Questioning Philosophy

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis develops a nominalist approach for the area of Philosophical Logic that deals with existence and identity.
It does so from an epistemic perspective, with the objective of describing what a philosopher can call knowledge when discussing such topics.
One of the features of the Philosophical Inquisitive Logic (PhIL) here outlined is given by techniques from Inquisitive Logic. Such semantics is expressive enough to characterize not only declarative statements but also questions, in the optics that within philosophy a good unanswered question is as important as a good answer.
The other non-classical approach used is Partial Semantics, that makes it possible to deal with multivalued logics.

Acknowledgments

To quote Leopardi, what you are about to read are:

> "The toilsome papers where
> my prime was being consumed,
> the best of me"
> -A Silvia, Giacomo Leopardi

But no person is an island. Or, if they are, they probably live in an archipelagos.
What I mean is that surviving this Master and Thesis had no little effect on my surroundings, and it greatly affected my work and my life.

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"Not all questions are answered, commander,
but fortunately some answers are questioned."

- Snuff, Terry Pratchett

Contents

Introduction

The objective of this thesis is to develop a formal semantics that can merge a nominalist approach to Philosophical Logic and Epistemic Logic, in order to have a philosophically satisfactory characterization of knowledge within a given philosophical theory as ontologically neutral as possible.
The aim of such approach is to be metaphysically innocent, in order to provide a flexible *"logical toolkit"* for counterfactual and modal reasoning.
Two different non-classical approaches converge in this work: Inquisitive Semantics, worthy of a brief introduction since it is an ever expanding and recent field, and Partial Logic, which will be employed non-standardly.
The former will be introduced in chapter 1 on page 7 and I will be particularly interested in its Epistemic expansion, that will serve as core of this project. Inquisitive Semantics has been developed mainly by Ivano Ciardelli, Jeroen Groenendijk, and Floris Roelofsen; its expressive power makes it possible to treat not only declarative sentences but also questions. Partial Logic will be introduced in chapter 3 on page 31 in response to the problems and proposed solutions discussed in chapter 2 on page 21.

In the rest of this chapter I will discuss how the primitive concepts of "knowledge" and "agent" are interpreted in standard Epistemic Logic and how they should be read within the system that will be discussed in chapter 4 on page 59: Philosophical Inquisitive Logic (PhIL).

Reasoning Agents

The first line of the voice "Epistemic Logic" in the Stanford Encyclopedia of Philosophy says:

> "Epistemic logic is the logic of knowledge[...]."

However it seems quite important for *knowledge*, in the narrow understanding of the word, to have *a knower*. This is why after reviewing the properties classically associated with knowledge I will consider which sort of agent they portray and from there I will discuss how different they are from an agent of PhIL. If Aesop said "A man is known by the company he keeps", I say "An Epistemic Logic is known by the agents it describes".

The Properties of Knowledge

A standard approach to Epistemic Logic is described in the introductory sections of [Hans Van Ditmarsch, 2015]. I will assume familiarity with the formal aspects of Modal Logic and standard Epistemic Logic here and just discuss why a certain formalization is employed and which kind of philosophical interpretation it has.

The modal system $\mathcal{S}5$, a proper extension of $\mathcal{K}$, is used to capture what Van Ditmarsch calls the 'properties of knowledge'. The Rules and axioms of $\mathcal{S}5$ are the following, for every agent $a \in \mathcal{A}$:

Nec. Rule	if $\vdash \varphi$ then $\vdash K_a\varphi$	*Necessitation*
Axiom $\mathcal{K}$	$\vdash K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	*Distributivity*
Axiom $\mathcal{T}$	$\vdash K_a\varphi \to \varphi$	*Veridicity*
Axiom 4	$\vdash K_a\varphi \to K_aK_a\varphi$	*Positive Introspection*
Axiom 5	$\vdash \neg K_a\varphi \to K_a\neg K_a\varphi$	*Negative Introspection*

Figure 1: Axioms and Rules in Classical Epistemic Logic

Given such a characterization, these agents have of course a very peculiar type of knowledge.

In particular *Necessitation* tells us any agent knows every theorem provable in their theory, while *Distributivity* let them know any consequence that their personal knowledge entails.

Veridicity says that we can call knowledge only what actually happens to be true, *Positive* and *Negative Introspection* that our agents have a perfectly clear access to their own body of knowledge, and they are always able to specify what they know and what they do not know. From this brief description we may notice that the knowledge we are addressing is both *potential* and *ideal* in the literal sense.

This conception is similar to a Platonic Epistemology: the most iconic example of it can be met in the Socratic dialogue *Meno* [Bluck, 1961]. There Socrates shows that a slave can understand how to draw a square twice the area of another, even if he lacks of *mathematical maturity*.
For Plato, Meno's slave *remembers* theorems that he knew a priori rather than *discovering* or *learning* them, proving that the knowledge of mathematical truth was inside him all along. The provable mathematical truth "The area of a square b built on the diagonal of a square a is twice the area of a" is everyone's potential knowledge.

The *Veridicity* of knowledge appears to be a really intuitive constraint. Consider the following sentence:

(1) "Psyche knows that the creature who kidnapped her is a monster. However, he is actually Eros the god of love."

The sentence as it is sounds semantically infelicitous at the very least, if not straightforwardly contradictory. What we are far more prone to admit is that "Psyche believes that the creature who kidnapped her is a monster". *Positive*

Introspection is similar to the example of Meno's slave: given enough time and help remembering he will be able to know that he knows *many cheerful facts about the square of the hypotenuse.* This confirms the idea of potentiality of knowledge.

Negative Introspection is probably the black sheep among the properties of knowledge. It states that if an agent does not know something then they know that they do not know it. Of the aforementioned properties of knowledge it is probably the most debatable axiom, even when considering an ideal and potential interpretation of knowledge. Consider for example the sentence:

(2) "King Priam does not know that the Trojan Horse is full of Greek warriors.
Therefore, King Priam knows that he does not know whether the Trojan Horse is full of Greek warriors."

It seems too strong to be forced to admit that just from being ignorant about some proposition we can deductively know we are ignorant, even if we have no disposition toward the object of our ignorance. I will address this problem again in section 1.3.4 on page 17 along with an alternative proposal that leverages on Inquisitive Semantics.

Inquisitive and Classical Agents

We can conclude that this general account of Classical Epistemic Logic is at its core the "logic of perfectly self aware logicians".
It is not particularly problematic if some predictions are not met by actual epistemic agents or if they are far too ideal: *an agent in our theory is to an actual logician what a Turing Machine is to any actual machine able to manipulate symbols.*

In INQ Epistemic Logic the properties of knowledge in figure 1 on the preceding page are preserved unchanged in the declarative fragment. The main difference is in term of the modal operator K_a is expressibility: it is possible, for example to say that an INQ agent can know the answer of a question.
Moreover, it is possible to introduce an inquisitive modality $E_a\varphi$, read as "agent a entertains φ". This modality is equivalent to knowledge for declarative formulas; however, an agent may *entertain* a question without *knowing* how to answer it.
These concepts will be clarified in section 1.3.4 on page 15, and for an even more detailed account the reader can refer to [Ciardelli, 2014]. In the article the properties of K_a and E_a are characterized as shown in figure 2 on the next page.

For $\oplus \in \{K_a, E_a \mid a \in \mathcal{A}\}$ and α declarative

Nec. Rule	if $\vdash \varphi$ then $\vdash \oplus\varphi$	*Necessitation*
Axiom $\mathcal{K}$	$\vdash \oplus(\varphi \to \psi) \to (\oplus\varphi \to \oplus\psi)$	*Distributivity*
Axiom $\mathcal{T}$	$\vdash E_a\alpha \to \alpha$	*Veridicity*
Axiom 4	$\vdash E_a\varphi \to E_aE_a\varphi$	*Positive Introspection*
Axiom 5	$\vdash \neg E_a\varphi \to E_a\neg E_a\varphi$	*Negative Introspection*
Equivalence	$\vdash E_a\alpha \leftrightarrow K_a\alpha$	*K-E equivalence on declaratives*
K? distrib.	$\vdash K_a?\{\alpha_1, \ldots, \alpha_n\} \to K_a\alpha_1 \vee \ldots \vee K_a\alpha_n$	*K distribution over interrogatives*

Figure 2: Axioms and Rules in INQ Epistemic Logic

Philosophically Inclined Agents

Once the standard account has been clarified we can analyze how Philosophically Inclined Agents (PIA) are different from their classical counterparts.

Even assuming that questions in philosophy are as important as answers there are other reasons why "going inquisitive" is the best choice.

To answer the question "What is a Philosophically Inclined Agent?" is to answer the question "What can a Philosophically Inclined Agent know?". Well, I can start with an easy remark: *if they know more it is only because they can express more, and they see the world as infinitely more complex than their non-philosophical counterparts* (nPIA) *do.*

A PIA uses concept like "necessary" and "contingent", entertains theories about counterfactual reasoning, is able to reason about what exists and what does not and says that something is true only if it is verifiable.

When a PIA and a nPIA have a conversation, the latter believes that the informative content of whatever the formers adds is either trivial or meaningless, but they share a common ground on *every-day life facts.*

Most of the things that a PIA can express in the fragment that is theirs alone is 'armchair knowledge' in the most classical sense.
Assume for example they do not remember every detail from their class in Philosophy of Language, but they know:

(3) "necessarily, either Hesperus is equal to Phosphorus or Hesperus is different from Phosphorus"

Rather than considering this as a statement about the objects involved they would consider it a fact about how the names involved are used in counterfactual arguments.

They also heard about Descartes' *"cogito, ergo sum"*, and now they know they exist. However, following Kripke they also know their existence is not necessary.

All this body of *philosophical knowledge* they have depends of course on their standpoint on any of these debates: rather than keepers of a **Philosophical Truth** they express consistently their own **Philosophical perspective**.
They know they are, as regular philosophers, committed to their ideas, and what I mean by this is saying that they describe how a philosopher committed to such and such view should reason.

1. Informative and Inquisitive Content

There are various way to describe the meaning of a declarative sentence in a truth-conditional semantics. In particular, in Possible Worlds (PW) semantics we are given a *Logical Space* $\mathcal{W}$, a set of worlds we *could* be living in, among which we have to find which one is the real one.
Given a sentence in our language its intension is classically taken to be the set of all PW where said sentence is true. When we seek and use information we can interpret that as a way of locating ourself in the Logical Space with more precision.
For example, take an action like looking outside the window and recognizing that the sentence "It is raining" is true. Once we do it, we can conclude that we are located in that portion of the logical space where the sentence is true. In this way we can link sentences to the information that a speaker conveys in uttering them.
The goal of inquisitive semantics is to extend such account and deal not only with purely informative statements, but also with questions. This means that a sentence may not only contains information about how the world is, but also raise an issue about it.

Definition 1 [Informative and Inquisitive Content]

Given a Language $\mathcal{L}$, For any sentence φ we define

***Informative Content* of** φ the intension of φ in the logical space

***Inquisitive Content* of** φ the issue that φ raises

Take two sentences like the following, where the disjunction is interpreted as inclusive:

(4) Marco plays baseball on Thursday and on Tuesday

(5) Marco plays baseball on Thursday or on Tuesday

The two sentences have of course a different informative content. While example 4 is true if and only if Marco plays baseball both on Thursday and on Tuesday, for example 5 to be true we only require that he plays on either of those days.
However, one may say that while the first sentence has nothing more to it than what it is asserting, example 5 leaves open an issue: namely, which disjunct makes it true.

Similarly, consider sentences involving quantification:

(6) Everyone plays baseball

(7) Someone plays baseball

Once again we can see how given the truth of the former there seems to be no issue worth to be discussed, while the truth of the latter raises the question "Can you give me an example?".

The interpretation of such "open questions" left by disjunction and existential quantification in INQ is that such operators introduce inquisitiveness. However, before getting to the technical notions of INQ it is useful to introduce some more general concepts and terms that will be used even further on in the thesis.

1.1 General Notions

1.1.1 Information State

An information state is defined standardly as a set of *Possible Words*. What a set of PW represent is a body of information: namely, those statements that are true in all the worlds contained in it. As seen in the previous section we can denote the entirety of our Logical Space with $\mathcal{W}$, the set of all PW.

Definition 2 [Information State]

s is an information state iff $s \subseteq \mathcal{W}$

It can help to think about states as ways of finding the actual world in a more specific location of the logical space. Under such interpretation it is easy to see how an informative state t contains at least as much information as s if it locates the actual world with at least as much precision. To express this concept we say that t is an enhancement of s.

Definition 3 [Enhancement]

t enhances s iff $t \subseteq s$

From the few definitions we have seen so far we can see why $\mathcal{W}$ can be referred to as the *ignorant state*, since every state enhances it. Another state worth addressing is the empty state $\emptyset$, that is enhancement of any other state and is called the *inconsistent state*.

1.1.2 Issue

An issue is identified by the set of all the information states that contain enough information to resolve the issue itself. It is assumed that if s resolves an issue I, then also every enhancement of s does. This is expressed using the operation of downward-closure; if $A \subseteq \mathbb{P}(\mathcal{W})$:

$$A^{\downarrow} = \{s \subseteq \mathcal{W} \mid \exists s' \in A.\ s \subseteq s'\}$$

Definition 4 [Issue]
An issue is a non-empty and downward-closed set of information states.
I is an issue iff for some $S \subseteq \mathbb{P}(\mathcal{W})$, $S \neq \emptyset$ and $I = S^{\downarrow}$

Definition 5 [Resolving a Issue]
An information state s resolves an issue I if and only if s is an element of I.
We say that I is *settled* in s iff $s \in I$

For example the issue embodied by example 5 on page 7 can be solved not only by the state s that confirms Marco plays on Thursday, but also by any more specific state $t \subseteq s$ (Ex. He plays only on Thursday).
I will provide more examples of issues and resolutions in figure 1.2 and 1.3 on page 15.

We also have for issues the equivalent of what enhancement was for information states. We call this relation *refinement*, it compares two issues based on which information states would settle them.

Definition 6 [Issue Refinement]
An issue J is at least as inquisitive as I if every state that settles J also settles I.
J *refines* I iff $J \subseteq I$

Given an issue I we call *alternatives* in I all the maximal information states s that settle it, i.e. the least amount of information we need to settle the issue. Set theoretically, these are called the maximal elements of I. f

Definition 7 [Alternatives in an issue]
The alternatives in I are all maximal elements of an issue I
s is an *alternative* in I iff $s \in I$ and there is no $t \in I$ such that $s \subsetneq t$

1.2 Content

Going back to the notion of informative and inquisitive content we can now characterize it in a given language $\mathcal{L}$ of Inquisitive Semantics.

The Inquisitive Content of φ, denoted as $[\varphi]$ is a non-empty, downward closed set of information states.
By definition we have that $[\varphi]$ is always an issue.

The Informative Content of φ, denoted as $|\varphi|$ is the union of all the information states in $[\varphi]$.
By definition we have that $|\varphi|$ is always an information state.

1.2.1 Informativeness and Inquisitiveness

A sentence φ will be said to be inquisitive if its informative content is not enough to resolve the issue it raises. In the following sections I will sometimes refer to inquisitive sentences as *inquisitive statements* or *questions* while I call non-inquisitive sentences also *declarative statements*.
Further distinctions among sentences can be made. A sentence can be informative in case its informative content is not trivial, i.e. more informative than the ignorant state.
We call *tautology* a proposition that is neither inquisitive nor informative.
More formally we have the following equivalences:

Definition 8 [Informative and Inquisitive sentences]

For any $\varphi \in \mathcal{L}$,

- φ is inquisitive iff $|\varphi| \notin [\varphi]$ iff $[\varphi]$ has no supremum
- φ is informative iff $|\varphi| \subsetneq \mathcal{W}$
- φ is a tautology iff $\mathcal{W} \in [\varphi]$

1.2.2 Algebraic Notions

Now it is possible to define entailment and algebraic operations on informative and inquisitive content. For a thorough discussion on the topic the reader is invited to check [Ciardelli and Roelofsen, 2011]

In short, we can define *entailment* between sentences as inclusion in the set-theoretic sense, both for states and for issues.

Definition 9 [Classical Entailment]
We say that φ classically entails ψ iff $|\varphi|$ enhances $|\psi|$
For any $\varphi, \psi \in \mathcal{L}$, $\varphi \Vdash \psi$ iff $|\varphi| \subseteq |\psi|$

Definition 10 [Inquisitive Entailment]
We say that φ entails in INQ ψ iff $[\varphi]$ refines $[\psi]$
For any $\varphi, \psi \in \mathcal{L}$, $\varphi \models \psi$ iff $[\varphi] \subseteq [\psi]$.

The algebra for inquisitive semantic can be characterized defining *meet* and *join* in terms of intersection and union. Furthermore, since it forms a complete Heyting algebra, we have that for every two issues $[\varphi]$ and $[\psi]$, there is a unique weakest sentence $[\rho]$ such that $[\varphi] \cap [\rho]$ entails $[\psi]$. We call such $[\rho]$ the *Relative Pseudo-Complement* of $[\varphi]$ relative to $[\psi]$, and we write it as $[\varphi] \Rightarrow [\psi]$. The existence of the Relative Pseudo-Complement follows from its set-theoretic definition:

Definition 11 [Meet, Join and Relative Pseudo-Complement]

For any $\varphi, \psi \in \mathcal{L}$,

$$[\varphi] \cap [\psi] \overset{def}{=} \{s | s \in [\varphi] \text{ and } s \in [\psi]\}$$

$$[\varphi] \cup [\psi] \overset{def}{=} \{s | s \in [\varphi] \text{ or } s \in [\psi]\}$$

$$\varphi \Rightarrow \psi \overset{def}{=} \{s | \text{ For all } t \subseteq s; \text{ if } t \in [\varphi] \text{ then } t \in [\psi]\}$$

Applying the definition $|\varphi| \overset{def}{=} \bigcup[\varphi]$ it is possible to recover the operators on informative content. With such characterization of entailment we have that, for an inquisitive language $\mathcal{L}$, $\langle [\varphi]_{\mathcal{L}}, \models \rangle$ forms a complete Heyting algebra, while $\langle |\varphi|_{\mathcal{L}}, \Vdash \rangle$ is a Boolean Algebra.
Now that the terminology and intended reading of the primitive concepts is fixed it is possible to talk about how *Semantic Contents* and *Support Conditions* are defined in standard *First-Order INQ Epistemic Logic.*

1.3 First-Order Inquisitive Logic

1.3.1 The Language

The language $\mathcal{L}$ for standard First-Order Inquisitive Logic (FOINQ) as described in [Ciardelli, 2015] is based on a signature that consists in a set of function symbols $\mathcal{F}$ and a set of relation symbols $\mathcal{R}$, with arity $n \geq 0$. For simplicity in this thesis I will consider a language with only zero place functions $n \in \mathfrak{N}$, that we can call names, and variables $x \in \texttt{Var}$. The logical base are the connectives $\vee\!\!\!\vee$ (aka, the intuitionist/inquisitive disjunction) and $\rightarrow$ (implication); the constant $\bot$ (false), the quantifier $\overline{\exists}$ (aka, the intuitionist/inquisitive existential) and the families of modal operators K_a (Agent a knows) and E_a (Agent a entertains), with index $a \in \mathfrak{Ag}$.

We can recursively define *terms* t and *well formed formulas* φ as follows, assuming $n \in \mathfrak{N}$, $R^n \in \mathcal{R}$, $a \in \mathfrak{Ag}$, $\vec{t}^{\,n}$[1]:

$$t \overset{def}{=} n \mid x$$

$$\varphi \overset{def}{=} R^n(\vec{t}^{\,n}) \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid \varphi \rightarrow \varphi \mid \overline{\exists} x.\varphi \mid K_a\varphi \mid E_a\varphi$$

Other operators can be defined as follows:

$\neg\varphi \overset{def}{=} \varphi \rightarrow \bot$	$\top \overset{def}{=} \neg\bot$	$\forall x.\varphi \overset{def}{=} \neg\overline{\exists} x.\neg\varphi$	$\varphi \leftrightarrow \psi \overset{def}{=} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
$!\varphi \overset{def}{=} \neg\neg\varphi$	$?\varphi \overset{def}{=} \varphi \vee\!\!\!\vee \neg\varphi$	$\varphi \vee \psi \overset{def}{=} !(\varphi \vee\!\!\!\vee \psi)$	$\exists x.\varphi \overset{def}{=} !\overline{\exists} x.\varphi$

In section 1.3.3 on page 14, after having introduced the semantics, I will discuss how the different operators introduce and preserve inquisitiveness.

[1] $\vec{t}^{\,n}$ is an abbreviation for $\langle t_1, \ldots, t_n \rangle$

$$\mathcal{M} = \langle \mathcal{W}, \{\Sigma_a | a \in \mathfrak{Ag}\}, \mathcal{D}, \mathcal{I} \rangle$$

$\mathcal{W}$ non-empty set of possible worlds.

$\mathcal{D}$ non empty set whose elements are called individuals

$\Sigma_{\mathfrak{Ag}}$ $=\{\Sigma_a | a \in \mathfrak{Ag}\}$ a set of *inquisitive state maps* for every agent a in $\mathfrak{Ag}$, such that $\Sigma_a : \mathcal{W} \to \mathbb{P}(\mathbb{P}(\mathcal{W}))$.

$\Sigma_a(w)$ a *downward-closed* set of information states: a's inquisitive state at w

$\sigma_a(w) \overset{def}{=} \bigcup \Sigma_a(w)$: $a's$ informative (or epistemic) state at w.

$\mathcal{I}$ is a map $\mathcal{W} \to \mathcal{I}_w : w \mapsto I_w$ s.t.:

- for every name $n \in \mathfrak{N}_L$, $I_w(n) \in \mathcal{D}$
  Moreover, for every $w, v \in \mathcal{W}$ and any $n \in \mathfrak{N}_L$,
  $I_w(n) = I_v(n)$
- for every n-ary relation symbol $R \in \mathcal{R}_L$, $I_w(R) \subseteq \mathcal{D}^n$

Figure 1.1: First-Order INQ Epistemic Model

1.3.2 Semantics

A First-Order INQ Epistemic model is defined as the quadruple $\mathcal{M} = \langle \mathcal{W}, \{\Sigma_a | a \in \mathfrak{Ag}\}, \mathcal{D}, \mathcal{I} \rangle$, described in figure 1.1. On such model we can specify in which set of worlds $w \in \mathcal{W}$ a well formed formula φ is *classically true*, i.e. φ's truth-set.
The truth-set of φ corresponds to what we have been calling φ's Informative Content.
In this chapter I will only introduce truth-conditions for the primitive operators and treat the truth-set of an expression as function of the truth-conditions. It is however possible to give them a separate characterization and prove their correspondence, as done in previously mentioned articles.

In general, for any $\varphi \in \mathcal{L}$:

We write $\mathcal{M}, w \Vdash_g \varphi$ iff in $\mathcal{M}$, φ is true at w under the assignment g

We call φ's truth-set $|\varphi|^g_{\mathcal{M}}$ s.t $|\varphi|^g_{\mathcal{M}} \overset{def}{=} \{w \in \mathcal{W} | \mathcal{M}, w \Vdash_g \varphi\}$

The assignment $g : \mathtt{Var} \to \mathfrak{N}$ fixes the interpretation of $x \in \mathtt{Var}$; we write $g[x \mapsto n]$ for the assignment that maps x to n. The referent of a term t in w under an assignment g is defined as follows:

$[x]^g_w$ coincides with $g(x)$
$[n]^g_w$ coincides with n

Definition 12 [Truth-Set of Atom]

For any $R^n(\vec{t}^n) \in$ `Atom`; the truth set of $R^n(\vec{t}^n)$ in $\mathcal{M}$ relative to the assignment g is denoted as $|R^n(\vec{t}^n)|^g_{\mathcal{M}}$.

- $|R^n(\vec{t}^n)|^g_{\mathcal{M}} = \{w \in \mathcal{W} | \langle I_w([t_1]^g_w), \ldots, I_w([t_n]^g_w)\rangle \in I_w(R^n)\}$

For brevity I will specify the model $\mathcal{M}$ and assignment g only if needed or in case a new notation is introduced.

The inductive clauses for the base of operators are the following:

$\mathcal{M}, w \Vdash_g R^n \vec{t}^n$	iff	$w \in |R^n \vec{t}^n|^g_{\mathcal{M}}$
$\mathcal{M}, w \Vdash_g \bot$		never
$\mathcal{M}, w \Vdash_g \varphi \wedge \psi$	iff	$\mathcal{M}, w \Vdash_g \varphi$ and $\mathcal{M}, w \Vdash_g \psi$
$\mathcal{M}, w \Vdash_g \varphi \vee\!\!\!\vee \psi$	iff	$\mathcal{M}, w \Vdash_g \varphi$ or $\mathcal{M}, w \Vdash_g \psi$
$\mathcal{M}, w \Vdash_g \varphi \to \psi$	iff	if $\mathcal{M}, w \Vdash_g \varphi$ then $\mathcal{M}, w \Vdash_g \psi$
$\mathcal{M}, w \Vdash_g \overline{\exists} x.\varphi$	iff	There is a $n \in \mathfrak{N}$ such that $\mathcal{M}, w \Vdash_{g[x \mapsto n]} \varphi$
$\mathcal{M}, w \Vdash_g K_a\varphi$	iff	for all $w \in \sigma_a(w)$ $\mathcal{M}, s \Vdash_g \varphi$
$\mathcal{M}, w \Vdash_g E_a\varphi$	iff	for all for all $t \in \Sigma_a(w)$, $t \models_g \varphi$

These truth conditions for the modal K_a and the other non-modal operators are the 'classical'. By which I mean that under a natural *renaming* every formula PW-valid in INQ is also valid in classical first-order logic, and vice-versa. In particular we can define classical entailment and validity.

Definition 13 [Classical Entailment and Classical Validity]

$\varphi \Vdash \psi$ iff For every $\mathcal{M}$, g and $w \in \mathcal{W}$, if $\mathcal{M}, w \Vdash_g \varphi$ then $\mathcal{M}, w \Vdash_g \psi$

$\Vdash \varphi$ iff For every $\mathcal{M}$, g and $w \in \mathcal{W}$, $\mathcal{M}, w \Vdash_g \varphi$

However, the notion characterizing Inquisitive Semantic is that of *support-set* of φ, i.e. the set of information states $s \subseteq \mathcal{W}$ such that s supports φ. The support-set of φ corresponds to what we have been calling φ's Inquisitive Content.
For any $\varphi \in \mathcal{L}$:

We write $\mathcal{M}, s \models_g \varphi$ iff in $\mathcal{M}$, φ is supported by s under the assignment g

We call φ's support-set $[\varphi]^g_{\mathcal{M}}$ s.t $[\varphi]^g_{\mathcal{M}} \stackrel{def}{=} \{s \subseteq \mathcal{W} | \mathcal{M}, s \models_g \varphi\}$

The support conditions in a state s are then defined as follows:

$$
\begin{array}{lll}
\mathcal{M}, s \models_g R^n\vec{t}^{\,n} & \text{iff} & s \subseteq |R^n\vec{t}^{\,n}|^g_{\mathcal{M}} \\
\mathcal{M}, s \models_g \bot & \text{iff} & s = \emptyset \\
\mathcal{M}, s \models_g \varphi \wedge \psi & \text{iff} & \mathcal{M}, s \models_g \varphi \text{ and } \mathcal{M}, s \models_g \psi \\
\mathcal{M}, s \models_g \varphi \vee\!\!\!\vee\, \psi & \text{iff} & \mathcal{M}, s \models_g \varphi \text{ or } \mathcal{M}, s \models_g \psi \\
\mathcal{M}, s \models_g \varphi \rightarrow \psi & \text{iff} & \text{For all } t \subseteq s, \text{ if } \mathcal{M}, t \models_g \varphi \text{ then } \mathcal{M}, t \models_g \psi \\
\mathcal{M}, s \models_g \overline{\exists} x.\varphi & \text{iff} & \text{There is a } n \in \mathfrak{N} \text{ such that } \mathcal{M}, s \models_{g[x \mapsto n]} \varphi \\
\mathcal{M}, s \models_g K_a\varphi & \text{iff} & \text{for all } w \in s,\ \sigma_a(w) \models_g \varphi \\
\mathcal{M}, s \models_g E_a\varphi & \text{iff} & \text{for all } w \in s, \text{ for all } t \in \Sigma_a,\ t \models_g \varphi
\end{array}
$$

With these definitions it is also possible to prove that truth in a PW w is equivalent to support in the singleton state $\{w\}$, i.e.:

$$\text{For all } \mathcal{M}, g, w \text{ and } \varphi \in \mathcal{L};\ \mathcal{M}, w \Vdash_g \varphi \text{ iff } \mathcal{M}, \{w\} \models_g \varphi$$

Once we define entailment and validity we obtain a logic that is called *intermediate*, being strictly stronger than intuitionistic logic and weaker than classical logic.

Definition 14 [INQ Entailment and INQ Validity]

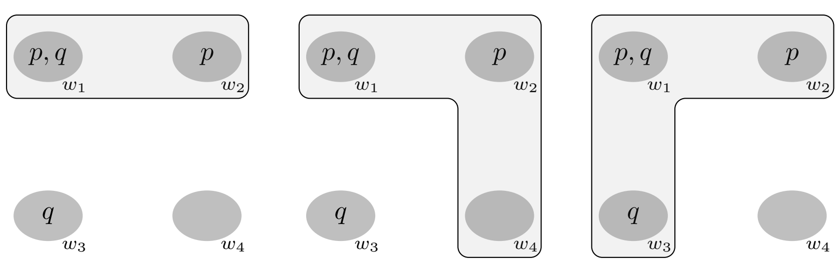
$\varphi \models \psi$ iff For every $\mathcal{M}$, g and $s \subseteq \mathcal{W}$, if $\mathcal{M}, s \models_g \varphi$ then $\mathcal{M}, s \models_g \psi$

$\models \varphi$ iff For every $\mathcal{M}$, g and $s \subseteq \mathcal{W}$, $\mathcal{M}, s \models_g \varphi$

1.3.3 The novelty of Questions

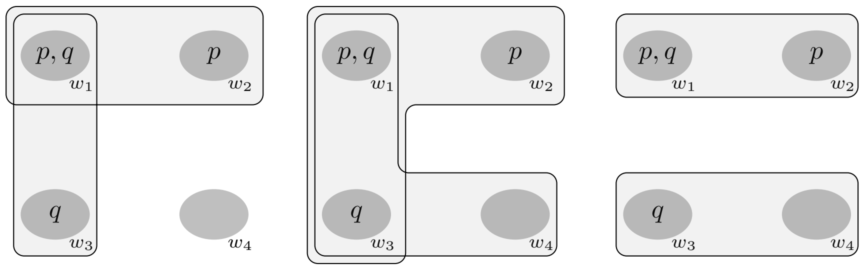
As anticipated, the operators that introduce questions in this semantics are $\vee\!\!\!\vee$ and $\overline{\exists}$ at the level of states. In order to better understand how that is done consider the example shown in figure 1.2 and 1.3.

The gray areas in evidence are the *maximal states* that support the formulas



(a) $[p]$ (b) $[q \rightarrow p]$ (c) $[p \vee q]$

Figure 1.2: non-inquisitive propositions



(a) $[p \vee q]$ (b) $[\neg q \to ?p]$ (c) $[p \vee \neg p]$

Figure 1.3: inquisitive propositions

written below them, they are also called *alternatives* (see definition 7). Therefore, the support-set for said formulas would be the downward closure of such alternatives, while the truth-set their union. The difference between declaring that "p or q" ($p \vee q$) and asking "p or q?" ($p \vee q$) can be seen in 1.2(c) and 1.3(a). They are equally informative, since the union of the alternatives in 1.3(a) is equal to the one in 1.2(c), however 1.3(a) is a refinement of 1.2(c).

In the same example we can see that only the alternative of 1.2(a) enhances 1.2(b) and 1.2(c), and settles every issue in example 1.3, and that the issue 1.3(a) is a refinement of 1.3(b).

Other interesting facts are that though 1.3(b) and 1.3(c) are both not informative they are not tautologies in INQ. In fact they do have an inquisitive content strictly more refined than the powerset of the whole space.

1.3.4 Modalities and Questions

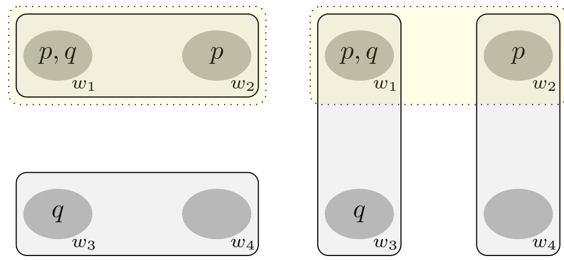
Knowledge and Questions: While for declaratives the modal operator K_a behaves exactly as it does in classical Epistemic Logic it is interesting to mention how it interacts with questions.

Take for example figure 1.4 and 1.5. There we have that a's epistemic states in w_1 and w_2 (i.e. the union over the inquisitive state) are such that $\bigcup \Sigma_a(w_1) = \bigcup \Sigma_a(w_2) = \sigma_a(w_1) = \sigma_a(w_2) = \{w_1, w_2\}$.

Now consider the expression $K_a(p \vee \neg p)$. This formula turns to be supported in a state s if and only if the agent a's epistemic state is an element of $[p \vee \neg p]$, i.e if and only if a knows *the answer* to the question whether p or $\neg p$. In 1.4(a) and 1.5(a) this requirement is met: $\{w_1\} \models K_a(p \vee \neg p)$, $\{w_2\} \models K_a(p \vee \neg p)$ therefore $\{w_1, w_2\} \models K_a(p \vee \neg p)$.

In general we say that $s \models K_a\varphi$ if and only if in every wold $w \in s$ the epistemic state of a is informative enough for a to settle the issue raised by φ.

Entertaining an issue: The modality called *Entertain* is one of the most interesting novelty in Inquisitive Epistemic Logic. It is read as "Agent a entertains the issue φ", and the intended meaning is: a is able to distinguish between the alternatives that may resolve the issue raised by φ, even though they **may** not



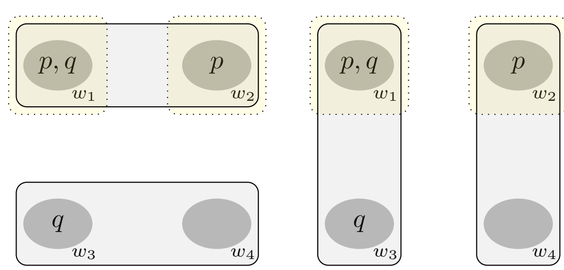
(a) $K_a?p$ (b) $\neg E_a?q$

Figure 1.4: $\Sigma_a(w_1) = \{\{w_1, w_2\}\}^{\downarrow}$

know which of those alternatives **is** the case. Again take figure 1.4 and 1.5 as an example. There we have that a's inquisitive state in w_1 is $\Sigma_a(w_1) = \{\{w_1, w_2\}\}^{\downarrow}$ while in w_2 it is $\Sigma_a(w_2) = \{\{w_1\}, \{w_2\}\}^{\downarrow}$.
Now take the expression $E_a(q \vee\!\!\!\vee \neg q)$. This formula turns to be supported in a state s if and only if the agent a's inquisitive state is a subset of $[q \vee\!\!\!\vee \neg q]$. In figure 1.4(b) this requirement is not met, while it is in 1.5(b):
thus $\{w_1\} \models \neg E_a(q \vee\!\!\!\vee \neg q)$ and $\{w_2\} \models E_a(q \vee\!\!\!\vee \neg q)$.

If we consider a non-inquisitive formula $!\varphi$ it turns out that $s \models E_a!\varphi$ if and only $s \models K_a!\varphi$. However, with inquisitive propositions φ we only have that if $s \models K_a\varphi$ then $s \models E_a\varphi$. This reflects the idea that knowing a fact is equivalent to entertain it, and knowing the answer to a question entails being able to entertain its question.

