theorem 5. $U \vDash^? \widehat{x}(x \notin x) \in \widehat{x}(x \notin x)$

Note that we take the extension of a set, say a, to consists of its members, i.e. $\{\overline{b} | b \in a\}$, and its antiextension to be everything else, i.e. $T^* \setminus \{\overline{b} | b \in a\}$. The definition of equality we gave earlier is not extensional (by extensional we just mean that if collections A and B have the same extension and antiextension, then $A = B$), since the set $\overline{a}$ and the class $\hat{x}(x \in \overline{a})$ have the same extension and antiextension but are not identical (it is of course extensional when it comes to dealing exclusively with sets, since coextensive sets are identical).

Maddy seems to find this outcome welcome since one of the aims of her theory is to render sets and classes as two clearly distinct entities. Moreover, identifying coextensional and antiextensional sets and classes would mean that we import the membership gaps present in classes onto the realm of sets, e.g. whether $\{\overline{a}\}$ has $\hat{z}(z \in \overline{a} \,\&\, \widehat{x}(x \notin x) \in \widehat{x}(x \notin x))$ as a member would be indeterminate, this would indeed be an unwelcome result. Note that failures of extensionality are not only observed across the set/class boundary, classes that are picked out by different terms can be coextensive without coinciding. Again, Maddy thinks this is an advantage of the theory since for her classes are to be understood as tightly tied to the properties that single them out and coextensive properties needn't be identified. However, she concedes that the distinctions in place may be too fine-grained since they distinguish between classes such as $\hat{x}(x \in \overline{a}$ and $\hat{y}(y \in \overline{a}$ or even between $\hat{x}(x \in y \vee x \in z$ and $\hat{x}(x \in z \vee x \in y$.

We now look at a different notion of identity.

Definition 11 ($\simeq$)**.** Let $t, t' \in T$, and z be a variable not in t or t', then if $\forall z(z \in t \equiv z \in t')$ we write $t \simeq t'$.

Note that it is now indeed the case that this relation holds between coextensional sets and classes, e.g. $U \vDash \{\hat{a}\} \simeq \hat{x}(x \in \overline{a})$, and between very similar classes, e.g. $U \vDash \hat{x}(x \in \overline{a}) \simeq \hat{x}(x \in \overline{a} \,\&\, x \in \overline{a})$. Recall that given the semantics of $\equiv$, for U to be decided about $u \in t \equiv u \in t'$ it cannot be undecided about $u \in t$ or $u \in t'$, hence, $U \vDash t \simeq t'$ is a way of expressing that t is total, i.e. $U \vDash \forall x(x \in t \vee x \notin t)$, and so as a result we have that $U \vDash^? \widehat{x}(x \notin x) \simeq \widehat{x}(x \notin x)$.

2.8.1 Looking for the axioms

We now look at the issue of finding an axiomatisation for the truths of our structure U, the goal would be to get as much information about sets as ZFC and a large amount of information about classes. The main obstacle here is the non-classical context created by the membership gaps, indeed, consider a comprehension axiom $x \in \hat{x}\phi \equiv \phi$, what Maddy calls Frege's principle, then given that the biconditional will, as remarked above, by indeterminate if either side is we cannot hope for this to be an axiom. However as the next result shows, the problem here is more general than our particular choice of conditional:

Proposition 5.1 (Curry's paradox)**.** *Any system with the following properties:*

1. $\Gamma \vdash x \in \widehat{x}\phi$ *iff* $\Gamma \vdash \phi$ *(Frege's principle)*
2. $\Gamma \cup \{\phi\} \vdash \psi$ *iff* $\Gamma \vdash \phi \supset \psi$ *(Deduction theorem)*
3. $\Gamma \vdash \phi$ *and* $\Gamma \vdash \psi$*, then* $\Gamma \vdash \psi$ *(Modus ponens)*

is inconsistent.

Maddy then considers the proof system M (after John Myhill) which counts with an infinitary axiomatisation and where the deduction theorem fails, moreover, due to the inability of $\equiv$ to express the comprehension axiom we will have a series of rules in addition to the axioms.

Definition 12 (The proof system M)**.**

- Axioms:
  1. Membership: For any $a, b \in V$, and any $t \in C^*$
     (a) $\overline{a} \in \overline{b}$ when $a \in b$
     (b) $\overline{a} \notin \overline{b}$ when $a \notin b$
     (c) $\overline{a} \in \overline{V}$
     (d) $t \notin \overline{a}$
     (e) $t \notin \overline{V}$
  2. Equality: For any distinct $t, t' \in T^*$
     (a) $t = t$
     (b) $t \neq t'$
- Rules of inference:
  1. Double negation
     $$\frac{\phi}{\sim\sim\phi}$$
  2. Conjunction
     (a) $\dfrac{\phi \qquad \psi}{\phi \,\&\, \psi}$ (b) $\dfrac{\sim\phi}{\sim(\phi \,\&\, \psi)}$ (c) $\dfrac{\sim\psi}{\sim(\phi \,\&\, \psi)}$
  3. Universal quantification
     (a) $\dfrac{\phi(t/x) \text{ for all } t \in T^*}{\forall x \phi}$ (b) $\dfrac{\sim\phi(t/x) \text{ for some } t \in T^*}{\sim\forall x \phi}$
  4. Frege's principle
     (a) $\dfrac{\phi(t/x)}{t \in \widehat{x}\phi}$ (b) $\dfrac{\sim\phi(t/x)}{t \notin \widehat{x}\phi}$

When σ can be derived in this system we write $\vdash_M \sigma$. M would be successful if it would be able to proof what U thinks true and only this, i.e. $\vdash_M \sigma$ iff $U \models \sigma$ (and $\vdash_M \sim\sigma$ iff $U \not\models \sigma$).

Proposition 5.2 (α-completeness of M w.r.t. U)**.** *For all sentences σ, and any ordinal α, if $\mathfrak{C}_\alpha \models \sigma$, then $\vdash_M \sigma$ and if $\mathfrak{C}_\alpha \not\models \sigma$, then $\vdash_M \sim\sigma$.*

To get from this to the completeness of M with respect to U it is enough that this construction reached a fixed point, i.e. $\mathfrak{C}_\alpha = \mathfrak{C}_{\alpha+1}$ for some α for then $U \models \tau$, would reduce to $\mathfrak{C}_\alpha \models \tau$.

Since we know that for any α, if $\mathfrak{C}_\alpha \models \tau$, then $U \models \tau$, to get from here to soundness of M with respect to U it suffices to know that if $\vdash_M \tau$ there is some α such that $\mathfrak{C}_\alpha \models \tau$. Since all the axioms are true in all the $\mathfrak{C}_\alpha$s we need to show that if the antecedent of a rule is true in some $\mathfrak{C}_\alpha$, then there is some α' such that $\mathfrak{C}_{\alpha'}$ forces its consequent. The problem comes when dealing with the rule for universal quantification and again the existence of a fixed point would solve the problem. Although advantageous for our purposes there is no such a fixed point in the construction

Theorem 6 (Tait)**.** *For any formula $\phi(y,x)$, there is a formula $\psi(z)$ s.t. $\widehat{z}\psi^+ = \widehat{x}\phi(\widehat{z}\psi, x)^+$ and $\widehat{z}\psi^- = \widehat{x}\phi(\widehat{z}\psi, x)^-$. In fact, for all α, $\widehat{z}\psi^+_{\alpha+1} = \widehat{x}\phi(\widehat{z}\psi, x)^+_\alpha$ and $\widehat{z}\psi^-_{\alpha+1} = \widehat{x}\phi(\widehat{z}\psi, x)^-_\alpha$.*

This theorem provides us with interesting examples, including one undermining the possibility of a fixed point.

Proposition 6.1. *Suppose that $\phi(y,x)$ is $\forall w(w \in x \supset w \in y)$. If ψ is formed as in the proof of Tait's theorem,*[33] *then:*

1. *For any α, $\alpha \notin \widehat{z}\psi^+_\alpha$.*
2. *For any α, $\exists n \in \mathbb{N}$ s.t. $\alpha \subseteq \widehat{z}\psi^+_{\alpha+n}$ and $\overline{\alpha} \in \widehat{z}\psi^+_{(\alpha+n)+2}$*
3. *$\widehat{z}\psi^-$ is empty.*

This shows that new ordinals will be entering $\widehat{z}\psi^+$ at arbitrarily high up stages, and so the construction cannot become constant and so no fixed point exists. Even if we cannot solve question about soundness and completeness of M with respect to U thanks to a fixed point, we can do this directly. First we turn our attention to the following class:

Proposition 6.2. *Suppose that $\phi(y,x)$ is $x \notin y$. If ψ is formed as in the proof of Tait's theorem (call such a class E, we also write $Ord(x)$ to say that x is transitive and $\in$-connected, also recall $\hat{z}\psi$ is the class we considered in the previous proposition), then:*

1. *$E^+ = E^- = \emptyset$*
2. *$U \models^? Ord(E)$*
3. *$U \models^? E \in \widehat{z}\psi$*

So for all $t \in T^*$, $U \models^? t \in E$.

We also note that the proofs in M and the construction of U go along hand in hand.

Proposition 6.3 (α-soundness of M w.r.t. U)**.** *For all sentences σ, if σ is provable in α steps, then $\mathfrak{C}_\alpha \models \sigma$, and if $\sim\sigma$ is provable in α steps, then $\mathfrak{C}_\alpha \not\models \sigma$.*

[33] Again, the reader is referred to (Maddy, 2000, p. 308), for such a proof.

But recall that our model of interest is U, note then that M would be sound in the sense that if $\vdash_M \tau$, then $U \vDash \tau$, but what we are really interested in is to know whether the rules are a dependable way to move from truths in U to truths in U (this is what we will mean in the sequent when we refer to soundness), and for this we would need that if the antecedents of a rule are true in U so it is its consequent.

Proposition 6.4. *If there is a $\phi(x)$ be s.t.*

1. $\forall t \in T^* \, \exists \, \alpha_t$ *s.t.* $\mathfrak{C}_{\alpha_t} \vDash \phi(t/x)$.
2. $\forall \alpha \, \mathfrak{C}_\alpha \vDash^? \forall x \phi$.

it follows that M *is both incomplete and unsound w.r.t. U.*

We now turn to the task of finding such a formula, since we know that ordinals enter $\hat{z}\phi$ at cofinal stages we might think of $Ord(x) \supset x \in \hat{z}\psi$ as a candidate, however the problem is that the strong Kleene interpretation of $\supset$ requires that $Ord(t)$ is false for any t of which $Ord(t)$ is not in $\hat{z}\psi$, E is a counterexample to this, since we saw that $U \vDash^? Ord(E)$ and $U \vDash^? E \in \hat{z}\psi$.

We now consider a variant of this formula, making $Ord(x)$ total by making it false of all non-ordinals for this we relativise it to $\overline{V}$, let then $Ord^*(x)$ be $x \in \overline{V} \,\&\, [Ord(x)]^{\overline{V}}$.

Proposition 6.5. *Suppose that $\phi(x)$ is $Ord^*(x) \supset x \in \widehat{z}\psi$, then*

1. $\forall t \in T^* \, \exists \, \alpha_t$ *s.t.* $\mathfrak{C}_{\alpha_t} \vDash \phi(t/x)$.
2. $\forall \alpha \, \mathfrak{C}_\alpha \vDash^? \forall x \phi$.

And thus we see that the system M is both incomplete and unsound.

2.9 Summary

In this chapter we saw a variety of different of formal theories of collections that although trying to still capture some of our pre-theoretical notions of collection, are careful through very different devices not to fall prey of the paradox that brought down their Fregean predecessor explored in §1.3.

We in §2.2 by looking at ST, a typed theory which imposes restrictions in the language in order to limit the instances of the comprehension schema in a way that excludes the problematic cases. In §2.3 we looked at NF, as well as its class extension ML, a system that encodes the solution to the paradoxes offered by type theories through the notion of a stratified formula, but is otherwise type-free. In §2.4 we looked at the theory of sets whose use is most widespread in the mathematics of today, ZFC were we saw how the paradoxes are avoided by imposing the doctrine of limitation of size, were we saw that if some too big things such as the domain are a set then contradiction ensues.

In §2.5 we looked at the extension of ZFC with classes, namely, NBG. Here some collections, the proper classes fail to be sets, and in particular the problematic classes will fall into this category, here a set is demarcated as such entity that belongs to some other collection, and comprehension is limited through the notion of predicative formula. We also looked at MK a similar theory which allows for unrestricted comprehension, but where again the diagnosis is that the Russell class fails to be a set.

In §2.6 we studied theories such as A, S and P this theories emphasise the idea that sets are generated through an open iterative process and so at no given point we can fix for good what is the meaning of talk of all the sets, hence the appeal from these to notions of imagination as well as their intuitionistic flavour. This idea was then given a formal framework in §2.6.1 when looking at intuitionistic semantics for set theory.

In §2.7 we looked at Schindler's theory. This gives us a class for every open formula, but although sticking to classical logic uses the notion of range of significance to block the paradoxical instances of comprehension, this is motivated in the idea that although always true or false some assertions incur in categorical mistakes. In the case of the paradoxes we saw how Schindler blames in a circularity phenomenon as seen by the fact that the Russell class is not in its own range of significance. In §2.7.1 we turned our attention to a consistency proof of the theory, as well as more briefly at the introduction of urelements.

Lastly, in §2.8 we looked at Maddy's theory of classes. We saw that as in the case of Schindler's here we will also have a class for every property. In contrast, now we drop classical logic and adopt a paracomplete system. We can then show using a hierarchy of structures that thinks that the paradoxical Russell statement is neither true nor false but undecided, and so no contradiction arises. In any case, in §2.8.1 we looked at an axiomatisation of the theory that was shown to be incomplete and unsound.

Chapter 3

What a theory of collections could not be

3.1 Introduction

In §1 we motivated the need to design formal theories of collections more nuanced than that originally created by Frege in order to deal with the antinomies that apply to it. In §2 we explored a series of such theories, these came in different groups such as type theories, like ST, theories of sets without classes, such as ZFC or NF, theories of sets with classes, like A or NBG or theories of classes, like that of Schindler.

In this chapter, we will examine some characteristics that we find a suitable theory of collections should meet. Our focus will be mainly on the issues of unrestricted quantification, §3.3, and self-instantiating properties, §3.2. The aim of the section will be then to set the stage for the remaining of the work which will focus on theories of sets and classes, by arguing that such systems fare better than theories without classes, such as type theories like ST and canonical theories of sets without classes ZFC, when it comes to dealing with issues of universal quantification and self-instantiation.

3.2 Self-instantiation

In §1.2 we remarked how to each property we can associate a unique collection namely that featuring as members the objects satisfying it, its extension in the Fregean vocabulary. So far we have been talking of the extension class of a property being associated or determined by the property itself. However, one might opt for a more parsimonious ontology and take the extension itself to be the property. As Lewis puts it: 'The simplest plan is to take a property just as the set of all its instances', [1]

[1] See (Lewis, 1986, p50).

This characterisation of predicates in terms of collections is called a reduction in the sense that we have now one sort of entity, namely objects, for recall that classes are just objects which are composed of other objects, when before we had two, objects and relations. Note that, if accused of arbitrariness in this move, one can justify such a reduction in the unique relationship between a predicate and its extension just indicated. One might object to this reduction on a more principled manner by pointing out that there are distinct properties, for their intension or sense differs, but are nevertheless coextensional. Hence this will be identified after the reduction and so this cannot be a felicitous endeavour, since in a sense the reduction process is not fine-grained enough. In that case the reductionist could clarify that what Lewis in the quote above, as well as them, means by *all its instances* are all the instances in all possible worlds. So although the properties might be coextensional when restricting their extension to a particular world, such as the actual. This will cease to be the case when we consider the entire extension.[2]

Leaving the worries just voiced aside, we see that if this reduction can indeed be successfully carried out we have reasons for rejecting frameworks that do not allow a property to instantiate itself. We look in particular here at two examples one from metaphysics the other from semantics of natural language. The idea is, when translated into the framework of the theories explained in §2, that we will need it to be allowed for a set, or class or other kind of aggregate, to be contained in itself. After the reduction, this will model the state of affairs where a property instantiates itself.

Now, before moving to these examples let us consider in more detail the framework we have in mind. Suppose we had in the universe of the theory of collections, not only pure sets, but also urelements, atoms, or in short objects that are not collections. Then, if we want to render in our reduced framework that *Moby Dick is a whale* is true, we begin by seeing that this means that the object named Moby Dick, say b, instantiates the property of being a whale, M, and so Mb holds. Now suppose we have reduced the property of being a whale to the set of whales, say m, now we render the instantiation of this predicate by this object in the reduced framework by saying that b belongs to m, it is a member of the collection to which the predicate has been reduced, i.e. $b \in m$.

Moreover, we see that this reductionist picture seems to be commonplace in all the theories of sets through the various comprehension schema we have been presenting, albeit with the qualm that in those schema the reduction to the collection takes place directly from the open sentence and not through the associated property. In any case, we have now been talking about objects instantiating properties, however, collection theories we have been considering are usually taken to be dealing with pure sets, i.e. sets that have as members only other collections. Now, if we interpret these collections as reduced properties these frameworks also allow us to speak of properties that instantiate other properties, so coming back from our reduced property of being a whale m, this property is identical to itself, and so it instantiates the property of being self-identical, say that this corresponds to the collection V, then we could express this formally as $m \in V$. However, presumably also the property of being

[2] (See Lewis, 1986, p51)

self-identical is self-identical to itself and so $V \in V$. For a less trivial example consider the property of being infinite, presumably there are infinitely many infinite things, indeed one could just think about co-finite subsets of the natural numbers to illustrate this point. So if this property seems again to instantiate itself. Now, this however is definitely something not expressible in the standard set-theoretic account, ZFC, or any other system adopting the axiom of regularity decreeing all its entities to be well-founded.

A final note of caution regarding self-instantiating properties seems in order. Indeed, although in the rest of this section we will provide some reasons to allow into our theory collections those aggregates corresponding to self-instantiating properties, one can immediately recall that these were precisely, the kind of property that self-refers, the ones that landed us in Russell's paradox and out of Frege's garden of Eden in the first place. Indeed, recall that in ST's synatctic constraints are set out precisely to rule out such instances. So what this means is that even if we have strong theoretical reasons to allow for such properties in our system, a felicitous paradox-free implementation of this desiderata in practice will be an endeavour requiring the utmost care.

3.2.1 A metaphysical excursion

We now return, as promised above, to some applications of a theory of collections, or perhaps more perspicuously of properties, that makes use of self-instantiantiation in the way we have just been discussing. The first concerns the metaphysical puzzle known as Bradley's regress, first formulated by the British idealist F.H. Bradley at the end of the XIX century. [34] We now give an outline of the problem, although Bradley originally used the example of a lump of sugar and its properties, we can here use the already familiar example of Moby Dick. As we pointed out, we say that Moby Dick instantiates the property of being a whale. However, one could note that this instantiation is just another relation, say I, and so we not only have that Mb, but also that IMb, or in Bradley's words: 'There is a relation C [I in our example], in which A [M in our example] and B [b in our example] stand; and it appears with both of them'[5] But if we now try to describe the situation as before when talking about Mb we must say something along the lines of that Moby Dick and being a whale are an instantiation of the relation I, which is itself some instantiation relation, and so following Bradley: 'If so, it would appear to be another relation, D [our second instantiation relation], in which C, on one side, and, on the other side, A and B, stand.' [6]. Note also that at face value this instantiation relations are different since one is a 2-place while the other is a 3-place relation. Bradley concludes that:

[3]See for the original formulation (Bradley, 1893, §2)

[4]The reader who has a preference for more contemporary metaphysical puzzles might want to consider the effort to explain what grounds the grounding relation, the interested reader able to parse the title are referred to (Litland, 2017), for an instance where self instantiating properties might also be helpful.

[5]See (Bradley, 1893, p. 19).

[6]See (Bradley, 1893, Ibid.)

> [S]uch a makeshift leads at once to the infinite process. The new relation D can be predicated in no way of C, or of A and B; and hence we must have recourse to a fresh relation, E, which comes between D and whatever we had before. But this must lead to another, F; and so on, indefinitely. (Bradley, 1893, Ibid.)

Indeed, the idea being that we need an infinite amount of instantiation relations to account for, what we could call, the metaphysical glue holding the object and property it instantiates together. Since for any series of properties or objects that instantiate some (n-place) property we can always take a further distinct ($n + 1$-place) instantiation relation as relating them and like this *ad infinitum*. On the metaphysical side, the upshot of the argument seems to be to undercut the idea that objects and properties that they seem to instantiate are really related since our attempt at explaining their interaction leads to an infinite number of relations and so cannot be taken as satisfactory. This is because we seem to be shifting the problem one level up each time without ever actually addressing it.

