# Polarization and Echo Chambers: A Logical Analysis of Balance and Triadic Closure in Social Networks

**MSc Thesis** *(Afstudeerscriptie)*

written by

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**Abstract**

Motivated by the danger of polarization and echo chambers in social networks, we develop several logics to analyze the social phenomena balance, triadic closure and homophily. We first expand on a logical framework known from the literature with several intentions in mind. To explore measures of how far a network is from polarization, we consider and compare a variation of distances between models in relation to balance. We introduce additional modalities to the language of positive and negative relations logic to define previously undefinable frame properties in the original work. We also include dynamic operators in this framework to investigate change in networks with respect to polarization. Then we move away from balance and present tied logic: a hybrid logic of strong and weak ties. We provide an axiomatization, prove soundness and strong completeness and relate our results to analyzing echo chambers. Inspired by work on social group formation we define a subclass of threshold frames in which relations are justified on the basis of features agents in a network share. Lastly, we extend tied logic with epistemic states and dynamic and epistemic modalities to examine the interplay between change and knowledge in networks of strong and weak ties.

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[1] A Russian card game of attack and defense.

# Contents

# Chapter 1

# Introduction: Polarization and Echo Chambers

The way in which we receive and exchange information changes rapidly with the advances of new technology in our current world. Simultaneously we are facing world issues that are driving our opinions to the extremes of the political landscape. Two increasingly dangerous social phenomena related to these trends are *group polarization* and *echo chambers*.

Group polarization, or polarization for short, is not a new concept but has adapted well to communication through social media [25]. The phenomenon has been extensively researched by, among others, Cass Sunstein (in e.g. [36] [37]). Polarization describes the tendency for people to develop more extreme views after deliberation within a group. Although issues up for debate often are complex and dependent on a number of factors, an effect of polarization is that fine lines are blurred and that answers to complicated questions are driven into opposing parties of either 'for' or 'against'. This applies to juries in court rooms and participants in political discussions, but can also find its way into mundane everyday social settings.

The reasons for polarization are taken to be a combination of peer pressure and the way information exchange is carried out within group settings. One important aspect of this process is that individuals with a weak inclination towards one opinion are likely be confronted with louder voices expressing a radicalized version of the same opinion. As a result of exposure to new arguments and the desire to be part of a community, unsure agents might leave their insecurities behind and adopt a stronger position.

Another indication of the potential danger of a group of likeminded people is the emergence of echo chambers. Echo chamber is not a formal terminology with an explicit definition, but is widely and vaguely referred to in several contexts. The phrase is often used as a derogatory term pointing to a situation where certain information within a group is contained and repeated inwards, and where challenging opinions are rejected. As an echo reflects sound, an echo chamber reflects similar opinions in a setting closed off from the outside world to a certain degree.

In this thesis, we develop several logics to analyze these social phenomena by working in a formal framework based on theories from social network analysis. In particular, we assess the social concepts of structural balance, triadic closure and homophily through a logical lens to develop a deeper understanding of these social network tendencies and their relation to polarization and echo chambers.

Our initial outset is the logic of positive and negative relations (**PNL**) first seen in the PhD thesis of Zuojun Xiong defended in 2017 [39]. **PNL** is a logic modeling two-sorted Kripke frames as networks where agents can be related positively or negatively, but not both, and is created for the purpose of analyzing balance in social networks. Balance is a local property of groups of three agents equivalent to a global property of the network: that all agents can be divided into two internally positive groups where all relations to members of the other group are negative. This equivalence between local and global balance properties is a known result by Frank Harary called the Balance Theorem [22].

There are a number of questions we seek to answer in this thesis. One concerns the close relationship between balance and polarization. By examining the distance from any model to a balanced model, we aim to interpret how far a social network is from being polarized.

Two essential motivations then leads us to build upon and extend the language and semantics of **PNL**. The first is again related to polarization. We add dynamic operators to **PNL** with the goal of analyzing change in the networks. These operators are inspired by local dynamic modalities known from sabotage modal logic [3] [28] and enables us to review in a stepwise manner how a social setting can change from imbalanced to balanced; unpolarized to polarized, or vice versa.

Not only are we interested in developing **PNL** to study polarization, but also for the sake of investigating the expressive power of the logic itself. A strong motivation to expand the logical framework is to be able to define vital frame properties in this context with corresponding axioms. **PNL** has been axiomatized in the literature, yet a number of important properties are axiomatized as rules, not axioms. By including new elements in the language, we discuss different approaches that enables us to define the formulas we need. Most notably we introduce a dynamic characterization of the balance property.

Balance is, as mentioned, not the only social concept we intend to study in a logical framework. Another prospect of this thesis is to analyze triadic closure in social networks and its connection to echo chambers. Triadic closure is meant to formalize the phenomenon where one is likely to know the friends of one's friends. Based on a socio-psychological background proposed by Mark Granovetter [21], we introduce the novel tied logic (**TL**). Like **PNL**, this logic models social networks as two-sorted frames, although in terms of strong and weak ties instead of positive and negative relations. Furthermore, **TL** is a hybrid logic that lets us name agents in the network. We show that Granovetter's most important result is a validity in **TL** and provide an axiomatization as well as proof of soundness and strong completeness.

The last social notion we consider in this thesis is homophily; the tendency that we are like our friends. Of several reasons behind homophily, our topic of interest is social selection: the idea that we become friends with people who are similar to us. This concept is particularly relevant to reason formally about echo chambers in combination with triadic closure. To implement the idea of social selection into our formal framework we borrow from existing literature on social group formation [34] [35] and define threshold models of **PNL** and **TL**, respectively. The threshold models make up a subclass of models where relations between agents are justified on the basis of properties they have in common with respect to a given threshold.

The final angle from which we will approach social phenomena in this context is to evaluate our formal concepts of strong and weak ties in an epistemic and dynamic setting. By investigating the interplay between knowledge and change in these networks, we gain an extensive picture of how agents might reason in specific social situations. We extend the frames and models of **TL** with epistemic states and add a knowledge and dynamic modalities to the language, thereby defining tied epistemic logic (**TEL**).

## 1.1 Thesis Outline

The structure of the thesis will be as follows. We begin in Chapter 2 by presenting an overview of structural balance theory including the Balance Theorem. The logic of positive and negative relations from the literature is also introduced in this chapter. We show the axiomatization of **PNL** and discuss some details we find unsatisfactory.

Chapter 3 is devoted to measures of distance in terms of balance to analyze how close a network is to polarization. We present three metrics and discuss strengths and weaknesses before using an example for comparison.

In Chapter 4 we present a series of suggestive additions to **PNL** in order to characterize balance, non-overlapping and collective connectedness. We show that this can be done in several ways and discuss benefits and disadvantages of each approach. We then continue by extending **PNL** with local dynamic adding and link changing modalities with the purpose of studying stepwise change in our networks. A summary of all possible additions to **PNL** is given together with an example for intuitive explanation.

In Chapter 5 we present tied logic (**TL**) to reason about strong and weak ties in relation to triadic closure and echo chambers. We outline the socio-psychological framework behind the strength of weak ties theory and present the syntax and semantics of **TL**. We show that we can formalize a well-known claim as a validity in tied logic. Then, we give an axiomatization of **TL** as well as a proof of soundness and strong completeness.

Chapter 6 concerns the implementation of the social selection aspect of homophily into our logical framework. We define a subclass of threshold models for both logic of positive and negative relations and tied logic, before presenting some observations on the latter.

In the final Chapter 7 we extend tied logic to tied epistemic logic (**TEL**) by including knowledge and dynamics in the established system of the previous chapters. We present a new syntax and semantics and reason about axioms we can embrace to restrict the class of tied epistemic frames. Provided is also a discussion on validities, both of frames restricted and unrestricted by thresholds. Finally, we give an example of a tied epistemic threshold model and reason about the truth value of formulas depending on what axioms we choose to adopt.

## 1.2 Contributions to the Field

This thesis is a contribution to the emerging field of social network logic. The main results are listed below.

- We provide a novel definition of line index of imbalance in relation to **PNL**.
- We prove that we can define a dynamic characterization of balance by extending **PNL** with a global modality and global adding modalities.
- We prove that non-overlapping can be defined by adding either the $[D]$ operator, the $\langle + \cap - \rangle$ modality or nominals to **PNL**, and that collective connectedness can be defined by including the global $[A]$ modality.
- We extend **PNL** with dynamic modalities to analyze change in relation to balance and polarization in a dynamic framework.
- We define a new logic **TL** where we show that known properties from social network analysis can be formalized with hybrid formulas. We also give a full axiomatization and prove soundness and strong completeness of **TL** with respect to tied frames.
- We extend **TL** to **TEL** and present validities of this novel logic.
- We implement definitions from social group formation theory in a new setting and define subclasses of threshold models of **PNL**, **TL** and **TEL**.

# Chapter 2

# Preliminaries: Logic of Balance in Social Networks

In this preliminary chapter, we informally explain the social network theory behind the idea of structural balance in signed networks and its close connection to polarization. We then introduce the syntax and semantics of the *logic of positive and negative relations (**PNL**)* which we will extend and use in later chapters. We formally present the Balance Theorem in this framework to give the reader a proper understanding of the concepts we will be working with. The chapter is concluded by a discussion of the axiomatization, soundness and weak completeness of **PNL**.

## 2.1 Structural Balance Theory

The aim of this section is to give the reader an informal introduction to the social concept of balance. We will define properties and theorems in a conceptual and intuitive fashion and leave formal definitions of the same properties to the succeeding section.

### 2.1.1 The enemy of my enemy is my friend

The notion of *structural balance*, often referred to as *balance* for short, is defined in the context of *signed graphs* [16]. A signed graph is a two-sorted undirected graph structure where nodes are related by either positive or negative edges (see Figure 2.1).

As motivation for the concept of structural balance we can interpret signed graphs as social networks [16]. Here, the nodes are viewed as agents where edges represent positive and negative relations these agents have to each other. In this context the graph is restricted such that there cannot be both a positive and a negative edge between two nodes. We can also think of all agents having a positive relation to themselves.

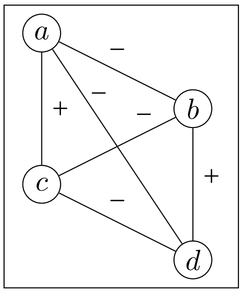


Figure 2.1: Example of a signed graph.

The idea behind structural balance in a signed graph originates from theories in social psychology [24], and it also carries empirical support [27]. Intuitively, a structurally balanced signed graph is depicting a scenario where relations between agents are somewhat stable. Consider the four following signed graphs in Figure 2.2 (example from [16]). The four signed graphs represent all possible combinations of edges for a set of 3 connected nodes. We will examine them all and explain that $(a)$ and $(c)$ are balanced, whereas $(b)$ and $(d)$ are not.

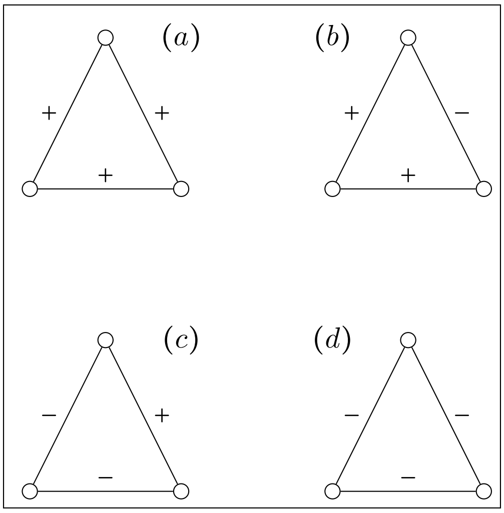


Figure 2.2: Signed graphs $(a)$ and $(c)$ are balanced, $(b)$ and $(d)$ are not.

In graph $(a)$, all nodes have positive relations to each other. No reason to suspect there to be any sudden changes to the structure. This also holds for $(c)$. Here, two of the nodes have a positive relation between them, and they have a common "enemy" in the third node. In graph $(b)$ and $(d)$, the situation is not as stable. In $(b)$ one of the agents is positively related to both of the other agents while they are negatively related to each other. We can imagine a situation where one of the "enemies" tries to pull the "friend" over to their side, or where the mutual "friend" wants the others to reconcile. We also observe that $(d)$ share a similar instability. As all relations between nodes are negative, two of the three agents might have an incentive to gang up on the third.

The structural balance property is a local property based on this relationship between three nodes in a signed graph, but it is extended to hold for a graph of any size. The succeeding definition holds that the graph in question is *complete*. A graph is complete when there are edges connecting every single pair of nodes in the graph. We define structural balance in the following way.

**Definition 1** *(Structural Balance Property [16]) A complete signed graph has the structural balance property iff for every set of three nodes, if we consider the three edges connecting them, either all three of these edges are labeled +, or exactly one of them is labeled +.*

We return to Figure 2.1 and see that it is both complete and structurally balanced.

It is important to already note that looking at relations between humans in this way is a significant over-simplification at the very least. The complete graph assumption does not seem to make much sense as a general rule. It is also rarely such that one can view relations between people as purely negative or positive. Another way to look at the signed edges in a social network is by agreement or disagreement in context of a certain debate or political issue. We will get back to this at a later stage, and to balance in incomplete graphs shortly.

### 2.1.2 The Balance Theorem

Before we begin to define the logical framework we will analyze balance in, we introduce the *Balance Theorem.* The Balance Theorem was proved in 1953 by Frank Harary [22] and claims an equivalence between the structural balance property in Definition 1 and a global property of the graph.

**Theorem 1** *(*Complete* Balance Theorem [16]) A complete signed graph has the structural balance property iff the nodes in the graph can be divided into two sets $X$ and $Y$ where all nodes in $X$ have a positive relation to each other and negative relations to the nodes in $Y$, and all nodes in $Y$ have a positive relation to each other and negative relations to the nodes in $X$.*

To illustrate the Balance Theorem we return again to Figure 2.1 from the beginning of this section, now depicted in Figure 2.3. We see that the theorem indeed holds for this graph. We can divide the nodes into the sets $X = \{a, c\}$ and $Y = \{b, d\}$ such that $a$ and $c$ have a positive relation to each other, while both have a negative relation to the nodes $b$ and $d$ in $Y$, and vice versa. We also observe that for all sets of three agents, the edges relating them correspond to the local balance property of the signed graph.

The connection to polarization is already evident. A balanced graph is a polarized one, where all agents find themselves in a group of peers with negative relations to all agents in another group. It is also suggested in the literature (e.g. [16] [26]) that like polarization, balance is a property that social networks seem to converge towards. We will therefore use the terms balanced and polarized interchangeably when the context disallow any confusion.

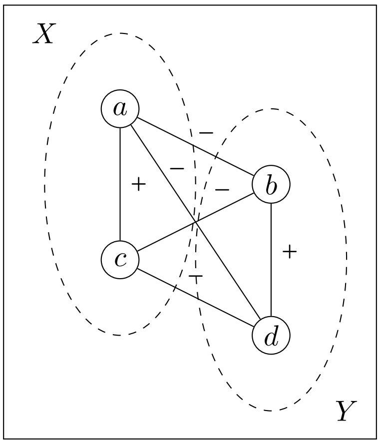


Figure 2.3: Division of nodes into sets $X$ and $Y$.

As mentioned earlier, modeling social relations as complete graphs is only relevant in particular cases. In this thesis we will juggle between these two graph types as both have advantages and disadvantages in terms of formalization through logic and application to human interaction. We therefore introduce a general Balance Theorem. This is the standard we adopt in later sections, as it also implies the complete version.

**Theorem 2** *(*General* Balance Theorem [16]) The nodes in a signed graph can be divided into two sets $X$ and $Y$ where all relations between nodes in $X$ are positive and all relations between nodes in $Y$ are positive, while relations between nodes in $X$ and nodes in $Y$ are negative iff it is possible to fill in edges in such a way that the resulting complete signed graph has the structural balance property.*

This theorem states that any incomplete graph which has the *potential* of being structurally balanced by adding positive and negative edges has the mentioned global property. We turn to Figure 2.4 for an example and recognize the network from the previous Figure 2.1 and Figure 2.3 with several edges excluded. We observe again that the nodes can be divided into sets $X = \{a, c\}$ and $Y = \{b, d\}$ where relations within the sets are positive and the relation across sets between $b$ and $c$ is negative. As we know from the earlier figures, we can "fill in" edges to get a complete signed graph with the structural balance property. Since the two are equivalent, we will refer to a signed graph with the two mentioned balance properties as *balanced*.

What is especially interesting about the Balance Theorem is that it states an equivalence between a *local* property of sets of three nodes and a *global* property of the whole graph; that all nodes can be divided into two groups, where everyone within the group are "friends", while "enemies" towards everyone in the other group. There is, however, another property equivalent to these two: that there are no *simple cycles* with an odd number of negative edges. We will refer to these cycles as *negative cycles*. A simple cycle, often and especially in this thesis just called a *cycle*, is defined in graph theory as a path of nodes and at least three edges, in which the first and last nodes are the same and visited exactly twice [16]. Otherwise all nodes are distinct. We distinguish between simple cycles and *closed walks*, where the latter

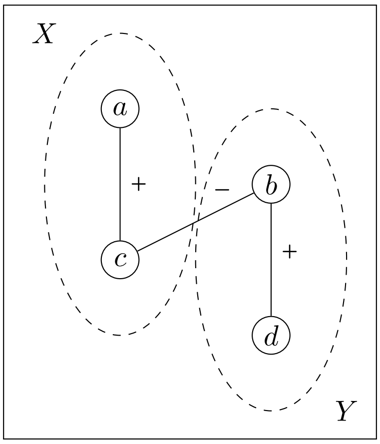


Figure 2.4: A balanced incomplete signed graph.

allows the path to visit the start/end node more than twice [26]. The cycle-version of the Balance Theorem is presented below.

**Theorem 3** *(*Cyclic* Balance Theorem [11] [22]) A signed graph is balanced iff it contains no negative cycles.*

As it will become necessary to explain the formal properties of the logic presented to reason about balance, we also define the concept of *n-balance*.

**Definition 2** *(n-Balance [40]) Let $n \in \mathbb{N}^+$. A signed graph is n-balanced iff it has no simple cycle of length less than or equal to $n$ with an odd number of negative edges.*

The motivation behind the n-balance property is that studies (such as [17]) show that short cycles have more effect upon a person's tension than longer cycles. We are also more likely to observe longer cycles of odd negative length in a social network than short ones. It follows that the n-balance definition yields an alternative version of the cyclic Balance Theorem.

**Theorem 4** *(n-Balance Theorem) A signed graph is balanced iff it is n-balanced for all $n \in \mathbb{N}^+$.*

### 2.1.3 Weak Balance

Balance as we have defined it so far describes a fairly strong graph property. There is also a weaker notion of balance appropriately called *weak balance*, introduced by James A. Davis in 1967 [13]. Weakly balanced graphs are supersets of balanced graphs that disallow only one type of triangle:

**Definition 3** *(Weak Structural Balance [16]) A complete signed graph has the weak structural balance property iff for every set of three nodes, if we consider the three edges connecting them, either all three of these edges are labeled +, all three edges labeled − or exactly one of them is labeled +.*

Davis [13] also proved a similar Balance Theorem for weak balance, although in this case the global property of weak balance characterizes the possibility of dividing nodes into not just two, but any number of sets of "friends". In other words, weakly balanced signed graphs are polarized with respect to a collection of groups, where the relations within each group are positive and all relations between agents in different groups are negative.

Analogous to (strong) balance, a weakly balanced signed graph also has a cycle property: it cannot contain a simple cycle with only *one* negative edge. As before, there is a complete and general version of the weak Balance Theorem, but as the latter implies the former we present only the general theorem.

**Theorem 5** *(Weak Balance Theorem [13]) Let $G$ be a signed graph. The following properties are equivalent:*

1. *It is possible to fill in edges in $G$ in such a way that the resulting complete signed graph has the weak structural balance property.*
2. *The nodes in $G$ can be divided into sets of nodes where all edges between nodes within each set are labeled + and all edges between nodes across different sets are labeled −.*
3. *$G$ has no simple cycles with exactly one negative edge.*

The theorem is exemplified in Figure 2.5. We observe that all three weak balance properties hold in this network. Adding a negative edge between $a$ and $d$ constructs a complete graph with the weak structural balance property. Illustrated are the sets $X = \{a, c\}$, $Y = \{b\}$ and $Z = \{d\}$ where relations between sets are negative and the only relation within a set, between $a$ and $c$, is positive. Moreover, there are no simple cycles with exactly one negative edge. We also note that although this signed graph is weakly balanced, it is not balanced: the three edges between agents $b$, $c$ and $d$ are all negative.

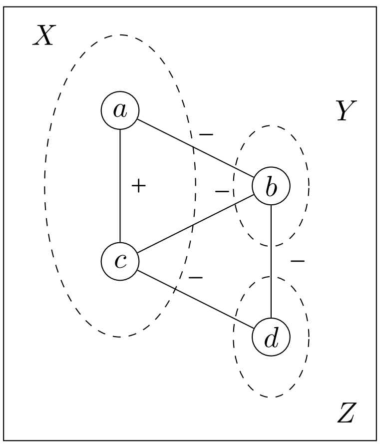


Figure 2.5: A weakly balanced signed graph.

Studies, such as [26], has found strong balance to be too restrictive as a common property for real-world social networks and propose weak balance as a more likely alternative. In this thesis we will keep both definitions as they serve different purposes. A network of football fans might converge to a weakly balanced graph structure where supporters of the same team agree and disagree with supporters from other teams. In the context of particular political issues, like Brexit or anti-vaccination, the same network could converge to a strongly polarized network in camps of 'yes' and 'no'. Depending on social context and research goal both balance definitions are valuable in their own respect.