With the epistemic modality of Knowledge and Entertain it is possible to define the Epistemic attitude *Wonder*:



(a) $K_a?p$ (b) $E_a?q$

Figure 1.5: $\Sigma_a(w_2) = \{\{w_1\}, \{w_2\}\}^{\downarrow}$

Definition 15 [Wondering]
"Agent a Wonders whether φ" if and only if
"Agent a Entertains the issue φ even though they do not know how it is settled"

$$W_a\varphi \overset{def}{=} E_a\varphi \wedge \neg K_a\varphi$$

Inquisitive Negative Introspection

In light of the inquisitive expressiveness, consider again example 2 on page 3 phrased with *wondering* as defined above:

(8) "(Having heard Cassandra's prophecy) King Priam wonders whether the Trojan Horse is full of Greek warriors.
i.e. He does not know whether Trojan Horse is full of Greek warriors, but he is entertaining the question.
Therefore, King Priam knows that he does not know whether the Trojan Horse is full of Greek warriors."

If we feel like the this inference is more valid than the one in example 2 it could be interesting to substitute Axiom 5 with what I call Axiom 5*i*:

Axiom 5*i* $\vdash (E_a\varphi \wedge \neg K_a\varphi) \rightarrow K_a\neg K_a\varphi$ *Inquisitive Introspection*

Being an instantiation of the weakening rule it is easy to see that "Axiom 5 $\not\dashv\vdash$ Axiom 5*i*".

The notion of *wondering* employed in this example should be read in a weaker way than the one used in the standard INQ Semantics. In fact, when agents in INQ wonder about an issue they want to know settle it.
On the other hand, the reading I propose is much more focused on the potential disposition we may have toward asking ourselves a certain question.

For example, in standard INQ it would be wrong to say "King Priam *wonders* whether the Trojan Horse is full of warriors", simply because he will not try to open the horse and control if it is. However, I say that for a philosophical analysis it is not contradictory to wonder about issues that are not to be settled. We can find other examples in which Axiom 5*i* seems a good alternative to Axiom 5. I can at any time entertain the question the question "What did my brother have for breakfast?", of course without the urgency of calling him to have an answer. Moreover, every time I wonder about such question I would also admit that I know I do not know what he had for breakfast.
On the other hand, I can safely assume the reader does not know what my brother had for breakfast[2]. However, it would be far fetched to conclude they *knew* they did not know what my brother had for breakfast before reading this section, possibly because they were not able to entertain the idea that I have a brother in the first place.

[2] unless my brother happens to be the reader

1.4 Identity in FOIEL

Let us now consider identity. The main problem with identity in FOIEL is that it can not be introduced directly.

For example, assume that the support condition for identity was the following (I will call this toy-identity $\overset{toy}{=}$):

$$\mathcal{M}, s \models t \overset{toy}{=} t' \quad \text{iff} \quad \text{For all } w \in s;\ I_w(t) = I_w(t')$$

Due to the fact that names are treated as rigid designators in our models (see def. in figure 1.1 on page 12), we would have to conclude from such a definition that (toy-)identity is always trivial.

FACT 1 [Triviality of toy-identity]
If $\mathcal{M}, s \models (h \overset{toy}{=} p)$ *then* $\mathcal{M} \models (h \overset{toy}{=} p)$

Proof. Assume that there is a Model $\mathcal{M}$ such that for a non empty state $s \subseteq \mathcal{W}$, $\mathcal{M}, s \models_g (h \overset{toy}{=} p)$. By toy-identity definition this means that (I) For all $w \in s$ $I_w(h) = I_w(p)$. Take an arbitrary $w' \in \mathcal{W}$. By the constraint on $\mathcal{I}$ we have that (II) $I_w(h) = I_{w'}(h)$ and (III) $I_w(p) = I_{w'}(p)$. By (I), (II) and (III) we have that $I_{w'}(h) = I_{w'}(p)$, thus by toy-identity definition $\mathcal{M}, w' \Vdash_g (h \overset{toy}{=} p)$. Since this is true for any $w' \in \mathcal{W}$ and $(h \overset{toy}{=} p)$ is atomic $\mathcal{M}, \mathcal{W} \models_g (h \overset{toy}{=} p)$. □

That is why, if the constraint on rigidity is not dropped, identity has to be treated differently. The way it is done in [Ciardelli, 2015] is by introducing *equivalence classes* on $\mathcal{D}$. This means that a model for epistemic INQ semantics with equality is going to look like the following:

$$\mathcal{M} = \langle \mathcal{W}, \{\Sigma_a \mid a \in \mathfrak{Ag}\}, \mathcal{D}, \mathcal{I}, \sim \rangle$$

$\sim$: a map assigning to each $w \in \mathcal{W}$ an equivalence relation $\sim_w \subseteq \mathcal{D} \times \mathcal{D}$ s.t.

- for any $R \in \mathcal{R}$, if $d \sim_w d'$:

$$d \in I_w(R) \text{ iff } d' \in I_w(R)$$

The domain $\mathcal{D}$ equipped with such equivalence relation makes it a domain of "epistemic" individuals rather than a domain of existing objects; individuals that may turn out to be different or equal as more information is acquired. Names and variables refer to these epistemic individuals rather than to objects themselves.

An "epistemic individual" can be interpreted as a way in which a certain object is conceptualized. Take for example *Mattia Pascal*, the main character of Pirandello's "The Late Mattia Pascal"; who, after faking his death, escaped his life in Miragno to live in Rome under the pseudonym *Adriano Meis*.
Both names refer to the same person, but to two different epistemic individuals. Mattia's fellow citizens talking about 'Mattia Pascal' address the individual they believed to be dead, whilst people in Rome call 'Adriano Meis' who they see as a foreigner arrived few months before.

Identity is not trivial since there are worlds in which such epistemic individuals are not in the same equivalence class, and worlds where they are: in the former an identity statements is true, while in others it is not.

$$\mathcal{M}, s \models_g t = t' \quad \text{iff} \quad \text{For all } w \in s;\ I_w(t) \sim_w I_w(t')$$

A reason why this approach is not particularly satisfactory is that whenever we discuss entities we refer directly to objects, not to conceptualizations. When we use the names "Hesperus" and "Phosphorus" we are talking about the same planet even though we can believe we are not. The interpretation of epistemic individuals is, in this respect, too bounded to the subjective use of names.
Moreover, when we assert an identity statement like "Hesperus is Phosphorus" what we do is not only claiming that individuals picked by the names are one and the same, but also that the way the two names pick them are the same, not unlike two different proofs with the same conclusion. With epistemic individual is difficult to have a direct interpretation of such claims.
Another consequence of this approach, that seems far more counterintuitive, is that every way of referring to these clusters of epistemic individuals is always felicitous, i.e. picks up existing objects.
Moreover, having to refer to objects in a domain would expose any philosophical account to the question "what are exactly these objects?" and "how do we say that they have such-and-such property?". Said question is even more difficult to answer if we have to explain how abstract entities like epistemic individuals have properties.
This is of course shared by every classical approach that uses domains, not only a problem of this particular account, though the use of epistemic individuals makes the problem even more puzzling. To understand how these problems can be overcome I will address them along with other topics from a strictly philosophical perspective in the next Chapter.

2. Questions in Philosophy

In this chapter I will discuss some issues that motivates the expansions and variations that will be introduced in Philosophical Inquisitive Logic (PhIL).

It is no secret, the term philosophy is broad; and so is the notion of philosophical issue. And a good question in philosophy is often more interesting than a good answer. One of the most interesting questions is of course "How many interesting questions are out there?". By interesting I mean worthy of a philosophical endeavors and sufficiently distinct to others.
For example, there are just so many disguises that Russell's Paradox can use before we are able to recognize its pattern. A barber with an insanely strict rule, a librarian with the passion of cataloguiong catalogues, a Cretan criticizing his fellow citizens, the halting problem, Tarski's Truth predicate and many others are substantially the same cup of tea.

What we discussed so far concerns mainly epistemology, therefore it seems natural to include problems that deal with what we *know* within philosophy. For this reason it seems natural to leave out paradoxes like the liar and the sorites, since they seem to defy the notion of knowledge and valid reasoning.
As toolbox devoted to describe reasoning and arguing within a given philosophical system, PhIL wants to be able to entertain different standpoints concerning ontological status and what they entail once endorsed rather than a specific philosophical position. Particular attention will be devoted to describe logical arguments and expressions with as little ontological commitment as possible.
For this reason, PhIL models are structured to accommodate a nominalist interpretation: i.e. names and predicates are taken to be primitive entities, rather than objects and extensions. Using these primitive notions it is possible for both realists and anti-realists to interpret entities as they see fit. While a realist would assume that names are nothing but pointers to real objects and will use predicates to define the extensions of properties, an anti-realist would regard any interpretation of this kind as too committing.

Some may say that such a move is like putting the cart before the horse. I feel that such approach is more flexible, since the primitive entities are abstract enough to allow an arbitrary interpretation, and more epistemologically adequate, since it focuses on *what can be said* rather than on *what there is*.

It is therefore important to focus on *Names* and *Predicates*, how we use them in arguments and how they behave when they are infelicitous. Other topics particularly important in epistemology are *Identity* and *Necessity*, and how they are interconnected.

2.1 To be or not to be?

A consequence of the standard INQ, alongside many other classical accounts, is that it is only possible to address existing objects.

Consider the following expressions:

(9) "Cicero is Tully" $(cicero = tully)$

(10) "There is a Roman orator who served as consul in the year 63BC" $\exists x.\text{Roman Orator}(x) \wedge \text{Consul in 63BC}(x)$

(11) "Cicero is the Roman orator who served as consul in the year 63BC" $\text{Roman Orator}(cicero) \wedge \text{Consul in 63BC}(cicero)$

(12) "Cicero exists" $\exists x.(cicero = x)$

For our purpose we can treat these all there sentences as declaratives and look at their informative content, i.e. set of worlds where the sentence is true.

In Ciardelli's account example 9 is informative, since the name Cicero and Tully can refer to different epistemic individuals, that may be in the same equivalence class only in certain worlds. Formally, a subset of the logical space A s.t.:

$$A = \{w \in \mathcal{W} \mid I_w(tully) \sim I_w(cicero))\}$$

The sentence expressed in example 10 is also informative, and it identifies the portion B s.t.:

$$B = \{w \in \mathcal{W} \mid \exists d \in \mathcal{D}.d \in I_w(\text{Roman Orator}) \wedge d \in I_w(\text{Consul in 63BC})\}$$

And of course, saying that Cicero was a Roman Orator, as in example 11, is also informative with the following intension:

$$C = \{w \in \mathcal{W} \mid I_w(cicero) \in I_w(\text{Roman Orator}) \wedge I_w(cicero) \in I_w(\text{Consul in 63BC})\}$$

However, we can consider the plausible scenario in which some historian finds a body of evidences Γ supporting the falsehood of example 10: there was no consul in $63BC$ that also happened to be an orator, because some laws would not allow this specific combination.
At the same time our current body of evidences Δ tells us example 11 cannot possibly be rejected, i.e. every text mentioning Cicero says he had to be a consul in $63BC$ and an orator.
What we can deductively conclude from such inquiry is that, if we cannot distrust neither Γ nor Δ, we have conclude that Cicero did not exist, thus rejecting example 12.

The problem is that example 12 turns out to be a tautology, since there is always an assignment $g[x \mapsto Cicero]$ that, for every w, maps the variable $[x]^{\mathcal{M}}_g$ to the name *Cicero*.
Therefore, since we cannot falsify (12), we are left only with the option of distrusting either Γ or Δ, thus rejecting either $\neg$(10) or (11).

More generally, every claim of existence like example 12 on the facing page is non-informative.

$$\begin{aligned}
|\exists x.x = \mathrm{t}| &= \{w \mid w \Vdash_g \exists x.x = \mathrm{t}\} \\
&= \bigcup_{n \in \mathfrak{N}} \{w \mid w \Vdash_{g[x \mapsto n]} x = \mathrm{t}\} \\
&= \bigcup_{n \in \mathfrak{N}} \{w \mid I_w(g(x)) \sim_w I_w(t)\} \\
&= \{w | \exists n \in \mathfrak{N}.I_w(n) \sim_w I_w(t)\} \\
&= \{w | I_w(t) \sim_w I_w(t)\} \\
&= \mathcal{W}
\end{aligned}$$

From this proof follows that the informative content of $|\exists x.x = \mathrm{t}|$ is equal to the ignorant state.
An unwanted corollary of this fact is that whenever we introduce a knowledge or belief operator we are forced to conclude that, at any time, we know or believe that everything that exists ought to exist.

A way of circumventing this problem is to be able to treat existence as a *special* predicate, opposed to *standard* predications (i.e. $P\vec{n}$). I will call such unary predicate $\mathcal{E}$ and, for an arbitrary name n, $\mathcal{E}(n)$ will read within a model as "n exists". Within the theory "not $\mathcal{E}(n)$" will mean "n does not exist"; however, for ontological neutrality, in the meta-theory I will use the expression *"n is infelicitous"*.
Employing an existence predicate is neither a new move in philosophical logic nor free of consequences, in fact there are some issues to determine:

- What is the truth-value of a First-Order predication with infelicitous names?
- Which quantifiers can we have in this theory, and how do they behave?
- How are identity statements evaluated?

2.1.1 Predication

In this fragment of PhIL any standard predication that includes an infelicitous name is meaningless. By meaningless (i.e. N) I mean neither true (1) nor false (0), but rather a truth-value in-between. In fact, the ordering relation on truth-values will be the following: $1 > N > 0$. This choice can be motivated through a *"verificationist"* optic: To say that a predicate is true or false I must find a way to verify it. To assert that "Ann is a lawyer" is true or false I should check Ann's whereabout, but if "Ann" is infelicitous there is no way to prove it nor to disprove it.

This concept can be expanded to account for fictional statements, where the truth of "Harry Potter is a Wizard" and "Harry Potter is Not a Wizard",

evaluated in the actual world, have the same strength since we have no mean of verifying either claim.
In section 5.1.2 on page 80 I will argue that while the *meaningless* truth-value can be employed for atomic fictional statements and "Story telling" (i.e. listing coherent but unverifiable claims) there is a way to bring falsehood and truth back to the fictional table: what we can do is respectively checking for fictional coherence ("Harry Potter is a Wizard and Harry Potter is Not a Wizard" would be false) and reason on the premises ("If Harry Potter is a Wizard then Harry Potter is a Wizard" would be true).
For these prediction to work the account in Chapter 3 and 4 will not do the job; however they are going to be the middleman for further developments, as I will stress in the conclusion.

The semantics I will be using to characterize complex sentences is a partial (bilateral) system, with truth ($\Vdash$) and falsehood ($\dashv\!|$). Given a sentence φ, we can talk about its extension (truth-value) in a possible world w s.t.if $[\![\varphi]\!]_w \in \{1, N, 0\}$ is an evaluation function.

$$
\begin{array}{lll}
[\![\varphi]\!]_w = 1 & \text{iff} & w \Vdash \varphi \\
[\![\varphi]\!]_w = 0 & \text{iff} & w \dashv\!| \varphi \\
[\![\varphi]\!]_w = N & \text{iff} & \text{neither } w \Vdash \varphi \text{ nor } w \dashv\!| \varphi
\end{array}
$$

The requirement that only "verifiable" standard predications have a truth is equivalent to the following constraint:

$$[\![P\vec{n}\,]\!]_w \in \{1, 0\} \text{ iff For all } n \in \vec{n}\,, [\![\mathcal{E}(n)]\!]_w = 1$$

2.1.2 Quantification

This approach leads to two different kinds of quantification: one is a "*weak quantification*", and it is purely nominal, while the other is the "*strong quantification*", and it reflects the classical one. It is often stated in philosophy that calling the quantifier $\exists$ the *Existential Quantifier* can sometimes be an abuse of meaning since the dual of Universal Quantification should be called Particular Quantification.
For example in [Orenstein, 1995] 'How to Get Something from Nothing' the author notices how from a valid sentence like '$P(n) \vee \neg P(n)$' it is possible, via Introduction of the Existential, to conclude that '$\exists x.P(x) \vee \neg P(x)$'. However this statement is true only in models with non-empty domain.
To avoid this problem we will say that from any valid sentence 'φ' we are allowed to derive a weak 'for a particular n, $\varphi[^n/_x]$', rather than the much stronger 'There exists an n, $\varphi[^n/_x]$', which will also require n to be felicitous.

Weak and Strong Quantifiers

Consider the following definitions and their intended reading in light of the aforementioned distinction:

Weak Universal Quantifier

- $\mathcal{M}, w \Vdash \nabla x.\varphi$ iff For all names n, $\mathcal{M}, w \Vdash \varphi[^n/_x]$
- $\mathcal{M}, w \dashV \nabla x.\varphi$ iff There is a name n s.t. $\mathcal{M}, w \dashV \varphi[^n/_x]$

Weak Particular Quantifier

- $\mathcal{M}, w \Vdash \rotatebox[origin=c]{180}{E} x.\varphi$ iff There is a name n s.t. $\mathcal{M}, w \Vdash \varphi[^n/_x]$
- $\mathcal{M}, w \dashV \rotatebox[origin=c]{180}{E} x.\varphi$ iff For all names n, $\mathcal{M}, w \dashV \varphi[^n/_x]$

Strong Universal Quantifier

- $\mathcal{M}, w \Vdash \forall x.\varphi$ iff For all names n, if $\mathcal{M}, w \Vdash \mathcal{E}(n)$ then $\mathcal{M}, w \Vdash \varphi[^n/_x]$
- $\mathcal{M}, w \dashV \forall x.\varphi$ iff There is a name n s.t. $\mathcal{M}, w \Vdash \mathcal{E}(n)$ and $\mathcal{M}, w \dashV \varphi[^n/_x]$

Strong Particular Quantifier

- $\mathcal{M}, w \Vdash \exists x.\varphi$ iff There is a name n s.t. $\mathcal{M}, w \Vdash \mathcal{E}(n)$ and $\mathcal{M}, w \Vdash \varphi[^n/_x]$
- $\mathcal{M}, w \dashV \exists x.\varphi$ iff For all names n, if $\mathcal{M}, w \Vdash \mathcal{E}(n)$ then $\mathcal{M}, w \dashV \varphi[^n/_x]$

With the difference introduced by weak and strong quantification using the term “Particular Quantifier” is extremely important. In fact only the Strong Particular Quantifier $\exists$ is, strictly speaking, an existential quantifier.

2.1.3 Identity

Identity in PhIL differs from standard predications in that it cannot be undefined. Within a nominalist approach identity statements are claims about how names are used. Considering a statement like $n = m$ there are three different situations that may occur:

- Both n and m are felicitous, therefore the way we use them is influenced by their reference: $n = m$ is true if their reference is indeed the same, and false if it is not.

- A name is felicitous and the other is not, therefore the identity statement is false in virtue of this difference.

- Both n and m are infelicitous, therefore once we understand that they are infelicitous we can analyze their use through counterfactual statements and establish conventionally if we use them in the same way or not.

The third point is of course the most vulnerable to controversy, but it is quite a conventional linguistic practice.
Consider the fictional Headmaster of Hogwarts, from the fantasy novel *Harry Potter*: (Albus Percival Wulfric Brian) 'Dumbledore'.
With the current definitions of existence every name he can have is undoubtedly infelicitous. Since the name has been translated along with the text, readers from different nations know him with different aliases: in Dutch he is 'Perkamentus', in Italian 'Silente' and in Norwegian 'Humlesnurr'.
However, once it is established that the actual world has no referent to support or reject an identity statement like *"Dumbledore is Silente"* it is reasonable to assert that this statement is true from purely linguistic conventions.

2.1.4 NON-TRIVIAL EXISTENCE

Going back to example 12 on page 22 it is possible to show that under the aforementioned conditions the strong particular quantifier (aka existential quantifier) is no longer trivial. In fact we have the following:

$$\begin{array}{lll} w \Vdash \exists x.x = \text{cicero} & \text{iff} & \text{There is a name } n \text{ s.t. } w \Vdash \mathcal{E}(n) \text{ and } w \Vdash (x = cicero)[^n/_x] \\ & \text{thus} & w \Vdash \mathcal{E}(cicero) \text{ and } w \Vdash (cicero = cicero) \\ & \text{iff} & w \Vdash \mathcal{E}(cicero) \end{array}$$

What is trivial is the weak particular quantifier $\Sigma x.x = \text{cicero}$, but its truth simply implies that Cicero has a name.

2.2 Agents within the system

Once it is possible for an epistemic agent to know if a name is felicitous or not, it seems equally interesting to be able to talk about the existence of oneself and other agents as well.

This topic is particularly important when addressed in conjunction with self-locating beliefs [Perry, 1979]: i.e. sentences like 'I am in Wisconsin' and 'John Perry believes he is Adam Smith'.

In order to do so I will have in the language $\mathcal{L}$, as a subset of the names $\mathfrak{N}$, the set of the agents' names $\mathfrak{Ag}$, that will also serve as index for the modal operators K_x and E_x. It will be possible to express for example that "Agent a knows agent b exists" ($K_a\mathcal{E}(b)$).

Rather than considering Veridicity as axiom I will assume a weakened version called Conditional Veridicity, i.e. "if an agents' epistemic state is consistent then it is factual". I call this Axiom $D \rightarrow T$: $\neg K_a \bot \rightarrow (K_a\varphi \rightarrow \varphi)$

Conditional Reflexivity: if Exists $w' \in \mathcal{W}; w' \in \sigma_a(w)$ then $w \in \sigma_a(w)$

The reason for this weakening is because the converse of Factuality implies that "If φ then a does not know $\neg\varphi$".
However, if we assume $\varphi \stackrel{def}{=} \neg\mathcal{E}(a)$ we would have that "if an agent does not

exist then they do not know they exist". To prevent this, I will say that any agent's knowledge $K_a\psi$ is determined at w if and only if a exists in w.

Moreover, an agent's epistemic state will be consistent if and only if they exist. So that with Conditional Reflexivity we will be able to recover Veridicity if only if the agent exists.
This idea is particularly close to the Cartesian *'Cogito, ergo sum'*, which is going to be the name of the restriction on the epistemic accessibility relation σ_a that will determine the validity of Axiom $\mathcal{E}(D)$: $\neg K_a \bot \leftrightarrow \mathcal{E}(a)$:

Cogito, ergo sum: $[\![A(a)]\!]_w = 1$ iff Exists $w' \in \mathcal{W}; w' \in \sigma_a(w)$

2.3 Necessary *vs* Necessarily Known

One of the most interesting distinctions in philosophy of language was introduced in 1892 by the German philosopher and mathematician Gottlob Frege, in his article "Sense and Reference" [Frege, 1948].
I will consider the example introduced by Frege in its version used in Naming and Necessity [Kripke, 1980]. The reason why I prefer Kripke's setting is that it focuses on names rather than on definite descriptions, but it still discusses the same issues. Kripke discusses three different approaches to the status of identity in natural language: (i) identical objects are necessarily identical, (ii) identity statements between rigid designators are necessary and (iii) identity statements using what we call 'names' are necessary.

Firstly, we have to outline the difference between Kripke's rigid designators and a nominalist "rigid behaviors". The formers are by definition designators that have the same referent in every possible world, therefore we have that "x is a rigid designator" if and only if "x has the same referent in every possible world". It is easy to see that this *property* of a designator is functional to its referent.
However, without a notion of objects or domain such a definition cannot be applied. Instead, we will say that "two designators n and m *behave rigidly*" if and only if "They are co-extensional in every ***metaphysically possible world***[1]". Therefore rigidity is taken to be a relation between referents across worlds rather than between a referent and a reference across worlds.

I will not discuss in this thesis (iii), assuming that when a Philosophically Inclined Agent (PIA) uses names they behave rigidly. Moreover, as previously pointed out, with a nominalist approach we do not have to worry about thesis (i): from an epistemological point of view necessity will be understood with respect to identity statements rather than ontological necessity of identity. Thesis (ii) appears to be a requirement dictated by norms of reasoning and understanding. If I say that x and y are co-extensional in the actual world, it would be nothing

[1] A subset of what we have implicitly called up until now ***logically possible worlds***. This distinction will be discussed in section 2.3.1

but talking of linguistic conventions to say that they *could* have referred to different objects. Possibly a linguistic convention that would not reflect the way we use names in counterfactual arguments and that would make them even more obscure.

Think about the two names given by the Greeks to Venus: Hesperus (h), the Evening Star, and Phosphorus (p), the Morning Star. The two names refer to the same object, namely Venus, and we consider them to behave rigidly. Now, we can say "Hesperus could have been a star", but if that was the case how could we assert that "Phosphorus could not have been a star" without causing confusion? Following a reference across worlds is nothing but following its referent. That said, take the following derivation:

$$\frac{\text{Alice knows that } p \text{ is a planet} \qquad \text{Necessarily } p = h}{\text{Alice knows that } h \text{ is a planet}}$$

This would be classically a logically valid inference, by the meaning of identity and application of substitution. However, Alice could have read that p is indeed Venus, without knowing that the same holds for h, and that it holds trivially since the two names refer to the same object. Maybe Alice read that p is Venus in an article, but she thinks that h is actually a star. In other words, Alice could not *know* the identity statement $p = h$, even though it is necessary. In order to describe this situation we need to to distinguish between agents' knowledge and metaphysical necessities.

As argued by Kripke the fact that "Hesperus is equal to Phosphorus" is an example of a statement that is indeed necessary, but not a priori. In fact, in order to know that they are the same object we need a scientific inquiry.

Consider figure 2.1. The case we want to describe surely must have two worlds $w, v \in \sigma_a(w)$ such that $w \Vdash (a = b)$ and $v \Vdash (a \neq b)$, in this way Alice's epistemic state will be such that $\sigma_a(w) \not\models (a = b)$ and $\sigma_a(w) \not\models \neg(a = b)$. However, I call these two worlds not *Metaphysically Compatible* since, as previously stated, it is not reasonable to say that in w it is true that $(a = b)$

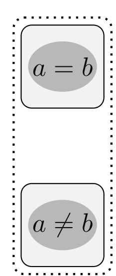


Figure 2.1: An example of a model where the agent is not Metaphysically Omniscient

but it could have been true that $(a \neq b)$.
In technical terms, if we call $\mathcal{N}(w)$ the set of all worlds that are *metaphysically accessible* to w, we want to exclude v from $\mathcal{N}(w)$.
More formally we can introduce some definitions:

Definition 16 **[Metaphysical Accessibility]**

We say that w' is metaphysically accessible to w
if and only if $w' \in \mathcal{N}(w)$

Definition 17 **[Possibility]**
We say that in a world w it is possible that φ iff
w has a metaphysically accessible word where φ is true

$$w \Vdash \Diamond\varphi \quad \text{iff} \quad \text{There is a } w' \in \mathcal{N}(w);\ w' \Vdash \varphi$$

Definition 18 **[Necessity]**
We say that in a world w it is necessary that φ iff
in all metaphysically accessible worlds φ is true

$$w \Vdash \Box\varphi \quad \text{iff} \quad \text{For all } w' \in \mathcal{N}(w);\ w' \Vdash \varphi$$

In the previous example, we can explain the knowledge that Alice lacks of by saying that she does not know which metaphysical necessity is actual. Once she will discover that in fact $h = p$, she will be able to rule out both the factual statement $h \neq p$ and the modal statement $\Diamond(h \neq p)$, meaning that not only she will have a better understanding of her position in the *logical space*, but also how to use those names in counterfactual and modal statements in virtue of their identity.
For example, if Ann knows that, for certain names $x\&\ y$, $x = y$ then she knows that "a does not exists necessarily" implies "b does not exists necessarily".
Notice that such conclusion does not follow from discovering facts with no metaphysical impact: i.e. if she discovers that Venus is yellowish-white, she cannot conclude that it is necessarily so.

2.3.1 On Metaphysical Compatibility

On one hand, we can regard the given definition of Metaphysical Accessibility as purely descriptive: it is used to tell us what we mean when we say *possible* and *necessary.*
On the other hand, the notion of *Metaphysical Compatibility* is normative. It would tell whether a certain world w **can** access another.

For example, assume that in a certain world Platonism is the true ontology. It would be contradictory to say in that world that it could be the case that anti-realism was the true ontology. To express this we say in general that two worlds with conflicting ontologies are always metaphysically incompatible. We can extend this account in order to fit our Philosophical Theory (**PT**).

For example, if we were to assume the previously mentioned "necessity of identity statements", every identity statement would be necessary, i.e. it is always the case that "if $a = b$ then $\Box(a = b)$". Similarly, if we want to model the way Essentialists reason we will have that *some* properties are *necessary.*

In our case we want identity statements to restrict the notion of metaphysical compatibility in the theory: i.e. if w and v are metaphysically compatible then for every identity statement $x = y$, $w \Vdash x = y$ iff $v \Vdash x = y$.

We can force such condition saying that arbitrary names in $\mathfrak{N}$ always behave rigidly. I call this *Nominal Rigidity*, and it limits compatibility as follows:

Definition 19 [Nominal Rigidity]

v and w are compatible under Nominal Rigidity if and only if
For all $f, g \in \mathfrak{N}$, $[\![g = f]\!]_w = [\![g = f]\!]_v$.

Now all we need to do is to specify that the Accessibility Relation $\mathcal{N}$ of every acceptable model must follow the norm that the compatibility relation induces. In particular, we are going to require that if $v \in \mathcal{N}(w)$ then v is compatible with w.

To summarize , the philosophical motivations and background for PhIL include being able to discuss infelicitous names (and subsequently their impact on predication, quantification and identity statements), being able to approach counterfactual/modal reasoning and consider the epistemic agents as 'living' parts of our model rather than just external entities.

In the next Chapter I will start by introducing a possible world semantic to address the first two problem, while in chapter 4 on page 59 I will explore its State Based Epistemic expansion.

3. Partial Modal Logic

In this Chapter I will introduce the Partial Modal Semantics that will be used in PhIL.
Modal Logic is usually presented as a conservative expansion of classical logic. Here I will use it in combination with my proposed Partial Semantics.
Partial Semantics is a wide field, where different approaches are motivated both by philosophical reasons and practical application.
In this proposal below the modal operator will be treated differently from other partial approaches, like [Blamey, 2002] and [Johannesson, 2018]. If these authors treat the modal $\Diamond$ *weakly* ("$\Diamond\varphi$ is undefined in w" if and only if "φ is undefined in some PW in $\mathcal{N}(w)$ and false or undefined elsewhere") I will use it *strongly*, i.e. true if true somewhere in $\mathcal{N}(w)$ and false if never true in $\mathcal{N}(w)$.
For brevity the modal $\Box$ will be treated in this thesis as dual of $\Diamond$, therefore $\Box\varphi$ will be true in w if φ is never false in $\mathcal{N}(w)$, and false otherwise. I believe that this treatment is not adequate, since $\Box\varphi$ should be true in w if φ is always true $\mathcal{N}(w)$, and false otherwise. However, this would mean to drop duality and the system would be far more complex and the thesis would go off track. The way partial semantics is employed in this chapter is to deal with a multi-valued system. Multi-valued Semantics will be used to treat standard predication, existence and identity in the way described in section 2.1 on page 22.

3.1 Partial Semantics

Language

The language $\mathcal{L}^{PW}$ consists of a set of relation symbols $\mathcal{R}$, with arity $m \geq 0$ and constant functions $n \in \mathfrak{N}$ (i.e. 0-place function symbols) that we call names and variables $x \in \texttt{Var}$. The primitive operators are the connective $\rightarrow$ (implication); the constants $\bot$ (false) and $\bigstar$ (undefined), the quantifier Σ (weak particular), the unary predicate $\mathcal{E}$ (existence) and the modal operator $\Diamond$ (possible).

Terms t and *well formed formulas* φ are defined standardly:

$$t \stackrel{def}{=} n \mid x$$

$$\mathcal{L}^{PW} \stackrel{def}{=} R^m(\vec{t}^{\,m}) \mid \mathcal{E}(t) \mid t = t \mid \bot \mid \bigstar \mid \varphi \rightarrow \varphi \mid \Sigma x.\varphi \mid \Diamond\varphi$$

The *classical* existential quantifier in section 2.1.2 on page 24 can be defined using the weak particular and the predicate of existence, while the *classical* universal quantifier can be defined using the non-primitive weak universal ∇

and the predicate of existence.
Other common operators can be defined on the base as usual, in particular we have:

$\neg\varphi \overset{def}{=} \varphi \to \bot$	$\top \overset{def}{=} \neg\bot$	$\varphi \wedge \psi \overset{def}{=} \neg(\varphi \to \neg\psi)$
$\varphi \vee \psi \overset{def}{=} (\neg\varphi \to \psi)$	$\nabla x.\varphi \overset{def}{=} \neg\mathfrak{Z}x.\neg\varphi$	$\exists x.\varphi \overset{def}{=} \mathfrak{Z}x.\mathcal{E}(x) \wedge \varphi$
$\forall x.\varphi \overset{def}{=} \nabla x.\mathcal{E}(x) \to \varphi$	$\Box\varphi \overset{def}{=} \neg\Diamond\neg\varphi$	

While in this section I will specify the conditions for the primitive operators the defined one are spelled out in chapter A on page 87, section A.1.
We call the set of atomic formulas `Atom`. Such set comprehends all and only the standard predications, the predicate of existence and identity statements.

Substitution

The rules for terms substitution, assuming t is a term, φ a well formed formula, $n \in \mathfrak{N}$, $y, x \in$ `Var`, are the following:

$$\begin{aligned}
\text{For } t \text{ non identical to } x,\ t &\Leftrightarrow t[^n/_x] \\
n &\Leftrightarrow x[^n/_x] \\
\langle t[^n/_x], \ldots, t[^n/_x]\rangle &\Leftrightarrow \vec{t}^{\,m}[^n/_x] \\
\mathcal{R}^m(\vec{t}^{\,m}[^n/_x]) &\Leftrightarrow R^m(\vec{t}^{\,m})[^n/_x] \\
\mathcal{E}(t[^n/_x]) &\Leftrightarrow \mathcal{E}(t)[^n/_x] \\
t[^n/_x] = t[^n/_x] &\Leftrightarrow (t = t)[^n/_x] \\
\bot &\Leftrightarrow \bot[^n/_x] \\
\bigstar &\Leftrightarrow \bigstar[^n/_x] \\
\varphi[^n/_x] \to \varphi[^n/_x] &\Leftrightarrow (\varphi \to \varphi)[^n/_x] \\
\mathfrak{Z}x.\varphi &\Leftrightarrow (\mathfrak{Z}x.\varphi)[^n/_x] \\
\text{For } y \text{ non identical to } x,\ \mathfrak{Z}y.(\varphi[^n/_x]) &\Leftrightarrow (\mathfrak{Z}y.\varphi)[^n/_x] \\
\Diamond(\varphi[^n/_x]) &\Leftrightarrow (\Diamond\varphi)[^n/_x]
\end{aligned}$$

In section 2.2 on page 26 we saw that the index $\mathfrak{Ag}$ of the family of Epistemic operators K_a will be a subset of the names $\mathcal{N}$. For this reason it may be interesting to define a substitution for such operator so that:

$$\begin{aligned}
K_a\varphi[^a/_x] &\Leftrightarrow (K_x\varphi)[^a/_x] \\
\text{For } y \text{ non identical to } x,\ K_y(\varphi[^a/_x]) &\Leftrightarrow (K_y\varphi)[^a/_x]
\end{aligned}$$

This would allow quantifications on the set of agents like $\forall x.K_x\varphi$ ("Every agent knows φ") and $\exists x.K_x\psi$ ("Some agent knows ψ").
This is an interesting development since it captures respectively *Common Knowledge* and *Distributed Knowledge* as defined operators. However this topic goes beyond the scope of the thesis, and I will only address it as further development.

$$\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, [\![-]\!] \rangle$$

$\mathcal{W}$ a non-empty set of possible worlds.