The literature offers several attempted solutions of this puzzle. Firstly, one could bite the bullet and accept that such a regress is vicious but that it does not arise in the first place. The mischaracterisation in the way we have described the situation lies in the fact that, although we have talked about instantiation as being a relation, it really is not, as David Armstrong puts it:'We have to allow the introduction of a fundamental tie or nexus: instantiation.'[7] Instead of a common or garden property, instantiation is the brute metaphysical glue that binds a relation with its relata. And so there is an infinite regress but this is logical and not ontological, since: 'As we go on expanding the regress, our statements remain true, but no new truth-maker, or ontological ground, is required for all these statements to be true.', [8] but just this nexus provided by the porperty-like entity of instantiation.[9]

Another solution stems from a distinction drawn by Francesco Orilia in the ways one can read the regress: an externalist and an internalist one.[10] In the internalist reading we have a single state of affairs containing an infinity of different instantiation relations, however in the externalist reading we have an infinity of states of affairs and these, in turn have finitely many instantiation relations each.[11] For Orilia only the internalist regress is a vicious one,[12] since this requires us to have states of affairs with infinitely many constituents, as opposed to simply infinitely many finitely-constituted states of affairs. This would force us to admit that the world must have infinite complexity, and this 'is in conflict (...) with the basic intuition that a simple fact like Fa must have a finite number of primary constituents',[13]whereas admitting infinite chains of evidence as in the externalist reading does not contradict any basic intuition:

> That at any given stage we can continue the explanatory task does not show that no

[7](Armstrong, 1989, p. 109)
[8](Armstrong, 1989, p. 110)
[9]For a discussion of such an approach consult (Allen, 2016, p. 32).
[10]See (Orilia, 2006, p. 216)
[11]See (Orilia, 2006) section 3.
[12]See (Orilia, 2006) sections 6 and 7.
[13]See (Orilia, 2006, p. 228)

knowledge or no understanding is provided at any stage. It merely shows that at no stage we know/understand everything that there is to know/understand about the explicandum which gives rise to the explanatory chain. (Orilia, 2006, p. 232)[14]

More relevant for our focus on self-instantiating properties are approaches that try, as Armstrong does, to stop the regress, but unlike him do take instantiation to be a genuine relation. Indeed, the idea here would be to push-back on our assertion above that the several instances of an instantiation relation present in the regress must be different since they have different arities, in the sense that we are dealing with a single relation albeit an unusual one in the sense that it is variably polyadic. This would mean that it can relate different numbers of objects in different instances, and so firstly the instantiation relation is diadic, then, triadic, and so on. Hence we see that, if we allow properties to self-instantiate what we have is that there is no infinite regress since we have that there are only three entities present here, Moby Dick, the property being a whale and the instantiation relation, or better its instances. Even though this relation or, again, its infinite chain of instances of increasing arity, instantiates itself infinitely often.[15]

3.2.2 An expedition into natural language semantics

Leaving the metaphysical camp for another more worldly area where self-instantiating properties are useful, we turn our attention to the semantics of natural language. Indeed, if we want our reduced properties to be the basis of a formal semantics system that adequately models natural language inferences it seems plain that self-instantiation must be a feature if we want to accommodate inferences like the following:[16]

1. Everything has the property of being self-identical.

∴ The property of being self-identical has the property of being self-identical.

Indeed, this inference could be rendered as the valid inference from $\forall x(x \in V)$ to $V \in V$, via the universal instantiation rule. Recall V is the reduction of the property of being self-identical, i.e. the universe of discourse. Note also that, as Menzel points out[17], this inference is exactly analogous to the one that is indeed warranted by well-founded theories of sets, namely

2. Everything has the property of being self-identical.

[14] Again, for a discussion of such an approach consult (Allen, 2016, p. 30).

[15] Note that a similar solution would be to take the several instantiation relations in the regress not as equal but as inexactly resembling instances of a single relation. The idea can be illustrated thinking that aloe, tulip or petunia are instances of the property being a flower, for more details on inexact resemblance consult (Allen, 2016, §2.3.3). In any case, it does nOt really matter whether we opt for a multigrade relation or a group of inexactly resembling instances of a relation since in both cases we would need these properties to self-instantiate.

[16] This example is taken from (Chierchia & Turner, 1988, p. 253).

[17] See (Menzel, 1986, p. 2)

∴ Moby Dick has the property of being self-identical.

Or, formally from $\forall x(x \in V)$ to $b \in V$, via the universal instantiation rule. So one proponent of well-founded theories of sets that rejects the first inference as illegitimate while accepting the second might be legitimately accused of cherry-picking.

Now, putting aside self-reference for a moment, we look at some specific challenges that face typed theories such as ST (Russell's theory discussed in §2.2). We have already mentioned some problems for these approaches when it comes to universal quantification, we now turn to issues with existential quantification when we come to analysing natural language. Consider the following inference:[18]

3. Being a whale is less common that being an atom.[19]

∴ Something is less common that being an atom.

Indeed, this inference could be rendered as a (second-order) valid inference from $(B, A) \in L$ to $\exists X^1((X^1, A) \in L)$, via the existential weakening rule. Now, this would indeed be valid in a typed theoretic framework. But note that the existential quantifier in the conclusion is ranging over entities of type 1, since we take objects like Moby Dick to be of type 0 and so predicates, or their reductions containing objects of type 1, incidentally the predicate L will be of type 2. Now consider the following inference which as in the case of (1) and (2), is exactly analogous to (3):

4. Moby Dick is a whale.

∴ Something is a whale.

This inference is also valid in a typed framework using existential weakening. However, we see a disanalogy with the formalisation of (3), since here we would write that from $b \in M$ we conclude $\exists x^0(x^0 \in M)$. And so we see that whereas there on the surface no difference between the natural language deductions (3) and (4), there is indeed one when rendering it in a typed framework. Indeed there is a divergence in the formal regimentation since in the one case we use the existential quantifier of type 1 and in the other of type 0. In fact, there is nothing particular to these two instances, but we see that whenever we use existential weakening in a deduction there is an ambiguity with respect to which typed quantifier we will use. In fact, notice that here it was easy to write down the inferences since the entities involved were just objects and properties of objects, but if the entities in use are of a very high type we might have *actual* problems as to how to transcribe them. This is because we must first know which is the

[18]Here we are following (Menzel, 1986, pp. 3-5).

[19]Here by X being less common than Y I mean that after a reduction of properties X and Y to their extensions one cannot find a surjection from X to Y. And so (3) is indeed true since there are more atoms than whales.

type corresponding to the reduction of the natural language properties alluded to, which might be nontrivial business.

In any case, the type-theorist, would then have to, if taking his translation seriously, be prepared to sustain some kind of error-theory with regards to existential quantification. In the sense that the speaker is using a single quantifier when he should used instead a family of these. Moreover, this of ambiguity problem is not just confined to quantifiers but also permeates to relations, for consider the following statements one, familiar one new (in fact, we could have also used the conclusion of (1) here but we prefer not to in order to avoid any qualms with self-instantaition):

2C. Moby Dick has the property of being self-identical.

6. The property of being a whale has the property of being self-identical.

Here again the type-theory proponent should endorse some kind of error theory with respect to natural language and defend that, contrary to what appears, there are two predicates in use $self - identical_1$, a type 1 property, and $self - identical_2$ a type 2. Actually, we see that for any type n, there will be a different property $self - identical_{n+1}$, and so when the natural language speaker thought to be just one predicate the type-theorist ought to affirm that there are infinitely many. In fact this entire ambiguity problem might be seen as the natural language analogue of the mathematical fact observed before that in ST there is not *an* emptyset but infinitely many, and the same goes for many of the mathematical constructions, one for each (non-zero) type.

3.3 Universal quantification

We begin this section by noting, first that a prominent feature of ST is that we take objects in the domain of quantification to be stratified into different types and so in particular whenever writing a quantified statement we must specify over which of these types we intend to be quantifying over. Clearly then what such a system does not warrant us to do is say, when prompted over which types we intend to quantify, that over all types, namely over all objects in the universe. However, this seems like a shortcoming of the theory since in mathematical contexts,[20] we seem to be doing just that, namely when asserting that there is a set that is empty or infinite, ordinarily, but also more generally that everything is self-identical. As Quine forcefully asserts about ST: 'Not only are all these cleavages and reduplications intuitively repugnant, but they call continually for more or less elaborate technical manoeuvres by way of restoring severed connections.'[21]

Indeed, it seems that when using these expressions we do not intend to be understood in terms of stratified domains. Indeed we would not take it as

[20] Although one could argue that in some other contexts such as computer science typed languages are indeed a natural choice, see for instance the discussion in (Turner & Eden, 2008, §6.2).

[21] See (Quine, 1937, p. 79).

acceptable that if someone asserted that *All numbers are even* and we pointed out to them that 3 is not, they replied that this is not a counterexample since they were not meaning to quantify over the type of 3. This does not seem a valid not a valid reply to his universal statement.

More generally, what we are doubting here is the distrust of unrestricted quantification, the worries however are not merely the ones attaining to our ordinary use of the quantifiers sketched above but, might even rest in the logical inconsistency of the position. This comes from observing that when one says that we cannot quantify over everything, this entails that there is something we cannot quantify over.[22]

3.3.1 Against unrestricted quantification

We concluded the previous section by doubting the coherence of rejecting unrestricted quantification after giving some reasons in favour of universal quantification. However, in this section we give an argument that doubts the coherence of this position. Unsurprisingly, this argument will have a very similar appearance as Russell's paradox but in a more general fashion since, for instance, we will not be referring here to extensions of predicates. Indeed, this similarity is not surprising since the type theory proposed by Russell as well as Zermelo's restricted comprehension principle sidestep the paradoxes by preventing universal quantification. [23]

Now, for the argument. Here we will follow Williamson's presentation[24] and start by noticing that when talking about logically valid inferences, such as *modus ponens*:

1. $\forall x Px$
2. $\forall x(Px \supset Qx)$

$\therefore$ $\forall x Qx$

The validity of such inference is preserved regardless of how we interpret the predicate letters P and Q. So it must be possible to interpret these predicate letters by some (legitimate) interpretation of any predicate of our meta-language, put more formally:

3. For everything x, $I(F)$ is an interpretation in which P applies to x iff Fx.

Now, at this point we make one of the key observations of the argument, namely that since quantifiers are unbounded here they range also over interpre-

[22] Strictly speaking what is self-defeating is to utter is that *It is impossible to quantify in my current language over everything*, see (Willamson, 2003, §v) for details.

[23] This idea that when we quantify over everything we are really quantifying over the members of some domain, that which is not itself in the scope of the quantifier amounts, as we will see in §3.3.3, to a denial of the All-in-One principle.

[24] See (Willamson, 2003, §IV)

tations, and so these entities are after all objects in the domain, as Williamson puts it:

> On the naive theorist's maximally liberal understanding of 'thing', even an interpretation such as I(F) counts as a thing: to claim that it is not a thing would be self-defeating. (Willamson, 2003, p. 426)

The only thing that is left in order to complete the argument is to be able to define the following predicate R, which we do as follows:

4. $\forall x(Rx$ iff x is not an interpretation in which P applies to $x)$.

Since the ability to form R is key for the argument we need to be sure that such definition is legitimate. For Williamson this is the case since: 'it is well-formed out of materials entirely drawn from the naive theory [the theory of unrestricted quantification] itself.', [25] and so for him there is no room for a supporter of unrestricted quantification to complain since all the elements in the predicate are accepted as part of their theory.

To continue with the argument we apply (3) to R:

5. For everything x, $I(R)$ is an interpretation in which P applies to x iff Rx.

Now we apply the definition in (4) so:

6. For everything x, $I(R)$ is an interpretation in which P applies to x iff x is not an interpretation in which P applies to x.

But since we take interpretations to be objects we can universally instantiate (6) with $I(R)$:

7. $I(R)$ is an interpretation in which P applies to $I(R)$ iff $I(R)$ is not an interpretation in which P applies to $I(R)$.

Contradiction!

Now, how can the supporter of unrestricted quantification respond to this argument? According to Linnebo,[26] one thing we can do is turn our attention to the key observation above endorsed by Williamson that interpretations of our language are just objects like any other, since after all we are able to quantify over them using the same (first-order) quantifiers as for any other entity. The move then is to introduce higher order variables to range over interpretations and hence deny that these are objects. As we will see in the next section, this will entail the introduction of ST back into the picture.

[25] See (Willamson, 2003, ibid.)
[26] See (Linnebo, 2006, §6.3).

3.3.2 ST revisited

We concluded the previous section by indicating that a way to block the argument against the coherence of universal quantification by insisting that interpretations are not in the range our first-order quantifiers since these are second-order entities and so no instantiation of such a variable as it is done in the derivation of the paradox is acceptable. In fact, it is no longer the case that the problematic predicate R can be defined since it would incur in the mistake of taking a first order variable x to be a second-order entity such as an interpretation.

However, the story is more complicated than what we just presented since it is not only that we need second-order entities to block the paradox but arbitrarily many finite types, and so a theory like ST. Indeed, as Linnebo observes:

> The Semantic Argument challenges us to develop a general semantics for some first-order language $\mathcal{L}_1$, (...) The response just outlined develops a general semantics for $\mathcal{L}_1$ in a second order language $\mathcal{L}_2$. (Linnebo, 2006, p. 152)

And so it is clear that when developing the semantics of our language we run into trouble with the concept of interpretation of that language. This we take, to avoid problems, to be an entity of some other second order language, and so to explain the semantics of our first order language we go to a second order language. Now, the obvious worry presents itself by asking how to explain the semantics of this second order language, and the answer as before that we will now need a third order language and so on until we have the full typed language of ST available.

One might find curious that we bring in type theory into the picture to save a theory that was seen, as we noted above, as a reaction against such typed frameworks in the first place. In fact the worry is deeper, since it is the case that the semantic picture sketched above that prompted the adoption of a typed homework entails some consequences which cannot be expressed in a such a framework. For instance, consider the assertion:

Unique Existence. Every expression of every syntactic category has a semantic value which is unique, not just within a particular type, but across all types.[27]

Indeed, according to our response, expressions of type 1 will have a semantic value (interpretation) of type 2, those of type 2 of type 3 and successively and so in this sense we say that the value is unique across types, since it will indeed be found in only one such entity, but again since as we pointed out before type theories do not allow to quantify over all types this principle is not expressible in the system.

Hence, it seems that such a proposed solution is of not a lot of help since one of the main aims of allowing for unrestricted quantification was to gain in

[27] For this and other examples as well as a discussion of possible replies by the type theorist look at (Linnebo, 2006, pp. 154-55)

expressive power. However this proposal will make us adopt in order to attain this a framework which carries its own limitations of expressive power. We seem to be between Scylla and Charybdis, in some sense, it would be much better if we did not need to appeal to a framework with its own manifest expressive shortcomings in order to achieve our aim of defending a system we prefer for its improved expresssive power. Hence, it would be preferable to solve the challenges presented without appealing to a typed framework is called for[28] Note, however, that, as Linnebo points out, this freedom of type restrictions comes at a cost:

> The cost is the reinstatement of the first premise (...) of the Semantic Argument [interprtations are taken to be objects]; for the first-order variables of the meta-language will then be allowed to range over interpretations. This will remove the type-theorists' defense against paradox. (Linnebo, 2006, p. 156)

Indeed, since now again all entities in our language will be objects the only way to block the paradoxical argument from §3.3.1 will be then to deny that such a predicate R in operation there is well-formed, the challenge of course will be to do so in a principled and non ad-hoc manner. Some ways to do so were surveyed in §2 when explaining how type-free collections avoid the paradox, the degree of success of such attempts will be judged in more detail in the remainder of this work. For now, the remarks here were aimed to give some reasons why a responses to the challenge presented using type theory should be rejected.

3.3.3 All in One principle

We turn our attention to the notion of universe of discourse or universe of quantification itself which. This is of course relevant to our discussion since we have been defending throughout universal quantification and so one question we ought to ask ourselves is whether this domain will itself constitute an object. As Catwright puts it this notion of domain ought to be used cautiously:

> [Speaking of the universe of discourse] involve[s] a certain risk, the risk of being understood to imply that the universe of discourse is an object-a set, or class, or collection-of which the values of the variables of the language are the members. The implication must simply be disavowed: to say that the universe of discourse of a language comprises the ordinal numbers is to say no more than that the ordinal numbers are the values of the variables of the language.(see R. Cartwright, 1994, p3)

The inference that Catwright tells us to guard ourselves against is what he calls the All-in-One principle (AiO), namely that the objects in a domain of discourse constitute an object. In fact this principle is familiar from mathematics, for instance when taking the domain of a model to be a set, or the worlds in a Kripke-frame to be the members of some set.

Now, recall here that what we are trying to argue is that their inability to quantify over everything is a problem for typed theories. The idea then is that

[28] Of course we already gave independent reasons not to adopt such a framework in section §3.2 when dealing with self-instantiating properties.

the proponent of typed theories will point out to the All-in-One principle as an unwanted consequence of the view and hence as a tool to make us abandon it. One could first meet the challenge and affirm that there is indeed a universal collection. The issue is then to characterise the nature of the collection. It cannot be a set, at least as ordinarily understood in the context of ZFC, the usual theory of sets, since in this theory there is a doctrine of limitation of size in operation. It could be however a set in the sense of some non-classical theories like NF, there are of course questions that such a position will have to answer, for instance whether such an understanding of set as different from the more common one is warranted in the first place, we will return to these issues when we discuss NF and the notion of set in more detail in subsequent sections. All we should note at the moment is that defending that such a universal set is, at least at the outset, a reply open to the supporter of universal quantification.

Another, perhaps more appealing, way of proceeding after accepting the All-in-One principle is to assert that this collection in question is not a set but rather a (proper) class. And so we would be supplementing our ontology, if we take the domain of discourse to be the universe of sets, with an object containing all sets but not itself a set. Again, the notion of class as well as the philosophical merits of different theories of classes explored in §2 will be explored in more detail in subsequent chapters. What is relevant now is that the adherent to the AiO principle need not be committed to the existence of a universal set.

What must however be noted is that any adherence to AiO will entail a departure from ZFC the usual theory of understanding sets. Hence, one might think this is sufficient reason to prefer to reject the principle altogether. Indeed, Cartwright himself boldly remarks that 'There would appear to be every reason to think it false.', [29] as for him it seems to rely on the equivocation that whenever there is a group of some objects (the objects quantified over in the given domain) there is some corresponding object that has all of them as members (the purported universal collection). Indeed, even if we can find many instances were this is the case, the week collecting the different days or ω the different natural numbers, it seems a generalisation requiring further support that this is a must. This distinction seems in line with the contrast between Russell's use of the terms class-as-one and class-as-many or Cantor's consistent and inconsistent multiplicities.[30]

Nevertheless, in what follows we will disagree with Cartwright's indication above and abide by the AiO, since this seems to follow from taking our reduction of properties seriously,[31] again quoting Linnebo:

> Moreover, given that we want to allow quantification over absolutely everything, we have no choice but to accept that a predicate can be true of absolutely everything. Such a predicate must thus have as its semantic value an object that somehow collects or represents absolutely all objects, including itself.(Rayo & Uzquiano, 2006, p. 156)

[29] See (R. Cartwright, 1994, p8)

[30] See (Russell, 1937, p. 104) and (Cantor, 1899, pp. 113-7), respectively.

[31] In fact, it suffices to take the more modest position that for every property there corresponds an object, namely its extension, without having to endorse that the property *is* the extension.

Indeed, since we do take self-identity to be a property, in fact it is not even an outright paradoxical property like that of Russell, as well as the idea that for each property, or if the reader prefers non-paradoxical property at least, there corresponds an object with the objects instantiating it as members, or for the reductionist that such object is the property. We do accept that there is some object that will be some aggregate containing all self-identical things, i.e. everything.

3.4 Summary

In this chapter we have presented some considerations that a satisfactory theory of collections must take into account. We have focused in the need to accommodate self-instantiating properties, in order to account for natural language inferences, §3.3.2, and metaphysical puzzles, §3.3.1, as well as unrestricted quantification. We noted that theories like ST or ZFC are not the best suited to deal with these desiderata, for instance in §3.3.2 we suggested that types theories are not the best ways to deal with the problems posed for unrestricted quantification in §3.3.1, in , §3.3.3 we highlited how our endorsement of the All-in-One principle is incompatible with a theory like ZFC where there are only sets and these are size-limited. The task at hand seems then to opt for theories that allow us to avoid paradox but not at a price we are not prepared to pay, i.e. giving up on our desiderata. As suggested by our catalogue of theories we now turn to theories of sets and classes in order to get this job done, the basic idea being that, as we saw in §2, it will be classes and not sets the entities associated to the most ill-behaved properties. Hence, in the next chapter we will explore more closely the central notions of these, namely sets and classes.

Chapter 4

Sets and classes

4.1 Introduction

In the previous chapter we gave some reasons in favour of allowing our formalisation of a theory of properties and objects to be type-free or allow for a universal collection, due to issues having to do with, for instance unrestricted quantification or self-instantiation. As we saw in chapter §2 most of the theories that allow such features are theories of sets and classes. This chapter aims to clarify the notions of set and class, and, hence, identify and substantiate their differences.

We thus begin in §4.2 with some historical remarks regarding sets and classes. Since the notion of set arose and developed in the context of work by mathematicians at the end of the XIX century, in particular that of Georg Cantor, we can trace the original motivation of this notion with great precision. With regards to the notion of class, there is no such a precise point of origin in time as with sets, since this is tied to the more general notions of property, predicate or condition. We will however look at the role properties had in the approach of Russell to the work of Cantor, again at the turn of the XX century, and more specifically springing from the notion of Fregean concept that we already discussed in §1, this will be the focus of section §4.3. Next, in §4.4 we continue this journey through Cantor's paradise and focus in some novel existence principle he introduces, that of the powerset which is present in the diagonal argument investigated by Russell. This historical excursion will be worth pursuing for its own sake, however it will also serve to pinpoint very clearly that sets and classes are different entities, several ways in which the original motivations for their use makes them distinct will be dealt with in §4.5.

