# THE MCKINSEY-TARSKI THEOREM FOR LOCALLY COMPACT ORDERED SPACES

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Abstract. We prove that the modal logic of a crowded locally compact generalized ordered space is $\mathsf{S4}$. This provides a version of the McKinsey-Tarski theorem for generalized ordered spaces. We then utilize this theorem to axiomatize the modal logic of an arbitrary locally compact generalized ordered space.

## 1. Introduction

In topological semantics of modal logic $\Box$ is interpreted as topological interior and hence $\Diamond$ as topological closure. The famous McKinsey-Tarski theorem [17] states that under such interpretation the modal logic of an arbitrary crowded (that is, dense-in-itself) metrizable space is Lewis's well-known modal system $\mathsf{S4}$. The original McKinsey-Tarski theorem had an additional assumption of separability, which was shown to be redundant by Rasiowa and Sikorski [18]. On the other hand, if a metric space is not crowded, it can give rise to other modal logics. A full axiomatization of such logics was given in [5]. To describe this result, for a topological space $X$, let $\mathsf{L}(X)$ be the modal logic of $X$; that is, $\mathsf{L}(X)$ is the set of modal formulas valid in $X$. Let also $\mathsf{iso}X$ be the set of isolated points of $X$. We then have the following result, where all the undefined notions can be found in Section 2.

**Theorem 1.1.** [5, Thm. 3.8] *Let $X$ be a nonempty metrizable space.*

1. *If $\mathsf{iso}X$ is not dense in $X$, then $\mathsf{L}(X) = \mathsf{S4}$.*
2. *If $\mathsf{iso}X$ is dense in $X$, but $X$ is not scattered, then $\mathsf{L}(X) = \mathsf{S4.1}$.*
3. *If $X$ is scattered and the Cantor-Bendixson rank of $X$ is infinite, then $\mathsf{L}(X) = \mathsf{S4.Grz}$.*
4. *If $X$ is scattered and of Cantor-Bendixson rank $n \geq 1$, then $\mathsf{L}(X) = \mathsf{S4.Grz}_n$.*

It is unclear whether the McKinsey-Tarski theorem holds for a larger class of spaces. For example, a natural generalization of the class of metrizable spaces is that of paracompact spaces. But the McKinsey-Tarski theorem does not hold for crowded paracompact spaces as it already fails for crowded compact Hausdorff spaces. Indeed, the modal logic of an arbitrary infinite crowded extremally disconnected compact Hausdorff space is $\mathsf{S4.2}$ [7, Prop. 4.3].

Our aim is to obtain a version of the McKinsey-Tarski theorem for a different class of spaces, which also plays an important role in topology, and has numerous applications. One could think of metrizable spaces as a natural generalization of the topology of the real line $\mathbb{R}$, which is induced by the metric $d(x, y) = |x - y|$. But this topology is also induced by the ordering $\leq$ of $\mathbb{R}$. Thus, the concept of a linearly ordered topological space (or LOTS for short) is another natural generalization of $\mathbb{R}$ (see, e.g., [11, p. 56]). Unlike the class of metrizable spaces, the class of LOTS is not closed under subspaces. Closing the class of LOTS under subspaces leads to the notion of a generalized ordered space (or GO-space for short); see, e.g., [16].

2010 *Mathematics Subject Classification.* 03B45; 54F05; 54D45; 54G12; 03B55.

*Key words and phrases.* modal logic; topological semantics; linearly ordered space; generalized ordered space; locally compact space; scattered space.

The classes of metrizable spaces and GO-spaces are incomparable. Each Euclidean space of dimension $\geq 2$ is an example of a metrizable space that is not a GO-space. The circle is an example of a one-dimensional metrizable space that is not a GO-space. Examples of GO-spaces that are not metrizable include $\omega_1$, the Sorgenfrey line, and the long line (see, e.g., [11, p. 237]). More generally, typical examples of non-metrizable GO-spaces are topologies of "long" lexicographic orders.

Our first main result establishes the McKinsey-Tarski theorem for an arbitrary crowded locally compact GO-space. If the real line is the guiding example in proving the McKinsey-Tarski theorem, the guiding example for our version of McKinsey-Tarski theorem is the long line.

Our strategy is based on the modern proof of the McKinsey-Tarski theorem presented in [2], which is based on the partition and mapping lemmas. Starting from a closed nowhere dense set $N$, the partition lemma builds a partition of a space consisting of $N$ and finitely many open sets such that $N$ is in the closure of each. Utilizing such partitions, the mapping lemma delivers a refutation for each non-theorem of $\mathsf{S4}$ via an interior map onto an arbitrary finite rooted $\mathsf{S4}$-frame. We develop these results for crowded locally compact GO-spaces.

As a brief survey aimed at providing intuition, we sketch how the partition lemma works for $\mathbb{R}$, then for an arbitrary crowded metrizable space, and finally for a crowded locally compact GO-space. Rather than working directly with $\mathbb{R}$, consider the real open unit interval $(0, 1)$. Start with $N$ equal to the Cantor set (excluding 0 and 1) constructed through the well-known recursion of deleting open middle thirds. Then the open sets of the partition are obtained by taking appropriate unions of the deleted open thirds. Since an arbitrary crowded metrizable space need not have an immediate analogue of the Cantor set, Bing's metrization theorem is utilized in a nontrivial way to prove a much more elaborate version of the partition lemma in [2]. The situation for a crowded locally compact GO-space is simpler because it contains a nowhere dense preimage $N$ of the Cantor set. This yields a starting point, which relies nontrivially on local compactness (see Section 3), that is analogous to the above construction for $\mathbb{R}$. Moreover, the complement of $N$ contains enough open sets to realize the desired partition. It remains an interesting open problem whether we can drop local compactness from our assumptions.

Our second main result axiomatizes the modal logics arising as $\mathsf{L}(X)$ for some locally compact GO-space $X$. In particular, we obtain an analogue of Theorem 1.1 for locally compact GO-spaces. While our proof technique is similar to that of [5], there is one important difference. Namely, the proof of [5] requires that each scattered metrizable space is strongly zero-dimensional, which is achieved by utilizing Telgarsky's theorem [20]. However, Telgarsky's theorem is not applicable to every locally compact GO-space. Instead we use Herrlich's theorem [12] that a hereditarily disconnected LOTS is strongly zero-dimensional, and generalize it to the setting of GO-spaces. This yields that each scattered GO-space is strongly zero-dimensional.

The paper is organized as follows. In Section 2 we recall some basic definitions and facts about modal logic and its topological semantics. We also provide the necessary background on LOTS and GO-spaces. Section 3 is dedicated to proving the McKinsey-Tarski theorem for crowded locally compact GO-spaces. In Section 4 we generalize Herrlich's result on hereditarily disconnected LOTS to hereditarily disconnected GO-spaces. Finally, in Section 5 we prove an analogue of Theorem 1.1 for locally compact GO-spaces.

## 2. Background

In this section we recall basic definitions and facts about modal logic and topology that play a key role in the paper. As basic references we use [9] for modal logic, [11] for topology, and [16] for GO-spaces.

2.1. **Modal logic.** Lewis's modal system $\mathsf{S4}$ is the least set of modal formulas containing

- the classical tautologies,
- $\Box(p \to q) \to (\Box p \to \Box q)$,
- $\Box p \to p$,
- $\Box p \to \Box\Box p$,

and closed under the inference rules

- modus ponens $\frac{\varphi,\ \varphi\to\psi}{\psi}$,
- substitution $\frac{\varphi(p_0,\dots,p_n)}{\varphi(\psi_0,\dots,\psi_n)}$,
- necessitation $\frac{\varphi}{\Box\varphi}$.

As is common, we use $\Diamond\varphi$ as an abbreviation for $\neg\Box\neg\varphi$.

We call a set $\mathsf{L}$ of modal formulas a *normal extension* of $\mathsf{S4}$ if $\mathsf{S4} \subseteq \mathsf{L}$ and $\mathsf{L}$ is closed under the above three inference rules. The following well-known normal extensions of $\mathsf{S4}$ play a key role in the paper:

$$\begin{aligned} \mathsf{S4.1} &= \mathsf{S4} + \Box\Diamond p \to \Diamond\Box p \\ \mathsf{S4.Grz} &= \mathsf{S4} + \Box(\Box(p \to \Box p) \to p) \to p \\ \mathsf{S4.Grz}_n &= \mathsf{S4.Grz} + \mathsf{bd}_n \ (n \geq 1) \end{aligned}$$

where

$$\begin{aligned} \mathsf{bd}_1 &= \Diamond\Box p_1 \to p_1 \\ \mathsf{bd}_{n+1} &= \Diamond(\Box p_{n+1} \wedge \neg\mathsf{bd}_n) \to p_{n+1} \ (n \geq 1) \end{aligned}$$

In relational semantics of modal logic, an $\mathsf{S4}$-*frame* is a pair $\mathfrak{F} = (W, R)$ consisting of a nonempty set $W$ and a reflexive and transitive binary relation $R$ on $W$. Let $\mathfrak{F} = (W, R)$ be an $\mathsf{S4}$-frame. Then $\mathfrak{F}$ is a partially ordered $\mathsf{S4}$-frame if $R$ is additionally antisymmetric. For $A \subseteq W$, define

$$R(A) = \{w \in W \mid \exists a \in A \text{ with } aRw\} \text{ and } R^{-1}(A) = \{w \in W \mid \exists a \in A \text{ with } wRa\}.$$

If $A = \{w\}$, then we simply write $R(w)$ and $R^{-1}(w)$. Call $\mathfrak{F}$ *rooted* if there is $r \in W$, called a *root* of $\mathfrak{F}$, such that $R(r) = W$. A *cluster* of $\mathfrak{F}$ is an equivalence class of the equivalence relation $\equiv$ on $W$ defined by $w \equiv v$ iff $wRv$ and $vRw$. The *skeleton* $\rho\mathfrak{F}$ of $\mathfrak{F}$ is the quotient of $\mathfrak{F}$ by $\equiv$. Then $\rho\mathfrak{F}$ is a partially ordered $\mathsf{S4}$-frame whose order is induced by $R$ in the natural way.