$\mathcal{N}$: $\mathcal{W} \to \mathbb{P}(\mathcal{W})$ a function from PW to their successors, s.t.:

Reflexive For all $w \in \mathcal{W}.w \in \mathcal{N}(w)$

Euclidean For all $w, v, u \in \mathcal{W}$. if $v \in \mathcal{N}(w)$ and $u \in \mathcal{N}(w)$ then $v \in \mathcal{N}(u)$

$[\![-]\!]$: $\mathcal{W} \mapsto [\![-]\!]^{\mathcal{M}}_w$ are interpretation maps from atoms and possible words into denotations, such that for every $\alpha \in \texttt{Atom}$ and $w \in \mathcal{W}$ an interpretation is a map:

$$[\![-]\!]^{\mathcal{M}}_w : \texttt{Atom} \to \{0, N, 1\} : \alpha \mapsto [\![\alpha]\!]^{\mathcal{M}}_w$$

Moreover, we have the following constraints:

Existence Predicate For every term t:

$$[\![\mathcal{E}(t)]\!]^{\mathcal{M}}_w \in \{0, 1\}$$

Definiteness
$[\![P\vec{t}]\!]^{\mathcal{M}}_w \in \{0, 1\}$ iff for all $t \in \vec{t}$; $[\![\mathcal{E}(t)]\!]^{\mathcal{M}}_w = 1$

Identity For every name $c, b \in \mathfrak{N}$:

$$[\![c = b]\!]^{\mathcal{M}}_w \in \{0, 1\}$$

Nominal Rigidity For every $w, v \in \mathcal{W}$ and $c, b \in \mathfrak{N}$;
if $v \in \mathcal{N}(w)$ then $[\![c = b]\!]^{\mathcal{M}}_w = [\![c = b]\!]_v$

Self-identity For every $w \in \mathcal{W}$ and $n \in \mathfrak{N}$; $[\![n = n]\!]^{\mathcal{M}}_w = 1$

Leibniz's Laws If $[\![c = b]\!]^{\mathcal{M}}_w = 1$ then
For every atom $\alpha \in$ Atom, $[\![\alpha[{}^b/{}_a]]\!]^{\mathcal{M}}_w = [\![\alpha[{}^c/{}_a]]\!]^{\mathcal{M}}_w$

Figure 3.1: First-Order Bilateral Modal Logic Model

3.1.1 Models for Partial Logic

A model for this logic is a triple as in figure 3.1 on the previous page. Given a frame, the requirement of Nominal Rigidity restricts the possible interpretation functions. In particular the relation of metaphysical accessibility $\mathcal{N}$ creates classes of worlds that we can call metaphysical clusters. An identity statement "$a = b$" in a world w has the same extension in every world v in w's metaphysical cluster.

Alternative accounts instead of preserving co-extension of identity statements preserve the extension of a name across possible worlds (for example if they embrace *cross-world identity*) or give different domains to every possible world and require a *counter-part relation* between objects in order to understand modal clams such "Al Gore could have won the elections".

However, within a more general approach of modal reasoning both cross-world identities and counter-part relations are dispensable readings of what I would generally call cross-world identifications. The requirement that we really need to understand a modal/counterfactual argument is that names that we establish to be co-extensional in the actual world, either for by evidence we have or by linguistic conventions, are still co-extensional across worlds.

3.1.2 Algebraic approach

As anticipated, the objective of the bilateral approach is to evaluate sentences allowing them to be true (1), false (0) or undefined (N) in a given world w.
The truth-value N appears at the atomic level if and only if the atom is a standard predication and at least one of the names mentioned in the predicate is infelicitous while it is carried in more complex formulas following Strong Kleene operators' truth tables. In particular we can characterize the algebra of well-formed expressions once we establish an ordering as in Priest's and Kleene's three-lattice (figure 3.2).

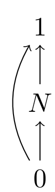


Figure 3.2: Three-lattice (with $<$ represented as $\rightarrow$)

In the algebraic approach, outlined in section A.2 on page 87, the notation for the evaluation of a formula φ in a model $\mathcal{M}$ at the possible w is $[\![\varphi]\!]^{\mathcal{M}}_w$.

The way we read an expression like, for $\mathcal{M}$ and $w \in \mathcal{W}$, $[\![\varphi]\!]^{\mathcal{M}}_w < [\![\psi]\!]^{\mathcal{M}}_w$ is "in $\mathcal{M}$ at w, ψ is strictly stronger than φ", while for $\mathcal{M}$ and $w \in \mathcal{W}$, $[\![\varphi]\!]^{\mathcal{M}}_w \leq [\![\psi]\!]^{\mathcal{M}}_w$ means that "in $\mathcal{M}$ at w, ψ is at least as strong as φ".

Taking two arbitrary truth values a and b, their joint $a \sqcup b$ is defined as their maximal element $\max(a, b)$, while the negation is a switch operation that send false to true, and vice-versa but maps N into itself.

While this algebraic approach is further discussed in the appendix, along with its correspondence results, I will focus here on truth (and falsehood) conditions; considering the algebra as a consequence of such conditions. In particular we have that for any $\varphi \in \mathcal{L}^{PW}$:

φ **is true at** w (notation $[\![\varphi]\!]_w^{\mathcal{M}} = 1$) iff $w \Vdash \varphi$

φ **is false at** w (notation $[\![\varphi]\!]_w^{\mathcal{M}} = 0$) iff $w \dashv\Vdash \varphi$

φ **is undefined at** w (notation $[\![\varphi]\!]_w^{\mathcal{M}} = N$) iff
neither $w \Vdash \varphi$ nor $w \dashv\Vdash \varphi$

TRUTH AND FALSEHOOD CONDITIONS

We can now spell out the truth (and falsehood) conditions for every operator in the logical base. With $\alpha \in \texttt{Atom}$ we have that:

$\mathcal{M}, w \Vdash \alpha$	iff	$[\![\alpha]\!]_w^{\mathcal{M}} = 1$
$\mathcal{M}, w \dashv\Vdash \alpha$	iff	$[\![\alpha]\!]_w^{\mathcal{M}} = 0$
$\mathcal{M}, w \Vdash \bot$		never
$\mathcal{M}, w \dashv\Vdash \bot$		always
$\mathcal{M}, w \Vdash \bigstar$		never
$\mathcal{M}, w \dashv\Vdash \bigstar$		never
$\mathcal{M}, w \Vdash \varphi \to \psi$	iff	$\mathcal{M}, w \dashv\Vdash \varphi$ or $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \dashv\Vdash \varphi \to \psi$	iff	$\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \dashv\Vdash \psi$
$\mathcal{M}, w \Vdash \mathfrak{Z}x.\varphi$	iff	There is a $n \in \mathfrak{N}$; $\mathcal{M}, w \Vdash \varphi[^n/_x]$
$\mathcal{M}, w \dashv\Vdash \mathfrak{Z}x.\varphi$	iff	For all $n \in \mathfrak{N}$; $\mathcal{M}, w \dashv\Vdash \varphi[^n/_x]$
$\mathcal{M}, w \Vdash \Diamond\varphi$	iff	There is a $w' \in \mathcal{N}(w)$; $\mathcal{M}, w' \Vdash \varphi$
$\mathcal{M}, w \dashv\Vdash \Diamond\varphi$	iff	For all $w' \in \mathcal{N}(w)$; $\mathcal{M}, w' \nVdash \varphi$

It useful, both to test the given definitions and to shorten proofs, to introduce the following lemma:

Lemma 3.1.1. *Given any model $\mathcal{M}$, possible world w and formula φ it cannot be the case that $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \dashv\Vdash \varphi$*

Proof. We can prove this lemma by induction on φ

Base case

$\alpha \in \texttt{Atom}$ For every atom since the interpretation $[\![-]\!]$ is a function it cannot be the case that $[\![\alpha]\!]_w^{\mathcal{M}} = 0 = 1$.

$\bot$ and $\bigstar$ Trivially since both $\mathcal{M}, w \Vdash \bot$ and $\mathcal{M}, w \Vdash \bigstar$ are never the case.

Inductive step (With IH being "for any φ less complex than ρ, it is not the case that $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \dashv\vert \varphi$")

$\rho \stackrel{def}{=} \varphi \to \psi$ Assume for contradiction that (I) $\mathcal{M}, w \Vdash \varphi \to \psi$ and (II) $\mathcal{M}, w \dashv\vert \varphi \to \psi$, by definition from (I) we have that it must be the case that either $\mathcal{M}, w \dashv\vert \varphi$ or $\mathcal{M}, w \Vdash \psi$ and from (II) we also have that $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \dashv\vert \psi$. However, from the inductive hypothesis (IH) we have that neither $\mathcal{M}, w \dashv\vert \varphi$ and $\mathcal{M}, w \Vdash \varphi$ nor $\mathcal{M}, w \Vdash \psi$ and $\mathcal{M}, w \dashv\vert \psi$ is possible, therefore by contradiction it cannot be the case that (I) and (II).

$\rho \stackrel{def}{=} \mathcal{E}x.\varphi$ Assume for contradiction that (I) $\mathcal{M}, w \Vdash \mathcal{E}x.\varphi$ and (II) $\mathcal{M}, w \dashv\vert \mathcal{E}x.\varphi$. By definition from (I) we have that there is a $n \in \mathfrak{N}$ such that $\mathcal{M}, w \Vdash \varphi[^n/_x]$ and from (II) we have that for all $n' \in \mathfrak{N}$, $\mathcal{M}, w \dashv\vert \varphi[^{n'}/_x]$. However, by IH we have that it cannot be the case that there is an n such that $\mathcal{M}, w \Vdash \varphi[^n/_x]$ and for all n' $\mathcal{M}, w \dashv\vert \varphi[^{n'}/_x]$, therefore by contradiction it cannot be the case that (I) and (II).

$\rho \stackrel{def}{=} \Diamond\varphi$ Assume for contradiction that (I) $\mathcal{M}, w \Vdash \Diamond\varphi$ and (II) $\mathcal{M}, w \dashv\vert \Diamond\varphi$. Thus we have from (I) that there is a $w' \in \mathcal{N}(w)$; $\mathcal{M}, w' \Vdash \varphi$ and from (II) that For all $w'' \in \mathcal{N}(w)$; $\mathcal{M}, w'' \nVdash \varphi$. However, we know that in w' it cannot be the case that $\mathcal{M}, w' \Vdash \varphi$ and $\mathcal{M}, w' \nVdash \varphi$, therefore by contradiction it cannot be the case that (I) and (II).

□

3.1.3 Substitutivity of Identicals

With the given definitions we can check that substitutivity of identical terms preserve truth value.

FACT 2 [Substitutivity of Identicals]
If $\mathcal{M}, w \Vdash b = c$ *then*
$\mathcal{M}, w \Vdash \varphi[^b/_x]$ *iff* $\mathcal{M}, w \Vdash \varphi[^c/_x]$ *and*
$\mathcal{M}, w \dashv\vert \varphi[^b/_x]$ *iff* $\mathcal{M}, w \dashv\vert \varphi[^c/_x]$

Proof. We can prove this lemma by induction on φ

Base case

$\alpha \in$ `Atom` Assume $\mathcal{M}, w \Vdash b = c$ and $\alpha \in$ `Atom`. By Leibniz's Laws $\mathcal{M}, w \Vdash \alpha[^b/_x]$ iff $\mathcal{M}, w \Vdash \alpha[^c/_x]$ and $\mathcal{M}, w \dashv\vert \alpha[^b/_x]$ then $\mathcal{M}, w \dashv\vert \alpha[^c/_x]$

$\bot, \bigstar$ Trivially

Inductive step (With IH being "for any φ less complex than ρ, If $\mathcal{M}, w \Vdash b = c$ then $\mathcal{M}, w \Vdash \varphi[^b/_x]$ iff $\mathcal{M}, w \Vdash \varphi[^c/_x]$ and $\mathcal{M}, w \dashv\vert \varphi[^b/_x]$ iff $\mathcal{M}, w \dashv\vert \varphi[^c/_x]$")

$\rho \stackrel{def}{=} \varphi \to \psi$ Assume $\mathcal{M}, w \Vdash b = c$.

$$\begin{aligned}
\mathcal{M}, w \Vdash (\varphi \to \psi)[^b/_x] \ &\text{iff} \ \ \mathcal{M}, w \Vdash \varphi[^b/_x] \to \psi[^b/_x] \\
&\text{iff} \ \ \mathcal{M}, w \dashv\!\!\vdash \varphi[^b/_x] \text{ or } \mathcal{M}, w \Vdash \psi[^b/_x] \\
&\text{iff (By IH)} \ \mathcal{M}, w \dashv\!\!\vdash \varphi[^c/_x] \text{ or } \mathcal{M}, w \Vdash \psi[^c/_x] \\
&\text{iff} \ \ \mathcal{M}, w \Vdash (\varphi \to \psi)[^c/_x]
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}, w \dashv\!\!\vdash (\varphi \to \psi)[^b/_x] \ &\text{iff} \ \ \mathcal{M}, w \dashv\!\!\vdash \varphi[^b/_x] \to \psi[^b/_x] \\
&\text{iff} \ \ \mathcal{M}, w \Vdash \varphi[^b/_x] \text{ and } \mathcal{M}, w \dashv\!\!\vdash \psi[^b/_x] \\
&\text{iff (By IH)} \ \mathcal{M}, w \Vdash \varphi[^c/_x] \text{ and } \mathcal{M}, w \dashv\!\!\vdash \psi[^c/_x] \\
&\text{iff} \ \ \mathcal{M}, w \dashv\!\!\vdash (\varphi \to \psi)[^c/_x]
\end{aligned}$$

$\rho \stackrel{def}{=} \mathfrak{Z} x.\varphi$ Trivial if $(\mathfrak{Z} x.\varphi)[^n/_x]$, analogous to the previous proof if y not identical to x and $(\mathfrak{Z} y.\varphi)[^n/_x]$

$\rho \stackrel{def}{=} \Diamond\varphi$ Assume $\mathcal{M}, w \Vdash b = c$

$$\begin{aligned}
\mathcal{M}, w \Vdash (\Diamond\varphi)[^b/_x] \ &\text{iff} \ \ \text{There is } w' \in \mathcal{N}(w) \text{ s.t. } \mathcal{M}, w' \Vdash \varphi[^b/_x] \\
&\text{iff (by Nominal Rigidity)} \ \ \text{There is } w' \in \mathcal{N}(w) \text{ s.t. } \mathcal{M}, w' \Vdash \varphi[^b/_x] \wedge b = c \\
&\text{iff (By IH) There is } w' \in \mathcal{N}(w) \text{ s.t. } \mathcal{M}, w' \Vdash \varphi[^c/_x] \wedge b = c \\
&\text{iff} \ \ \text{There is } w' \in \mathcal{N}(w) \text{ s.t. } \mathcal{M}, w' \Vdash \varphi[^c/_x] \\
&\text{iff} \ \ \mathcal{M}, w \Vdash (\Diamond\varphi)[^c/_x]
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}, w \dashv\!\!\vdash (\Diamond\varphi)[^b/_x] \ &\text{iff} \ \ \text{For all } w' \in \mathcal{N}(w); \ \mathcal{M}, w' \nVdash \varphi[^b/_x] \\
&\text{iff (by Nominal Rigidity)} \ \ \text{For all } w' \in \mathcal{N}(w); \ \mathcal{M}, w' \nVdash \varphi[^b/_x] \text{ and } \mathcal{M}, w' \Vdash b = c \\
&\text{iff (By IH) For all } w' \in \mathcal{N}(w); \ \mathcal{M}, w' \nVdash \varphi[^c/_x] \text{ and } \mathcal{M}, w' \Vdash b = c \\
&\text{iff (by Nominal Rigidity)} \ \ \text{For all } w' \in \mathcal{N}(w); \ \mathcal{M}, w' \nVdash \varphi[^c/_x] \\
&\text{iff} \ \ \mathcal{M}, w \dashv\!\!\vdash (\Diamond\varphi)[^c/_x]
\end{aligned}$$

□

3.1.4 Algebraic approach for Informative Content

In the first chapter we introduced the notion of informative content or truth-set of a sentence. In a bilateral system it is no longer the case that the complement of the truth-set is the set where the proposition is false.

The algebra of informative content in Partial Modal Logic uses the notion of *truth-set of* φ (i.e. $|\varphi|^{\top}$) and *falsehood-set of* φ (i.e. $|\varphi|^{\perp}$). The former is the set of those world where φ is true, the latter were it is false. In general we can define it as follows:

Definition 20 [Truth and falsehood set]
For every well-formed formula $\varphi \in \mathcal{L}^{PW}$:

- $|\varphi|^{\top}_{\mathcal{M}} \stackrel{def}{=} \{w \in \mathcal{W} \mid \mathcal{M}, w \Vdash \varphi\}$
- $|\varphi|^{\bot}_{\mathcal{M}} \stackrel{def}{=} \{w \in \mathcal{W} \mid \mathcal{M}, w \dashV \varphi\}$

The algebra of truth/falsehood-set is discussed in section A.3 on page 89, alongside with the algebra of well-formed expressions and their correspondence results with the truth/falsehood conditions.

3.2 Entailment and Validity in the Bilateral Approach

In this section I will introduce the definition of validity and various definitions of entailments along with their properties. All such notions are to be read as "Possible Worlds validity" and "Possible Worlds entailments", as oppose to next Chapter's "States validity" and "States entailments"

3.2.1 Entailments

The notion of Possible Worlds entailment is particularly interesting in this approach, since there are definitions of different strength that can be employed. In this thesis I will show results for only some of them; in particular the ones I call *Strong, Intermediate and Weak.* The intermediate level is additionally divided in *Intermediate Positive Entailment* and *Intermediate Negative Entailment.*

The Strong Entailment (SE) is surely the closest notion to the classical one, therefore, following the definition of chapter 1 on page 7 I also call it "PhIL Enhancement" from now on. Moreover, where there is no ambiguity I will call the Intermediate Positive Entailment just Positive Entailment (PE) and the Intermediate Negative Entailment just Negative Entailment (NE) while the Weak Entailment will be abbreviated WE.

Definition 21 [Strong Entailment]
We say that ψ PhIL enhances φ if and only if ψ is always at least as strong as φ

- $\varphi \stackrel{strong}{\Vdash} \psi$ iff for all $\mathcal{M}$ and w; $[\![\varphi]\!]^{\mathcal{M}}_{w} \leq [\![\psi]\!]^{\mathcal{M}}_{w}$

Definition 22 [Intermediate Positive Entailment]
We say that φ positively entails ψ if and only if when φ is true also ψ is true.

- $\varphi \Vdash \psi$ iff for all $\mathcal{M}$ and w; if $\mathcal{M}, w \Vdash \varphi$ then $\mathcal{M}, w \Vdash \psi$

Definition 23 [Intermediate Negative Entailment]
We say that ψ negatively entails φ if and only if when ψ is false also φ is false

- $\psi \dashv\!\mid \varphi$ iff for all $\mathcal{M}$ and w; if $\mathcal{M}, w \dashv\!\mid \psi$ then $\mathcal{M}, w \dashv\!\mid \varphi$

Definition 24 [Weak Entailment]
We say that φ weakly entails ψ if and only if when φ is true ψ is not false

- $\varphi \overset{weak}{\Vdash} \psi$ iff for all $\mathcal{M}$ and w; if $w \Vdash \varphi$ then $w \not\dashv\!\mid \psi$

The Enhancement is the most important of the three, since we will see that validity is closed with respect to it under *Modus Ponens* and *Modus Tollens* as a corollary of 8 and fact 9 on page 43.

Moreover, it is possible to show that PhIL's enhancement is equivalent to many other important notions previously introduced.

FACT 3 [Equivalent notions of PhIL enhancement]
Given φ and ψ the following are equivalent:

- $\varphi \overset{strong}{\Vdash} \psi$
- *for all $\mathcal{M}$ and w;* $[\![\varphi]\!]^{\mathcal{M}}_w \leq [\![\psi]\!]^{\mathcal{M}}_w$
- $\varphi \Vdash \psi$ *and* $\psi \dashv\!\mid \varphi$
- *for all $\mathcal{M}$;* $|\varphi|^{\top}_{\mathcal{M}} \subseteq |\psi|^{\top}_{\mathcal{M}}$ *and* $|\psi|^{\bot}_{\mathcal{M}} \subseteq |\varphi|^{\bot}_{\mathcal{M}}$

Proof. Follows from the definitions and the correspondence proof on section A.4 on page 89. □

A corollary of said fact is that Strong Entailment entails both the Intermediate Entailments.

$$\frac{\varphi \Vdash \psi \qquad \psi \dashv\!\mid \varphi}{\varphi \overset{strong}{\Vdash} \psi} \qquad\qquad \frac{\varphi \overset{strong}{\Vdash} \psi}{\psi \dashv\!\mid \varphi} \qquad\qquad \frac{\varphi \overset{strong}{\Vdash} \psi}{\varphi \Vdash \psi}$$

Additionally, we can prove that the Intermediate Entailments independently entail Weak Entailment.

FACT 4 [If $\varphi \Vdash \psi$ then $\varphi \overset{weak}{\Vdash} \psi$]

$$\frac{\varphi \Vdash \psi}{\varphi \overset{weak}{\Vdash} \psi}$$

Proof. Assume $\varphi \Vdash \psi$, by def. for all $\mathcal{M}$& $w \in \mathcal{W}$ (I) if $\mathcal{M}, w \Vdash \varphi$ then $\mathcal{M}, w \Vdash \psi$.

Now assume $\mathcal{M}, w \Vdash \varphi$, by Modus Ponens with (I) we have that $\mathcal{M}, w \Vdash \psi$, and by Lemma 3.1.1 that implies $\mathcal{M}, w \not\Dashv \psi$; therefore by definition $\varphi \overset{weak}{\Vdash} \psi$. ☐

FACT 5 [If $\psi \Dashv \varphi$ then $\varphi \overset{weak}{\Vdash} \psi$]

$$\frac{\psi \Vdash \varphi}{\varphi \overset{weak}{\Vdash} \psi}$$

Proof. Assume $\psi \Dashv \varphi$, by def. for all $\mathcal{M}$& $w \in \mathcal{W}$ (I) if $\mathcal{M}, w \Dashv \psi$ then $\mathcal{M}, w \Dashv \varphi$.

Now assume $\mathcal{M}, w \not\Dashv \varphi$, from this by Lemma 3.1.1 we know that $\mathcal{M}, w \nVdash \varphi$, and by Modus Tollens with (I) we have that $\mathcal{M}, w \not\Dashv \psi$; therefore by definition $\varphi \overset{weak}{\Vdash} \psi$. ☐

This means that in general these definitions have the following strength ordering:

$$\begin{array}{ccc} & \varphi \overset{strong}{\Vdash} \psi & \\ \swarrow & & \searrow \\ \varphi \Vdash \psi & & \psi \Dashv \varphi \\ \searrow & & \swarrow \\ & \varphi \overset{weak}{\Vdash} \psi & \end{array}$$

Figure 3.3: Entailment relations in order of strength

Some of the most important properties of these entailments are proved across this section and summarized in figure 3.4 on page 42.

3.2.2 Valid and Invalid formulas

Now we can define Possible World validity. We have two main alternatives: one considers valid any sentence that is always true (in every world of every model), the other any sentence that is never false.

In this thesis I will employ the second definition and show how under it the valid sentences in the fragment $\mathcal{L}^{PW} \setminus \{\mathcal{E}, \bigstar\}$ is sound with respect to Classical Validity.

Definition 25 [PW validity]
A sentence φ is PhIL PW valid if and only if φ enhances $\bigstar$.

$\Vdash \varphi$ iff $\bigstar \overset{strong}{\Vdash} \varphi$

We know that in Classical Logic a formula is valid if and only if it is a tautology. The antonym of tautology is contradiction, and in PhIL the definition of contradiction is the dual of the tautology, i.e. a formula that is never true.

FACT 6 [Negation Switch]
$\Vdash \varphi$ *iff* $\dashv\!\vert\; \neg\varphi$ *and* $\dashv\!\vert\; \psi$ *iff* $\Vdash \neg\psi$

Proof.

$$\begin{aligned}
\Vdash \varphi \text{ iff } & \text{ for all } \mathcal{M} \text{ and } w \in \mathcal{W}; \mathcal{M}, w \not\dashv\!\vert\; \varphi \\
\text{iff } & \text{ for all } \mathcal{M} \text{ and } w \in \mathcal{W}; \mathcal{M}, w \not\dashv\!\vert\; \varphi \text{ or } \mathcal{M}, w \Vdash \bot \\
\text{iff } & \text{ for all } \mathcal{M} \text{ and } w \in \mathcal{W}; \mathcal{M}, w \nVdash \varphi \to \bot \\
\text{iff } & \text{ for all } \mathcal{M} \text{ and } w \in \mathcal{W}; \mathcal{M}, w \nVdash \neg\varphi \\
\text{iff } & \dashv\!\vert\; \neg\varphi
\end{aligned}$$

$$\begin{aligned}
\dashv\!\vert\; \varphi \text{ iff } & \quad \text{for all } \mathcal{M} \text{ and } w \in \mathcal{W}; \mathcal{M}, w \nVdash \varphi \\
\text{iff } & \quad \text{for all } \mathcal{M} \text{ and } w \in \mathcal{W}; \mathcal{M}, w \not\dashv\!\vert\; \neg\varphi \\
\text{iff } & \quad \Vdash \neg\varphi
\end{aligned}$$

□

Under the aforementioned definition of validity we can see a reason why the Weak Entailment is an important notion: it is the only entailment that proves the Deduction Theorem, as shown in fact 11 on page 44.

Equivalences and Correspondence

Applying the definitions of Entailment and the equivalences of fact 3 on page 39 it is possible to provide many equivalent definitions of valid and invalid formulas:

	MP	MT	Contr.	Ded.Th.	Ref.	Mod. Weak.	Rad. Weak.	Trans.	Abs.
$\overset{strong}{\Vdash}$	✓	✓	✓	✗	✓	✓	✗	✓	✓
$\Vdash$	✗	✓	$\dashV$	✗	✓	✓	✗	✓	✗
$\dashV$	✓	✗	$\Vdash$	✗	✓	✓	✗	✓	✓
$\overset{weak}{\Vdash}$	✗	✗	✓	✓	✓	✓	✓	✗	✓

Figure 3.4: SE, PE, NE and WE and their properties

FACT 7 [Equivalent notions of validity and invalidity]

Given φ the following are equivalent:

- $\Vdash \varphi$
- *for all* $\mathcal{M}$ *and* w; $N \leq [\![\varphi]\!]_w^{\mathcal{M}}$
- *for all* $\mathcal{M}$ *and* w; $\mathcal{M}, w \not\dashV \varphi$
- $\varphi \dashV \bigstar$
- $\top \overset{weak}{\Vdash} \varphi$
- *for all* $\mathcal{M}$; $|\varphi|^{\perp}_{\mathcal{M}} = \emptyset$

Given ψ the following are equivalent:

- $\dashV \psi$
- *for all* $\mathcal{M}$ *and* w; $N \geq [\![\psi]\!]_w^{\mathcal{M}}$
- *for all* $\mathcal{M}$ *and* w; $\mathcal{M}, w \nVdash \psi$
- $\psi \Vdash \bigstar$
- $\psi \overset{weak}{\Vdash} \perp$
- *for all* $\mathcal{M}$; $|\psi|^{\top}_{\mathcal{M}} = \emptyset$

Proof. Follows from the definitions. □

3.2.3 Properties and Closure

In this section I will prove which classical properties hold for the four entailments and under which classical rules validity is preserved. In figure 3.4 it is possible to see a summary of the results of this section.

Modus Tollens and Modus Ponens

We can prove that with the aforementioned approach validity is not preserved under Modus Ponens (MP) nor Modus Tollens (MT) with respect to WE. However, validity is preserved under Modus Ponens (MP) and Modus Tollens (MT) with respect to Enhancement. This can be proved to be a consequence of the fact that validity is preserved under MP wrt the Negative Entailment and under MT wrt the Positive.

FACT 8 [Validity is closed over MP wrt NE]

$$\frac{\psi \dashV \varphi \qquad \Vdash \varphi}{\Vdash \psi}$$

Proof. Assume that (I) $\Vdash \varphi$ and (II) $\psi \dashV \varphi$. From (II) follows that for all $\mathcal{M}$ and w if $\mathcal{M}, w \dashV \psi$ then $\mathcal{M}, w \dashV \varphi$. By contraposition for all $\mathcal{M}$ and w if $\mathcal{M}, w \not\dashV \varphi$ then $\mathcal{M}, w \not\dashV \psi$. By (I) we have that for all $\mathcal{M}$ and w, $\mathcal{M}, w \not\dashv\!\!\mid \varphi$, therefore for all $\mathcal{M}$ and w, $\mathcal{M}, w \not\dashv\!\!\mid \psi$. By definition this means that $\Vdash \psi$. □

FACT 9 [Validity is closed over MT wrt PE]

$$\frac{\varphi \Vdash \psi \qquad \dashv\!\!\mid \psi}{\dashv\!\!\mid \varphi}$$

Proof. Assume that (I) $\dashv\!\!\mid \psi$ and (II) $\varphi \Vdash \psi$. From (II) follows that for all $\mathcal{M}$ and w if $\mathcal{M}, w \Vdash \varphi$ then $\mathcal{M}, w \Vdash \psi$. By contraposition for all $\mathcal{M}$ and w if $\mathcal{M}, w \nVdash \psi$ then $\mathcal{M}, w \nVdash \varphi$. By (I) we have that for all $\mathcal{M}$ and w, $\mathcal{M}, w \nVdash \psi$, therefore for all $\mathcal{M}$ and w, $\mathcal{M}, w \nVdash \varphi$. By definition this means that $\dashv\!\!\mid \varphi$. □

So in particular we also have that Validity is closed over MP and MT wrt Enhancement.

$$\frac{\dfrac{\varphi \overset{strong}{\Vdash} \psi}{\psi \dashv\!\!\mid \varphi} \qquad \Vdash \varphi}{\Vdash \psi} \qquad\qquad \frac{\dfrac{\varphi \overset{strong}{\Vdash} \psi}{\varphi \Vdash \psi} \qquad \dashv\!\!\mid \psi}{\dashv\!\!\mid \varphi}$$

FACT 10 [Validity is not closed over MP (nor over MT) wrt WE]

Proof. Consider the formula $\bigstar \to \bot$. Since $\bigstar$ is always undefined it is never false. For the same reason $\bigstar \to \bot$ itself can never be false, thus $\Vdash \bigstar$ and $\bigstar \overset{weak}{\Vdash} \bot$. However, it is of course false that $\Vdash \bot$.
For MT consider $\top \to \bigstar$. The proof is analogous.
The proof also works without $\bigstar$ as follows. Consider the formula $(P(n) \vee \neg P(n)) \to \mathcal{E}(n)$.
Assume that for a certain $\mathcal{M}$ and $w \in \mathcal{W}$ we have that $\mathcal{M}, w \dashv\!\!\mid (P(n) \vee \neg P(n))$, by definition that would mean that $\mathcal{M}, w \dashv\!\!\mid P(n)$ and $\mathcal{M}, w \dashv\!\!\mid \neg P(n))$, thus $\mathcal{M}, w \dashv\!\!\mid P(n)$ and $\mathcal{M}, w \Vdash P(n))$. By Lemma 3.1.1 that is not possible. Since $\mathcal{M}$ and w are arbitrary we have that $\Vdash (P(n) \vee \neg P(n))$.
Assume that for a certain $\mathcal{M}$ and $w \in \mathcal{W}$ we have that $\mathcal{M}, w \Vdash (P(n) \vee \neg P(n))$, this mean that either $\mathcal{M}, w \Vdash P(n)$ or $\mathcal{M}, w \dashv\!\!\mid P(n)$. In any case that implies that $[\![\mathcal{E}(n)]\!]^{\mathcal{M}}_{w} = 1$, thus $\mathcal{M}, w \Vdash \mathcal{E}(n)$. This shows that it cannot be the case that if $\mathcal{M}, w \Vdash P(n) \vee \neg P(n)$ then $\mathcal{M}, w \not\dashv\!\!\mid \mathcal{E}(n)$. Since $\mathcal{M}$ and w are arbitrary in general $P(n) \vee \neg P(n) \overset{weak}{\Vdash} \mathcal{E}(n)$.
However, $\nVdash \mathcal{E}(n)$. In fact it is sufficient to consider a model $\mathcal{M}$ and

world w such that $[\![\mathcal{E}(n)]\!]_w^{\mathcal{M}} = 0$.

For MT consider $\neg\mathcal{E}(n) \overset{weak}{\Vdash} P(n)\wedge\neg P(n)$. The proof is analogous. $\square$

Contraposition

The classical rule of contraposition is respected by the WE and the Enhancement, as a corollary of the fact that PE is equivalent to its contraposition in NE, and vice versa.

$$\begin{aligned}\varphi \overset{weak}{\Vdash} \psi \text{ iff } & \text{ For all } \mathcal{M}\&\ w \in \mathcal{W} \text{ if } \mathcal{M}, w \Vdash \varphi \text{ then } \mathcal{M}, w \not\dashv\!\!\mid \psi \\ \text{iff } & \text{ For all } \mathcal{M}\&\ w \in \mathcal{W} \text{ if } \mathcal{M}, w \dashv\!\!\mid \psi \text{ then } \mathcal{M}, w \nVdash \varphi \\ \text{iff } & \text{ For all } \mathcal{M}\&\ w \in \mathcal{W} \text{ if } \mathcal{M}, w \Vdash \neg\psi \text{ then } \mathcal{M}, w \not\dashv\!\!\mid \neg\varphi \\ \text{iff } & \neg\psi \overset{weak}{\Vdash} \neg\varphi\end{aligned}$$

$$\begin{aligned}\varphi \Vdash \psi \text{ iff } & \text{For all } \mathcal{M}\&\ w \in \mathcal{W}, \text{ if } \mathcal{M}, w \Vdash \varphi \text{ then } \mathcal{M}, w \Vdash \psi \\ \text{iff } & \text{For all } \mathcal{M}\&\ w \in \mathcal{W}, \text{ if } \mathcal{M}, w \dashv\!\!\mid \neg\varphi \text{ then } \mathcal{M}, w \dashv\!\!\mid \neg\psi \\ \text{iff } & \neg\varphi \dashv\!\!\mid \neg\psi\end{aligned}$$

$$\begin{aligned}\neg\psi \Vdash \neg\varphi \text{ iff } & \text{For all } \mathcal{M}\&\ w \in \mathcal{W}, \text{ if } \mathcal{M}, w \Vdash \neg\psi \text{ then } \mathcal{M}, w \Vdash \neg\varphi \\ \text{iff } & \text{For all } \mathcal{M}\&\ w \in \mathcal{W}, \text{ if } \mathcal{M}, w \dashv\!\!\mid \psi \text{ then } \mathcal{M}, w \dashv\!\!\mid \varphi \\ \text{iff } & \psi \dashv\!\!\mid \varphi\end{aligned}$$

$$\begin{aligned}\varphi \overset{strong}{\Vdash} \psi \text{ iff } & \varphi \Vdash \psi \text{ and } \psi \dashv\!\!\mid \varphi \\ \text{iff } & \neg\varphi \dashv\!\!\mid \neg\psi \text{ and } \neg\psi \Vdash \neg\varphi \\ \text{iff } & \neg\psi \overset{strong}{\Vdash} \neg\varphi\end{aligned}$$

Deduction Theorem and Implication-In

As mentioned before an important aspect of the Weak Entailment is that it not only proves the introduction rule of the implication, as every other entailment, but also its "dual": the Deduction Theorem. A corollary of the Deduction theorem and fact 10 on the preceding page is that also valid implications do not preserve validity under MP and MT.