4.2 Cantorian sets

In 1869, Georg Cantor was a young mathematician and a former student of Karl Weirstrass in Berlin that had started working at the university of Halle-

Wittenberg and was about to obtain his *Habilitation*.[1] Cantor was challenged by his colleague Eduard Heine to fix the proof that the trigonometric series representing a function is, if it exists, unique provided this converges everywhere.[2]

Cantor managed to show the uniqueness of such series, the Fourier series, by the next year, and continued his work on the topic by generalising the result to series that diverge at finitely many points, that is to series with finitely many exceptional points, in 1871. In 1872, Cantor provided a further generalisation for series with infinitely many exceptional points, provided these are isolated, in the sense that there for any such point there are always two other points for which this is the only exceptional point between them. Moreover, if a point is not isolated we say it is a limit or accumulation point. In his proof of this latter result, Cantor defined a new notion, that of the derived set operation, which just takes a set, say E, as input and returns the set of its accumulation points, ∂E. For an example note that if E is the unit interval $(0, 1)$, i.e. $r \in E$ iff $0 < r < 1$, $\partial E = [0, 1]$. Thus, the condition for the theorem to hold can be expressed by the expression $\partial E = \emptyset$, with E of course the appropriate set of exceptional points.

Cantor also showed that the result holds when ∂E is finite and more generally whenever ∂E is infinite but these exceptional points are isolated. This fact can be easily expressed using the derived set operation, namely by $\partial(\partial E) = \emptyset$, or more succinctly by $\partial^2 E = \emptyset$. For this to hold it is again sufficient that $\partial^2 E$ is finite or if infinite its accumulation points are, zero, finite or again its derived set fulfills the aforementioned conditions. So the upshot is that it is enough if for some $n \in \mathbb{N}$, $\partial^n E$ is finite.

We then see that by studying the properties of Fourier series, Cantor had arrived to the study of sets of points with surprising characteristics. In the sense that these were big since infinite but still not too big or chaotic in order to compromise the uniqueness results, as Maddy eloquently points out: 'But what an odd set of points it was: infinite, and quite complex, yet still somehow small enough, or well-behaved enough, in relationship to all the reals, to do no damage!'[3]

This result led Cantor to the study of the real numbers in two ways. On a more practical level, since he needed a precise formulation of these numbers in order to define and iterate the derived sets of infinite sets of accummulation points, making sure these higher order sets are themselves sets of accummulation points.[4] But secondly on a more abstract note, connected to the peculiar nature of these sets we pointed out above, and relating to the comparisons of size between several infinite collections, which lead to his results regarding the equinumerosity of the rationals and the natural numbers in 1874. As well as his celebrated proof that the naturals are not equinumerous to the reals, and

[1] For a biography of Cantor consult (O'Connor & Robertson, 1998).

[2] The problem with the proof was that it computed the coefficients of the series by integrating the different terms separately, but as already overlooked by Cauchy, the notion required for such a procedure to be legitimate is not convergence but uniform convergence. (See (Lavine, 1994, pp. 33,39)).

[3] See (Maddy, 1990, p. 108).

[4] See (Lavine, 1994, p. 40).

the equinumerosity of the closed unit interval with a plane, or an n-dimensional space for that matter, whose surprise upon finding this result in 1877 was expressed by: 'Je le vois, mais je ne le crois pas' in a letter to Richard Dedekind.[5]

These series of discoveries lead Cantor to define notions familiar to contemporary mathematicians, like that of closed set or, more relevant for set theory that of cardinality in terms of existence of a bijection in 1879. Even though the study of sets was just beginning, at least that of subsets of points of the real line, his research so far already took Cantor to push the field to its limits by considering the question of whether every subset of the reals is equal to the size of the naturals or the reals themselves. Namely, that there are no intermediate cardinalities between these two, or more precisely still that the size of the continuum is the immediately succeeding cardinality to that of the natural numbers. This is what Cantor supposed, today we call this hypothesis the continuum hypothesis. Quoting Cantor himself:

> And now that we have proved, for a very rich and extensive field of manifolds [subsets], the property of being capable of correspondence with the points of a continuous straight line or with a part of it (...) the question arises ...: Into how many and what classes (if we say that manifolds of the same or different power [cardinality] are grouped in the same or different classes respectively) do linear manifolds [subsets of the real numbers] fall? By a process of induction, into the further description of which we will not enter here, we are led to the theorem that the number of classes is two (...) (Cantor, 1915, p. 45)

If we now turn our attention back to the derived set operation defined above, we see that the condition regarding the structure of the set of exceptional points of a series makes use of finitely many iterations of the operation. However, the existence of a finite iteration of the operation where the derived set is finite is equivalent to saying that after performing the operation infinitely many times (or more precisely the intersection of all the $\partial^n E$ for n finite, or the points that belong to the derived set no matter how many finite iterations we carry out), name this set ∂^ω, this derived set will be empty.[6] However, as Cantor noticed, this point set operation can be iterated even after infinitely many applications, indeed just apply the operation again to $\partial^\omega E$, to get $\partial^{\omega+1} E$ and even further into what what Cantor called the transfinite.

This is how the sequence of ordinal numbers, whose symbols were introduced by Cantor in 1880, was born, by shifting attention from the derived sets to the sequence of symbols of the sets, namely:

$$0, 1, 2, \ldots, \omega, \omega+1, \ldots, \omega+\omega, \ldots, \omega\cdot 3, \ldots, \omega\cdot\omega, \ldots, \omega^3, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots$$

Notice that in this sequence every ordinal has an immediate successor and that every infinite increasing sequence of ordinals has some ordinal as a limit. Abstracting from the ordinal sequence Cantor obtained the notion of a well-ordered set. This is a set such that any of its subsets has a least element and is

[5]See (Dauben, 1990, p. 55) and also for the negative impact this discovery had in his relation with David Kronecker see (O'Connor & Robertson, 1998)

[6]As an example note that if $E := \{0\} \cup \{\frac{1}{n} : n \geq 2\} \cup \{\frac{1}{n} + \frac{1}{n^m} : n, m \geq 2\} \cup [1, 2]$, $\partial E = \{0\} \cup \{\frac{1}{n} : n \geq 2\} \cup [1, 2]$, $\partial^2 E = \{0\} \cup [1, 2]$ and $\partial^\omega E = [1, 2]$.

totally ordered. Indeed, notice that in such a set (if infinite) every element has an immediate successor and that if we take an infinite ascending sequence of elements in the set these will have a least upper bound. We say that two well-ordered sets have the same order type if there is an order isomorphism between them, one can then think of the ordinal numbers as representatives of each of these order-types.

Cantor drew a parallel between the notion of counting an infinite collection and well-ordering it, in other words finding an order isomorphism between the collection and one of the ordinal numbers. This generalises counting a finite set since when we do so we begin by picking an element and labelling it as the first, then a second from the remaining ones and so on, this orders the elements of the set in the process. So for any subset of the set there is always an element with the smallest label and for any proper sequence of elements, there is always an element that follows these elements in the counting, unless of course the sequence exhausts the collection. Conversely, if one has a well ordered set one can read-off this order as a way of counting the set, namely start by the smallest element, and continue with the immediate successors and the least upper bounds in case of sequences.[7] The centrality of the ordinals is now apparent since for him that a collection can be counted, in this formalised sense through well-orders, is part and parcel of being a well-defined set:

> The concept of well-ordered set turns out to be essential to the entire theory of point-sets. It is always possible to bring any well-defined set into the form of a well-ordered set. (Cantor, 1883, p. 550) *as cited in* (Moore, 1982, p. 42)

Note also that it is already apparent that the theory of Cantor is not one that is to be applied to all collections. This signals a contrast with theories of classes as we will see below, these begin by trying to associate an object to any property. Indeed, if we consider the collection of ordinal numbers it is clear that this cannot be well-ordered since suppose this would be counted by an ordinal, then notice that the order type of any initial segment of the ordinal sequence is the successor ordinal of this initial segment. So the collection of ordinals preceding the ordinal counting the class of ordinals is well ordered by it, but then since this ordinal well orders a proper initial segment of the sequence of ordinals, this cannot well-order the entire sequence, contradiction![8] Hence, the distinction that Cantor draws between the Absolute and the transfinite. The former being radically different from the transfinite numbers, in the sense that these can be increased unboundedly by finding bigger and bigger successors in the sequence, but precisely because of this cofinal nature, these numbers cannot reach the Absolute infinite represented by the entirety of the sequence itself. This reasoning leads Cantor to mystically assert that: 'The absolute can only be recognised, never known, not even approxinately.'[9][10]

Cantor is thus not offering us a theory of all collections, only of those infinite collections that are sufficiently similar to finite ones in his sense of being able

[7]See (Lavine, 1994, p. 53).
[8]Incidentally, this is the reasoning behind the Burali-Forti paradox.
[9]See (Hallett, 1986, p. 42).
[10]For the theological underpinnings of Cantors notion of the Absolute, including a proof of the existence of God based in his ordinal sequence, see (Dauben, 1990, pp. 143-44).

to be, like the former, counted in the sense explained above. Cantor remains silent about collections like the class of ordinals that symbolise the Absolute and so cannot be understood by us. These he also calls inconsistent multiplicities. Since this is precisely the kind of collection that will land the class theorist in trouble with the paradoxes, it is important to ask if Cantor is warranted in taking our inability to count them, in his sense of the term, as a good reason to bar these collections from set theory.[11] Indeed, one could turn the argument against Cantor by saying that, since not all infinite collections can be counted this notion is not a good foundation for such a theory and take instead, like Frege, the notion of cardinality as foundational. According to this approach, one has not to pay attention to the ordering of the elements of a set in order to determine its number, but merely on how many of these there are. Then, no distinction between infinite sets that can and cannot be counted would seem to be called for.[12]

In 1882 Cantor also realised that there is a transfinite hierarchy of infinte sizes of sets, starting from the size of the natural numbers, by considering the set of all ordinal numbers equinumerous with ω. Adapting his proof of the distinct cardinality of the reals and the rationals he showed that this set is not equinumerous to any of its members, and more generally that for each ordinal, there corresponds a cardinality.[13] In 1891 Cantor presented a new proof of the difference in power between the reals and the rationals, that based on his celebrated diagonalisation. This method was also used to formalise the result just mentioned that there are arbitrarily large cardinalities, and shows that given any set we can construct one of larger power.[14] Also notice that the order types discussed above are distinct from cardinality, in the sense that collections with the same size can be counted in different ways. For instance, consider the sets $\{0, 1, \ldots\}$ and $\{3, 4 \ldots 1, 2\}$ both have the same number of elements as the natural numbers, but the first set is counted by ω and the second by $\omega + 2$, they have different order types. In 1883 he related these cardinalities, or powers, with cardinal numbers and so to the cardinality of ω there corresponds the cardinal number $\aleph_0$ [15], to that of the sets equinumerous to ω, $\aleph_1$, to the set of ordinals of cardinality $\aleph_1$, the cardinal $\aleph_2$, and so on, obtaining a sequence of cardinal numbers similar to that of ordinals beginning with the finite cardinals:

$$0, 1, 2, \ldots, \aleph_0, \aleph_1, \ldots, \aleph_\omega, \ldots, \aleph_{\omega+1}, \ldots, \aleph_{\omega+\omega}, \ldots, \aleph_{\omega\cdot 3}, \ldots, \aleph_{\omega\cdot\omega}, \ldots, \aleph_{\omega^\omega}, \ldots$$

Cantor also assumed that the real numbers are sets and so well-ordered, i.e. can be counted by some ordinal, and so that their cardinality is that of one of the symbols in the sequence above. In 1895 he defined exponentiation for

[11] In particular, it seems troublesome that even after admitting that a collection such as the transfinite sequence of ordinals can be a symbol for the Absolute we are still unable to know it even approximately.

[12] See (Hallett, 1986, pp. 151-153) for a discussion.

[13] From this and the fact observed above that the sequence of ordinals cannot be counted we conclude that the class of cardinalities will also be another example of inconsistent multiplicity.

[14] This article is reprinted in English as appendix B to chapter 4 in (Lavine, 1994).

[15] This was the notation introduced by Cantor in 1886, earlier this number was denoted by (I), see (Lavine, 1994, pp. 45, 50).

cardinal numbers and noticed that the cardinality of the real numbers is that of the set of functions from the natural numbers to a set with two members, i.e. $2^{\aleph_0}$.

Note that in this section the theme has been to use the history of Cantor's development of set theory to emphasise that this arose from within mathematics and not as some foundational project for mathematics, as Frege's attempt to do so with the tools of logic. Nevertheless, set theory has developed to offer a foundation to the mathematical edifice and this potential was already present in Cantor's thinking in 1885 when he remarked that:

> Sie [Cantor's theory of ordered types] bildet einen wichtigen und grossen Theil der reinen Mengenlehre (Thèorie des ensembles), also auch der reinen Mathematik, denn letztere ist nach meiner Auffassung nichts Anderes als reine Mengenlehre. (Grattan-Guinness, 1970, p. 84)[16]

It is also worth mentioning that Frege wrote approvingly of a paper by Cantor on 1887 developing the arithmetic of transfinite cardinal numbers. Although complaining of the lack of rigour in his exposition, Lavine points out[17] Frege could have taken the theory developed in his Grundgesetze, whose first volume was published in 1893, as able to play the foundational role in Cantor's theory of the infinite that he found lacking in its original presentation.

4.3 Russellian classes and sets

In 1895, while Cantor was publishing his last papers in set theory, Bertrand Russell was writing his fellowship dissertation in Cambridge, after he knew of Cantor's writings Russell was opposed to the idea of Cantor that the real numbers form a set, taking it to be contradictory. By 1899, following Leibniz, he accepted the idea of actual infinity but rejected that of infinite number, since if this was the case given, that as we saw in §1 and following Frege, he took any extension associated to a concept to constitute an object, paradox arises when we try to determine the number associated to the extension of the predicate *being a number*. However, by 1900, after meeting Peano, he became convinced that to every aggregate there corresponds a number, and so pointed out an alleged error in Cantor's diagonalisation argument by pointing out that the cardinal of the set of all classes is indeed the biggest such number. Taking this to mean that there are some classes like that of all individuals or that of all classes, which he took as equinumerous with the former, such that one cannot diagonalise their way out of it. In the sense that the class of all subclasses of the class of all classes cannot be bigger than the class of all classes itself, since this is a member of it. [18]

More formally Russell took Cantor's diagonalisation to tell us that for any map d from a class to its class of subclasses the class of members, x, of the class

[16]'It constitutes an important and large part of pure set theory (Thèorie des ensembles), thus of pure mathematics, for the latter is, in my opinion, nothing other than pure set theory.' (Our translation)

[17]See (Lavine, 1994, p. 49)

[18]Russell also noted that there is indeed a biggest order type that of the sequence of ordinals itself, and took this to mean that this sequence is not itself well-ordered after all.

such that x is not in $d(x)$ is a member of the class of subclasses not in the range of d. Then put V for the class of all classes, and for any map d from V to its class of subclasses, say $\mathbb{P}(V)$, note that the cardinality of V is not larger than that of $\mathbb{P}(V)$ since for any x in V, the class only containing x is in $\mathbb{P}(V)$ for it is a subclass of V. But as we mentioned, the diagonal argument shows we cannot establish a bijection from V to $\mathbb{P}(V)$, and so $\mathbb{P}(V)$ is bigger than V. However, Russell thought that in the case of V, one could indeed produce a surjection from V to $\mathbb{P}(V)$. He defined d as follows:

$$d(x) = \begin{cases} x, & \text{if } x \text{ is a pure class} \\ \{x\}, & \text{otherwise} \end{cases}$$

But then note that the class of members of V such that x is not in $d(x)$, is in the range of the function since this will just be a fixpoint, say r, and so diagonalisation seems to fail. Thus, for Russell, turning our attention to the universal class we see that Cantor's argument that there is no largest cardinality is flawed. Note that, with some ingenuity we see that r is the familiar Russell class since it is the class of all classes such that $x \notin d(x)$, and so we must have that $d(x) = x$ and so of (pure) classes that do not belong to themselves. This was the very same class that was causing so much trouble to the Fregean project in §1.3.2. In fact this seems to be the origin of Russell's paradox since after sketching the argument above in a 1900 draft of his *Principles of Mathematics*, he notes the impossibility of applying the diagonalisation procedure to the class r just defined:

> In fact, the procedure is, in this case, impossible; for if we apply it to [r] itself, we find that [r] is a [d(r)] and therefore not a [r]; but from the definition, [r] should be a [r]. (As quoted in (Coffa, 1979, pp. 35-6))

Indeed, even if these remarks seem quite cryptic, we see that one can both deduce that $r \in r$ iff $r \notin r$. Thus, for Russell, turning our attention to the universal class we see that Cantor's argument that there is no largest cardinality is flawed.

We then see that there is a gap between the objects that Russell and Cantor are talking about. Indeed, Russell seems to be failing to appreciate that Cantor's arguments do not apply to all classes, but only to those countable in the sense explained above and so, in particular, not to his paradoxical class of all numbers. Indeed this is too close to the Cantorian Absolute as we saw in §4.2 and not in the transfinite, the region of the infinite which is the focus of his theory. In fact, these inconsistent multiplicities cannot be individuated and so in particular cannot be members of consistent multiplicities, thus $\mathbb{P}(V)$ is not a legitimate mathematical object. This was the diagnosis of Cantor himself regarding Russell's argument, as he puts it in a letter to Jourdain from 1904:

> Were we now, as Mr. Russell proposes, to replace $\mathfrak{M}$ by an inconsistent multiplicity (perhaps by the totality of all transfinite ordinal numbers, which you call $\mathfrak{W}[V]$), then a totality corresponding to $[\mathbb{P}(\mathfrak{M})]$) could by no means be formed. The impossibility rests upon this: an inconsistent multiplicity because it cannot be understood as a whole, thus as a thing,

cannot be used as an element of a multiplicity. (Reprinted in English as appendix A to chapter 4 in (Lavine, 1994).)

Hence, Cantor's results escape the antinomies of Russell discussed above, and also the paradox named after him discussed at length in §1 regarding the property of *being a class*. Russell seemed to realise this by 1901 when in a letter to Courat he said that: 'Je croyais pouvoir refuter Cantor; maintenant je vois qu'il est irrefutable'[19]

However, the situation with Frege's system printed in the second volume of his *Begriffsschrift* is different from that of Cantor's and his insistence that to all predicates there corresponds an extension, also if this turns to be a Cantorian inconsistent multiplicity as happens to be the Russell class, as was made known to him by Russell to him in his famous letter of 1902.

4.4 Powerset, a novel inhabitant of Cantor's heaven

We now return to the quote from Cantor discussed at the end of §4.3. Cantor points out that the collection of subcollections, more precisely of functions from the collection to a tuple, is not a set whenever this collection is an inconsistent multiplicity. However, Cantor seems to be endorsing the set existence principle that when a collection is a set, this collection of functions, what its known as the powerset, is also a set. This principle is in operation on his diagonalisation argument for the existence of the cardinal number hierarchy, which remarkably, does not require transfinite ordinals, and so is independent of his notion of set as something that can be associated with an ordinal. Thus the existence principle discussed here was not apparent until the publication of that article in 1891. [20]

This powerset axiom is what enabled Cantor to show his result about the cardinality of the continuum in 1895 mentioned above, and thus prove that this is a set, and so we come to realise its great importance in the Cantorian theory. Note however that the existence of the set $2^{\aleph_0}$, as well of course of the other powersets, seems to be in tension with the notion of set as well-orderable collection espoused by Cantor since for the first time Cantor did not know how proceed with the definition of the well-ordering of such set. Indeed, in the case of the reals at least he held the position that such a well-ordering could be found.[21] Hence, we see that Cantor's theory is in trouble, not the kind of trouble that the paradoxes caused Frege, but a more conceptual trouble. Namely, that of fitting the powerset operation within his framework where the main notion was that of a well-order, as Lavine puts it:

Cantor's theory was in trouble, but it was not trouble caused by the paradoxes. It was

[19] As quoted in (Coffa, 1979, p. 37).

[20] A less precise principle justifying the existence of the powerset is what Hallett calls Cantor's domain principle(Hallett, 1986, p. 7) and is adapted by Lavine (Lavine, 1994, p. 90) as saying that the domain of a mathematical variable is a set. Indeed, this principle can justify the adoption of the existence of the the set of reals, as well as being a source of justification for the set of natural numbers.