Let $\mathfrak{F} = (W, R)$ be a partially ordered $\mathsf{S4}$-frame. A subset $C$ of $W$ is a *chain* in $\mathfrak{F}$ if $wRv$ or $vRw$ for all $w, v \in C$. The *depth* of $\mathfrak{F}$ is $n \geq 1$ provided there is a chain in $\mathfrak{F}$ consisting of $n$ elements but no chain in $\mathfrak{F}$ has $n+1$ elements. Call $\mathfrak{F}$ a *tree* if $\mathfrak{F}$ is rooted and $R^{-1}(w)$ is a finite chain for each $w \in W$. Let $\mathfrak{F} = (W, R)$ be a tree and $w, v \in W$. Call $v$ a *child* of $w$ and $w$ the *parent* of $v$ provided $v$ covers $w$; that is, $wRv$, $w \neq v$, and $wRuRv$ implies $u = w$ or $u = v$ for each $u \in W$.

Let $\mathfrak{F} = (W, R)$ be an $\mathsf{S4}$-frame. The *depth* of $\mathfrak{F}$ is $n$ provided the depth of $\rho\mathfrak{F}$ is $n$. Let $A \subseteq W$. We call $w \in A$ *quasi-maximal* (resp. *maximal*) in $A$ if $wRv$ implies $vRw$ (resp. $w = v$) for each $v \in A$. The concept of quasi-minimal (resp. minimal) is defined dually. Let $\mathrm{qmax}A$ (resp. $\max A$) be the set of quasi-maximal (resp. maximal) points in $A$. Call $\mathfrak{F}$ a *quasi-tree* whenever $\rho\mathfrak{F}$ is a tree. Let $\mathfrak{F} = (W, R)$ be a quasi-tree. Then $\mathfrak{F}$ is a *top-thin-quasi-tree* provided that $\text{qmax}W = \max W$ and each maximal cluster is the unique child of its parent cluster in $\rho\mathfrak{F}$; see Figure 1.

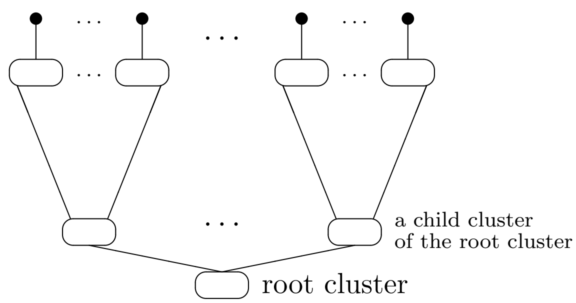


FIGURE 1. A top-thin-quasi-tree.

The modal language is interpreted in an $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$ by associating to each propositional letter a subset of $W$. This extends to all modal formulas by interpreting the classical connectives as Boolean operations and the modal box by setting

$$w \vDash \Box\varphi \text{ iff } (\forall v \in W)(wRv \text{ implies } v \vDash \varphi),$$

and hence

$$w \vDash \Diamond\varphi \text{ iff } (\exists v \in W)(wRv \text{ and } v \vDash \varphi).$$

A formula $\varphi$ is *valid* in $\mathfrak{F}$, written $\mathfrak{F} \vDash \varphi$, provided under every valuation of the propositional letters we have $w \vDash \varphi$ for each $w \in W$. Let $\mathsf{L}(\mathfrak{F})$ be the set of modal formulas valid in $\mathfrak{F}$, and for a class $\mathcal{K}$ of $\mathsf{S4}$-frames, let $\mathsf{L}(\mathcal{K}) = \bigcap\{\mathsf{L}(\mathfrak{F}) \mid \mathfrak{F} \in \mathcal{K}\}$. It is well known that $\mathsf{L}(\mathfrak{F})$ and hence $\mathsf{L}(\mathcal{K})$ are normal extensions of $\mathsf{S4}$. We call $\mathsf{L}(\mathfrak{F})$ the *modal logic* of $\mathfrak{F}$ and $\mathsf{L}(\mathcal{K})$ the *modal logic* of $\mathcal{K}$. The following result is well known; see, e.g., [9] (or [7, Prop. 2.5]).

**Lemma 2.1.**
(1) $\mathsf{S4}$ *is the logic of the class of all finite quasi-trees.*
(2) $\mathsf{S4.1}$ *is the logic of the class of all finite top-thin-quasi-trees.*
(3) $\mathsf{S4.Grz}$ *is the logic of the class of all finite trees.*
(4) $\mathsf{S4.Grz}_n$ *is the logic of the class of all finite trees of depth* $\leq n$.

2.2. **Topological semantics.** As in relational semantics of modal logic, in topological semantics we assume that the modal language is interpreted in nonempty topological spaces. Let $X$ be a nonempty topological space. We interpret propositional letters as subsets of $X$, classical connectives as the corresponding Boolean operations, $\Box$ as interior, and hence $\Diamond$ as closure. Therefore, for $x \in X$, we have

$$x \vDash \Box\varphi \text{ iff there is an open neighborhood } U \text{ of } x \text{ such that } y \vDash \varphi \text{ for all } y \in U,$$

and hence

$$x \vDash \Diamond\varphi \text{ iff for every open neighborhood } U \text{ of } x \text{ there is } y \in U \text{ such that } y \vDash \varphi.$$

A modal formula $\varphi$ is *valid* in $X$, written $X \vDash \varphi$, provided under all valuations we have $x \vDash \varphi$ for each $x \in X$. The *modal logic* $\mathsf{L}(X)$ of $X$ is the set of formulas valid in $X$, and the *modal logic* $\mathsf{L}(\mathcal{K})$ of a class of spaces is $\bigcap\{\mathsf{L}(X) \mid X \in \mathcal{K}\}$. It is well known that $\mathsf{L}(X)$ and hence $\mathsf{L}(\mathcal{K})$ are normal extensions of $\mathsf{S4}$.

Topological semantics generalizes relational semantics of $\mathsf{S4}$ since each $\mathsf{S4}$-frame can be viewed as a special topological space, in which an arbitrary intersection of open sets is open. Such spaces are known as *Alexandroff spaces.* For an $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$, call $U \subseteq W$ an

*R-upset* if $R(U) = U$. An *R-downset* is defined dually, and for a partially ordered set we simply say an *upset* or *downset*. The collection $\tau_R$ of all $R$-upsets of $\mathfrak{F}$ is an Alexandroff topology on $W$ such that $R^{-1}$ is the closure operator, $\{R(w) \mid w \in W\}$ is a basis for $\tau_R$, and $\mathfrak{F} \vDash \varphi$ iff $(W, \tau_R) \vDash \varphi$ for each modal formula $\varphi$. Consequently, if a normal extension of $\mathsf{S4}$ is complete with respect to its relational semantics, then it is also complete with respect to its topological semantics.

For a topological space $X$, we denote the closure and derivative operators by $\mathsf{c}$ and $\mathsf{d}$, respectively. We recall that a point $x \in X$ is *isolated* if $\{x\}$ is open. Let $\mathsf{iso}X$ be the set of isolated points of $X$. Then $X$ is *crowded* (or *dense-in-itself*) if $\mathsf{iso}X = \varnothing$, and $X$ is *scattered* if every nonempty subspace of $X$ has an isolated point (in the relative topology).

Following a suggestion of Archangel'skii (see [4, Sec. 2.2]), we call $X$ *densely discrete* provided $\mathsf{iso}X$ is dense in $X$ (that is, $\mathsf{c}(\mathsf{iso}X) = X$). It is easy to see that every scattered space is densely discrete, but that the converse is not true in general.

By the famous Cantor-Bendixson theorem, each space is decomposed into the disjoint union of a closed crowded subspace $D$ and an open scattered subspace $S$. For $A \subseteq X$ and ordinal $\alpha$, define recursively $\mathsf{d}^\alpha A$ by setting

$$\begin{array}{lcl} \mathsf{d}^0 A & = & A \\ \mathsf{d}^{\alpha+1} A & = & \mathsf{d}\,(\mathsf{d}^\alpha A) \\ \mathsf{d}^\alpha A & = & \bigcap_{\beta<\alpha} \mathsf{d}^\beta A \text{ if } \alpha \text{ is a limit ordinal.} \end{array}$$

There is a least ordinal $\varrho$, called the *Cantor-Bendixson rank* of $X$, such that $\mathsf{d}^\varrho X = \mathsf{d}^{\varrho+1} X$. The Cantor-Bendixson decomposition $X = D \cup S$ is then realized by $D = \mathsf{d}^\varrho X$ and $S = X \setminus D$. It is well known that $X$ is scattered iff $D = \varnothing$, and that $X$ is crowded iff $S = \varnothing$. For scattered spaces, the Cantor-Bendixson rank is the topological analogue of the depth of an $\mathsf{S4}$-frame. It is well known that an $\mathsf{S4}$-frame $\mathfrak{F}$ is of depth $\leq n$ iff $\mathfrak{F} \vDash \mathsf{bd}_n$ (see, e.g., [9, Prop. 3.44]). Likewise, it follows from [5, Lem. 3.6] that a nonempty scattered space $X$ is of Cantor-Bendixson rank $\leq n$ iff $X \vDash \mathsf{bd}_n$. The following topological analogue of Lemma 2.1 is well known (see, for example, [7, Prop. 2.2]).

**Lemma 2.2.**

(1) $\mathsf{S4}$ *is the logic of the class of all topological spaces.*
(2) $\mathsf{S4.1}$ *is the logic of the class of all densely discrete spaces.*
(3) $\mathsf{S4.Grz}$ *is the logic of the class of all scattered spaces.*
(4) $\mathsf{S4.Grz}_n$ *is the logic of the class of all scattered spaces of Cantor-Bendixson rank* $\leq n$.

Let $f : X \to Y$ be a map between topological spaces. We recall that $f$ is

- *continuous* if $f^{-1}(V)$ is open in $X$ for each open $V \subseteq Y$,
- *open* if $f(U)$ is open in $Y$ for each open $U \subseteq X$,
- *interior* if it is both continuous and open.

It is well known that $f$ is interior iff $f^{-1}$ commutes with closure. We will use this in Section 5. If $f$ is an onto interior map, then we call $Y$ an *interior image* of $X$. Because an $\mathsf{S4}$-frame is equivalently an Alexandroff space, these definitions make sense if either $X$ or $Y$ is an $\mathsf{S4}$-frame. Indeed, an interior map is the topological analogue of a p-morphism. As such, onto interior maps preserve validity; that is, if $Y$ is an interior image of $X$, then $\mathsf{L}(X) \subseteq \mathsf{L}(Y)$.

Let $Y$ be an open subspace of $X$. Then the inclusion $Y \to X$ is an interior map. As with interior images, we have that open subspaces preserve validity; that is, if $Y$ is an open subspace of $X$, then $\mathsf{L}(X) \subseteq \mathsf{L}(Y)$.