## 2.2 PNL

An aim of the next chapters is to analyze change in signed networks within a logical framework. To do this we build upon *positive and negative relation logic (**PNL**)* developed in the PhD thesis of Zuojun Xiong defended in 2017 [39] and further studied in [40] in which the latter paper constitutes the basis of our presentation. **PNL** is a fully axiomatized sound and weakly complete logic using two-sorted Kripke frames to account for social networks of positive and negative relations. In this section we introduce **PNL** and formally define the properties related to balance and weak balance in a logical context. Then, we present the axiomatization of **PNL** as well as a brief assessment of the soundness and weak completeness results known from the literature.

### 2.2.1 Syntax and Semantics

We begin by defining the syntax of **PNL**.

**Definition 4** *(Syntax) Let* $\boldsymbol{At}$ *be a countable set of propositional letters. We define the well-formed formulas of the language* $\mathcal{L}_{PNL}$ *to be generated by the following grammar:*

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid \langle + \rangle\phi \mid \langle - \rangle\phi$$

*where* $p \in \boldsymbol{At}$*. We define propositional connectives like* $\vee, \rightarrow$ *and the formulas* $\top, \bot$ *as usual. Further, we define the duals as standard* $\boxplus := \neg \langle + \rangle \neg$ *and* $\boxminus := \neg \langle - \rangle \neg$*.*

To rightfully depict relations $R^+$ and $R^-$ as edges in a signed graph, we want both relations to be symmetric as signed graphs are undirected. We also want $R^+$ to be reflexive and $R^-$ to be irreflexive to account for the reasonable restriction of agents having a positive relation to themselves. Moreover, we want the accessibility relation property that in [40] and [39] is called *non-overlapping*: no two agents can be related by both a positive and negative relation. To reason about complete graphs we also occasionally want to reason about *collectively connected* relations. Formal definitions of the non-overlapping and collective connectedness properties follow.

**Definition 5** *(Non-overlapping and Collective Connectedness) Let $A$ be a set of agents and $R^+$ and $R^-$ be two binary relations on $A$. We define the following properties of $R^+$ and $R^-$:*

- *$R^+$ and $R^-$ are non-overlapping iff $\forall a, b \in A : \neg(aR^+b)$ or $\neg(aR^-b)$.*
- *$R^+$ and $R^-$ are collectively connected iff $\forall a, b \in A : aR^+b$ or $aR^-b$.*

We can now define signed frames and models, and the semantics of **PNL**.

**Definition 6** *(Signed Frame) Let $A$ be a set of agents and $R^+$ and $R^-$ be two symmetric and non-overlapping binary relations on $A$ where $R^+$ is reflexive. We call the tuple $\mathbb{F} = \langle A, R^+, R^- \rangle$ a signed frame.*

For $a, b \in A$ we will denote members of $R^+$ as $(a, b)_+$, members of $R^-$ as $(a, b)_-$.

**Definition 7** *(Signed Model) A signed model $\mathbb{M} = \langle A, R^+, R^-, V \rangle$ is a tuple where $\langle A, R^+, R^- \rangle$ is a signed frame and $V : \boldsymbol{At} \to \wp(A)$ is a valuation function.*

**Definition 8** *(Semantics) Let $\mathbb{M}$ be a signed model and $a$ be an agent in $A$. We define the truth conditions for **PNL** as follows:*

$$\begin{aligned}
&\mathbb{M}, a \Vdash p \textit{ iff } a \in V(p)\\
&\mathbb{M}, a \Vdash \neg\phi \textit{ iff } \mathbb{M}, a \nVdash \phi\\
&\mathbb{M}, a \Vdash \phi \wedge \psi \textit{ iff } \mathbb{M}, a \Vdash \phi \textit{ and } \mathbb{M}, a \Vdash \psi\\
&\mathbb{M}, a \Vdash \oplus\phi \textit{ iff } \exists b \in A \textit{ such that } aR^+b \textit{ and } \mathbb{M}, b \Vdash \phi\\
&\mathbb{M}, a \Vdash \ominus\phi \textit{ iff } \exists b \in A \textit{ such that } aR^-b \textit{ and } \mathbb{M}, b \Vdash \phi
\end{aligned}$$

Intuitively, we read $\oplus\phi$ to hold at an agent if and only if the current agent is positively related to an agent where $\phi$ is true. Similarly, we read $\ominus\phi$ to be forced at an agent if and only if the current agent is related negatively to an agent where $\phi$ holds.

### 2.2.2 The Balance Theorem

We proceed to define three definitions of balance, which we will call *local balance*, *global balance* and *cyclic balance*. Local balance is capturing the structural balance property in Definition 1, now applied to signed frames. Global balance is the global property of being able to divide all agents into two groups of "friends" presented in the Balance Theorems 1 and 2. Cyclic balance is the property of a network having no simple cycles with an odd number of negative edges depicted in the cyclic Balance Theorem 3.

**Definition 9** *(Local Balance) A signed frame $\mathbb{F} = \langle A, R^+, R^- \rangle$ has the local balance property iff $\forall a, b, c \in A$ :*

- *if $aR^+b$ and $bR^+c$, or $aR^-b$ and $bR^-c$, then $aR^+c$, and*
- *if $aR^+b$ and $bR^-c$, or $aR^-b$ and $bR^+c$, then $aR^-c$.*

**Definition 10** *(Global Balance) A signed frame* $\mathbb{F} = \langle A, R^{+}, R^{-}\rangle$ *has the global balance property iff* $\exists S \subseteq A$ *such that* $\forall a, b \in A$ :

- *if* $aR^{+}b$*, then* $a, b \in S$ *or* $a, b \in A \setminus S$*, and*
- *if* $aR^{-}b$*, then* $a \in S$ *and* $b \in A \setminus S$*, or* $a \in A \setminus S$ *and* $b \in A$*.*

**Definition 11** *(Cyclic Balance) A signed frame* $\mathbb{F} = \langle A, R^{+}, R^{-}\rangle$ *has the cyclic balance property iff* $\forall a_1, \dots, a_r \in A$*: if* $a_1 R^{x_1} \dots R^{x_{i-1}} a_m R^{x_i} a_1$ *for* $x_n \in \{+, -\}$*, then* $|\{(a_s, a_t)_{-} \mid 1 \leq s < t \leq m\}| = 2n$ *for* $n \in \mathbb{N}_0$*.* [1]

For any signed frame $\mathbb{F}$ with balance properties, we will call it locally balanced, globally balanced or cyclic balanced if we want to make it explicit that we are addressing a specific balance property. However, we will mostly just call the signed frame balanced. Further, for any valuation $V$ on a balanced frame, we will also call the signed model $\mathbb{M} = \langle \mathbb{F}, V\rangle$ balanced. We go on to state the Balance Theorem again, in more formal terms now than in the previous section.

**Theorem 6** *(The Balance Theorem) Let* $\mathbb{F} = \langle A, R^{+}, R^{-}\rangle$ *be a signed frame. The following properties are equivalent:*

1. *There exists a collectively connected signed frame* $\mathbb{F}' = \langle A', R^{+'}, R^{-'}\rangle$ *such that* $A = A'$*,* $R^{+} \subseteq R^{+'}$ *and* $R^{-} \subseteq R^{-'}$ *that has the local balance property.*
2. $\mathbb{F}$ *has the global balance property.*
3. $\mathbb{F}$ *has the cyclic balance property.*

*Proof.* See [39], or [11] for the original proof. Additionally, parts of the proof are included in Appendix A.1.1.

**Corollary 1** *A locally balanced signed frame* $\mathbb{F} = \langle A, R^{+}, R^{-}\rangle$ *is balanced.*

When a signed frame $\mathbb{F}$ is locally balanced *and* collectively connected, Corollary 1 is the Balance Theorem. A frame can however have the local balance property while not being collectively connected. It might have disconnected elements, i.e. agents in $A$ not related by any relation to any other agent. Similarly, it might consist of several smaller frames that individually are collectively connected, but not related to each other. Yet, this locally balanced signed frame will still be generally balanced. By the Balance Theorem, every disconnected component of the frame will have the cyclic balance property. Having them together in the same frame will not affect this cyclicness. Hence, the corollary follows directly from the Balance Theorem.

### 2.2.3 The Weak Balance Theorem

We turn to the notion of weak balance and define properties accordingly.

[1] We denote $\mathbb{N} \cup \{0\}$ as $\mathbb{N}_0$.

**Definition 12** *(Weak Local Balance) A signed frame* $\mathbb{F} = \langle A, R^+, R^- \rangle$ *has the weak local balance property iff* $\forall a, b, c \in A$ :

- *if* $aR^+b$ *and* $bR^+c$*, then* $aR^+c$*, and*

- *if* $aR^+b$ *and* $bR^-c$*, or* $aR^-b$ *and* $bR^+c$*, then* $aR^-c$*.*

**Definition 13** *(Weak Global Balance) A signed frame* $\mathbb{F} = \langle A, R^+, R^- \rangle$ *has the weak global balance property iff* $\exists S_1, \dots, S_n \subseteq A$ *such that* $\forall a, b \in A$ :

- *if* $aR^+b$*, then* $a, b \in S_m$ *for* $1 \leq m \leq n$*, and*

- *if* $aR^-b$*, then* $a \in S_s$ *and* $b \in S_t$ *for* $1 \leq s < t \leq n$ *and* $s \neq t$*.*

**Definition 14** *(Weak-Cyclic Balance) A signed frame* $\mathbb{F} = \langle A, R^+, R^- \rangle$ *has the weak-cycled balance property iff* $\forall a_1, \dots, a_r \in A$*: if* $a_1 R^{x_1} \dots R^{x_{i-1}} a_m R^{x_i} a_1$ *for* $x_n \in \{+, -\}$*, then* $|\{(a_s, a_t)_- \mid 1 \leq s < t \leq m\}| \neq 1$*.*

Note that all classical, or strong, balance definitions directly entail the corresponding weak balance properties. We now present the weak Balance Theorem.

**Theorem 7** *(Weak Balance Theorem) Let* $\mathbb{F} = \langle A, R^+, R^- \rangle$ *be a signed frame. The following properties are equivalent:*

1. *There exists a collectively connected signed frame* $\mathbb{F}' = \langle A', R^{+'}, R^{-'} \rangle$ *such that* $A = A'$*,* $R^+ \subseteq R^{+'}$ *and* $R^- \subseteq R^{-'}$ *that has the weak local balance property.*

2. $\mathbb{F}$ *has the weak global balance property.*

3. $\mathbb{F}$ *has the weak-cyclic balance property.*

*Proof.* See [13].

By a similar reasoning as in the case of balance in the last section, we have a corresponding weak version of Corollary 1.

**Corollary 2** *A weak locally balanced signed frame* $\mathbb{F} = \langle A, R^+, R^- \rangle$ *is weakly balanced.*

### 2.2.4 Axiomatization, Soundness and Weak Completeness

In [40] an axiomatic system called $\mathbf{pnl}_n$ over the language $\mathcal{L}_{PNL}$ is introduced for a given number $n \in \mathbb{N}^+$.[2] Included in the axiomatization as the only component dependent on $n$ is the rule $\mathbf{Nb}_n$. To define this rule, we first need to get acquainted with the following definitions.

[2] We denote $\mathbb{N} \setminus \{0\}$ as $\mathbb{N}^+$.

**Definition 15** *[40] Let $x, y, z \in \mathbb{N}$ and $\phi \in \mathcal{L}_{PNL}$. We define the following:*

- $(\boxplus;\boxminus)^{x,y}_{\phi}$ *is the set of all formulas that are obtained by prefixing $\phi$ with a sequence of $x$ positive ($\boxplus$) and $y$ negative ($\boxminus$) box modalities in some order;*
- $\bigwedge(\boxplus;\boxminus)^{x,y}_{\phi}$ *is the conjunction of all elements in the set $(\boxplus;\boxminus)^{x,y}_{\phi}$;*
- $\bigwedge\limits_{n}(\boxplus;\boxminus)^{O}_{\phi}$ *is the conjunction of all $\bigwedge(\boxplus;\boxminus)^{x,o}_{\phi}$ such that $x + o = n$ and $o$ is an odd number;*
- $\bigwedge\limits_{n}(\boxplus;\boxminus)^{E}_{\phi}$ *is the conjunction of all $\bigwedge(\boxplus;\boxminus)^{x,e}_{\phi}$ such that $x + e = n$ and $e$ is an even number.*

There is no formula corresponding to the property "there are no negative cycles of length $n$ at the current agent" [40], although there are formulas sufficient but not necessary for this property to hold. We define the class of formulas $name_n$. Note that this class of formulas also is sufficient for the non-overlapping property to hold.

**Definition 16** *($name_n(\phi,\psi)$ [40]) Let $n \in \mathbb{N}^+$. For any $\phi, \psi \in \mathcal{L}_{PNL}$, we define:*

$$name_n(\phi,\psi) = \boxplus(\phi \wedge \neg\psi) \wedge \bigwedge_{n-1}(\boxplus;\boxminus)^{O}_{\neg\phi\vee\psi} \wedge \boxminus(\neg\phi \wedge \psi) \wedge \bigwedge_{n-1}(\boxplus;\boxminus)^{E}_{\phi\vee\neg\psi}.$$

**Lemma 1** *[40] Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame, $V$ a valuation on $\mathbb{F}$, $a \in A$, $n \in \mathbb{N}^+$ and $\phi, \psi \in \mathcal{L}_{PNL}$. If $\langle \mathbb{F}, V \rangle, a \Vdash name_n(\phi,\psi)$, then $\mathbb{F}$ has no negative cycle of length $n$ starting in $a$.*

*Proof.* See [40].

We now have enough background to present the rule $\mathbf{Nb}_n$ and the subsequent proposition.

**Definition 17** *($\mathbf{Nb}_n$ [40]) Let $n \in \mathbb{N}^+$ and let $P(\phi)$ denote the set of propositional atoms in $\phi$ for any $\phi \in \mathcal{L}_{PNL}$. We define the rule $\mathbf{Nb}_n$ as follows:*

$$\vdash name_n(p,q) \to \chi \text{ implies } \vdash \chi, \text{ where } p, q \notin P(\chi) \text{ and } p \neq q.$$

**Proposition 1** *$\mathbf{Nb}_n$ preserves validity with respect to the class of $n$-balanced signed frames.*

*Proof.* See [40].

We summarize the axioms of $\mathbf{pnl}_n$ given by [40] in Table 2.1.

Table 2.1: Axiomatization $\mathbf{pnl}_n$ where $p, q \in \mathbf{At}$, $\Diamond \in \{\langle + \rangle, \langle - \rangle\}$ and $\Box \in \{\boxplus, \boxminus\}$.

| | | |
|---|---|---|
| $(PC)$ | all substitution instances of propositional tautologies | |
| $(T^+)$ | $\vdash p \to \langle + \rangle p$ | (Positive-reflexivity) |
| $(B^\pm)$ | $\vdash p \to (\boxplus \langle + \rangle p \wedge \boxminus \langle - \rangle p)$ | (Symmetry) |
| $(Dual)$ | $\vdash \Box p \leftrightarrow \neg \Diamond \neg p$ | (Duality) |
| $(K_s)$ | $\vdash \Box(p \to q) \to (\Box p \to \Box q)$ | (Signed K-axiom) |
| (**MP**) | $\vdash \phi \to \psi \;\&\; \vdash \phi \Rightarrow\; \vdash \psi$ | (Modus Ponens) |
| (**Nec**) | $\vdash \phi \Rightarrow\; \vdash \Box \phi$ | (Signed Necessitation) |
| (**Us**) | $\vdash \phi \Rightarrow\; \vdash \phi(\psi_1/p_1, \ldots, \psi_n/p_n)$ | (Universal Substitution) |
| ($\mathbf{Nb}_n$) | $\vdash name_n(p,q) \to \chi \Rightarrow\; \vdash \chi$, where $p, q \notin P(\chi)$ and $p \neq q$ | ($n$-balanced) |

With these axioms, [40] prove soundness and weak completeness of $\mathbf{pnl}_n$ with respect to the class of $n$-balanced models.

**Theorem 8** *[40] For any $n \in \mathbb{N}^+$, $\boldsymbol{pnl}_n$ is sound and weakly complete with respect to the class of $n$-balanced models.*

We make some important observations of the work on **PNL** in addition to the satisfactory results we have been supplied with for using the logic as a tool to analyze social networks. Firstly, there is no formula in $\mathcal{L}_{PNL}$ that defines the frame property of being balanced for all signed frames; as we have seen, balance is axiomatized as a rule, and not an axiom. Secondly, the non-overlapping property is as mentioned also axiomatized in the $\mathbf{Nb}_n$ rule. Moreover, it has been proved that non-overlapping is not modally definable in **PNL**.

### 2.2.5 Collectively Connected

Recall that collective connectedness is the property of a signed frame $\mathbb{F} = \langle A, R^+, R^- \rangle$ if and only if $\forall a, b \in A : aR^+b$ or $aR^-b$. Although there can be no formula defining the balance property for all signed frames, there is a formula defining balance given that the signed frame is collectively connected. This formula is in [39] named **4B**.

$$((\langle + \rangle \langle + \rangle p \vee \langle - \rangle \langle - \rangle p) \to \langle + \rangle p) \wedge ((\langle + \rangle \langle - \rangle p \vee \langle - \rangle \langle + \rangle p) \to \langle - \rangle p) \tag{4B}$$

**Lemma 2** *For any signed frame $\mathbb{F}$, $\mathbb{F} \Vdash$ **4B** iff $\mathbb{F}$ has the local balance property.*

*Proof.* See [40].

Collective connectedness, like non-overlapping, has in [39] been proved modally undefinable in **PNL**. These properties and formulas that can define them will be addressed later in this thesis.

## 2.3 Summary

In this chapter we presented what constitutes as preliminaries of this thesis in two parts: structural balance theory and introduction of the logic **PNL**. The concept of structural balance in graph theory is motivated by conflict-stable situations of relations between agents in a social network. The most notable result in this context is the Balance Theorem which states the equivalence between local and global properties of a network. To reason about graphs with these properties, the logic of positive and negative relations, **PNL**, has been developed in recent literature. This is a sound and weakly complete logic that is fully axiomatizable. We noted that the balance property is axiomatized as a rule, not an axiom and that the properties of non-overlapping and collective connectedness are not modally definable in **PNL**.

# Chapter 3

# Distance

The goal of this chapter is to investigate networks changing from imbalanced to balanced, and in particular analyze how far a network is to being polarized. To do this, we begin by assessing different properties a measure of distance might have and discuss the usefulness of the criteria for our purpose. Then we introduce several measures of distance from a balanced model found in the literature, but accommodated to **PNL**. We review gains and losses of each metric and compare the measures in an example.

## 3.1 Distance Properties

There is already a body of literature on distance. Some covers a measure of distance between judgment sets for the purpose of judgment aggregation [33], others between models for use in belief revision [10] or for modeling implementation of social laws in multi-agent systems [1]. Although we will build upon important features from existing measurements, we need to adapt the methods of distance in terms of balance.

In the literature (e.g. [1], [10]), distance between two "ordinary" Kripke models[1] is defined as a mapping from an ordered pair of two models to a real number. This mapping usually has to satisfy certain properties:

**Definition 18** *(Distance [10]) Let $\mathcal{K}$ be the class of Kripke models. The distance between two Kripke models $\mathbb{K}, \mathbb{K}' \in \mathcal{K}$ is a mapping $d : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ which satisfies the following properties:*
***[indistinguishability]*** $d(\mathbb{K}, \mathbb{K}') = 0$ *iff* $\mathbb{K} \sim \mathbb{K}'$
***[symmetry]*** $d(\mathbb{K}, \mathbb{K}') = d(\mathbb{K}', \mathbb{K})$
***[subadditivity]*** $d(\mathbb{K}, \mathbb{K}'') \leq d(\mathbb{K}, \mathbb{K}') + d(\mathbb{K}', \mathbb{K}'')$
***[nonnegativity]*** $d(\mathbb{K}, \mathbb{K}') \geq 0$,
*where $\sim \subseteq \mathcal{K} \times \mathcal{K}$ is an indistinguishability relation.*

[1]We assume the reader will be familiar with Kripke semantics for modal logic. For clarification, see [9].

These properties for distance are relatively non-controversial. Yet, we need to modify these properties to accommodate for what we will call *balanced distance*. The core feature of balanced distance should be that it tells us how far a signed model is from being balanced. For this purpose, we will first and foremost consider single signed models and measure how far they are from the closest balanced model. Therefore, we might as well define balanced distance as a mapping from *one* signed model to a real number. This has the following consequence for the classical distance properties. The indistinguishability, symmetry and subadditivity constraints are not necessary in this context. Instead we need another constraint that we call *balance indistinguishability*. This property tells us that a signed model has balanced distance 0 if and only if the model is balanced. We summarize the characteristics of a balanced distance in the following definition.