FACT 11 [Deduction Theorem for WE]

$$\frac{\varphi \overset{weak}{\Vdash} \psi}{\Vdash \varphi \to \psi} \qquad\qquad \frac{\Vdash \varphi \to \psi}{\varphi \overset{weak}{\Vdash} \psi}$$

Proof.

$\varphi \overset{weak}{\Vdash} \psi$ iff For all $\mathcal{M}\&\ w \in \mathcal{W}$, if $\mathcal{M}, w \Vdash \varphi$ then $\mathcal{M}, w \not\dashv\!\vert\ \psi$
iff does not Exist $\mathcal{M}\&\ w \in \mathcal{W}$ such that $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \dashv\!\vert\ \psi$
iff does not Exist $\mathcal{M}\&\ w \in \mathcal{W}$ such that $\mathcal{M}, w \dashv\!\vert\ \varphi \to \psi$
iff $\Vdash \varphi \to \psi$

□

$$\dfrac{\dfrac{\varphi \Vdash \psi}{\varphi \overset{weak}{\Vdash} \psi}}{\Vdash \varphi \to \psi} \qquad \dfrac{\dfrac{\psi \dashv\!\vert\ \varphi}{\varphi \overset{weak}{\Vdash} \psi}}{\Vdash \varphi \to \psi} \qquad \dfrac{\dfrac{\dfrac{\varphi \overset{strong}{\Vdash} \psi}{\varphi \Vdash \psi}}{\varphi \overset{weak}{\Vdash} \psi}}{\Vdash \varphi \to \psi}$$

Reflexivity

It is easy to show that any formula Enhances itself, as consequence of this we also have that PE, NE and WE are reflexive:

FACT 12 [Ref]

$$\dfrac{}{\varphi \overset{strong}{\Vdash} \varphi}$$

Proof. Trivially, since for any $\mathcal{M}\&\ w \in \mathcal{W}$ if $\mathcal{M}, w \Vdash \varphi$ then $\mathcal{M}, w \Vdash \varphi$ and if $\mathcal{M}, w \dashv\!\vert\ \varphi$ then $\mathcal{M}, w \dashv\!\vert\ \varphi$ □

$$\dfrac{\dfrac{}{\varphi \overset{strong}{\Vdash} \varphi}}{\varphi \Vdash \varphi} \qquad \dfrac{\dfrac{}{\varphi \overset{strong}{\Vdash} \varphi}}{\varphi \dashv\!\vert\ \varphi} \qquad \dfrac{\dfrac{\dfrac{}{\varphi \overset{strong}{\Vdash} \varphi}}{\varphi \Vdash \varphi}}{\varphi \overset{weak}{\Vdash} \varphi}$$

Moderate and Radical Weakening

We can prove that if a formula φ Positively Entails ψ then also any other stronger formula $\varphi \wedge \rho$ Positively Entails ψ, the same hold for WE while the opposite is true with NE (i.e. if ψ Negatively Entails φ then also any other ψ Negatively Entails any stronger formula $\varphi \wedge \rho$). I call this property *Moderate Weakening*, since it goes from a certain entailment to the weakened version of the same entailment. We can prove from the previous claims that Mod. Weakening holds for Enhancement too.

It is distinct from the *Radical Weakening*, that says that if $\Vdash \varphi$ then any formula entails φ, therefore if goes from a validity to a weakened entailment. Radical Weakening holds only for WE.

FACT 13 [PE, NE and WE Moderate Weakening]

$$\frac{\varphi \Vdash \psi}{\varphi \wedge \rho \Vdash \psi} \qquad \frac{\psi \dashv\Vdash \varphi}{\psi \dashv\Vdash \varphi \wedge \rho} \qquad \frac{\varphi \overset{weak}{\Vdash} \psi}{\varphi \wedge \rho \overset{weak}{\Vdash} \psi}$$

Proof. Assume $\varphi \Vdash \psi$, by definition for any $\mathcal{M}\&\ w \in \mathcal{W}$ (1) if $\mathcal{M}, w \Vdash \varphi$ then $\mathcal{M}, w \Vdash \psi$. Now assume for arbitrary $\mathcal{M}'\&\ w' \in \mathcal{W}$ $\mathcal{M}', w' \Vdash \varphi \wedge \rho$, by definition we have that (2) $\mathcal{M}', w' \Vdash \varphi$ and $\mathcal{M}', w' \Vdash \rho$, from (1) and (2) we have that $\mathcal{M}', w' \Vdash \psi$. Wlog, for any $\mathcal{M}\&\ w \in \mathcal{W}$ if $\mathcal{M}, w \Vdash \varphi \wedge \rho$ then $\mathcal{M}', w' \Vdash \psi$, therefore $\varphi \wedge \rho \Vdash \psi$.

Analogue proofs hold for "if $\psi \dashv\Vdash \varphi$ then $\psi \dashv\Vdash \varphi \wedge \rho$" and "if $\varphi \overset{weak}{\Vdash} \psi$ then $\varphi \wedge \rho \overset{weak}{\Vdash} \psi$" □

$$\cfrac{\cfrac{\cfrac{\varphi \overset{strong}{\Vdash} \psi}{\varphi \Vdash \psi}}{\varphi \wedge \rho \Vdash \psi} \qquad \cfrac{\cfrac{\varphi \overset{strong}{\Vdash} \psi}{\psi \dashv\Vdash \varphi}}{\psi \dashv\Vdash \varphi \wedge \rho}}{\varphi \wedge \rho \overset{strong}{\Vdash} \psi}$$

FACT 14 [WE Radical Weakening]

$$\frac{\Vdash \varphi}{\rho \overset{weak}{\Vdash} \varphi}$$

Proof. From fact 7 on page 42 we have that $\Vdash \varphi$ is equivalent to $\top \overset{weak}{\Vdash} \varphi$, and by Moderate Weakening we can prove $\top \wedge \rho \overset{weak}{\Vdash} \varphi$ which is equivalent to $\rho \overset{weak}{\Vdash} \varphi$. □

Transitivity

PE and NE (and as a consequence Enhancement) are transitive, while WE is not.

FACT 15 [PE and NE are Transitive]

$$\frac{\varphi \Vdash \psi \qquad \psi \Vdash \rho}{\varphi \Vdash \rho} \qquad\qquad \frac{\rho \mathrel{\dashv\mid} \psi \qquad \psi \mathrel{\dashv\mid} \varphi}{\rho \mathrel{\dashv\mid} \varphi}$$

Proof. Follows from definition. □

$$\dfrac{\dfrac{\dfrac{\psi \overset{strong}{\Vdash} \rho}{\rho \mathrel{\dashv\mid} \psi} \quad \dfrac{\varphi \overset{strong}{\Vdash} \psi}{\psi \mathrel{\dashv\mid} \varphi}}{\varphi \mathrel{\dashv\mid} \rho} \qquad \dfrac{\dfrac{\varphi \overset{strong}{\Vdash} \psi}{\varphi \Vdash \psi} \quad \dfrac{\psi \overset{strong}{\Vdash} \rho}{\psi \Vdash \rho}}{\rho \Vdash \varphi}}{\rho \overset{strong}{\Vdash} \varphi}$$

FACT 16 [WE is not Transitive]

Proof. Consider $\varphi \overset{def}{=} P(n) \vee \neg P(n)$, $\psi \overset{def}{=} P(m) \vee \neg P(m)$ and $\rho \overset{def}{=} \mathcal{E}(m)$. Since $\Vdash \psi$, by Radical Weakening $\varphi \overset{weak}{\Vdash} \psi$.
As proved in fact 10 on page 43 $\Vdash \psi \to \rho$, thus by Deduction Theorem $\psi \overset{weak}{\Vdash} \rho$.
However, $\nVdash \varphi \to \rho$. In fact we can consider a Model and world s.t. $\mathcal{M}, w \Vdash \mathcal{E}(n)$ and $\mathcal{M}, w \mathrel{\dashv\mid} \mathcal{E}(m)$. Thus by Deduction Theorem $\varphi \overset{weak}{\nVdash} \rho$, □

Absurdum Rule

A similar distinction to Moderate and Radical Weakening can be done with the Reductio ad Absurdum. While the Absurdum rule always holds in the Radical case (i.e. from an entailment to a validity), in the Moderate case it holds regularly for the Enhancement, NE and WE but not for PE.

FACT 17 [Radical Absurdum]

$$\frac{\neg\varphi \overset{weak}{\Vdash} \bot}{\Vdash \varphi}$$

Proof. Assume $\neg\varphi \overset{weak}{\Vdash} \bot$. By def. for all $\mathcal{M}$& $w \in \mathcal{W}$ $\mathcal{M}, w \nVdash \neg\varphi$, thus for all $\mathcal{M}$& $w \in \mathcal{W}$ $\mathcal{M}, w \not\mathrel{\dashv\mid} \varphi$, which by def is $\Vdash \varphi$. □

By facts 4 and 5 we also have that Enhancement, NE and PE prove Radical Absurdum:

$$\dfrac{\dfrac{\neg\varphi \Vdash \bot}{\neg\varphi \overset{weak}{\Vdash} \bot}}{\Vdash \varphi} \qquad \dfrac{\dfrac{\bot \dashV \neg\varphi}{\neg\varphi \overset{weak}{\Vdash} \bot}}{\Vdash \varphi} \qquad \dfrac{\dfrac{\neg\varphi \overset{strong}{\Vdash} \bot}{\neg\varphi \overset{weak}{\Vdash} \bot}}{\Vdash \varphi}$$

FACT 18 [Moderate Absurdum]

$$\dfrac{\bot \dashV \rho \wedge \neg\varphi}{\rho \overset{strong}{\Vdash} \varphi} \qquad \dfrac{\rho \wedge \neg\varphi \overset{weak}{\Vdash} \bot}{\rho \overset{weak}{\Vdash} \varphi}$$

Proof. Assume $\bot \dashV \rho \wedge \neg\varphi$. By def. for all $\mathcal{M}\&\ w \in \mathcal{W}\ \mathcal{M}, w \dashV \rho \wedge \neg\varphi$, thus (1) for all $\mathcal{M}\&\ w \in \mathcal{W}$ if $\mathcal{M}, w \not\dashV \rho$ then $\mathcal{M}, w \Vdash \varphi$ and equivalently (2) for all $\mathcal{M}\&\ w \in \mathcal{W}$ if $\mathcal{M}, w \nVdash \varphi$ then $\mathcal{M}, w \dashV \rho$. Assume for an arbitrary $\mathcal{M}'\&\ w' \in \mathcal{W}\ \mathcal{M}', w' \Vdash \rho$, by Lemma 3.1.1 $\mathcal{M}', w' \not\dashV \rho$ thus by (1) $\mathcal{M}', w' \Vdash \varphi$. Wlog for all $\mathcal{M}\&\ w \in \mathcal{W}$ if $\mathcal{M}, w \Vdash \rho$ then $\mathcal{M}, w \Vdash \varphi$, therefore (i) $\rho \Vdash \varphi$. Assume for an arbitrary $\mathcal{M}''\&\ w'' \in \mathcal{W}\ \mathcal{M}'', w'' \dashV \varphi$, by Lemma 3.1.1 $\mathcal{M}'', w'' \nVdash \varphi$ thus by (2) $\mathcal{M}'', w'' \dashV \rho$. Wlog for all $\mathcal{M}\&\ w \in \mathcal{W}$ if $\mathcal{M}, w \dashV \varphi$ then $\mathcal{M}, w \dashV \rho$, therefore (i) $\varphi \dashV \psi$. ☐

We also have that Enhancement alone proves the Absurdum, that NE alone prove its own Absurdum and that PE proves WE's Absurdum:

$$\dfrac{\dfrac{\rho \wedge \neg\varphi \overset{strong}{\Vdash} \bot}{\bot \dashV \rho \wedge \neg\varphi}}{\rho \overset{strong}{\Vdash} \varphi} \qquad \dfrac{\dfrac{\bot \dashV \rho \wedge \neg\varphi}{\rho \overset{strong}{\Vdash} \varphi}}{\varphi \dashV \rho} \qquad \dfrac{\dfrac{\rho \wedge \neg\varphi \Vdash \bot}{\rho \wedge \neg\varphi \overset{weak}{\Vdash} \bot}}{\rho \overset{weak}{\Vdash} \varphi}$$

3.2.4 Parallelisms with Other Non-Classical Logics

It is interesting to notice that the notions of Entailments, Validity and Invalidity in PhIL are deeply connected with other standard accounts in classical logic. The definition of Positive Entailment is equivalent to Łukasiewicz's notion of Entailment, the definition of Negative Entailment is equivalent to the entailment in Priest's Logic of Paradox while PhIL's validity and the definition of Weak Entailment are equivalent to the Strict to Tolerant Validity and Entailment in [Cobreros et al., 2012].
However, some properties that currently do not hold for PhIL's Weak Entailment can be recovered in a fragment of the language that will be explored in the next section.

Notably, WE does not preserve validity over MP, MT and it is not transitive, while the aim of the restricted language is to recover these features.

3.3 Classical Logic in a restricted Language

3.3.1 Recovering Modus Ponens, Modus Tollens and Transitivity

The fact that implication and thus WE are not close under MP and MT applies only to the fragment of formulas with the constant ★ and the predicate of existence $\mathcal{E}$, while both can be recovered focusing only on $\mathcal{L}^{PW} \setminus \{\bigstar, \mathcal{E}\}$. To prove it we need a definition and few additional Lemmas:

Definition 26 [Inflated models]

Given an arbitrary model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, [\![-]\!] \rangle$ we say that $\mathcal{M}' = \langle \mathcal{W}', \mathcal{N}', [\![-]\!] \rangle$ is an inflated model of $\mathcal{M}$ (notation $\mathcal{M}' \in \mathfrak{I}(\mathcal{M})$) if and only if:

- $\mathcal{W} = \mathcal{W}'$.
- $\mathcal{N} = \mathcal{N}'$
- For every $w \in \mathcal{W}$

  Existence Predicate For every name $t \in \mathfrak{N}$: $[\![\mathcal{E}(t)]\!]_w^{\mathcal{M}'} = 1$

  Identity For every name $c, b \in \mathfrak{N}$: $[\![c = b]\!]_w^{\mathcal{M}} = [\![c = b]\!]_w^{\mathcal{M}'}$

  Propositions For every property and list of terms,
  if $[\![R(\vec{t})]\!]_w^{\mathcal{M}} = 1$ then $[\![R(\vec{t})]\!]_w^{\mathcal{M}'} = 1$ and
  if $[\![R(\vec{t})]\!]_w^{\mathcal{M}} = 0$ then $[\![R(\vec{t})]\!]_w^{\mathcal{M}'} = 0$

Lemma 3.3.1. *If every atom $\alpha \in \texttt{Atom}$ is defined everywhere in a model $\mathcal{M}$ then every formula $\varphi \in \mathcal{L}^{PW} \setminus \{\bigstar, \mathcal{E}\}$ is also defined in $\mathcal{M}$ and $w \in \mathcal{W}$.*

Proof. This proof is an induction on φ. The base case is trivial since it means to prove that if φ is defined then φ is defined.

Inductive step (With IH being "for any φ less complex than ρ, φ is defined in $\mathcal{M}$ and $w \in \mathcal{W}$")

$\rho \stackrel{def}{=} \varphi \to \psi$ Since φ and ψ are defined in $\mathcal{M}$ and $w \in \mathcal{W}$ either (I) $M, w \Vdash \varphi$ and $M, w \dashv\!\Vert \psi$, (II) $M, w \dashv\!\Vert \varphi$ (and ψ arbitrary) or (III) $M, w \Vdash \psi$ (and φ arbitrary).
If (I) is the case then $M, w \dashv\!\Vert \varphi \to \psi$ and if (II) or (III) $M, w \Vdash \varphi \to \psi$, either way $\varphi \to \psi$ is defined in w.

$\rho \stackrel{def}{=} \Sigma x.\varphi$ By IH for all $n \in \mathfrak{N}$, $\varphi[^n/_x]$ is defined. Therefore, either for some n $\mathcal{M}, w \Vdash \varphi[^n/_x]$, thus $\mathcal{M}, w \Vdash \Sigma x.\varphi$, or for every n $\mathcal{M}, w \dashv\!\Vert \varphi[^n/_x]$, thus $\mathcal{M}, w \dashv\!\Vert \Sigma x.\varphi$.

$\rho \stackrel{def}{=} \Diamond\varphi$ By IH for every $v \in \mathcal{N}(w)$, φ is defined in v. This means that either for some v, $\mathcal{M}, v \Vdash \varphi$, thus $\mathcal{M}, w \Vdash \Diamond\varphi$, or for every v, $\mathcal{M}, v \nVdash \varphi$, thus $\mathcal{M}, w \dashv\!| \Diamond\varphi$. In either case $\Diamond\varphi$ is defined.

□

As a corollary this lemma tells us that in any inflated model $\mathcal{M}' \in \mathfrak{I}(\mathcal{M})$ every formula is defined, i.e. either true or false.
The following lemma tells us that every formula $\varphi \in \mathcal{L}^{PW} \setminus \{\bigstar, \mathcal{E}\}$ that is true in a model $\mathcal{M}$ is also true in every inflated model $\mathcal{M}'$, and every formula false in $\mathcal{M}$ is also false in $\mathcal{M}'$.

Lemma 3.3.2. *For every formula $\varphi \in \mathcal{L}^{PW} \setminus \{\bigstar, \mathcal{E}\}$ and every $\mathcal{M}' \in \mathfrak{I}(\mathcal{M})$, if $\mathcal{M}, w \dashv\!| \varphi$ then $\mathcal{M}', w \dashv\!| \varphi$ and if $\mathcal{M}, w \dashv\!| \varphi$ then $\mathcal{M}', w \dashv\!| \varphi$.*

Proof. We can prove this with induction on the formulas.

Base case

$\varphi \stackrel{def}{=} a = b$ By definition $[\![a = b]\!]^{\mathcal{M}}_w = [\![a = b]\!]^{\mathcal{M}'}_w$

$\varphi \stackrel{def}{=} \bot$ trivially since $\bot$ is always accepted and always rejected

$\varphi \stackrel{def}{=} P(\vec{v}\,)$ Follows from the definition of Inflated model

Inductive step (With IH being "for any φ less complex than ρ, if φ was defined in $\mathcal{M}$, then it retains its truth value in $\mathcal{M}'$")

$\rho \stackrel{def}{=} \psi \to \chi$ Assume $\mathcal{M}, w \Vdash \psi \to \chi$, thus $\mathcal{M}, w \dashv\!| \psi$ or $\mathcal{M}, w \Vdash \chi$. By IH $\mathcal{M}', w \dashv\!| \psi$ or $\mathcal{M}', w \Vdash \chi$, therefore $\mathcal{M}', w \Vdash \psi \to \chi$.
Assume $\mathcal{M}, w \dashv\!| \psi \to \chi$, thus $\mathcal{M}, w \Vdash \psi$ and $\mathcal{M}, w \dashv\!| \chi$. By IH $\mathcal{M}', w \Vdash \psi$ and $\mathcal{M}', w \dashv\!| \chi$, therefore $\mathcal{M}', w \dashv\!| \psi \to \chi$.

$\rho \stackrel{def}{=} \Diamond\psi$ Assume $\mathcal{M}, w \Vdash \Diamond\psi$, thus there is a $w' \in \mathcal{N}(w)$ such that $\mathcal{M}, w' \Vdash \psi$. By IH $\mathcal{M}', w' \Vdash \psi$, by definition of Inflated Model $w' \in \mathcal{N}'(w)$ therefore $\mathcal{M}', w \Vdash \Diamond\psi$.
Similarly, if $\mathcal{M}, w \dashv\!| \Diamond\psi$ for all $w' \in \mathcal{N}(w)$; $\mathcal{M}, w' \dashv\!| \psi$. By IH $\mathcal{M}', w' \dashv\!| \psi$, by definition of Inflated Model all and only w' are in $\mathcal{N}'(w)$, therefore $\mathcal{M}', w \dashv\!| \Diamond\psi$

$\rho \stackrel{def}{=} \mho x.\psi$ Assume $\mathcal{M}, w \Vdash \mho x.\psi$, therefore there is a $n \in \mathfrak{N}$ such that $\mathcal{M}, w \Vdash \psi[^n/_x]$. By IH there is $n \in \mathfrak{N}$ such $\mathcal{M}', w \Vdash \psi[^n/_x]$, therefore $\mathcal{M}', w \Vdash \mho x.\psi$
Similarly, if $\mathcal{M}, w \dashv\!| \mho x.\psi$ then for all $n \in \mathfrak{N}$, $\mathcal{M}, w \dashv\!| \psi[^n/_x]$. By IH for all $n \in \mathfrak{N}$, $\mathcal{M}', w \dashv\!| \psi[^n/_x]$, therefore $\mathcal{M}', w \Vdash \mho x.\psi$

□

FACT 19 [Implication is closed under MP and MT with $\mathcal{L}^{PW} \setminus \{\bigstar, \mathcal{E}\}$ formulas]

$$\frac{\Vdash \varphi \to \psi \qquad \Vdash \varphi}{\Vdash \psi}$$

Proof. Assume that for formulas $\varphi, \psi \in \mathcal{L}^{PW} \setminus \{\bigstar, \mathcal{E}\}$ we have that (1) $\Vdash \varphi$, (2) $\Vdash \varphi \to \psi$ and, for the sake of Reductio, (3) $\nVdash \psi$.
By (3) there must be a $\mathcal{M}$ and $w \in \mathcal{W}$ such that $\mathcal{M}, w \dashv\vdash \psi$. However by Lemma 3.1.2 we also have that there is an Inflated model $\mathcal{M}' \in \mathcal{I}(\mathcal{M})$ were φ is defined, and by (1) we have that if φ is defined it has to be true, thus $\mathcal{M}', w \Vdash \varphi$.
In $\mathcal{M}'$, by Lemma 3.1.3, ψ retains the truth value it had in $\mathcal{M}$, i.e. $\mathcal{M}', w \dashv\vdash \psi$.
However, this means that $\mathcal{M}', w \dashv\vdash \varphi \to \psi$, that by Lemma 3.1.1 is in contradiction with (2). Therefore, it must be the case that $\Vdash \psi$.

With the same strategy, if $\dashv\vdash \psi$ and $\Vdash \varphi \to \psi$ then $\dashv\vdash \varphi$. □

FACT 20 [Implication is transitive with $\mathcal{L}^{PW} \setminus \{\bigstar, \mathcal{E}\}$ formulas]

$$\frac{\Vdash \varphi \to \psi \qquad \Vdash \psi \to \rho}{\Vdash \varphi \to \rho}$$

Proof. Assume that for formulas $\varphi, \psi, \rho \in \mathcal{L}^{PW} \setminus \{\bigstar, \mathcal{E}\}$ we have that (1) $\Vdash \varphi \to \psi$, (2) $\Vdash \psi \to \rho$ and, for the sake of Reductio, (3) $\nVdash \varphi \to \rho$.
By (3) there must be a $\mathcal{M}$ and $w \in \mathcal{W}$ such that $\mathcal{M}, w \dashv\vdash \varphi \to \rho$ thus $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \dashv\vdash \rho$. By (1) and (2) we have that $[\![\varphi \to \psi]\!]^{\mathcal{M}}_w = N$ and $[\![\psi \to \rho]\!]^{\mathcal{M}}_w = N$, therefore $[\![\psi]\!]^{\mathcal{M}}_w = N$. However by Lemma 3.1.2 we also have that there is an Inflated model $\mathcal{M}' \in \mathcal{I}(\mathcal{M})$ were ψ is defined.
By (1) and (2) we have that if ψ is defined then $\mathcal{M}', w \Vdash \varphi \to \psi$ and $\mathcal{M}', w \Vdash \psi \to \rho$.
In $\mathcal{M}'$, by Lemma 3.1.3, ρ and φ retain the truth values they had in $\mathcal{M}$, i.e. $\mathcal{M}', w \Vdash \varphi$ and $\mathcal{M}', w \dashv\vdash \rho$.
Thus, $\mathcal{M}', w \Vdash \varphi$ and $\mathcal{M}', w \dashv\vdash \rho$ and $\mathcal{M}', w \Vdash \varphi \to \psi$ and $\mathcal{M}', w \Vdash \psi \to \rho$.
Therefore, (i) $\mathcal{M}', w \Vdash \varphi$ and (ii) $\mathcal{M}', w \dashv\vdash \rho$ and $\big[$(iii) $\mathcal{M}', w \dashv\vdash \varphi$ or (iv) $\mathcal{M}', w \Vdash \psi\big]$ and $\big[$(v) $\mathcal{M}', w \dashv\vdash \psi$ or (vi) $\mathcal{M}', w \Vdash \rho\big]$.
However, Lemma 3.1.1 (i) & (iii), (ii) & (vi) and (iv) & (v) are contradictory.
Therefore, it must be the case that $\Vdash \varphi \to \rho$. □

By Deduction Theorem the same holds for WE.

We can analyze these facts on different levels. First, we can focus on the difference between the meaning of formulas like $\Xi x.\varphi$ and $\exists x.\varphi$. As previously discussed, the interpretation of the weak particular is "There is a name such that $\varphi[^n/_x]$", while the strong particular states that "There is a *felicitous* name such that $\varphi[^n/_x]$".

It is only to be expected that MP fails in some instances of the strong particular, namely of the form $\Vdash \psi \to \exists x.\varphi$. In fact we are not ensured that ψ contain witnesses for the existence of referents. In general, this confirms that existence is independent from deductive reasoning, i.e. there is no way to prove the validity of $\mathcal{E}(n)$ for any n nor should a conditional be considered valid if its consequent makes a strong particular claim not justified in its antecedent. This fact just confirms that logic alone cannot "[...]Get Something from Nothing" [Orenstein, 1995].

3.3.2 Rules and Entailments in the Fragment

Thanks to the results of Lemma 3.1.1 we were able to prove that in the restricted fragment $\mathcal{L}^{PW} \setminus \{\bigstar, \mathcal{E}\}$ WE is Transitive and validity is closed over MP and MT. Moreover, WE trivially retains the properties shown in figure 3.4 on page 42.
Now it is possible under the same restriction to prove that WE supports the Rules of introduction and elimination for the implication and the weak particular. That along with Rad. Absurdum, Rad. Weakening and the equivalence $\top \overset{weak}{\Vdash} \varphi$ iff $\Vdash \varphi$ shows that every PhIL's validity in $\mathcal{L}^{PW} \setminus \{\bigstar, \mathcal{E}\}$ is a classical validity.

Implication Rules

FACT 21 [Implication-el]

$$\frac{\rho \wedge \varphi \overset{weak}{\Vdash} \psi}{\rho \overset{weak}{\Vdash} \varphi \to \psi}$$

Proof. Assume $\rho \wedge \varphi \overset{weak}{\Vdash} \psi$. By def. for all $\mathcal{M}$& $w \in \mathcal{W}$ if $\mathcal{M}, w \Vdash \rho$ and $\mathcal{M}, w \Vdash \varphi$ then $\mathcal{M}, w \not\dashv\!\!\mid \psi$.
Thus, for all $\mathcal{M}$& $w \in \mathcal{W}$ if $\mathcal{M}, w \Vdash \rho$ then if $\mathcal{M}, w \Vdash \varphi$ then $\mathcal{M}, w \not\dashv\!\!\mid \psi$. This is equivalent to for all $\mathcal{M}$& $w \in \mathcal{W}$ if $\mathcal{M}, w \Vdash \rho$ then it is not the case that both $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \dashv\!\!\mid \psi$. Therefore,for all $\mathcal{M}$& $w \in \mathcal{W}$ if $\mathcal{M}, w \Vdash \rho$ then $\mathcal{M}, w \not\dashv\!\!\mid \varphi \to \psi$, which by definition is $\rho \overset{weak}{\Vdash} \varphi \to \psi$. □

FACT 22 [Implication-in]

$$\frac{\rho \overset{weak}{\Vdash} \varphi \to \psi \qquad \rho \overset{weak}{\Vdash} \varphi}{\rho \overset{weak}{\Vdash} \psi}$$

Proof. Assume $\rho \overset{weak}{\Vdash} \varphi \to \psi$ and $\rho \overset{weak}{\Vdash} \varphi$, thus by def.:
(i) for all $\mathcal{M}\&\ w \in \mathcal{W}$ if $\mathcal{M}, w \Vdash \rho$ and $\mathcal{M}, w \Vdash \varphi$ then $\mathcal{M}, w \not\dashv\!\vdash \psi$.
(ii) for all $\mathcal{M}\&\ w \in \mathcal{W}$ if $\mathcal{M}, w \Vdash \rho$ then $\mathcal{M}, w \not\dashv\!\vdash \varphi$. Now assume for Reductio $\rho \overset{weak}{\nVdash} \psi$, thus there is $\mathcal{M}'\&w' \in \mathcal{W}$ such that $\mathcal{M}', w' \Vdash \rho$ and $\mathcal{M}', w' \dashv\!\vdash \psi$. By Lemma 3.1.2 there is an Inflated Model $\mathcal{M}''$ s.t. (1) $\mathcal{M}'', w' \Vdash \rho$ and (2) $\mathcal{M}'', w' \dashv\!\vdash \psi$. By (ii) and (1) $\mathcal{M}'', w' \not\dashv\!\vdash \varphi$, thus by Lemma 3.1.3 and (3) $\mathcal{M}'', w' \Vdash \varphi$. By (i), (1) and (3) $\mathcal{M}'', w' \not\dashv\!\vdash \psi$, but this contradicts (2). Since this is a contradiction $\rho \overset{weak}{\Vdash} \psi$. □

Weak Particular Rules

FACT 23 [Weak Particular-in]

$$\frac{\rho \overset{weak}{\Vdash} \psi[^n/_x]}{\rho \overset{weak}{\Vdash} \Xi x.\psi}$$

Proof. Assume that (1) $\rho \overset{weak}{\Vdash} \psi[^n/_x]$. Take an arbitrary $\mathcal{M}$ and $w \in \mathcal{W}$ such that $\mathcal{M}, w \Vdash \rho$. By (1) there is an n s.t. $\mathcal{M}, w \not\dashv\!\vdash \psi[^n/_x]$, thus $\mathcal{M}, w \not\dashv\!\vdash \Xi x.\psi$. Therefore $\rho \overset{weak}{\Vdash} \Xi x.\psi$ □

FACT 24 [Weak Particular-el]

$$\frac{\rho \wedge \varphi[^n/_x] \overset{weak}{\Vdash} \psi \qquad \rho \overset{weak}{\Vdash} \Xi x.\varphi}{\rho \overset{weak}{\Vdash} \psi}$$

Proof. Assume (1) $\rho \wedge \varphi[^n/_x] \overset{weak}{\Vdash} \psi$ and (2) $\rho \overset{weak}{\Vdash} \Xi x.\varphi$, thus by def.:
(i) for all $\mathcal{M}\&\ w \in \mathcal{W}$ if $\mathcal{M}, w \Vdash \rho$ and $\mathcal{M}, w \Vdash \varphi[^n/_x]$ then $\mathcal{M}, w \not\dashv\!\vdash \psi$.
(ii) for all $\mathcal{M}\&\ w \in \mathcal{W}$ if $\mathcal{M}, w \Vdash \rho$ then $\mathcal{M}, w \not\dashv\!\vdash \Xi x.\varphi$. Assume for Reductio that $\rho \overset{weak}{\nVdash} \psi$, thus there is $\mathcal{M}'\&w' \in \mathcal{W}$ such that $\mathcal{M}', w' \Vdash \rho$ and $\mathcal{M}', w' \dashv\!\vdash \psi$. By Lemma 3.1.2 there is an Inflated Model $\mathcal{M}''$ s.t.

(1) $\mathcal{M}'', w' \Vdash \rho$ and (2) $\mathcal{M}'', w' \dashV \psi$. By (i) and (1) $\mathcal{M}, w \not\dashV \exists x.\varphi$ thus by Lemma 3.1.3 and $\mathcal{M}, w \Vdash \exists x.\varphi$, therefore for some $n \in \mathcal{N}$ (3) $\mathcal{M}, w \Vdash \varphi[^n/_x]$. By (i), (1) and (3) $\mathcal{M}'', w' \not\dashV \psi$, but this contradicts (2). Since this is a contradiction $\rho \overset{weak}{\Vdash} \psi$. □

IDENTITY

FACT 25 [Self-Identity]

$$\frac{}{\rho \overset{weak}{\Vdash} n = n}$$

Proof. By the Self-identity constraint, for every $\mathcal{M}$& $w \in \mathcal{W}$, $[\![n = n]\!]^{\mathcal{M}}_w = 1$, thus by def. $\mathcal{M}, w \Vdash n = n$ and by Lemma 3.1.1 $\mathcal{M}, w \not\dashV n = n$.
Therefore $\overset{weak}{\Vdash} n = n$, and by Radical Weakening $\rho \overset{weak}{\Vdash} n = n$. □

FACT 26 [Indiscernibility of Identicals]

$$\frac{\rho \overset{weak}{\Vdash} c = b \qquad \rho \overset{weak}{\Vdash} \varphi[^c/_x]}{\rho \overset{weak}{\Vdash} \varphi[^b/_x]}$$

Proof. Follows from fact 2 on page 36. □

3.3.3 Classical validity and PhIL validity

Another way to prove that every valid formula in Classical Logic is PhIL valid is to show that the rules and axiom schemata used in Hilbert-Style Deduction are PhIL valid and that MP preserves validity. The former has been proved in fact 19 on page 51. This can be also done for the Modal expansion of the language.

PROPOSITIONAL VALIDITY

FACT 27 [The Propositional Axiom Schemata are PhIL valid]

Proof.

$\Vdash \varphi \to (\psi \to \varphi)$ Assume for contradiction that there is an arbitrary $\mathcal{M}$& $w \in \mathcal{W}$ $\mathcal{M}, w \dashV \varphi \to (\psi \to \varphi)$. By definition $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \dashV \psi \to \varphi$, thus (I) $\mathcal{M}, w \Vdash \varphi$ and (II) $\mathcal{M}, w \dashV \varphi$ and

$\mathcal{M}, w \Vdash \psi$. By Lemma 3.1.1 that (I) and (II) cannot be the case. Therefore there is no $\mathcal{M}$& $w \in \mathcal{W}$ s.t. $\mathcal{M}, w \dashV \varphi \to (\psi \to \varphi)$

$\Vdash (\varphi \to (\psi \to \rho)) \to ((\varphi \to \psi) \to (\varphi \to \rho))$ Assume for contradiction that there is an arbitrary $\mathcal{M}$& $w \in \mathcal{W}$ $\mathcal{M}, w \dashV (\varphi \to (\psi \to \rho)) \to ((\varphi \to \psi) \to (\varphi \to \rho))$. By definition that means (1) $[\mathcal{M}, w \dashV \varphi$ or $\mathcal{M}, w \dashV \psi$ or $\mathcal{M}, w \Vdash \rho]$ and (2) $[\mathcal{M}, w \dashV \varphi$ or $\mathcal{M}, w \Vdash \psi]$ and (3) $\mathcal{M}, w \dashV \varphi$ or (4) $\mathcal{M}, w \dashV \rho$. By Lemma 3.1.1 (4) and (3) implies that neither $\mathcal{M}, w \dashV \varphi$ not $\mathcal{M}, w \Vdash \rho$ cannot be the case. However, (1) and (2) are true only if $\mathcal{M}, w \dashV \psi$ and $\mathcal{M}, w \Vdash \psi$ which by Lemma 3.1.1 cannot be the case. Therefore there is no $\mathcal{M}$& $w \in \mathcal{W}$ s.t. $\mathcal{M}, w \dashV (\varphi \to (\psi \to \rho)) \to ((\varphi \to \psi) \to (\varphi \to \rho))$

$\Vdash (\neg\psi \to \neg\varphi) \to ((\neg\psi \to \varphi) \to \psi)$ Assume for contradiction that there is an arbitrary $\mathcal{M}$& $w \in \mathcal{W}$ $\mathcal{M}, w \dashV (\neg\psi \to \neg\varphi) \to ((\neg\psi \to \varphi) \to \psi)$. By definition that means (i) $[\mathcal{M}, w \Vdash \psi$ or $\mathcal{M}, w \dashV \varphi]$ and (ii) $[\mathcal{M}, w \Vdash \psi$ or $\mathcal{M}, w \Vdash \varphi]$ and (iii) $\mathcal{M}, w \dashV \psi$. By Lemma 3.1.1 (iii) implies that $\mathcal{M}, w \Vdash \psi$ cannot be the case. However, (i) and (ii) are true only if $\mathcal{M}, w \dashV \varphi$ and $\mathcal{M}, w \Vdash \varphi$ which by Lemma 3.1.1 cannot be the case. Therefore there is no $\mathcal{M}$& $w \in \mathcal{W}$ s.t. $\mathcal{M}, w \dashV (\neg\psi \to \neg\varphi) \to ((\neg\psi \to \varphi) \to \psi)$

□

First-Order Validity

FACT 28 [The First-Order Axiom Schemata and Rules are PhIL valid]

Proof.