2.3. **LOTS and GO-spaces.** We recall that a partially ordered set $(X, \leq)$ is *linearly ordered* if $x \leq y$ or $y \leq x$ for each $x, y \in X$ (that is, $X$ is a chain). We write $x < y$ provided $x \leq y$

and $x \neq y$. The intervals

$$(x, y), [x, y], [x, y), (y, x]$$

are defined as usual, and so are the intervals

$$(x, \rightarrow), [x, \rightarrow), (\leftarrow, y), (\leftarrow, y].$$

Therefore,

$$(x, y) = \{z \in X \mid x < z < y\}, \quad (x, \rightarrow) = \{z \in X \mid x < z\}, \quad \text{etc.}$$

We say that $C \subseteq X$ is *convex* provided $x, y \in C$ and $x \leq y$ imply $[x, y] \subseteq C$. Clearly each interval is convex. Let $\varnothing \neq Y \subseteq X$. A *convex component* $C$ of $Y$ is a maximal convex subset of $Y$. Each convex component of $Y$ is an interval (possibly a singleton). For each $x \in Y$, there is a unique convex component $C_x$ of $Y$ containing $x$; namely

$$C_x = \bigcup \{C \subseteq Y \mid x \in C \text{ and } C \text{ is convex}\},$$

and the convex components of $Y$ yield a partition of $Y$. The next definition is well known (see, e.g., [11, pp. 56–57]).

**Definition 2.3.** A topological space $(X, \tau)$ is called a *linearly ordered topological space* or simply a *LOTS* provided there is a linear order $\leq$ on $X$ such that the family

$$\mathscr{B}_{\leq} := \{(x, \rightarrow), (x, y), (\leftarrow, y) \mid x, y \in X\}$$

is a basis for $\tau$. We call $\tau$ the *interval topology*, and say that $\leq$ *induces* $\tau$.

Typical examples of LOTS include the real numbers $\mathbb{R}$, rationals $\mathbb{Q}$, the Cantor space $\mathsf{C}$, etc. On the other hand, the subspace $X := (0, 1) \cup \{2\}$ of $\mathbb{R}$ is not a LOTS (see, e.g., [16, Rem. 6.2]). This shows that the class of LOTS is not closed under taking subspaces.

**Definition 2.4.** A topological space $X$ is called a *generalized ordered space* or simply a *GO-space* provided it is homeomorphic to a subspace of some LOTS.

The next theorem is well known (see, e.g., [16, Sec. 2]).

**Theorem 2.5.** *A topological space $(X, \tau)$ is a GO-space iff there is a linear ordering $\leq$ of $X$ such that $\mathscr{B}_{\leq} \subseteq \tau$ and each $x \in X$ has a local basis consisting of intervals in $X$.*

We conclude this section with a lemma in which we collect together some well-known facts about GO-spaces that will be useful in the rest of the paper. Where we were unable to find an exact reference, we briefly sketch a proof. We recall that a subset of a topological space is *clopen* if it is simultaneously closed and open, and that a property is *hereditary* if every subspace has it. We also recall that the concept of a *collectionwise normal* space is a strengthening of the concept of a normal space; see [11, p. 305] for details.

**Lemma 2.6.** *Let $X$ be a GO-space.*

(1) *The convex components of an open subset $U$ of $X$ are open, and hence $U$ is uniquely represented as a disjoint union of open convex sets in $X$.*
(2) *If $X$ is compact, then $X$ is a LOTS.*
(3) *If $K \subseteq X$ is compact and nonempty, then* $\max K$ *and* $\min K$ *exist.*
(4) *If $X$ is separable, then $X$ is first-countable.*
(5) *$X$ is separable iff $X$ is hereditarily separable.*
(6) *$X$ is hereditarily collectionwise normal.*

*Proof.* (1) Let $C$ be a convex component of $U$ and $x \in C$. As $x \in U$, there is an open convex subset $V$ of $X$ such that $x \in V$ and $V \subseteq U$. By the maximality of $C$ we see that $V \subseteq C$. Therefore, $C$ is open in $X$.

(2) See, e.g., [16, Lem. 6.1].

(3) If $\max K$ does not exist, then $\{(\leftarrow, x) \mid x \in K\}$ is an open covering of $K$ with no finite subcover. That $\min K$ exists is proved similarly.

(4) Let $D$ be a countable dense subset of $X$ and $x \in X$. If $x$ is isolated in $X$ then $\{\{x\}\}$ is a countable local basis at $x$. Suppose that $x$ is not isolated. Then

$$x \in \mathsf{c}\,(X \setminus \{x\}) = \mathsf{c}(\leftarrow, x) \cup \mathsf{c}(x, \rightarrow).$$

If $x \in \mathsf{c}(\leftarrow, x) \cap \mathsf{c}(x, \rightarrow)$, then

$$\{(d, e) \mid d, e \in D \text{ and } d < x < e\}$$

is a countable local basis at $x$. Suppose that $x \in \mathsf{c}(\leftarrow, x) \setminus \mathsf{c}(x, \rightarrow)$. Then $\mathsf{c}(\leftarrow, x) = (\leftarrow, x]$ and $\mathsf{c}(x, \rightarrow) = (x, \rightarrow)$. Therefore, $(\leftarrow, x]$ is a clopen downset, implying that $\{(a, x] \mid a < x\}$ is a local basis at $x$. Thus,

$$\{(d, x] \mid d \in D \cap (\leftarrow, x)\}$$

is a countable local basis at $x$. Similarly, if $x \in \mathsf{c}(x, \rightarrow) \setminus \mathsf{c}(\leftarrow, x)$, then $\{[x, d) \mid d \in D \cap (x, \rightarrow)\}$ is a countable local basis at $x$.

(5) See, e.g., [16, Prop. 2.10(a)].

(6) By [16, Prop. 4.1], each GO-space is collectionwise normal. Since a subspace of a GO-space is a GO-space, each GO-space is hereditarily collectionwise normal. □

## 3. The McKinsey-Tarski Theorem for crowded locally compact GO-spaces

In this section we prove our first main result, that the McKinsey-Tarski Theorem holds for an arbitrary crowded locally compact GO-space. The section is divided into three subsections. The first subsection consists of several auxiliary lemmas, the second subsection proves the Partition Lemma, the key tool in proving the Mapping Lemma, which is done in the third subsection. The Mapping Lemma then easily delivers the main result of the section, that $\mathsf{S4}$ is the logic of an arbitrary crowded locally compact GO-space.

3.1. **Auxiliary lemmas.** We recall that a space $X$ is *locally compact* provided for each $x \in X$ there is an open neighborhood $U$ of $x$ such that $\mathsf{c}U$ is a compact Hausdorff subspace of $X$. By [11, Thm. 3.3.1], each locally compact space is Tychonoff. We also recall that a continuous onto map between compact Hausdorff spaces is *irreducible* provided the image of a closed proper subset is proper. The following fact is well known. Since we were unable to find a reference, we give a short proof.

**Lemma 3.1.** (Folklore) *Let $X$ be a nonempty crowded locally compact space. Then there is a compact subspace $Y$ of $X$ and an irreducible map $f$ from $Y$ onto the Cantor space $\mathsf{C}$.*

*Proof.* Since $X$ is nonempty locally compact, there is a nonempty open subset $U$ of $X$ such that $\mathsf{c}U$ is compact. Because $X$ is crowded, $\mathsf{c}U$ is not scattered. Therefore, [19, Thm. 8.5.4] yields a continuous onto map $f : \mathsf{c}U \to [0, 1]$. Thus, $f^{-1}(\mathsf{C})$ is a closed, hence compact subspace of $\mathsf{c}U$, and the restriction of $f$ to $f^{-1}(\mathsf{C})$ is a continuous map onto $\mathsf{C}$. Finally, apply [15, p. 102] to deliver a closed, hence compact subspace $Y$ of $f^{-1}(\mathsf{C})$ such that the restriction of $f$ to $Y$ is an irreducible map onto $\mathsf{C}$. □

**Lemma 3.2.** *For $X$, $Y$, and $f : Y \to \mathsf{C}$ as in Lemma 3.1, there is a compact nowhere dense $Z \subseteq X$ and an irreducible map $g : Z \to \mathsf{C}$.*

*Proof.* Let $\mathsf{C}_0$ be a nowhere dense closed subspace of $\mathsf{C}$ that is homeomorphic to $\mathsf{C}$. Since $f$ is irreducible, $f^{-1}(\mathsf{C}_0)$ is closed and nowhere dense in $Y$, hence closed and nowhere dense in $X$. There is a closed subspace $Z$ of $f^{-1}(\mathsf{C}_0)$ such that $f|_Z : Z \to \mathsf{C}_0$ is an irreducible map (see, e.g., [15, p. 102]). As $Y$ is compact and $Z$ is closed in $Y$, we have that $Z$ is compact. Moreover, $Z$ is nowhere dense in $X$ since $Z$ is nowhere dense in $f^{-1}(\mathsf{C}_0)$, which is nowhere dense in $X$. The desired irreducible map $g : Z \to \mathsf{C}$ is then the composition of $f|_Z$ and a homeomorphism of $\mathsf{C}_0$ onto $\mathsf{C}$. □

**Lemma 3.3.** *If $X$, $Z$, and $g : Z \to \mathsf{C}$ are as in Lemma 3.2, then $Z$ is separable and crowded.*

*Proof.* Let $D$ be a countable dense subset of $\mathsf{C}$. Because $g$ is onto, we may choose $z_x \in g^{-1}(x)$ for each $x \in D$. Let $E = \{z_x \mid x \in D\}$. Then $E$ is dense in $Z$ since $g$ is irreducible. Therefore, $Z$ is separable. If $x$ is an isolated point of $Z$, then $Z \setminus \{x\}$ is a proper closed subset of $Z$, and hence $g(Z \setminus \{x\})$ is a proper closed subset of $\mathsf{C}$. This is a contradiction since $g(Z \setminus \{x\}) = \mathsf{C} \setminus \{g(x)\}$. Thus, $Z$ is crowded. □

**Lemma 3.4.** *Let $X$ be a crowded locally compact GO-space and $C$ a nonempty convex open subset of $X$. Then there is a nonempty crowded compact separable nowhere dense subspace of $C$.*

*Proof.* Being a nonempty convex open subset of a crowded Hausdorff space, there are $x, y \in C$ such that $\varnothing \neq (x, y) \subseteq C$. Since $(x, y)$ is open in $X$, which is crowded and locally compact, we have that $(x, y)$ is a crowded locally compact subspace of $X$. Lemmas 3.2 and 3.3 then yield a nonempty crowded compact separable nowhere dense subspace $Z$ of $(x, y)$. Since $Z$ is nowhere dense in $(x, y)$, it is nowhere dense in $C$. □

The nonempty crowded compact separable nowhere dense subspaces play the same role in our construction as the Cantor space plays in the construction of [6, Sec. 3].