**Definition 19** *(Balanced Distance) Let $\mathcal{M}$ be the class of signed models. The balanced distance of a signed model $\mathbb{M} \in \mathcal{M}$ is a mapping $d : \mathcal{M} \to \mathbb{R}$ which satisfies the following properties:*

***[nonnegativity]*** $d(\mathbb{M}) \geq 0$,

***[balance indistinguishability]*** $d(\mathbb{M}) = 0$ *iff* $\mathbb{M}$ *is balanced.*

In addition to these standard properties of balanced distance, there are other restrictions we can impose on a metric of balanced distance depending on our motivation and purpose. The first one is *long cycle discrimination*. As mentioned, there are studies showing that longer cycles have less effect on people's tension than shorter cycles [17]. Moreover, the number of cycles in a network of a given length generally increases with length [26]. A count of cycles in a network would therefore be dominated by long cycles. This might motivate the need for a metric that downplays the role of longer cycles in the calculation.

It is evident that transition from imbalance to balance is closely related to the number of negative cycles. The relationship between balance and negative cycles can however turn out less straightforward than one might think. By simply counting the number of negative cycles, we do not distinguish between cases where the cycles overlap and cases where they do not. Imagine a network containing only two overlapping negative cycles. There is only need of a single link change for the network to become balanced. In a network of the same two negative cycles, however in this case not overlapped, we require two link changes for the purpose of a balanced network. Counting the number of bad cycles determines the same balanced distance between these two networks. This problem might provoke the need for an *overlapping cycle discrimination*.

Lastly, note that for all measures of balanced distance there is a corresponding weakly balanced version. As balance always entails weak balance, balanced and weakly balanced distance measures might, but not necessarily, output the same number. With all possible properties in mind, we turn to examine some options for a concrete notion of balanced distance.

## 3.2 Counting Cycles

By the Balance Theorem, imbalance is directly related to negative cycles. This observation was applied to balanced distance already in a paper by Cartwright and Harary in 1956 [11] and realized as *degree of balance*. Recall that we, as is often custom, will refer to simple cycles as just cycles. Degree of balance in its original form is the number of balanced cycles, that is, cycles that are not negative, divided on the number of cycles. In our context, this would make a balanced model have degree 1 of balance. As we have defined a measure of balanced distance to be 0 when the model is balanced, we define a variation of degree of balance to accommodate to our needs. This variation simply subtracts the degree of balance from 1. Therefore we appropriately rename this variation *degree of imbalance*. We define it concretely and also consider the weak version in the following definition.

**Definition 20** *(Degree of Imbalance) Let* $c^+(\mathbb{M})$ *denote the number of cycles in* $\mathbb{M}$ *that are not negative, and* $c(\mathbb{M})$ *denote the total number of cycles in* $\mathbb{M}$*. Denote* $c^{+W}(\mathbb{M})$ *the number of cycles in* $\mathbb{M}$ *that do not have exactly one single negative edge. Note that* $c^+(\mathbb{M}) \subseteq c^{+W}(\mathbb{M}) \subseteq c(\mathbb{M})$*.*

*Let* $\mathcal{M}$ *be the class of signed models and let* $\mathbb{M} = \langle A, R^+, R^-, V\rangle \in \mathcal{M}$*. The degree of imbalance of* $\mathbb{M}$ *is a map* $d_{DB} : \mathcal{M} \to \mathbb{R}$ *such that* $d_{DB}(\mathbb{M}) = 1 - \frac{c^+(\mathbb{M})}{c(\mathbb{M})}$*. The degree of weak imbalance of* $\mathbb{M}$ *is a map* $d_{DBW} : \mathcal{M} \to \mathbb{R}$ *such that* $d_{DBW}(\mathbb{M}) = 1 - \frac{c^{+W}(\mathbb{M})}{c(\mathbb{M})}$*.*

We observe that although this simple measure of distance is a balanced distance metric by Definition 19, it does not satisfy neither the long cycle nor the overlapping cycle discrimination property. [26] defines another cycle counting measure of balance that is motivated by long cycle discrimination, called *level of imbalance*.

**Definition 21** *(Level of Imbalance) Let* $\mathcal{M}$ *be the class of signed models and let* $\mathbb{M} = \langle A, R^+, R^-, V\rangle \in \mathcal{M}$*. The level of imbalance of* $\mathbb{M}$ *is a map* $d_{Bz} : \mathcal{M} \to \mathbb{R}$ *such that* $d_{Bz}(\mathbb{M}) = \sum_{k=1}^{\infty} \dfrac{I_k}{z^k}$ *where* $I_k$ *is the number of negative cycles of length* $k$ *and* $z > 1$ *is a free parameter. The weak level of imbalance of* $\mathbb{M}$ *is a map* $d_{BzW} : \mathcal{M} \to \mathbb{R}$ *such that* $d_{BzW}(\mathbb{M}) = \sum_{k=1}^{\infty} \dfrac{I_{Wk}}{z^k}$ *where* $I_{Wk}$ *is the number of cycles with a single negative edge of length* $k$*.*

The level of imbalance satisfies the long cycle discrimination property in addition to being a measure of balanced distance. The measure divides the number of negative cycles by a free parameter that increases by the negative cycle's length. Like degree of imbalance, this metric does not satisfy the overlapping cycle discrimination property. We turn to the last measure of distance in this chapter called *line index of imbalance* for a balanced distance that discriminates overlapping cycles.

## 3.3 Line Index of Imbalance

Line index of imbalance is a measure of distance in terms of balance proposed by Harary in 1959 [23]. The idea is simple: the line index of imbalance of a network measures the minimal number of edges deleted for the network to be balanced. The measure has also been implemented in terms of weak balance by [15], and discussed further in [38].

The transition from a signed model to a submodel of fewer edges can seem unintuitive when we imagine the links between agents to model positive and negative relations. Where it is easy to imagine relations in a network to be created, it might be slightly harder to think of situations where agents completely lose touch. We can still of course regard line index of imbalance as a fruitful measurement, although we also remark that the minimal number of edges deleted is the same number as the smallest number of edges changing signs in order to make the network balanced. The reasoning is as follows. By the general Balance Theorem 2, we have that a network is balanced if and only if it has the potential to have the local structural balance property for each set of three agents. That is, as long as it is possible to fill in missing edges such as to create a collectively connected model where all triangles have either three positive signs or one positive and two negative, the signed model is balanced. Thus, changing signs in an imbalanced network have the same purpose as deleting edges in terms of balance: each edge needed to change signs could be deleted and now have the potential of the desired sign.

We present a novel definition of line index of imbalance suitable for the signed models of **PNL**. For this purpose, we turn back to the literature on distance between Kripke models for inspiration. [1] and [10] introduce *Kripke distance* as a measurement mapping that judges distance between two Kripke models by the number of elements in the relations in which they are dissimilar.

**Definition 22** *(Kripke Distance) Let* $\mathbb{M} = \langle A, R^+, R^-, V\rangle$ *and* $\mathbb{M}' = \langle A', R^{+'}, R^{-'}, V'\rangle$ *be signed models. We define the Kripke distance between* $\mathbb{M}$ *and* $\mathbb{M}'$ *as* $\delta(\mathbb{M}, \mathbb{M}') = \sum_{i\in\{+,-\}} |R^i / R^{i'}|$.

There are several versions of Kripke distance, especially presented in [1], although none of them specified according to our requirements. We briefly list the properties in which we need to modify this metric to accommodate. Firstly, a measurement of balanced distance is a map taking one signed model. Secondly, Kripke distance is introduced in the literature as a measure between two Kripke models where one is a subset of the other. It is not necessarily so in our case. Thus for all signed models $\mathbb{M} = \langle A, R^+, R^-, V\rangle$ and $\mathbb{M}' = \langle A', R^{+'}, R^{-'}, V'\rangle$, for $i \in \{+,-\}$ if $R^i \subseteq R^{i'}$, then $d(\mathbb{M}, \mathbb{M}') = 0$. In this case we would still want a number of difference between the elements in $R^i$ and $R^{i'}$.

Thirdly, we have two sets of relations that are both symmetric. Fourthly, for comparison purposes we want to normalize the metric. Fifthly, we want a measure of distance that judges no distance between two balanced models. Sixthly, but related to the fifth, there are many combinations of relation sets on the nodes that constitute a balanced signed model. We need a measurement that can give us the distance from any imbalanced model to the closest balanced model.

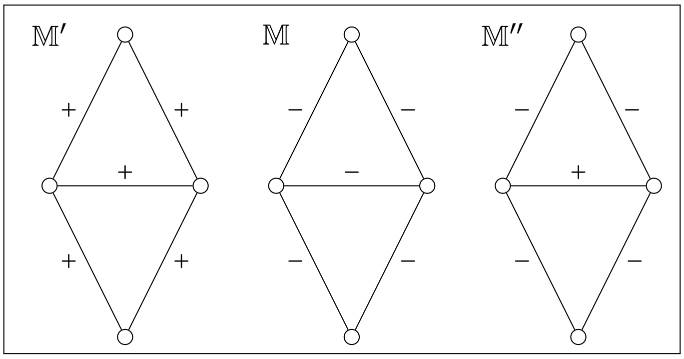


Figure 3.1: $\mathbb{M}'$ and $\mathbb{M}''$ are two alternatives for balance by relation change in $\mathbb{M}$.

As an example, all imbalanced models can turn balanced by changing all the negative relations to positive relations. It does not mean that this is the fastest way towards balance. Consider the signed models in Figure 3.1 (for simplicity, we have omitted the reflexive positive relations). We observe that $\mathbb{M}$ is imbalanced. $\mathbb{M}'$ and $\mathbb{M}''$ are only two of several ways of altering the relations on $\mathbb{M}$ for balance, especially when we also consider the possibility of adding and deleting relations. What we see here is nevertheless two equally balanced models with a different Kripke distance from $\mathbb{M}$. We observe that the Kripke distance from $\mathbb{M}$ to $\mathbb{M}'$ is 10 whereas the distance from $\mathbb{M}$ to $\mathbb{M}''$ is 2.

**Definition 23** *(Line Index of Imbalance) Let $\mathcal{M}$ be the class of signed models and let $\mathbb{M} = \{A, R^+, R^-, V\} \in \mathcal{M}$. The line index of imbalance of $\mathbb{M}$ is a map $d_{LI} : \mathcal{M} \to \mathbb{R}$ such that $d_{LI}(\mathbb{M}) = min\{\sum_{i\in\{+,-\}} |\frac{|R^i| - |R^{i'}|}{2(|R^+ \cup R^-| - |A|)}| \mid \mathbb{M}' = \langle A', R^{+'}, R^{-'}, V'\rangle$ where $A' = A$ and $\mathbb{M}'$ is balanced$\}$.*
*The line index of weak imbalance of $\mathbb{M}$ is a map $d_{LIW} : \mathcal{M} \to \mathbb{R}$ such that $d_{LIW}(\mathbb{M}) = min\{\sum_{i\in\{+,-\}} |\frac{|R^i| - |R^{i'}|}{2(|R^+ \cup R^-| - |A|)}| \mid \mathbb{M}' = \langle A', R^{+'}, R^{-'}, V'\rangle$ where $A' = A$ and $\mathbb{M}'$ is weakly balanced$\}$.*

As an example of line index of imbalance in action, consider again the signed models in Figure 3.1. We decide the line index of imbalance $d_{LI}(\mathbb{M})$. We first calculate with respect to $\mathbb{M}'$:

$$\begin{aligned}\sum_{i\in\{+,-\}} |\frac{|R^i| - |R^{i'}|}{2(|R^+ \cup R^-| - |A|)}| &= |\frac{|R^+| - |R^{+'}|}{2(|R^+ \cup R^-| - |A|)}| + |\frac{|R^-| - |R^{-'}|}{2(|R^+ \cup R^-| - |A|)}| \\ &= |\frac{4 - 14}{2(14 - 4)}| + |\frac{10 - 0}{2(14 - 4)}| = \frac{20}{20} = 1.\end{aligned}$$

Then, similarly with respect to $\mathbb{M}''$:

$$\sum_{i \in \{+,-\}} |\frac{|R^i| - |R^{i''}|}{2(|R^+ \cup R^-| - |A|)}| = |\frac{|R^+| - |R^{+''}|}{2(|R^+ \cup R^-| - |A|)}| + |\frac{|R^-| - |R^{-''}|}{2(|R^+ \cup R^-| - |A|)}|$$
$$= |\frac{4-6}{2(14-4)}| + |\frac{10-8}{2(14-4)}| = \frac{4}{20} = 0.2.$$

It becomes clear that $d_{LI}(\mathbb{M}) = min\{1, 0.2\} = 0.2$.

Note that there are of course other balanced signed models than $\mathbb{M}'$ and $\mathbb{M}''$ with the same number of agents as $\mathbb{M}$. However, $\mathbb{M}''$ already guarantees $d_{LI}(\mathbb{M}) = 0.2$. This number is the smallest possible line index of imbalance of $\mathbb{M}$ as it is imbalanced and $0.2 = \frac{1}{2(|R^+ \cup R^-| - |A|)}$.

Line index of imbalance satisfies the properties to be a balanced distance. It does not discriminate long cycles; in a network with both shorter and longer negative cycles, line index of imbalance will output a number independent on the ratio between short and long negative cycles. As mentioned, line index of imbalance satisfies the overlapping cycle property. In networks where cycles overlap, this metric will not count twice any edges needed to be changed for the purpose of balance.

## 3.4 Comparing Measurements: How Far From Polarization?

We will now look at an example to compare the different measurements we have considered in this chapter. How far the network is from being polarized or weakly polarized is decided with respect to the metric one chooses to adopt. Consider the network in Figure 3.2. Positive reflexive arrows are omitted for simplicity.

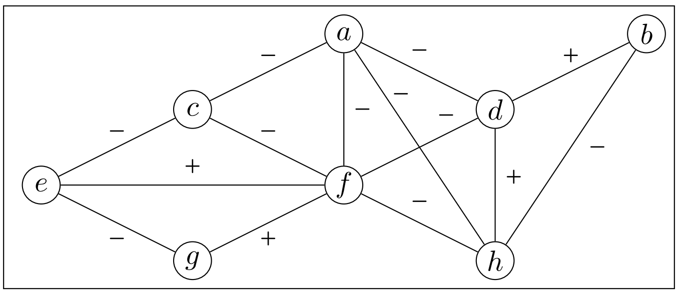


Figure 3.2: A network that is not yet polarized.

Call the signed model depicted in the figure $\mathbb{M}$. We calculate and compare the distance towards polarization in Table 3.1.

Table 3.1: How far is $\mathbb{M}$ in Figure 3.2 from being polarized?.

| Strong Polarization | Weak Polarization |
|---|---|
| Degree of Imbalance | |
| $d_{DB}(\mathbb{M}) = 1 - \dfrac{c^{+}(\mathbb{M})}{c(\mathbb{M})} = \frac{15}{27} \approx 0.556$ | $d_{DBW}(\mathbb{M}) = 1 - \dfrac{c^{+W}(\mathbb{M})}{c(\mathbb{M})} = \frac{2}{27} \approx 0.074$ |
| Level of Imbalance | |
| $d_{Bz}(\mathbb{M}) = \sum_{k=1}^{\infty} \dfrac{I_k}{z^k} = \frac{183}{729} \approx 0.251$ for $z = 3$ | $d_{BzW}(\mathbb{M}) = \sum_{k=1}^{\infty} \dfrac{I_{Wk}}{z^k} = \frac{2}{27} \approx 0.074$ for $z = 3$ |
| Line Index of Imbalance | |
| $d_{LI}(\mathbb{M}) = \frac{1}{4} = 0.25$ | $d_{LIW}(\mathbb{M}) = \frac{1}{6} \approx 0.167$ |

We make some observations from the comparison. In level of imbalance, $z$ is a free parameter. The choice of $z = 3$ was made deliberately for comparison. We see that the degree of imbalance is higher than the level of imbalance and line index of imbalance for strong polarization. We also note that line index of imbalance has a slightly higher measure with respect to weak polarization. As a general analysis of the signed model in Figure 3.2 over all three measurements, we see that the network is relatively far from being strongly polarized. Nevertheless, we observe that this is a social setting quite close to being weakly polarized. Recall that this would indicate a situation where the agents are divided into groups where there would be friendships within, but hostility towards all other groups.

## 3.5 Summary

In this chapter we investigated measures of distance of signed models in terms of balance. We proposed a definition of properties a map should inhabit to be classified as a balanced distance. Three metrics were presented: degree of imbalance, level of imbalance and line index of imbalance. The two former measurements were modified to accommodate signed models of **PNL**, whereas the latter is novelly defined in this thesis based on ideas from social psychology and literature on distance between Kripke models. We end by a comparison of the three with an example, in turn showing how we can analyze a degree of polarization with the given distance measurements.

# Chapter 4

# Extending PNL

We have now reached the point where we extend the logic of positive and negative relations with the following aims. There is a clear incentive to investigate what additions we need to the language to be able to define properties such as balance, non-overlapping and collective connectedness. We will present some likely candidates and discuss what other purpose these operators can carry in our context. Furthermore, we want to analyze transition to, and from, polarization and balance in the networks we are formalizing. With this intention we introduce dynamic modalities for local link change and local link addition to examine step-by-step change in our signed models.

## 4.1 Speaking of Balance

Finding an axiom to define the signed frame property of being balanced is not only desirable for the sake of its own. In beginning to formalize change in signed models, we are faced with an immediate problem. We have a strong motivation to be able to write formally that *after a network has changed, it is now balanced*. Recall that the only formula we have in **PNL** to define balance on a signed frame is axiom **4B** [39].

$$((\oplus \oplus p \vee \ominus \ominus p) \to \oplus p) \wedge ((\oplus \ominus p \vee \ominus \oplus p) \to \ominus p) \qquad \textbf{(4B)}$$

This axiom defines the local balance property, yet is only really relevant when we are working with collectively connected signed frames. Collective connectedness has as we know been shown to be modally undefinable in **PNL** [39]. Also recall that balance in the general sense is axiomatized as a rule, namely $\mathbf{Nb}_n$, and not as an axiom. We have the formula $name_n(\phi, \psi)$ sufficient to express that there are no negative cycles starting at the current agent, although this is not necessarily true if the property holds.

To begin to resolve this issue, we introduce a global modality $[A]$ and global adding-modalities $[/\!\!\wedge\!\!\backslash+]_G$ and $[/\!\!\wedge\!\!\backslash-]_G$. The global adding-modalities take inspiration from sabotage modal logic [3] [19], although sabotage modal logic is traditionally equipped with a deleting modality instead of an adding modality.

Intuitively, the formula $[A]\phi$ states that $\phi$ is true at all agents in the network. The formulas $[/\!\!\wedge\!\!\backslash+]_G\phi$ and $[/\!\!\wedge\!\!\backslash-]_G\phi$ are forced at an agent if and only if $\phi$ is true at the current agent after adding a positive or negative link anywhere in the network, respectively. We go on to present the semantics in a more formal way.

**Definition 24** *(Semantics of Global Addition Modalities) Let* $\mathbb{M} = \langle A, R^+, R^-, V\rangle$ *be a signed model and* $a \in A$. *We define truth conditions for the global addition modalities as follows:*

$$\begin{aligned}
&\mathbb{M}, a \Vdash [A]\phi \textit{ iff } \forall b \in A : \mathbb{M}, b \Vdash \phi\\
&\mathbb{M}, a \Vdash [/\!\!\wedge\!\!\backslash+]_G\phi \textit{ iff } \exists b, c \in A \textit{ such that } \langle A, R^+ \cup \{(b,c),(c,b)\}, R^-, V\rangle, a \Vdash \phi\\
&\mathbb{M}, a \Vdash [/\!\!\wedge\!\!\backslash-]_G\phi \textit{ iff } \exists b, c \in A \textit{ such that } \langle A, R^+, R^- \cup \{(b,c),(c,b)\}, V\rangle, a \Vdash \phi
\end{aligned}$$

This gives us enough to develop the following axiom $\mathbf{B}_G$ as a dynamic characterization of balance:

$$[/\!\!\wedge\!\!\backslash x_1]_G \ldots [/\!\!\wedge\!\!\backslash x_n]_G[A]\mathbf{4B} \text{ for } x_i \in \{+,-\} \qquad (\mathbf{B}_G)$$

What this formula states is that it holds at any agent in the network if and only if axiom **4B** will be forced at all agents after adding positive and negative edges anywhere in the signed model. This is essentially characterizing the general Balance Theorem: the formula holds at an agent in a signed model $\mathbb{M}$ if and only if there exists a supermodel of $\mathbb{M}$ where the local balance property holds.

Another way to define $\mathbf{B}_G$ is by adding the following *choice* and *iteration* modalities inspired by known dynamic logics, e.g. propositional dynamic logic (PDL) [9]. We accommodate them to the global adding modalities of our language and define them accordingly.

**Definition 25** *(Semantics of Choice and Iteration Modalities) Let* $\mathbb{M} = \langle A, R^+, R^-, V\rangle$ *be a signed model and* $a \in A$. *We define truth conditions for the global addition choice and iteration modalities as follows:*

$$\begin{aligned}
&\mathbb{M}, a \Vdash [/\!\!\wedge\!\!\backslash + \cup /\!\!\wedge\!\!\backslash -]_G\phi \textit{ iff } [/\!\!\wedge\!\!\backslash+]_G\phi \textit{ or } [/\!\!\wedge\!\!\backslash-]_G\phi\\
&\mathbb{M}, a \Vdash [(/\!\!\wedge\!\!\backslash + \cup /\!\!\wedge\!\!\backslash -)*]_G\phi \textit{ iff } \exists n \geq 0 \textit{ such that}\\
&\mathbb{M}, a \Vdash [/\!\!\wedge\!\!\backslash + \cup /\!\!\wedge\!\!\backslash -]_{G1} \cdots [/\!\!\wedge\!\!\backslash + \cup /\!\!\wedge\!\!\backslash -]_{Gn-1}[/\!\!\wedge\!\!\backslash + \cup /\!\!\wedge\!\!\backslash -]_{Gn}\phi
\end{aligned}$$

The modality $[/\!\!\wedge\!\!\backslash + \cup /\!\!\wedge\!\!\backslash -]_G$ is a version of the *change* operator known from the literature and is to be understood as that $[/\!\!\wedge\!\!\backslash + \cup /\!\!\wedge\!\!\backslash -]_G\phi$ is true at an agent if and only if $\phi$ is true at the current agent after adding a positive or negative link anywhere in the network. We read the iterated modality $[(/\!\!\wedge\!\!\backslash + \cup /\!\!\wedge\!\!\backslash -)*]_G\phi$ to be true at an agent if and only if $\phi$ holds at the current agent after adding a finite number of positive or negative edges to the signed frame. This lets us, perhaps unsurprisingly, have the following definition of the $\mathbf{B}_G$ axiom:

$$[(\mathbin{\text{⩕}}+\cup\mathbin{\text{⩕}}-)*]_G[A]\mathbf{4B} \tag{$\mathbf{B}_G$}$$

We now have the following lemma.