$\Vdash \varphi[^n/_x] \to \exists x.\varphi$ Assume for contradiction that for a certain $\mathcal{M}$& $w \in \mathcal{W}$ $\mathcal{M}, w \dashV \varphi[^n/_x] \to \exists x.\varphi$, thus (i) $\mathcal{M}, w \Vdash \varphi[^n/_x]$ and (ii) $\mathcal{M}, w \dashV \exists x.\varphi$. From (ii) we have that for all n', $\mathcal{M}, w \dashV \varphi[^{n'}/_x]$, but by Lemma 3.1.1 that and (i) cannot be the case. Therefore there is no $\mathcal{M}$& $w \in \mathcal{W}$ s.t. $\mathcal{M}, w \dashV \varphi[^n/_x] \to \exists x.\varphi$, thus $\Vdash \varphi[^n/_x] \to \exists x.\varphi$.

$\dfrac{\Vdash \varphi \to \psi}{\Vdash (\exists x.\varphi) \to \psi}$ Assume that $\Vdash \varphi \to \psi$. By Deduction Theorem that means (1) $\varphi \overset{weak}{\Vdash} \psi$. Assume for an arbitrary $\mathcal{M}$& $w \in \mathcal{W}$ $\mathcal{M}, w \dashV \exists x.\varphi$, thus (2) there is an n such that $\mathcal{M}, w \dashV \varphi[^n/_x]$. From (1) and (2) follows $\mathcal{M}, w \not\dashV \psi$. Since $\mathcal{M}$& $w \in \mathcal{W}$ were arbitrary wlog we can say that for all $\mathcal{M}$& $w \in \mathcal{W}$ if $\mathcal{M}, w \dashV \exists x.\varphi$

then $\mathcal{M}, w \not\dashV \psi$. Therefore, $\exists x.\varphi \overset{weak}{\Vdash} \psi$ and by Deduction Theorem $\Vdash (\exists x.\varphi) \rightarrow \psi$.

Identity

For all x**,** $\Vdash x = x$
Trivially from the Self-Identity rule in figure 3.1 on page 33.

For all x, y**,** $\Vdash x = y \rightarrow y = x$ Follows from fact 2 on page 36

For all x, y, z**,** $\Vdash (x = y \wedge y = z) \rightarrow x = z$ Follows from fact 2 on page 36

For all x, y**,** $\Vdash x = y \rightarrow (\varphi[^x/_a] \leftrightarrow \varphi[^y/_a])$ Follows from fact 2 on page 36

□

Modal Logic

To show that every formula valid in First-Order Modal logic is valid we need to show that Necessitation, $\mathcal{K}$ are valid.

FACT 29 [The Modal Logic Axiom Schemata and Rule are PhIL valid]

Proof.

$\dfrac{\Vdash \neg\varphi}{\Vdash \neg\Diamond\varphi}$ Assume $\Vdash \neg\varphi$, thus (I) $\dashV \varphi$ and for contradiction that for a certain $\mathcal{M}$& $w \in \mathcal{W}$ $\mathcal{M}, w \dashV \neg\Diamond\varphi$, thus $\mathcal{M}, w \Vdash \Diamond\varphi$. By definition that means that there is a $w' \in \mathcal{N}(w)$; $\mathcal{M}, w' \Vdash \varphi$, however by (I) there is no $\mathcal{M}$& $w \in \mathcal{W}$ s.t. $\mathcal{M}, w \Vdash \varphi$. Thus, wlog we can say that there is no $\mathcal{M}$& $w \in \mathcal{W}$ such that $\mathcal{M}, w \dashV \neg\Diamond\varphi$.

$\Vdash \neg\Diamond\neg(\varphi \rightarrow \psi) \rightarrow (\neg\Diamond\neg\varphi \rightarrow \neg\Diamond\neg\psi)$ Assume for contradiction that for $\mathcal{M}$& $w \in \mathcal{W}$ $\mathcal{M}, w \dashV \neg\Diamond\neg(\varphi \rightarrow \psi) \rightarrow (\neg\Diamond\neg\varphi \rightarrow \neg\Diamond\neg\psi)$. Thus $\mathcal{M}, w \dashV \Diamond\neg(\varphi \rightarrow \psi)$ and $\mathcal{M}, w \dashV (\neg\Diamond\neg\varphi \rightarrow \neg\Diamond\neg\psi)$. Thus for all $w' \in \mathcal{N}(w)$ $\mathcal{M}, w' \not\dashV (\varphi \rightarrow \psi)$, $\mathcal{M}, w \dashV \Diamond\neg\varphi$ and $M, w \Vdash \Diamond\neg\psi$. Therefore, (1) for all $w' \in \mathcal{N}(w)$, $\mathcal{M}, w' \nVdash \varphi$ or $\mathcal{M}, w' \not\dashV \psi$ and (2) for all $w'' \in \mathcal{N}(w)$, $\mathcal{M}, w'' \Vdash \varphi$ and there is a $w''' \in \mathcal{N}(w)$ s.t. $M, w''' \dashV \psi$. By (1) and (2) $\mathcal{M}, w''' \nVdash \varphi$ and $\mathcal{M}, w''' \Vdash \varphi$, which cannot be the case. Thus, wlog we can say that there is no $\mathcal{M}$& $w \in \mathcal{W}$ such that $\mathcal{M}, w \dashV \neg\Diamond\neg(\varphi \rightarrow \psi) \rightarrow (\neg\Diamond\neg\varphi \rightarrow \neg\Diamond\neg\psi)$.

□

The proofs for the $S5$ characterization are analogous to the standard ones with the same proof strategy of the Distribution Axiom.

Necessity of Identity

Another validity that can be proved in PhIL is the necessity of identity,

FACT 30 [Necessity of Identity is PhIL valid]

Proof.

$\Vdash (a = b) \rightarrow \neg\Diamond\neg(a = b)$ Assume for contradiction that for $\mathcal{M}\&\ w \in \mathcal{W}$ $\mathcal{M}, w \dashv\vdash (a = b) \rightarrow \neg\Diamond\neg(a = b)$. Thus (1) $\mathcal{M}, w \Vdash (a = b)$ and (2) $\mathcal{M}, w \Vdash \Diamond\neg(a = b)$. From (1) by Nominal Rigidity for all $w' \in \mathcal{N}(w)$ $\mathcal{M}, w' \Vdash (a = b)$ and from (2) there is $w'' \in \mathcal{N}(w)$ $\mathcal{M}, w'' \dashv\vdash (a = b)$. By Lemma 3.1.1 this cannot be the case. Thus, wlog we can say that there is no $\mathcal{M}\&\ w \in \mathcal{W}$ such that $\mathcal{M}, w \dashv\vdash (a = b) \rightarrow \neg\Diamond\neg(a = b)$.

□

4. Philosophical Inquisitive Logic

4.1 PhIL's Semantics

In this Chapter I will present Philosophical Inquisitive Logic. TI will be using the concepts of support state and of epistemic modality introduced in chapter 1 on page 7 alongside with the Partial Modal Semantics presented in the previous chapter.
This will introduce inquisitiveness in partial semantics, that we can apply to question about existence ($?\mathcal{E}(n)$: "Does n exist?") and necessity ($?\Box a = b$); similarly we will be able to express that an agent may know the answer of a question ($K_a?\varphi$). Moreover, it will enable us to make explicit what we can call the *presupposition of existence*: i.e. from a question like "Does Elizabet II have veto power?" we can infer that "Elizabet II exists".
The same remark done for $\Box$ in the previous chapter holds here for the modal operator K_a: $K_a\varphi$ will be true in w if φ is never false in $\sigma_a(w)$, and false otherwise.
In further works K_a should be treated not as an almost-dual of $\Diamond$: $K_a\varphi$ should be true in w if φ is always true $\sigma_a(w)$, and false otherwise.

The Language

I will refer to the language of Philosophical Inquisitive Logic (PhIL) as $\mathcal{L}_{PhIL}$. It consists of two sets of function symbols: $\mathfrak{N}$ is the set of Names, and its subset $\mathfrak{Ag}$ represents the set of the names of epistemic agents; $\mathcal{R}$ is the set of n-ary relations.
The logical base is composed by the connectives $\rightarrow$ (flat implication) and $\twoheadrightarrow$ (inquisitive implication); the constant $\bot$ (false), the quantifier $\overline{\mathfrak{Z}}$ (inquisitive soft particular), the unary predicate $\mathcal{E}$ (existence) and the modal operators $\Diamond$ (possible) and K_a (a knows) with index $a \in \mathfrak{Ag}$.

Terms and formulas in $\mathcal{L}_{PhIL}$ are defined as follows, with $n \in \mathfrak{N}$, $P \in \mathcal{R}$, $a \in \mathfrak{Ag}$, and z free in φ:

$$t \overset{def}{=} n \mid x$$

$$\mathcal{L}_{PhIL} \overset{def}{=} P(\vec{t}) \mid \mathcal{E}(n) \mid t = t \mid \bot \mid \varphi \rightarrow \varphi \mid \varphi \twoheadrightarrow \varphi \mid \overline{\mathfrak{Z}}z.\varphi \mid \Diamond\varphi \mid K_a\varphi$$

Using the primitive operators we have a multitude of operators that can be defined. They can be broadly divided in three categories: inquisitive, classical and hybrid.

$\neg\varphi \overset{def}{=} \varphi \twoheadrightarrow \bot$	$\varphi \bigwedge \psi \overset{def}{=} \neg(\varphi \twoheadrightarrow (\neg\psi))$	$\varphi \bigvee \psi \overset{def}{=} (\neg\varphi \twoheadrightarrow \psi)$
$\underline{\nabla} x.\varphi \overset{def}{=} \neg\overline{\Xi} x.\neg\varphi$	$\top \overset{def}{=} \bot \twoheadrightarrow \bot$	

(a) Defined Inquisitive Operators

$\neg\varphi \overset{def}{=} \varphi \to \bot$	$!\varphi \overset{def}{=} \neg\neg\varphi$	$\varphi \vee \psi \overset{def}{=} !(\varphi \bigvee \psi)$
$\Box\varphi \overset{def}{=} \neg\Diamond\neg\varphi$	$\varphi \wedge \psi \overset{def}{=} !(\varphi \bigwedge \psi)$	$\Xi x.\varphi \overset{def}{=} !\overline{\Xi} x.\varphi$

(b) Defined Classical Operators

$?\varphi \overset{def}{=} \neg\varphi \bigvee \varphi$	$\overline{\exists} x.\varphi \overset{def}{=} \overline{\Xi} x.(\mathcal{E}(x) \bigwedge \varphi)$	$\underline{\forall} x.\varphi \overset{def}{=} \underline{\nabla} x.(\mathcal{E}(x) \to \varphi)$
$\exists x.\varphi \overset{def}{=} !\overline{\exists} x.\varphi$	$\forall x.\varphi \overset{def}{=} !\underline{\forall} x.\varphi$	

(c) Defined hybrid Operators

The conditions if these operators are explored in section B.1 on page 95, while their expressiveness is discussed in section B.1.1 on page 96.
Moreover, we can establish a translation, or natural renaming, from $\varphi \in \mathcal{L}_{PhIL}$ to their non-inquisitive counterpart $\varphi^{cl} \in \left(\mathcal{L}^{PW} \setminus \{\bigstar\} \cup \{K_a\}\right)$[1]. If $\alpha \in \texttt{Atom} \cup \{\bot\}$ we have that:

$$
\begin{aligned}
\alpha^{cl} &= \alpha & (\varphi \twoheadrightarrow \psi)^{cl} &= \varphi^{cl} \to \psi^{cl} & (\varphi \to \psi)^{cl} &= \varphi^{cl} \to \psi^{cl} \\
(\overline{\Xi} x.\varphi)^{cl} &= \Xi x.(\varphi^{cl}) & (\Diamond\varphi)^{cl} &= \Diamond(\varphi^{cl}) & (K_a\varphi)^{cl} &= K_a(\varphi^{cl})
\end{aligned}
$$

4.1.1 The Model

A model for PhIL is the quadruple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \{\sigma_a \mid a \in \mathfrak{Ag}\}, [\![-]\!] \rangle$ in figure 4.1 on the next page, where the relation of metaphysical accessibility $\mathcal{N}$ and the interpretation function $[\![-]\!]$ behave as in the previous chapter.

The family of modal operators σ_a represents the agents' information states in a given world w. As described in section 2.2 on page 26, $\sigma_a(w)$ is empty if and only if a is not a felicitous name in w, we call this constraint *Cogito, ergo sum*. Moreover, whenever the agent's name is felicitous, thanks to Conditional Reflexivity and the Euclidean restriction we recover the usual properties of knowledge. Since the support conditions for states s will be partial, rather than using the term "support" I will say that a state s can "accept" or "reject" a formula.
In the previous chapter we saw how if φ is neither true nor false in a possible world then it is there undefined. In this approach we will have that a state can neither accept nor reject φ for other reasons too. For example, it could be everywhere undefined, somewhere undefined and somewhere true or just somewhere true and somewhere false.

[1]The constant $\bigstar$ is not really interesting and it was used only in the previous chapter to have a direct example of a formula always undefined.
The behavior of the operator K_a in Partial Modal Logic will be stated explicitly in few pages.

$$\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \{\sigma_a \mid a \in \mathfrak{Ag}\}, [\![-]\!] \rangle$$

$\mathcal{W}$ a non-empty set of possible worlds.

$\mathcal{N}$: $\mathcal{W} \to \mathbb{P}(\mathcal{W})$ a function from Possible Worlds to the set of their successors, s.t.:

Reflexive For all $w \in \mathcal{W}.w \in \mathcal{N}(w)$

Euclidean For all $w, v, u \in \mathcal{W}$. if $v \in \mathcal{N}(w)$ and $u \in \mathcal{N}(w)$ then $v \in \mathcal{N}(u)$

σ_a : $\mathcal{W} \to \mathbb{P}(\mathcal{W})$ a function from Possible Worlds to agent a's informative state:

Euclidean (Axiom 5**)** For all $w, v, u \in \mathcal{W}$, $a \in \mathfrak{Ag}$ if $v \in \sigma(w)$ and $u \in \sigma(w)$ then $v \in \sigma(u)$

"Cogito, Ergo sum" (Axiom $\mathcal{E}(D)$**)** For all $w \in \mathcal{W}$, $a \in \mathfrak{Ag}$ $[\![\mathcal{E}(a)]\!]_w = 1$ iff Exists $w' \in \mathcal{W}; w' \in \sigma_a(w)$

Conditional Reflexivity (Axiom $D \to T$**)** For all $w \in \mathcal{W}$ If Exists $w' \in \mathcal{W}; w' \in \sigma_a(w)$ then $w \in \sigma_a(w)$

$[\![-]\!]$: $\mathcal{W} \mapsto [\![-]\!]_w$ are interpretation maps from atoms and possible words into denotations, such that for every $\alpha \in \texttt{Atom}$ an interpretation is the map:

$$[\![\ ]\!]_w : \texttt{Atom} \to \{0, N, 1\} : \alpha \mapsto \{0, N, 1\}$$

Moreover, we have the following constraints:

Existence Predicate For every term t:

$$[\![\mathcal{E}(t)]\!]_w \in \{0, 1\}$$

Definiteness
$[\![P\vec{t}\,]\!]_w \in \{0, 1\}$ iff for all $t \in \vec{t}$; $[\![\mathcal{E}(t)]\!]_w = 1$

Identity For every name $c, b \in \mathfrak{N}$:

$$[\![c = b]\!]_w \in \{0, 1\}$$

Nominal Rigidity For every $w, v \in \mathcal{W}$ and $c, b \in \mathfrak{N}$; if $v \in \mathcal{N}(w)$ then $[\![c = b]\!]_w = [\![c = b]\!]_v$

Self-identity For every $w \in \mathcal{W}$ and $n \in \mathfrak{N}$; $[\![n = n]\!]_w = 1$

Leibniz's Laws If $[\![c = b]\!]_w = 1$ then For every $\alpha \in$ Atom, $[\![\alpha[^b/_a]]\!]_w = [\![\alpha[^c/_a]]\!]_w$

Figure 4.1: First-Order Bilateral Modal Logic Model

4.1.2 Support conditions

The bilateral treatment for states inherits its PW partiality as follows. For any $\varphi \in \mathcal{L}_{PhIL}$ we say that:

φ **is accepted by** s (notation $s \models \varphi$) iff For all $w \in s$, $[\![\varphi]\!]_w = 1$

φ **is rejected by** s (notation $s \dashv \varphi$) iff For all $w \in s$, $[\![\varphi]\!]_w = 0$

φ **is undefined in** s iff For all $w \in s$, $[\![\varphi]\!]_w = N$

Acceptance and Rejection Conditions

We can spell out the Acceptance (and Rejection) conditions for every operator in the logical base. Considering $\alpha \in \texttt{Atom}$ we have that:

$\mathcal{M}, s \models \alpha$	iff	For all $w \in s$, $[\![\alpha]\!]_w = 1$
$\mathcal{M}, s \dashv \alpha$	iff	For all $w \in s$, $[\![\alpha]\!]_w = 0$
$\mathcal{M}, s \models \bot$	iff	$s = \emptyset$
$\mathcal{M}, s \dashv \bot$		always
$\mathcal{M}, s \models \varphi \twoheadrightarrow \psi$	iff	$\mathcal{M}, s \dashv \varphi$ or $\mathcal{M}, s \models \psi$
$\mathcal{M}, s \dashv \varphi \twoheadrightarrow \psi$	iff	$\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \dashv \psi$
$\mathcal{M}, s \models \varphi \rightarrow \psi$	iff	For all $t \subseteq s$; if For all $w \in t$ $\mathcal{M}, w \not\dashv \varphi^{cl}$ then $\mathcal{M}, t \models \psi$
$\mathcal{M}, s \dashv \varphi \rightarrow \psi$	iff	$\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \dashv \psi$
$\mathcal{M}, s \models \overline{\exists} x.\varphi$	iff	There is a $n \in \mathfrak{N}$; $\mathcal{M}, s \models \varphi[^n/_x]$
$\mathcal{M}, s \dashv \overline{\exists} x.\varphi$	iff	For all $n \in \mathfrak{N}$; $\mathcal{M}, s \dashv \varphi[^n/_x]$
$\mathcal{M}, s \models \Diamond\varphi$	iff	For all $w \in s$; There is a $t \subseteq \mathcal{N}(w)$; $\mathcal{M}, t \not\models \bot$ and $\mathcal{M}, t \models \varphi$
$\mathcal{M}, s \dashv \Diamond\varphi$	iff	For all $w \in s$; For all $t \subseteq \mathcal{N}(w)$; $\mathcal{M}, t \not\models \varphi$
$\mathcal{M}, s \models K_a\varphi$	iff	$\mathcal{M}, s \models \mathcal{E}(a)$ and For all $w \in s$; For all $t \subseteq \sigma_a(w)$; $\mathcal{M}, t \not\dashv \varphi$
$\mathcal{M}, s \dashv K_a\varphi$	iff	$\mathcal{M}, s \models \mathcal{E}(a)$ and For all $w \in s$; There is a $t \subseteq \sigma_a(w)$ s.t. $\mathcal{M}, t \not\models \bot$ and $\mathcal{M}, t \dashv \varphi$

Moreover since we did not treat the Knowledge operator in the previous chapter we can define its truth conditions. If $\varphi \in \mathcal{L}^{PW} \cup \{K_x\}$:

$\mathcal{M}, w \Vdash K_a\varphi$	iff	$\mathcal{M}, w \Vdash \mathcal{E}(a)$ and For all $w' \in \sigma_a(w) \not\dashv\!\mid \varphi$
$\mathcal{M}, w \dashv\!\mid K_a\varphi$	iff	$\mathcal{M}, w \Vdash \mathcal{E}(a)$ and There is a $w' \in \sigma_a(w)$; $\mathcal{M}, w' \dashv\!\mid \varphi$

As in the previous Chapter we can introduce a "non contradiction" lemma and prove that Downward Monotonicity and World-Singleton Correspondence are respected.

Facts and Lemmas

FACT 31 [Downward Monotonicity]
If $\mathcal{M}, s \models \varphi$ *then for all* $t \subseteq s$, $\mathcal{M}, t \models \varphi$
If $\mathcal{M}, s \mathrel{=}\!\!| \varphi$ *then for all* $t \subseteq s$, $\mathcal{M}, t \mathrel{=}\!\!| \varphi$

Proof. In chapter B on page 95, Theorem B.2.1. □

FACT 32 [World-Singleton Correspondence]
$\mathcal{M}, \{w\} \models \varphi$ *iff* $\mathcal{M}, w \Vdash \varphi^{cl}$
$\mathcal{M}, \{w\} \mathrel{=}\!\!| \varphi$ *iff* $\mathcal{M}, w \dashv\!\!| \varphi^{cl}$

Proof. In chapter B on page 95, Theorem B.3.1. □

As a corollary of the World-Singleton Correspondence we also have that $\twoheadrightarrow$ and $\rightarrow$ are equivalent in Singleton states:

$$\begin{aligned} \mathcal{M}, \{w\} \models \varphi \rightarrow \psi \quad &\text{iff} \quad \mathcal{M}, w \Vdash \varphi \rightarrow \psi^{cl} \\ &\text{iff} \quad \mathcal{M}, \{w\} \models \varphi \twoheadrightarrow \psi \end{aligned}$$

$$\begin{aligned} \mathcal{M}, \{w\} \mathrel{=}\!\!| \varphi \rightarrow \psi \quad &\text{iff} \quad \mathcal{M}, w \dashv\!\!| \varphi \rightarrow \psi^{cl} \\ &\text{iff} \quad \mathcal{M}, \{w\} \mathrel{=}\!\!| \varphi \twoheadrightarrow \psi \end{aligned}$$

Now it is possible to prove the INQ equivalent of Lemma 3.1.1:

Lemma 4.1.1. *Given any model* $\mathcal{M}$*, state* s *and formula* φ *if* $\mathcal{M}, s \models \varphi$ *and* $\mathcal{M}, s \mathrel{=}\!\!| \varphi$ *then* $s = \emptyset$

Proof. We can prove this lemma by induction on φ

Base case

$\alpha \in \texttt{Atom}$ Assume for some $\alpha \in \texttt{Atom}$, $\mathcal{M}, s \models \alpha$ and $\mathcal{M}, s \mathrel{=}\!\!| \alpha$, thus For all $w \in s$, $[\![\alpha]\!]_w = 1$ and For all $w \in s$, $[\![\alpha]\!]_w = 0$.

Assume for contradiction that there is a $w' \in s$, thus $[\![\alpha]\!]_{w'} = 1$ and $[\![\alpha]\!]_{w'} = 0$. From Lemma 3.1.1 we know that it cannot be the case that $[\![\alpha]\!]_{w'} = 1$ and $[\![\alpha]\!]_{w'} = 0$, thus by contradiction $w' \notin s$.

Therefore there is no $w \in s$, i.e. $s = \emptyset$

$\bot$ By definition since $\mathcal{M}, s \Vdash \bot$ if and only if $s = \emptyset$

Inductive step (With IH being "for any φ less complex than ρ, if $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \mathrel{=}\!\!| \varphi$ then $s = \emptyset$")

$\rho \stackrel{def}{=} \varphi \twoheadrightarrow \psi$ Assume that for an arbitrary $\mathcal{M}\&\ s \subseteq \mathcal{W}$, $\mathcal{M}, s \models \varphi \twoheadrightarrow \psi$ and $\mathcal{M}, s \dashv \varphi \twoheadrightarrow \psi$. By definition we have that either (i) $[\mathcal{M}, s \models \varphi$ or $\mathcal{M}, s \dashv \psi]$ and (ii) $\mathcal{M}, s \models \varphi$ and (iii) $\mathcal{M}, s \dashv \psi$.

If (i) is true since $\mathcal{M}, s \models \varphi$ then along with (ii) and the Inductive Hypothesis $s = \emptyset$.

If (i) is true since $\mathcal{M}, s \dashv \psi$ then along with (iii) and the Inductive Hypothesis $s = \emptyset$.

Thus, for every $\mathcal{M}\&\ s \subseteq \mathcal{W}$, if $\mathcal{M}, s \models \varphi \twoheadrightarrow \psi$ and $\mathcal{M}, s \dashv \varphi \twoheadrightarrow \psi$ then $s = \emptyset$.

$\rho \stackrel{def}{=} \varphi \to \psi$ Assume that for an arbitrary $\mathcal{M}\&\ s \subseteq \mathcal{W}$, $\mathcal{M}, s \models \varphi \to \psi$ and $\mathcal{M}, s \dashv \varphi \to \psi$. By definition we have that either (i) [For all $t \subseteq s$; if for all $w \in t$ $\mathcal{M}, w \not\dashv\!\!\dashv \varphi$ then $\mathcal{M}, t \models \psi$] and (ii) $\mathcal{M}, s \models \varphi$ and (iii) $\mathcal{M}, s \dashv \psi$.

By (ii), Downward Monotonicity and Singleton-World Correspondence we have that For all $t \subseteq s$; for all $w \in t$ $\mathcal{M}, w \Vdash \varphi$, which by Lemma 3.1.1 implies For all $t \subseteq s$; for all $w \in t$ $\mathcal{M}, w \not\dashv\!\!\dashv \varphi$. That alongside with (i) implies that For all $t \subseteq s$, $\mathcal{M}, t \models \psi$ and in particular $\mathcal{M}, s \models \psi$. That with (iii) by IH implies that $s = \emptyset$.

Wlog, $\mathcal{M}\&\ s \subseteq \mathcal{W}$, if $\mathcal{M}, s \models \varphi \to \psi$ and $\mathcal{M}, s \dashv \varphi \to \psi$ then $s = \emptyset$.

$\rho \stackrel{def}{=} \exists x.\varphi$ Assume that for an arbitrary $\mathcal{M}\&\ s \subseteq \mathcal{W}$, $\mathcal{M}, s \models \exists x.\varphi$ and $\mathcal{M}, s \dashv \exists x.\varphi$. By definition (1) there is a $n \in \mathfrak{N}$ such that $\mathcal{M}, s \models \varphi[^n/_x]$ and (2) for all $n' \in \mathfrak{N}$, $\mathcal{M}, s \dashv \varphi[^{n'}/_x]$.

By (2) it must also be the case that $\mathcal{M}, s \dashv \varphi[^n/_x]$, which along with (1) by IH implies $s = \emptyset$.

Wlog, for every $\mathcal{M}\&\ s \subseteq \mathcal{W}$, if $\mathcal{M}, s \models \exists x.\varphi$ and $\mathcal{M}, s \dashv \exists x.\varphi$ then $s = \emptyset$.

$\rho \stackrel{def}{=} \Diamond\varphi$ Assume that for an arbitrary $\mathcal{M}\&\ s \subseteq \mathcal{W}$, $\mathcal{M}, s \models \Diamond\varphi$ and $\mathcal{M}, s \dashv \Diamond\varphi$. Assume for Reductio that non-vacuously For all $w \in s$ There is a $t \subseteq \mathcal{N}(w)$ s.t. $t \neq \emptyset$ and $\mathcal{M}, t \models \varphi$ and For all $t' \subseteq \mathcal{N}(w)$; $\mathcal{M}, t' \not\models \varphi$, that is contradictory.

Therefore $w \notin s$ and wlog $s = \emptyset$.

$\rho \stackrel{def}{=} K_a\varphi$ Analogue to $\Diamond\varphi$.

□

Lemma 4.1.2. *If $\mathcal{M}, s \models \varphi$ then For all $w \in s$; $\mathcal{M}, w \Vdash \varphi^{cl}$ and If $\mathcal{M}, s \dashv \varphi$ then For all $w \in s$; $\mathcal{M}, w \dashv\!\!\dashv \varphi^{cl}$*

Proof. Assume $\mathcal{M}, s \models \varphi$. By Downward Monotonicity For all $\{w\} \subseteq s$, $\mathcal{M}, \{w\} \models \varphi$, thus by World-Singleton Correspondence For all $w \in s$ $\mathcal{M}, w \Vdash \varphi^{cl}$. Assume $\mathcal{M}, s \dashv \varphi$. By Downward Monotonicity For all $\{w\} \subseteq s$, $\mathcal{M}, \{w\} \dashv \varphi$, thus by World-Singleton Correspondence For all $w \in s$ $\mathcal{M}, w \dashv\!\!\dashv \varphi^{cl}$. □

ALGEBRAIC APPROACH FOR INQUISITIVE CONTENT

In Chapter One the inquisitive content of a proposition φ was defined as $[\varphi]$: i.e. the set of informative states s that support φ.

In the bilateral setting we ought once again to distinguish acceptance and rejection, therefore we say that:

In the model $\mathcal{M}$, φ is accepted in s (notation $\mathcal{M}, s \models \varphi$) iff $s \in [\varphi]^a_{\mathcal{M}}$

In the model $\mathcal{M}$, φ is rejected in s (notation $\mathcal{M}, s \mathrel{=\!\!|} \varphi$) iff $s \in [\varphi]^r_{\mathcal{M}}$

For a detailed characterization of the algebra for Inquisitive Content refer to section B.4 on page 100.

The Informative Content of a proposition coincides with the union over its inquisitive content, always considering separately rejection and acceptance:

FACT 33 [Informative Content]

- $|\varphi^{cl}|^{\top} = \bigcup [\varphi]^a$
- $|\varphi^{cl}|^{\bot} = \bigcup [\varphi]^r$

Proof. In chapter B on page 95, Theorem B.6.1 □

4.2 Entailments and Validity

4.2.1 Entailment in PhIL

The notion of PhIL's Positive Inquisitive Entailment (PIE), Negative Inquisitive Entailment (NIE) and Strong Inquisitive Entailment (SIE), also called *Refinement* in accordance with the definitions in the first chapter, are similar to PE, NE and SE in chapter 4 on page 59, but formulated at the state level. Weak Inquisitive Entailment has to be less direct and slightly different from the previous notion, since if a state does not reject a proposition it does not imply that there is no world in the state that reject it.

Definition 27 [Strong Inquisitive Entailment]
We say that φ refines ψ if and only if whenever φ is accepted so is ψ and whenever ψ is rejected so is φ
$\varphi \overset{strong}{\models} \psi$ iff for all $\mathcal{M}$ and s;
if $\mathcal{M}, s \models \varphi$ then $\mathcal{M}, s \models \psi$ and if $\mathcal{M}, s \mathrel{=\!\!|} \psi$ then $\mathcal{M}, s \mathrel{=\!\!|} \varphi$

Definition 28 [Positive Inquisitive Entailment]
We say that φ positively INQ entails ψ if and only if whenever φ is accepted so is ψ
$\varphi \models \psi$ iff for all $\mathcal{M}$ and s;
if $\mathcal{M}, s \models \varphi$ then $\mathcal{M}, s \models \psi$

Definition 29 [Negative Inquisitive Entailment]
We say that ψ negatively INQ entails φ if and only if whenever ψ is rejected so is φ
$\psi \dashv \varphi$ iff for all $\mathcal{M}$ and s;
if $\mathcal{M}, s \dashv \psi$ then $\mathcal{M}, s \dashv \varphi$

Definition 30 [Weak Inquisitive Entailment]
We say that φ Weakly INQ entails ψ if and only if φ is rejected and ψ is accepted only in the inconsistent state
$\varphi \overset{weak}{\models} \psi$ iff For all $\mathcal{M}$ and s; if $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \dashv \psi$ then $s = \emptyset$

Once again we can show that the Refinement (SIE) has many equivalent notions and that the same ordering that we had at the Possible Worlds level holds in the States level:

FACT 34 [Equivalent notions of PhIL enhancement]
Given φ and ψ the following are equivalent:

- $\varphi \overset{strong}{\models} \psi$
- $\varphi \models \psi$ *and* $\psi \dashv \varphi$
- *for all* $\mathcal{M}$; $[\varphi]^a_{\mathcal{M}} \subseteq [\psi]^a_{\mathcal{M}}$ *and* $[\psi]^r_{\mathcal{M}} \subseteq [\varphi]^r_{\mathcal{M}}$

Proof. Follows from the definition and the correspondence proved in section B.5 on page 100. ☐

$$\frac{\varphi \models \psi \qquad \psi \dashv \varphi}{\varphi \overset{strong}{\models} \psi} \qquad \frac{\varphi \overset{strong}{\models} \psi}{\psi \dashv \varphi} \qquad \frac{\varphi \overset{strong}{\models} \psi}{\varphi \models \psi}$$

FACT 35 [If $\varphi \models \psi$ then $\varphi \overset{weak}{\models} \psi$]

$$\frac{\varphi \models \psi}{\varphi \overset{weak}{\models} \psi}$$

Proof. Assume $\varphi \models \psi$, by def. for all $\mathcal{M}\&\ s \subseteq \mathcal{W}$, (I) if $\mathcal{M}, s \models \varphi$ then $\mathcal{M}, s \models \psi$.
Take an arbitrary $\mathcal{M}\&\ s \subseteq \mathcal{W}$ s.t. $\mathcal{M}, s \models \varphi$ and (II) $\mathcal{M}, s \Dashv \psi$. By (I) we have that $\mathcal{M}, s \models \psi$. Thus with (II) by Lemma 4.1.1 $s = \emptyset$
Wlog, for all $\mathcal{M}\&\ s \subseteq \mathcal{W}$, if $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \Dashv \psi$ then $s = \emptyset$, thus $\varphi \overset{weak}{\models} \psi$ □

FACT 36 [If $\psi \Dashv \varphi$ then $\varphi \overset{weak}{\models} \psi$]

$$\frac{\psi \Dashv \varphi}{\varphi \overset{weak}{\models} \psi}$$

Proof. Analogous to the previous one. □

This means that with the given definitions we still have the following strength ordering:

$$\begin{array}{ccc} & \varphi \overset{strong}{\models} \psi & \\ \swarrow & & \searrow \\ \varphi \models \psi & & \psi \Dashv \varphi \\ \searrow & & \swarrow \\ & \varphi \overset{weak}{\models} \psi & \end{array}$$

Figure 4.2: Inquisitive Entailment relations in order of strength

4.2.2 Validity in PhIL

Similarly to the previous chapter we have that formulas are valid if the only state that can reject them is the empty set and are invalid if the only state that accept them is the empty state.