**Lemma 3.5.** *Let $X$ be a crowded compact separable GO-space and*

$$L = \{x \in X \mid x \in \mathsf{c}(\leftarrow, x)\}.$$

1. *$L$ is dense in $X$.*
2. *There is a countable dense subset $D$ of $X$ such that $D \subseteq L$ and each $x \in D$ is the supremum of a strictly increasing sequence in $D$.*

*Proof.* (1) If $L$ is not dense, then there is a nonempty open subset $G$ of $X$ such that $G \cap L = \varnothing$. Since $G \neq \varnothing$ and $X$ is normal (see Lemma 2.6(6)), there is a nonempty open subset $U$ of $X$ such that $\mathsf{c}U \subseteq G$. Because $U$ is a nonempty open subspace of a crowded space, $U$ is crowded, so $\mathsf{c}U$ is also crowded. Moreover, $\mathsf{c}U$ is compact and $\mathsf{c}U \cap L = \varnothing$. Let $x = \max \mathsf{c}U$ (see Lemma 2.6(3)). As $x \notin L$, we have that $x \notin \mathsf{c}(\leftarrow, x)$. Therefore, there is an open neighborhood $V$ of $x$ such that $V \cap (\leftarrow, x) = \varnothing$. Thus, $V \cap \mathsf{c}U = \{x\}$, implying that $x$ is an isolated point of $\mathsf{c}U$. The obtained contradiction proves that $L$ is dense in $X$.

(2) By Lemma 2.6(5), $L$ is separable. Let $D$ be a countable dense subset of $L$. Then $D$ is dense in $X$ since $L$ is dense in $X$ by (1). Let $x \in D$. It follows from Lemma 2.6(4) that there is a countable local basis $\{U_n \mid n \in \omega\}$ at $x$. Without loss of generality we may assume $U_{n+1} \subset U_n$ for each $n \in \omega$. We recursively define a sequence $\{x_n\}_{n\in\omega}$ utilizing both that $x \in \mathsf{c}(\leftarrow, x)$ (since $x \in L$) and that $D$ is dense in $X$. As $U_0$ is an open neighborhood of $x$, we have that $U_0 \cap (\leftarrow, x) \neq \varnothing$. Because $U_0 \cap (\leftarrow, x)$ is open in $X$, we may choose $x_0 \in U_0 \cap (\leftarrow, x) \cap D$. For $n \in \omega$, assume that $x_n \in U_n \cap D$ has been chosen so that $x_n < x$. Noting that $(x_n, \rightarrow) \cap U_{n+1}$ is an open neighborhood of $x$, we have that $(x_n, \rightarrow) \cap U_{n+1} \cap (\leftarrow, x)$ is a nonempty open subset of $X$. Thus, we may choose $x_{n+1} \in (x_n, \rightarrow) \cap U_{n+1} \cap (\leftarrow, x) \cap D$.

By construction, we have that $\{x_n\}_{n\in\omega}$ is a strictly increasing sequence in $D$ bounded above by $x$. Let $y \in X$ be such that $y < x$. Then $(y, \rightarrow)$ is an open neighborhood of $x$. So there is $n \in \omega$ such that $U_n \subseteq (y, \rightarrow)$. Since $x_n \in U_n$, we have that $y < x_n$. Therefore, $y$ is not an upper bound of $\{x_n\}_{n\in\omega}$, and hence $x$ is the supremum of $\{x_n\}_{n\in\omega}$. □

3.2. **The Partition Lemma.** Let $\kappa \geq 1$ be a cardinal. We recall that a space $Y$ is $\kappa$-*resolvable* if there is a partition of $Y$ into $\kappa$ subsets that are each dense in $Y$. Clearly being 1-resolvable merely means that the space is nonempty. Nevertheless, this notion is useful for inductive arguments. The following lemma on resolvability is a straightforward consequence of the work of Hewitt, Ceder, Illanes, and Eckertson.

**Lemma 3.6.** *Let $Y$ be a nonempty crowded Hausdorff space and $1 \leq \kappa \leq \omega$.*

(1) *If $Y$ is first-countable, then $Y$ is $\kappa$-resolvable.*
(2) *If $Y$ is locally compact, then $Y$ is $\kappa$-resolvable.*

*Proof.* (1) Since $Y$ is a nonempty crowded first-countable space, $Y$ is 2-resolvable by [13, p. 331]. Because the property of being first-countable and crowded is preserved by dense subsets of $Y$, a straightforward induction yields that $Y$ is $n$-resolvable for each $n \geq 2$. By [14, Thm. 5], $Y$ is also $\omega$-resolvable.

(2) Recall that the dispersion character of $Y$ is

$$\Delta(Y) := \min\{|U| \mid U \text{ is nonempty open in } Y\}.$$

Since $Y$ is a nonempty crowded Hausdorff space, $\Delta(Y) \geq \omega$. Because $Y$ is also locally compact, $Y$ is $\Delta(Y)$-resolvable by [8, Thm. 7]. Now apply [10, Prop. 1.1(b)]. □

**Lemma 3.7.** [Partition Lemma] *Let $X$ be a crowded locally compact GO-space, $F$ a nonempty crowded compact separable nowhere dense subset of $X$, and $k \in \omega$. Then there is a partition $\{F, U_0, \dots, U_k\}$ of $X$ such that each $U_i$ is open in $X$ and $\mathsf{c}U_i = U_i \cup F$.*

*Proof.* Since $F$ satisfies the conditions of Lemma 3.5, there is a countable dense subset $D$ of $F$ as in Lemma 3.5(2). Being a compact subspace of a GO-space, $F$ is closed in $X$. Let $\mathscr{C}$ be the collection of all convex components of $X \setminus F$. By Lemma 2.6(1), each element of $\mathscr{C}$ is open in $X$. For each $x \in X \setminus F$ let $C_x$ be the unique element of $\mathscr{C}$ such that $x \in C_x$.

For each $x \in D$, we build a countably infinite pairwise disjoint subcollection $\mathscr{C}'_x$ of $\mathscr{C}$. Let $x \in D$. By Lemma 3.5(2), there is a strictly increasing sequence $\{x_n\}_{n\in\omega}$ in $D$ whose supremum is $x$. Let $n \in \omega$ and consider the open interval $(x_n, x_{n+1})$, which is nonempty by Lemma 3.5(2) because $x_{n+1} \in D$. Since $F$ is nowhere dense, $(x_n, x_{n+1}) \not\subseteq F$. Choose $y_n \in (x_n, x_{n+1}) \setminus F$ and consider $C_{y_n} \in \mathscr{C}$. Since $x_n, x_{n+1} \in F$, it must be the case that $C_{y_n} \subseteq (x_n, x_{n+1})$. Set $\mathscr{C}'_x = \{C_{y_n} \mid n \in \omega\}$.

Let $x, y \in D$ be such that $y < x$. As $x$ is the supremum of the strictly increasing sequence $\{x_n\}_{n\in\omega}$, there is $N \in \omega$ such that $y < x_n$ for all $n \geq N$. Thus, all but finitely many members of $\mathscr{C}'_x$ are contained in $(y, \rightarrow)$. This implies that $\mathscr{C}'_x \cap \mathscr{C}'_y$ is finite since $\bigcup \mathscr{C}'_y \subseteq (\leftarrow, y)$. Let $\{x_m \mid m \in \omega\}$ be an enumeration of $D$. For each $m \in \omega$ put

$$\mathscr{C}_{x_m} = \mathscr{C}'_{x_m} \setminus \bigcup_{i<m} \mathscr{C}'_{x_i}$$

Then $\{\mathscr{C}_x \mid x \in D\}$ is a pairwise disjoint family of countably infinite subcollections of $\mathscr{C}$.

Since $F$ is a crowded separable GO-space and $D$ is a dense subspace of $F$, we have that $D$ is crowded and separable (see Lemma 2.6(5)). By Lemma 2.6(4), $D$ is first-countable. By Lemma 3.6(1), there is a partition $\{D_0, \dots, D_k\}$ of $D$ consisting of dense subsets of $D$. For

$0 \le i \le k$, set $U_i = \bigcup \mathscr{C}_i$ where

$$\mathscr{C}_i = \begin{cases} \bigcup\{\mathscr{C}_x \mid x \in D_i\} & \text{if } i < k \\ \mathscr{C} \setminus \bigcup\{\mathscr{C}_x \mid x \in D \setminus D_k\} & \text{if } i = k \end{cases}$$

Since $\mathscr{C}$ consists of open convex sets in $X$, for each $0 \le i \le k$ we have that $U_i$ is open in $X$ and its set of convex components is $\mathscr{C}_i$. Moreover, $\{\mathscr{C}_0, \dots, \mathscr{C}_k\}$ is a partition of $\mathscr{C}$. Therefore, $\{U_0, \dots, U_k\}$ is pairwise disjoint and

$$X \setminus F = \bigcup \mathscr{C} = \bigcup_{i=0}^{k} \bigcup \mathscr{C}_i = \bigcup_{i=0}^{k} U_i.$$

Thus, $\{F, U_0, \dots, U_k\}$ is a partition of $X$.

Let $0 \le i \le k$. To see that $\mathsf{c}U_i = U_i \cup F$, we first observe that $U_i \cup F$ is closed since

$$X \setminus (U_i \cup F) = U_0 \cup \dots \cup U_{i-1} \cup U_{i+1} \cup \dots \cup U_k$$

is open. Thus, $\mathsf{c}U_i \subseteq U_i \cup F$ and it is sufficient to show that $F \subseteq \mathsf{c}U_i$. Let $y \in F$ and $U$ be an open neighborhood of $y$. Since open convex sets form a basis of $X$, without loss of generality we may assume that $U$ is convex. Because $D_i$ is dense in $F$, there is $x \in D_i \cap U$. Since $x \in D$ and $F$ is closed in $X$, we have that $x \in \mathsf{c}(\leftarrow, x) \cap F = \mathsf{c}\left((\leftarrow, x) \cap F\right)$. Thus, $U \cap (\leftarrow, x) \cap F$ is a nonempty open subset of $F$, and so there is $a \in D \cap U \cap (\leftarrow, x)$. By Lemma 3.5(2), there is $N \in \omega$ such that $a < x_n$ for $n \ge N$. Therefore, $C_{y_n} \subseteq (x_n, x_{n+1}) \subseteq (x_N, x) \subseteq (a, x) \subseteq U$ for $n \ge N$. Since $\mathscr{C}'_x \setminus \mathscr{C}_x$ is finite, there is $n \in \omega$ with $n \ge N$ and $C_{y_n} \in \mathscr{C}_x \subseteq \bigcup_{z \in D_i} \mathscr{C}_z = \mathscr{C}_i$. Thus, $C_{y_n} \subseteq \bigcup \mathscr{C}_i = U_i$, which yields that $\varnothing \ne C_{y_n} \subseteq U_i \cap U$. Consequently, $F \subseteq \mathsf{c}U_i$. $\square$

**Remark 3.8.** The last paragraph of the proof of the Partition Lemma shows that for each open neighborhood $U$ of $y \in F$ and $0 \le i \le k$, there is a convex component $C_i$ of $U_i$ such that $C_i \subseteq U$.