**Lemma 3** *For any signed frame $\mathbb{F}$, $\mathbb{F}\Vdash \boldsymbol{B}_G$ iff $\mathbb{F}$ has the balance property.*

*Proof.* ($\Rightarrow$) Let $\mathbb{F}=\langle A,R^+,R^-\rangle$ be a signed frame such that $\mathbb{F}\Vdash\mathbf{B}_G$. Fix an arbitrary valuation $V$ on $\mathbb{F}$ and let $\mathbb{M}=\langle\mathbb{F},V\rangle$. Also let $a\in A$. Since the formula $\mathbf{B}_G$ is valid on the signed frame, we have $\mathbb{M},a\Vdash\mathbf{B}_G$. Thus, $\mathbb{M},a\Vdash[\text{⩕}x_1]_G\cdots[\text{⩕}x_n]_G[A]\mathbf{4B}$. Then there exists $b_1,\dots,b_j\in A$ such that $\langle A,R^+\cup\{(b_m,b_n),\dots\},R^-\cup\{(b_s,b_t),\dots\},V\rangle\Vdash[A]\mathbf{4B}$. For simplicity, call $\langle A,R^+\cup\{(b_m,b_n),\dots\},R^-\cup\{(b_s,b_t),\dots\}\rangle=\mathbb{F}'$ and $\langle\mathbb{F}',V\rangle=\mathbb{M}'$. We now have that $\forall b\in A$, $\mathbb{M}',b\Vdash\mathbf{4B}$. Since we fixed an arbitrary valuation $V$, it follows that $\mathbb{F}'\Vdash\mathbf{4B}$. By Lemma 2, also known as Lemma 8 in [40], $\mathbb{F}'$ has the local balance property. By Corollary 1, $\mathbb{F}'$ has the balance property and thus by the Balance Theorem 6 there exists a collectively connected signed frame $\mathbb{F}''=\langle A'',R^{+''},R^{-''}\rangle$ such that $A''=A'$, $R^{+'}\subseteq R^{+''}$ and $R^{-'}\subseteq R^{-''}$ that has the local balance property. Now, since $\mathbb{F}\subseteq\mathbb{F}'$ and $\mathbb{F}'\subseteq\mathbb{F}''$, we have $\mathbb{F}\subseteq\mathbb{F}''$ and again by the Balance Theorem $\mathbb{F}$ has the balance property.

($\Leftarrow$) Let $\mathbb{F}=\langle A,R^+,R^-\rangle$ be a signed frame with the balance property. By the Balance Theorem 6 there exists a collectively connected signed frame $\mathbb{F}'=\langle A',R^{+'},R^{-'}\rangle$ such that $A'=A$, $R^+\subseteq R^{+'}$ and $R^-\subseteq R^{-'}$ that has the local balance property. By Lemma 2, it follows that $\mathbb{F}'\Vdash\mathbf{4B}$. Now fix an arbitrary element $a\in A=A'$ and an arbitrary valuation $V$ on $\mathbb{F}$ and $\mathbb{F}'$ and let $\mathbb{M}=\langle\mathbb{F},V\rangle$ and $\mathbb{M}'=\langle\mathbb{F}',V\rangle$. Since $\mathbb{F}'\Vdash\mathbf{4B}$, we have that $\mathbb{M}',a\Vdash[A]\mathbf{4B}$. Thus, as $R^+\subseteq R^{+'}$ and $R^-\subseteq R^{-'}$ it follows directly that $\mathbb{M},a\Vdash[\text{⩕}x_1]_G\cdots[\text{⩕}x_n]_G[A]\mathbf{4B}$ and hence $\mathbb{M},a\Vdash\mathbf{B}_G$. Since we fixed an arbitrary valuation $V$ and $a\in A$, we conclude that $\mathbb{F}\Vdash\mathbf{B}_G$. $\square$

Note that this means that we now have an axiom not only for balance in a collectively connected signed model, modeling a complete signed graph, but for any signed model. In fact, we have that for any signed model, $\mathbf{B}_G$ is forced at *any* agent if and only if the model is balanced.

**Corollary 3** *Let $\mathbb{M}=\langle A,R^+,R^-,V\rangle$ be a signed model and $a\in A$. $\mathbb{M},a\Vdash\boldsymbol{B}_G$ if and only if $\mathbb{M}$ is balanced.*

## 4.2 Speaking of Weak Balance

By modifying the **4B** axiom to adapt to the local weak balance conditions, we present the axiom **4W** for local weak balance.

$$(\oplus\oplus p\to\oplus p)\wedge((\oplus\ominus p\vee\ominus\oplus p)\to\ominus p) \tag{\textbf{4W}}$$

We now also have the analogous lemma for weak locally balanced signed frames.

**Lemma 4** *For any signed frame $\mathbb{F}$, $\mathbb{F} \Vdash$ **4W** iff $\mathbb{F}$ has the weak local balance property.*

*Proof.* ($\Rightarrow$) Proof by contraposition. Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame without the weak local balance property. Then, without loss of generality $\exists a, b, c \in A$ such that $aR^+b$, $bR^+c$ and $aR^-c$. Now, let $V$ be a valuation on $\mathbb{F}$ such that $V(p) = \{c\}$. It follows that $\langle \mathbb{F}, V \rangle, a \Vdash \diamondplus \diamondplus p$. However, by the non-overlapping property, we have that $\neg(aR^+c)$. Thus $\langle \mathbb{F}, V \rangle, a \nVdash \diamondplus p$. We have that $\langle \mathbb{F}, V \rangle, a \nVdash \diamondplus \diamondplus p \to \diamondplus p$ and hence $\mathbb{F} \nVdash \mathbf{4W}$.

($\Leftarrow$) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame with the weak local balance property and fix an arbitrary valuation $V$ and $a \in A$. Assume that $\langle \mathbb{F}, V \rangle, a \Vdash \diamondplus \diamondplus p$. Then $\exists b \in A$ such that $aR^+b$ and $\langle \mathbb{F}, V \rangle, b \Vdash \diamondplus p$. Thus it follows that $\exists c \in A$ such that $bR^+c$ and $\langle \mathbb{F}, V \rangle, c \Vdash p$. By the weak local balance property $aR^+c$ and hence $\langle \mathbb{F}, V \rangle, a \Vdash \diamondplus \diamondplus p \to \diamondplus p$. Now assume that $\langle \mathbb{F}, V \rangle, a \Vdash \diamondplus \diamondminus p$. Then $\exists b, c \in A$ such that $aR^+b$ and $bR^-c$ where $\langle \mathbb{F}, V \rangle, c \Vdash p$. The weak local balance property of $\mathbb{F}$ demand $aR^-c$ and therefore $\langle \mathbb{F}, V \rangle, a \Vdash \diamondminus p$. By similar reasoning $\langle \mathbb{F}, V \rangle, a \Vdash \diamondminus p$ if we assume $\langle \mathbb{F}, V \rangle, a \Vdash \diamondminus \diamondplus p$. Hence it follows that $\langle \mathbb{F}, V \rangle, a \Vdash (\diamondplus \diamondplus p \to \diamondplus p) \wedge ((\diamondplus \diamondminus p \vee \diamondminus \diamondplus p) \to \diamondminus p)$ and as we fixed an arbitrary $V$ and $a \in A$ we conclude that $\mathbb{F} \Vdash \mathbf{4W}$. $\square$

As in the case of strong balance, we can use the global modalities to axiomatize the property of general weak balance. We name the axiom $\mathbf{B}_W$.

$$[(\mathbin{\wedge\!\!\!\wedge} + \cup \mathbin{\wedge\!\!\!\wedge} -)*]_G[A]\mathbf{4W} \qquad (\mathbf{B}_W)$$

Thus we have the following lemma.

**Lemma 5** *For any signed frame $\mathbb{F}$, $\mathbb{F} \Vdash \mathbf{B}_W$ iff $\mathbb{F}$ has the weak balance property.*

*Proof.* ($\Rightarrow$) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame such that $\mathbb{F} \Vdash \mathbf{B}_W$. Let $V$ be an arbitrary valuation on $\mathbb{F}$ and name $\mathbb{M} = \langle \mathbb{F}, V \rangle$. Let $a \in A$. We have that $\mathbb{M}, a \Vdash \mathbf{B}_W$. Then there exists $b_1, \ldots, b_j \in A$ such that $\langle A, R^+ \cup \{(b_m, b_n), \ldots\}, R^- \cup \{(b_s, b_t), \ldots\}, V \rangle \Vdash [A]\mathbf{4W}$. For simplicity, call $\langle A, R^+ \cup \{(b_m, b_n), \ldots\}, R^- \cup \{(b_s, b_t), \ldots\} \rangle = \mathbb{F}'$ and $\langle \mathbb{F}', V \rangle = \mathbb{M}'$. We now have that $\forall b \in A$, $\mathbb{M}', b \Vdash \mathbf{4W}$. Since we fixed an arbitrary valuation $V$, it follows that $\mathbb{F}' \Vdash \mathbf{4W}$. By Lemma 4 we have that $\mathbb{F}'$ has the weak local balance property. By Corollary 2, $\mathbb{F}'$ is weakly balanced and thus by the weak Balance Theorem 7 there exists a collectively connected frame $\mathbb{F}'' = \langle A'', R^{+''}, R^{-''} \rangle$ with the weak local balance property such that $A'' = A'$, $R^{+'} \subseteq R^{+''}$ and $R^{-'} \subseteq R^{-''}$. Since $\mathbb{F} \subseteq \mathbb{F}' \subseteq \mathbb{F}''$ it follows that $\mathbb{F} \subseteq \mathbb{F}''$ and hence again by the weak Balance Theorem $\mathbb{F}$ has the weak balance property.

($\Leftarrow$) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame with the weak balance property. Then by the weak Balance Theorem 7 there exists a collectively connected frame $\mathbb{F}' = \langle A', R^{+'}, R^{-'} \rangle$ such that $A = A'$, $R^+ \subseteq R^{+'}$ and $R^- \subseteq R^{-'}$ that has the local balance property. It follows from Lemma 4 that $\mathbb{F}' \Vdash \mathbf{4W}$. Fix an arbitrary valuation $V$ and an arbitrary $a \in A, A'$. It follows that $\langle \mathbb{F}', V \rangle, a \Vdash [A]\mathbf{4W}$. Since $A = A'$, $R^+ \subseteq R^{+'}$ and $R^- \subseteq R^{-'}$, it follows directly that $\langle \mathbb{F}, V \rangle, a \Vdash [(\mathbin{\wedge\!\!\!\wedge} + \cup \mathbin{\wedge\!\!\!\wedge} -)*]_G[A]\mathbf{4W}$. As we chose an arbitrary $V$ and $a \in A$, we conclude that $\mathbb{F} \Vdash \mathbf{B}_W$. $\square$

The subsequent corollary follows directly from Lemma 5.

**Corollary 4** *Let $\mathbb{M} = \langle A, R^+, R^-, V\rangle$ be a signed model and $a \in A$. $\mathbb{M}, a \Vdash \boldsymbol{B}_W$ if and only if $\mathbb{M}$ is weakly balanced.*

## 4.3 Collective Connectedness

Recall that the collective connectedness property is modally undefinable in **PNL**. It just so happens that by adding the global modality $[A]$ we can have an axiom for collective connectedness. Call it **C**:

$$(\boxplus p \to [A]p) \vee (\boxminus p \to [A]p) \qquad (\mathbf{C})$$

We prove the following lemma:

**Lemma 6** *For any signed frame $\mathbb{F}$, $\mathbb{F} \Vdash \boldsymbol{C}$ iff $\mathbb{F}$ has the collective connectedness property.*

*Proof.* ($\Rightarrow$) Let $\mathbb{F} = \langle A, R^+, R^-\rangle$ be a signed frame and $\mathbb{F} \Vdash (\boxplus p \to [A]p) \vee (\boxminus p \to [A]p)$. Fix $a \in A$ arbitrarily. For any $V : \langle \mathbb{F}, V\rangle, a \Vdash (\boxplus p \to [A]p) \vee (\boxminus p \to [A]p)$. Then $\langle \mathbb{F}, V\rangle, a \Vdash \boxplus p \to [A]p$ or $\langle \mathbb{F}, V\rangle, a \Vdash \boxminus p \to [A]p$. Let $V(p) = \{b \mid aR^+b \text{ or } aR^-b\}$. Fix $c \in A$ arbitrarily. We want to prove that $\langle \mathbb{F}, V\rangle, c \Vdash p$. Assume that $\langle \mathbb{F}, V\rangle, a \Vdash \boxplus p \to [A]p$. By $V$, we have $\langle \mathbb{F}, V\rangle, a \Vdash \boxplus p$ and thus $\langle \mathbb{F}, V\rangle, a \Vdash [A]p$. Therefore $\langle \mathbb{F}, V\rangle, c \Vdash p$. Similarly for the case where $\langle \mathbb{F}, V\rangle, a \Vdash \boxminus p \to [A]p$. Since we fixed $a, c \in A$ arbitrarily, we conclude that $\mathbb{F}$ is collectively connected.

($\Leftarrow$) Let $\mathbb{F} = \langle A, R^+, R^-\rangle$ be a signed frame with the collective connectedness property. Then $\forall a, b \in A : aR^+b$ or $aR^-b$. Suppose for reductio that $\exists a \in A$ and $V$ such that $\langle \mathbb{F}, V\rangle, a \nVdash (\boxplus p \to [A]p) \vee (\boxminus p \to [A]p)$. Then $\langle \mathbb{F}, V\rangle, a \Vdash \neg(\boxplus p \to [A]p) \wedge \neg(\boxminus p \to [A]p)$. Thus $\langle \mathbb{F}, V\rangle, a \Vdash (\boxplus p \wedge \boxminus p) \wedge \neg[A]p$. Then $\exists b \in A$ such that $\langle \mathbb{F}, V\rangle, b \nVdash p$. As $\langle \mathbb{F}, V\rangle, a \Vdash \boxplus p \wedge \boxminus p$ and $aR^+b$ or $aR^-b$, this is a contradiction. Hence $\mathbb{F} \Vdash (\boxplus p \to [A]p) \vee (\boxminus p \to [A]p)$. □

The following corollaries follow directly from this theorem.

**Corollary 5** *For any signed frame $\mathbb{F}$, $\mathbb{F} \Vdash \boldsymbol{4B} + \boldsymbol{C}$ iff $\mathbb{F}$ is locally balanced and has the collective connectedness property.*

**Corollary 6** *For any signed frame $\mathbb{F}$, $\mathbb{F} \Vdash \boldsymbol{4W} + \boldsymbol{C}$ iff $\mathbb{F}$ is weak locally balanced and has the collective connectedness property.*

Perhaps more interestingly we can now also express the following results.

**Corollary 7** *For any signed frame $\mathbb{F} = \langle A, R^+, R^-\rangle$, $\mathbb{F} \Vdash \boldsymbol{B}_G$ iff $\exists \mathbb{F}' = \langle A', R^{+'}, R^{-'}\rangle$ such that $A = A'$, $R^+ \subseteq R^{+'}$, $R^- \subseteq R^{-'}$ and $\mathbb{F}' \Vdash \boldsymbol{C} + \boldsymbol{4B}$.*

**Corollary 8** *For any signed frame $\mathbb{F} = \langle A, R^+, R^-\rangle$, $\mathbb{F} \Vdash \boldsymbol{B}_W$ iff $\exists \mathbb{F}' = \langle A', R^{+'}, R^{-'}\rangle$ such that $A = A'$, $R^+ \subseteq R^{+'}$, $R^- \subseteq R^{-'}$ and $\mathbb{F}' \Vdash \boldsymbol{C} + \boldsymbol{4W}$.*

## 4.4 Non-Overlapping

Like collective connectedness, the property of non-overlapping is modally undefinable in **PNL**. Recall that it is axiomatized with $n$-balance in the $\mathbf{Nb}_n$ rule. [39] also proposes another rule that captures the property of non-overlapping for the possibility to axiomatize **PNL** without $n$-balance. Although, like $\mathbf{Nb}_n$, this is a rule, and not an axiom. In this section, we propose some options of additions to the language such that we can have an axiom for non-overlapping and briefly discuss strengths and weaknesses of each approach.

### 4.4.1 Difference

One possible solution is the difference operator $[D]$. This is not a commonly used operator and requires the inequality relation $\neq$. We the define the semantics of the modality in the following definition.

**Definition 26** *(Semantics of Difference Operator [9]) Let $\mathbb{M} = \langle A, R^{+}, R^{-}, V\rangle$ be a signed model and let $a \in A$. We define the semantics of the difference operator $[D]$ as follows:*

$$\mathbb{M}, a \Vdash [D]\phi \textit{ iff } \exists b \in A \textit{ such that } b \neq a \textit{ and } \mathbb{M}, b \Vdash \phi.$$

With this definition, we introduce the axiom $\mathbf{N}_D$ for the non-overlapping property:

$$(p \wedge \neg[D]p) \rightarrow (\boxplus(\diamondminus p \rightarrow p) \wedge \boxminus(\diamondplus p \rightarrow p)) \qquad (\mathbf{N}_D)$$

Inclusion of the $[D]$ modality is not hard to motivate in connection to this thesis. It simply states that $[D]\phi$ holds at an agent if and only if there is another agent in the network where $\phi$ is true. A problem with this operator is that it is not modally definable in **PNL**. We let $\mathcal{L}_{PNL_{[D]}}$ be the language of **PNL** including $[D]\phi$ and prove the following lemma.

**Lemma 7** *For any symmetric frame $\mathbb{F} = \langle A, R^{+}, R^{-}\rangle$ of $\mathcal{L}_{PNL_{[D]}}$, $\mathbb{F} \Vdash \boldsymbol{N}_D$ iff $\mathbb{F}$ has the non-overlapping property.*

*Proof.* ($\Rightarrow$) Let $\mathbb{F} = \langle A, R^{+}, R^{-}\rangle$ be a frame of $\mathcal{L}_{PNL_{[D]}}$ such that $\mathbb{F} \Vdash \mathbf{N}_D$. Let $a, b \in A$ and without loss of generality assume that $aR^{+}b$. We want to prove that $\neg(aR^{-}b)$. Let $V$ be a valuation on $\mathbb{F}$ such that $V(p) = \{a\}$. It follows that $\langle \mathbb{F}, V\rangle, a \Vdash p \wedge \neg[D]p$. Since $\mathbb{F} \Vdash (p \wedge \neg[D]p) \rightarrow (\boxplus(\diamondminus p \rightarrow p) \wedge \boxminus(\diamondplus p \rightarrow p))$, we have that $\langle \mathbb{F}, V\rangle, a \Vdash \boxplus(\diamondminus p \rightarrow p) \wedge \boxminus(\diamondplus p \rightarrow p)$. As $aR^{+}b$, then $\langle \mathbb{F}, V\rangle, b \Vdash \diamondminus p \rightarrow p$. We know that $\langle \mathbb{F}, V\rangle, b \nVdash p$, thus $\langle \mathbb{F}, V\rangle, b \nVdash \diamondminus p$. Hence, $\neg(bR^{-}a)$ and by symmetry $\neg(aR^{-}b)$.

($\Leftarrow$) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a symmetric frame of $\mathcal{L}_{PNL_{[D]}}$ with the non-overlapping property. Fix an arbitrary valuation $V$ on $\mathbb{F}$ and $a \in A$. Assume that $\langle \mathbb{F}, V \rangle, a \Vdash p \wedge \neg [D]p$. Then $\neg \exists b \in A$ such that $b \neq a$ and $\langle \mathbb{F}, V \rangle, b \Vdash p$. It follows that $V(p) = \{a\}$. Let $c \in A$ such that $aR^+c$. By symmetry and non-overlapping $\neg(cR^-a)$. Thus $\langle \mathbb{F}, V \rangle, c \nVdash \diamondminus p$ and hence $\langle \mathbb{F}, V \rangle, c \Vdash \diamondminus p \to p$. Then $\langle \mathbb{F}, V \rangle, a \Vdash \boxplus(\diamondminus p \to p)$. Now, let $d \in A$ such that $aR^-d$. By similar reasoning $\langle \mathbb{F}, V \rangle, d \Vdash \diamondplus p \to p$ and thus $\langle \mathbb{F}, V \rangle, a \Vdash \boxminus(\diamondplus p \to p)$. It follows that $\langle \mathbb{F}, V \rangle, a \Vdash (p \wedge \neg [D]p) \to (\boxplus(\diamondminus p \to p) \wedge \boxminus(\diamondplus p \to p))$ and as we chose an arbitrary $V$ and $a \in A$, we conclude that $\mathbb{F} \Vdash \mathbf{N}_D$. $\square$

We note that we can also define collective connectedness with this operator, as the axiom $\mathbf{C}_D$:

$$((p \vee [D]p) \to \boxplus p) \vee ((p \vee [D]p) \to \boxminus p) \qquad (\mathbf{C}_D)$$

We leave the proof to the reader.