Definition 31 [Inquisitive Validity]
We say that φ is valid in PhIL if it is rejected only in the inconsistent state
$\models \varphi$ iff for all $\mathcal{M}$ and s; if $\mathcal{M}, s \Dashv \varphi$ then $s = \emptyset$

Definition 32 [Inquisitive Invalidity]
We say that ψ is invalid in PhIL if it is accepted only in the inconsistent state
$\Dashv \psi$ iff for all $\mathcal{M}$ and s; if $\mathcal{M}, s \models \psi$ then $s = \emptyset$

Equivalence and Correspondence

Applying the definition of the Refinement and the equivalences of fact 34 on page 66 it is possible to provide many equivalent definitions of validity

FACT 37 [Equivalent notions of validity and invalidity]

Given φ *the following are equivalent:*

- $\models \varphi$
- $\top \overset{weak}{\models} \varphi$
- *for all* $\mathcal{M}$*;* $[\varphi]^r_{\mathcal{M}} = \{\emptyset\}$

Given ψ *the following are equivalent:*

- $\dashv \psi$
- $\psi \overset{weak}{\models} \bot$
- *for all* $\mathcal{M}$*;* $[\psi]^a_{\mathcal{M}} = \{\emptyset\}$

Proof. Follows from the definitions and the correspondence proved in section B.5 on page 100. □

The Negation Switch of fact 6 on page 41 has an INQ correspondence only with the negation that preserves inquisitiveness, i.e. $\neg\varphi \overset{def}{=} (\varphi \twoheadrightarrow \bot)$:

FACT 38 [INQ Negation Switch]

$\models \varphi$ *iff* $\dashv \neg\varphi$ *and* $\dashv \psi$ *iff* $\models \neg\psi$

Proof.

$$\begin{aligned}
\models \varphi \text{ iff } & \text{ for all } \mathcal{M} \text{ and } s \subseteq \mathcal{W};\ \text{if } \mathcal{M}, s \dashv \varphi \text{ then } s = \emptyset \\
\text{iff } & \text{ for all } \mathcal{M} \text{ and } s \subseteq \mathcal{W};\ \text{if } \mathcal{M}, s \dashv \varphi \text{ or } \mathcal{M}, s \models \bot \text{ then } s = \emptyset \\
\text{iff } & \text{ for all } \mathcal{M} \text{ and } s \subseteq \mathcal{W};\ \text{if } \mathcal{M}, s \models \varphi \twoheadrightarrow \bot \text{ then } s = \emptyset \\
\text{iff } & \text{ for all } \mathcal{M} \text{ and } s \subseteq \mathcal{W};\ \text{if } \mathcal{M}, s \models \neg\varphi \text{ then } s = \emptyset \\
\text{iff } & \dashv \neg\varphi
\end{aligned}$$

$$\begin{aligned}
\dashv \psi \text{ iff } & \text{ for all } \mathcal{M} \text{ and } s \subseteq \mathcal{W};\ \text{if } \mathcal{M}, s \models \psi \text{ then } s = \emptyset \\
\text{iff } & \text{ for all } \mathcal{M} \text{ and } s \subseteq \mathcal{W};\ \text{if } \mathcal{M}, s \dashv \neg\psi \text{ then } s = \emptyset \\
\text{iff } & \models \neg\psi
\end{aligned}$$

□

Inquisitive Validity to World Validity

It is easy to show that Inquisitive Validity is preserved in Possible world validity:

FACT 39 [Preserved Validity]

Proof.

$\models \varphi$ iff For all $\mathcal{M}\&\ s$; if $\mathcal{M}, s \,=\!\!|\, \varphi$ then $s = \emptyset$
thus (by Mon.) For all $\mathcal{M}\&\ s; \{w\} \subseteq s$ if $\mathcal{M}, \{w\} \,=\!\!|\, \varphi$ then $\{w\} = \emptyset$
thus (by Corr.) For all $\mathcal{M}\&\ s; \{w\} \subseteq s$ if $\mathcal{M}, w \dashv\!\!\!| \varphi^{cl}$ then $\bot$
iff For all $\mathcal{M}\&w; \mathcal{M}, w \not\dashv\!\!\!| \varphi^{cl}$
iff $\Vdash \varphi^{cl}$

□

Deduction Theorem

FACT 40 [Deduction Theorem for WIE]

$$\frac{\varphi \overset{weak}{\models} \psi}{\models \varphi \rightarrow \psi} \qquad\qquad \frac{\models \varphi \rightarrow \psi}{\varphi \overset{weak}{\models} \psi}$$

Proof.

$\varphi \overset{weak}{\models} \psi$ iff For all $\mathcal{M}\&s \subseteq \mathcal{W}$, if $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \,=\!\!|\, \psi$ then $s = \emptyset$
iff For all $\mathcal{M}\&s \subseteq \mathcal{W}$, if $\mathcal{M}, s \,=\!\!|\, \varphi \rightarrow \psi$ then $s = \emptyset$
iff $\models \varphi \rightarrow \psi$

□

4.3 Standard INQ in the restricted Language

We can prove that the Rules employed by Ciardelli to give a proof-theoretic account of First-Order INQ also preserve PhIL's validity wrt Weak Inquisitive Entailment in a special fragment of the Language called $\mathcal{L}^{\top}_{PhIL}$.
This restricted language, much like in the previous chapter, drops the Existent predicate. Moreover, rather than having as primitive the inquisitive arrow $\twoheadrightarrow$ (with all its family of "purely inquisitive" operators, like $\wedge\!\!\!\wedge$ and $\neg\!\!\!\neg$) maintains only the inquisitive disjunction $\vee\!\!\!\vee$ $(\varphi \vee\!\!\!\vee \psi \overset{def}{=} \neg\varphi \twoheadrightarrow \psi)$ and of course the inquisitive weak particular quantifier $\overline{\exists}$.
From this proof we can deduce that all the valid patterns of inference in standard INQ are valid patterns of inference for PhIL under Weak Inquisitive Entailment in this fragment.

$$\mathcal{L}^{\top}_{PhIL} \overset{def}{=} P(\vec{t}) \mid t = t \mid \bot \mid \varphi \rightarrow \varphi \mid \neg\varphi \twoheadrightarrow \varphi \mid \overline{\exists} z.\varphi$$

This restriction is used to show exactly in which fragment of $\mathcal{L}_{PhIL}$ we have that every INQ valid inference is also a PhIL valid inference under WIE. However, since PhIL's *flat* implication $\to$ is strictly stronger than INQ's, the opposite result cannot be proved.

4.3.1 Propositional Rules

Implication

FACT 41 [Implication-el]

$$\frac{\rho \wedge \varphi \overset{weak}{\models} \psi}{\rho \overset{weak}{\models} \varphi \to \psi}$$

Proof. Assume (1) $\rho \wedge \varphi \overset{weak}{\models} \psi$ and for Reductio $\rho \overset{weak}{\not\models} \varphi \to \psi$, thus there is an $\mathcal{M}\&\ s \neq \emptyset$ s.t. $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \mathrel{=\!|} \varphi \to \psi$. Since s is not empty, by Lemma 4.1.2, there is a $w \in s$ s.t. $\mathcal{M}, w \Vdash \rho$ and $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \dashv\!\mid \psi$. Thus by World-Singleton Correspondence $\mathcal{M}, \{w\} \models \rho$ and $\mathcal{M}, \{w\} \models \varphi$ and $\mathcal{M}, \{w\} \mathrel{=\!|} \psi$. In particular, (2) $\mathcal{M}, \{w\} \models \rho \wedge \varphi$ and (3) $\mathcal{M}, \{w\} \dashv\!\mid \psi$. However, by (1), (2) and (3) we have that $\{w\} = \emptyset$, since this is a contradiction we have by Absurdum that $\rho \overset{weak}{\models} \varphi \to \psi$. □

FACT 42 [Implication-in]

$$\frac{\rho \overset{weak}{\models} \varphi \to \psi \qquad \rho \overset{weak}{\models} \varphi}{\rho \overset{weak}{\models} \psi}$$

Proof. Assume (1) $\rho \overset{weak}{\models} \varphi \to \psi$, (2) $\rho \overset{weak}{\models} \varphi$ and for Reductio $\rho \overset{weak}{\not\models} \psi$, thus there is an $\mathcal{M}\&\ s \neq \emptyset$ s.t. $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \mathrel{=\!|} \psi$. Since s is not empty, by Lemma 4.1.2, there is a $w \in s$ s.t. $\mathcal{M}, w \Vdash \rho$ and $\mathcal{M}, w \dashv\!\mid \psi$. By Lemma 3.1.2 there is a $\mathcal{M}', w \Vdash \rho$ and $\mathcal{M}', w \dashv\!\mid \psi$.

Thus by World-Singleton Correspondence (3) $\mathcal{M}', \{w\} \models \rho$ and (4) $\mathcal{M}', \{w\} \mathrel{=\!|} \psi$. However, by (2) and (3) $\mathcal{M}', \{w\} \not\mathrel{=\!|} \varphi$ thus by Lemma 3.1.3 $\mathcal{M}', \{w\} \models \varphi$ and by (1) and (3) $\mathcal{M}', \{w\} \not\mathrel{=\!|} \varphi \to \psi$ thus by Lemma 3.1.3 $\mathcal{M}', \{w\} \models \varphi \to \psi$. Therefore, $\mathcal{M}', \{w\} \models \psi$ which with (4) entails that $\{w\} = \emptyset$, since this is a contradiction we have by Absurdum that $\rho \overset{weak}{\models} \psi$. □

False

FACT 43 [Falsum]

$$\frac{\rho \overset{weak}{\models} \bot}{\rho \overset{weak}{\models} \varphi}$$

Proof. Assume $\rho \overset{weak}{\models} \bot$, thus (1) For all $\mathcal{M}\&\ s \subseteq \mathcal{W}$ if $\mathcal{M}, s \models \rho$ then $s = \emptyset$. Take an arbitrary $\mathcal{M}'\&s' \subseteq \mathcal{W}$ s.t. $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \dashv \varphi$. By (1) $s' = \emptyset$. Wlog, if $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \dashv \varphi$ then $s = \emptyset$ □

FACT 44 [¬¬-el]

$$\frac{\rho \overset{weak}{\models} \alpha}{\rho \overset{weak}{\models} (\alpha \to \bot) \to \bot}$$

Proof. Given α is a classical formula assume $\rho \overset{weak}{\models} \alpha$, thus (1) For all $\mathcal{M}\&\ s \subseteq \mathcal{W}$ if $\mathcal{M}, s \models \rho$ and $s \dashv \alpha$ then $s = \emptyset$. Assume for reduction $\rho \overset{weak}{\not\models} (\alpha \to \bot) \to \bot$. Therefore there is a state $\mathcal{M}\&\ s \neq \emptyset$ s.t. $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \dashv (\alpha \to \bot) \to \bot$. Thus $\mathcal{M}, s \models \rho$ and since α is a classical formula For all $w \in s$ $\mathcal{M}, w\neg \Vdash \alpha$. Since s is not empty, by Lemma 4.1.2 there is $w' \in s$ s.t. $\mathcal{M}, w' \Vdash \rho$ and $\mathcal{M}, w' \dashv\vdash \alpha$. By Lemma 3.1.2 there is an $\mathcal{M}'$ s.t. $\mathcal{M}, w' \Vdash \rho$ and $\mathcal{M}', w' \dashv\vdash \alpha$. By World-Singleton Correspondence $\mathcal{M}, \{w'\} \Vdash \rho$ and $\mathcal{M}', \{w'\} \dashv\vdash \alpha$, but by (1) that means that $\{w'\} = \emptyset$. Since this is a contradiction we have by Absurdum that $\rho \overset{weak}{\models} (\alpha \to \bot) \to \bot$. □

Inquisitive Disjunction

FACT 45 [Inquisitive Disjunction-in]

$$\frac{\rho \overset{weak}{\models} \varphi}{\rho \overset{weak}{\models} \varphi \vee\!\!\!\vee \psi}$$

Proof. Assume $\rho \overset{weak}{\models} \varphi$, thus by def. (i) For all $\mathcal{M}\&\ s \subseteq \mathcal{W}$ if $\mathcal{M}, s \models \rho$ and $s \dashv \varphi$ then $s = \emptyset$. Take an arbitrary $\mathcal{M}'\&s' \subseteq \mathcal{W}$ s.t.

(ii) $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \Dashv \varphi \vee \psi$, i.e. (iii) $\mathcal{M}, s \Dashv \varphi$ and $\mathcal{M}, s \Dashv \psi$. By (i) from (ii) and (iii) we have that $s = \emptyset$. Wlog, For all $\mathcal{M}$& $s \subseteq \mathcal{W}$ if $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \Dashv \varphi \vee \psi$ then $s = \emptyset$. □

FACT 46 [Inquisitive Disjunction-el]

$$\frac{\rho \overset{weak}{\models} \varphi \vee \psi \qquad \rho, \psi \overset{weak}{\models} \chi \qquad \rho, \varphi \overset{weak}{\models} \chi}{\rho \overset{weak}{\models} \chi}$$

Proof. Assume $\rho \overset{weak}{\models} \varphi \vee \psi$, $\rho, \psi \overset{weak}{\models} \chi$, $\rho, \varphi \overset{weak}{\models} \chi$, i.e.:
(i) if $\mathcal{M}, s \models \rho$ and $[\mathcal{M}, s \Dashv \varphi$ and $\mathcal{M}, s \Dashv \psi]$ then $s = \emptyset$
(ii) if $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \models \psi$ and $\mathcal{M}, s \Dashv \chi$ then $s = \emptyset$
(iii) if $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \Dashv \chi$ then $s = \emptyset$
Assume for Reduction that $\rho \overset{weak}{\not\models} \chi$, thus for some $s \neq 0$, $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \Dashv \chi$. Since s is not empty, By Lemma 4.1.2, there is $w \in s$ s.t. $\mathcal{M}, w \Vdash \rho$ and $\mathcal{M}, w \dashV \chi$. By World-Singleton Correspondence $\mathcal{M}, \{w\} \models \rho$ and $\mathcal{M}, \{w\} \Dashv \chi$. By Lemma 3.1.2 there is a model $\mathcal{M}'$ s.t. (1) $\mathcal{M}', \{w\} \models \rho$ and $\mathcal{M}', \{w\} \Dashv \chi$, and by Lemma 3.1.3 either (2) $\mathcal{M}', \{w\} \Dashv \varphi$ or (3) $\mathcal{M}', \{w\} \models \varphi$ and (4) $\mathcal{M}', \{w\} \Dashv \psi$ or (5) $\mathcal{M}', \{w\} \models \psi$. However, if (3) then by (1) and (iii) $s = \{w\}$, if (5) then by (1) and (iii) $s = \{w\}$ and if (4) and (2) then by (1) and (i) $s = \{w\}$. Either way we have a contradiction, thus by Absurdum $\rho \overset{weak}{\models} \chi$. □

Inquisitive Disjunction Split

FACT 47 [Inquisitive Disjunction Split]

$$\frac{\rho \overset{weak}{\models} \alpha \to (\varphi \vee \psi)}{\rho \overset{weak}{\models} (\alpha \to \varphi) \vee (\alpha \to \psi)}$$

Proof. Given that α is a classical formula, assume $\rho \overset{weak}{\models} \alpha \to (\varphi \vee \psi)$, thus if $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \Dashv \alpha \to (\varphi \vee \psi)$ then $s = \emptyset$.
Therefore if $\mathcal{M}, s \models \rho$, $\mathcal{M}, s \models \alpha$, $\mathcal{M}, s \Dashv \varphi$ and $\mathcal{M}, s \Dashv \psi$ then $s = \emptyset$
That is equivalent to if $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \Dashv \alpha \to \varphi$ and $\mathcal{M}, s \Dashv \alpha \to \psi$ then $s = \emptyset$. Thus if $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \Dashv (\alpha \to \varphi) \vee (\alpha \to \psi)$ then $s = \emptyset$. Therefore, By def. $\rho \overset{weak}{\models} (\alpha \to \varphi) \vee (\alpha \to \psi)$. □

4.3.2 First-Order Rules

Inquisitive Weak Particular

FACT 48 [Inquisitive Weak Particular-in]

$$\frac{\rho \overset{weak}{\models} \varphi[^n/_x]}{\rho \overset{weak}{\models} \overline{\exists} x.\varphi}$$

Proof. Assume $\rho \overset{weak}{\models} \varphi[^n/_x]$, thus if $\mathcal{M}, s \models \rho$ and for all $n \in \mathfrak{N}$ $\mathcal{M}, s \rightleftharpoons \varphi[^n/_x]$ then $s = \emptyset$. Therefore, if $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \rightleftharpoons \overline{\exists} x.\varphi$ then $s = \emptyset$. Thus, $\rho \overset{weak}{\models} \overline{\exists} x.\varphi$. □

FACT 49 [Inquisitive Weak Particular-el]

$$\frac{\rho \overset{weak}{\models} \overline{\exists} x.\varphi \qquad \rho \wedge \varphi[^n/_x] \overset{weak}{\models} \psi}{\rho \overset{weak}{\models} \psi}$$

Proof. Assume $\rho \overset{weak}{\models} \overline{\exists} x.\varphi$ and $\rho \wedge \varphi[^n/_x] \overset{weak}{\models} \psi$, i.e.:
(i) if $\mathcal{M}, s \models \rho$ and For All $n \in \mathfrak{N}$ $\mathcal{M}, s \rightleftharpoons \varphi[^n/_x]$ then $s = \emptyset$.
(ii) if $\mathcal{M}, s \models \rho$ and For Some $n \in \mathfrak{N}$ $\mathcal{M}, s \models \varphi[^n/_x]$ and $\mathcal{M}, s \rightleftharpoons \psi$ then $s = \emptyset$.
Assume for Reductio $\rho \overset{weak}{\not\models} \psi$, thus there is an $\mathcal{M}$& $s \neq \emptyset$ s.t. $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \rightleftharpoons \psi$.
Since s is not empty, by Lemma 4.1.2, there is a $w \in s$ s.t. $\mathcal{M}, w \Vdash \rho$ and $\mathcal{M}, w \dashV \psi$. By World-Singleton Correspondence $\mathcal{M}, \{w\} \models \rho$ and $\mathcal{M}, \{w\} \rightleftharpoons \psi$. By Lemma 3.1.2 there is a Model $\mathcal{M}'$ (1) $\mathcal{M}', \{w\} \models \rho$ and $\mathcal{M}', \{w\} \rightleftharpoons \psi$.
By Lemma 3.1.3 either (2) $\mathcal{M}', \{w\} \rightleftharpoons \overline{\exists} x.\varphi$ or (3) $\mathcal{M}', \{w\} \models \overline{\exists} x.\varphi$. However, if (2) then by (1) and (i) $\{w\} = \emptyset$ and if (3) then by (1) and (ii) $\{w\} = \emptyset$.
Either way we have a contradiction, therefore $\rho \overset{weak}{\models} \psi$.

□

Inquisitive Weak Particular Split

FACT 50 [Inquisitive Weak Particular Split]

$$\frac{\rho \overset{weak}{\models} \alpha \to \overline{\exists} x.\varphi}{\rho \overset{weak}{\models} \overline{\exists} x.\alpha \to \varphi}$$

Proof. Given x is not free in α assume $\rho \overset{weak}{\models} \alpha \to \overline{\exists} x.\varphi$, thus: if $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \models \alpha$ and $\mathcal{M}, s \dashv \overline{\exists} x.\varphi$ then $s = \emptyset$. Therefore, if $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \models \alpha$ and For a $n \in \mathfrak{N}$ $\mathcal{M}, s \dashv \varphi[^n/_x]$ then $s = \emptyset$. Since x is not free in α, if $\mathcal{M}, s \models \rho$ and For a $n \in \mathfrak{N}$ $\mathcal{M}, s \models \alpha$ and $\mathcal{M}, s \dashv \varphi[^n/_x]$ then $s = \emptyset$. Thus if $\mathcal{M}, s \models \rho$ and For a $n \in \mathfrak{N}$ $\mathcal{M}, s \overset{strong}{\models} (\alpha \to \varphi)[^n/_x]$ then $s = \emptyset$. Therefore, if $\mathcal{M}, s \models \rho$ and $\mathcal{M}, s \overset{strong}{\models} \overline{\exists} x.(\alpha \to \varphi)$ then $s = \emptyset$, which by definition means that $\rho \overset{weak}{\models} \overline{\exists} x.\alpha \to \varphi$ □

Identity

FACT 51 [Ref.]

$$\frac{}{\rho \overset{weak}{\models} n = n}$$

Proof. Follows from Self-identity □

FACT 52 [Ref.]

$$\frac{\rho \overset{weak}{\models} \varphi[^c/_x] \qquad \rho \overset{weak}{\models} c = b}{\rho \overset{weak}{\models} [^b/_x]\varphi}$$

Proof. Follows from Leibniz's Laws □

4.3.3 Modal Fragment

For lack of time in this thesis I will not explore the soundness for the Modal Fragment. However, since every modal is "truth-conditional", (it is accepted in a state if and only if it is true on every world on the state and it is rejected if and only if it is false on every world on the state), it can be derived by Monotonicity and World-State Correspondence.

4.4 Results

Now that the core concepts of PhIL have been outlined it is possible to show how it meets the motivations that were mentioned in the Chapters 1 and 2: The relations between knowledge, existence, identity and necessity.

Cogito, Ergo Sum

In section 2.2 on page 26 we discussed the possibility for agents to use other agents' names and, as a corollary of it, their own. That feature also grounded the need for self-aware agents, that know they exist.

We can in fact prove that for all $a \in \mathcal{A}$, $\models K_a\mathcal{E}(a)$:

FACT 53 [Self-awareness]

Proof. Assume for Reductio there is a $\mathcal{M}\&\ s \subseteq \mathcal{W}$ such that $s \neq \emptyset$ $\mathcal{M}, s \mathrel{=\!\!|} K_a\mathcal{E}(a)$. Therefore $\mathcal{M}, s \models \mathcal{E}(a)$ and (non vacuously) for all $w \in s$, there is a $t \subseteq \sigma_a(w)$ s.t. (I) $\mathcal{M}, t \not\models \mathcal{E}(a)$.
By Cogito, Ergo Sum $\sigma_a(w) \neq \emptyset$ iff $[\![\mathcal{E}(a)]\!]^{\mathcal{M}}_w = 1$, thus for all $w \in s$, $\sigma_a(w) \neq \emptyset$. By Conditional Reflexivity $w \in \sigma_a(w)$ and by the Euclidean property if $w' \in \sigma_a(w)$ also $w' \in \sigma_a(w')$, thus again by Cogito Ergo Sum if $w' \in \sigma_a(w)$ then $[\![\mathcal{E}(a)]\!]^{\mathcal{M}}_w = 1$. Therefore for every non-empty $t' \subseteq \sigma_a(w)$, (I) $\mathcal{M}, t' \models \mathcal{E}(a)$. (1) and (2) are in contradiction, thus $s = \emptyset$. Therefore, for all $\mathcal{M}\&s \subseteq \mathcal{W}$ if $\mathcal{M}, s \mathrel{=\!\!|} K_a\mathcal{E}(a)$ then $s = \emptyset$, i.e. $\models K_a\mathcal{E}(a)$ □

Non-trivial Knowledge of Existence

To show that existence is not trivially known it is sufficient to show that it is possible for an epistemic state $\sigma_a(w)$ to not support $\mathcal{E}(n)$, provided that $n \neq a$. Consider for example the model in figure 4.3, such that (i) $\sigma_a(w_1) = \sigma_a(w_2) = \mathcal{W}$, $[\![\mathcal{E}(n)]\!]^{\mathcal{M}}_{w_1} = 1$ and $[\![\mathcal{E}(n)]\!]^{\mathcal{M}}_{w_2} = 0$. By "Cogito, Ergo Sum" $\mathcal{M}, \mathcal{W} \models \mathcal{E}(a)$. Since $\mathcal{M}, \mathcal{W} \not\models \mathcal{E}(n)$, thus by (i) and the definition of K_a in the ignorant state we have that $\mathcal{M}, \mathcal{W} \models \neg K_a\mathcal{E}(n)$.

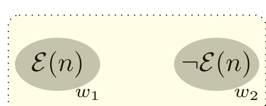


Figure 4.3: A PhIL model where $\mathcal{M} \models \neg K_a\mathcal{E}(n)$

Moreover, assume that for a certain unary predicate $P \in \mathcal{R}$ we have that $[\![P(n)]\!]^{\mathcal{M}}_{w_1} = 1$ and, by Definiteness, $[\![P(n)]\!]^{\mathcal{M}}_{w_2} = N$.

Thus, $\mathcal{M}, w_2 \dashv\Vdash \mathcal{E}(n)$, $\mathcal{M}, \{w_1\} \models \mathcal{E}(n)$ and $\mathcal{M}, \{w_1\} \models P(n)$.
From these assumptions we have the following:

$$\begin{aligned}
&\mathcal{M}, \mathcal{W} \models \mathcal{E}(a) \text{ and For all } w \in \mathcal{W} \text{ For all } t \subseteq \sigma_a(w),\\
&\quad \text{For all } w' \in t,\ \mathcal{M}, w' \dashv\Vdash \mathcal{E}(n) \text{ or } \mathcal{M}, t \models P(n)\\
\text{iff } &\mathcal{M}, \mathcal{W} \models \mathcal{E}(a) \text{ and For all } w \in \mathcal{W} \text{ For all } t \subseteq \sigma_a(w),\\
&\quad \text{if For all } w' \in t,\ \mathcal{M}, w' \not\dashv\Vdash \mathcal{E}(n) \text{ then } \mathcal{M}, t \models P(n)\\
\text{iff } &\mathcal{M}, \mathcal{W} \models \mathcal{E}(a) \text{ and For all } w \in \mathcal{W}, \mathcal{M}, \sigma_a(w) \models \mathcal{E}(n) \to P(n)\\
\text{iff } &\mathcal{M}, \mathcal{W} \models K_a(\mathcal{E}(n) \to P(n))
\end{aligned}$$

Which means "Agent a knows that if n exists then $P(n)$".

KNOWLEDGE OF NECESSITY OF IDENTITY

In section 2.3 on page 27 it was pointed out the difference between knowing whether $a = b$ or $a \neq b$ and knowing that either necessarily $a = b$ or necessarily $a \neq b$.
While the knowledge of identity has to be non-trivial, as it can be shown with a model similar to figure 4.3, we can prove that in PhIL ever agent knows that $\Box?(a = b)$ (i.e. the true answer to the question "$a = b$ or $a \neq b$" is necessary true), even though they may fail to know whether $a = b$.

FACT 54 [Necessity of Identity]
True Identity statements are necessarily true:
i.e. $a = b \overset{strong}{\models} \Box(a = b)$

Proof. Assume there is a $\mathcal{M}\&\ s \subseteq \mathcal{W}$ such that and $\mathcal{M}, s \models a = b$. Thus, For every $w \in s$, $\mathcal{M}, w \Vdash a = b$. By nominal rigidity, For all $w' \in s$; For all $w' \in \mathcal{N}(w')$; $\mathcal{M}, w \Vdash a = b$. Since $a = b$ is not-inquisitive, for all $w \in s$ $\mathcal{M}, \mathcal{N}(w) \models a = b$, thus $\mathcal{M}, s \models \Box(a = b)$.
Assume there is a $\mathcal{M}\&\ s \subseteq \mathcal{W}$ such that and $\mathcal{M}, s \Dashv \Box(a = b)$. Thus $\mathcal{M}, s \models \Diamond(a \neq b)$. Therefore for every $w \in s$ there is a $t \subseteq \mathcal{N}(w)$ s.t. $\mathcal{M}, t \models (a \neq b)$ and $t \neq \emptyset$. By Nominal Rigidity $\mathcal{M}, \mathcal{N}(w) \models (a \neq b)$, and by reflexivity $\mathcal{M}, \{w\} \models (a \neq b)$. Therefore for every $w \in s$, $\mathcal{M}, \{w\} \models (a \neq b)$, and since $a \neq b$ is not inquisitive $\mathcal{M}, s \models (a \neq b)$, therefore $\mathcal{M}, s \Dashv (a = b)$.
Thus, $a = b \overset{strong}{\models} \Box(a = b)$ ∎

From previously proved relations of entailments and deduction theorem we also have the following:

$$\dfrac{\dfrac{a = b \overset{strong}{\models} \Box(a = b)}{a = b \overset{weak}{\models} \Box(a = b)}}{\models a = b \to \Box(a = b)}$$

FACT 55 [Triviality of Necessity of Identity]
Necessarily either an identity statement is true or it is false:
i.e. $\top \overset{strong}{\models} \Box?(a = b)$

Proof. By definition of identity we have that for every $\mathcal{M}\&\ w \in \mathcal{W}$, either $\mathcal{M}, w \Vdash a = b$ or $\mathcal{M}, w \Vdash a \neq b$. By the previous proof we also have that if $\mathcal{M}, w \Vdash a = b$ then $\mathcal{M}, \mathcal{N}(w) \models a = b$ and if $\mathcal{M}, w \Vdash a \neq b$ then $\mathcal{M}, \mathcal{N}(w) \models a \neq b$, therefore for every $w \in \mathcal{W}$, $\mathcal{M}, \mathcal{N}(w) \models a = b$ or $\mathcal{M}, \mathcal{N}(w) \models a \neq b$. By definition, for every $w \in \mathcal{W}$, $\mathcal{M}, \mathcal{N}(w) \models a = b \vee a \neq b$, thus $\mathcal{M}, \mathcal{W} \models \Box(a = b \vee a \neq b)$, therefore $\Box(a = b \vee a \neq b)$ is not only valid (i.e. rejected only in $\emptyset$) but also tautological (i.e. accepted in every state). $\Box$

FACT 56 [Knowledge of Necessity of Identity]
Every agent knows that Identity is necessary

Proof. Assume for Reductio that there is a $\mathcal{M}\&s \neq \emptyset$ s.t. $\mathcal{M}, s \dashv K_a\Box?(a = b)$. This meant that For all $w \in s$; there is a $t \subseteq \sigma_a(w)$ s.t. $\mathcal{M}, t \not\models \Box?(a = b)$. However, this is in contradiction with the previous Fact, thus $\models K_a\Box?(a = b)$ $\Box$

5. Conclusion

To conclude I will focus on some issues that further developments of PhIL could include or explore. Moreover, I will sketch how I would intend to address them and how that approach departs from the current literature.
Then I will briefly go over the results and goals achieved in this thesis.

5.1 Other topics of interest

5.1.1 Analyticity and Indexicals

The standard account for indexicals [Kaplan, 1979][Stalnaker, 1970], commonly known as two-dimensional semantic, deals with Indexicals and Analyticity through a semantic that uses possible worlds and contexts of evaluation.

The meaning of a sentence in $2D$-semantic can be represented as a two dimensional matrix, like the one in the figure below. The whole matrix represents a propositional concept. Given that i is the actual world, the upper row of the matrix represents the intension expressed by a sentence from i's, j's and k's "perspective".
Take for example the sentence 'He runs' as represented in figure 5.1(a). The horizontal intension is called 2-intension. The diagonal, or 1-intension, of the matrix is what Stalnaker calls "diagonal proposition", and it should represent the meaning of the utterance.

Take the sentence 'Tully=Cicero', in figure 5.1(b). While the 2-intension shows that it is a necessary statement, from its 1-intension we can see that the statement is contingent: meaning that the sentence could have expressed with a different and possibly false proposition. Therefore we conclude that 'Tully=Cicero' is true if i is our would of evaluation, and necessary (true or false) if uttered in other worlds, but it is not analytic.

	i	j	k
i	***T***	*F*	*T*
j	T	**F**	T
k	F	T	**F**

(a) He runs

	i	j	k
i	***T***	*T*	*T*
j	F	**F**	F
k	T	T	**T**

(b) Tully=Cicero

	i	j	k
i	***T***	*F*	*F*
j	F	**T**	F
k	F	F	**T**

(c) I am here now

On the other hand, a sentence like 'I am here now' can be represented as in figure 5.1(c). The 1-intension shows that the sentence expresses a *metaphysically* *contingent* fact, while the 2-intension says it is analytic: it's guaranteed by semantic rules to be true in every possible context in which it is uttered.

An expansion of the language and semantic of PhIL with indexicals and the notion of analyticity could be particularly interesting. This expansion should take into account not only the previously mentioned analysis, but also the problem of the essential indexicals [Perry, 1979] and their epistemic significance. Therefore, it should be possible to have agents that do not know a priori what expressions like "I", "here" and "now" refer to.

As for analyticity my plan is still unrefined, but I believe it should depart from the current literature at least in some aspects. For example, I do not believe that analyticity should be taken as a standalone primitive concept, since it is a consequence of the agents' knowledge.
Roughly speaking, using the previously existent apparatus of Epistemic Logic, an idea could be to define analytically true a statement that is known as true by every (possible?) agent. This would reflect the idea that an analytic statement can be known from "one agent's armchair".

For example, given that $a \in \mathfrak{Ag}$, p_a is a position (etc...) we could say that a context c is an array of the form $c \overset{def}{=} \langle a, p_a, \ldots \rangle$.
A context c would proper in w *iff* the agent a exists in w uniquely at the position p_a (etc...). In this way, it could be possible to evaluate sentences with respect to pairs $\langle c, w \rangle$, such that c is an admissible context for w (notation $c \in \mathcal{C}(w)$). Then we can say that φ is analytic if and only if, for every context $c' \in \mathcal{C}(w)$, if a is the agent of the context c' then $\mathcal{M}, \langle c', w \rangle \Vdash K_a\varphi$:

$$\mathcal{M}, \langle c', w \rangle \Vdash \dagger\varphi \quad \text{iff} \quad \text{For All } c' \in \mathcal{C}(w); \text{ if } a \in c' \text{ then } \mathcal{M}, \langle c', w \rangle \Vdash K_a\varphi$$

5.1.2 Story telling vs Reasoning on the Premises

The last issue I would like to address both logically and philosophically is how we could make sense of fictional statements. The literature on the matter is wide.

On one extreme, the Russellian analysis [Russell, 2005] tells us that every statement must be quantified. Therefore, saying that 'Sherlock Holmes is intelligent' is false in virtue of his non existence, and equally false would be saying that he is not.

On the other side of the board, Lewis' Possibilism [Lewis, 1978] claims that every statement can be properly true with the right context-switching operator if we refer to the right cluster of possible worlds. Thus, 'Cassandra is a prophet' must be prefixed with the modal 'In Greek mythology', which will turn our attentions to all those disconnected space-times where it is known as a true fact that Cassandra exists and she is a fortune teller.

On one hand, I believe that the former approach demeans what can be truthfully said about fiction, and impose meaninglessness where some would be able to see that a rich and diverse universe of discourse exists. On the other hand, the latter solves the problem asking us to consider fiction as not-fiction in order to understand it, and does not address problems like the *cross-worlds reference* that we allegedly use works when we say 'Sherlock Holmes is the popularized stereotype of a private detective'.

The account I propose lives in-between these two approaches, and its slogan could be

> 'All [...] myths are *true*, for a given value of *'true'*.'
> - Terry Pratchett

The claim is that there are two substantially different attitudes we can have when we talk about fiction: story telling and reasoning on the premises.

In the first case we ought to distinguish between fictional claims and actual claims, and see how they interact. Consider this list of fictional and actual claims:

(13) 'Sherlock Holmes does not exist'

(14) 'Sherlock Holmes exists'

(15) 'Sherlock Holmes is intelligent'

(16) 'Sherlock Holmes is not intelligent'

(17) 'Sherlock Holmes is intelligent, though he does not exist'

(18) 'If Sherlock Holmes is intelligent, then he has an his IQ score'

While most would agree with me when I say that 14 and 13 have to be considered as respectively true and false, the attitude towards 17 are surely less unanimous. However, I say that we are able to recognize that a complex sentence like '(15) and (16)' is nonsense no matter the existential whereabout of the private detective, therefore we should treat it as properly false.
Moreover, I claim that a sentence like 18 should be considered *properly true on the premises* of the fiction. Such analysis is not too different from a structuralist approach to mathematics.

A way to distinguish story telling and arguing about fiction could be the following: the former is neither true nor false, though it ought to be self-consistent; the latter, if carried out reasonably, expresses true statement, while it is properly false if it is contradictory.