3.3. **The Mapping Lemma.** As we pointed out in Section 2.2, we view $\mathsf{S4}$-frames as Alexandroff spaces.

**Lemma 3.9.** [Mapping Lemma] *Let $X$ be a nonempty crowded locally compact GO-space and let $\mathfrak{T} = (W, R)$ be a finite quasi-tree. Then $\mathfrak{T}$ is an interior image of $X$.*

*Proof.* Our proof is by strong induction on the depth $n \ge 1$ of $\mathfrak{T}$. Let the root cluster of $\mathfrak{T}$ be $C = \{r_j \mid 0 \le j \le m\}$ for some $m \in \omega$.

**Base case:** Suppose $n = 1$. By Lemma 3.6(2), $X$ is $(m+1)$-resolvable. Therefore, $\mathfrak{T}$ is an interior image of $X$ by [1, Lem. 5.9].

**Inductive step:** Suppose $n \ge 1$, the depth of $\mathfrak{T}$ is $n+1$, and each finite quasi-tree of depth $\le n$ is an interior image of any nonempty crowded locally compact GO-space.

Let $w_0, \dots, w_k$ be representatives of the children clusters of the root cluster $C$ of $\mathfrak{T}$. For each $i \le k$ put $W_i = R(w_i)$ and $\mathfrak{T}_i = (W_i, R_i)$, where $R_i$ is the restriction of $R$ to $W_i$. Then each $\mathfrak{T}_i$ is a quasi-tree of depth $\le n$; see Figure 2.

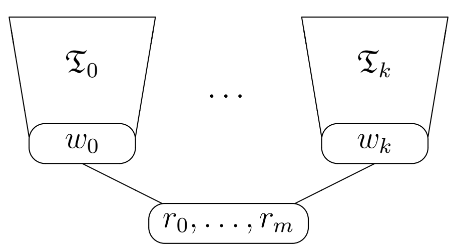


FIGURE 2. The quasi-trees $\mathfrak{T}$ and $\mathfrak{T}_0, \dots, \mathfrak{T}_k$.

Because $X$ is nonempty, by Lemma 3.4, there is a nonempty crowded compact separable nowhere dense subspace $F$ of $X$. In particular, $F$ is locally compact, so Lemma 3.6(2) delivers a partition $\{F_j \mid 0 \le j \le m\}$ of $F$ such that each $F_j$ is dense in $F$.

Let $\{F, U_0, \dots, U_k\}$ be as in the Partition Lemma and $0 \le i \le k$. Adopting the notation in the proof of the Partition Lemma, we have $U_i = \bigcup \mathscr{C}_i$ where $\mathscr{C}_i$ is the set of convex components of $U_i$. Let $C \in \mathscr{C}_i$. Then $C$ is a convex component of $X \setminus F$. Because $F$ is closed, $C$ is open by Lemma 2.6(1). Thus, the subspace $C$ is a nonempty crowded locally compact GO-space. By the inductive hypothesis, there is an onto interior map $f_{C,i} : C \to \mathfrak{T}_i$.

Define $f : X \to \mathfrak{T}$ by

$$f(x) = \begin{cases} f_{C,i}(x) & \text{if } x \in C \text{ for } 0 \le i \le k \text{ and } C \in \mathscr{C}_i \\ r_j & \text{if } x \in F_j \text{ for } 0 \le j \le m \end{cases}$$

Then $f$ is a well-defined onto map since $\{F, U_0, \dots, U_k\}$ is a partition of $X$, $\{F_0, \dots, F_m\}$ is a partition of $F$, $\mathscr{C}_i$ is a partition of $U_i$ for each $0 \le i \le k$, and each mapping $f_{C,i}$ is onto. Figure 3 depicts the mapping $f$ where the set $F$ is represented by bullets, the convex components of some $U_i$ are depicted with angled brackets, and $C$ is a convex component of $U_i$.

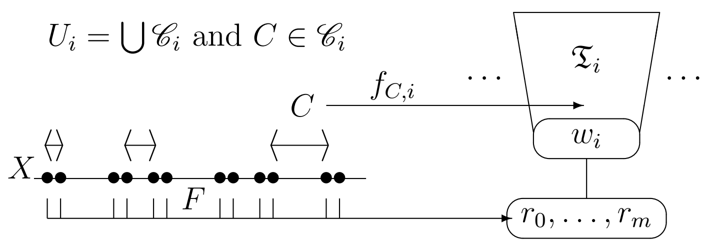


FIGURE 3. The mapping $f : X \to \mathfrak{T}$.

**Claim 3.10.** *$f$ is continuous.*

*Proof.* Let $w \in W$. If $w$ is a root of $\mathfrak{T}$, then $f^{-1}(R(w)) = f^{-1}(W) = X$ is open in $X$. Suppose that $w$ is not a root of $\mathfrak{T}$. Since $\mathfrak{T}$ is a quasi-tree, there is a unique $0 \le i \le k$ such that $w_i R w$. Let $C \in \mathscr{C}_i$. Since $f_{C,i}$ is continuous, $f_{C,i}^{-1}(R(w))$ is open in $C$, and hence open in $X$ (because $C$ is open in $X$). Thus, $f^{-1}(R(w)) = \bigcup_{C \in \mathscr{C}_i} f_{C,i}^{-1}(R(w))$ is open in $X$. Because $\{R(w) \mid w \in W\}$ is a basis for the Alexandroff topology on $\mathfrak{T}$, it follows that $f$ is continuous. $\square$

**Claim 3.11.** *$f$ is open.*

*Proof.* Let $U$ be a nonempty open subset of $X$. Since convex open subsets form a basis for $X$ and the direct image of a function commutes with arbitrary unions, without loss of generality we may assume that $U$ is convex. For each $0 \le i \le k$ and $C \in \mathscr{C}_i$, the set $U \cap C$ is open in $C$. As $f_{C,i}$ is open, it follows that $f_{C,i}(U \cap C)$ is open in $\mathfrak{T}_i$, and hence open in $\mathfrak{T}$. We have

$$X = F \cup \bigcup\nolimits_{i=0}^{k} U_i = \left(\bigcup\nolimits_{j=0}^{m} F_j\right) \cup \left(\bigcup\nolimits_{i=0}^{k} \bigcup \mathscr{C}_i\right)$$

Therefore,

$$U = U \cap X = \left(\bigcup\nolimits_{j=0}^{m} (U \cap F_j)\right) \cup \left(\bigcup\nolimits_{i=0}^{k} \bigcup\nolimits_{C \in \mathscr{C}_i} (U \cap C)\right)$$

Thus,

$$\begin{aligned} f(U) &= \left(\bigcup_{j=0}^{m} f\left(U\cap F_j\right)\right)\cup\left(\bigcup_{i=0}^{k}\bigcup_{C\in\mathscr{C}_i} f\left(U\cap C\right)\right) \\ &= \left(\bigcup_{j=0}^{m} f\left(U\cap F_j\right)\right)\cup\left(\bigcup_{i=0}^{k}\bigcup_{C\in\mathscr{C}_i} f_{C,i}\left(U\cap C\right)\right). \end{aligned}$$

If $U\cap F=\varnothing$, then $U\cap F_j=\varnothing$ for all $0\le j\le m$, which yields that

$$f(U)=\bigcup_{i=0}^{k}\bigcup_{C\in\mathscr{C}_i} f_{C,i}\left(U\cap C\right)$$

is open in $\mathfrak{T}$ since each $f_{C,i}\left(U\cap C\right)$ is open in $\mathfrak{T}$.

Suppose that $U\cap F\neq\varnothing$. This yields that $U\cap F_j\neq\varnothing$ for all $0\le j\le m$ since each $F_j$ is dense in $F$ and $U\cap F$ is a nonempty open subset of $F$. Therefore, $f(U\cap F_j)=\{r_j\}$ for all $0\le j\le m$. Let $0\le i\le k$. Because $U\cap F\neq\varnothing$, Remark 3.8 implies that there is a convex component $C_i$ of $U_i$ contained in $U$. Since $f_{C_i,i}(C_i)=\mathfrak{T}_i$, we have

$$\begin{aligned} f(U) &= \left(\bigcup_{j=0}^{m} f\left(U\cap F_j\right)\right)\cup\bigcup_{i=0}^{k}\bigcup_{C\in\mathscr{C}_i} f_{C,i}\left(U\cap C\right) \\ &\supseteq \left(\bigcup_{j=0}^{m}\{r_j\}\right)\cup\bigcup_{i=0}^{k} f_{C_i,i}\left(U\cap C_i\right) \\ &= \{r_0,\dots,r_m\}\cup\bigcup_{i=0}^{k} f_{C_i,i}\left(C_i\right) \\ &= \{r_0,\dots,r_m\}\cup\bigcup_{i=0}^{k}\mathfrak{T}_i=\mathfrak{T}. \end{aligned}$$

Thus, $f$ is open. □

Consequently, $\mathfrak{T}$ is an interior image of $X$. □

We are ready to prove an analogue of the McKinsey-Tarski Theorem for crowded locally compact GO-spaces.

**Theorem 3.12.** *If $X$ is a nonempty crowded locally compact GO-space, then* $\mathsf{L}(X)=\mathsf{S4}$.

*Proof.* Since $\mathsf{S4}\subseteq\mathsf{L}(X)$, it is sufficient to prove that if $\mathsf{S4}\not\vdash\varphi$, then $\varphi$ is refuted on $X$. By Lemma 2.1(1), $\varphi$ is refuted on some finite quasi-tree $\mathfrak{T}$. By the Mapping Lemma, $\mathfrak{T}$ is an interior image of $X$. As interior images preserve validity, $X\not\models\varphi$. Thus, $\mathsf{L}(X)=\mathsf{S4}$. □

Since a LOTS is a GO-space, we immediately obtain the following corollary.