### 4.4.2 Intersection

Another possible option is to include an intersection modality in our language. The intersection modality is perhaps most commonly used as a distributed knowledge operator known in the literature of Dynamic Epistemic Logic, e.g. [14] and [30]. We modify it to our purpose and define the semantics in the following way.

**Definition 27** *(Semantics of Intersection Modality) Let $\mathbb{M} = \langle A, R^+, R^-, V \rangle$ be a signed model and let $a \in A$. We define the semantics of the intersection modality $\langle + \cap - \rangle$ as follows:*

$$\mathbb{M}, a \Vdash \langle + \cap - \rangle \phi \text{ iff } \exists b \in A \text{ such that } aR^+b, aR^-b \text{ and } \mathbb{M}, b \Vdash \phi$$

By including this operator, the axiom for non-overlapping $\mathbf{N}_I$ would simply be:

$$\langle + \cap - \rangle \perp \qquad (\mathbf{N}_I)$$

Let $\mathcal{L}_{PNL_{\langle + \cap - \rangle}}$ be the language of **PNL** including $\langle + \cap - \rangle \phi$. The proof of the following lemma is trivial.

**Lemma 8** *For any frame $\mathbb{F} = \langle A, R^+, R^- \rangle$ of $\mathcal{L}_{PNL_{\langle + \cap - \rangle}}$, $\mathbb{F} \Vdash \boldsymbol{N}_I$ iff $\mathbb{F}$ has the non-overlapping property.*

This modality shares with the difference operator that it is modally undefinable in **PNL**. The intersection modality is however not either easy to motivate in connection to the theme of this thesis. We read $\langle + \cap - \rangle \phi$ to hold at an agent if and only if there exists another agent that is both a friend and an enemy of the current agent where $\phi$ is true. That two agents *cannot* be both friends and enemies is a property assumed in the original work on signed graphs, and it is therefore difficult to see how the intersection operator would have any application outside axiomatizing the non-overlapping property.

### 4.4.3 Nominals

A third option is to add nominals in the hybrid tradition. We will leave the formal discussion of nominals rather brief and suggest that the reader turn to [2] for further details. A more extensive explanation of elements from hybrid logic will also be given in later sections.

Nominals are a set of propositional variables where output of the valuation function is a singleton. In other words, nominals are propositional variables that can only be true at exactly one world. In our context, this lets us *name* individual agents in the network.

To do this, we extend the set of propositional variables to be the union of two sets **At** and **Nom** with an empty intersection. **At** is the set of propositional atoms, whereas **Nom** is the set of *nominals*. We also extend our valuation function and call it $V_H$ such that $V_H : \mathbf{At} \cup \mathbf{Nom} \to \wp(A)$ satisfies the property: for all $i \in \mathbf{Nom}$, $|V_H(i)| = 1$. We denote members of $\mathbf{At} = \{p, q, r, \dots\}$ and $\mathbf{Nom} = \{i, j, k, \dots\}$. Satisfaction of nominals in a signed model with nominals $\mathbb{M} = \langle A, R^+, R^-, V_H \rangle$ with $a \in A$ is defined as we are used to with propositional variables:

$$\mathbb{M}, a \Vdash i \text{ iff } a \in V_H(i)$$

Before we present a nominal axiom for non-overlapping we show that we can now also define negative irreflexivity. Irreflexivity is property known to be modally undefinable in the Kripke semantics for modal logic [9]. We call the language of **PNL** including nominals $\mathcal{L}_{PNL_i}$ and the nominal axiom for negative irreflexivity $\mathbf{IrrT}^-$:

$$i \to \neg \Diamond^- i \qquad (\mathbf{IrrT}^-)$$

**Lemma 9** *For any frame $\mathbb{F} = \langle A, R^+, R^- \rangle$ of $\mathcal{L}_{PNL_i}$, $\mathbb{F} \Vdash \mathbf{IrrT}^-$ iff $\mathbb{F}$ is negative irreflexive.*

The proof of previous lemma is trivial. We go on to present the nominal axiom for non-overlapping $\mathbf{N}_H$:

$$i \to (\boxplus(\Diamond^- i \to i) \wedge \boxminus(\Diamond^+ i \to i)) \qquad (\mathbf{N}_H)$$

We conclude this section with the following lemma.

**Lemma 10** *For any negative irreflexive and symmetric frame $\mathbb{F} = \langle A, R^+, R^- \rangle$ of $\mathcal{L}_{PNL_i}$, $\mathbb{F} \Vdash \mathbf{N}_H$ iff $\mathbb{F}$ has the non-overlapping property.*

*Proof.* ($\Rightarrow$) Proof by contraposition. Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a symmetric and negative irreflexive frame of $\mathcal{L}_{PNL_i}$ such that $\mathbb{F}$ does not have the non-overlapping property. Then $\exists a, b \in A$ such that $aR^+b$ and $aR^-b$. By negative irreflexivity $a \neq b$. Now, let $V$ be a valuation on $\mathbb{F}$ such that $V(i) = \{a\}$. By symmetry $bR^-a$. It follows that $\langle \mathbb{F}, V \rangle, b \Vdash \Diamond^- i$ whereas $\langle \mathbb{F}, V \rangle, b \nVdash i$. As $aR^+b$, we have that $\langle \mathbb{F}, V \rangle, a \Vdash \Diamond^+(\Diamond^- i \to \neg i)$. Thus $\langle \mathbb{F}, V \rangle, a \nVdash i \to (\boxplus(\Diamond^- i \to i) \wedge \boxminus(\Diamond^+ i \to i))$ and hence $\mathbb{F} \nVdash \mathbf{N}_H$.

($\Leftarrow$) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a symmetric and negative irreflexive frame of $\mathcal{L}_{PNL_i}$ with the non-overlapping property. Fix an arbitrary valuation $V$ and $a \in A$. Assume that $\langle \mathbb{F}, V \rangle, a \Vdash i$. Let $b \in A$ such that $aR^+b$. By symmetry and non-overlapping it follows that $\neg(bR^-a)$. Thus $\langle \mathbb{F}, V \rangle, b \nVdash \ominus i$. Hence we have $\langle \mathbb{F}, V \rangle, b \Vdash \ominus i \to i$ and therefore also $\langle \mathbb{F}, V \rangle, a \Vdash \boxplus(\ominus i \to i)$. Now, let $c \in A$ such that $aR^-c$. By similar reasoning as before $\langle \mathbb{F}, V \rangle, c \Vdash \oplus i \to i$ and we have $\langle \mathbb{F}, V \rangle, a \Vdash \boxminus(\oplus i \to i)$. Hence $\langle \mathbb{F}, V \rangle, a \Vdash \mathbf{N}_H$ and as we fixed $V$ and $a$ arbitrarily we conclude that $\mathbb{F} \Vdash \mathbf{N}_H$. $\square$

Including nominals greatly extends the expressivity of a logic. However, it is not always evident what the motivation is beyond simply being allowed to express otherwise undefinable properties like irreflexivity, asymmetry, antisymmetry and intransitivity, to mention some. That being said, when we are modeling agent based networks with a logic like **PNL**, we already have an incentive to add nominals to make it clear *who* we are modeling. This is not a novel approach to social network logics, and can be seen in work such as [12], [31] and [32].

## 4.5 Local Link Change

The natural first step in observing and analyzing change in a signed social network is to look at how we can formalize *link change*. This depicts the action of agents changing their views towards one another, and is seemingly the most crucial action of transition in this context. Since we indeed want to model the fact that agents can affect relationships they have with other agents, we add modalities of *local* link change to our language.

Again, we take inspiration from sabotage modal logic, most notably [3] and [28]. As mentioned, sabotage modal logic usually includes modalities on link deletion, but we adapt the operators to create a language suited to our needs. We will add two modalities $[\oplus]_L$ and $[\ominus]_L$. In intuitive terms, we read $[\oplus]_L\phi$ to be true at an agent if and only if $\phi$ holds at the current agent after changing one edge connected to this agent from *negative to positive*. Similarly, we read $[\ominus]_L\phi$ to be true at an agent if and only if $\phi$ holds at the current agent after changing one edge connected to this agent from *positive to negative*. We proceed to a more formal definition.

**Definition 28** *(Semantics of Local Link Change Modalities) Let $\mathbb{M} = \langle A, R^+, R^-, V \rangle$ be a signed model and $a \in A$. We define truth conditions for the local link change modalities as follows:*

$$\mathbb{M}, a \Vdash [\oplus]_L\phi \text{ iff } \exists b \in A \text{ such that } aR^-b \text{ and}$$
$$\langle A, R^+ \cup \{(a,b),(b,a)\}, R^- \setminus \{(a,b),(b,a)\}, V \rangle, a \Vdash \phi$$
$$\mathbb{M}, a \Vdash [\ominus]_L\phi \text{ iff } \exists b \in A \text{ such that } aR^+b,\ a \neq b \text{ and}$$
$$\langle A, R^+ \setminus \{(a,b),(b,a)\}, R^- \cup \{(a,b),(b,a)\}, V \rangle, a \Vdash \phi$$

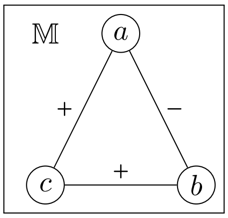


Figure 4.1: An imbalanced signed model.

Consider the imbalanced triangle $\mathbb{M}$ in Figure 4.1. Note again that positive reflexive arrows are omitted for simplicity, and that they will continue to be in examples to come. We can now state the following claims about $\mathbb{M}$, among many:

$$
\begin{aligned}
&\mathbb{M}, a \nVdash \mathbf{B}_G \\
&\mathbb{M}, a \Vdash [\oplus]_L \mathbf{B}_G \\
&\mathbb{M}, c \nVdash [\oplus]_L \mathbf{B}_G \\
&\mathbb{M}, c \Vdash [\ominus]_L [\ominus]_L \mathbf{B}_W \\
&\mathbb{M}, c \Vdash [\ominus]_L \mathbf{B}_G
\end{aligned}
$$

## 4.6 Adding Local Links

Change in relations between agents in a social network is not captured in its entirety as adjustments from positive to negative, or from negative to positive. As we have mentioned in earlier sections, collective connectedness is a strong restriction when modeling relations in a social network. There are only in specific social settings we can assume that all agents have a relation to each other. It might therefore be useful to introduce local edge addition modalities.

We need to clarify that adding edges between agents in an imbalanced signed model can never make the model balanced. The reasoning goes as follows: by the Balance Theorem, an imbalanced signed model contains at least one negative cycle. Adding relations between agents in the network cannot get rid of any negative cycles. Link addition in a balanced network can however make the model imbalanced. We present the local adding modalities for the purpose of analyzing these situations.

We include the modalities $[\Lambda\!\!\!\Lambda+]_L$ and $[\Lambda\!\!\!\Lambda-]_L$. These operate in a close similarity to $[\Lambda\!\!\!\Lambda+]_G$ and $[\Lambda\!\!\!\Lambda-]_G$. Although where $[\Lambda\!\!\!\Lambda+]_G\phi$ holds at an agent if and only if $\phi$ is true at the current agent after *any* link addition to the signed model, $[\Lambda\!\!\!\Lambda+]_L\phi$ holds if and only if $\phi$ is true at the current agent after a link addition to the signed model at that specific agent. Note that $[\Lambda\!\!\!\Lambda+]_L\phi$ implies $[\Lambda\!\!\!\Lambda+]_G\phi$, but the converse does not hold. The analogous implication holds for $[\Lambda\!\!\!\Lambda-]_G$ and $[\Lambda\!\!\!\Lambda-]_L$. We follow up with a formal definition of the semantics of the new operators.

**Definition 29** *(Semantics of Local Addition Modalities) Let* $\mathbb{M} = \langle A, R^+, R^-, V \rangle$ *be a signed model and* $a \in A$. *We define truth conditions for the local link addition modalities as follows:*

$$
\begin{aligned}
\mathbb{M}, a \Vdash [\bigwedge\!\!\!\backslash +]_L \phi \ &\textit{iff } \exists b \in A \textit{ such that } \neg(aR^+b),\ \neg(aR^-b) \textit{ and} \\
&\qquad \langle A, R^+ \cup \{(a,b),(b,a)\}, R^-, V \rangle, a \Vdash \phi \\
\mathbb{M}, a \Vdash [\bigwedge\!\!\!\backslash -]_L \phi \ &\textit{iff } \exists b \in A \textit{ such that } \neg(aR^+b),\ \neg(aR^-b) \textit{ and} \\
&\qquad \langle A, R^+, R^- \cup \{(a,b),(b,a)\}, V \rangle, a \Vdash \phi
\end{aligned}
$$

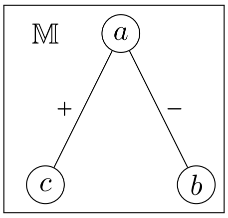


Figure 4.2: A collectively non-connected balanced signed model.

We can now state the following selected facts about the signed model $\mathbb{M}$ in Figure 4.2:

$$
\begin{aligned}
&\mathbb{M}, b \Vdash \mathbf{B}_G \\
&\mathbb{M}, b \Vdash [\bigwedge\!\!\!\backslash +]_L \neg \mathbf{B}_G \\
&\mathbb{M}, c \Vdash [\ominus]_L [\bigwedge\!\!\!\backslash -]_L \neg \mathbf{B}_G \\
&\mathbb{M}, c \Vdash [\bigwedge\!\!\!\backslash +]_L \Diamond\!\!\!\!\!+\ [\oplus]_L \mathbf{B}_G
\end{aligned}
$$

## 4.7 Summing-up Definitions and Example

To clear up any confusion and to give a comprehensive view of suggested additions to the language of **PNL**, we display a full syntax and semantics of the modalities presented in this chapter. We also present a more extensive example to give an intuitive representation of the extended **PNL**. As previously discussed, some additions to the language such as $\langle + \cap - \rangle$ are not particularly fruitful in this context, however included in the following definitions. The extended language we call $\mathcal{L}_{PNL+}$ is the language including all possible extensions given in this thesis, not a syntax meant to be adopted as is. The idea is rather to embrace a subset according to needs and ambitions of a particular context. As $\mathcal{L}_{PNL+}$ includes nominals, the semantics of all operators will be defined on a signed model with a nominal valuation.

**Definition 30** *(Syntax of* ***PNL+****) Let* $\boldsymbol{At}$ *be a set of propositional atoms and* $\boldsymbol{Nom}$ *be a set of nominals. Let* $\boldsymbol{At}$ *and* $\boldsymbol{Nom}$ *be countable and pairwise disjoint. We define the well-formed formulas of the language* $\mathcal{L}_{PNL+}$ *to be generated by the following grammar:*

$$\phi ::= p \mid i \mid \neg\phi \mid (\phi \wedge \phi) \mid \diamondplus\phi \mid \diamondminus\phi \mid [A]\phi \mid [/\!\!\backslash\!\!\backslash +]_G\phi \mid [/\!\!\backslash\!\!\backslash -]_G\phi \mid [D]\phi \mid \langle + \cap - \rangle\phi$$

$$[\oplus]_L\phi \mid [\ominus]_L\phi \mid [/\!\!\backslash\!\!\backslash +]_L\phi \mid [/\!\!\backslash\!\!\backslash -]_L\phi$$

*where* $p \in \boldsymbol{At}$ *and* $i \in \boldsymbol{Nom}$*. We define propositional connectives like* $\vee, \rightarrow$ *and the formulas* $\top, \bot$ *as usual. Further, we define the duals as standard* $\boxplus := \neg \diamondplus \neg$ *and* $\boxminus := \neg \diamondminus \neg$.

We will continue to denote members of $\mathbf{At} = \{p, q, r, \dots\}$ and $\mathbf{Nom} = \{i, j, k, \dots\}$.

**Definition 31** *(Signed Model with Nominals of* ***PNL+****) Let* $A$ *be a set of agents and* $R^+$ *and* $R^-$ *be two symmetric and non-overlapping binary relations on* $A$ *where* $R^+$ *is reflexive. A signed model with nominals is a tuple* $\mathbb{M} = \langle A, R^+, R^-, V_H \rangle$ *where* $V_H : \boldsymbol{At} \cup \boldsymbol{Nom} \rightarrow \wp(A)$ *is a valuation function such that* $|V_H(i)| = 1$ *for* $i \in \boldsymbol{Nom}$*.*

*We define a signed frame with nominals* $\mathbb{F} = \langle A, R^+, R^- \rangle$ *as a signed model with nominals without valuation.*

**Definition 32** *(Semantics of* ***PNL+****) Let* $\mathbb{M} = \langle A, R^+, R^-, V_H \rangle$ *be a signed model with nominals and a an agent in* $A$*. We inductively define the truth conditions as follows:*

$\mathbb{M}, a \Vdash p$ *iff* $a \in V(p)$ *for* $p \in \boldsymbol{At}$

$\mathbb{M}, a \Vdash i$ *iff* $a \in V(i)$ *for* $i \in \boldsymbol{Nom}$

$\mathbb{M}, a \Vdash \neg\phi$ *iff* $\mathbb{M}, a \nVdash \phi$

$\mathbb{M}, a \Vdash \phi \wedge \psi$ *iff* $\mathbb{M}, a \Vdash \phi$ *and* $\mathbb{M}, a \Vdash \psi$

$\mathbb{M}, a \Vdash \diamondplus\phi$ *iff* $\exists b \in A$ *such that* $aR^+b$ *and* $\mathbb{M}, b \Vdash \phi$

$\mathbb{M}, a \Vdash \diamondminus\phi$ *iff* $\exists b \in A$ *such that* $aR^-b$ *and* $\mathbb{M}, b \Vdash \phi$

$\mathbb{M}, a \Vdash [A]\phi$ *iff* $\forall b \in A : \mathbb{M}, b \Vdash \phi$

$\mathbb{M}, a \Vdash [/\!\!\backslash\!\!\backslash +]_G\phi$ *iff* $\exists b, c \in A$ *such that* $\langle A, R^+ \cup \{(b,c),(c,b)\}, R^-, V \rangle, a \Vdash \phi$

$\mathbb{M}, a \Vdash [/\!\!\backslash\!\!\backslash -]_G\phi$ *iff* $\exists b, c \in A$ *such that* $\langle A, R^+, R^- \cup \{(b,c),(c,b)\}, V \rangle, a \Vdash \phi$

$\mathbb{M}, a \Vdash [D]\phi$ *iff* $\exists b \in A$ *such that* $b \neq a$ *and* $\mathbb{M}, b \Vdash \phi$

$\mathbb{M}, a \Vdash \langle + \cap - \rangle\phi$ *iff* $\exists b \in A$ *such that* $aR^+b, aR^-b$ *and* $\mathbb{M}, b \Vdash \phi$

$\mathbb{M}, a \Vdash [\oplus]_L\phi$ *iff* $\exists b \in A$ *such that* $aR^-b$ *and*
$\langle A, R^+ \cup \{(a,b),(b,a)\}, R^- \setminus \{(a,b),(b,a)\}, V \rangle, a \Vdash \phi$

$\mathbb{M}, a \Vdash [\ominus]_L\phi$ *iff* $\exists b \in A$ *such that* $aR^+b$*,* $a \neq b$ *and*
$\langle A, R^+ \setminus \{(a,b),(b,a)\}, R^- \cup \{(a,b),(b,a)\}, V \rangle, a \Vdash \phi$

$\mathbb{M}, a \Vdash [/\!\!\backslash\!\!\backslash +]_L\phi$ *iff* $\exists b \in A$ *such that* $\neg(aR^+b)$*,* $\neg(aR^-b)$ *and*
$\langle A, R^+ \cup \{(a,b),(b,a)\}, R^-, V \rangle, a \Vdash \phi$

$\mathbb{M}, a \Vdash [/\!\!\backslash\!\!\backslash -]_L\phi$ *iff* $\exists b \in A$ *such that* $\neg(aR^+b)$*,* $\neg(aR^-b)$ *and*
$\langle A, R^+, R^- \cup \{(a,b),(b,a)\}, V \rangle, a \Vdash \phi$

Consider now the example in Figure 4.3, reused from the last chapter, although modified slightly in terms of signs for our purposes. We call the network $\mathbb{M}$ and make the following observations.

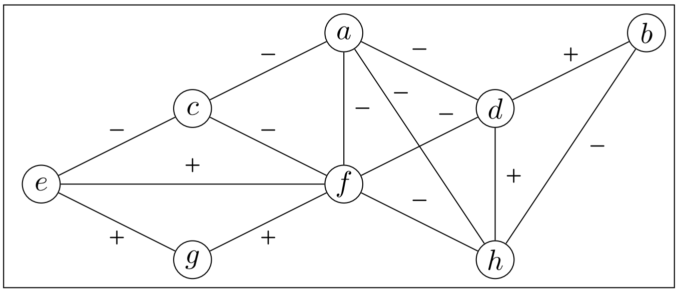


Figure 4.3: A network $\mathbb{M}$ with 8 agents.