[21] For interesting remarks regarding Cantor's concept of well-ordering and definable well orderings consult the technical note in (Lavine, 1994, p. 96).

trouble caused by trying to fit the Power Set Axiom into a theory that took well-orderings to be primary. (Lavine, 1994, p. 97)

As an example of this conceptual strife, we can take that after the introduction of cardinal exponentianion through the powerset Cantor does no longer take as immediate the well ordering of the cardinalities, as he says:

> On the other hand, the theorem that, with any two cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ one of these three relations [trichotomy] must necessarily be realized is by no means self-evident and can hardly be proved at this stage. (Cantor, 1915, p. 90)

Since this result follows from the fact that all-sets are well ordered, it is also conceivable that he came to doubt such otherwise central feature of his theory. However, Cantor did produce a proof of the well ordering of the sequence of cardinals in a letter to Dedekind:

> If we take a definite multiplicity V and assume that no aleph corresponds to it as its cardinal number, we conclude that V must be inconsistent. For we readily see that, on the assumption made, the whole system Ω [the transfinite sequence of ordinals] is projectible into the multiplicity V, that is, there must exist a submultiplicity V' of V that is equivalent to the system Ω. V' is inconsistent because Ω is, and the same must therefore be asserted of V. (Cantor, 1899, pp. 116-7)

Cantor also communicated this proof to Jourdain four years later, but refused him permission to publish it.[22] Thus, he might have had doubts about the proof, however, what is clear is that the problem with such a cardinality is that one would fail to count it using the transfinite sequence of ordinals. So in a sense we must be able to 'project' the sequence into a subcollection of this cardinal, but being this as we saw above an inconsistent multiplicity, this cardinal number would have the transfinite as a proper part and so would be itself inconsistent.

This seems to be the origin of the notion of limitation of size, namely, that some collections, such as the one under discussion, are too big to be sets. As we have already seen, this is encoded in modern set theories, for instance by the Z notion that new sets are separated from already existing sets. More generally, any collection that fails to be a set is bigger than any set for if we begin counting it and can finish this at some transfinite ordinal this was a set, and so not inconsistent after all. So in each of these multiplicities one can project, as in Cantor's reasoning, the entire ordinal sequence. Hence, this would be larger than every set for these can be counted. So we see that Cantor characterises inconsistent multiplicities as those things too big to be a set, or alternatively sets as those collections that are small enough, in the precise sense of being able to be enumerated by an ordinal.

Note *in passim* that the logical notion of class has nothing to do with size matters and so to identify Cantor's inconsistent multiplicities with classes seems to evidence some degree of lack of awareness with regards to the historical development of these notions. Note also that this doctrine of limitation of size does not originally arise as an attempt to solve the paradoxes, although the

[22] See (Grattan-Guiness, 1971, pp. 115-8).

term was apparently firstly introduced by Russell when describing a theory that intended to do precisely so.[23]

We then see that even though Cantor had a procedure for showing that some collection is not a set, after the introduction of the powerset operation there is no unified characterisation of what is a set. Indeed, we are left with just a negative characterisation of the notion of set, which does indeed sound less appealing than the original position of Cantor in which sets were just obtained by following the transfinite sequence of ordinals and finding canonical well orderings with it. Indeed, if the cardinals form a well-order then every set can be counted, but this does not tell us what is a set in the first place. Consider for instance the reals, if they are a set then they can be well-ordered but are they a set in the first place? Well, Cantor can just tell us that if it is too big it will not be, but given our inability to find a explicit well-order that is all that can be said about such entity. Indeed, We cannot rule out its status as set yet for it could be, as it is indeed the case, one of the sets given by the new power set operation. Surely, more is required for a satisfactory theory of sets, this was the task taken up by Zermelo and others that culminated in ZFC.

4.5 The real gap between sets and classes

It seems that a clear difference now emerges between the Cantorian sets and Russellian classes, for the former existence is governed by the ability to be well-ordered, but the latter are ruled just by a comprehension principle, quoting Russell:[24]

> The values of x which render a propositional function ϕx true are like the roots of an equation—indeed the latter are a particular case of the former—and we may consider all the values of x which are such that ϕx is true. In general, these values form a class, and in fact a class may be defined as all the terms satisfying some propositional function (Russell, 1937, §23)

This notion of class is what Maddy calls the logical notion of collection, in the sense that the way we determine the members of this class is by checking for all objects in the domain of discourse whether they satisfy some open formula of the language, i.e. a logical requirement, as she puts it:

> The logical notion, beginning with Frege's extension of a concept, (...) [is characterised by]the idea of dividing absolutely everything into two groups according to some sort of rule. (Maddy, 1990, p. 121).

Indeed, note that the picture suggested by the remarks by Russell is familiar from our earlier survey of the Fregean theory if we replace the talk of propositional functions by concepts and that of classes by their extensions. However, as Lavine indicates, [25] there exist relevant differences between the notion of extension of Frege and that of class of Peano and Russell, these include their

[23] See (Lavine, 1994, Ch. 4, f.43).

[24] Such a principle seems to have been formulated firstly by Giuseppe Peano in his Principles of Arithmetic, see (Peano, 1889, p. 90).

[25] Consult (Lavine, 1994, f.2, pp 63-5).

differences when it comes to the property of *being true*, the ontological dependence of extensions on concepts and not in objects, or a proposal to solve the paradox by Frege assigning the same extension to two concepts with different collections of objects satisfying them.

As pointed out above, Russell was aware that the Cantorian notion of set, as opposed to his classes, avoided paradox, and so were different entities. He characterises his understanding of this difference in the following way:

> [W]hen mathematicians deal with (...) [a set], it is common, especially where the number of terms involved is finite, to regard the object in question (...) as defined by the enumeration of its terms (...). Here it is not predicates and denoting that are relevant, but terms connected by the word *and*, in the sense in which this word stands for a numerical conjunction. (Russell, 1937, §68)

This passage is of remarkable interest. Firstly, note how Russell starts by pointing out that the Cantorian sets are the formalisation of collections employed by mathematicians. Indeed, as we pointed out in the previous section that a difference in a sociological level between sets and classes is that the former emerged from the within mathematics and the latter outwith mathematics in the logicist endeavour of reducing it to logic of Frege and later Whitehead and Russell. After noticing this purpose it is understandable why the classes are precisely those that can be individuated through first-order definable properties, and why the notion of number is here defined in terms of the definable property of bijective correspondence. As has already widely discussed both formally and philosophically in this work, Russell's attempt to give a paradox-free account of a classes is his theory of types. Note also that it is the approach of Cantor, later axiomatised by Zermelo and discussed in the first chapter has become, through time, in the mathematicians' theory of choice, as opposed that of the typed frameworks of Russell and Whitehead.

Secondly, on a more conceptual level, Russell seems to be giving a combinatorial characterisation of Cantorian sets. The notion of counting in use is already familiar, namely that what matters is not that a property picks out the members of the set as in the case of the classes that are defined via logical properties, but merely that the elements of a set are arbitrarily put together. In the finite case, as Russell points out this would be accomplished by defining the set as a conjunction of its elements, since there need not exist any other predicate individuating such a collection. Hence we see that at the heart of the distinction between sets and classes lies in that, in the former case its elements are allowed to be pick out by arbitrary functions but in the latter we require some kind of explicitly defined rule for this job. That the notion of arbitrary selection of elements was understood as a key feature of sets is even more explicitly asserted in the following remarks by Paul Bernays:

> [Sets] are used in a 'quasi-combinatorial' sense, by which I mean: (...) one views a set of integers as the result of infinitely many independent acts deciding for each number whether it should be included or excluded. (...) Sequences of real numbers and sets of real numbers are envisaged in an analogous manner. From this point of view, constructive definitions of (...) sets are only ways to pick out an object which exists independently of, and prior to, the

construction. (Bernays, 1964, pp. 275-6)

Of especial interest is the last part of the passage, which makes clear that in the case of sets, as opposed to classes, we do not require a constructive specification of the collection in order for it to exist. Hence Bernays label of Platonistic for this notion of set as opposed to the more intuitionistic flavour of classes, which seem to depend on us and our language.

Another difference between the notions of set and class seems to be classes can be bigger than sets. Indeed, consider the transfinite sequence of ordinals, these sequence is, as we saw, too large to be counted and so does not constitute a set. However, it seems perfectly licit to collect it under the logical property of *being an ordinal*. As we remarked above limitation of size applies to sets and not to classes. It also seems, however, that given a domain there are more sets than classes. Indeed there are usually more subsets of the universe than collections that can be individuated through definable properties. It is clear that this claim will depend on our notion of definable logical property, but it seems that unless we already accept the combinatorial notion as providing legitimate collections, and so we allow for properties with set parameters such as *being a member of the set A*, this will be the case. Indeed, consider the discussion regarding the axiom of choice below. Moreover, note that the restrictions on the classes that exist in the different theories we surveyed in chapter 2 came, not like in Cantor's cases from size matters, but from restricting the properties that are successful participants in a comprehension principle. On a related point, note also that while the notion of set is not subject to paradoxical conclusion this is not the case with classes.

We also note some intriguing differences between the individuation conditions of sets and classes. These stem from the fact that sets are just given by some arbitrary function and so they are in this sense arbitrary (not-too big) collections of objects, there is no more to them than their members, in Cantor's words they are: '(...) consisting of clearly differentiated, conceptually separated elements $m, m', \ldots$ and which is thereby determined and delimited'[26]

Hence, even if we used a property to discover the existence of a set. For instance, from being a finite number we arrive to the cofinal sequence of natural numbers and after some observations about well-orders to its existence, this property was just a heuristic device in our grasp of the set and not an essential feature of it. Quoting Hallett: 'while an intension is what pointed out the set to us, the set itself is something quite separate from the intension.'[27]

On the other hand, classes are more closely tied to the definable properties specifying them, in the sense that there is nothing more to the class than the property, no further independent Cantorian considerations are required to ensure its existence. The result of these remarks is that although extensionality suffices to determine the identity of sets, this is not the case with classes, since their specification conditions make them inherit the intensional flavour of properties. Indeed, the class of all sets is co-extensional as the class of all

[26]See ((Cantor, 1932, p. 387), as quoted in (Hallett, 1986, p. 34))
[27]See (Hallett, 1986, ibid.)

well-founded sets in a theory with the axiom of regularity but these are distinct properties. In short, co-extensionality is a sufficient condition for the identity of sets but not of classes, as Maddy puts it:

> [C]lasses can be coextensive without coinciding if they are picked out by different terms. To a certain extent, this seems appropriate, because classes are understood as closely tied to the properties that determine them, and coextensive properties are not identified. (Maddy, 2000, p. 305)

Note however, that as Maddy also points out immediately after this quote, it is a question that requires some subtle choices that of when to decide that the predicates defining the classes are similar enough to determine the same class. Indeed, the properties being a whale and being a whale or being a whale are strictly speaking *not identical*, but it is not clear that one should be willing to take their characteristic classes as distinct.

After the remarks above one might now, in turn, have concerns regarding the necessity of co-extensionality when it comes to the individuation of sets and classes. Again, when it comes to sets since there is no more to them than their members this seems to be the case. However, the case of classes seems more subtle. Again noting that perhaps the key notion of the concept of class is its close connection to that of property and that the extension of a property changes without this doing so, indeed the property of being a citizen of Amsterdam changes extension with each update of the municipal census, but the property remains unchanged. Similarly, to avoid reference to time, consider the property of being a set, and the intuitionistic model discussed in §2.6.1. Here the extension of the property contains at the stage $\omega + 1$ the set of naturals but not in the levels below it. And so, again, taking the close relationship between classes and properties seriously moves us to accept that there are classes that have different members and are nevertheless the same. In fact this seems a natural consequence of the fact that we take classes as extensions of properties, not as the particular extension of the property at a given point. Hence, since the objects instantiating the property change so does the membership of the class. In sum, we see that sets are individuated extensionally while classes are not. This is summarised in the following remarks by Reinhardt:

> A proper class P may however be distinguished from a set x in the following way (if the reader will indulge another counterfactual conditional): If there were more ordinals $(\dots)$, x would have exactly the same members, whereas P would necessarily have new elements. We could say that the extension of x is fixed but that of P depends on what sets exist. Roughly, x is its extension, whereas P has more to it than that. (Reinhardt, 1974a, p. 196)

Note that here instead of classifying classes as intensional entities Reinhardt seems to be seeing them as hyperintensional. Indeed, since it is usually accepted that in all possible worlds the same sets exist necessarily, for the class of ordinals to vary its extension we would need to consider impossible worlds. Of course, intensionality is enough if we reject Platonism and adopt a constructive framework in which sets can exist in one possible world and not in another. Even a Platonist might regard classes as genuinely intensional if the talk of possible worlds here is understood in terms of epistemic states of the speaker, or community of speakers, rather than ontologically.[28] In any case, even in a purely metaphysical Platonistic reading some classes would be genuinely intensional, unless we press the more hardline view that classes can only contain mathematical objects, and so that a a property such as being a student of the University of Amsterdam fails to determine a class in good standing.

Note also, that here we are not making a distinction between classes and proper classes, those that violate limitation of size. Consider again our example above of the citizens of Amsterdam. Presumably, these do not violate limitation of size, but again we take the class to have varying members, as Maddy puts it: 'though small enough to be sets, are individuated differently, so they are classes, too, without being proper.'[29] Moreover, note also that the remarks we have made regarding classes having different members at different points leads, quite naturally to a modal treatment with regards to possible worlds, leaving aside the remarks about impossible worlds following Reinhardt's quote above, so that we can talk of sets as rigid designators or in a more Lewisian treatment of them being their own counterparts in all possible worlds, and of classes as functions from possible worlds to extensions.[30]

Another difference that Maddy points out between sets and classes is that the latter can be members of themselves while the latter do not, indeed this claim about classes follows naturally from the remarks made in §3.2 regarding self-instantiating properties and the close relation between sets and properties. However, one should be cautious about the claim that sets are not self-membered. Indeed, this is the case under the iterative conception of set, that takes sets to be generated in successive stages, and always after all their members have already been formed through different operations. So the axiom of foundation, banning for instance such self-membered sets and sets with infinitely descending chains of members, is a natural principle formulating this picture. Note then that such conception has not been mentioned while discussing Cantor's notion of set, but was first officially introduced in Zermelo's 1930 axiomatisation of set theory and is since a trademark feature of ZFC the dominant theory of sets. Indeed, Lavine remarks that 'Cantor never, so far as I know, commented on whether a set can be a member of itself.'[31]

The point is that even if the most popular theory of collections in use by mathematicians today takes sets to be well-founded, this was not a fundamental feature of the notion of set as conceived by Cantor. Since Cantor did not seem to take non-wellfoundedness as an essential feature of sets we will here remain agnostic about this point and be silent in this respect when it comes to the requirement well-foundedness for an adequate theory of sets. To be clear we require (some) classes to be non-wellfounded but we are silent about this requirement for sets, hence if we are to agree with Maddy in that this feature marks a difference between sets and classes is only in a weaker sense than she maintains. Namely that classes require non-wellfoundedness but sets do not,

[28] See for instance, (Incurvati, 2008, p. 93)

[29] See (Maddy, 1990, p. 102).

[30] Such a treatment is offered by Charles Parsons in (Charles Parsons, 1983, §III-V).

[31] See (Lavine, 1994, p. 145), note that, as he makes clear immediately after this quote, he believes the Cantorian theory to enable the existence of such sets.

which does not mean as she takes to that sets are are well-founded. Of course this is just due to the fact that she is working with the more developed notion of ZFC set while here we are guided by a less fine-grained Cantorian notion.

Finally, we turn our attention to some axioms in which the difference between sets and classes becomes relevant in the debate regarding their acceptability. First we look at the axiom of choice, as we mentioned in the first chapter, this principle tells us that for every family of non-empty collections there is a choice-collection that is constituted by a member of each of the collections in the family. Now, in the Cantorian notion of collection such a principle seems completely harmless for, as Bernays pointed out, the elements of a set can be chosen from other sets by an arbitrary function. However on the notion of a Russellian class, such a collection will be highly suspicious since there is no way, in general, to provide an effective procedure to pick the elements of such a class from each of the classes in the family. That the axiom of choice is nowadays widely accepted correlates with the idea stressed before that the concept of set encodes the collection of choice for mathematicians.[32]

Similarly, consider the axiom of constructibility, which takes all the existing sets to be constructible.[33] Then one can see as a reason to reject the principle the fact that the notion of set has at its core the notion of arbitrary selection of elements and so this axiom's constraint regarding definability over a first order language seems out of place. On the contrary, in the understanding of collections as classes one seems to have fewer reasons to consider these restrictions as arbitrary but rather necessary for the adequate understanding of the objects under consideration.[34]

4.6 Summary

In this chapter we have offered a historical account of Cantor's notion of set as a well-orderable collection (§4.2), we then turned to examining Russell's reaction to Cantor's diagonal argument noting his mistaken suspicion of an error in the prove owing to his failure to distinguish the notion of set from that of class, entities very related to Fregean concepts and their extensions (§4.3). Next we looked at how the diagonal argument, with its acceptance of the powerset, undermines Cantor's unified and positive picture of sets that we had before, for a negative one in which inconsistent multiplicities will be those collections bigger than any set and thus violating limitation of size (§4.4). Finally (§4.5), we were in a position to assert several differences between sets and

[32] This is the diagnosis of Donald Martin, as quoted by Maddy in (Maddy, 1997, p. 55)

[33] The constructible sets are the inhabitants of the hierarchy L, whose stages are defined as follows:

$$L_0 = \{x : x \text{ is an individual}\}$$

$$L_{\alpha+1} = \text{def}(L_\alpha), \text{ i.e. the set of sets definable over } L_\alpha \text{ in } FOL$$

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$$

Where α is an ordinal and λ a limit ordinal.

[34] Reasons along these lines are given, for instance by Moschovakis and Devlin against and in favour of this axiom. See (Moschovakis, 2009, p. 610) and (Devlin, 1977, pp. iv, 13-8) respectively.

classes, from the combinatorial nature of sets to the logical of classes, to issues regarding size and number, through some deeper issues like those relating to individuating conditions, and some less significant for the Cantorian sets related to non-wellfoundedness of some classes. We also provided some comments as to how these two notions can inform the debate regarding some mathematical axioms.

Chapter 5

Defending classes

5.1 Introduction

In §4 we offered a mostly historical account that showed the differences in the notion of Cantorian set and Russellian class and pointed out several differences of these two entities that result from their distinct developmental stories. In the present chapter we will make use of these facts about sets and classes in order to address some philosophical arguments against the use of both sets and classes in our theories of collections.

The chapter begins by addressing in §5.2 the very direct objection that the usual theories employed in mathematics only employ first order logic and so the burden of proof is on us if we want to introduce second order resources such as classes. Next we move in §5.3 to the criticism that the classes used in theories of collections are not well-motivated entities, as opposed to sets, and thus should not be used. Thirdly, in §5.4 we address the worry that even if warranted in using sets and classes, since one of the entities is reducible to the other, we could obtain a more parsimonious ontology by giving up the reducible entities. Finally we address the worry that such a simpler ontology could be obtained, not by reduction, but by interpreting the second order resources in play in a more parsimonious and so paraphrasing away the use of classes in the discourse, and so in §5.5 we look at taking talk of classes as merely plural quantification over sets.

5.2 Second order resources

In this section we comment on perhaps the more blunt objection to the employment of classes, or more generally of second order quantification, in our formalisation of ideas about objects, properties collections and so on. The idea is simple and seems quite persuasive, especially perhaps for the more (classically) mathematically minded.

Indeed, suppose that we deal with someone who is informed in their ontology and language by ZFC, the most common theory of collections in use by mathematicians. Now, since this is a first order theory, in particular, classes or other devices of second order logic are completely absent from its picture. The question then seems to be why do we want to add them.Indeed, the burden of justification seems to be on the proponent of classes. Now, I think that throughout this work we have developed some tools that allow us to begin to reply to this challenge in a satisfactory fashion.

Note that in this section we will usually talk about classes as representative example of the use second order resources. However, the reader might prefer to replace our reference to classes here with their favourite interpretation of second order quantification, for instance pluralities of sets. What matters here is the need for these resources rather than their interpretation.

5.2.1 Not all collections

First, we take issue with the idea that set theory, or more precisely a set theory that omits any mention of the resources of second order logic, succeeds at being an adequate theory. To begin with we saw in §4 how there are two formal notions of collections that of Cantorian set and that of Russellian class, that have not only a distinct historical origin but also different ontological flavour, one tied to properties and so to second order machinery, the other to counting. These have as we saw in particular in §4.5 several differences, such as the limitations of sets with respect to size, or their different individuation conditions, i.e. classes seem to be intensional, or even hyperintensional, entities while sets are purely extensional. So these remarks give us *prima facie* reasons to resist that a first order theory of sets tells us the entire story about collections.

Moreover, if as it seems to be the case here one takes the sets to be the well-founded entities of ZFC, the remarks in §3 seem particularly concerning. First, with this limited in size notion of set since in §3.3.3 we argued for the inclusion of a universal collection. This is a collection that the set theorist might feel more pressured to accept than some other classes tied to other properties since taking the set theorist seriously seems to be on a par with taking the quantifiers to range over absolutely all sets[1]. Moreover, if as it seems to be the case here too, one takes the sets to be the well-founded, we have given some reasons to allow some collections to be self-membered in §3.2, for instance in order to account for some metaphysical puzzles or natural language inferences. Thus, we do not take sets to be all collections on pain of contenting ourselves with a suboptimal theory.

5.2.2 Not all mathematical collections

The critic might reply that what they meant by saying that a first order theory was sufficient to deal perhaps not with all the requirements posed by trying

[1] More on this in the §5.4.2

to model natural language inferences, or solve metaphysical puzzles, but it is after all more than enough to conduct all the mathematical business. So, the motivations just alluded to would be irrelevant for what is meant by an adequate theory in this context. These are simply not things that they are trying to solve. However, we would like to maintain that there are still some practices of a distinctly mathematical flavour not captured by set theories devoid of second order resources. Indeed, from a strictly Cantorian conception of set one could think of things such as the empty set or the singleton set,[2] from a more modern understanding of the term though there are still some mathematical entities that fail to be sets such as the collection of ordinals, cardinals or that of infinite (mathematical) collections.