**Corollary 3.13.** *If $X$ is a nonempty crowded locally compact LOTS, then* $\mathsf{L}(X)=\mathsf{S4}$.

**Remark 3.14.**

(1) Since $\mathbb{R}$ and $\mathsf{C}$ are crowded locally compact LOTS, it follows from Corollary 3.13 that $\mathsf{S4}$ is the logic of both $\mathbb{R}$ and $\mathsf{C}$.
(2) The Euclidean spaces $\mathbb{R}^n$ for $n\ge 2$ are not GO-spaces. Nevertheless, it is an easy consequence of Corollary 3.13 that $\mathsf{L}(\mathbb{R}^n)=\mathsf{S4}$. Indeed, since the projection map from $\mathbb{R}^n$ onto $\mathbb{R}$ is an onto interior map, every formula $\varphi$ refuted on $\mathbb{R}$ is also refuted on $\mathbb{R}^n$. This implies that $\mathsf{L}(\mathbb{R}^n)=\mathsf{S4}$.
(3) On the other hand, since $\mathbb{Q}$ is not locally compact, our results do not yield that $\mathsf{L}(\mathbb{Q})=\mathsf{S4}$.
(4) Local compactness is essential for our proof as it produces our basic building block for the recursive step in the Mapping Lemma. Without the locally compact assumption it is unclear how to construct such a building block.

**Open Problem:** Is $\mathsf{S4}$ the logic of an arbitrary nonempty crowded GO-space?

## 4. Zero-dimensional GO-spaces

In this section we recall Herrlich's result about hereditarily disconnected LOTS, and then utilize a result of Čech to generalize Herrlich's result to GO-spaces. We start by the following well-known definition (see, e.g., [11, Sec. 6.2]).

**Definition 4.1.** Let $X$ be a topological space.

(1) $X$ is *hereditarily disconnected* if the only nonempty connected subsets of $X$ are singletons.
(2) $X$ is *zero-dimensional* if $X$ is $T_1$ and has a basis of clopen sets.
(3) $X$ is *strongly zero-dimensional* if $X$ is Tychonoff and the Čech-Stone compactification $\beta X$ of $X$ is zero-dimensional.

Every strongly zero-dimensional space is zero-dimensional (see, e.g., [11, Thm. 6.2.6]), and every zero-dimensional space is hereditarily disconnected (see, e.g., [11, Thm. 6.2.1]).

**Theorem 4.2.** (Herrlich [12, Lem. 1]) *A LOTS is strongly zero-dimensional iff it is hereditarily disconnected.*

To generalize Herrlich's result to GO-spaces, we use Lutzer's modification of Čech's construction.

**Definition 4.3.** (Lutzer [16, Def. 2.5]) Let $X$ be a GO-space with order $\leq$ and topology $\tau$, and let $\sigma$ be the interval topology induced by $\leq$. Define $X^* \subseteq X \times \mathbb{Z}$ by

$$\begin{aligned} X^* = (X \times \{0\}) \quad &\cup \quad \{(x,n) \mid [x,\rightarrow) \in \tau \setminus \sigma \text{ and } n \leq 0\} \\ &\cup \quad \{(x,m) \mid (\leftarrow, x] \in \tau \setminus \sigma \text{ and } m \geq 0\}\,. \end{aligned}$$

We view $X^*$ as a LOTS whose interval topology is induced by the restriction of the lexicographic order on $X \times \mathbb{Z}$.

**Remark 4.4.** We can think of $X^*$ as being obtained from $X$ by inserting a decreasing sequence of isolated points below each $x \in X$ satisfying $[x,\rightarrow) \in \tau \setminus \sigma$, and an increasing sequence of isolated points above each $x \in X$ satisfying $(\leftarrow, x] \in \tau \setminus \sigma$. Each such sequence does not have a limit in $X^*$. For the subspace $X := \{-1\} \cup (0,1) \cup \{2\}$ of the LOTS $\mathbb{R}$, we have that $X^*$ is homeomorphic to

$$\left\{-\frac{1}{m+1} \mid m \geq 0\right\} \cup (0,1) \cup \left\{\frac{2-n}{1-n} \mid n \leq 0\right\}$$

The next theorem is attributed to Čech in [16, Prop. 2.7].

**Theorem 4.5.** *Let $X$ be a GO-space and let $X^*$ be as in Definition 4.3. The mapping $f : X \to X^*$ given by $f(x) = (x, 0)$ is an order-isomorphism and homeomorphism of $X$ and the subspace $X \times \{0\}$ of $X^*$.*

From now on we identify $X$ with the subspace $X \times \{0\}$ of $X^*$. It follows from the proof of [16, Thm. 2.9] that $X^* \setminus X$ consists of isolated points of $X^*$, which yields that $X$ is a closed subspace of $X^*$.

**Lemma 4.6.** *If $X$ is a hereditarily disconnected GO-space, then $X^*$ is a hereditarily disconnected LOTS.*

*Proof.* Let $A$ be a nonempty connected subspace of $X^*$. If $A \subseteq X$, then $A$ is a singleton since $X$ is hereditarily disconnected. Suppose $A \not\subseteq X$. Then $A \cap (X^* \setminus X) \neq \varnothing$. Therefore, $A$ contains an isolated point $x$ of $X^*$. Because $\{x\}$ is clopen and $A$ is connected, we conclude that $A = \{x\}$. Thus, $X^*$ is hereditarily disconnected. □

We are ready to generalize Herrlich's result to GO-spaces.

**Theorem 4.7.** *A GO-space is strongly zero-dimensional iff it is hereditarily disconnected.*

*Proof.* It is sufficient to show that every hereditarily disconnected GO-space is strongly zero-dimensional. Let $X$ be a hereditarily disconnected GO-space and let $X^*$ be as in Definition 4.3. By Lemma 4.6, $X^*$ is hereditarily disconnected. Therefore, by Theorem 4.2, $X^*$ is strongly zero-dimensional. By [11, Thm 6.2.11], strong zero-dimensionality is a hereditary property for closed subspaces of a normal space. Thus, $X$ is strongly zero-dimensional because $X$ is a closed subspace of the strongly zero-dimensional normal space $X^*$. □

**Corollary 4.8.** *Let $X$ be a GO-space.*

1. *$X$ is zero-dimensional iff $X$ is strongly zero-dimensional.*
2. *If $X$ is scattered, then $X$ is strongly zero-dimensional.*

*Proof.* (1) This is immediate from Theorem 4.7.

(2) We show that $X$ is hereditarily disconnected. Let $A$ be a nonempty connected subspace of $X$. Since $X$ is scattered, $A$ has an isolated point, so $\{x\}$ is a clopen subset of $A$. Because $A$ is connected, $A = \{x\}$. Thus, $X$ is hereditarily disconnected, and applying Theorem 4.7 finishes the proof. □

## 5. Logics arising from locally compact GO-spaces

In this final section we axiomatize all logics arising as $\mathsf{L}(X)$ for some nonempty locally compact GO-space. The main result is that

$$\mathsf{S4} \subset \mathsf{S4.1} \subset \mathsf{S4.Grz} \subset \cdots \subset \mathsf{S4.Grz}_3 \subset \mathsf{S4.Grz}_2 \subset \mathsf{S4.Grz}_1$$

are exactly the logics obtained this way, thus yielding an analogue of Theorem 1.1 for locally compact GO-spaces.

Let $X$ be a nonempty locally compact GO-space. If $X$ is not densely discrete, then Theorem 3.12 yields that $\mathsf{L}(X) = \mathsf{S4}$, as we next show.

**Theorem 5.1.** *If $X$ is a locally compact GO-space that is not densely discrete, then $\mathsf{L}(X) = \mathsf{S4}$.*

*Proof.* We only need to show $\mathsf{L}(X) \subseteq \mathsf{S4}$ since the other inclusion always holds. Suppose $\mathsf{S4} \not\vdash \varphi$. Because $X$ is not densely discrete, $U := X \setminus \mathsf{c}(\mathsf{iso}X)$ is a nonempty open subset of $X$. Let $X = D \cup S$ be the Cantor-Bendixson decomposition of $X$. Since $S \subseteq \mathsf{c}(\mathsf{iso}X)$, we have $U \subseteq D$. Therefore, $U$ is a nonempty crowded locally compact GO-space. By Theorem 3.12, $U$ refutes $\varphi$. As open subspaces preserve validity, $X$ also refutes $\varphi$. Thus, $\mathsf{L}(X) = \mathsf{S4}$. □

Next suppose that $X$ is densely discrete. To determine $\mathsf{L}(X)$ we need one more mapping lemma (Lemma 5.3) for which we recall the following two results about normal spaces. The first one follows from a straightforward inductive argument from the well-known fact that if $F_1, F_2$ are disjoint closed subsets of a normal space, then there exist open subsets $U_1, U_2$ such that $F_1 \subseteq U_1$, $F_2 \subseteq U_2$, and $\mathsf{c}U_1, \mathsf{c}U_2$ are disjoint. The second one can, for example, be found in [5, Lem. 3.2].

**Lemma 5.2.** *Let $X$ be a normal space, $n \geq 1$, and $\{F_i \mid i < n\}$ a pairwise disjoint family of nonempty closed subsets of $X$.*

1. *There is a family $\{U_i \mid i < n\}$ of open subsets of $X$ such that $F_i \subseteq U_i$ for each $i < n$ and $\{\mathsf{c}U_i \mid i < n\}$ is pairwise disjoint.*
2. *If in addition $X$ is strongly zero-dimensional, then $\{U_i \mid i < n\}$ can be chosen to be a partition of $X$.*

For a finite top-thin-quasi-tree $\mathfrak{T} = (W, R)$ let $\mathfrak{T}^-$ be the quasi-tree obtained from $\mathfrak{T}$ by deleting $\max W$; see Figure 4.

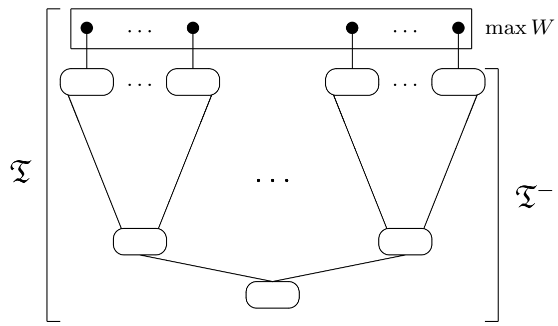


FIGURE 4. The quasi-trees $\mathfrak{T}$ and $\mathfrak{T}^-$.