- $\mathbb{M}$ is neither balanced nor weakly balanced.

$$\mathbb{M}, a \nVdash \mathbf{B}_G \lor \mathbf{B}_W$$

- $\mathbb{M}$ can become weakly balanced after changing a link from negative to positive, or vice versa, at agent $b$.

$$\mathbb{M}, b \Vdash [\oplus]_L \mathbf{B}_W \land [\ominus]_L \mathbf{B}_W$$

- There exists an agent other than $c$ where after changing a link from negative to positive, $\mathbb{M}$ can become weakly balanced.

$$\mathbb{M}, c \Vdash [D][\oplus]_L \mathbf{B}_W$$

- A positive edge can be added from agent $g$, such that after it has been added, then at no agent is it possible to make $\mathbb{M}$ weakly balanced by changing one link from negative to positive or vice versa.

$$\mathbb{M}, g \Vdash [\rlap{/}\setminus +]_L [A] \neg([\oplus]_L \mathbf{B}_W \lor [\ominus]_L \mathbf{B}_W)$$

## 4.8 Summary

This chapter was devoted to extending the logic **PNL** for two purposes. The first was to examine options of additions to the language that could give us axioms to define otherwise modally undefinable properties in **PNL**. By including dynamic edge adding operators, we presented the $\mathbf{B}_G$ and $\mathbf{B}_W$ axioms as dynamic characterizations of balance and weak balance, respectively. We also introduced a set of possible candidates of extensions to $\mathcal{L}_{PNL}$ to get axioms corresponding to the frame properties of collective connectedness and non-overlapping. Our second motivation for extending the logic of positive and negative relations was to analyze network change in particular regards to balance and polarization. We approached this challenge by including dynamic local link change and link adding modalities. The chapter ended with a review of the syntax and semantics of all the discussed additions to **PNL** with an example for clarification.

# Chapter 5

# Strong and Weak Ties

In this chapter we move away from positive and negative relations and introduce strong and weak ties. To reason about strong and weak ties we develop *tied logic (**TL**)*, a hybrid logic inspired by former work on **PNL** and our extensions to its framework. We begin the chapter by assessing the socio-psychological background in which we will reason about strong and weak ties: Granovetter's theory on the strength of weak ties and its close relation to echo chamber formation in social networks. We then present the syntax and semantics of tied logic, and show that the most notable claim in this context is a validity in **TL**. We also offer a full axiomatization of **TL** and prove that it is sound and strongly complete with respect to the class of what we call tied frames.

## 5.1 The Strength of Weak Ties

Where we formerly considered relations between agents in a social network to be either positive or negative, we now divide edges into strong and weak ties. Strong ties represent friends, and weak ties model acquaintances such that one tie cannot be both strong and weak. It is of course impossible to give a strict characterization of what is required for a tie between two agents to be either strong or weak. We will therefore rather think of the distribution of ties to function as a simplification, one that hopefully can give us some important indications of essential behavior in social networks.

Central to the implementation of strong and weak ties is the concept of *triadic closure*. Triadic closure is meant to formalize the phenomenon where one is likely to know the friends of one's friends. The formalization was made popular by Mark Granovetter in the 1970s as part of his theory of the strength of weak ties [21]. Triadic closure, or *Strong Triadic Closure* as it will be called in this context, is a property of agents in the network. The property holds of an agent if and only if its strong ties are subsequently tied together by a weak or a strong tie.

As an example we turn to Figure 5.1 where nodes depict agents and strong ties are labeled '$S$' and weak ties are labeled '$W$'. We observe that all agents except $b$ have the Strong Triadic Closure property, as $b$ is strongly tied to $a$ and $d$ who are not related.

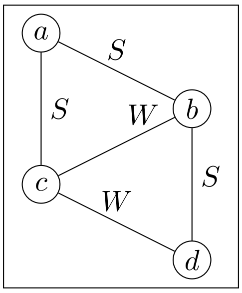


Figure 5.1: All agents except '$b$' have the Strong Triadic Closure property.

The strength of weak ties theory by Granovetter suggests that agents in social networks have a high chance of being strongly triadic closed. Further, as a result of Strong Triadic Closure, a network of strong and weak ties are likely to be made up of clusters with a high density of strong ties which are individually tied by weak ties. See Figure 5.2, taken from [16] for an illustration. The argument goes that weak ties have an important role channeling information between clusters of strong ties. This is the strength of weak ties.

We propose that these clusters where information to a great degree is preserved within the strong ties essentially are echo chamber-like structures. Assumed in the division between strong and weak ties is that we share information with our strong ties more frequently than with our weak ties [21]. Moreover, we are more likely to share the views of our strong ties. Therefore, there seems to be a great likelihood that echo chambers occur in such a cluster. In Figure 5.2 the echo chamber clusters are marked with dashed ellipses.

## 5.2 Syntax and Semantics

As in other hybrid logics,[1] the language of **TL** includes operators $@_i$ and $\downarrow x$. Intuitively, $@_i$ lets us shift the evaluation to the agent where name $i$ is true. $\downarrow x$ names the current agent '$x$'. These operators are closely related, but serve different purposes. By including both, we allow formulas where naming agents lets us later return the evaluation to the same agent. The language of **TL** includes the two diamond modalities $\langle S \rangle$ and $\langle W \rangle$. They are read intuitively as $\langle S \rangle \phi$ when the current agent has a strong tie where $\phi$ holds. A strong tie is replaced by a weak tie for $\langle W \rangle \phi$. The reader will recognize the association to positive and negative relations. We define the syntax, frames and models of **TL** formally as follows.

[1]For further details on hybrid logics beyond the scope of this paper, we again recommend turning to [2].

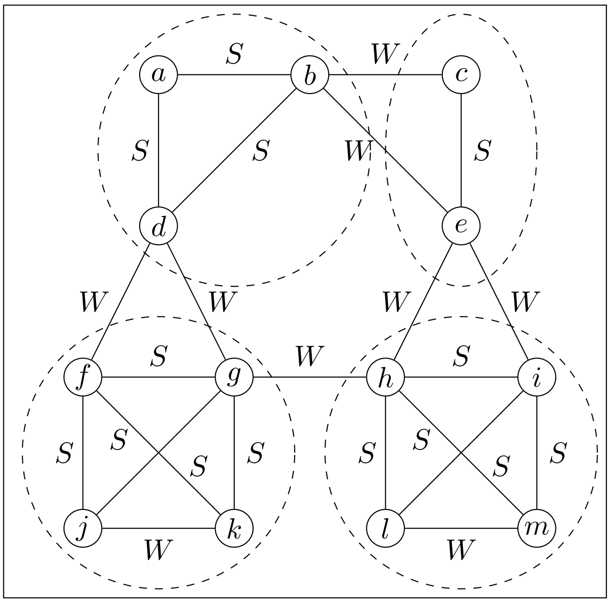


Figure 5.2: A network where all agents have the Strong Triadic Closure property.

**Definition 33** *(Syntax of **TL**) Let **At** be a set of propositional atoms and **Nom** be a set of nominals. Further, let **Var** be a set of agent variables. Let **At**, **Nom** and **Var** be countable and pairwise disjoint. We define the well-formed formulas of the language $\mathcal{L}_{\boldsymbol{TL}}$ to be generated by the following grammar:*

$$\phi ::= p \mid s \mid \neg\phi \mid (\phi \wedge \phi) \mid \langle S \rangle \phi \mid \langle W \rangle \phi \mid @_s \phi \mid \downarrow x.\phi$$

*where $p \in$ **At**, $s \in$ **Nom** $\cup$ **Var** and $x \in$ **Var**. We define propositional connectives like $\vee, \rightarrow$ and the formulas $\top, \bot$ as usual. Further, we define the duals as standard $[S] := \neg\langle S \rangle \neg$ and $[W] := \neg\langle W \rangle \neg$.*

We will denote members of **At** = $\{p, q, r, \dots\}$, **Nom** = $\{i, j, k, \dots\}$ and **Var** = $\{x, y, z, \dots\}$.

**Definition 34** *(Tied Model and Frame) Let $A$ be a set of agents and $R^S$ and $R^W$ be two symmetric and non-overlapping binary relations on $A$ where $R^S$ is reflexive and $R^W$ is irreflexive. A tied model is a tuple $\mathbb{M} = \langle A, R^S, R^W, V \rangle$ where $V$ : **At**$\cup$**Nom** $\rightarrow \wp(A)$ is a valuation function such that $\forall i \in$ **Nom**: $|V(i)| = 1$.*

*We define a tied frame $\mathbb{F} = \langle A, R^S, R^W \rangle$ as a tied model without valuation.*

Our two relations $R^S$ and $R^W$ define strong and weak ties, respectively. We assume reflexivity of $R^S$, irreflexivity of $R^W$ and symmetry of both relations. As in **PNL** we also require the property of non-overlapping: no two agents can be related by both a strong and a weak tie. For $a, b \in A$ we will denote members of $R^S$ as $(a, b)_S$, members of $R^W$ as $(a, b)_W$. As tied models describe social networks, we will as before sometimes refer to tied models as networks or social networks. Moreover, we view all formulas as propositions about agents. For instance we read $a \in V(p)$ as proposition $p$ or feature $p$ holds of agent $a$.

To define truth in a tied model $\mathbb{M} = \langle A, R^S, R^W, V\rangle$, we need to include an assignment function $g : \mathbf{Var} \to A$ that assigns agents to variables. Further, define the *x-variant* of $g$ to be $g^x_a(x) = a$ and $g^x_a(y) = g(y)$ for all $y \neq x$. We also define $[s]^{\mathbb{M},g}$ for $s \in \mathbf{Nom} \cup \mathbf{Var}$. For $i \in \mathbf{Nom}$, $[i]^{\mathbb{M},g}$ is the state $a \in A$ called 'i', i.e. the unique $a$ such that $a \in V(i)$. For $x \in \mathbf{Var}$, $[x]^{\mathbb{M},g} = g(x)$. We can now present satisfaction in a tied model.

**Definition 35** *(Semantics of **TL**) Let $\mathbb{M} = \langle A, R^S, R^W, V\rangle$ be a tied model, a an agent in A and $g : \boldsymbol{Var} \to A$ an assignment function. We inductively define the truth conditions as follows:*

$$\mathbb{M}, g, a \Vdash p \textit{ iff } a \in V(p) \textit{ for } p \in \boldsymbol{At}$$
$$\mathbb{M}, g, a \Vdash s \textit{ iff } a = [s]^{\mathbb{M},g} \textit{ for } x \in \boldsymbol{Nom} \cup \boldsymbol{Var}$$
$$\mathbb{M}, g, a \Vdash \neg\phi \textit{ iff } \mathbb{M}, g, a \nVdash \phi$$
$$\mathbb{M}, g, a \Vdash \phi \land \psi \textit{ iff } \mathbb{M}, g, a \Vdash \phi \textit{ and } \mathbb{M}, g, a \Vdash \psi$$
$$\mathbb{M}, g, a \Vdash \langle S\rangle\phi \textit{ iff } \exists b \in A \textit{ such that } aR^S b \textit{ and } \mathbb{M}, g, b \Vdash \phi$$
$$\mathbb{M}, g, a \Vdash \langle W\rangle\phi \textit{ iff } \exists b \in A \textit{ such that } aR^W b \textit{ and } \mathbb{M}, g, b \Vdash \phi$$
$$\mathbb{M}, g, a \Vdash @_s\phi \textit{ iff } \mathbb{M}, g, [s]^{\mathbb{M},g} \Vdash \phi \textit{ for } s \in \boldsymbol{Nom} \cup \boldsymbol{Var}$$
$$\mathbb{M}, g, a \Vdash \downarrow x.\phi \textit{ iff } \mathbb{M}, g^x_a, a \Vdash \phi$$

### 5.2.1 Strong Triadic Closure and Local Bridges

As mentioned earlier, the formation of echo chambers is tightly connected to the property of Strong Triadic Closure. Recall that an agent $a$ is strongly triadic closed when all its strong ties are related by a strong or a weak tie. A formal definition follows.

**Definition 36** *(Strong Triadic Closure [16], [21]) Let $\mathbb{M} = \langle A, R^S, R^W, V\rangle$ be a tied model. An agent $a \in A$ has the strong triadic closure property iff $\forall b, c \in A$:*

- *if $aR^S b$ and $aR^S c$, then $bR^S c$ or $bR^W c$.*

Strong Triadic Closure is closely related to Euclidicity in the standard Kripke semantics. [2] It is important to note that where Euclidicity is a frame property, Strong Triadic Closure is defined as a property of agents in the network. However, the property of *all* agents being strongly triadic closed is indeed a frame property which we prove in the following lemma.

Define the axiom called $\mathbf{STC}_G$ for *global* strong triadic closure:

$$\langle S\rangle p \to [S](\langle S\rangle p \lor \langle W\rangle p) \qquad (\mathbf{STC}_G)$$

[2] See [9] for details.

**Lemma 11** *For any tied frame $\mathbb{F}$, $\mathbb{F} \Vdash \boldsymbol{STC}_G$ iff all agents in $\mathbb{F}$ have the strong triadic closure.*

*Proof.* ($\Rightarrow$) Proof by contraposition. Let $\mathbb{F} = \langle A, R^S, R^W \rangle$ be a tied frame such that there exists an agent $a \in A$ that does not have the Strong Triadic Closure property. Then $\exists b, c \in A$ such that $aR^S b$ and $aR^S c$, but $\neg(bR^S c)$ and $\neg(bR^W c)$. Consider now the valuation $V(p) = \{b\}$. Since $\neg(bR^S c)$ and $\neg(bR^W c)$, it follows that $(\mathbb{F}, V), c \Vdash \neg\langle S\rangle p \wedge \neg\langle W\rangle p$. Thus, as $aR^S c$, we know that $(\mathbb{F}, V), a \Vdash \langle S\rangle(\neg\langle S\rangle p \wedge \neg\langle W\rangle p)$. As $aR^S b$, we have that $(\mathbb{F}, V), a \Vdash \langle S\rangle p$. Hence $(\mathbb{F}, V), a \nVdash \langle S\rangle p \to [S](\langle S\rangle p \vee \langle W\rangle p)$ and we conclude that $\mathbb{F} \nVdash \langle S\rangle p \to [S](\langle S\rangle p \vee \langle W\rangle p)$.

($\Leftarrow$) Let $\mathbb{F} = \langle A, R^S, R^W \rangle$ be a tied frame where all agents in $A$ have the Strong Triadic Closure property. Fix an arbitrary $a \in A$ and let $V$ be a valuation on $\mathbb{F}$ such that $(\mathbb{F}, V), a \Vdash \langle S\rangle p$. Then $\exists b \in A$ such that $aR^S b$ and $(\mathbb{F}, V), b \Vdash p$. Let $c \in A$ be an arbitrary agent such that $aR^S c$. By the Strong Triadic Closure property of $a$, it follows that $bR^S c$ or $bR^W c$. Thus $(\mathbb{F}, V), c \Vdash \langle S\rangle p \vee \langle W\rangle p$. Hence, as $c$ was chosen arbitrarily $(\mathbb{F}, V), a \Vdash \langle S\rangle \to [S](\langle S\rangle p \vee \langle W\rangle p)$. As we fixed $a \in A$ and $V$ arbitrarily too, we have that $\mathbb{F} \Vdash \langle S\rangle \to [S](\langle S\rangle p \vee \langle W\rangle p)$ which concludes the proof. □

Granovetter's theory that networks with a high occurrence of Strong Triadic Closure have a tendency to form weakly tied clusters of strong ties, is demonstrated in a known claim. Before we present this claim, we introduce the concept of a local bridge.

**Definition 37** *(Local Bridge [16]) Let $\mathbb{M} = \langle A, R^S, R^W, V \rangle$ be a tied model. Let $a, b \in A$. An edge $(a, b)_\circ$ for $\circ \in \{S, W\}$ is a local bridge iff $\forall c \in A$ such that $c \neq a$, $c \neq b$: $\neg(aR^S c)$ and $\neg(aR^W c)$, or $\neg(bR^S c)$ and $\neg(bR^W c)$.*

A local bridge is tie between two agents such that these two agents have no other friends or acquaintances in common. Agents in a social network that are related by a local bridge are in an important position when it comes to distribution of information. In a clustered network, local bridges are essential carriers of outside information. As we argue, clusters create echo chamber-like situations. Local bridges carrying new information are crucial in dissolving dangerous situations such as radicalization. We return to Figure 5.2 to observe that there is a local bridge between agents $g$ and $h$.

The claim is stated informally as follows.

**Claim.**[16][21]*If an agent in a network satisfies the Strong Triadic Closure property and is connected to other agents by at least two strong ties, then any local bridge it is related with must be a weak tie.*[3]

[3]For proof of the claim see Granovetter's original paper [21].

To formalize this claim as a validity in **TL**, we introduce formulas corresponding to the relevant properties. For simplicity, we first define the following abbreviation $\langle S \cup W\rangle\phi := \langle S\rangle\phi \vee \langle W\rangle\phi$. We read $\langle S \cup W\rangle\phi$ as true at an agent $a$ if and only if $a$ is connected to an agent by a strong or a weak tie where $\phi$ holds. The reader might recognize a similar operator in the rewritten $\mathbf{B}_G$ axiom in earlier chapters. We now present the formula $\mathbf{STC}_L$ which holds at an agent $a$ if and only if $a$ has the Strong Triadic Closure property.

$$\downarrow x.[S] \downarrow y.@_x[S](\neg y \to \langle S \cup W\rangle y) \tag{$\mathbf{STC}_L$}$$

Then we propose the following formula named $\mathbf{S}_2$ which holds at an agent $a$ if and only if $a$ is strongly tied to at least two other agents.

$$\downarrow x.\langle S\rangle \downarrow y.(\neg x \wedge @_x\langle S\rangle(\neg x \wedge \neg y)) \tag{$\mathbf{S}_2$}$$

The formula **LB** holds at an agent $a$ if and only if $a$ is related by a local bridge.

$$\downarrow x.\langle S \cup W\rangle \downarrow y.(\neg x \wedge \neg\langle S \cup W\rangle(\neg x \wedge \neg y \wedge \langle S \cup W\rangle x)) \tag{$\mathbf{LB}$}$$

Lastly, we introduce the formula $\mathbf{LB}_W$ which holds at an agent $a$ if and only if any local bridge $a$ is related with must be a weak tie.

$$\downarrow x.[S] \downarrow y.((\langle S \cup W\rangle(\neg x \wedge \neg y \wedge (\langle S \cup W\rangle x)) \tag{$\mathbf{LB}_W$}$$

We can now present the following corollary; that Granovetter's claim is a validity of **TL**.

**Corollary 9** *$(\boldsymbol{STC}_L \wedge \boldsymbol{S}_2) \to \boldsymbol{LB}_W$ is a validity of **TL**.*

*Proof.* Follows by the original work by Granovetter [21].

### 5.2.2 Axiomatization

To account for strong and weak ties in a social network, we assume strong reflexivity and weak irreflexivity, as previously noted. These frame properties are defined by the following two axioms $T^S$ and $IrrT^W$, respectively.

$$i \to \langle S\rangle i \tag{$T^S$}$$

$$i \to \neg\langle W\rangle i \tag{$IrrT^W$}$$

Symmetry of both relations is preserved in the following axiom $B^{SW}$.

$$i \to ([S]\langle S\rangle i \wedge [W]\langle W\rangle i) \tag{$B^{SW}$}$$

Non-overlapping can also be defined with the hybrid axiom $NonO$, stated directly below. As we know from earlier chapters, non-overlapping, like weak irreflexivity is modally undefinable in the standard Kripke semantics considering two-sorted Kripke frames.

$$i \to ([S](\langle W\rangle i \to i) \wedge [W](\langle S\rangle i \to i)) \tag{$NonO$}$$

The final axiomatixation of **TL** is the axiomatization of the standard normal hybrid logic $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$ [2] together with our recently presented axioms. A full list of axioms and rules is found in Table 5.1.

Table 5.1: Axiomatization of **TL**, where $\Diamond \in \{\langle S\rangle, \langle W\rangle\}$ and $\Box \in \{[S],[W]\}$.

| | |
|---|---|
| $(CT)$ | All classical tautologies |
| $(K_\Box)$ | $\vdash \Box(\phi \to \psi) \to \Box\phi \to \Box\psi$ |
| $(K_@)$ | $\vdash @_i(\phi \to \psi) \to @_i\phi \to @_i\psi$ |
| $(Selfdual)$ | $\vdash @_i\phi \leftrightarrow \neg @_i \neg\phi$ |
| $(Ref_@)$ | $\vdash @_i i$ |
| $(Agree)$ | $\vdash @_i @_j \phi \leftrightarrow @_j\phi$ |
| $(Intro)$ | $\vdash i \to (\phi \leftrightarrow @_i\phi)$ |
| $(Back)$ | $\vdash \Diamond @_i\phi \to @_i\phi$ |
| $(DA)$ | $\vdash @_i(\downarrow x.\phi \leftrightarrow \phi[x/i])$ |
| $(T^S)$ | $\vdash i \to \langle S\rangle i$ |
| $(IrrT^W)$ | $\vdash i \to \neg\langle W\rangle i$ |
| $(B^{SW})$ | $\vdash i \to ([S]\langle S\rangle i \wedge [W]\langle W\rangle i)$ |
| $(NonO)$ | $\vdash i \to ([S]\langle W\rangle i \to i) \wedge [W](\langle S\rangle i \to i))$ |
| (**MP**) | If $\vdash \phi$ and $\vdash \phi \to \psi$ then $\vdash \psi$ |
| (**Subst**) | If $\vdash \phi$ then $\vdash \phi^\sigma$, for $\sigma$ a substitution |
| ($\mathbf{Gen}_@$) | If $\vdash \phi$ then $\vdash @_i\phi$ |
| ($\mathbf{Gen}_\Box$) | If $\vdash \phi$ then $\vdash \Box\phi$ |
| (**Name**) | If $\vdash @_i\phi$ and $i$ does not occur in $\phi$, then $\vdash \phi$ |
| (**BG**) | If $\vdash @_i \Diamond j \to @_j\phi, j \neq i$ and $j$ does not occur in $\phi$, then $\vdash @_i \Box \phi$ |

### 5.2.3 Soundness and Strong Completeness

We will now prove that **TL** is sound and strongly complete with respect to the class of tied frames.