For example, we can take a fictional narrative F and map all the standard predication that are said to be true only within F to a truth value a, such that $1 > a > 0$ ($[\![F]\!] = a$).
Similarly, we can take all the standard predication that are said to be false within F to a truth value a^c, such that $1 > a^c > 0$, but neither $a > a^c$ nor $a^c > a$. This truth value will define what we could call F's Counter-Narrative, or Counter-Fiction ($[\![F^c]\!] = a$).
We should also have that there is a truth value N such that $1 > N$, but $N > a$ and $N > a^c$, so that the disjunction between fictionally contradictory statement

is not properly true. Consider for example the lattice in figure 5.1.

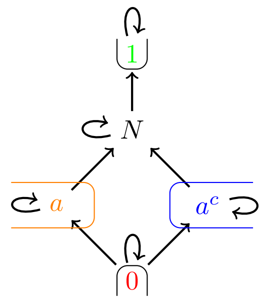


Figure 5.1: Truth values and ordering

The "gap" value N can also be used to map all those predications that are neither strictly-speaking fictional, nor true properly (maybe by virtue of non-existent reference).
For example, the previously mentioned claim 'Sherlock Holmes is the popularized stereotype of a private detective'; it can be interpreted as *stronger* than any false/fictional statement, but it is strictly *weaker* than a properly true statement.

At this point all we need to do is to define different truth/falsehood conditions, that checks which degree of truth a formula φ has:

$w \Vdash \varphi$	$[\![\varphi]\!]_w = 1$	iff φ is true at w.
$w \dashV \varphi$	$[\![\varphi]\!]_w = 0$	iff φ is false at w.
$w \Vdash_F \varphi$	$[\![\varphi]\!]_w = a$	iff φ is true in the fiction F at w
$w \dashV_F \varphi$	$[\![\varphi]\!]_w = a^c$	iff φ is false in the fiction F at w

Now for an arbitrary fiction/counter-fiction $\bigstar$ we can define truth and falsehood in it as follows:

$\mathcal{M}, w \Vdash_\bigstar P\vec{x}$	iff	$[\![P\vec{x}]\!]_w = [\![\bigstar]\!]$
$\mathcal{M}, w \dashV_\bigstar P\vec{x}$	iff	$[\![P\vec{x}]\!]_w = [\![\bigstar^c]\!]$
$\mathcal{M}, w \Vdash_\bigstar \varphi \wedge \psi$	iff	$\mathcal{M}, w \Vdash_\bigstar \varphi$ and $\mathcal{M}, w \Vdash_\bigstar \psi$
$\mathcal{M}, w \dashV_\bigstar \varphi \wedge \psi$	iff	$\mathcal{M}, w \dashV_\bigstar \varphi$ or $\mathcal{M}, w \dashV_\bigstar \psi$
$\mathcal{M}, w \Vdash_\bigstar \neg\varphi$	iff	$\mathcal{M}, w \dashV_\bigstar \varphi$
$\mathcal{M}, w \dashV_\bigstar \neg\varphi$	iff	$\mathcal{M}, w \Vdash_\bigstar \varphi$

The conditions for the *actual truth* are then defined taking into account narrative non-contradiction and reasoning on the premises.

$\mathcal{M}, w \Vdash \varphi \wedge \psi$	iff	$\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \dashv\vert \varphi \wedge \psi$	iff	$\mathcal{M}, w \dashv\vert \varphi$ or $\mathcal{M}, w \dashv\vert \psi$ or
		For some $\Vdash_\bigstar$; ($\mathcal{M}, w \Vdash_\bigstar \varphi$ and $\mathcal{M}, w \dashv\vert_\bigstar \psi$)
$\mathcal{M}, w \Vdash \varphi \rightarrow \psi$	iff	$\mathcal{M}, w \dashv\vert \varphi$ or $\mathcal{M}, w \Vdash \psi$ or
		For all $\Vdash_\bigstar$; (if $\mathcal{M}, w \Vdash_\bigstar \varphi$ then $\mathcal{M}, w \Vdash_\bigstar \psi$)
$\mathcal{M}, w \dashv\vert \varphi \rightarrow \psi$	iff	$\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \dashv\vert \psi$ or
		For some $\Vdash_\bigstar$; ($\mathcal{M}, w \Vdash_\bigstar \varphi$ and $\mathcal{M}, w \dashv\vert_\bigstar \psi$)

5.2 Conclusion

In this thesis we first surveyed Classical and standard Inquisitive Epistemic Logic, and seen the properties that knowledge has in these theories.
We then saw how these approaches cannot express non-existence and other concepts commonly used in philosophical logic. That is why a different approach, called Philosophical Inquisitive Logic, was brought forth. PhIL was in fact outlined to have the expressive power to deal non-trivially with Identity, Existence and Necessity. The final objective was to bind them altogether within an Epistemic Logic where not only agents are able to address each other but they are also themselves “Philosophically Inclined”.
Since this semantics is bilateral and inquisitive it let us not only express what happens when we ask counterfactual/modal questions, but also what we can learn from them.
All of this has been done in a nominalist context, showing how such topics can be dealt with without ontological commitment on objects and extensions but only looking at how we use names and predicates in philosophical arguments.
Moreover, with the proof-theoretic results we saw how every classically valid formula is valid in the“non-existential” fragment of Partial Modal Logic, while every INQ valid formula is valid in the $\mathcal{L}^{\top}_{PhiL}$ fragment of PhIL.
Outside those fragments we saw which properties are respected by some of the possible definitions of entailment, how they interact with a soft notion of validity and what more can be expressed.

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A. Partial Modal Logic

A.1 Defined Logical Operators

$\neg\varphi \stackrel{def}{=} \varphi \to \bot$	$\top \stackrel{def}{=} \neg\bot$	$\varphi \wedge \psi \stackrel{def}{=} \neg(\varphi \to \neg\psi)$
$\varphi \vee \psi \stackrel{def}{=} (\neg\varphi \to \psi)$	$\nabla x.\varphi \stackrel{def}{=} \neg\Game x.\neg\varphi$	$\Box\varphi \stackrel{def}{=} \neg\Diamond\neg\varphi$
$\exists x.\varphi \stackrel{def}{=} \Game x.(\mathcal{E}(x) \wedge \varphi)$	$\forall x.\varphi \stackrel{def}{=} \nabla x.\mathcal{E}(x) \to \varphi$	

$\mathcal{M}, w \Vdash \neg\varphi$	iff	$\mathcal{M}, w \dashV \varphi$
$\mathcal{M}, w \dashV \neg\varphi$	iff	$\mathcal{M}, w \Vdash \varphi$
$\mathcal{M}, w \Vdash \top$	iff	always
$\mathcal{M}, w \dashV \top$	iff	never
$\mathcal{M}, w \Vdash \varphi \wedge \psi$	iff	$\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \dashV \varphi \wedge \psi$	iff	$\mathcal{M}, w \dashV \varphi$ or $\mathcal{M}, w \dashV \psi$
$\mathcal{M}, w \Vdash \varphi \vee \psi$	iff	$\mathcal{M}, w \Vdash \varphi$ or $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \dashV \varphi \vee \psi$	iff	$\mathcal{M}, w \dashV \varphi$ and $\mathcal{M}, w \dashV \psi$
$\mathcal{M}, w \Vdash \nabla x.\varphi$	iff	For all $n \in \mathfrak{N}$; $\mathcal{M}, w \Vdash \varphi[^n/_x]$
$\mathcal{M}, w \dashV \nabla x.\varphi$	iff	There is a $n \in \mathfrak{N}$ s.t $\mathcal{M}, w \dashV \varphi[^n/_x]$
$\mathcal{M}, w \Vdash \Box\varphi$	iff	For all $w' \in \mathcal{N}(w)$; $\mathcal{M}, w' \not\dashV \varphi$
$\mathcal{M}, w \dashV \Box\varphi$	iff	There is a $w' \in \mathcal{N}(w)$; $\mathcal{M}, w' \dashV \varphi$
$\mathcal{M}, w \Vdash \exists x.\varphi$	iff	There is a $n \in \mathfrak{N}$ s.t $\mathcal{M}, w \Vdash \mathcal{E}(n)$ and $\mathcal{M}, w \Vdash \varphi[^n/_x]$
$\mathcal{M}, w \dashV \exists x.\varphi$	iff	For all $n \in \mathfrak{N}$; $\mathcal{M}, w \Vdash \mathcal{E}(n)$ and $\mathcal{M}, w \dashV \varphi[^n/_x]$
$\mathcal{M}, w \Vdash \forall x.\varphi$	iff	For all $n \in \mathfrak{N}$; $\mathcal{M}, w \dashV \mathcal{E}(n)$ or $\mathcal{M}, w \Vdash \varphi[^n/_x]$
$\mathcal{M}, w \dashV \forall x.\varphi$	iff	There is a $n \in \mathfrak{N}$ s.t $\mathcal{M}, w \Vdash \mathcal{E}(n)$ and $\mathcal{M}, w \dashV \varphi[^n/_x]$

A.2 Algebraic approach for the Valuation Function

In the following section I will introduce an algebraic approach to the Partial Modal Logic treated in chapter 3 on page 31.
The valuation function is a natural expansion of the interpretation function. With a Lilliputian abuse of notation we can use the same symbolism for the two functions knowing that if $\alpha \in \texttt{Atom}$ then the the interpretation function and the valuation function are one and the same:

$$[\![-]\!]_w^{\mathcal{M}} : \mathcal{L}^{PW} \to \{0, N, 1\} : \varphi \mapsto [\![\varphi]\!]_w^{\mathcal{M}}$$

For brevity in the following definitions I will never specify the model $\mathcal{M}$. Since this system does not have dynamic operators such omission is inconsequential.

A.2.1 Basic Algebraic Operations

Maximal Element

$$[\![\varphi]\!]_w \sqcup [\![\psi]\!]_w \stackrel{def}{=} \max\left([\![\varphi]\!]_w, [\![\psi]\!]_w\right)$$

Complementation

$$[\![\varphi]\!]^*_w \stackrel{def}{=} \begin{cases} 0 & \text{if } [\![\varphi]\!]_w = 1 \\ N & \text{if } [\![\varphi]\!]_w = N \\ 1 & \text{if } [\![\varphi]\!]_w = 0 \end{cases}$$

N-Elimination

$$\overline{[\![\varphi]\!]}_w \stackrel{def}{=} \begin{cases} 1 & \text{if } [\![\varphi]\!]_w = 1 \\ 0 & \text{otherwise} \end{cases}$$

A.2.2 Defined Algebraic Operators

$$[\![\varphi]\!]_w \Rightarrow [\![\psi]\!]_w \stackrel{def}{=} [\![\varphi]\!]^*_w \sqcup [\![\psi]\!]_w$$

$$[\![\varphi]\!]_w \sqcap [\![\psi]\!]_w \stackrel{def}{=} \left([\![\varphi]\!]^*_w \sqcup [\![\psi]\!]^*_w\right)^*$$

$$\bigsqcup_{n \in X} \left([\![\varphi[^n/_x]]\!]_w\right) \stackrel{def}{=} \begin{cases} \bigsqcup_{n \in X} [\![\varphi[^n/_x]]\!]_w = 0 & \text{if } X = \emptyset \\ \bigsqcup_{n \in X \cup \{y\}} [\![\varphi[^n/_x]]\!]_w = [\![\varphi[^y/_x]]\!]_w \sqcup \left(\bigsqcup_{n \in X \setminus \{y\}} [\![\varphi[^n/_x]]\!]_w\right) \end{cases}$$

$$\biguplus_{w \in Y} \left([\![\varphi]\!]_w\right) \stackrel{def}{=} \begin{cases} \biguplus_{w \in Y} [\![\varphi]\!]_w = 0 & \text{if } Y = \emptyset \\ \biguplus_{w \in Y \cup \{w'\}} [\![\varphi]\!]_w = \overline{[\![\varphi]\!]}_{w'} \sqcup \left(\biguplus_{w \in Y \setminus \{w'\}} [\![\varphi]\!]_w\right) \end{cases}$$

A.2.3 Recursive Definition of the Valuation Function

$$[\![\varphi]\!]_w = \begin{cases} [\![\varphi]\!]_w = [\![\alpha]\!]_w & \text{if } \varphi = \alpha \text{ and } \alpha \in \texttt{Atom} \\ [\![\star]\!]_w = N \\ [\![\bot]\!]_w = 0 \\ [\![\varphi \to \psi]\!]_w = [\![\varphi]\!]_w \Rightarrow [\![\psi]\!]_w \\ [\![\exists x.\varphi]\!]_w = \bigsqcup_{n \in \mathfrak{N}} \left([\![\varphi[^n/_x]]\!]_w\right) \\ [\![\Diamond\varphi]\!]_w = \biguplus_{w' \in \mathcal{N}(w)} [\![\varphi]\!]_{w'} \end{cases}$$

A.3 Algebraic approach for Informative Content

Similarly we can provide an algebra for the Informative Content of well formed formulas described in section 3.1.4 on page 37, such that:

- $|\varphi|^{\top}_{\mathcal{M}} \stackrel{def}{=} \{w \in \mathcal{W} \mid \mathcal{M}, w \Vdash \varphi\}$
- $|\varphi|^{\bot}_{\mathcal{M}} \stackrel{def}{=} \{w \in \mathcal{W} \mid \mathcal{M}, w \dashv\Vert \varphi\}$

As in the previous section, I will not specify the model $\mathcal{M}$ in the following definitions

A.3.1 Truth-set and Falsehood-set

$$
\begin{aligned}
|\alpha|^{\top} =& \{w \in \mathcal{W} \mid [\![\alpha]\!]_w = 1\} \\
|\alpha|^{\bot} =& \{w \in \mathcal{W} \mid [\![\alpha]\!]_w = 0\} \\
|\bot|^{\top} =& \emptyset \\
|\bot|^{\bot} =& \mathcal{W} \\
|\bigstar|^{\top} =& \emptyset \\
|\bigstar|^{\bot} =& \emptyset \\
|\varphi \to \psi|^{\top} =& |\varphi|^{\bot} \cup |\psi|^{\top} \\
|\varphi \to \psi|^{\bot} =& |\varphi|^{\top} \cap |\psi|^{\bot} \\
|\Xi x.\varphi|^{\top} =& \bigcup_{n \in \mathfrak{N}} \left(|\varphi[^{n}/_{x}]|^{\top}\right) \\
|\Xi x.\varphi|^{\bot} =& \bigcap_{n \in \mathfrak{N}} \left(|\varphi[^{n}/_{x}]|^{\bot}\right) \\
|\Diamond\varphi|^{\top} =& \{w \in \mathcal{W} \mid \left(\mathcal{N}(w) \cap |\varphi|^{\top}\right) \neq \emptyset\} \\
|\Diamond\varphi|^{\bot} =& \{w \in \mathcal{W} \mid \left(\mathcal{N}(w) \cap |\varphi|^{\top}\right) = \emptyset\}
\end{aligned}
$$

A.4 Partial Modal Logic Correspondence

We can now prove the correspondence between Truth/Falsehood conditions, Valuation function and Informative Content. Namely, for all $\mathcal{M}$& $w \in \mathcal{W}$:

$$\mathcal{M}, w \Vdash \varphi \text{ iff } [\![\varphi]\!]^{\mathcal{M}}_w = 1 \text{ iff } w \in |\varphi|^{\top}_{\mathcal{M}}$$

$$\mathcal{M}, w \dashv\Vert \varphi \text{ iff } [\![\varphi]\!]^{\mathcal{M}}_w = 0 \text{ iff } w \in |\varphi|^{\bot}_{\mathcal{M}}$$

I will divide the proof in two sub-case that altogether, by transitivity, will be equivalent to the aforementioned claim.

section A.4.1:
$\mathcal{M}, w \Vdash \varphi$ iff $[\![\varphi]\!]_w^{\mathcal{M}} = 1$ and $\mathcal{M}, w \dashv\!\mid \varphi$ iff $[\![\varphi]\!]_w^{\mathcal{M}} = 0$

section A.4.2 on page 92:
$[\![\varphi]\!]_w^{\mathcal{M}} = 0$ iff $w \in |\varphi|_{\mathcal{M}}^{\bot}$ and $[\![\varphi]\!]_w^{\mathcal{M}} = 1$ iff $w \in |\varphi|_{\mathcal{M}}^{\top}$

As always I will not specify the model $\mathcal{M}$, which will be always arbitrary in the proofs.

A.4.1 truth/falsehood and Valuation

FACT 57 [truth/falsehood & Valuation Correspondence]
$w \Vdash \varphi$ *iff* $[\![\varphi]\!]_w = 1$ *and* $w \dashv\!\mid \varphi$ *iff* $[\![\varphi]\!]_w = 0$

Proof. We can prove this Fact by induction:

Base case

Atomic Sentences
If $\alpha \in \texttt{Atom}$ by definition of truth/falsehood and Valuation we have that:
$w \Vdash \alpha$ iff $[\![\alpha]\!]_w = 1$ and
$w \dashv\!\mid \alpha$ iff $[\![\alpha]\!]_w = 0$

False
By definition of truth/falsehood and Valuation we have that:
never $w \Vdash \bot$ and never $[\![\bot]\!]_w = 1$ and
always $w \dashv\!\mid \bot$ and always $[\![\bot]\!]_w = 0$

Undefined
By definition of truth/falsehood and Valuation we have that:
never $w \Vdash \bigstar$ and never $[\![\bigstar]\!]_w = 1$ and
never $w \dashv\!\mid \bigstar$ and never $[\![\bigstar]\!]_w = 0$

Inductive Step (With IH being "for any φ less complex than ρ, $w \Vdash \varphi$ iff $[\![\varphi]\!]_w = 1$ and $w \dashv\!\mid \varphi$ iff $[\![\varphi]\!]_w = 0$ ")

Implication: $\rho \stackrel{def}{=} \varphi \to \psi$

$$\begin{aligned} w \Vdash \varphi \to \psi \ &\text{iff} \ w \dashv\!\mid \varphi \text{ or } w \Vdash \psi \\ &\text{iff} \ \text{(by IH)} \ [\![\varphi]\!]_w = 0 \text{ or } [\![\psi]\!]_w = 1 \\ &\text{iff} \ [\![\varphi]\!]_w^* = 1 \text{ or } [\![\psi]\!]_w = 1 \\ &\text{iff} \ ([\![\varphi]\!]_w^* \sqcup [\![\psi]\!]_w) = 1 \\ &\text{iff} \ ([\![\varphi]\!]_w \Rightarrow [\![\psi]\!]_w) = 1 \\ &\text{iff} \ [\![\varphi \to \psi]\!]_w = 1 \end{aligned}$$

$$
\begin{aligned}
w \dashv\!\!\mid \varphi \to \psi \ \text{ iff } & \ w \Vdash \varphi \text{ and } w \dashv\!\!\mid \psi \\
\text{iff } & \ \text{(by IH)} \ \ [\![\varphi]\!]_w = 1 \text{ and } [\![\psi]\!]_w = 0 \\
\text{iff } & \ [\![\varphi]\!]_w = 1 \text{ and } [\![\psi]\!]^*_w = 1 \\
\text{iff } & \ ([\![\varphi]\!]_w \sqcap [\![\psi]\!]^*_w) = 1 \\
\text{iff } & \ ([\![\varphi]\!]^*_w \sqcup [\![\psi]\!]^{**}_w)^* = 1 \\
\text{iff } & \ ([\![\varphi]\!]^*_w \sqcup [\![\psi]\!]_w) = 0 \\
\text{iff } & \ ([\![\varphi]\!]_w \Rightarrow [\![\psi]\!]_w) = 0 \\
\text{iff } & \ [\![\varphi \to \psi]\!]_w = 0
\end{aligned}
$$

Particular Quantifier: $\rho \stackrel{def}{=} \mathfrak{Z}x.\varphi$

$$
\begin{aligned}
w \Vdash \mathfrak{Z}x.\varphi \ \text{ iff } & \ \text{there is a } n \in \mathfrak{N} \ \text{ s.t. } \ w \Vdash \varphi[{}^n/_x] \\
\text{iff } & \ \text{(by IH)} \ \ \text{there is a } n \in \mathfrak{N} \ \text{ s.t. } \ [\![\varphi[{}^n/_x]]\!]_w = 1 \\
\text{iff } & \ \bigsqcup_{n \in \mathfrak{N}} ([\![\varphi[{}^n/_x]]\!]_w) = 1 \\
\text{iff } & \ [\![\mathfrak{Z}x.\varphi]\!]_w = 1
\end{aligned}
$$

$$
\begin{aligned}
w \dashv\!\!\mid \mathfrak{Z}x.\varphi \ \text{ iff } & \ \text{For all } n \in \mathfrak{N}, \ w \dashv\!\!\mid \varphi[{}^n/_x] \\
\text{iff } & \ \text{(by IH)} \ \ \text{For all } n \in \mathfrak{N}, \ [\![\varphi[{}^n/_x]]\!]_w = 0 \\
\text{iff } & \ \bigsqcup_{n \in \mathfrak{N}} ([\![\varphi[{}^n/_x]]\!]_w) = 0 \\
\text{iff } & \ [\![\mathfrak{Z}x.\varphi]\!]_w = 0
\end{aligned}
$$

Possible $\rho \stackrel{def}{=} \Diamond\varphi$

$$
\begin{aligned}
w \Vdash \Diamond\varphi \ \text{ iff } & \ \text{there is a } v \in \mathcal{N}(w) \ \text{ s.t. } \ v \Vdash \varphi \\
\text{iff } & \ \text{(by IH)} \ \ \text{there is a } v \in \mathcal{N}(w) \ \text{ s.t. } \ [\![\varphi]\!]_v = 1 \\
\text{iff } & \ \text{there is a } v \in \mathcal{N}(w) \ \text{ s.t. } \ \overline{[\![\varphi]\!]}_v = 1 \\
\text{iff } & \ \biguplus_{v \in \mathcal{N}(w)} ([\![\varphi]\!]_v) = 1 \\
\text{iff } & \ [\![\Diamond\varphi]\!]_w = 1
\end{aligned}
$$

$$
\begin{aligned}
w \dashv\!\!\mid \Diamond\varphi \ \text{ iff } & \ \text{For all } v \in \mathcal{N}(w), \ w \not\Vdash \varphi \\
\text{iff } & \ \text{(by IH)} \ \ \text{For all } v \in \mathcal{N}(w), [\![\varphi]\!]_w \neq 1 \\
\text{iff } & \ \text{For all } v \in \mathcal{N}(w), \overline{[\![\varphi]\!]}_w = 0 \\
\text{iff } & \ \biguplus_{v \in \mathcal{N}(w)} ([\![\varphi]\!]_w) = 0 \\
\text{iff } & \ [\![\Diamond\varphi]\!]_w = 0
\end{aligned}
$$

□

A.4.2 Informative Content and Valuation

FACT 58 [Informative Content & Valuation Correspondence]
$w \in |\varphi|^{\top}$ *iff* $[\![\varphi]\!]_w = 1$ *and* $w \in |\varphi|^{\perp}$ *iff* $[\![\varphi]\!]_w = 0$

Proof. We can prove this Fact by induction:

Base case

Atomic Sentences If $\alpha \in \texttt{Atom}$ by definition of Informative Content and Valuation we have that:
$w \in |\alpha|^{\top}$ iff $w \in \{w \in \mathcal{W} \mid [\![\alpha]\!]_w = 1\}$ iff $[\![\alpha]\!]_w = 1$ and
$w \in |\alpha|^{\perp}$ iff $w \in \{w \in \mathcal{W} \mid [\![\alpha]\!]_w = 0\}$ iff $[\![\alpha]\!]_w = 0$

False By definition of Informative Content and Valuation we have that:
since $\emptyset = |\bot|^{\top}$, never $w \in \emptyset$ and never $[\![\bot]\!]_w = 1$ and
since $\mathcal{W} = |\bot|^{\perp}$, always $w \in \mathcal{W}$ and always $[\![\bot]\!]_w = 0$

Undefined By definition of Informative Content and Valuation we have that:
since $\emptyset = |\bigstar|^{\top}$, never $w \in \emptyset$ and never $[\![\bigstar]\!]_w = 1$ and
since $\emptyset = |\bigstar|^{\perp}$, never $w \in \emptyset$ and never $[\![\bigstar]\!]_w = 0$

Inductive Step (With IH being "for any φ less complex than ρ, $w \in |\varphi|^{\top}$ iff $[\![\varphi]\!]_w = 1$ and $w \in |\varphi|^{\perp}$ iff $[\![\varphi]\!]_w = 0$ ")

Implication: $\rho \overset{def}{=} \varphi \to \psi$

$$\begin{aligned}
w \in |\varphi \to \psi|^{\top} \quad & \text{iff} \quad w \in (|\varphi|^{\perp} \cup |\psi|^{\top}) \\
& \text{iff} \quad w \in |\varphi|^{\perp} \text{ or } \ w \in |\psi|^{\top} \\
& \text{iff} \quad \text{(by IH)} \ \ [\![\varphi]\!]_w = 0 \text{ or } [\![\psi]\!]_w = 1 \\
& \text{iff} \quad [\![\varphi]\!]^*_w = 1 \text{ or } [\![\psi]\!]_w = 1 \\
& \text{iff} \quad ([\![\varphi]\!]^*_w \sqcup [\![\psi]\!]_w) = 1 \\
& \text{iff} \quad ([\![\varphi]\!]_w \Rightarrow [\![\psi]\!]_w) = 1 \\
& \text{iff} \quad [\![\varphi \to \psi]\!]_w = 1
\end{aligned}$$

$$\begin{aligned}
w \in |\varphi \to \psi|^{\perp} \quad & \text{iff} \quad w \in (|\varphi|^{\top} \cap |\psi|^{\perp}) \\
& \text{iff} \quad w \in |\varphi|^{\top} \text{ and } \ w \in |\psi|^{\perp} \\
& \text{iff} \quad \text{(by IH)} \ \ [\![\varphi]\!]_w = 1 \text{ and } [\![\psi]\!]_w = 0 \\
& \text{iff} \quad [\![\varphi]\!]_w = 1 \text{ and } [\![\psi]\!]^*_w = 1 \\
& \text{iff} \quad ([\![\varphi]\!]_w \sqcap [\![\psi]\!]^*_w) = 1 \\
& \text{iff} \quad ([\![\varphi]\!]^*_w \sqcup [\![\psi]\!]^{**}_w)^* = 1 \\
& \text{iff} \quad ([\![\varphi]\!]^*_w \sqcup [\![\psi]\!]_w) = 0 \\
& \text{iff} \quad ([\![\varphi]\!]_w \Rightarrow [\![\psi]\!]_w) = 0 \\
& \text{iff} \quad [\![\varphi \to \psi]\!]_w = 0
\end{aligned}$$

Particular Quantifier: $\rho \stackrel{def}{=} \mathfrak{Z}x.\varphi$

$$
\begin{aligned}
w \in |\mathfrak{Z}x.\varphi|^{\top} \quad &\text{iff} \quad w \in \bigcup_{n \in \mathfrak{N}} (|\varphi[^n/_x]|^{\top}) \\
&\text{iff} \quad \text{There is a } n \in \mathfrak{N} \text{ s.t. } w \in |\varphi[^n/_x]|^{\top} \\
&\text{iff} \quad \text{(by IH)} \quad \text{there is a } n \in \mathfrak{N} \text{ s.t. } [\![\varphi[^n/_x]]\!]_w = 1 \\
&\text{iff} \quad \bigsqcup_{n \in \mathfrak{N}} ([\![\varphi[^n/_x]]\!]_w) = 1 \\
&\text{iff} \quad [\![\mathfrak{Z}x.\varphi]\!]_w = 1
\end{aligned}
$$

$$
\begin{aligned}
w \in |\mathfrak{Z}x.\varphi|^{\perp} \quad &\text{iff} \quad w \in \bigcap_{n \in \mathfrak{N}} (|\varphi[^n/_x]|^{\perp}) \\
&\text{iff} \quad \text{For all } n \in \mathfrak{N},\ w \in |\varphi[^n/_x]|^{\perp} \\
&\text{iff} \quad \text{(by IH)} \quad \text{For all } n \in \mathfrak{N},\ [\![\varphi[^n/_x]]\!]_w = 0 \\
&\text{iff} \quad \bigsqcup_{n \in \mathfrak{N}} ([\![\varphi[^n/_x]]\!]_w) = 0 \\
&\text{iff} \quad [\![\mathfrak{Z}x.\varphi]\!]_w = 0
\end{aligned}
$$

Possible $\rho \stackrel{def}{=} \Diamond\varphi$

$$
\begin{aligned}
w \in |\Diamond\varphi|^{\top} \quad &\text{iff } w \in \{u \in \mathcal{W} |\ (\mathcal{N}(u) \cap |\varphi|^{\top}) \neq \emptyset\} \\
&\text{iff } (\mathcal{N}(w) \cap |\varphi|^{\top}) \neq \emptyset \\
&\text{iff } \text{there is a } v \in \mathcal{N}(w) \text{ s.t. } v \in |\varphi|^{\top} \\
&\text{iff } \text{(by IH)} \quad \text{there is a } v \in \mathcal{N}(w) \text{ s.t. } [\![\varphi]\!]_v = 1 \\
&\text{iff } \text{there is a } v \in \mathcal{N}(w) \text{ s.t. } \overline{[\![\varphi]\!]}_v = 1 \\
&\text{iff } \Cup_{v \in \mathcal{N}(w)} ([\![\varphi]\!]_w) = 1 \\
&\text{iff } [\![\Diamond\varphi]\!]_w = 1
\end{aligned}
$$

$$
\begin{aligned}
w \in |\Diamond\varphi|^{\perp} \quad &\text{iff } w \in \{u \in \mathcal{W} |\ (\mathcal{N}(u) \cap |\varphi|^{\top}) = \emptyset\} \\
&\text{iff } (\mathcal{N}(w) \cap |\varphi|^{\top}) = \emptyset \\
&\text{iff } \text{For all } v \in \mathcal{N}(w), v \notin |\varphi|^{\top} \\
&\text{iff } \text{(by IH)} \quad \text{For all } v \in \mathcal{N}(w), [\![\varphi]\!]_w \neq 1 \\
&\text{iff } \text{For all } v \in \mathcal{N}(w), \overline{[\![\varphi]\!]}_w = 0 \\
&\text{iff } \Cup_{v \in \mathcal{N}(w)} ([\![\varphi]\!]_w) = 0 \\
&\text{iff } [\![\Diamond\varphi]\!]_w = 0
\end{aligned}
$$

□

B. Philosophical Inquisitive Logic

In this section the acceptance/rejection conditions of the operators defined in section 4.1 on page 59 are listed.

B.1 Defined Logical Operators in PhIL

$\neg\varphi \stackrel{def}{=} \varphi \twoheadrightarrow \bot$	$\varphi \owedge \psi \stackrel{def}{=} \neg(\varphi \twoheadrightarrow (\neg\psi))$	$\varphi \ovee \psi \stackrel{def}{=} (\neg\varphi \twoheadrightarrow \psi)$
$\underline{\nabla} x.\varphi \stackrel{def}{=} \neg\overline{\text{Ʒ}} x.\neg\varphi$	$\top \stackrel{def}{=} \bot \twoheadrightarrow \bot$	

(a) Defined Inquisitive Operators

$\neg\varphi \stackrel{def}{=} \varphi \to \bot$	$!\varphi \stackrel{def}{=} \neg\neg\varphi$	$\varphi \vee \psi \stackrel{def}{=} !(\varphi \ovee \psi)$
$\Box\varphi \stackrel{def}{=} \neg\Diamond\neg\varphi$	$\varphi \wedge \psi \stackrel{def}{=} !(\varphi \owedge \psi)$	$\text{Ʒ} x.\varphi \stackrel{def}{=} !\overline{\text{Ʒ}} x.\varphi$

(b) Defined Classical Operators

$?\varphi \stackrel{def}{=} \neg\varphi \ovee \varphi$	$\overline{\exists} x.\varphi \stackrel{def}{=} \overline{\text{Ʒ}} x.(\mathcal{E}(x) \owedge \varphi)$	$\underline{\forall} x.\varphi \stackrel{def}{=} \underline{\nabla} x.(\mathcal{E}(x) \to \varphi)$
$\exists x.\varphi \stackrel{def}{=} !\overline{\exists} x.\varphi$	$\forall x.\varphi \stackrel{def}{=} !\underline{\forall} x.\varphi$	

(c) Defined hybrid Operators

$\mathcal{M}, s \models \neg\varphi$	iff	$\mathcal{M}, s \mathrel{=\!\mid} \varphi$
$\mathcal{M}, s \mathrel{=\!\mid} \neg\varphi$	iff	$\mathcal{M}, s \models \varphi$
$\mathcal{M}, s \models \varphi \owedge \psi$	iff	$\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \models \psi$
$\mathcal{M}, s \mathrel{=\!\mid} \varphi \owedge \psi$	iff	$\mathcal{M}, s \mathrel{=\!\mid} \varphi$ or $\mathcal{M}, s \mathrel{=\!\mid} \psi$
$\mathcal{M}, s \models \varphi \ovee \psi$	iff	$\mathcal{M}, s \models \varphi$ or $\mathcal{M}, s \models \psi$
$\mathcal{M}, s \mathrel{=\!\mid} \varphi \ovee \psi$	iff	$\mathcal{M}, s \mathrel{=\!\mid} \varphi$ and $\mathcal{M}, s \mathrel{=\!\mid} \psi$
$\mathcal{M}, s \models \underline{\nabla} x.\varphi$	iff	For all $n \in \mathfrak{N}$; $\mathcal{M}, s \models \varphi[^n/_x]$
$\mathcal{M}, s \mathrel{=\!\mid} \underline{\nabla} x.\varphi$	iff	There is a $n \in \mathfrak{N}$ s.t. $\mathcal{M}, s \mathrel{=\!\mid} \varphi[^n/_x]$
$\mathcal{M}, s \models \top$	iff	always
$\mathcal{M}, s \mathrel{=\!\mid} \top$	iff	$s = \emptyset$

$\mathcal{M}, s \models \neg\varphi$	iff	For all $w \in s$; $\mathcal{M}, w \dashV \varphi^{cl}$
$\mathcal{M}, s \Dashv \neg\varphi$	iff	$\mathcal{M}, s \models \varphi$
$\mathcal{M}, s \models !\varphi$	iff	For all $w \in s$; $\mathcal{M}, w \Vdash \varphi^{cl}$
$\mathcal{M}, s \Dashv !\varphi$	iff	For all $w \in s$; $\mathcal{M}, w \dashV \varphi^{cl}$
$\mathcal{M}, s \models \varphi \wedge \psi$	iff	For all $w \in s$; $\mathcal{M}, w \Vdash \varphi^{cl}$ and $\mathcal{M}, w \Vdash \psi^{cl}$
$\mathcal{M}, s \Dashv \varphi \wedge \psi$	iff	For all $w \in s$; $\mathcal{M}, w \dashV \varphi^{cl}$ or $\mathcal{M}, w \dashV \psi^{cl}$
$\mathcal{M}, s \models \varphi \vee \psi$	iff	For all $w \in s$; $\mathcal{M}, w \Vdash \varphi^{cl}$ or $\mathcal{M}, w \Vdash \psi^{cl}$
$\mathcal{M}, s \Dashv \varphi \vee \psi$	iff	For all $w \in s$; $\mathcal{M}, w \dashV \varphi^{cl}$ and $\mathcal{M}, w \dashV \psi^{cl}$
$\mathcal{M}, s \models \Box\varphi$	iff	For all $w \in s$; For all $t \subseteq \mathcal{N}(w)$; $\mathcal{M}, t \not\Dashv \varphi$
$\mathcal{M}, s \Dashv \Box\varphi$	iff	For all $w \in s$; There is a $t \subseteq \mathcal{N}(w)$; $\mathcal{M}, t \not\models \bot$ and $\mathcal{M}, t \Dashv \varphi$
$\mathcal{M}, s \models \text{Ʒ}x.\varphi$	iff	For all $w \in s$; There is a $n \in \mathfrak{N}$ s.t $\mathcal{M}, w \Vdash \varphi[^n/_x]^{cl}$
$\mathcal{M}, s \Dashv \text{Ʒ}x.\varphi$	iff	For all $w \in s$; For all $n \in \mathfrak{N}$; $\mathcal{M}, w \dashV \varphi[^n/_x]^{cl}$

$\mathcal{M}, s \models ?\varphi$	iff	$\mathcal{M}, s \models \varphi$ or For all $w \in s$; $\mathcal{M}, w \Vdash \varphi^{cl}$
$\mathcal{M}, s \Dashv ?\varphi$	iff	$s = \emptyset$
$\mathcal{M}, s \models \overline{\exists}x.\varphi$	iff	There is a $n \in \mathfrak{N}$ s.t $\mathcal{M}, s \models \mathcal{E}(n)$ and $\mathcal{M}, s \models \varphi[^n/_x]$
$\mathcal{M}, s \Dashv \overline{\exists}x.\varphi$	iff	For all $n \in \mathfrak{N}$; $\mathcal{M}, s \Dashv \mathcal{E}(n)$ or $\mathcal{M}, s \Dashv \varphi[^n/_x]$
$\mathcal{M}, s \models \underline{\forall}x.\varphi$	iff	For all $n \in \mathfrak{N}$; For all $t \subseteq s$, if For all $\mathcal{M}, w \in t$; $w \Vdash \mathcal{E}(x)$ then $\mathcal{M}, t \models \varphi[^n/_x]$
$\mathcal{M}, s \models \underline{\forall}x.\varphi$	iff	There is a $n \in \mathfrak{N}$ s.t. $\mathcal{M}, s \Dashv \mathcal{E}(n)$ and $\mathcal{M}, s \Dashv \varphi[^n/_x]$
$\mathcal{M}, s \models \exists x.\varphi$	iff	For all $w \in s$; There is a $n \in \mathfrak{N}$ s.t $\mathcal{M}, w \Vdash \mathcal{E}(n)$ and $\mathcal{M}, w \Vdash \varphi[^n/_x]^{cl}$
$\mathcal{M}, s \Dashv \exists x.\varphi$	iff	For all $w \in s$; For all $n \in \mathfrak{N}$; $\mathcal{M}, w \dashV \mathcal{E}(n)$ and $\mathcal{M}, w \dashV \varphi[^n/_x]^{cl}$
$\mathcal{M}, s \models \forall x.\varphi$	iff	For all $w \in s$; For all $n \in \mathfrak{N}$, $\mathcal{M}, w \dashV \mathcal{E}(n)$ or $\mathcal{M}, w \Vdash \varphi[^n/_x]^{cl}$
$\mathcal{M}, s \Dashv \forall x.\varphi$	iff	For all $w \in s$; There is a $n \in \mathfrak{N}$ s.t. $\mathcal{M}, s \Vdash \mathcal{E}(n)$ and $\mathcal{M}, w \dashV \varphi[^n/_x]^{cl}$

B.1.1 Expressiveness

In standard INQ inquisitiveness is introduced by the inquisitive disjunction $\mathbin{\vee\!\!\!\vee}$ and the inquisitive existential $\overline{\exists}$.
For example, $\varphi \mathbin{\vee\!\!\!\vee} \psi$ is supported by all those states that bear the witness to which disjunct makes the disjunction true, while $\overline{\exists}x.\varphi$ is supported by all the states that bear the witness to which object makes φ true.
Inquisitiveness is eliminated through the negation, since $\neg\varphi$ is supported in all the states $s \subseteq (\mathcal{W} \setminus \bigcup[\varphi]_{\mathcal{M}})$. Since $(\mathcal{W} \setminus \bigcup[\varphi]_{\mathcal{M}}) \in [\neg\varphi]_{\mathcal{M}}$, the set has a supremum, and it is therefore not-inquisitive. Consider the previously mentioned examples, using a negation on an inquisitive disjunction or an inquisitive existential is equivalent to "losing the witnesses".
In this way it is possible to define the classical conjunction, disjunction and the quantifications as seen in section 1.3.1 on page 11. The same operators can be defined in PhIL as seen above, with the difference that the notion of support becomes that of acceptance and the addition of the "nominalist" inquisitive weak quantifier $\overline{\text{Ʒ}}$.