**Lemma 5.3.** *Let $X$ be a non-scattered densely discrete locally compact GO-space. Then every finite top-thin-quasi-tree $\mathfrak{T} = (W, R)$ is an interior image of $X$.*

*Proof.* Let $X = D \cup S$ be the Cantor-Bendixson decomposition of $X$. Because $X$ is nonempty and densely discrete, $S \supseteq \mathsf{iso}X \neq \varnothing$. Also since $X$ is not scattered, $D$ is nonempty, and so $D$ is a crowded locally compact GO-space. By the Mapping Lemma (Lemma 3.9), there is an onto interior map $g : D \to \mathfrak{T}^-$. We show by strong induction on the depth $n$ of $\mathfrak{T}$ that each such map $g$ can be extended to an onto interior map $f : X \to \mathfrak{T}$ so that $f(S) = \max W$. It follows from the definition of a top-thin-quasi-tree that the depth of $\mathfrak{T}$ is $\geq 2$. Therefore, the base case for induction is $n = 2$. Let $C_r$ be the root cluster of $\mathfrak{T}$.

**Base case:** Suppose that $n = 2$. Then $W = C_r \cup \{m\}$ where $m$ is the maximum element of $\mathfrak{T}$; see Figure 5.

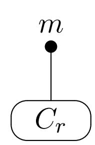


FIGURE 5. A top-thin-quasi-tree of depth 2.

Let $g : D \to \mathfrak{T}^-$ be an onto interior map. Define $f : X \to \mathfrak{T}$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in D \\ m & \text{if } x \in S \end{cases}$$

Because $\{D, S\}$ is a partition of $X$ and $g$ is onto, it follows that $f$ is a well-defined onto map that extends $g$. Clearly $f(S) = \{m\} = \max W$.

Since $\{m\}$ is the only nonempty proper open subset of $\mathfrak{T}$ and $f^{-1}(m) = S$ is open in $X$, we have that $f$ is continuous. Let $U$ be a nonempty open subset of $X$. As $X$ is densely discrete, $\varnothing \neq U \cap \mathsf{iso}X \subseteq U \cap S$. If $U \subseteq S$, then $f(U) = \{m\}$ is open in $\mathfrak{T}$. Suppose $U \not\subseteq S$. Then $U \cap D$ is a nonempty open subset of $D$. Because $g$ is interior and $\mathfrak{T}^-$ consists of a single cluster, namely $C_r$, we have

$$f(U) = f(U \cap S) \cup f(U \cap D) = \{m\} \cup g(U \cap D) = \{m\} \cup C_r = W.$$

Thus, $f$ is open, and hence $\mathfrak{T}$ is an interior image of $X$.

**Inductive step:** Suppose that the depth of $\mathfrak{T}$ is $n + 1$, where $n \geq 2$. By inductive hypothesis, for each top-thin-quasi-tree $\mathfrak{F} = (V, S)$ of depth $\leq n$, a non-scattered densely discrete locally compact GO-space $Y$ whose Cantor-Bendixson decomposition is $Y = D' \cup S'$, and $G : D' \to \mathfrak{F}^-$ an onto interior map, there is an onto interior map $F : Y \to \mathfrak{F}$ extending $G$ such that $F(S') = \max V$.

Let $g : D \to \mathfrak{T}^-$ be an onto interior map. We must extend $g$ to an onto interior map $f : X \to \mathfrak{T}$ so that $f(S) = \max W$. Let $C_0, \dots, C_k$ be the children clusters of the root cluster $C_r$ of $\mathfrak{T}$. For $i = 0, \dots, k$, let $\mathfrak{T}_i = (R(C_i), R_i)$ where $R_i$ is the restriction of $R$ to $R(C_i)$. Because $\mathfrak{T}$ is a finite top-thin-quasi-tree of depth $n+1$, each $\mathfrak{T}_i$ is a finite top-thin-quasi-tree of depth $\leq n$. Therefore, $\mathfrak{T}_i^- = (W_i, Q_i)$ where $W_i = R(C_i) \setminus \max W$ and $Q_i$ is the restriction of $R$ to $W_i$. Observe that $\{C_r, R(C_0), \dots, R(C_k)\}$ is a partition of $W$ and $\{C_r, W_0, \dots, W_k\}$ is a partition of $W \setminus \max W$. Set $F = g^{-1}(C_r)$, $D_i = g^{-1}(W_i)$, and $Y = X \setminus F$. Then $\{D_0, \dots, D_k\}$ is a partition of $D \setminus F$ and $Y = S \cup (D \setminus F)$; see Figure 6.

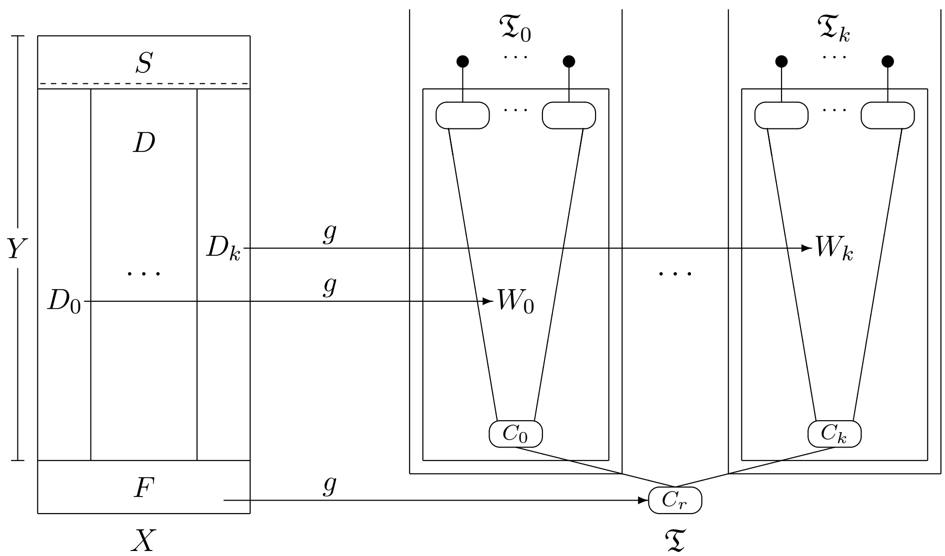


FIGURE 6. The mapping $g$ and the partition of $D$ it induces.

Because $g$ is interior, each $D_i$ is open in $D$ and $F$ is closed in $D$. As $D$ is closed in $X$, we have that $F$ is closed in $X$, which yields that $Y$ is open in $X$. In addition, the closure of $A \subseteq D$ relative to $D$ is $\mathsf{c}A$. Thus, $D_i$ is closed in $Y$ since

$$\mathsf{c}D_i = \mathsf{c}g^{-1}(W_i) = g^{-1}(R^{-1}W_i) = g^{-1}(W_i \cup C_r) = g^{-1}(W_i) \cup g^{-1}(C_r) = D_i \cup F$$

implies that $\mathsf{c}(D_i) \cap Y = (D_i \cup F) \cap Y = D_i$. Being a GO-space, $Y$ is normal. Therefore, we may apply Lemma 5.2(1) to $\{D_0, \dots, D_k\}$ to obtain a family $\{U_0, \dots, U_k\}$ of open subsets of $Y$, and hence of $X$, such that $D_i \subseteq U_i$ and $\{\mathsf{c}(U_i) \cap Y \mid i = 0, \dots, k\}$ is pairwise disjoint. Then $\{U_0, \dots, U_k\}$ is pairwise disjoint since $U_i \subseteq \mathsf{c}(U_i) \cap Y$.

We clearly have that each $D_i \subseteq U_i \cap D$. For the converse, let $x \in U_i \cap D$. Then $x \notin S$ and $x \in U_i \subseteq Y = X \setminus F$. So there is $j$ such that $x \in D_j$. Therefore, $x \in U_j$, which implies $U_i \cap U_j \neq \varnothing$. Thus, $j = i$, so $x \in D_i$ and hence $D_i = U_i \cap D$.

The family $\{\mathsf{c}(U_i) \cap S \mid i = 0, \dots, k\}$ is pairwise disjoint and consists of closed subsets of $S$. Since $\mathsf{iso}X \subseteq S$ and $X$ is densely discrete, $S$ is dense in $X$. Because each $U_i$ is a nonempty open subset of $X$, we have that $U_i \cap S \neq \varnothing$. Therefore, each $\mathsf{c}(U_i) \cap S$ is nonempty. Since $S$ is a scattered GO-space, Corollary 4.8(2) implies that $S$ is a strongly zero-dimensional normal space. By Lemma 5.2(2), there is a partition $\{S_0, \dots, S_k\}$ of $S$ consisting of open subsets of $S$ (which are also open in $X$) such that $\mathsf{c}(U_i) \cap S \subseteq S_i$.

For each $i$ put $Y_i = D_i \cup S_i$; see Figure 7.

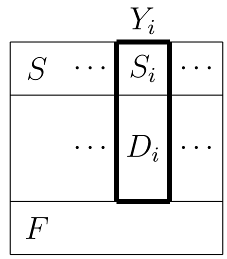


FIGURE 7. The subspace $Y_i = D_i \cup S_i$.

Because $D_i = U_i \cap D$ and $U_i \cap S \subseteq \mathsf{c}(U_i) \cap S \subseteq S_i$, we have

$$Y_i = D_i \cup S_i = (U_i \cap D) \cup (U_i \cap S) \cup S_i = (U_i \cap (D \cup S)) \cup S_i = (U_i \cap X) \cup S_i = U_i \cup S_i.$$

Therefore, $Y_i$ is open in $X$ since $U_i$ and $S_i$ are open in $X$. Thus, being an open subspace of a densely discrete space, $Y_i$ is densely discrete. It is obvious that $\{Y_0, \dots, Y_k\}$ is a partition of $Y$.