**Theorem 9** ***TL*** *is sound and strongly complete with respect to the class of tied frames.*

*Proof.* (Soundness) Let $\mathcal{F}$ be the class of tied frames. Since $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$ is sound with respect to the class of all hybrid frames, we know that $\mathcal{F} \Vdash \mathbf{K}_{\mathcal{H}(@,\downarrow)}$. Thus it suffices to show the validity of the axioms $T^S, IrrT^W, B^{SW}$ and $NonO$. Validity of $T^S, IrrT^W$ and $B^{SW}$ is trivial. To show the validity of $NonO$, we refer to Lemma 10 where we observe that axiom $\mathbf{N}_H$ is $NonO$ where instead of $\boxplus, \diamondplus$ we substitute $[S], \langle S\rangle$ and replace $\boxminus, \diamondminus$ with $[W], \langle W\rangle$.

(Completeness) Note again that **TL** is $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \{T^S, IrrT^W, B^{SW}, NonO\}$. The Sahlqvist-like theorem proved in [8] states that if $\Sigma$ is a set of pure $\mathcal{H}(@,\downarrow)$-formulas, then $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$ is strongly complete for the class of frames defined by $\Sigma$. It follows directly that if we can show that $\{T^S, IrrT^W, B^{SW}, NonO\}$ is a set of pure $\mathcal{H}(@,\downarrow)$-formulas, then **TL** is strongly complete with respect to the class of tied frames. The result follows straightforwardly from the fact that none of the axioms contain any propositional variables and that they can all be formulated in the language $\mathcal{H}(@,\downarrow)$. Proof of soundness and strong completeness follows.

**Corollary 10** *$\boldsymbol{TL} + \langle S \rangle i \to [S](\langle S \rangle i \vee \langle W \rangle i)$ is sound and strongly complete with respect to the class of tied frames where all agents have the Strong Triadic Closure property.*

Proof of Corollary 10 follows from Lemma 11 and Theorem 9 given that the above formula is a nominal version of $\mathbf{STC}_G$. Whereas this is perhaps not a surprising result, completeness on this class of frames is worth taking note of. Recall that according to Granovetter's theory, Strong Triadic Closure is a property we often observe across social networks. We now have the ability to reason and conduct a logical analysis directly within networks with this property; where the following corollary is a favorable example.

**Corollary 11** *$\boldsymbol{S}_2 \to \boldsymbol{LB}_W$ is a validity of $\boldsymbol{TL} + \langle S \rangle i \to [S](\langle S \rangle i \vee \langle W \rangle i)$.*

## 5.3 Summary

In this chapter we introduced a novel logic called tied logic, abbreviated **TL**. Tied logic is a hybrid logic based on theories originating from Mark Granovetter where relations in social networks are divided into strong and weak ties. According to this theory, agents in social networks are likely to have the Triadic Closure property, namely that their strong ties are subsequently strongly or weakly tied. This leads to the formation of strongly tied clusters that are tied by weak ties. We argued that these clusters are closely related to the emergence of echo chambers. We presented syntax and semantics of **TL** before showing that Granovetter's most notable claim is a validity in **TL**. Then we provided a full axiomatization of tied logic and proved that the logic is sound and strongly complete with respect to tied frames.

# Chapter 6

# Homophily

In the preceding chapters, we have been considering change and knowledge in social networks with agents related by different types of relations. We have however not provided any justification behind these relations, nor said anything about how they might have come to be. *Homophily* is known as the tendency of being similar to one's friends [16]. Perhaps, among many, there are two particularly prominent reasons behind this tendency. One is social influence: the habit of becoming like the people we surround ourselves with. Another is social selection; that we form friendships with others who are alike us. In this chapter we integrate our work with research on social group formation [34] [35] to illustrate the latter phenomenon. By defining a subclass of previously introduced models similar to the threshold models known from the literature, we present a support for relation making by properties agents have in common. We will integrate homophily in both logic of positive and negative relations and tied logic in successive sections. In the tied logic section we also make some observations to explore the attributes this subclass holds.

## 6.1 Positive and Negative Relations

Before we begin assessing positive and negative relations in terms of properties the agents share, we need to clarify exactly what subset of $\mathcal{L}_{PNL+}$ we will be working with. A natural choice seems to be $\mathcal{L}_{PNL+}$ without nominals or the $\langle + \cap - \rangle$ modality. This gives us the logic $\mathbf{PNL}_D$ as in $D$ for both dynamic and the $[D]$ operator. As we know, $\mathbf{PNL}_D$ is a dynamic logic that guarantees axioms for balance, collective connectedness and non-overlapping. To confirm that the reader is up to speed, we define $\mathcal{L}_{PNL_D}$.

**Definition 38** *(Syntax of $\boldsymbol{PNL}_D$) Let $\boldsymbol{At}$ be a countable set of propositional atoms. We define the well-formed formulas of the language $\mathcal{L}_{PNL_D}$ to be generated by the following grammar:*

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Diamond\!\!\!+\phi \mid \Diamond\!\!\!-\phi \mid [A]\phi \mid [\text{⩕}+]_G\phi \mid [\text{⩕}-]_G\phi \mid [D]\phi \mid$$

$$[\oplus]_L\phi \mid [\ominus]_L\phi \mid [\text{⩕}+]_L\phi \mid [\text{⩕}-]_L\phi$$

*where $p \in \boldsymbol{At}$. We define propositional connectives like $\vee, \to$ and the formulas $\top, \bot$ as usual. Further, we define the duals as standard $\boxplus := \neg \Diamond\!\!\!+ \neg$ and $\boxminus := \neg \Diamond\!\!\!- \neg$.*

To simplify notions to come, we will in this section work with signed models $\mathbb{M} = \langle A, R^+, R^-, V\rangle$ as defined in Definition 7 with a slight modification. The valuation function is now defined $V : A \to \wp(\mathbf{At})$. Recall that we consider propositions in our language to be propositions of agents, or rather certain features agents in the network can have. The set of properties of an agent $a$ is now captured in $V(a)$.

We define the semantics of $\mathbf{PNL}_D$ as a fragment of Definition 32 with a minor change in the truth of a propositional atom in a model, according to the redefinition of the valuation function:

$$\mathbb{M}, a \Vdash p \text{ iff } p \in V(a) \text{ for } p \in \mathbf{At}.$$

### 6.1.1 Mismatch and Distance

We now define the notions *mismatch* and *distance*.

**Definition 39** *(MSMTCH and DIST [34]) Let $\mathbb{M} = \langle A, R^+, R^-, V\rangle$ be a signed model. We define the set of features distinguishing agents $a, b \in A$ in $\mathbb{M}$ as:*

$$MSMTCH^{\mathbb{M}}(a,b) := \boldsymbol{At} \setminus \{p \in \boldsymbol{At} : p \in V(a) \textit{ iff } p \in V(b)\}.$$

*Further, we define the distance between $a$ and $b$ in $\mathbb{M}$ to be:*

$$DIST^{\mathbb{M}}(a,b) := |MSMTCH^{\mathbb{M}}(a,b)|.$$

Intuitively, the mismatch of agents $a$ and $b$ in signed model $\mathbb{M}$ is the set of all features or properties that the agents do not share. This also includes properties that none of them have. The distance of $a$ and $b$ with respect to the same tied model is the cardinality of the mismatch, i.e. the number of properties $a$ and $b$ do not share. We read that agents with a small distance are more similar and have a higher degree of homophily than agents with a larger distance. Important to note here is that $DIST$ is a distance between *agents*, not *models* as we have considered in earlier chapters. We also observe that $DIST$ is non-negative, symmetric and subadditive.

### 6.1.2 Signed Threshold Models

Using the distance between agents, we define *signed threshold models* as a subclass of signed models in close relation to the similarity updates of [34] and [35]. Now positive and negative relations between agents are decided based on the number of properties they have in common with respect to certain natural number thresholds $\theta_+$ and $\theta_-$. We define signed threshold models accordingly.

**Definition 40** *Let $\theta_+, \theta_- \in \mathbb{N}$ be two given thresholds such that $\theta_+ < \theta_- \leq |\boldsymbol{At}|$. We define a signed threshold model $\mathbb{M}_{\theta+-} = \langle A, R^{\theta+}, R^{\theta-}, V \rangle$ where:*

- *$A$ is a set of agents*
- *$R^{\theta+}$ is a symmetric and reflexive binary relation such that*

$$R^{\theta+} := \{(a,b) \in A \times A : DIST^{\mathbb{M}}(a,b) \leq \theta_+\}$$

- *$R^{\theta-}$ is a symmetric and $R^{\theta+}$-non-overlapping binary relation such that*

$$R^{\theta-} := \{(a,b) \in A \times A : \theta_- < DIST^{\mathbb{M}}(a,b)\}$$

- *$V : A \to \wp(\boldsymbol{At})$ is a valuation function*

*A signed threshold frame $\mathbb{F}_\theta = \langle A, R^{\theta+}, R^{\theta-} \rangle$ is a signed threshold model without valuation.*

We read $aR^{\theta+}b$ as agent $a$ and $b$ have enough in common to be positively related. Similarly, we read $aR^{\theta-}b$ as $a$ and $b$ does not have enough in common to be positively related, and furthermore have so few common properties that they are negatively related instead.

## 6.2 Strong and Weak Ties

The introduction of the social selection aspect of homophily into tied logic is highly related to echo chamber formation, particularly in the context of information exchange through social media. Most social media platforms give users or platform operators the ability to filter out annoying and/or incompatible voices. This is directly related to social selection, and can promote the creation of echo chambers where a shortage of new and opposing information can lead to fragmentation in society [25].

We will as in the former section define a threshold model subclass, now of tied models. Implemented in networks where we have formalized Strong Triadic Closure, strongly tied clusters are now made up of agents not only strongly related, but also similar to a certain degree. The resemblance to echo chambers is only strengthened.

We bring back **TL** and define a subclass of tied models called tied threshold models. As in the case of signed models, we modify the valuation function of a tied model. Recall the definition of a tied model $\mathbb{M} = \langle A, R^S, R^W, V \rangle$ in Definition 34. We alter the valuation function to be defined as $V : A \to \wp(\mathbf{At} \cup \mathbf{Nom})$ such that $\forall i \in \mathbf{Nom}$: if $i \in V(a)$ and $i \in V(b)$, then $a = b$. Thus, for $i \in \mathbf{Nom}$, $[i]^{\mathbb{M},g}$ is now the unique $a$ such that $i \in V(a)$. This again modifies the semantics of **TL** given in Definition 35 in the following fashion.

$$\mathbb{M}, g, a \Vdash p \text{ iff } p \in V(a) \text{ for } p \in \mathbf{At}.$$

### 6.2.1 Tied Threshold Models

Let the definitions of $MSMTCH$ and $DIST$ be as in Definition 39, however now specified for a tied model instead of a signed model. We define the subclass of tied threshold models.

**Definition 41** *Let $\theta_S, \theta_W \in \mathbb{N}$ be two thresholds such that $\theta_S < \theta_W \leq |\boldsymbol{At}|$. We define a tied threshold model $\mathbb{M}_{\theta SW} = \langle A, R^{\theta S}, R^{\theta W}, V \rangle$ where:*

- *$A$ is a set of agents*
- *$R^{\theta S}$ is a symmetric and reflexive binary relation such that*

$$R^{\theta S} := \{(a,b) \in A \times A : DIST^{\mathbb{M}}(a,b) \leq \theta_S\}$$

- *$R^{\theta W}$ is an symmetric, irreflexive and $R^{\theta S}$-non-overlapping binary relation such that*

$$R^{\theta W} := \{(a,b) \in A \times A : \theta_S < DIST^{\mathbb{M}}(a,b) \leq \theta_W\}$$

- *$V : A \to \wp(\boldsymbol{At} \cup \boldsymbol{Nom})$ is a valuation function*

We read $aR^{\theta S}b$ as agent $a$ and $b$ have enough in common to be connected by a strong tie. Similarly, we read $aR^{\theta W}b$ as $a$ and $b$ have enough in common to be connected by a weak tie, but not enough in common to be connected by a strong tie.

We observe that the thresholds interact differently in tied threshold models in comparison to signed threshold models. In the former, for two agents $a$ and $b$ that are not related by either weak or strong tie $\theta_W < DIST^{\mathbb{M}}$, i.e. the agents will not have enough in common to be related by even a weak tie, let alone a strong tie. In the latter we also operate with two thresholds $\theta_+$ and $\theta_-$, however for two agents $a$ and $b$ not related by either positive or negative relation $\theta_+ < DIST^{\mathbb{M}}(a,b) \leq \theta_-$. In other words, $a$ and $b$ do not have enough in common to be positively related, yet too much in common to be negatively related.

### 6.2.2 Observations

To familiarize the reader with the newly defined tied threshold models, we make the following observations.

- $@_i\langle S\rangle j \to (@_i p \leftrightarrow @_j p)$ is valid on tied threshold frames where $\theta_S = 0$;
- $@_i\neg\langle S \cup W\rangle j \to (@_i p \to @_j \neg p)$ is valid on tied threshold frames where $\theta_W = |\mathbf{At}| - 1$.

The first formula expresses that when $\theta_S = 0$, if two agents $i$ and $j$ are strongly tied, then the property $p$ holds at $i$ if and only if $p$ holds at $j$. When $\theta_W = |At| - 1$, two agents that are neither tied strongly nor weakly do not share any properties. The latter formula asserts that if this is the case for two agents $i$ and $j$, if the property $p$ holds at $i$, then it does not hold at $j$.

Further observations where homophily and tied threshold models come into play will be presented in the next chapter where we explore the extension of knowledge and dynamics to **TL**.

## 6.3 Summary

In the penultimate chapter of this thesis before concluding remarks, we implemented the social selection element of homophily in our formal frameworks. Homophily is the social concept that we are like our friends. Social selection is one aspect of homophily suggesting that we choose to form friendships with others who are similar to us. To incorporate these ideas into our previous work, we combined our results with research on social group formation based on agent similarity. We defined a subclass of threshold models for each logic where relations and ties between agents are justified in terms of properties agents have in common. In the section on tied threshold models we also presented some observations of this subclass in particular.

# Chapter 7

# Ties, Knowledge and Dynamics

In this chapter we add a knowledge modality and dynamic operators to **TL** and in consequence, extending it to *tied epistemic logic (**TEL**)*. Our motivation is to investigate the interplay between social concepts like echo chambers and local bridges, change in the network, and how this relates to agents' knowledge of their surrounding context. We present the syntax and semantics of **TEL**, threshold frames and models, and discuss a number of possible axioms up for adoption to further specify the frames within which we want to reason. We also introduce some validities of tied epistemic logic on both frames with and without thresholds, depending on what axioms we choose to adopt. The chapter concludes with an example to accustom the reader to this extensive logic.

## 7.1 Syntax and Semantics

**TEL** is inspired by other epistemic logics for social networks, such as [31] and [32]. Still, **TEL** differs from these on some notable accounts. Firstly, our valuation function in which the range includes nominals depends on epistemic states. This is as we do not want the underlying assumption that every agent in the network knows the name of all other agents. Secondly, we have included dynamic local adding modalities to the language of **TEL**, similar to those of the dynamic extension of **PNL**. Local dynamic operators of this kind are as far as we know yet to be seen in a social network context. We introduce the syntax of **TEL**.

**Definition 42** *(Syntax of **TEL**) Let **At** be a set of propositional atoms and **Nom** be a set of nominals. Further, let **Var** be a set of agent variables. Let **At**, **Nom** and **Var** be countable and pairwise disjoint. We define the well-formed formulas of the language $\mathcal{L}_{TEL}$ to be generated by the following grammar:*

$$\phi ::= p \mid s \mid \neg\phi \mid (\phi \wedge \phi) \mid \langle S \rangle \phi \mid \langle W \rangle \phi \mid [A]\phi \mid K\phi \mid @_s\phi \mid\downarrow x.\phi \mid [\mathbin{/\!\!\!\backslash}S]_L\phi \mid [\mathbin{/\!\!\!\backslash}W]_L\phi$$

*where $p \in$ **At**, $s \in$ **Nom**$\cup$ **Var** and $x \in$ **Var**. We define propositional connectives like $\vee, \rightarrow$ and the formulas $\top, \bot$ as usual. Further, we again define the duals as standard $[S] := \neg\langle S \rangle \neg$ and $[W] := \neg\langle W \rangle \neg$.*

We observe that there are four new operators in the language. The intuitive reading of them are as follows. We read $[A]\phi$ to hold at the current agent if and only if $\phi$ is universally true at all agents in the network. $K\phi$ is intuitively read as the current agent knows that $\phi$. The dynamic modality $[\mathbin{/\!\!\!\backslash}S]_L\phi$ holds at agent $a$ if and only if after adding a strong tie that $a$ previously did not have, $\phi$ is true at $a$. $[\mathbin{/\!\!\!\backslash}W]_L\phi$ is read similarly, although by replacing a strong tie with a weak tie. Before we present the semantics of these operators, we define *tied epistemic models*.

**Definition 43** *(Tied Epistemic Frames and Models) A tied epistemic model is a tuple $\mathbb{M} = \langle W, A, \sim, R^S, R^W, V \rangle$ where:*

- *$W$ is a set of epistemic alternatives,*
- *$A$ is a set of agents,*
- *$\sim$ is a family of equivalence relations $\sim_a$ on $W$ for every $a \in A$,*
- *$R^S$ is a family of symmetric and reflexive relations $R^S_w$ on $A$ for each $w \in W$,*
- *$R^W$ is a family of $R^S$-non-overlapping and symmetric relations $R^W_w$ on $A$ for each $w \in W$, and*
- *$V : W \times A \rightarrow \mathcal{P}($**At**$\cup$ **Nom**$)$ is a valuation function, assigning each agent to a unique name and a set of properties in an epistemic state. I.e, for each $i \in$ **Nom** and for all $w \in W$ and all $a, b \in A$: if $i \in V(w, a)$ and $i \in V(w, b)$, then $a = b$. Additionally, all names correspond to an agent and an epistemic state. That is $\forall i \in$ **Nom**: $\exists a \in A$ and $\exists w \in W$ such that $i \in V(w, a)$.*

*We define a frame $\mathbb{F} = \langle W, A, \sim, R^S, R^W \rangle$ in the usual way, as a model without valuations.*

Again, let $g : \textbf{Var} \rightarrow A$ be an assignment function assigning agents to variables. Furthermore, define the *x-variant* of $g$ to be $g^x_a(x) = a$ and $g^x_a(y) = g(y)$ for all $y \neq x$. We now define the semantics of **TEL**.

**Definition 44** *(Semantics of* ***TEL****) Let* $\mathbb{M}$ *be a model,* $a$ *an agent in* $A$*,* $w \in W$ *an epistemic state and* $g : \boldsymbol{Var} \to A$ *an assignment function. We inductively define the truth conditions as follows:*

$$\mathbb{M}, g, w, a \Vdash p \textit{ iff } p \in V(w,a) \textit{ for } p \in \boldsymbol{At}$$
$$\mathbb{M}, g, w, a \Vdash i \textit{ iff } i \in V(w,a) \textit{ for } i \in \boldsymbol{Nom}$$
$$\mathbb{M}, g, w, a \Vdash x \textit{ iff } a = g(x) \textit{ for } x \in \boldsymbol{Var}$$
$$\mathbb{M}, g, w, a \Vdash \neg\phi \textit{ iff } \mathbb{M}, g, w, a \nVdash \phi$$
$$\mathbb{M}, g, w, a \Vdash \phi \wedge \psi \textit{ iff } \mathbb{M}, g, w, a \Vdash \phi \textit{ and } \mathbb{M}, g, w, a \Vdash \psi$$
$$\mathbb{M}, g, w, a \Vdash \langle S\rangle\phi \textit{ iff } \exists b \in A \textit{ such that } aR^S_w b \textit{ and } \mathbb{M}, g, w, b \Vdash \phi$$
$$\mathbb{M}, g, w, a \Vdash \langle W\rangle\phi \textit{ iff } \exists b \in A \textit{ such that } aR^W_w b \textit{ and } \mathbb{M}, g, w, b \Vdash \phi$$
$$\mathbb{M}, g, w, a \Vdash [A]\phi \textit{ iff } \forall b \in A : \mathbb{M}, g, w, b \Vdash \phi$$
$$\mathbb{M}, g, w, a \Vdash K\phi \textit{ iff } \forall v \in W \textit{ such that } w \sim_a v : \mathbb{M}, g, v, a \Vdash \phi$$
$$\mathbb{M}, g, w, a \Vdash @_s\phi \textit{ iff } \forall b \in A : \mathbb{M}, g, w, b \Vdash s \to \phi \textit{ for } s \in \boldsymbol{Nom} \cup \boldsymbol{Var}$$
$$\mathbb{M}, g, w, a \Vdash \downarrow x.\phi \textit{ iff } \mathbb{M}, g^x_a, w, a \Vdash \phi$$
$$\mathbb{M}, g, w, a \Vdash [\wedge\!\!\!\wedge S]_L\phi \textit{ iff } \exists b \in A \textit{ such that } \neg(aR^S_w b),\ \neg(aR^W_w b) \textit{ and}$$
$$\langle W, A, \sim, R^S_w \cup \{(a,b),(b,a)\}, R^W_w, V\rangle, g, w, a \Vdash \phi$$
$$\mathbb{M}, g, w, a \Vdash [\wedge\!\!\!\wedge W]_L\phi \textit{ iff } \exists b \in A \textit{ such that } \neg(aR^S_w b),\ \neg(aR^W_w b) \textit{ and}$$
$$\langle W, A, \sim, R^S_w, R^W_w \cup \{(a,b),(b,a)\}, V\rangle, g, w, a \Vdash \phi$$

The two dynamic modalities $[\wedge\!\!\!\wedge S]_L$ and $[\wedge\!\!\!\wedge W]_L$ are model changing operators. Their semantics, similar as in Dynamic Epistemic Logic [4] [14], are evaluated by taking into account an updated model in which only the relations $R^S$ or $R^W$ are changed.