However, in PhIL it is possible to define a different operator called inquisitive negation $\overline{\neg}$: a negation that does not lose the witnesses.

The difference between $s \models \neg\varphi$ and $s \models \neg\varphi$ is that the former checks whether φ is classically false in every possible world $w \in s$ while the latter just “flips” the acceptance to rejection. This crucial difference allows the inquisitive negation to always preserve inquisitiveness and makes it possible to define the inquisitive conjunction $\Wedge$ and the inquisitive existential $\underline{\forall}$.
These two operators are the duals of $\Vee$ and $\overline{\exists}$, in that they have similar properties to their counterparts when rejected.
Namely, $\varphi \Wedge \psi$ is rejected by all those states that bear the witness to which conjunct makes the conjunction false and $\underline{\forall} x.\varphi$ is rejected by all the states that bear the witness to which name makes φ false.
On one hand, where $[\varphi \Vee \psi]^a_{\mathcal{M}}$ is equivalent to the question “φ or ψ?” we have that $[\varphi \Wedge \psi]^r_{\mathcal{M}}$ can be read as “Why not φ and ψ?” (aka $[\neg\varphi \Vee \neg\psi]^a_{\mathcal{M}}$).
On the other hand, if $[\overline{\exists} x.\varphi]^a_{\mathcal{M}}$ is equivalent to the question “Who is φ?” we have that $[\underline{\forall} x.\varphi]^r_{\mathcal{M}}$ is “Who is not φ?” (aka $[\overline{\exists} x.\neg\varphi]^a_{\mathcal{M}}$).

B.2 Downward Monotonicity

Theorem B.2.1. *aka fact 31 on page 63.*
If $\mathcal{M}, s \models \varphi$ then for all $t \subseteq s$, $\mathcal{M}, t \models \varphi$ and
If $\mathcal{M}, s \Dashv \varphi$ then for all $t \subseteq s$, $\mathcal{M}, t \Dashv \varphi$

Proof. We can prove this theorem by induction on φ:

Base case

$\alpha \in$ `Atom` By Downward Monotonicity of “For All”.

$\bot$ Trivially.

Inductive step (With IH being “for any φ less complex than ρ,
If $\mathcal{M}, s \models \varphi$ then for all $t \subseteq s$, $\mathcal{M}, t \models \varphi$ and
If $\mathcal{M}, s \Dashv \varphi$ then for all $t \subseteq s$, $\mathcal{M}, t \Dashv \varphi$”)

$\rho \overset{def}{=} \varphi \twoheadrightarrow \psi$

$\mathcal{M}, s \models \varphi \twoheadrightarrow \psi$ Assume that for an arbitrary $\mathcal{M}$& $s \subseteq \mathcal{W}$, $\mathcal{M}, s \models \varphi \twoheadrightarrow \psi$. By definition we have that either $\mathcal{M}, s \Dashv \varphi$ or $\mathcal{M}, s \models \psi$. By IH either for all $t \subseteq s$, $\mathcal{M}, t \Dashv \varphi$ or for all $t \subseteq s$, $\mathcal{M}, t \models \psi$. Thus by definition, for all $t \subseteq s$, $\mathcal{M}, t \models \varphi \twoheadrightarrow \psi$.

$\mathcal{M}, s \Dashv \varphi \twoheadrightarrow \psi$ Assume that for an arbitrary $\mathcal{M}$& $s \subseteq \mathcal{W}$, $\mathcal{M}, s \Dashv \varphi \twoheadrightarrow \psi$. By definition we have that $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \Dashv \psi$. By IH for all $t \subseteq s$, $\mathcal{M}, t \models \varphi$ and for all $t \subseteq s$, $\mathcal{M}, t \Dashv \psi$. Thus by definition, for all $t \subseteq s$, $\mathcal{M}, t \models \varphi \twoheadrightarrow \psi$.

$\rho \overset{def}{=} \varphi \to \psi$

$\mathcal{M}, s \models \varphi \to \psi$ Assume that for an arbitrary $\mathcal{M}$& $s \subseteq \mathcal{W}$, $\mathcal{M}, s \models \varphi \to \psi$.
By definition we have that For all $t \subseteq s$ if For all $w \in t$ $M, w \not\Dashv \varphi^{cl}$ then $\mathcal{M}, t \models \psi$.

By IH For all $t \subseteq s$ if For all $w \in t$ $M, w \not\dashv\!\!\!| \varphi^{cl}$ then For all $t' \subseteq t$ $\mathcal{M}, t' \models \psi$ which is equivalent to say that For all $t \subseteq s$ For all $t' \subseteq t$ if For all $w \in t$ $M, w \not\dashv\!\!\!| \varphi^{cl}$ then $\mathcal{M}, t' \models \psi$.
By Downward Monotonicity of "For all", For all $t \subseteq s$ For all $t' \subseteq t$ if For all $w' \in t'$ $M, w' \not\dashv\!\!\!| \varphi^{cl}$ then $\mathcal{M}, t' \models \psi$,
i.e. For all $t \subseteq s$ $\mathcal{M}, t \models \varphi \to \psi$.

$\mathcal{M}, s \rightleftharpoons \varphi \to \psi$ Analogous to $\mathcal{M}, s \rightleftharpoons \varphi \twoheadrightarrow \psi$

$\rho \stackrel{def}{=} \overline{\exists} x.\varphi$

$\mathcal{M}, s \models \overline{\exists} x.\varphi$ Assume that for an arbitrary $\mathcal{M}$& $s \subseteq \mathcal{W}$, $\mathcal{M}, s \models \overline{\exists} x.\varphi$.
By definition we have that there is a $n \in \mathfrak{N}$ s.t. $s \models \varphi[^n/_x]$. By IH implies that for all $t \subseteq s$ there is a $n \in \mathfrak{N}$ s.t. $t \models \varphi[^n/_x]$.
i.e. For all $t \subseteq s$, $\mathcal{M}, t \models \overline{\exists} x.\varphi$.

$\mathcal{M}, s \rightleftharpoons \overline{\exists} x.\varphi$ Assume that for an arbitrary $\mathcal{M}$& $s \subseteq \mathcal{W}$, $\mathcal{M}, s \rightleftharpoons \overline{\exists} x.\varphi$.
By definition we have that For all $n \in \mathfrak{N}$, $s \rightleftharpoons \varphi[^n/_x]$. By IH implies that For all $t \subseteq s$ For all $n \in \mathfrak{N}$, $t \rightleftharpoons \varphi[^n/_x]$.
i.e. For all $t \subseteq s$, $\mathcal{M}, t \rightleftharpoons \overline{\exists} x.\varphi$.

$\rho \stackrel{def}{=} \Diamond\varphi$ Follows directly from Downward Monotonicity of "For all".

$\rho \stackrel{def}{=} K_a\varphi$ Analogous to $\rho \stackrel{def}{=} \Diamond\varphi$

□

B.3 Singleton-World Correspondence

Theorem B.3.1. *aka fact 32 on page 63.*
$\mathcal{M}, \{w\} \models \varphi$ *iff* $\mathcal{M}, w \Vdash \varphi^{cl}$ *and* $\mathcal{M}, \{w\} \rightleftharpoons \varphi$ *iff* $\mathcal{M}, w \dashv\!\!| \varphi^{cl}$

Proof. We can prove this theorem by induction on φ.

Base case

$\alpha \in$ `Atom` $\mathcal{M}, \{w\} \models \alpha$ iff For all $w' \in \{w\}$, $[\![\alpha]\!]^{\mathcal{M}}_{w'} = 1$ iff $[\![\alpha]\!]^{\mathcal{M}}_{w} = 1$ iff $\mathcal{M}, w \Vdash \alpha$.
$\mathcal{M}, \{w\} \rightleftharpoons \alpha$ iff For all $w' \in \{w\}$, $[\![\alpha]\!]^{\mathcal{M}}_{w'} = 0$ iff $[\![\alpha]\!]^{\mathcal{M}}_{w} = 0$ iff $\mathcal{M}, w \dashv\!\!| \alpha$.

$\bot$ $\{w\} \neq s$ thus never $\mathcal{M}, \{w\} \models \bot$ and never $\mathcal{M}, w \Vdash \bot$.
Always $\mathcal{M}, \{w\} \rightleftharpoons \bot$ and always $\mathcal{M}, w \dashv\!\!| \bot$.

Inductive step (With IH being "for any φ less complex than ρ, $\mathcal{M}, \{w\} \models \varphi$ iff $\mathcal{M}, w \Vdash \varphi^{cl}$ and $\mathcal{M}, \{w\} \rightleftharpoons \varphi$ iff $\mathcal{M}, w \dashv\!\!| \varphi^{cl}$")

$\rho \stackrel{def}{=} \varphi \twoheadrightarrow \psi$

$$\begin{aligned}
\mathcal{M}, \{w\} \models \varphi \twoheadrightarrow \psi \text{ iff } & \quad \mathcal{M}, \{w\} \rightleftharpoons \varphi \text{ or } \mathcal{M}, \{w\} \models \psi \\
\text{iff (By IH) } & \quad \mathcal{M}, w \dashv\!\!| \varphi^{cl} \text{ or } \mathcal{M}, \{w\} \Vdash \psi^{cl} \\
\text{iff } & \quad \mathcal{M}, w \Vdash \varphi \to \psi^{cl}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{M},\{w\} \mathrel{=\!\!\mid} \varphi \twoheadrightarrow \psi \text{ iff} \quad & \mathcal{M},\{w\} \models \varphi \text{ and } \mathcal{M},\{w\} \mathrel{=\!\!\mid} \psi \\
\text{iff (By IH)} \quad & \mathcal{M}, w \Vdash \varphi^{cl} \text{ and } \mathcal{M},\{w\} \mathrel{\dashv\!\mid} \psi^{cl} \\
\text{iff} \quad & \mathcal{M}, w \mathrel{\dashv\!\mid} \varphi \to \psi^{cl}
\end{aligned}
$$

$\rho \overset{def}{=} \varphi \to \psi$

$$
\begin{aligned}
\mathcal{M},\{w\} \models \varphi \to \psi \text{ iff} \quad & \text{if } \mathcal{M}, w \not\mathrel{\dashv\!\mid} \varphi^{cl} \text{ then } \mathcal{M},\{w\} \mathrel{=\!\!\mid} \psi \\
\text{iff (By IH)} \quad & \text{if } \mathcal{M}, w \not\mathrel{\dashv\!\mid} \varphi^{cl} \text{ then } \mathcal{M}, w \mathrel{\dashv\!\mid} \psi^{cl} \\
\text{iff} \quad & \mathcal{M}, w \mathrel{\dashv\!\mid} \varphi^{cl} \text{ or } \mathcal{M}, w \mathrel{\dashv\!\mid} \psi^{cl} \\
\text{iff} \quad & \mathcal{M}, w \Vdash \varphi \to \psi^{cl}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M},\{w\} \mathrel{=\!\!\mid} \varphi \to \psi \text{ iff} \quad & \mathcal{M},\{w\} \models \varphi \text{ and } \mathcal{M},\{w\} \mathrel{=\!\!\mid} \psi \\
\text{iff (By IH)} \quad & \mathcal{M}, w \Vdash \varphi^{cl} \text{ and } \mathcal{M},\{w\} \mathrel{\dashv\!\mid} \psi^{cl} \\
\text{iff} \quad & \mathcal{M}, w \mathrel{\dashv\!\mid} \varphi \to \psi^{cl}
\end{aligned}
$$

$\rho \overset{def}{=} \overline{Ʒ}x.\varphi$

$$
\begin{aligned}
\mathcal{M},\{w\} \models \overline{Ʒ}x.\varphi \text{ iff} \quad & \text{There is a } n \in \mathfrak{N}; \mathcal{M},\{w\} \models \varphi[^{n}/_{x}] \\
\text{iff (By IH)} \quad & \text{There is a } n \in \mathfrak{N}; \mathcal{M}, w \Vdash \varphi[^{n}/_{x}]^{cl} \\
\text{iff} \quad & \mathcal{M}, w \Vdash Ʒx.\varphi^{cl}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M},\{w\} \mathrel{=\!\!\mid} \overline{Ʒ}x.\varphi \text{ iff} \quad & \text{For all } n \in \mathfrak{N}; \mathcal{M},\{w\} \mathrel{=\!\!\mid} \varphi[^{n}/_{x}] \\
\text{iff (By IH)} \quad & \text{For all } n \in \mathfrak{N}; \mathcal{M}, w \mathrel{\dashv\!\mid} \varphi[^{n}/_{x}]^{cl} \\
\text{iff} \quad & \mathcal{M}, w \mathrel{\dashv\!\mid} Ʒx.\varphi^{cl}
\end{aligned}
$$

$\rho \overset{def}{=} \Diamond\varphi$

$$
\begin{aligned}
\mathcal{M},\{w\} \models \Diamond\varphi \text{ iff} \quad & \text{There is a } t \subseteq \mathcal{N}(w); t \neq \emptyset \text{ and } \mathcal{M}, \mathcal{N}(w) \models \varphi \\
\text{iff} \quad & \text{There is a } w'' \in \mathcal{N}(w); \mathcal{M},\{w''\} \models \varphi \\
\text{iff (By IH)} \quad & \text{There is a } w'' \in \mathcal{N}(w); \mathcal{M}, w'' \Vdash \varphi^{cl} \\
\text{iff} \quad & \mathcal{M}, w \Vdash \Diamond\varphi^{cl}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M},\{w\} \mathrel{=\!\!\mid} \Diamond\varphi \text{ iff} \quad & \text{For all } t \subseteq \mathcal{N}(w);\ \mathcal{M}, t \not\models \varphi \\
\text{iff} \quad & \text{For all } \{w'\} \subseteq \mathcal{N}(w); \mathcal{M},\{w'\} \not\models \varphi \\
\text{iff (By IH)} \quad & \text{For all } w' \in \mathcal{N}(w); \mathcal{M}, w' \nVdash \varphi^{cl} \\
\text{iff} \quad & \mathcal{M}, w \mathrel{\dashv\!\mid} \Diamond\varphi^{cl}
\end{aligned}
$$

$\rho \overset{def}{=} K_a\varphi$ Analogous to $\rho \overset{def}{=} \Diamond\varphi$.

□

B.4 Algebraic approach for Inquisitive Content

We can provide an algebra for the Inquisitive Content of well formed formulas described in section 4.1.2 on page 65. We call $[\varphi]^a_{\mathcal{M}}$ and $[\varphi]^r_{\mathcal{M}}$ respectively *acceptance-set* and *rejection-set*, such that:

- $[\varphi]^a_{\mathcal{M}} \overset{def}{=} \{s \subseteq \mathcal{W} \mid \mathcal{M}, s \models \varphi\}$
- $[\varphi]^r_{\mathcal{M}} \overset{def}{=} \{s \subseteq \mathcal{W} \mid \mathcal{M}, s \dashv \varphi\}$

As in the previous section, I will not specify the model $\mathcal{M}$ in the following definitions

B.4.1 Defined Operator

$$[\varphi]^r \Rrightarrow [\psi]^a \overset{def}{=} \left\{s \subseteq \mathcal{W} \mid \text{ For all } t \subseteq s, \text{ if } \left(t \cap |\varphi^{cl}|^r\right) = \emptyset \text{ then } t \in [\psi]^a\right\}$$

B.4.2 Acceptance-set and Rejection-set

$$\begin{aligned}
[\alpha]^a &= \mathbb{P}(|\alpha|^\top) \\
[\alpha]^r &= \mathbb{P}(|\alpha|^\bot) \\
[\bot]^a &= \{\emptyset\} \\
[\bot]^r &= \mathbb{P}(\mathcal{W}) \\
[\varphi \twoheadrightarrow \psi]^a &= [\varphi]^r \cup [\psi]^a \\
[\varphi \twoheadrightarrow \psi]^r &= [\varphi]^a \cap [\psi]^r \\
[\varphi \rightarrow \psi]^a &= [\varphi]^r \Rrightarrow [\psi]^a \\
[\varphi \rightarrow \psi]^r &= [\varphi]^a \cap [\psi]^r \\
[\overline{\exists} x.\varphi]^a &= \bigcup_{n \in \mathfrak{N}} ([\varphi[^n/_x]]^a) \\
[\overline{\exists} x.\varphi]^r &= \bigcap_{n \in \mathfrak{N}} ([\varphi[^n/_x]]^r) \\
[\Diamond\varphi]^a &= \left\{s \subseteq \mathcal{W} \mid \text{ For all } w \in s;\ \left(\mathcal{N}(w) \cap |\varphi|^\top\right) \neq \emptyset\right\} \\
[\Diamond\varphi]^r &= \left\{s \subseteq \mathcal{W} \mid \text{ For all } w \in s;\ \left(\mathcal{N}(w) \cap |\varphi|^\top\right) = \emptyset\right\}
\end{aligned}$$

B.5 PhIL Correspondence

We can now prove the correspondence between Acceptance/Rejection Conditions and Inquisitive Content. Namely, for all $\mathcal{M}\&\ s \subseteq \mathcal{W}$:

$$\mathcal{M}, s \models \varphi \text{ iff } s \in [\varphi]^a_{\mathcal{M}}$$
$$\mathcal{M}, s \dashv \varphi \text{ iff } s \in [\varphi]^r_{\mathcal{M}}$$

FACT 59 [Acceptance/Rejection & Inquisitive Content Correspondence]
$\mathcal{M}, s \models \varphi$ *iff* $s \in [\varphi]^a_{\mathcal{M}}$ *and* $\mathcal{M}, s \mathrel{=\!\!|} \varphi$ *iff* $s \in [\varphi]^r_{\mathcal{M}}$

Proof. We can prove this Fact by induction:

Base case

Atomic Sentences If $\alpha \in \texttt{Atom}$ by definition of Inquisitive Content and Acceptance/Rejection we have that:
$s \in [\alpha]^a$ iff $s \in \mathbb{P}(|\alpha|^\top)$ iff $s \subseteq |\alpha|^\top$ iff For all $w \in s$, $[\![\alpha]\!]_w = 1$ iff $s \models \alpha$
$s \in [\alpha]^r$ iff $s \in \mathbb{P}(|\alpha|^\perp)$ iff $s \subseteq |\alpha|^\perp$ iff For all $w \in s$, $[\![\alpha]\!]_w = 0$ iff $s \mathrel{=\!\!|} \alpha$

False By definition of Inquisitive Content and Acceptance/Rejection we have that:
$s \in [\perp]^a$ iff $s \in \{\emptyset\}$ iff $s = \emptyset$ iff $s \models \perp$
$s \in [\perp]^r$ iff $s \in \mathbb{P}(\mathcal{W})$ iff $s \mathrel{=\!\!|} \perp$

Inductive Step (With IH being "for any φ less complex than ρ, $\mathcal{M}, s \models \varphi$ iff $s \in [\varphi]^a_{\mathcal{M}}$ and $\mathcal{M}, s \mathrel{=\!\!|} \varphi$ iff $s \in [\varphi]^r_{\mathcal{M}}$ ")

Inquisitive Implication: $\rho \overset{def}{=} \varphi \to \psi$

$$\begin{aligned} s \in [\varphi \twoheadrightarrow \psi]^a \;\; & \text{iff} \;\; s \in ([\varphi]^r \cup [\psi]^a) \\ & \text{iff} \;\; s \in [\varphi]^r \text{ or } \; s \in [\psi]^a \\ & \text{iff} \;\; \text{(by IH)} \;\; s \mathrel{=\!\!|} \varphi \text{ or } s \models \psi \\ & \text{iff} \;\; s \models \varphi \twoheadrightarrow \psi \end{aligned}$$

$$\begin{aligned} s \in [\varphi \twoheadrightarrow \psi]^r \;\; & \text{iff} \;\; s \in ([\varphi]^a \cap [\psi]^r) \\ & \text{iff} \;\; s \in [\varphi]^a \text{ and } \; s \in [\psi]^r \\ & \text{iff} \;\; \text{(by IH)} \;\; s \models \varphi \text{ and } s \mathrel{=\!\!|} \psi \\ & \text{iff} \;\; s \mathrel{=\!\!|} \varphi \twoheadrightarrow \psi \end{aligned}$$

Flat Implication: $\rho \overset{def}{=} \varphi \to \psi$

$$\begin{aligned} s \in [\varphi \to \psi]^a \;\; & \text{iff} \;\; s \in ([\varphi]^r \Rrightarrow [\psi]^a) \\ & \text{iff} \;\; s \in \left\{ s' \subseteq \mathcal{W} \mid \text{ For all } t \subseteq s'; \text{ if } \left(t \cap |\varphi^{cl}|^\perp\right) = \emptyset \text{ then } t \in [\psi]^a \right\} \\ & \text{iff} \;\; \text{For all } t \subseteq s; \text{ if } \left(t \cap |\varphi^{cl}|^\perp\right) = \emptyset \text{ then } t \in [\psi]^a \\ & \text{iff} \;\; \text{(by Falsehood-set/Falsehood Condition Correspondence)} \\ & \qquad \text{For all } t \subseteq s; \text{ if } \text{ For all } w \in t; \; w \not\mathrel{=\!\!|} \varphi^{cl} \text{ then } t \in [\psi]^a \\ & \text{iff} \;\; \text{(by IH)} \;\; \text{For all } t \subseteq s; \text{ if } \text{ For all } w \in t; \; w \not\mathrel{=\!\!|} \varphi^{cl} \text{ then } t \models \psi \\ & \text{iff} \;\; s \models \varphi \to \psi \end{aligned}$$

$$\begin{aligned} s \in [\varphi \to \psi]^r \;\; & \text{iff} \;\; s \in ([\varphi]^a \cap [\psi]^r) \\ & \text{iff} \;\; s \in [\varphi]^a \text{ and } s \in [\psi]^r \\ & \text{iff} \;\; \text{(by IH)} \;\; s \models \varphi \text{ and } s \rightleftharpoons\!| \; \psi \\ & \text{iff} \;\; s \dashv \varphi \to \psi \end{aligned}$$

Particular Quantifier: $\rho \overset{def}{=} \overline{\Xi} x.\varphi$

$$\begin{aligned} s \in [\overline{\Xi} x.\varphi]^a \;\; & \text{iff} \;\; s \in \bigcup_{n \in \mathfrak{N}} ([\varphi[^n/_x]]^a) \\ & \text{iff} \;\; \text{There is a } n \in \mathfrak{N} \text{ s.t. } s \in [\varphi[^n/_x]]^a \\ & \text{iff} \;\; \text{(by IH)} \;\; \text{There is a } n \in \mathfrak{N} \text{ s.t. } s \models \varphi[^n/_x] \\ & \text{iff} \;\; s \models \overline{\Xi} x.\varphi \end{aligned}$$

$$\begin{aligned} s \in [\overline{\Xi} x.\varphi]^r \;\; & \text{iff} \;\; s \in \bigcap_{n \in \mathfrak{N}} ([\varphi[^n/_x]]^r) \\ & \text{iff} \;\; \text{For all } n \in \mathfrak{N}, \; s \in [\varphi[^n/_x]]^r \\ & \text{iff} \;\; \text{(by IH)} \;\; \text{For all } n \in \mathfrak{N}, \; s \dashv \varphi[^n/_x] \\ & \text{iff} \;\; s \dashv \overline{\Xi} x.\varphi \end{aligned}$$

Possible $\rho \overset{def}{=} \Diamond\varphi$

$$\begin{aligned} s \in [\Diamond\varphi]^a \;\; & \text{iff} \;\; s \in \{s' \subseteq \mathcal{W} | \text{ For all } w \in s'; \; (\mathcal{N}(w) \cap |\varphi|^\top) \neq \emptyset\} \\ & \text{iff} \;\; \text{For all } w \in s; \; (\mathcal{N}(w) \cap |\varphi|^\top) \neq \emptyset \\ & \text{iff} \;\; \text{For all } w \in s; \; \text{There is a } v \in \mathcal{N}(w) \text{ s.t. } v \in |\varphi|^\top \\ & \text{iff} \;\; \text{(by Falsehood-set/Falsehood Condition Correspondence)} \\ & \quad \text{For all } w \in s; \text{ There is a } v \in \mathcal{N}(w) \text{ s.t. } v \Vdash \varphi \\ & \text{iff} \;\; \text{(by World/Singleton Correspondence) For all } w \in s; \\ & \quad \text{There is a } v \in \mathcal{N}(w) \text{ s.t. } \{v\} \models \varphi \\ & \text{iff} \;\; \text{For all } w \in s; \text{ There is a } t \subseteq \mathcal{N}(w) \text{ s.t. } t \not\models \bot \text{ and } t \models \varphi \\ & \text{iff} \;\; s \models \Diamond\varphi \end{aligned}$$

$s \in [\Diamond\varphi]^r$ iff $s \in \{s' \subseteq \mathcal{W} |$ For all $w \in s'$; $(\mathcal{N}(w) \cap |\varphi|^a) = \emptyset\}$
iff For all $w \in s$; $(\mathcal{N}(w) \cap |\varphi|^a) = \emptyset$
iff For all $w \in s$; For all $v \in \mathcal{N}(w)$; $v \notin |\varphi|^a$
iff (by Truth-set/Truth Condition Correspondence) For all $w \in s$; For all $v \in \mathcal{N}(w)$; $v \not\Vdash \varphi$
iff (by World/Singleton Correspondence) For all $w \in s$; For all $v \in \mathcal{N}(w)$; $\{v\} \not\models \varphi$
iff (by Downward Monotonicity) For all $w \in s$; For all $t \subseteq \mathcal{N}(w)$; $t \not\models \varphi$
iff $s \dashv \Diamond\varphi$

Knowledge $\rho \overset{def}{=} K_a\varphi$, analogous to $\rho \overset{def}{=} \Diamond\varphi$

□

B.6 Relation between Informative and Inquisitive Content

Theorem B.6.1. *aka fact 33 on page 65.*

- $|\varphi^{cl}|^\top = \bigcup [\varphi]^a$
- $|\varphi^{cl}|^\bot = \bigcup [\varphi]^r$

Proof. We can prove this theorem by induction on φ.

Base case

Atoms If $\alpha \in$ `Atom` by definition we have that:
$|\alpha^{cl}|^\top = |\alpha|^\top = \bigcup \mathbb{P}(|\alpha|^\top) = \bigcup [\alpha]^a$
$|\alpha^{cl}|^\bot = |\alpha|^\bot = \bigcup \mathbb{P}(|\alpha|^\bot) = \bigcup [\alpha]^r$

False If $\varphi = \bot$ by definition we have that:
$|\bot^{cl}|^\top = |\bot|^\top = \emptyset = \bigcup\{\emptyset\} = \bigcup [\bot]^a$
$|\bot^{cl}|^\bot = |\bot|^\bot = \mathcal{W} = \bigcup \mathbb{P}(\mathcal{W}) = \bigcup [\bot]^r$

Inductive step (With IH being "for any φ less complex than ρ, $|\varphi^{cl}|^\top = \bigcup [\varphi]^a$ and $|\varphi^{cl}|^\bot = \bigcup [\varphi]^r$ ")

Inquisitive Implication

$$\begin{aligned}
|(\varphi \twoheadrightarrow \psi)^{cl}|^{\top} &= |\varphi^{cl} \to \psi^{cl}|^{\top} \\
&= |\varphi^{cl}|^{\perp} \cup |\psi^{cl}|^{\top} \\
\text{(by IH)} \quad &= \bigcup[\varphi]^{r} \cup \bigcup[\psi]^{a} \\
&= \bigcup([\varphi]^{r} \cup [\psi]^{a}) \\
&= \bigcup[\varphi \twoheadrightarrow \psi]^{a}
\end{aligned}$$

$$\begin{aligned}
|(\varphi \twoheadrightarrow \psi)^{cl}|^{\perp} &= |\varphi^{cl} \to \psi^{cl}|^{\perp} \\
&= |\varphi^{cl}|^{\top} \cap |\psi^{cl}|^{\perp} \\
\text{(by IH)} \quad &= \bigcup[\varphi]^{a} \cap \bigcup[\psi]^{r} \\
&= \bigcup([\varphi]^{a} \cap [\psi]^{r}) \\
&= \bigcup[\varphi \twoheadrightarrow \psi]^{r}
\end{aligned}$$

Flat Implication

$$\begin{aligned}
|(\varphi \to \psi)^{cl}|^{\top} &= |\varphi^{cl} \to \psi^{cl}|^{\top} \\
&= |\varphi^{cl}|^{\perp} \cup |\psi^{cl}|^{\top} \\
\text{(by IH)} \quad &= |\varphi^{cl}|^{\perp} \cup \bigcup[\psi]^{a} \\
\text{iff} \quad & \text{if } w \notin |\varphi^{cl}|^{\perp} \text{ then } w \in \bigcup[\psi]^{a} \\
\text{iff} \quad & w \in \bigcup([\varphi]^{r} \Rrightarrow [\psi]^{a}) \\
&= \bigcup[\varphi \to \psi]^{a}
\end{aligned}$$

$$\begin{aligned}
|(\varphi \to \psi)^{cl}|^{\perp} &= |\varphi^{cl} \to \psi^{cl}|^{\perp} \\
&= |\varphi^{cl}|^{\top} \cap |\psi^{cl}|^{\perp} \\
\text{(by IH)} \quad &= \bigcup[\varphi]^{a} \cap \bigcup[\psi]^{r} \\
&= \bigcup([\varphi]^{a} \cap [\psi]^{r}) \\
&= \bigcup[\varphi \to \psi]^{r}
\end{aligned}$$

Particular Quantifier

$$\begin{aligned}
|(\overline{\Xi}x.\varphi)^{cl}|^{\top} &= |\Xi x.\varphi^{cl}|^{\top} \\
&= \bigcup_{n\in\mathfrak{N}} |\varphi[^{n}/_{x}]^{cl}|^{\top} \\
\text{(by IH)} \quad &= \bigcup_{n\in\mathfrak{N}} \bigcup [\varphi[^{n}/_{x}]]^{a} \\
&= \bigcup \bigcup_{n\in\mathfrak{N}} [\varphi[^{n}/_{x}]]^{a} \\
&= \bigcup [\overline{\Xi}x.\varphi]^{a}
\end{aligned}$$

$$\begin{aligned}
|(\overline{\Xi}x.\varphi)^{cl}|^{\perp} &= |\Xi x.\varphi^{cl}|^{\perp} \\
&= \bigcap_{n\in\mathfrak{N}} |\varphi[^{n}/_{x}]^{cl}|^{\perp} \\
\text{(by IH)} \quad &= \bigcap_{n\in\mathfrak{N}} \bigcup [\varphi[^{n}/_{x}]]^{r} \\
&= \bigcup \bigcap_{n\in\mathfrak{N}} [\varphi[^{n}/_{x}]]^{r} \\
&= \bigcup [\overline{\Xi}x.\varphi]^{r}
\end{aligned}$$

Possible

$$\begin{aligned}
|(\Diamond\varphi)^{cl}|^{\top} &= |\Diamond\varphi^{cl}|^{\top} \\
&= \{w \in \mathcal{W} \mid (\mathcal{N}(w) \cap |\varphi|^{\top}) \neq \emptyset\} \\
&= \bigcup \{s \subseteq \mathcal{W} \mid \text{ For all } w \in s;\ (\mathcal{N}(w) \cap |\varphi|^{\top}) \neq \emptyset\} \\
&= \bigcup [\Diamond\varphi]^{a}
\end{aligned}$$

$$\begin{aligned}
|(\Diamond\varphi)^{cl}|^{\perp} &= |\Diamond\varphi^{cl}|^{\perp} \\
&= \{w \in \mathcal{W} \mid (\mathcal{N}(w) \cap |\varphi|^{\top}) = \emptyset\} \\
&= \bigcup \{s \subseteq \mathcal{W} \mid \text{ For all } w \in s;\ (\mathcal{N}(w) \cap |\varphi|^{\top}) = \emptyset\} \\
&= \bigcup [\Diamond\varphi]^{r}
\end{aligned}$$

□