Note first that one might have doubts about why we take this collections to be mathematical, for sure their members are mathematical entities, but does this status transfer to the class collecting them? Of course, the pluralist should not worry at this point since they do not take it to be anything over and above the members of the collection, and so the question would make little sense to them. The friend of taking, like us, classes seriously ontologically might indeed think so. However, we might just want to, instead of trying to answer that seemingly obscure metaphysical question, insist that all we seem to mean by a mathematical collection is one that is, or was at some point, used by mathematicians in their everyday mathematical business. This seems to be enough though, since mathematicians do talk about theorems holding of all ordinals, cardinals or even all infinite sets. This coupled with our beliefs expressed in §3 of taking the all in one principle seriously, i.e. that for all property there corresponds an extension, a class, implies that not only these entities talked about are collections but are mathematical collections.

In what remains of this section we illustrate this point by giving some examples in which mathematicians employ second order resources in their investigations, or at least seem to do so, in a way that seems crucial to their endeavours. Note first that classes can be used to give a succinct and finite axiomatisation of theories such as ZFC, indeed the following schema:

Separation
$\forall x \exists y \forall u (u \in y \equiv (u \in x \,\&\, \phi(x)))$
For any ϕ in the language, with y not free in ϕ.
Replacement
$Fun(\phi) \supset \forall w \exists z \forall v (v \in z \equiv \exists u (u \in w \,\&\, \phi(u, v)))$
For any $\phi(x, y)$ in the language.

Can be rendered as axioms using classes as follows:

[2]Indeed, the emptyset and singleton sets, though commonplace in theories such as ZFC are not clearly part of the Cantorian notion of set, see (Oliver & Smiley, 2013, pp. 14.3-4)

Separation
$\forall z \forall x \exists y \forall u (u \in y \equiv (u \in x \,\&\, x \in \hat{z}\phi(z)))$
Replacement
$\forall x \forall y Fun(\hat{x}\hat{y}\phi) \supset \forall w \exists z \forall v (v \in z \equiv \exists u (u \in w \,\&\, (u, v) \in \hat{x}\hat{y}\phi))$

And so we see that, even though not strictly necessary for the mathematical discourse, one could still see that some mathematical entities that are classes can be coherently taken to form part of the mathematician's toolbox. Say for instance if this placed a great emphasis on the importance of finite axiomatisations. Note here that, importantly, we are not appealing necessarily to classes that are too big to be sets like in the case of cardinals above and so it would not be enough to use limitation of size. Indeed, take for instance the property $[\lambda x Fin(x) \,\&\, \exists z \in (z = \emptyset)]$ it seems not problematic to use the associated class in separation for size concerns, to block the existence of these classes, but something more general about them like that there are not such things as extensions associated to properties, which is the key feature of classes in use here. As we just said, talk of classes connected to the issue of finite axiomatisation is more a matter of conciseness or ease of expression rather than of necessity. However, there are instances in mathematics when the talk of classes seems more difficult to dispose of, here we look at the example of reflection arguments for large cardinal axioms. First recall that a strongly inaccessible cardinal, κ, is an uncountable cardinal such that $2^\lambda < \kappa$, for any $\lambda < \kappa$, i.e. it is a strong limit, and the least limit ordinal α such that there is an increasing sequence $(\beta_\eta)_{\eta<\alpha}$ converting to κ is κ, i.e. it is a regular cardinal. As Jech points out these cardinals get their name from the fact that they cannot be reached from below:

> The [strongly] inaccessible cardinals owe their name to the fact that they cannot be obtained from smaller cardinals by the usual set-theoretical operations.(Jech, 2003, p. 58)

Now from the fact that if κ is strongly inaccessible V_κ is, as we have already mentioned, a model of ZFC. It follows that this theory doesn't prove the existence of inaccessibles, for then violating Gödel's second incompleteness theorem, it would prove its own consistency. One way to motivate the existence of such cardinals is to use reflection principles, the idea motivating these is that the universe of sets is structurally undefinable, as Gödel explains further:

> One possibility of making this statement precise is the following: The universe of sets cannot be uniquely characterized (i.e. distinguished from all its initial segments) by any internal structural property of the e-relation in it, expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal number. (Wang, 1974, p. 189)

More concretely then the idea is that if we have some property that the universe of sets enjoys there must be some set that does too, so in a sense in contrast with the set theoretic operations here we approach the universe of sets from above.

Back into strong inaccessibles then, and consider the class of all ordinals.

Then we see that this is an inaccessible since any sequence of ordinals has an ordinal as limit and the application of the continuum function to any ordinal renders another ordinal. Hence by the reflection principle there is some set exhibiting this features and so there exists an inaccessible cardinal. So here we have seen an example of the usefulness of reflection principles in justifying our axioms, however the problem is that, as Uzquiano points out[3], it is challenging to encode this principles in the first order language of ZFC. In fact what we can prove in ZF is, as Jech puts it that 'for any finite number of formulas, there is a set M that is like an "elementary submodel" of the universe, with respect to the given formulas.' (Jech, 2003, p. 168), i.e. taking the case of a single formula that:

Reflection
$\forall \alpha \exists \beta > \alpha \forall x_1, \ldots, x_n \in V_\beta (\phi(x_1, \ldots, x_n) \equiv \phi_{V_\beta}(x_1 \ldots x_n))$
For any $\phi(y_1, \ldots, y_n)$ in the language.

We say that V_β reflects ϕ. This being provable in ZFC does of course not imply the existence of strongly inaccessibles but it does indeed prove the axioms of replacement and infinity. In any case, it seems clear that if we want to be able to use reflection principles strong enough to formalise the informal argument for the existence of inaccessibles, the talk of classes, like that of the ordinals or the universe, is not easily sidelined.[4]

Now, reflection principles are, as we mentioned, a way to investigate the (iterative) set-theoretic universe from above. However, taking seriously the idea that we also find in Cantor that the transfinite sequence of ordinals are a generalisation of the finite natural numbers gives us another way to investigate it, namely from below, via transfer principles. As Friedman puts it these encode the idea that:

'any assertion of a certain logical form that holds of all functions on $\mathbb{N}$ holds of all functions on On [the class of ordinals]. (Friedman, 1997, p. 1)'

And so we transfer properties of V_ω to V itself. It turns out, that these principles are equivalent to strong cardinal axioms. However, it also turns out that these principles are not fully expressible unless we add classes to the language of ZFC. Indeed, a transfer principle relating properties of functions on the naturals with those of functions on the ordinals is as follows:

Definition 13 (*Transfer principle*)**.** If for all appropriate functions $f_1, \ldots, f_n : \mathbb{N}^k \to \mathbb{N}$, $A(f_1, \ldots, f_n)$, with A an appropriate existential formula, then for all appropriate functions $f_1, \ldots, f_n : On^k \to On$, $A(f_1, \ldots, f_n)$.

However, as Friedman also points out, to formalise the content of the consequent of the principle we need to be working in a theory of classes. In fact,

[3]See (Uzquiano, 2012, §2), which we follow closely when exploring this point.

[4]Incidentally, it also proves together with Gödel's theorem that ZFC cannot have a finite axiomatisation, and so that if one is hoping for such an achievement one must take the introduction of classes as a serious possibility.

the use of classes seems to be stronger than in other cases since here we are not only referring to the class of ordinals but to that of ordinal functions.

Recall that here we are giving some examples of how the talk of classes, or more broadly the employment of second order resources, are used by mathematicians in their research and so we seem to have reasons to look at how classes are used in the study of large cardinals since, as Jech points out: 'The theory of large cardinals plays central role in modern set theory'[5] Indeed, classes are used prominently in the study of the hierarchy of large cardinals, through the employment of elementary embeddings from the universe of sets to an inner model of ZFC. Recall that such an embedding is just an injection that preserves truth, i.e. $f : V \to M$ s.t. if $V \vDash \phi(x_1, \ldots x_n)$, then $M \vDash \phi(f(x_1), \ldots f(x_n))$ for every formula of the language, and that an inner model is just a transitive $\in$-model that contains the ordinals. Now, it is the case that the existence of a measurable cardinal is equivalent to the existence of a (non-trivial) elementary embedding from the universe to some inner model.[6] We can investigate further large cardinals by imposing conditions on the model M, for instance we call a cardinal κ, λ-strong, with $\lambda > \kappa$ if there exists some (non-trivial) elementary embedding f, with critical point κ, i.e. κ is the smallest ordinal moved by the embedding, such that $f(\kappa) \geq \lambda$ and $V_\lambda \subset M$. Moreover, if κ is λ-strong for all $\lambda \geq \kappa$ we say κ is strong. If $V_{f(\lambda)} \subset M$ we say that κ is superstrong. If we keep imposing conditions on the structure of M we arrive at higher stages of the large cardinal hierarchy such as weakly compact, supercompact or huge cardinals. Now, a limit to this method was found by Kunen[7] when he showed that the only elementary embedding from the universe to itself is the identity. Incidentally, this was not proven in ZFC but rather in MK, a theory of classes.

Now, the problem here is that these principles regarding elementary embeddings are not expressible in a first order theory like ZFC, first of course, because of its satisfaction relation is not formalisable in itself, now this can be addressed by introducing an implicit satisfaction definition in the language of ZFC. However, here we are more interested with the problem that there is no universal set, V, in ZFC, and so of course no function with domain V as these principles discuss. Now, there are indeed devices to express the existence of such maps within the language of set theory, for instance in terms of ultrafilters or the rather technically ingenious theory of extenders[8]. However, that one can usually find such reformulations is no guarantee that will be able to do so for any principle regarding embeddings, for instance, as Welch and Vickers put it when discussing embeddings from a model to the universe:

> It is quite natural to study the properties of elementary embeddings $j : V \to M$ for M some inner model, since many such embeddings, if they exist, have first order formulations within ZFC. The question of reversing the arrow and looking at a non-trivial $j : M \to V$ in general does not readily admit of such formulations.(Vickers & Welch, 2001, p. 1090)

In view of this, one could try a bolder route to encode these embeddings

[5]See (Jech, 2003, p. 285).
[6]See for instance (Jech, 2003, §17)
[7]See (Kunen, 1971, §1)
[8]See for instance (Jech, 2003, §20)

into the language of ZFC. Namely, to include a function symbol in the language and add an axiom schema that reflects the fact that this map is an elementary embedding, quoting Jech:

> As the statement "there exists an elementary embedding of V " is not expressible in the language of set theory, the theorem needs to be understood as a theorem in the following modification of ZFC: The language has, in addition to $\in$, a function symbol j, the axioms include Separation and Replacement Axioms for formulas that contain the symbol j, and axioms that state that j is an elementary embedding of V (Jech, 2003, p. 290)

It is clear that, even if in this case the classes can be disposed of, this process seems rather artificial, for they are, through model theory, valuable in the development of principles regarding these cardinals, or as Uzquiano puts it although

> sometimes eliminable, but which nevertheless seem heuristically indispensable (...) Indeed, set theorists often begin to work within an informal theory of sets and classes, and then search for technical formulations within either ZFC or some schematic extension thereof. (Uzquiano, 2012, pp. 70,72)

Not only that, but these theories seem more cumbersome to use than a theory of classes where one can use the second order machinery available to quantify at will over functions such as the ones under discussion here. So one could press the point further of why one should not take classes seriously given their important role in the development of the theory, but be content with what seems a much more artificial and *ad hoc* formulation which seems to be preferable just because it does away with the classes.

In view of all these examples showing how mathematicians employ resources of second order logic in their daily business, one could concede that point while still insist that we needn't take them as committed to the existence of classes. Or press the point further by claiming that they are not really talking about classes when they make statements about all ordinals and the such, i.e. the use of classes must, or at least could, be paraphrased away. More precisely, the use of second order resources could be explained in some way that does not ontologically commit us to these entities distinct from sets. This will be explored in §5.4 − 5, below by means of reduction of classes to sets or the use of plurals. Nevertheless, our aim will be to convince the reader of the plausibility of accepting these ontological commitments. Now, one reason why one might not want to take classes as genuine entities that must be taken seriously is that, as opposed to sets, they are not well-motivated entities. They would be just some patches that we employ to fix a flawed theory, i.e. that classes are *ad hoc*, this is the focus of the next section.

5.3 Ad hocness

We take as a representative example of this argument against the use of classes the remarks Jonathan Lear makes in the introduction to his article *Sets and Semantics*. Lear points out that modern theories of collections usually employ both the notion of set and of proper class, in order: 'to reconcile the intuition that any well-determined objects can be collected together and the classical interpretation of the universal quantifier.'[9]. Indeed, the idea is that the universal quantifier is taken to range over all sets, and that any collection of well-determined sets can be collected into another set. Now, even if dismissing the last query by pointing out how Cantor's principle of limitation of size acknowledges the fact that some collections of well-formed entities such as the ordinal or cardinal numbers are too big to do so. The fact seen in §3 that we do accept the fact that the universal quantifier ranges over all sets, together with our endorsement of the All in one principle (AiO), does indeed prompt us to affirm that there is some object collecting all sets. However, this respect for the limitation of size principle does inform us that this will not be a set. This last idea is something that most theories of collections agree with and, as Lear points out, these usually take the position, either implicit such as ZFC, or explicitly, such as NBG, that this object is a proper class.

Before proceeding note that what Lear is talking about is a class of all sets, strictly speaking all we are sure to affirm is that there is a class that contains all sets, namely $[\lambda x x = x]$. But not only this, since it will also contain all classes, hence the existence of this class, given that we take the logical notion of class seriously, will be contingent upon having a language that is allowed to express the notion of being a set, for instance via a predicate as in ZFC or A, or a constant such as in Maddy's theory.

Lear thinks that this use of classes in order to account for the problem just mentioned is 'unacceptable'. Firstly, he says because:

> classically interpreted quantifier must indeed range over all sets. But it is not necessary for the quantifier to range over an object that contains all sets. (Lear, 1977, p. 87)

Indeed, this is just expressing his rejection of the AiO principle that we defended from Cartwright's remarks in §3, and so need not detain us here. Next, Lear seems to complain that the use of classes does not clarify the issue at stake since they seem an *ad hoc* object, given that the restrictions imposed on them in the usual theories of collections seem unmotivated, as he puts it:

> the standard restrictions imposed—e.g., that proper classes can only have sets as members, that one cannot perform set-theoretic operations on them—appear arbitrary.(Lear, 1977, ibid.)

Indeed, I agree with Lear that these usual restrictions on classes are deeply unsatisfactory, however, I think that with the discussion about the Russellian notion of class in §4, we can respond by saying that we do have a clear understanding of the notion of class or at least of *our* notion of class and that it is precisely our task to elucidate which of these conditions usually imposed on classes are acceptable or not. Indeed, it is precisely the close tie between properties and these entities on the logical notion of class which is historically very different from that of set that allows us to face any criticism of *ad-hocness*. For instance, we are ready to say that we must reject that classes can have only sets as members, as our remarks above about the universal class show. Of course,

[9]See (Lear, 1977, p. 86)

we should also reject that one can perform set-theoretic operations on them, since these are not sets, in fact these are entities of a distinctive logical flavour and so what we can indeed do is perform logical operations some of which, such as negation, which seem to have no counterpart among set-theoretic operations. Indeed, in ZFC for instance the complement of $\emptyset$ is not taken to be a set. In fact, in this way we can also respond to Lear's immediate comment that there is nothing preventing us from creating what he calls a layer of set-like objects using some analoguos of the power set operation for classes, which he calls the power class operation. Incidentally, this seems to be one of Boolos' worries regarding classes:

> If one admits that there are proper classes at all, oughtn't one to take seriously the possibility of an iteratively generated hierarchy of collection-theoretic universes in which the sets which ZF recognizes play the role of the ground-floor objects? (Boolos, 1998, p. 36)

The answer would indeed be that no. As we have just pointed out, we do not have any prima facie reason to allow set operations to be applied to classes, and as we saw in §4.4 the power set is a distinctively set-theoretic operation. Indeed, it is clear, as Lear points out, it seems that if allowed to do so: 'we have succeeded only in constructing another rank of the cumulative hierarchy.'[10], and so that we didn't take the universe of sets as argument of the operation in the first place, for here subclasses are taken to be just collections of sets, and so the idea would be that this new stage reveals some possible sets we were missing. Thus, classes could be seen as additional layers of sets if the hierarchy of sets went up high enough.[11] Indeed, I agree that this seems like a very serious objection to theories of collections that do allow for such a move, and so we would have that in sets and classes we would have a distinction without a difference. In any case this is definitely not what we would allow, under our logical conception of class, there is a real difference between sets and classes, indeed this was precisely the point that §4 wanted to make.

5.4 Reduction

Suppose then that we are persuaded by the remarks of the section above and we take classes to be well-motivated entities, now we look into a the worry that even if sets and classes are distinct entities, and so our theory of collections might employ both notions without fear of ad hocness, one could still maintain that our theory could be doing the same with less. So perhaps if not motivated by quantitative parsimony yes by qualitative parsimony, the idea is that if we could reduce sets to classes or viceversa we would have, at the very least a simpler theory in ontological terms. This is a question that occupies Charles Parsons, as he notes after pointing out the different combinatorial notion of set and the logical notion of class, one should indeed be careful when stating that classes are just new layers of sets. Indeed, this was our opinion in §5.3, as he puts it: 'The above discussion, (...), should have made clear that it is at least not obvious that extension and set are just one concept.'[12]. It is also immediate as

[10]see (Lear, 1977, ibid.)
[11]See §5.4.2$formoreonthisissue.$
[12]See (Charles Parsons, 1974, p. 8)

Parsons says and we already mentioned in §4.5 when considering the question of whether there are more sets than classes,[13] that this, as we will also see below, 'turns as much on the conception of extension as on that of set.'[14], or more precisely on the logical language we use. Now, following Parsons, we take ourselves to deal with two questions, first whether we can reduce sets to classes and secondly whether classes can be reduced to sets.

5.4.1 From sets to classes

It seems that our the best attempt of reducing sets to classes is by taking a language with a name for each set. Indeed, note that this would be a language very different from the ones we are used for instance in set theory since it will be an uncountable language. Say $\overline{a}$ is the name for the set a, we would then have the coextensional class $\hat{x}(x \in \overline{a})$ as a class surrogate for the set a. Now, it is immediate that although on some occasions such class surrogates can be found without introducing all the aforementioned constants, for instance in the case of the emptyset $\hat{x}(\sim x = x)$, its singleton $\hat{x}(\sim\exists y(y \in x))$ or the powerset of the latter $\hat{x}(\forall y(y \in x \supset \sim\exists z(z \in y)))$, it will clearly not be the case in general that for every set there is a coextensional property expressible in, say, $\text{FOL}^{=,\in}$. Think for instance of $\mathbb{P}(\omega)$, hence the bulk of the work is done by the constants, and so such a process can only be possible by presupposing set theory, and in so doing begging the question of the reduction.

Note that here we are already leaving aside worries that, even if we had a coextensional property with a set, this would be different since the individuation conditions of sets and classes are as we pointed out in §4.5 different. Though here one might reply that insofar as properties specified as being a member of set x, these inherit their rigidity of designation from the sets they mention and so in these particular cases extensionality does suffice for equality. Nevertheless, one could further reply to this by pointing out that these properties are somehow too close to sets to be full-blooded classes, or allowing for the terminological mismatch, to be *proper* classes.

Indeed, we return now to our familiar theme that classes as opposed to sets are closely tied to what is expressible given a language, while sets depend on some set operations, as well as, more generally, on the notion of well-order. This seems to be independent of our choice of language. To borrow Bernays' point in his quote from §4.5, these exist independent of us and so, in particular, of our language. Indeed, even the notion of counting meant, for Cantor, countable by God, as Lavine remarks,[15] and so is far removed from our fellow earthly mathematicians. So, as Parsons puts it such a purported reduction fails since when talking about what constitutes a set, as opposed to a class, there is an: 'absence of any specific role for language.'[16]

[13]This is of course closely tied to that of whether we can reduce classes to sets for if this was feasible the answer would be a clear no.
[14]See (Charles Parsons, 1974, ibid.)
[15]See, (Lavine, 1994, p. 55)
[16]See (Charles Parsons, 1974, p. 10)

Another problem for the reductionist seems to surface when we considering the distinctively set theoretic operation of the powerset [17]. Now, the idea here is that we suppose that we have obtained the class reduction of some set and we want to obtain by some analogous operation the class surrogates of all its subsets, i.e. a class containing all the subclasses. Now, here we brig into play a key different between sets and classes, for in the case of sets we now that all the members of a set are well defined sets. Indeed, as we now from Cantor's demonstration that some collections are not sets, a collection containing an inconsistent multiplicity is itself inconsistent, and so we can say that for this operation to be carried out successfully we need to be able to quantify over all members of the set in order to collect them into the definite subsets, and in particular over these subsets since this will be the well-defined members of the new set. In fact, for any set we must be able to quantify over its members since this will be a well-defined domain. Indeed, these are crucial features of sets, as Parsons puts it: ' the two assumptions that get real set theory off the ground-the extensional definiteness of quantification over all subsets of a given set, and the existence of the power set-'[18].