We have that $D_i$ is crowded since it is an open subspace of the crowded space $D$. Recalling that $\mathsf{c}D_i = D_i \cup F$, it follows that $\mathsf{d}D_i = D_i \cup F$, which implies that $\mathsf{d}^\alpha D_i = D_i \cup F$ for each nonzero ordinal $\alpha$. Let $\varrho$ be the Cantor-Bendixson rank of $X$. Then $\varrho \neq 0$ and $\mathsf{d}^\varrho S_i \subseteq \mathsf{d}^\varrho S \subseteq \mathsf{d}^\varrho X = D$. Because $U := \bigcup\{Y_j \mid j \neq i\}$ is open in $X$, we have that $Y_i \cup F = X \setminus U$ is closed in $X$. Therefore, $\mathsf{d}^\varrho S_i \subseteq \mathsf{d}S_i \subseteq \mathsf{c}S_i \subseteq Y_i \cup F$, yielding that

$$\mathsf{d}^\varrho S_i \subseteq (Y_i \cup F) \cap D = (D_i \cup S_i \cup F) \cap D = (D_i \cap D) \cup (S_i \cap D) \cup (F \cap D) = D_i \cup F$$

Thus,

$$\mathsf{d}^\varrho Y_i = \mathsf{d}^\varrho(D_i \cup S_i) = \mathsf{d}^\varrho D_i \cup \mathsf{d}^\varrho S_i = D_i \cup F \cup \mathsf{d}^\varrho S_i = D_i \cup F,$$

which implies $\mathsf{d}^\varrho Y_i \cap Y_i = D_i$. Therefore, the Cantor-Bendixson decomposition of $Y_i$ is $D_i \cup S_i$. Because $D_i \neq \varnothing$, it follows that $Y_i$ is a non-scattered densely discrete locally compact GO-space.

Let $g_i$ be the restriction of $g : D \to \mathfrak{T}^-$ to $D_i$. Since $g$ is an onto interior map and $D_i$ is open in $D$, we have that $g_i$ is an interior mapping of $D_i = g^{-1}(W_i)$ onto $\mathfrak{T}_i^- = (W_i, Q_i)$. By the inductive hypothesis, there is an interior mapping $f_i$ of $Y_i$ onto $\mathfrak{T}_i = (R(C_i), R_i)$ extending $g_i$ such that $f_i(S_i) = \max R(C_i)$; see Figure 8.

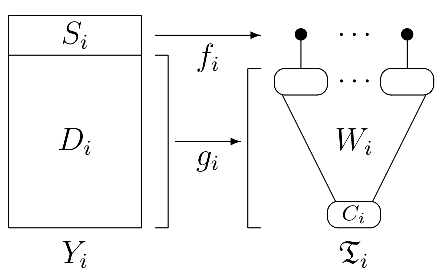


FIGURE 8. Extending $g_i$ to $f_i$.

Define $f : X \to \mathfrak{T}$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in F \\ f_i(x) & \text{if } x \in Y_i \end{cases}$$

Note that $f$ is a well-defined map since $\{F, Y_0, \dots, Y_k\}$ is a partition of $X$. It is clear that $f$ extends $g$.

**Claim 5.4.** *$f$ is onto.*

*Proof.*

$$\begin{aligned} f(X) &= f\left(F \cup Y_0 \cup \cdots \cup Y_k\right) = f(F) \cup f(Y_0) \cup \cdots \cup f(Y_k) \\ &= g(F) \cup f_0(Y_0) \cup \cdots \cup f_k(Y_k) = C_r \cup R(C_0) \cup \cdots \cup R(C_k) = W. \end{aligned}$$

□

**Claim 5.5.** $f(S) = \max W$.

*Proof.*

$$\begin{aligned} f(S) &= f\left(S_0 \cup \cdots \cup S_k\right) = f(S_0) \cup \cdots \cup f(S_k) \\ &= f_0(S_0) \cup \cdots \cup f_k(S_k) = \max R(C_0) \cup \cdots \cup \max R(C_k) = \max W. \end{aligned}$$

□

**Claim 5.6.** *$f$ is continuous.*

*Proof.* Let $w \in W$. If $w \in C_r$ then $f^{-1}(R(w)) = f^{-1}(W) = X$ is open in $X$. Suppose that $w \notin C_r$. Then there is a unique $i$ such that $w \in R(C_i)$. Therefore, $f^{-1}(R(w)) = f_i^{-1}(R_i(w))$, so is open in $Y_i$. As $Y_i$ is open in $X$, it follows that $f^{-1}(R(w))$ is open in $X$. Thus, $f$ is continuous. □

**Claim 5.7.** *$f$ is open.*

*Proof.* Let $U$ be a nonempty open subset of $X$. Then

$$\begin{aligned} f(U) = f(U \cap X) &= f\left(U \cap (F \cup Y_0 \cup \cdots \cup Y_k)\right) \\ &= f\left((U \cap F) \cup (U \cap Y_0) \cup \cdots \cup (U \cap Y_k)\right) \\ &= f(U \cap F) \cup f(U \cap Y_0) \cup \cdots \cup f(U \cap Y_k) \\ &= g(U \cap F) \cup f_0(U \cap Y_0) \cup \cdots \cup f_k(U \cap Y_k). \end{aligned}$$

Because each $f_i$ is interior, $f_i(U \cap Y_i)$ is open in $\mathfrak{T}_i$, and hence open in $\mathfrak{T}$. If $U \cap F = \varnothing$, then $f(U) = \bigcup_{i=0}^{k} f_i(U \cap Y_i)$ is a union of open subsets of $\mathfrak{T}$, and so is open in $\mathfrak{T}$. Suppose that $U \cap F \neq \varnothing$. Let $x \in U \cap F$. Then $g(x) \in C_r$ is a root of both $\mathfrak{T}$ and $\mathfrak{T}^-$. Since $g$ is an open map and $U \cap D$ is open in $D$, we have that $g(U \cap D)$ is an open subset of $\mathfrak{T}^-$ containing a root. Therefore, $g(U \cap D) = W \setminus \max W$, and hence $g(U \cap F) = C_r$. For each $i$ we have that $x \in g^{-1}(R^{-1}(C_i)) = \mathsf{c}g^{-1}(C_i)$, which implies that there is $y_i \in U \cap g^{-1}(C_i)$. Note that $f_i(y_i) = f(y_i) = g(y_i) \in C_i$ is a root of $\mathfrak{T}_i$. Being an open subset of $\mathfrak{T}_i$ containing a root, we have that $f_i(U \cap Y_i) = R_i(C_i) = R(C_i)$. Thus,

$$f(U) = g(U \cap F) \cup f_0(U \cap Y_0) \cup \cdots \cup f_k(U \cap Y_k) = C_r \cup R(C_0) \cup \cdots \cup R(C_k) = W,$$

and hence $f$ is open. □

Consequently, $\mathfrak{T}$ is an interior image of $X$. □

**Theorem 5.8.** *If $X$ is a non-scattered densely discrete locally compact GO-space, then* $\mathsf{L}(X) = \mathsf{S4.1}$.

*Proof.* Since $X$ is densely discrete, $\mathsf{S4.1} \subseteq \mathsf{L}(X)$ by Lemma 2.2(2). Suppose that $\mathsf{S4.1} \not\vdash \varphi$. By Lemma 2.1(2), there is a finite top-thin-quasi-tree $\mathfrak{T}$ refuting $\varphi$. By Lemma 5.3, $\mathfrak{T}$ is an interior image of $X$. Because interior images preserve validity, $\varphi$ is refuted on $X$. Thus, $\mathsf{L}(X) = \mathsf{S4.1}$. □

**Theorem 5.9.** *Let $X$ be a nonempty scattered locally compact GO-space and $n \geq 1$.*

(1) *If the Cantor-Bendixson rank of* $X$ *is* $n$*, then* $\mathsf{L}(X) = \mathsf{S4.Grz}_n$.
(2) *If the Cantor-Bendixson rank of* $X$ *is infinite, then* $\mathsf{L}(X) = \mathsf{S4.Grz}$.

*Proof.* Lemma 2.6(6) implies that every open subspace of $X$ is collectionwise normal. Since each subspace of a scattered space is scattered, Corollary 4.8(2) yields that every open subspace of $X$ is strongly zero-dimensional. Thus, it follows from [3, Thm. 4.9] that we may apply [3, Thm. 7.3] to obtain the result. □

Putting Theorems 5.1, 5.8, and 5.9, we arrive at the following axiomatization of $\mathsf{L}(X)$ for each nonempty locally compact GO-space.

**Theorem 5.10.** *Let* $X$ *be a nonempty locally compact GO-space.*
(1) *If* $X$ *is not densely discrete, then* $\mathsf{L}(X) = \mathsf{S4}$.
(2) *If* $X$ *is densely discrete but not scattered, then* $\mathsf{L}(X) = \mathsf{S4.1}$.
(3) *If* $X$ *is scattered and has infinite Cantor-Bendixson rank, then* $\mathsf{L}(X) = \mathsf{S4.Grz}$.
(4) *If* $X$ *is scattered and has Cantor-Bendixson rank* $n \geq 1$*, then* $\mathsf{L}(X) = \mathsf{S4.Grz}_n$.

**Remark 5.11.** Utilizing the well-known Gödel translation (see, e.g., [9, Sec. 3.9]), Theorem 5.10 yields a characterization of the superintuitionistic logics (si-logics for short) arising from nonempty locally compact GO-spaces. Let $\mathsf{IPC}$ be the intuitionistic propositional calculus and $\mathsf{IPC}_n := \mathsf{IPC} + \mathsf{ibd}_n$ where

$$\begin{aligned} \mathsf{ibd}_1 &= p_1 \vee \neg p_1 \\ \mathsf{ibd}_{n+1} &= p_{n+1} \vee (p_{n+1} \to \mathsf{ibd}_n) \end{aligned}$$

The formulas $\mathsf{ibd}_n$ are the intuitionistic version of the modal formulas $\mathsf{bd}_n$.

We recall (see, e.g., [9, Sec. 9.6]) that via the Gödel translation each si-logic $\mathsf{L}$ gives rise to an interval (with respect to $\subseteq$) of normal extensions of $\mathsf{S4}$ consisting of *modal companions* of $\mathsf{L}$. It is well known that the modal companions of $\mathsf{IPC}$ form the interval $[\mathsf{S4}, \mathsf{S4.Grz}]$. Thus, each of $\mathsf{S4}$, $\mathsf{S4.1}$, and $\mathsf{S4.Grz}$ is a modal companion of $\mathsf{IPC}$. Moreover, $\mathsf{S4.Grz}_n$ is a modal companion of $\mathsf{IPC}_n$. This together with Theorem 5.10 yields that the si-logic of a nonempty locally compact GO-space $X$ is:
(1) $\mathsf{IPC}_n$ if $X$ is scattered and has Cantor-Bendixson rank $n \geq 1$, and
(2) $\mathsf{IPC}$ otherwise.

Thus, the si-logics

$$\mathsf{IPC} \subset \cdots \subset \mathsf{IPC}_3 \subset \mathsf{IPC}_2 \subset \mathsf{IPC}_1$$

are exactly those that arise as the si-logic of a nonempty locally compact GO-space.

**Acknowledgement:** We thank Klaas Pieter Hart for kindly providing a copy of [16].

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