## 7.2 Possible Axioms

We might want to add some axioms to narrow down our class of tied epistemic frames. In this section we consider candidates corresponding to some properties we believe put natural constraints on agents in an epistemic context.

The first property we propose is that an agent knows it when it is strongly tied to another agent.

$$\downarrow x.[S] \downarrow y.@_x K\langle S\rangle y \tag{1}$$

Perhaps a bit less likely is the property that an agent knows it when it is weakly tied to another agent. This property depends on what we assign to the term ‘acquaintance’ and the meaning we expect of the knowledge modality.

$$\downarrow x.[W] \downarrow y.@_x K\langle W\rangle y \tag{2}$$

Another reasonable attribute to assume is that an agent knows it when they have a property $p$ or a name $i$, defined by the following two axioms.

$$p \to Kp \tag{3}$$

$$i \to Ki \tag{4}$$

It is also likely to assume that if an agent is strongly tied to another agent whose name is $i$, they know the other agent's name.

$$\downarrow x.[S] \downarrow y.(i \to @_x K @_y i) \tag{5}$$

A further axiom up for discussion is the one defining the property that an agent knows if its strong tie has the property $p$. This is a strong assumption that might only be relevant in certain contexts.

$$\downarrow x.[S] \downarrow y.(p \to @_x K @_y p) \tag{6}$$

Similarly, but perhaps a weaker assumption is that agents know the strong ties of their strong ties. This seems likely when defining strong ties as agents' closest friends.

$$\downarrow x.[S] \downarrow y.[S] \downarrow z.@_x K @_y \langle S \rangle z \tag{7}$$

The last axiom we will consider defines the likely property of agents knowing when they are acquainted by a new agent by a weak tie. Note that this would be implied by Axiom (2) and that the strong tie version of this axiom is implied by Axiom (1).

$$\downarrow x.[\not\!\!\!\bigwedge W]_L [W] \downarrow y.@_x K \langle W \rangle y \tag{8}$$

## 7.3 Tied Epistemic Threshold Models

We proceed to present the definitions of $DIST$ and $MSMTCH$ aligned with the inclusion of epistemic states to the models. Note that for simplicity we will write $V(w,a)$ instead of $V((w,a))$. The set $V(w,a)$ now denotes the set of features, or properties, of agent $a$ in state $w$.

**Definition 45** *(MSMTCH and DIST) Let* $\mathbb{M} = \langle W, A, \sim, R^S, R^W, V \rangle$ *be a tied epistemic model. We define the set of features distinguishing agents* $a, b \in A$ *in state* $w \in W$ *in* $\mathbb{M}$ *as:*

$$MSMTCH_w^{\mathbb{M}}(a,b) := \boldsymbol{At} \setminus \{p \in \boldsymbol{At} : p \in V(w,a) \textit{ iff } p \in V(w,b)\}.$$

*Further, we define the distance between* $a$ *and* $b$ *in state* $w \in W$ *in* $\mathbb{M}$ *to be:*

$$DIST_w^{\mathbb{M}}(a,b) := |MSMTCH_w^{\mathbb{M}}(a,b)|.$$

We can now continue by introducing the notion of tied epistemic threshold frames and models.

**Definition 46** *(Tied Epistemic Threshold Frames and Models) Let $\theta_S, \theta_W \in \mathbb{N}$ be two thresholds such that $\theta_S < \theta_W \leq |\boldsymbol{At}|$. A tied epistemic threshold model is a tuple $\mathbb{M}_{\theta SW} = \langle W, A, \sim, R^{\theta S}, R^{\theta W}, V \rangle$ where:*

- *$R^{\theta S}$ is a family of symmetric and reflexive relations $R^{\theta S}_w$ for each $w \in W$ such that*
$$R^{\theta S}_w := \{(a, b) \in A \times A : \forall u \sim_a w, DIST^{\mathbb{M}}_u(a, b) \leq \theta_S\},$$
- *$R^{\theta W}$ is a family of $R^{\theta S}$-non-overlapping, irreflexive and symmetric relations $R^{\theta W}_w$ for each $w \in W$ such that*
$$R^{\theta W}_w := \{(a, b) \in A \times A : \forall u \sim_a w, \theta_S < DIST^{\mathbb{M}}_u(a, b) \leq \theta_W\}, and$$
- *$W, A, \sim$ and $V$ are defined as in the case of a tied epistemic model.*

*Again, we define a tied threshold frame $\mathbb{F}_{\theta SW} = \langle W, A, \sim, R^{\theta S}, R^{\theta W} \rangle$ as a model without valuation.*

We read $aR^{\theta S}_w b$ as agent $a$ knows in state $w$ that $a$ and $b$ have enough in common to be connected by a strong tie. Similarly, we read $aR^{\theta W}_w b$ as $a$ knows in state $w$ that $a$ and $b$ have enough in common to be connected by a weak tie, but not enough in common to be connected by a strong tie.

## 7.4 Validities

We look at some validities of **TEL** depending on axioms we choose to embrace to get a better understanding of our logic in relation to echo chambers and related social phenomena. Firstly, if we would adopt Axiom (1), the following formula would be a validity on a tied frame.

$$\downarrow x.((\langle S \rangle \downarrow y.@_x \langle S \rangle \downarrow z.@_x K @_z \neg \langle S \cup W \rangle y) \to K \neg \mathbf{STC})$$

This validity says that "*If I am tied to any two successors y and z by strong ties and I know that y and z do not know each other, then I know I am not strongly triadic closed.*" The formula represents a relationship between knowledge and echo chambers. If an agent knows that they do not have the Strong Triadic Closure property, they can derive that it is less likely that they are participating in an echo chamber.

$$\downarrow x.(K \neg \langle S \cup W \rangle \downarrow y.(i \wedge [A](\langle S \cup W \rangle y \to \neg \langle S \cup W \rangle x))) \to K[\not\!\!\!\wedge W]_L(\langle W \rangle i \to \mathbf{LB})$$

This formula states that "*If I know that there is another agent y in the network in which we do not have any friends in common, then I know that if we become acquainted by a weak tie, then I am related to another agent by a weak local bridge.*" Imagine agents find themselves in a place where they suspect an echo chamber has been or were about to be formed. They might have an incentive to get acquainted by a local bridge to receive some new information and hear opposing opinions. Note that the above formula is also valid in the case of strong instead of a weak tie.

The next is a validity on tied threshold frames where we adopt Axiom (6) for all $p \in \mathbf{At}$.

$$\mathbf{STC} \to K\mathbf{STC}$$

This validity is a result of the homophily-motivated definitions of $R^{\theta S}$ and $R^{\theta W}$ and represents that the current agent knows whether they are strongly triadic closed. Strong Triadic Closure is closely related to echo chamber formation. If the agent knows whether they are strongly triadic closed, then principally they would know whether they could be in an echo chamber-like situation.

## 7.5 Example

In the concluding section we present an example of a tied epistemic threshold model and discuss what formulas might hold at specific agents depending on what axioms we adopt to restrict the model.

Consider the tied epistemic threshold model $\mathbb{M}_{\theta SW} = \langle W, A, \sim, R^{\theta S}, R^{\theta W}, V \rangle$ in Figure 7.1. In particular, we observe that $A = \{a, b, c, d, e, f\}$ and $W = \{w, v\}$. For simplicity the reflexive arrows are omitted for $\sim_x$ for all $x \in A$ as well as for $R^{\theta S}_w$ and $R^{\theta S}_v$.

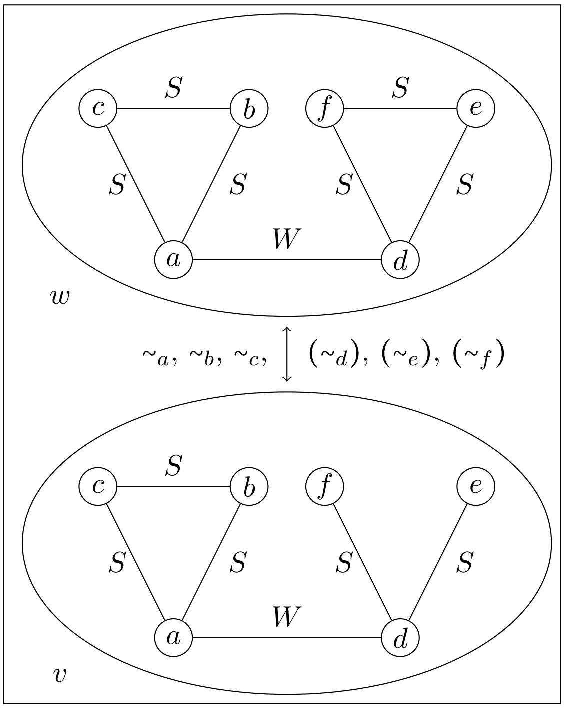


Figure 7.1: A tied epistemic threshold model $\mathbb{M}_{\theta SW}$.

We first regard the model $\mathbb{M}_{\theta SW}$ where $(w \sim_x v) \in \sim_x$ for all $x \in A$. Let $a, b, c, d, e, f \in$ **Nom** such that the corresponding 'name' is true for each agent in $A$ in each epistemic state in $W$. For instance $\mathbb{M}_{\theta SW}, g, w, c \Vdash c$ and $\mathbb{M}_{\theta SW}, g, v, e \Vdash e$ etc. We make, among many, the following observations.

- $\mathbb{M}_{\theta SW}, g, w, a \Vdash K\mathbf{STC}_L$

  Agent $a$ knows in $w$ that it has the Strong Triadic Closure property.

- $\mathbb{M}_{\theta SW}, g, w, d \Vdash \mathbf{STC}_L \wedge \neg K\mathbf{STC}_L$

  Agent $d$ has the Strong Triadic Closure property in state $w$, but does not know it.

- $\mathbb{M}_{\theta SW}, g, v, a \Vdash \mathbf{LB}$

  Agent $a$ is related by a local bridge in state $v$. We see that this also holds in both $w$ and $v$ for agents $a$ and $d$.

- $\mathbb{M}_{\theta SW}, g, w, f \Vdash \langle S\rangle e \wedge \neg K\langle S\rangle e$

  Agent $f$ is strongly tied to agent $e$ in state $w$, but does not know it.

- $\mathbb{M}_{\theta SW}, g, v, e \Vdash [\mathbin{/\!\!\!\backslash\!\!\!\backslash} S]_L(\langle S\rangle f \to @_d K\mathbf{STC}_L) \wedge [\mathbin{/\!\!\!\backslash\!\!\!\backslash} W]_L(\langle W\rangle f \to @_d K\mathbf{STC}_L)$

  If agent $e$ in state $v$ is strongly or weakly tied to $f$ after adding a strong or weak tie respectively, then agent $d$ will know that it then has the Strong Triadic Closure property.

We now notice the following regarding the axioms in the previous section.

- Axiom (1) does not hold in $\mathbb{M}_{\theta SW}$ in both states at $e$ and $f$ in particular. Adjusting $\sim$ such that $\neg(w \sim_e v)$ and $\neg(w \sim_f v)$ is one way to let Axiom (1) hold at all agents in both epistemic states. The reasoning is as follows. Axiom (1) lets agents $e$ and $f$ know that they are related in state $w$, and not in state $v$; $\mathbb{M}_{\theta SW}, g, w, e \Vdash K\langle S\rangle f$ and $\mathbb{M}_{\theta SW}, g, v, e \Vdash K\neg\langle S\rangle f$. Thus if both agents can distinguish between states $w$ and $v$, the axiom holds.

- Neither Axiom (6) nor Axiom (7) is forced at agent $d$ in either epistemic state. Letting $\neg(w \sim_d v)$ would make either axiom true. Axiom (7) restricts the model such that every agent knows the strong ties of their strong ties. If this is the case then $\mathbb{M}_{\theta SW}, g, w, d \Vdash K\mathbf{STC}$ while $\mathbb{M}_{\theta SW}, g, v, d \Vdash K\neg\mathbf{STC}$. Letting agent $d$ distinguish between $w$ and $v$ would solve this problem. Axiom (6) make agents know the properties of their strong ties. As $\mathbb{M}_{\theta SW}$ is a tied epistemic threshold model, agent $d$ would know by the properties of its strong ties $e$ and $f$ that $eR^S_w f$ whereas $\neg(eR^S_v f)$. Thus agent $d$ would again know whether it has the Strong Triadic Closure property. A contradiction is also avoided here when we let agent $d$ be able to distinguish between states $w$ and $v$.

## 7.6 Summary

In this chapter we presented the logic **TEL**, the epistemic and dynamic extension of **TL**. We introduced the language and semantics of a now complex and highly expressive logic and presented tied epistemic frames both restricted and unrestricted by thresholds. Then we discussed possible restrictions to tied epistemic frames and validities depending on these restrictions. We concluded the chapter with an example where we assessed what consequences adopting particular axioms would have in this setting.

# Chapter 8

# Further Directions and Concluding Remarks

This concludes our investigation thus far. After introducing the social network theory behind structural balance and the logic of positive and negative relations known from the literature, we set out to expand this logical framework with several intentions in mind. To explore measures of how far a network is from polarization, we considered and compared a variation of distances in relation to balance. We presented a number of additions to **PNL** to be able to define previously undefinable frame properties in the original work. In particular, we presented a dynamic characterization of the balance property. By further extension of dynamic modalities we developed a framework to analyze change with regards to signed social networks. Then, we moved away from positive and negative relations and defined tied logic: a hybrid logic of strong and weak ties. Closely related to the social phenomenon of echo chambers, we formalized essential properties from social psychology and showed that a known claim in this field is a validity in tied logic. We provided an axiomatization of tied logic and proved soundness and strong completeness with respect to tied frames. Inspired by work on social group formation we defined threshold frames in which relations are justified on the basis of features agents in a network share. Lastly, we introduced tied epistemic logic by extending tied frames with epistemic states and the language of tied logic with dynamic and epistemic operators. We discussed possible axioms to narrow down the class of tied epistemic frames, and presented valid formulas showing the interplay between dynamics and reasoning in significant social settings.

There are a number of paths in which we can continue the work of this thesis. We will address a selection of possible proposals for future work.

**Completeness Results for Extended PNL and TEL.** A natural place to continue this work is to explore further technical results of our extended dynamic logics, and in particular assess potential completeness proofs. It remains to see if the results are directly connected to the open problem of axiomatization of sabotage modal logic [3].

**Echo Chambers and Filtering Mechanisms.** Important to remark is that our presentation of echo chambers as strongly tied clusters is not the only way in which one can formalize the concept. Other formalizations (such as [18]) include a filtering mechanism, representing the habit of filtering out extraneous information within echo chambers. Such a mechanism could be engaging to implement into our framework and can possibly be done by taking into account the 'selective learning principle' that is formally explored in [5].

**Computational Results.** Beyond the scope of this thesis are various investigations of computational results related to the work. A suggestion for future work is the computational complexity of the different measures of balanced distance.

**Homophily and Social Influence.** In our work we implemented only the social selection component of homophily. Another angle to approach in a logical framework is the social influence side; that we become like our friends.

**Other Agent Attributes.** Interesting to explore are further attributes agents in a social network can hold in addition to knowledge, such as beliefs, preferences or trust. Trust would be especially interesting to investigate in relation to webpage rankings as explored in for instance [16]. We find several models of trust in the literature (e.g. [29]) for future application to our logical framework.

**Public Announcements and Common Knowledge.** On a final note, our work motivates a further investigation of change in social networks. This could be done by exploring other validities, but also by the inclusion of additional dynamic operators. A possible approach is to extend our multi-agent dynamic framework with features of communication such as public announcements and group knowledge such as common knowledge [4], [14].

# Appendix A

# Appendix

## A.1 Chapter 2

### A.1.1 Proof of Theorem 6

**Theorem 6**

*(Local Balance ⇒ Global Balance) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame. If there exists a collectively connected signed frame $\mathbb{F}' = \langle A', R^{+'}, R^{-'} \rangle$ such that $A = A'$, $R^+ \subseteq R^{+'}$ and $R^- \subseteq R^{-'}$ that has the local balance property, then $\mathbb{F}$ has the global balance property.*

*Proof.* Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame. Assume that there is a collectively connected signed frame $\mathbb{F}' = \langle A', R^{+'}, R^{-'} \rangle$ such that $A = A'$, $R^+ \subseteq R^{+'}$ and $R^- \subseteq R^{-'}$ that has the local balance property. If $\mathbb{F}'$ has the global balance property, it will stay globally balanced for any subset of $R^{+'}$ and $R^{-'}$. It follows that also $\mathbb{F}$ will have the global balance property. We will therefore prove that $\mathbb{F}'$ has the global balance property and our desired result will follow directly.

Pick an arbitrary $a \in A'$. Recall that as $\mathbb{F}'$ is collectively connected, $\forall b \in A'$ either $aR^{+'}b$ or $aR^{-'}b$. Now, define $S$ such that $\forall b \in A'$ : if $aR^{+'}b$, then $b \in A'$ and if $aR^{-'}b$, then $b \in A' \setminus S$.

Since $\mathbb{F}'$ is positive reflexive $aR^{+'}a$, and we have that $aR^+a$. Thus $a \in W$.

We want to prove that $\forall w, v \in A'$ :

- if $wR^{+'}v$, then $w, v \in S$ or $w, v \in A' \setminus S$, and
- if $wR^{-'}v$, then $w \in S$ and $v \in A' \setminus S$, or $w \in A' \setminus S$ and $v \in A'$.

Let $w$ and $v$ be arbitrary in $A'$.

- Assume that $wR^{+'}v$. Then by symmetry $wR^{+'}v$. There are two cases to consider:

  - $aR^{+'}w$. Then $w \in S$. By the local balance property, as $aR^{+'}w$ and $wR^{+'}v$, then $aR^{+'}v$. Thus $w, v \in S$.

  - $aR^{-'}w$. Then $w \in A' \setminus S$. By the local balance property, as $aR^{-'}w$ and $wR^{+'}v$, then $aR^{-'}v$. Thus $w, v \in A' \setminus S$.

- Assume that $wR^{-'}v$. Then $wR^{-'}v$. Again, there are two cases to consider:

  - $aR^{+'}w$. Then $w \in S$. By the local balance property, as $aR^{+'}w$ and $wR^{-'}v$, then $aR^{-'}v$. Thus $w \in S$ and $v \in A' \setminus S$.

  - $aR^{-'}w$. Then $w \in A' \setminus S$. By the local balance property, as $aR^{-'}w$ and $wR^{-'}v$, then $aR^{+'}v$. Thus $w \in A' \setminus S$ and $v \in S$.

We have proved that $\mathbb{F}'$ has the global balance property, and thus directly that $\mathbb{F}$ is globally balanced too. □

*(Global Balance ⇒ Cyclic Balance) If a signed frame $\mathbb{F} = \langle A, R^+, R^- \rangle$ has the global balance property, then it has the cyclic balance property.*

*Proof.* Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame. Assume that $\mathbb{F}$ is globally balanced. Suppose for reductio that $\mathbb{F}$ does not have the cyclic balance property. Then $\exists a_1, \ldots, a_m \in A$ such that $a_1 R^{x_1} \ldots R^{x_{i-1}} a_m R^{x_i} a_1$ for $x_n \in \{+, -\}$ and $|\{(a_s, a_t)_- \mid 1 \leq s < t \leq m\}| = \frac{2n}{1}$ for $n \in \mathbb{N}^+$.

Assume without loss of generality that $a_1 \in S$. By the global balance property of $\mathbb{F}$, if $a_1 R^+ a_2$ then $a_2 \in S$ and if $a_1 R^- a_2$ then $a_2 \in A \setminus S$. Similarly, if $a_1 R^+ a_2 R^- a_3$ then $a_1, a_2 \in S$ and $a_3 \in A \setminus S$, and if $a_1 R^- a_2 R^- a_3$ then $a_1, a_3 \in S$ and $a_2 \in A \setminus S$. And so on. In the cycle of relations from $a_1$, call the last negative relation in the cycle before reaching $a_1$ for $(a_j, a_k)_-$. Since the number of negative relations in the cycle is odd, it follows that $a_j \in S$ and $a_k \in A \setminus S$. We also have that $a_k R^+ \ldots R^+ a_1$, thus either $a_k, a_1 \in S$ or $a_k, a_1 \in A \setminus S$. This is a contradiction. Hence, we conclude that $\mathbb{F}$ must have the cyclic balance property. □

# Appendix B

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