Let's now return to our surrogate class for the reduced set, we want to find all the subclasses, in order to construct the power class which will serve as the class reduction of the power set. However, note that it is not clear what do we mean by *all* the subclasses since by using the logical operators we can find new classes out of the purported totality of subclasses. Indeed, just think about diagonalisation properties or properties that quantify over all such classes. Then, to get all the classes we should really need a hierarchy of languages that build up classes on stages by performing the class operations on the classes already formed in the previous stage and so on[19]. Then, we might hope for this hierarchy to reach a fixpoint at some high enough ordinal stage of iteration, or if this is not to happen, just take the union over all ordinals as our final language. However, such a thing will again be not a good basis for a reduction since we would then again be presupposing the set-theoretic operations or other notions such as the sequence of transfinite ordinals.

If, on the contrary, we take all classes to be 'all those that might be defined , independently of any specification of the means'[20], as Parsons puts it, it seems that doing this would be of no help since then we would not gain much in the determinacy of the quantification over the subclasses. Indeed, what does it even mean to talk about all classes that might be formed, and how is this determinate when, under this understanding it is not even determinate the language or group of languages under use? Moreover, recall that the notion of class is closely tied to that of definable property, not just that, but to a notion of logically definable in some given language, so it seems that to have a clear understanding of the totality of classes, and of just *a* class for that matter, one needs a clear picture of the languages (or sequence thereof) in use. This is not the case under such understanding of the totality of classes. Indeed, here the problem does not even seem to be that we lack an understanding of the totality of classes but of even

[17] Indeed, see §4.4.

[18] See (Charles Parsons, 1974, p. 9)

[19] See for instance (Maddy, 1983, pp. 126-7) for something along these lines.

[20] (Charles Parsons, 1974, p. 9)

any class. This is being forced upon by its close ties to language and our lack of clarity with respect to this.

5.4.2 From classes to sets

After giving some reasons for rejecting the possibility of reducing sets to classes in the previous section we now occupy ourselves with the converse notion, namely, can we reduce classes to sets? The idea of the reductionist here seems to be to understand statements regarding all sets as quantifying over some suitable large cardinal but such that it is, analogous to Russell's notion of typical ambiguity, not clear over which such cardinal this quantification is taking place. This is what Parsons calls a vague understanding of the quantifiers[21]. Given this understanding of the quantifiers, even when discussing properties that give rise to classes with an extension too big to be sets such as $[\lambda x(x = x)]$, once we make the statement more precise, ranging over some set, we can see that even these classes as a higher layer of sets. Indeed, suppose the statement is made precise by ranging over V_κ with κ some inaccessible strong enough that the speaker is not aware of its existence.[22] [23] Then, the class will be a set once we get to an interpretation of the quantifier when this ranges over $V_{\kappa+1}$, since the extension of $\hat{x}(x = x)$ is the domain of discourse, which was in that case κ, this being a set in $V_{\kappa+1}$. Now it is then very clear how this would reduce classes to sets, for what we take as classes are just sets we have not yet grasped.

There are several ways to resist this reduction. First, we could say that given our conception of class, the reduction rests in the concept we have of set, and so it is not sure that this could be obtained in general, but that only in certain theories of sets. Not, in particular, in a well-founded theory. Indeed, suppose we consider again $[\lambda x(x = x)]$, since clearly $[\lambda x(x = x)] = [\lambda x(x = x)]$, then $\hat{x}(x = x) \in \hat{x}(x = x)$, and suppose further that we have done the reduction of this class to κ, then we would have that $\kappa \in \kappa$, which is simply false in ZFC. That the reduction cannot be carried out in the most common theory of sets, which seems after all what Parsons tries to do since he talks about the von Neumann hierarchy as the universes of quantification, is not to say that this cannot be carried out at all, specially since we do not take sets to be necessarily well-founded, but perhaps indicates that classes of a more traditional theory such as NBG where self-membership of classes is disallowed would be less resistant to a purported reduction. Hence, we should pay special attention to make sure that reductionist arguments take into account, in order to be successful, the specifics of our notion of class.

Note also that the reduction that Parsons is suggesting would be perhaps not what we would at the outset thought of a successful reduction. Indeed, suppose we have the class $\hat{x}(x \neq x)$, then we would look at $\emptyset$ as the adequate reduction of

[21] See (Charles Parsons, 1974, p. 10) and (Charles Parsons, 1983, p. 521).

[22] Though, one must be, according to Parsons, careful since this precisification does not exactly capture the speaker's intention, for we seem to be sharpening the domain *too* much, see (Charles Parsons, 1983, p. 523).

[23] Moreover, the idea behind this proposal should remind the reader of the intuitionistic semantics for set theory explained in §2.

such class. Here we use a definite description since one can take the reductionist to be saying that to each class there corresponds one and only one set. However consider again our class $\hat{x}(x = x)$, this will be reduced to some κ. Now, what Parsons says is that which particular κ this turns out to be is ambiguous. But it would be some definite κ or another. However after realising this we again want to account for our universal quantification, accordingly, we will be now quantifying over some other inaccessible cardinal κ', such that $\kappa' > \kappa$, now then this class will be reduced to some κ'. The idea here might be that the class reduced to κ' is a new class and so it makes sense that it is assigned to a different set, since one class includes more sets than the other now that our understanding of the notion of sets has grown to a new inaccessible. However, note that, with our view of classes, this reply is not tenable since we take classes to be intensional entities closely tied to properties. Even if the extension of the class is different when taking the universe to be V_κ than when it is $V_{\kappa'}$, this is the same class since the property defining it $[\lambda x(x = x)]$ is the same in both cases. And so repeating this thought process we see that this class has been reduced to a infinite number of sets and all of them different. It is hard to see then how a picture like this can be seen as a satisfactory reduction since, it is not only that we do not know exactly to which set the class corresponds, but that we do not know to which infinite family of sets we have reduced the class. It is not clear how such a picture can give us a practical way to explain away our talk of classes. Indeed, if asked to reinterpret talk about the empty class I can just talk of the properties of the empty set, if asked about the universal class in what seems to be Parsons understanding I will just refer to some cardinal albeit not knowing which, but in our case we have lost also the definiteness of reference.

One might think that we are exaggerating the problems of such a reduction by drawing our attention to the case of structuralists about numbers[24], (at least when we are not talking about *ante-rem* structuralism) the idea here is that we reduce numbers to certain positions on isomorphic structures, hence as in the case we are interested with here, what seems like one object in our discourse has been reduced to a plurality of entities. However, note that the situation here looks more problematic since what for the structuralist brings these entities in the reduction together is to be certain positions they share on some isomorphic structure. So one might want to know what is the analogous thing these large cardinals share? Certainly nothing like that, for the only thing they have in common is that at some point the speaker had an understanding of set such that this could be formalised in a truth preserving way by a model of ZFC of that cardinality, V_κ. So unlike the structuralist reduction this seems a sui-generis reduction, relativised and particular to each of the mathematical subjects, and seemingly not capable of objective verification as is straightforward in the case of structuralists by means of finding an isomorphic map. In short, in the structuralist case there seems to be a clear tie between the entities that constitute the number, which seems to be lacking for the multiple sets corresponding to the class.

Now, the previous remarks were an attempt to undermine the reductionist

[24] For more information consult for instance Benacerraf, 1965.

picture by focusing on features of our conception of class, perhaps it is more direct to oppose Parsons idea that one is allowed to take talk of universal quantification over sets to be talk over some V_κ for a suitably large κ. Indeed, one might resist the idea that preservation of truth is enough to preserve the intended meaning of someone's utterances and assertions. As is familiar to readers acquainted with the debate regarding model-theoretic or Quinean arguments undermining the determinacy of reference, such positions come with a cost.[25]. Indeed, that this is the case is plain in the mathematical camp by thinking about the opposition to nonstandard models of arithmetic, these models preserve truth but might seem to be missing something significant nevertheless. As Boolos puts it in our case:

> reinterpreting what they say in such a way that it is not about all sets is changing the meaning of what was said if not the truth value; (...) [this] would be to misrepresent what they said. (Boolos, 1998, p. 31)

Hence, that Parsons reductionist account preserves truth is not a sufficient condition to take it as satisfactorily. In fact it seems to me that Parsons is conflating the distinction between not knowing that some set, like a large cardinal, is actually a set from the fact that this set is part of the universe. If my intention is to quantify over all sets, I will quantify over the inaccessible even though at some point on time I am ignorant of this fact. Upon becoming persuaded this, only my epistemic state with regards to my domain of quantification will change, not the domain itself. Quoting Boolos again:

> But why should we believe that this account of the matter is the correct one rather than the simpler one: that at to and t he quantified over the same sets and at ti believed something about those sets (viz., that one of them was an inaccessible) that he did not believe at to? (Boolos, 1998, p. 33)

Be that as it may, as Boolos also mentions[26] this relativisation of reference can be seen as going further than what Parsons seems prepared to accept. Indeed, he only talks of interpretations ranging over V_κ, for κ inacessible, indeed assuming the consistency of ZFC, according to the downward Löwenheim-Skolem theorem, there will be a countable model of this theory. Now, Parsons takes the universe to be some inaccessible cardinal but if this ambiguity is taken seriously and the only requirement is that the interpretation makes the statements true how could we rule out a model where the universe is a countable set? The dilemma seems to be that either we accept quantification over all sets, which seems to be also supported by set-theoretical practice. Indeed, there is no warning of ambiguity when a textbook presents a theorem as holding for all sets. Or we cannot rule out that the model we are quantifying over is a set we seem to be not intending to quantify over such as a countable set. This however seems to much ambiguity to accept happily, for one thing is that one does not grasp the theory of inaccessible cardinals, and so what he takes to be all sets is just one of those, but another is to make the speaker accept that they might be wrong in their understanding of Cantor's theorem and that actually they do not quantify

[25] The interested reader on this more general debate might consult (Williams, 2008) or (Mcgee, 2005).

[26] (Boolos, 1998, p. 31),

over uncountable sets.

Now a possible reply by Parsons would be to assert that any reasonable user of ZFC will have an adequate interpretation of Cantor's theorem and so, although his quantification will be ambiguous, it can rule out the possibility of quantifying over a countable set. Indeed, for him, upon being acquainted with an inaccessible the new domain cannot be this old inaccessible but something larger since it needs to include this cardinal. However, it is not then clear what is preventing a speaker to realise that the universe can contain larger and larger inaccessibles and so that, in order to make sure that all of these are encompassed in their speech, when they talk about all sets it is clear that the intended domain cannot be some set be this as big as one wants since a larger set is susceptible of being found. In a sense an understanding of this possibility is what would eventually remove the ambiguity, and so at some point any competent set theorist must be taken seriously as quantifying not over some given set but over the class $\hat{x}(x = x)$ which is not reducible to any large cardinal on pain of misreading the intended universality of the quantification. A user that would accept otherwise is, albeit maybe in a less flagrant way, suffering from the same incompetence as a speaker that accepted a countable model.

5.5 On the plurality of sets

In this section we consider the objection against classes that, any of the functions that our theory of collections assigns to them can be carried out without them by using plural quantification. The idea here is to substitute singular quantification over classes with plural quantification over sets in order to carry out the traditional mathematical tasks of classes explained in §5.2.2 such as achieving finite axiomatisations or motivate large cardinal axioms. The advantage then is one of qualitative parsimony, as already pointed out when talking about the reductionist in §5.4, quoting Burgess:

> The advantage of employing plural quantification in this way is that it leaves us able to maintain that there is just one kind of collection, and that set theory is the most general theory of collection. (Burgess, 2012, p. 201)

Then the question seems to be if we want to privilege a parsimonious ontology and so keep plurals, or a classical logic and so retain classes. This seems to be hence a matter of taste. Nevertheless, we can see that the worry as whether interpreting the use of classes as referring to a single object or simply to a plurality of sets is already present in Russell with the use of class as one and class as many mentioned in §4:

> Is a class which has many terms to be regarded as itself one or many? Taking the class as equivalent simply to the numerical conjunction "A and B and C and etc.", it seems plain that it is many; yet it is quite necessary that we should be able to count classes as one each, and we do habitually speak of a class. Thus classes would seem to be one in one sense and many in another.(Russell, 1937, §74)

Hence, for the supporter of plural interpretation of class talk, when mathematicians' talk quantifies singularly over classes, what is really happening is that they are using plural quantification over sets. Just as in the case of definite descriptions there is a mismatch between the grammatical structure and its logical or semantic content, as Cartwright puts it:

> the use of 'collection' I have in mind is one in which it serves only to singularize a plural nominal: they are some people iff they are a collection of people. They are the present kings of France iff they are the collection of present kings of France. (H. M. Cartwright, 1993, p. 213)

Hence, we would be in a position to paraphrase the truth conditions of a sentence regarding classes to one regarding pluralities of sets. Consider as an example under the plural reading that there are some sets that are well-founded sets and such that all such sets are one of them. Indeed, assume that ZFC deals with all sets and consider all sets this theory describes. This is enough to guarantee the truth of the assertion that the Russell class, namely $\hat{x}(x \notin x)$,[27] exists. Even as we might not stop using talk of classes since this singular nominal facilitates coreference and helps making statements about the collections in question more concise. Indeed, as Uzquiano points out:

> the purpose of the noun 'class' is to bring into play convenient and familiar devices of singular pronominal cross-reference (. . .) [the use of these nouns is useful since] plural paraphrases quickly become unwieldy and difficult to parse. (Uzquiano, 2012, p. 73)

In any case, the key idea is that even if very useful, this talk of classes can always be paraphrased in terms of plurals and, more importantly, that is only in that way that we understand the semantics of the expressions since we realise that no existence of a collection encompassing the terms of the plurality is required for the truth of such statements but only the existence of its components. So the Russellian class-as-many is the correct semantic reading even if the class-as-one is more friendly to the speaker.

One immediately sees the advantages of such an approach for a critic, such as Boolos in (Boolos, 1998), takes sets to be all collections. Be that because they are a friend of qualitative parsimony, as discussed in §5.4, or otherwise. As Burgess says:

> The advantage of employing plural quantification in this way is that it leaves us able to maintain that there is just one kind of collection, and that set theory is the most general theory of collection (Burgess, 2012, p. 201)

5.5.1 Membership matters

We want to point out several problems with this pluralist approach, that aim at persuading the reader not to follow the pluralist down their proposed route but to take the classes seriously both heuristically and semantically. As when dealing with the issue of reducing classes to sets in §5.4.2, we have some more general remarks and others more closely related to our notion of class. We begin with the latter and focus on some observations that Uzquiano makes with

[27] Note that here the variables are taken to range over the sets.

respect to the relation between sets and classes once we understand the latter pluralistically. It is perhaps worth noting how, as we will see in a moment, what Uzquiano characterises as a great advantage of the plural interpretation will be for us rather an inconvenience. This of course is a sign of how our notion of class is a revisionary one and departs from the way most usual theories characterise these objects.

The first thing that Uzquiano notices is that on the most usual understanding 'sets are a special case if classes'[28]. Indeed the usual classification seems to rule that classes encompass sets and proper classes, the latter being just those collections too big to be sets. It is of course immediate why this is the case, since in this framework sets are individual objects and classes are just collections of individual objects but that are members of a further singular entity existing over and above them, as it is indeed the case withe the members of sets. Recall that the closest we get to such an identification in our view is as seen in §5.4.1, through accepting a language with a name for each set and via the classes corresponding to the properties $[\lambda x.x \in \bar{a}]$. However, note that under the pluralist option the class we are looking for so that this is indistinguishable from the set a is not the one just mentioned but rather $[\lambda x.x = \bar{a}]$, which will have as member the set a i.e. under for the pluralist this will just be a. Uzquiano acknoledges the issue here by considering the relation between individual ordinals and the class On: 'nothing can literally be a member of them—even though every ordinal is one of them.'[29]. Uzquiano proposes thus that we understand the relation that a set bears to a class not as we ordinarily do in terms of membership, but in terms of correspondence:

> Let us say that a set x and a class X correspond if and only if for every set y, y is a member of x if and only if y is one of X. (Uzquiano, 2012, ibid.)

So coming back to our example, $[\lambda x.x \in \bar{a}]$ would correspond to a. Now, as we have made abundantly clear throughout this work, this is not our position since we take sets and classes to be very distinct entities and so the failure to identify sets with classes need not be a problem in our view, in fact it seems more an advantage of the pluralist position that in this view sets are not reducible to classes as we defended above, and not a drawback as Uzquiano thinks.

We now focus on a potential problem that pluralist approaches have with regards to membership. Indeed, the membership relation holds between two objects, not between an object and a plurality of objects or between two pluralities. Quoting Russell:

> And thus $\in$ cannot represent the relation of a term to its class as many; for this would be a relation of one term to many terms, not a two-term relation such as we want. This relation might be expressed by "Socrates is one among men"; but this, in any case, cannot be taken to be the meaning of $\in$. (Russell, 1937, §76)

The pluralist, thus has no problem in explaining membership between sets since these are singular objects. However, they must explain how we can have

[28]See (Uzquiano, 2012, p. 73)
[29]See (Uzquiano, 2012, ibid.)

a set belonging to a class, this is something that the pluralist is pressed to do since remember that he takes his view to vindicate the classical view of classes. Namely, that these do not contain other classes, as seen above this is a problem in our understanding of classes, and in this view sets are indeed the members of classes. Incidentally, this means that a class cannot be a member of a set, this is a common position regarding sets and classes that we do not have any reason to dispute. Similarly to the relation of correspondence introduced above, Uzquiano explains as follows the understanding of membership of some set of a class:

> The other item of business is to specify the conditions under which a formula of the form $x \in X$ is to be evaluated as true (relative to an assignment of values to the variables). The answer is that $x \in X$ will be true relative to an assignment when the set assigned to the set variable x is one of the sets assigned to the class variable X. (Uzquiano, 2012, p. 76)

Hence here membership just means that x is one of the sets of the plurality X. A natural question now is how would this framework handle membership of a class to another class, the answer is that it cannot do so, as Uzquiano observes: 'It is of interest to notice that the proposal to eliminate occurrences of the term 'class' in favor of plural noun phrases does not iterate.'[30]. Here we see that a class containing another class would be such iteration. Now, if classes would indeed iterate we would have such things as the class of non-self membered classes, and so on. The reason for this is that for the pluralist,[31] such iteration would be the same as having the plurality of pluralities of sets that are non-self membered, but this seems to be nothing else that the plurality of non-self membered sets, i.e. the pluralisation of classes is an idempotent device. Hence, the pluralist cannot distinguish between $X \in Y$ and $X \subset Y$. Uzquiano takes this to be an advantage of the pluralist account since in the usual theories of classes, say for instance NBG or MK, classes are not members of other classes. In these theory then, iterative classes, i.e. classes of classes are not a thing and so vindicate this plural reading that rejects so-called superplural quantification.

However, a theory of plurals that accepts this is not a permissible choice for our understanding of classes. Indeed, it is clear that given our conception of class where we have classes belonging to other classes and also potentially to themselves, this feature is not an advantage but rather a grave disadvantage.

Consider as an example the following property $[\lambda x.x \text{ is infinite}]$, since we take the variables to range over everything, and so in particular over individual objects and collections, be these sets or classes. And since presumably, as already mentioned we have reasons to believe that at least in the usual mathematical picture there are infinitely many of these, this property will be instantiated by infinitely many things and so its associated class will also satisfy it, i.e. $[\lambda x.x \text{ is infinite}]\hat{x}(x \text{ is infinite})$ and so $\hat{x}(x \text{ is infinite}) \in \hat{x}(x \text{ is infinite})$. Now, assume for simplicity that this class only contains the infinite sets and the class itself, then if we reject a superplural quantification one cannot distinguish this class from the class of all infinite sets, i.e.$\hat{x}(x \text{ is infinite} \,\&\, x \in V)$.

[30]See (Uzquiano, 2012, p. 74)

[31]Not for all pluralist though, for instance Eileen Wagner does advocate for a theory accommodating superplurals, see (Wagner, 2015).

Indeed, under the plural reading of classes, the class of infinite collections is just the collection of infinite sets, since ultimately all classes are just pluralities of these, and so this class will just consist of the collection of infinite sets twice counted which is still the same collection of infinite sets. But of course these classes are different since one contains itself and the other does not! Indeed while $\hat{x}(x \text{ is infinite}) \in \hat{x}(x \text{ is infinite})$ it is clear that $\hat{x}(x \text{ is infinite}) \notin \hat{x}(x \text{ is infinite} \,\&\, x \in V)$ as $\hat{x}(x \text{ is infinite}) \notin V$. Note that here we assume that the property of being a set is definable, if we do not assume this the problem could be even worse since we end up with something that according to our theory should not even be a class in the first place.

5.5.2 Where to hit the brakes

We now move to a more general worry regarding pluralist accounts, which however will have an impact for our choice of theory of classes. Indeed, the pluralist rejects that the members of a class form an individual object distinct from them. However, he seems to be perfectly happy that such individual objects collecting their members exists when we deal with sets. This is something that we agree with as explained in §4 when comparing sets to rigid designators, in the sense that whenever their members exist they exist, and so if we take the position that mathematical objects exist necessarily and we focus on the sets that only contain pure sets, as it is usually done in mathematics, we arrive at the conclusion that all mathematical sets exist in all possible worlds. In the case of non-mathematical sets, these that contain urelements, we get the weaker conclusion that whenever the urelements exist the set containing them also exists, as van Inwagen says:

> 'a position about sets that almost everyone holds: In every possible world in which, for instance, Tom, Dick, and Harry exist, there also exists a set that contains just them.' (Inwagen, 2006, p. 74)

and similarly Cartwright:

> The cookies in the jar are such that in every possible world in which they exist, there is a set of which they are the members (R. L. Cartwright, 2001, p. 30)

Now note first that although, as van Inwagen points out, the view expressed above is the most common one, one might nevertheless what we could call a nihilist about collections and insist that no plurality of objects forms a novel object namely its collection, be this sets or classes or otherwise. One could even spouse a more moderate version of the nihilist view, namely that the determination or not of a set by some objects may vary from possible world to possible world.[32] This is not the position with which we take issue here, although we of course are not nihilists about sets and take seriously the idea that whenever somethings can be collected in a set they are necessarily so collected (whenever their elements are present), here we follow Cantor as explained in §4.2, with regards to that the ability to count a finite collection is enough to have its set

[32] See for similar comments (Charles Parsons, 1983, p. 526), or (Boolos, 1984, p. 72).

and how we carry this idea on into the transfinite through the ordinal hierarchy or the powerset operation.[33]. Indeed, the position we want to focus on here is that which seems to be admitted by Uzquiano and Burgess, namely that the elements of sets, be this the strictly Cantorian ones or those more familiar to the mathematician of ZFC, succeed in determining a novel object, the set, while those in a class do not. The question is then, why these double standards? Why plurality principles apply to classes and not to sets?

It seems that the main problem with classes stems from the fact that as pointed out in §5.2, set theory ought to be the theory of all collections, echoing Boolos, Uzquiano says:

> The subject matter of set theory comprises all the collections there are (. . .). There are no well-founded collections that lie outside of its realm, and, in particular, no collections that fail to form a set, i.e., no proper classes. There is, as a consequence, no distinction to be made between sets and classes. There is no proper class of all sets. Nor is there a proper class of all ordinals. And, in general, one should not take at face value locutions that suggest that there are proper classes as well as sets. (Uzquiano, 2012, p. 68)

The idea here seems to be that the pluralist has no issue with real collections, or collections that are in good standing, the only such collections are sets and so they accept it without problem. Classes, however, are not in the same category and so this being unacceptable their use must be explained away, hence the introduction of plural quantification. Now, as we have mentioned abundantly we do not take set theory to encompass all collections, but those that were begun to be studied by Cantor at the turn of the XX century. There is other notion of collection namely those tied to extensions of logical properties that are also used in general and, at least before trying any paraphrasing or reductionist device seems also to be used in mathematics as indicated in §5.2. In fact, the main problem that the pluralist seems to have is that they take classes to be *ad hoc* devices to solve some problems of the theory of sets, but that since these problems can be solved without appeal to them by using plurals, the latter should be the preferred course of action. We, like the pluralist also reject the use of such a type of class, the more orthodox use of the term, as pointed out in §5.3.

That our classes are not the entities in *ill standing* that the pluralist wants to excise from the ontological picture can be justified indirectly by the fact noted in §5.5.1 that, when Uzquiano portrays as an advantage of the pluralist approach that it does not allow classes to contain other classes since the usual class theories also disallow such a move, this was taken by us as a big disadvantage. Indeed, we do insist that classes contain other classes, hence it seems that the classes Uzquiano is eager not to accept in his ontology are ones we also reject. We seem to be able to draw similar conclusions from the following remarks by Burgess:

> The word 'class' rather than 'set' was originally used by Frege for the extensions of concepts, but it has since come to be used for set-like entities that in some mysterious way fail

[33]For a concise argument for the conclusion that the falsity of the necessity of any group of individuals to constitute a set entails that there is a possible world devoid of sets consult (R. L. Cartwright, 2001, §6)

to be sets. (Burgess, 2012, p. 200)

Indeed, it is clear that our notion of class is much closer to that of a Fregean concept, given its key tie to logical properties, as opposed to the more commonplace notion of classes which, as we agree with Burgess make a distinction without a difference between sets and classes. That Burgess is not directing the pluralist account of classes to the entities we call classes is further demonstrated by the fact that he then offers two different accounts of what classes are, as we will see neither of this is an adequate description of our position. Burgess talks first of the splitters, which take no class to be a set. Indeed, this we also agree to, but then take every set to be coextensive with a class, now as we remarked in §5.4.1 we do not endorse such a view, indeed remember our remarks regarding the powerset operation, on pain of trivialising the language with a name for each set. Burgess then identifies the lumpers which takes sets not to be just coextensional but identical with some classes, the proper classes having no classes as other members. Now, this position is something we really do not endorse as can be clear from reading the previous chapter, we do take sets and classes to be very different entities and so not identical and we see no problem with properties instantiating other porperties.

In sum, the point that we are trying to make is that pluralists about classes such as Uzquiano or Burgess accept that the members of sets make up a collection over and above themselves, but reject this in the case of the usual classes of say NBG or MK. Since they see these entities as ill motivated, *ad hoc*, or just being entities that bring a distinction without a difference with respect to sets. Since our notion of class does not fall prey of any such charges as we have tried to show throughout this and the preceding chapter, we take the pluralists that takes sets as objects in good standing to have no reason not to do so with the classes that *we* propose. Failure to do so would result on being charged with arbitrariness since they would accept the existence of some well-standing collections such as sets but not of other such as classes also in perfectly good standing.

The previous defense of accepting classes has been focused in our notion of class, namely that the criticisms that the pluralist raise for classes do not apply to them and so they have no reason to reject our class offer. Now, it seems however that there is some strand of criticism to which notions of classes like ours that are close to Fregean extensions as pointed out by Burgess are specially vulnerable to. Namely, that one feature that distinguishes sets from classes, is that the latter are prone to paradox while the Cantorian notion of set is not. This reasoning is specially forceful since although a claim that the Russell class exists coupled with the possibility we accept of classes being members of themselves and other classes leads to paradox, there seem to be nothing more innocent that there are some objects such that all and only those objects that are sets and not members of themselves are among them, the plural reading of Russell's class existence.

The point here is that at some point we should limit our admission of constitution of new objects by their members or risk inconsistency. As Cartwright beautifully puts it: 'The brakes must thus be applied somewhere; but it is unclear where'[34] Note that in a sense Cantor gives us some indications as to when we ought to hit the breaks with regards to sets. For instance, we should observe the notion of limitation of size and so on, and thus since his theory of sets is taken to be paradox free the breaks seem to have been applied correctly there. However the issue seems trickier when we talk about classes, it seems we cannot allow all properties to determine a class, but it would be ad hoc to bar some properties to determine a class but not others. Even if asserting only to ban those arising from paradoxical properties, we cannot be sure which ones will turn out to be paradoxical and even this criterion seems prone to accusations of *ad hocness.* So, even if we knew, one ought to be consistent in the treatment of classes, i.e. so in any case one should not allow for any class to be a genuine individual object. Moreover, since before we accused the pluralist of arbitrariness when treating classes and sets differently, we would be blatantly falling on our own sword if discriminating among classes in these respect.

Since we accept this point but we still maintain that we take, a priori before paradoxical results appear, that both Cantorian sets and Russellian classes are notions of collection and that so these should be treated alike in terms of determination of novel objects in terms of their members, we opt to bite the pluralist bullet and accept that all properties, including the paradoxical ones will determine a class. Now, since of course we want to avoid paradox the reader can already suspect that if we are not willing to compromise in the camp of which properties determine a class, the compromise will have to come from elsewhere, as we have seen in theories that allow for such freedom in the construction of classes such as those of Maddy or Schindler, this will have to come from modifications to the logical system in use. These considerations will come to the forefront in the next chapter when we discuss how the different theories surveyed in §2 fare against the different criteria we have laid out throughout this work for our theory of sets and classes, and so this seems a fitting point to conclude the present chapter.

5.6 Summary

In this chapter made use of the facts about sets and classes explained in previous chapters in order to address some philosophical arguments against the use of classes in addition to sets in our theories of collections.

We began §5.2 by addressing the objection that the usual theories employed in mathematics do not employ second order resources, and so in particular do not employ classes and so we needn't adopt the use of such entities. The burden of proof is on us if we want to introduce second order resources and so the possibility of classes. Here we explained how both outside (§5.2.1), but also crucially within mathematics (§5.2.3) they seem to play an important role, if not officially for crucial heuristic purposes.

Next, in §5.3 we addressed the related criticism that our theories of collections needn't deal with classes since these are after all not well-motivated entities, as opposed to sets they are *ad hoc.* Against this charge we argued that

[34] See (R. L. Cartwright, 2001, p. 32).

even though this seems to be the case to more traditional conceptions of classes under our understanding classes are sufficiently well motivated and distinct from sets to avoid this charge.

In §5.4, we looked into the worry that even if warranted in using sets and classes, since one of the entities is reducible to the other, we would have a more parsimonious ontology by giving eliminating the reducible entities from our picture. First we looked into the possible reduction of sets to classes (§5.4.1) and noticed that these reductions are not satisfactory since they seem to presuppose the concept of set. Then we looked at the reduction of classes to sets (§5.4.2) in particular in Parsons proposal of doing so via the notion of ambiguity of the set-theoretical universe of quantification. Here again, we pointed out that this reduction is not satisfactory once we respect the intensional character of our classes, and more broadly, as well as by directly attacking the claim that our reference to the universe of sets is indeterminate.

Finally, §5.5 we addressed the worry that a simpler ontology could be obtained not by reductionist methods but by paraphrasing away the use of classes in the discourse as plural quantification over sets, this is the correct interpretation of th second order quntification in play. We took as an example of such an approach that of Uzquiano and then made some remarks to the effect that what he considers one of the main advantages of his approach, namely that as standard theories of classes it does not allow for classes to be members of others classes is in our view a disadvantage of the theory (§5.5.1). Next, we also raise the worry for the pluralist of why they accept plural quantification over sets for classes but they accept that objects can be collected into a set to form a novel object collecting them (§5.5.2). Since we take classes to be as sets collections in good standing we take the pluralist to be acting arbitrarily in this respect, however, being coherent with respect means that also us will have to accept that all properties determine a class even the ones leading to paradox, this will have revisoconsequences for the logic of our theory.

Chapter 6

Which classes?

6.1 Introduction

In this final chapter we will put together all the facts gathered in the preivous chapters of this work regarding sets and classes in order to go back to the different theories of collections surveyed in §2 and asses how they fullfil the different requirements we laid out for a satisfactory theory, offering the reader our advice about which theories ought to be prefered due to their better performance in meeting our desiderata.

The chapter begins by putting together the different features regarding sets and classes that we have been uncovering throughout the different sections of this work in §6.2. We hope that this will serve as a remainder to the reader about the desiderata we lay out for the theories studied as well as serve as some sort of checklist when we asses the different theories of collections in the remainder of the chapter. Indeed, this will be done in §6.3, were we will take the theories studied in §2 and filter them trough the requirements laid out in §6.2 in order to determine which system emerge as the more felicitous ones. We will look at theories such as ZFC and ST more briefly since their shortcomings have already been discussed extensively earlier in this work. We will pay more attention to systems like NBG, NF, A, or those of Maddy and Schindler. Finally, in §6.4 we will look back to the different chapters of this thesis and suggest to the reader some further avenues of research not covered there but that arise from their themes and we think are worth pursuing.

6.2 Taking stock of classes

During the course of this work we have been investigating sets and classes, including several ideas regarding the different characteristics that our notion of classes ought to fulfill. The aim of this section is to collect the different ideas developed about sets and classes, especially of classes given the revisionary view of these that we have taken here in order to carry out the review of the theories introduced in §2.

Notice then first that wen we think of collections of objects we might want to refer to the collection of objects that share together some property, this is in a sense a more subjective notion of collection tied to our linguistic abilities and in our formalisation of that of property expressible in our given logical language. Such extensions of properties is what we call classes, which are close to the notion of Fregean concept surveyed in §1. However, we also see that we might want to collect some objects that do not share any property, or that we do not take to share a property to the best of our knowldege, or that share a property we cannot express the with our language and so on. Here the idea is that if finite we can still take this objects to form a collection if we count them together.

This last idea is captured by the idea of Cantorian sets take into the infinite and transfinite realm. These are then, the sets and the classes, the two ways we have to approach the notion of putting together some objects into a collection. Thus we could say that we have a classical way to approach our notion of set but a revisionary way to approach the notion of class. Indeed, here we take sets to be those collections that can be well-ordered by establishing an order isomorphism between them and one of Cantor's ordinal numbers discovered in his investigations of the properties of Fourier series as discussed in §4.2. We also noted in §4.4 that following Cantor's powerset principle we take to be sets some collections that, at least before explicitly adopting the axiom of choice, are not as easy to well-order. We also see that from the idea of counting stems also the requirement of limitation of size. Namely, that if a collection is so big that one can project the transfinite sequence of ordinals this is somehow too big to be counted and so cannot be a set. From operations like the powerset in which we take all the subsets of a set, even if we have no way to refer to these in the language, we see that the notion of set is closely tied to that of an arbitrary function, as opposed to a definable one. And so in a sense we understand this objective standing of sets were their existence is not tied to what we can express and summarised by Lavine when remarking that: 'for Cantor, (...) "countable" meant countable by God'[1]

Regarding classes, our story is more heterodox. As mentioned above, as opposed to sets which are closely tied to arbitrary maps, classes are joined to definable properties. As seen in §4.3, we take the Russellian notion of class closely tied to that of a Fregean extension explained in §1 as a starting point. These are entities not bound by the limitation of size property as seen by the fact that Russell tried to show contradiction in Cantor's reasoning by applying his diagonalisation arguments to the universal class, of course although this is too big to be a set it is straightforward to associate it with a class to this via the property $[\lambda x.x = x]$.

Thus, we observed that while Cantor's notion of set is not prone to paradox this is the case with the Fregean extensions as belaboured in §1. We also require for the reasons outlined in §3 that classes can belong to other classes and to themselves. This was suggested by noticing how such approach is of great

[1]See (Lavine, 1994, p. 55)

advantage when one tries to solve puzzles in metaphysics for instance as we saw in §3.2.1 regarding Bradley's regress were the circle can be stopped by allowing self-instantiation and also when formalising natural language inferences as noted in §3.2.2 that deal with some property showing some other property. For instance, since the class arising from the aforementioned property $[\lambda x.x = x]$, $\hat{x}(x = x)$, is self-identical, i.e. $\hat{x}(x = x) = \hat{x}(x = x)$, then it follows that it instantiates the property $[\lambda x.x = x]\hat{x}(x = x)$, and so it is in its extension and so $\hat{x}(x = x) \in \hat{x}(x = x)$. Again, we see how the notion of class is prone to paradox since this kind of self-instantiation facilitates Russell's paradox. We also argued in §3.3 that when we use the property $[\lambda x.x = x]$ its corresponding class has as members all the objects in the domain, i.e. that we are really quantifying over the entire domain and endorsing the All-in-One principle, and so that our theory of classes will have such object as the universal class.

We also noticed in §4.5 that, although this will depend on our language, there might be in the same domain more sets than classes since the classes that exits will be limited by our ability to pick them out with the expressible properties of such a language. This is the sense in which classes can be seen as more subjective than sets since although the latter are independent from us in ontological terms, the former will depend on language. Related to this point is that as opposed to sets which will have the same members and exist necessarily in all possible worlds in which their members do, classes have an intensional, or perhaps even hyperintensional, character. Their membership depending on the world we the property they arise from is evaluated, unlike sets existence, as noted in §4.5, extensionality will not suffice to individuate them. Indeed, while the class of present emperors of France will be empty if evaluated in the actual world, it would not have been so if we had looked in 1854 for it would have contained Napoleon III. On the other hand ω would have had the same members regardless of the French constitutional arrangements of the day.

Finally we also noted in §5.5.2 that since we see some arbitrariness in that some friends of seeing talk of classes as just plural quantification over sets but not sets as talk of plural quantification over objects. As we saw this could be based in the fact that pluralists see sets but not classes as objects in good standing. Since we of course see classes as such and also see that we could be charged of the same offence if we took that some properties give rise to an actual class but some others would not, we bite this bullet and admit that for any definable property there correspond an object, namely its class. Even if this leads us to accept that there is, for instance, a Russell class $\hat{x}(x \notin x)$ associated to the paradox inciting Russell property $[x.x \notin x]$. Again this in contrast with sets were assortments of objects like the totality of ordinals that might look like candidates for this status are rejected due to their problematic size.

6.3 The theories revisited

With the remarks from the previous section regarding the features of our notions of set and class in place, we now move to an assessment of the theories introduced in §2. Here we will devote most of our space to deal with theories of sets and classes. Of course, these are not all the theories we have dealt with throughout this work. There is for instance ZFC a theory that only uses sets in its ontology, however we feel that the remarks made at length on §5 regarding the need to add classes to a theory of collections and especially to a mathematical theory of collections in which the basic theory for which supplementation with classes was being argued for was taken to be ZFC, should make clear to the reader why we do not endorse the use of ZFC alone. Indeed, this has no objects that are self membered, no objects that encompass the entire domain, or all objects are extensionally individuated. This is not to say that we reject it as a theory of sets, but that it is not an adequate theory of collections since it omits any mention of classes and so to be satisfactory it ought to be supplemented with a theory in which classes are treated appropriately, and for that a prerequisite is to be at least treated somehow.

Another theory that we already discussed at some length is ST, Russell's simple theory of types. First note that such a theory does not make the distinctions between the two types of collection that we have clearly drawn between sets and classes. For less general criticism, the reader is mostly referred to our remarks in §3 in which we insisted in the necessity of our collections to be capable of self-instantiation. We also gave some particular instances in which the formalisation of natural language inferences dealing with existential quantification seem to be ill-served with a typed framework, as well as how our defense of universal quantification over the entire domain does not fit well with the stratified framework of the theory of types. Hence, as we suggested in §3.4 we now turn, after the historical remarks of §4 and the philosophical arguments of §5, to examining the different theories that include the use of classes. We begin with Quine's new foundations.

6.3.1 Quine's New Foundations

The first thing we notice about Quine's NF, explored in §2.3, is that, even if this theory only recognises one type of collection, it is not entirely clear whether the objects dealt with here are classes or sets. For example, the fact that these collections are individuated extensionally might lead us to think we are dealing with sets. However, we also oberve after further scrutiny that there is a blatant lack of limitation of size doctrine for these entities and so the idea that these are sets becomes untenable. Indeed, as already discussed in §2.3 paradoxes are avoided in this theory through restricting the collections that exist in a type theoretic way through NF's axiom of comprehension which requires the properties that give raise to a class to be stratified. This will not bar collections such as the universal one V which is clearly too big to be a set. We also saw that in this theory $V \in V$ and this is the kind of self-membership that we want to guarantee for classes. Moreover that these objects are closer to classes is clear given our ability to prove the axiom of infinity through defining a stratified non-empty irreflexive transitive binary relation.

More evidence that these objects are not compatible with our Cantorian understanding of set comes when noticing that the stratification requirement blocks, as we remarked in §2.3, the proof of Cantor's theorem to the effect that

$\mathbb{P}(X) > X$. So it definitely seems that the entities defined here are closer to classes given the reliance in definition through properties definable through a stratified formula and we lack entities set-like enough. In the sense of being independent enough from the definable properties, further evidence from this overdependence on properties comes from the fact also mentioned above that the axiom of choice a distinctively set-theoretic principle is disprovable in NF. This is not what we desire for a theory of collections, we want a sharp distinction between sets and classes but of course for the distinction to be sharp there must be some distinction in place.

Note incidentally that this theory taken simply as a class theory would still fall short of our mark not only because of extensionality but because the notion of stratified formula lying at its core. Indeed this forces us to hit the ontological brake for classes such as the Russell class arising from a property which is not expressible in a stratified formula, <n otherwise syntactically well formed formula. Since we take seriously the slogan that no formula without its class, this restriction is not acceptable for a class theory that we would endorse.

So we see that NF is not an adequate theory of collections since it fails to distinguish among these sets and classes, although the objects posited are closer to classes than to sets, and so leaving aside the worry of lacking objects playing the role we demand from sets, the requirement of stratification still bars this from being considered as a satisfactory theory of classes.

We now briefly comment on ML the theory also introduced in §2.3. This introduces in NF class variables, although we now have unrestricted comprehenion for classes which was a criticsm of NF above, this theory is still deficient. First because it defines classes as those objects that do not have other classes as members, of course this is something we oppose as we have made abundantly clear. It has also the odd consequence that the universal class is considered a set. Still this approach does not solve one of the central problems of NF namely that its its sets, now ruled by comprehension of stratified formulas, are too different from the Cantorian sets we have in mind for us to be acceptable. Examples includes the aforementioned blatant violations of limitation of size as well as the refutation of AC, which ML conservatively extending NF carries with it, and so it is also not acceptable to us. This theory attempts a separation between sets and classes but not only are its classes not satisfactory, its sets look too unrecognasible to even be labelled as sets.

6.3.2 NBG and MK

We now turn our attention to NBG. This, as we commented on §2.5, is the usual expansion with classes of ZFC, this being the most common theory of sets, it can also be claimed that this is the most canonical theory of classes for the mathematician. As the reader can probably by this point accurately predict, we do not find this theory of sets and classes acceptable, even though we take ZFC as an acceptable development and axiomatisation of the principles behind the Cantorian notion of set. So here our problems will not be like in §6.3.1 with NF's notion of set, but purely with the notion class in place. Indeed that we do not approve of this theory is clear when noticing that this is the canonical theory of classes and we have labelled on multiple occasions on this work our approach to classes as revisionary precisely by comparing it to the usual conception of class which is embodied in NBG.

More concretely, we object to the fact that in NBG, like in ML above, all collections are classes and these are distinguished from sets and proper classes. We would rather prefer to label all objects as collections and proper classes just as classes and leave sets as sets. Putting these terminological issues aside, as in the case of ML, we do not accept that all elements of classes are sets. Indeed, as we know we take it as fundamental that some proper classes, using the NBG parlance, are elements of other proper classes. Here only sets are elements of other classes and this we reject, in fact note that here all collections are well-founded and again we do not take classes to obey this although we are open for sets to abide by it. Here both sets and proper classes are individuated extensionally, which again contradicts the intensional character that we attach to classes. Note on a positive note that some of the axioms of NBG such as that of intersection or complement do indeed encode the idea of classes obeying logical operations, in this case negation and conjunction. However, recall that classes includes sets so even if we are happy that these principles apply to proper classes we are more cautious with the prospect of sets having for instance a complement. Even if these turns out to be a proper class and not a set, we would prefer if sets did not have a complement at all since these objects are bound by the usual set-theoretic operations such as power set or pairing, which are indeed listed as axioms, but not the logical ones like negation. In short, one could say that here classes and sets are not properly well-distinguished entities.

We now turn our attention to the class existence theorem. If in NF we had a class for every stratified formula, here we will have a class for each predicative formula. Again as in the case of NF's comprehension, where we remarked that we do not find satisfactory to be limited in the classes that exist, here in particular by the fact that the property they arise from is such that only set variables are bound by quantifiers. Unlike in NF, NBG fares better in this respect since we do have the Russell class. This is after all tied to a predicative formula, however the paradox is blocked by using the fact that for something to be a member of this class it ought to be a set and so if the class itself is a set then contradiction would arise, again we do not agree with such a conditionaL. Indeed, we shouldn't assume that only sets can belong to classes and so again this doesn't seem to us a felicitous way to block the paradox. All the preceding remarks leads us to reject NBG as a satisfactory theory of classes.

We now briefly turn our attention to the other theory considered in §2.5, namely, MK. Recall that this is a theory stronger than NBG. In fact it is as strong as NBG plus the existence of an inaccessible cardinal. This is achieved by replacing the predicative comprehension schema of NBG in its infinite axiomatisation by an unrestricted comprehension schema. Given our principle of no property without class, it is clear that MK ought to be preferred to NBG. Still, we cannot take this theory as satisfactory with respect to its treatment of classes, since negative issues that get carried over from NBG include not allowing proper classes to contain other proper classes and solving the paradoxes through the same device using this fact regarding membership. In addition, the theory will not distinguish clearly enough between sets and classes with regards to issues like operations and individuaton.

6.3.3 Ackermann

In this section we deal with the theory A, Ackermann's set theory, as well as others introduced in §2.6. Although in A we also notice the problem that both sets and classes are individuated extensionally, we readily notice an advantage of this systme with respect to the previous theories of sets and classes just considered. Namely, that here, although we will also have a set predicate, this will not be definable. Recall that this was justified by Ackermann taking seriously the idea that the extension of the set predicate is never completed and varies with time and so we should not at any given point be allowed to fix its extension. We also saw this can be modelled using intutionistic Kripke frames. In particular it will not be definable in terms of membership as in ML, NBG, or MK and so we seem to have as desired a more genuine distinction between sets and proper classes. Indeed, as we also proved in §2.6 there are proper classes that contain other proper classes and so this is something that we welcome.

Notwithstanding the previous remarks, we see that this theory still has limitations that we cannot accept. For instance, consider the class comprehension principle, as we saw in the case of NBG it prevents Russell's paradox by specifying that the elements of this new class will just be sets. But of course, we do not endorse the restriction with regards the classes introduced that these should only be introduced through properties whose instances are sets. Our comprehension principle must not discriminate a priori between properties satisfied by sets and those by classes when it comes to deciding which entities are included in the new class. Indeed, although this principle would guarantee a class of all sets, it would not guarantee a universal class, i.e. a class including all sets and classes, as we require. Related to this point we see that since we cannot rely on this principle of comprehension to introduce classes containing other classes, we cannot prove the fact that some class contains some other class constructibly, by for instance considering the universal class. We must do this by appealing to a contradiction via the definability of sets. So in sum even if this theory allows a class to contain another class it does so in too an uninformative of a way to be considered satisfactory.

There are also serious worries here with the notion of set Ackermann is employing. Indeed, we see that as opposed to the Cantorian notion of set we espouse, for Ackermann the axiom guaranteeing the existence of sets is completely devoid of any flavour of limitation of size. Even leaving this aside, as we noted in §2.6, it seems arbitrary that since for Ackermann formulas with class parameters are ill formed sentences, since recall that since the notion of set is being always sharpened the range of these would not be stable enough, but allows quantification over all of them. Given our adoption of the all in one principle the domain must be a class and so ill formed as well according to him. In any case, his set comprehension schema seems much closer to a reflection principle. This makes sense when we consider his idea of the notion of set in continuous development, and so embodied in the fact that if at a given point some class has only sets as members, in the next step of the hierarchy it will be a itself coextensional with a set. Indeed, if the reader forgives the excursion into ZFC, in V_ω, there is an infinite class of sets, and this ω is a set in $V_{\omega+1}$. It seems an interesting question whether reflection principles fit in with the Cantorian idea of set, lacking a more exhaustive analysis of the issue, it seems however that closer to the spirit of Cantor's ideas would be the notion of transfer principles such as those of Friedman, as seen in §5.2.2. Indeed, reflection principles approach the hierarchy from above.This we see in the proof of the axiom of infinity of A through an appeal to the universe of sets. This would correspond to Cantor's transfinite and it seems at least dubious that appealing to such a, for him, unknowable entity to prove some facts about sets would be felicitous in this framework. Be this as it may, the notion of allowing a comprehension principle not alluding to size to be governing the existence of sets seems to us too close to the notion of class to find it satisfactory. In sum, in this theory even though the classes are closer to our idea of class, they seem rather obscure entities with regards to membership matters. It also seems that the notion of set is too close to that of class.

We now look briefly into Reinhardt's theory S. Note that one can interpret Reinhardt's talk of imagined entities as what we understand as classes. Even if not a technical problem this can already prompt a more ontological protest from us since we do not take classes as less real than sets as this denomination would suggest but rather as equally real albeit different. Remember, incidentally, that Ackermann takes the real subject of set theory to be sets and not classes, indeed for us a satisfactory theory ought to take into consideration both entities and do so seriously. In any case, the S schema characeristic of this theory is a reflection principle similar to the comprehension for sets in A, although stronger since here the objets satisfying the property are not required to be sets like in Ackermann's case, and so our worries doubting this is a principle respecting Cantor's notion of set limited by size are still a negative feature carried into this system. Also note that the class comprehension schema of S is less adequate than that of A since it embodies the idea of limitation of size, note in fact that the universal class of sets cannot be introduced through this principle, and so is added to the language as a constant. Hence it seems that this theory is actually less acceptable than A.

Finally, we also mentioned the stronger theory of Powell, P. This is an interesting theory in the sense that it adds a predication symbol, $\ni$, informing us that a class is predicated of a set. However, the fact that we have noted already in multiple occasions that sets and classes are not well-distingushed entities resurfaces, for here classes are just taken to be subsets of the universe of sets for which the language provides the constant V. Hence, although when talking about sets it is the same to say that a set belongs to a class or the class is predicated of the class. For classes no such axiom is present. Of course, we reject this since we seem to be taking classes as *too* close to properties since even if a class is predicated of the property giving rise to some class we want to say that it actually belongs to the given class not that it is predicated of it, this seems to be some kind of categorical mistake. In short, here sets seem to be taken more seriously than classes in ontological terms since only the former are allowed to belong to classes, this we do not take as a good principle. Other usual points of friction here include extensionality for classes as well as restrictions to the class comprehension of the theory that we do not accept, such as that classes can only appear to the left of predication.

6.3.4 Maddy and Schindler

We now look at the theories of classes of Maddy and Schindler described in §2.8 and §2.7 respectively. First notice that both of these theories satisfy one of our main desideratum regarding classes that we found wanting in many of the theories discussed above. Namely that every open formula in the language of the theory will give rise to a corresponding class. Now, our next question will be to determine how are the paradoxes blocked. Recall that in Schindler's case this is achieved by qualifying that the usual equivalence between satisfaction of a property and belonging to its corresponding class is restricted for those classes that are in the range of significance of the property. In the case of Maddy this is by making the logic of her system paracomplete and so by allowing the membership relation to have gaps. For Maddy's intended model then, whether the russell class is self-membered is neither true nor false and so the paradox does not ensue. In Schindler's case the point is that the Russell class is not in its own range of significance and so the comprehension schema leading to the paradox does not apply to it.

We note how these seem much more felicitous ways to avoid inconsistency than defining the sets in terms of the membership relation and so blocking the paradox by establishing that the Russell class is not a set, as we saw to be the case in some of the theories discussed above. First because we are not imposing restrictions on the membership of classes and perhaps also because, in a sense, it is unsatisfactory that something clear under our conception of set and class as this fact must be demonstrated thorugh a proof by contradiction. We also note that both the approaches of Schindler and Maddy are similar in their dealing with the paradoxical classes. So although to every property there corresponds a class, it is not necessarily the case that it makes sense for every object to be predicated of it and so that it is straightforward to determine its belonging to such a class through comprehension. Insisting in this would be incurring in some kind of categorical mistake. Schindler labels his theory as Gödelian in spirit given the following quote:

> a new idea for the solution of the paradoxes, especially suited to their intensional form. It consists in blaming the paradoxes not on the axiom that every propositional function defines a concept or class, but on the assumption that every concept gives a meaningful proposition, if asserted for any arbitrary object or objects as arguments.(Gödel, 1984, p. 228)

Incidentally, Gödel is referring in the above quote to the theory of types of Russell. Given our concerns with such a framework it is fortunate that both of the theories under consideration carry out the principle in a type-free way. The possibility of this achievement and that it would constitute an improvement on the typed solution is acknowledged subsequently by Gödel himself:

> It is not impossible that the idea of limited ranges of significance could be carried out without the above restrictive principle. It might even turn out that it is possible to assume every concept to be significant everywhere except for certain "singular points" or "limiting points," so that the paradoxes would appear as something analogous to dividing by zero. Such a system would be most satisfactory in the following respect: our logical intuitions would then remain correct up to certain minor corrections, i.e., they could then be considered to give an essentially correct, only somewhat "blurred," picture of the real state of affairs. (Gödel, 1984, p. 229)

Hence although a friend of bivalence might prefer Schindler's approach to Maddy's, and a friend of a the Kripkean apprach to the paradoxes of truth might prefer the opposite route, we see that both accounts solve the problem similarly. We also see that both theories have no problem with allowing other classes to belong to themselves or to others, in fact both take the universal class to belong to itself. Indeed, we take to be as Maddy also notices, a great advantage of her theory that, as we saw, it can prove that the class of infinite collections is self-membered as we have conjectured earlier in this work.

Coming back to Schindler's theory, we also take as positive feature that ranges of significance are closed under connectives. This takes seriously our idea that classes are a logical notion. However, it is not that positive that classes are, provided their ranges of significance coincide, extensionally individuated. Indeed, this is not the case in Maddy's theory in which, as discussed already in §2.7, although sets are coextensionally individuated, equality for classes only follows if the two terms are identical. Hence one might have here the opposite problem, namely that this criterion might be too extreme of a notion. She then introduces the idea of a different notion of identity in which we can express that a class is equivalent to its coextensional set or to classes given by very similar properties. Note also that this relation only holds between classes when these are total, and so Schindler's extensionality axiom would also apply here since these classes share the same range of significance, this is also a nice similarity between both theories.

Both of these theories are theories of classes, and so one should ask about the treatment of sets in these systems. As we saw in §2.8, Maddy suggest introducing a constant for the universe of sets and also for each set, and so we see that her theory is intended as a supplement for an already known such theory such as ZFC. The idea of a supplementary theory is also shared by Schindler's theory since he suggest introducing sets as urelements indicating that these will be in the range of significance of all the classes. Now here we find a problem since we do not take sets as urelements, indeed, this are extensionally individuated collections. However it seems easy to apply here some solution similar to Maddy by stipulating that the membership relations of the sets is determined already by some background theory like ZFC and even by introducing names for such entities, then since all of these will be total, they will indeed be extensionally individuated.

Now, although very philosophically satisfying the main problem of these theories compared to the previous ones examined seems to surface when we turn to their mathematical application, for instance as we saw in §2.8.1 the proposed axiomatisation of Maddy's theory is both unsound and incomplete, and the consistency strength of Schindler's theory is just $\omega + 1$ as seen in §2.7.1.

The issue now of which one of these to prefer over the other will depend largely on the reader's fondness or lack thereof of classical logic or Kripkean hierarchies. However, we would be satisfied with any of these moree than with any of the other proposals surveyed in theis work. These are, albeit not the most technically satisfying theories, the ones that fare best with respect to the desiderata laid out at the beginning of the chapter in their treatment of classes. They also seem able to piggyback on a theory of sets like ZFC which we find to give an acceptable picture of sets in order to deal with these entities. Finally, since we also see that their approach to dealing with the paradoxes is in both cases Gödelian in spirit in the sense described above, the fact of which one fares better could be largely seen as a matter of logical taste.

6.4 Looking backwards and looking forwards

In this last section we point out some topics that have emerged throughout this work about which we would have liked, time had permitted, to learn more about or regarding which we think that interesting work continuing the themes of the present work might be carried out in the future.

In §1 we explored Frege's formal theory as presented in (Frege, 1893) and (Frege, 1950), it would have been of course very interesting to delve more deeply into such a system especially given the importance of Frege's theorem in contemporary philosophy of mathematics, for instances in research programs such as neologicism. It would have also been interesting to look at Frege's own attempt to respond to Russell's paradox in the appendix he added to the second volume of (Frege, 1893), even if ultimately unsuccessful. In this chapter we also looked at possible diagnosis of where does the inconsistency arise, we only looked at this briefly and it would have been interesting to look more into this issue as is for instance discussed in (Boolos, 1986), (Boolos, 1993) or (Dummet, 1991, §17).

In §2 we looked at different theories of collection that try to salvage some of the ideas present in Frege's theory without incurring in contradiction. Of course our biggest regret in this chapter was not to include for reasons of time and space more theories to the discussion, although the possibilities are almost endless at this point, we would have liked to look with some attention to the theory of supersets of Toby Meadows proposed in (Meadows, 2016) and to the theory of properties presented by Linnebo in (Linnebo, 2006) and put them through the lens developed in this work with regards to sets and classes. Next to the desire to include more theories in the survey there is also that of looking in more detail to the ones included, especially those that we usually have less opportunities to encounter while stidying mathematics like tha or Ackermann or Reinhardt, and also that of Powell with its difference between the symbols for membership and predication. We would also have found interesting to look more into the relations that the different theories share with each other, since the comparisons drawn here were usually lim ited to ZFC.

In §3 we presented some considerations that a satisfactory theory of collections must take into account. The focus was in the need to accommodate self-instantiating properties, in order to account for natural language inferences, and metaphysical puzzles. It would seem natural now to come back and asses how well our conception of class accounts for the particular inferences and puzzles considered. It would have also been better of course if we had considered more applications of our notion of class, by looking at different areas of metaphysics or instances of inferences were their application could be helpful. In this chapter we also dealt with unrestricted quantification. We looked at the inadequacy of ST to deal with argument against restricted quantification presented there. It would be important to further look into the replies that a supporter of typed frameworks could offer to our criticism regarding the lack of expressive power of their language. We also defended the All-in-One principle, again it would have been better if more time had permitted dealing further with issues raised by the debate regarding whether the domain of discourse is an object. Similar remarks go to the objections levelled to our broad defense of property reductionism raised by the intensional or even hyperintensional character of properties.

In §4 we offered a historical account of Cantor's notion of set as a well-orderable collection, of course this account was quite brief and so we would have liked explore many features of his framework further. These include more central issues such as the importance of notion of counting and how this relates to ordinals or the tension in his conception of set after the introduction of the power. However are not limited to these since other areas that suggest themselves of interest include Cantor's theological motivations, his views in non-well founded collections or in particular enities like singletons and the emptyset. In this chapter we also dealt with Russell's reaction to Cantor's diagonal argument noting his mistaken suspicion of an error in the proof, it would be of interest to delve into the process that made Russell realise that his assumptions regarding the inconsistency of Cantor's system were misguided. Since we noted the similarity between Russellian classes and Fregean extension a detailed analysis of their differences also seems to be called for. Here we also looked into the differences between sets and classes. We for insance mentioned their different individuating conditions. This seems to be a very interesting issue since after rejecting extensionality and noting the issues regarding hyperintensionality one can ask what is the best way to individuate classes and how given the usual assumption that sets exist necessarily one can account for the intensional flavour of the entities. Suggestions here included looking into impossible worlds, or epistemic readings of possible worlds. Another fascinating topic that deserves attention and was briefly touched upon here is the use of our conception of set and classes in order to assess the justification of mathemaical axioms.

In §5 addressed some philosophical arguments against the use of classes in addition to sets in our theories of collections. We begun by giving instances of the use of in mathematics of second order resources, and so more time would have allowed to incorporate more examples to this small catalogue. Of particular interest seemed the use of such resources in the investigation of large cardinal principles. We also looked into the worry that even if warranted in using sets and classes, since one of the entities is reducible to the other, we would have a more parsimonious ontology by giving eliminating the reducible entities from our picture. Here we focused in, when looking at the reduction of classes to sets at Parsons proposal in (Charles Parsons, 1974), it would be of interest to compare his views there with the position he adopts in (Charles Parsons, 1983) and how this relates to our notion of class. Here we also addressed the worry that a simpler ontology could be obtained by paraphrasing away the use of classes in the discourse as plural quantification over sets. It looks worthy to investigate how the notion of superplurals as discussed in (Wagner, 2015) would have an impact in allowing the pluralist to make sense of our position requiring membership of classes to other classes. It would have also been important to look at other attemps to paraphrase away classes beyond plurals. We could for instance have looked at substitutional interpretations of classes as found for instance in (Quine, 1974) or (Charles Parsons, 1971). Of course we could have also looked at different objections to our use of classes and sets, for instance one might find it very important for a theory to give a uniform treatment of the collections in place and argue that this is lacking in our proposal. The criticism being that although we have a theory of classes and one of sets we do not have a single theory of collections. Indeed, for a physics analogy consider that one might think that having a theory of leptons and one of bosons does not amount to having a theory of particles. And so a theory like ZFC which for all its faults can acommodate this coherence requirement ought to be prefered.

Finally, in §6 we have put together the facts gathered in the preivous chapters regarding sets and classes and assesed how the theories of collections surveyed fullfil these requirements. Of course a more detailed and systematic analysis of each of the theories against the criteria summarised here would have been prefered. Especially, in the case of of those like ZFC and ST that were only superficially treated at this point. Regarding the methodology of our analysis note that we were assesing each theory mostly in individual terms. It would have perhaps been more illustrative in order to gain a more general picture to perform an analysis of a more comparative nature between the theories. Indeed, the usefulness of a comparative analysis seems to be patent when we noticed that our prefered theories of classes, those of Maddy and Schindler seem to adopt a similar apprach to the avoidance of paradox. Of particular interst would be to carry out an study of up to what extent these theories are similar. For instance, are the classes that Maddy's structure does not form a judgmenent regarding their membership of other class, precisely what for Schindler consitute the singularities of such class? And, if not how do these two notions relate? Note, of course, that the most ambitious task ahead of us would be to, instead of assesing how the theories of other authors fare when faced with our criteria as our present work does, produce our own theory of collections explicitly designed with the satisfaction of such criteria in mind.

6.5 Summary

In this final put together all the facts gathered in the preivous chapters of this work regarding sets and classes in order to asses how the theories of collections surveyed earlier fullfil the different requirements we laid out for a theory to be considered satsifactory.

The chapter begun by putting together the different features regarding sets and classes that we have been uncovering throughout this work in §6.2 and which can be broadly summarised as taking a Cantorian combinatorial view of sets and a Russellian logical position on classes. With these criteria in place we turned to the assesment of the theories in §6.3. Here we dealt in greater detail with the theories NF and ML (§6.3.1),NBG and MK (§6.3.2), A, S and P (§6.3.3) as well as those of Maddy and Schindler (§6.3.4). We also made some brief remarks about ZFC and ST recalling their already encountered shortcomings. We concluded that the theories better equipped to accommodate our notions of set and class are those of Maddy and Schindler, since they avoid common pitfalls of other discussed theories such as treating classes in a less serious way than sets, in terms of restrictons to their existence or their ability to be members of other classes, or drawing a difference without a distinction between the two catgories of entities. Finally we closed the chapter with some directions for future work arising from the topics present in this work in §6.4.

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