COMPLETENESS FOR AXIOMATIC EXTENSIONS OF MODAL LOGIC WITH THE MASTER MODALITY

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Abstract. We apply the technique of filtration to prove completeness for a class of axiomatic extensions of modal logic with the master modality. It suffices that the set of additional axioms is basic modal, canonical and admits filtration. We also take some steps towards generalising our results to sets of extra axioms formulated in the full language.

1. Introduction

Modal logic with master modality is the bimodal logic characterised by *regular* frames, *i.e.* those frames (S, R, R_*) for which R_* is the reflexive-transitive closure of R. Throughout this note this logic will be denoted mK and its language mML.

In different contexts, such as program logics and epistemic logics, there exist completeness results for extensions of mK characterised by *specific* classes of frames. But while there are many general completeness results for (axiomatic) extensions of basic modal logic, the authors are aware of no such results for mK. The biggest obstacle in proving such result is the fact that mK and most of its extensions lack the compactness property. One therefore has to resort to finitary methods.

In the following we shall use the technique of filtration for this purpose. We will show that for an extension generated by a set of additional axioms it suffices that this set is basic modal, canonical and admits filtration. We have not yet managed to extend our results to a general set of axioms, which may also contain the master modality $\diamond\!\!\!\!\diamond$, but we do take some steps towards this direction.

A key element of our method (Theorem 17) is based on the techniques used in the completeness proof for PDL given in [3].

2. Modal logic with the master modality

In this section we define the language mML as well as the logic mK and its axiomatic extensions. For the sake of simplicity, we confine ourselves to the case of one basic modality, but our results can easily be extended to the case where there are more. This means that for instance multi-agent epistemic logics with common knowledge fall within the scope of this note.

Let P be a fixed countable set of propositional variables.

Definition 1 The syntax mML of *modal m-formulas* over P is generated by:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\varphi \mid \diamond\!\!\!\!\diamond\varphi$$

where $p \in$ P.

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We interpret mML on Kripke frames for two modalities in the standard way. The intended interpretation of $\Diamond^*$ is as the reflexive-transitive closure of $\Diamond$. In terms of the modal μ-calculus this can be formulated as follows:

$$\Diamond^*\varphi \equiv \mu x.\varphi \vee \Diamond x.$$

The following proof system is based on Kozen's axiomatisation of the modal μ-calculus and the above equivalence.

Definition 2 The logic mK is axiomatised by the following Hilbert system:

$$\mathsf{K} \ + \ \varphi \vee \Diamond\Diamond^*\varphi \to \Diamond^*\varphi \ + \ \frac{(\varphi \vee \Diamond\psi) \to \psi}{\Diamond^*\varphi \to \psi}$$

We shall write pre for the additional axiom and min for the additional rule. The system mK^- is axiomatised by $\mathsf{K} + \mathsf{pre}$.

The dual of $\Diamond^*\varphi$ is defined in the standard way:

$$\Box^*\varphi := \neg\Diamond^*\neg\varphi.$$

In terms of the modal μ-calculus, we have

$$\Box^*\varphi \equiv \nu x.\varphi \wedge \Box x.$$

As one would expect, the following axiom pos and rule max are derivable in mK:

$$\Box^*\varphi \to \varphi \wedge \Box\Box^*\varphi \qquad \frac{\psi \to (\varphi \wedge \Box\psi)}{\psi \to \Box^*\varphi}$$

Let $\mathcal{F} = (S, R, R_*)$ be a Kripke frame. We say that $\mathcal{F}$ is *regular* if $R^* = R_*$, *i.e.* if R_* is the reflexive-transitive closure of R. We shall write RegFr for the class of regular frames. It is left to the reader to verify that mK is sound with respect to RegFr.

For Θ a set of modal m-formulas we let $\mathsf{RegFr}(\Theta)$ denote the restriction of RegFr to those frames on which all formulas in Θ are valid. On the syntactic side, we let $\mathsf{mK} \oplus \Theta$ be the logic axiomatised by mK with the formulas in Θ as additional axioms. Precisely, $\mathsf{mK} \oplus \Theta$ is the closure of $\mathsf{mK} \cup \Theta$ under modus ponens, uniform substitution and min. In the next sections we work with an arbitrary Θ and simply denote the resulting logic $\mathsf{mK} \oplus \Theta$ by L.

3. Compactness and consistency

Modal logic with the master modality is not compact on the class of regular frames, as witnessed by the following set of formulas:

$$\{\Diamond^* p, \neg p, \neg\Diamond p, \neg\Diamond\Diamond p, \ldots\}.$$

This set is clearly finitely satisfiable, yet not satisfiable. We can therefore not hope to prove strong completeness.

We shall write $\vdash \varphi$ to mean that φ is derivable (in L). For a set Γ of formulas, we write $\Gamma \vdash \varphi$ whenever $\vdash (\gamma_0 \wedge \ldots \wedge \gamma_n) \to \varphi$ for some finite subset $\{\gamma_0, \ldots, \gamma_n\} \subseteq \Gamma$. A set Γ is called *consistent* if $\Gamma \nvdash \bot$.

Definition 3 A set Γ is *maximal consistent* if it is consistent and maximal in that respect, *i.e.*

$$\Gamma \subset \Gamma' \text{ implies that } \Gamma' \text{ is inconsistent.}$$

We write MCS(L) for the collection of maximal L-consistent sets of modal m-formulas.

Lemma 4 *Let Γ be a maximal consistent set. Then:*

(i) *If $\vdash \varphi$, then $\varphi \in \Gamma$;*
(ii) *$\neg\varphi \in \Gamma$ if and only $\varphi \notin \Gamma$;*
(iii) *$\varphi \vee \psi \in \Gamma$ if and only $\varphi \in \Gamma$ or $\psi \in \Gamma$.*
(iv) *$\Diamond^*\varphi \in \Gamma$ if and only if $\varphi \in \Gamma$ or $\Diamond\Diamond^*\varphi \in \Gamma$.*

Lemma 5 (Lindenbaum) *Every consistent set can be extended to be maximal.*

The above lemma is less useful due to the lack of compactness. Indeed, a satisfiable set may very well be extended to a consistent, yet unsatisfiable set. This could indicate that we should strengthen the notion of consistency. We do, for instance, know the following.

Lemma 6 *Let Γ be a satisfiable maximal consistent set. Then:*

$$\Diamond^*\varphi \in \Gamma \Rightarrow \Diamond^n\varphi \in \Gamma \textit{ for some } n.$$

4. The Canonical model

Definition 7 The *canonical model* for L is given by $\mathcal{M}^{\mathsf{L}} := (S^{\mathsf{L}}, R^{\mathsf{L}}, R^{\mathsf{L}}_*, V^{\mathsf{L}})$, where:

$$\begin{aligned} S^{\mathsf{L}} &:= \mathrm{MCS}(\mathsf{L}) \\ \Gamma R^{\mathsf{L}}\Delta &:\Leftrightarrow \varphi \in \Delta \text{ implies } \Diamond\varphi \in \Gamma \\ \Gamma R^{\mathsf{L}}_*\Delta &:\Leftrightarrow \varphi \in \Delta \text{ implies } \Diamond^*\varphi \in \Gamma \\ V^{\mathsf{L}}(p) &:= \{\Gamma \mid p \in \Gamma\} \end{aligned}$$

When clear from the context we shall omit the superscript $\mathsf{L}'s$. As usual, we have the following lemma.

Lemma 8 (Truth Lemma) *For all φ, we have $\mathcal{M}, \Gamma \Vdash \varphi \Leftrightarrow \varphi \in \Gamma$.*

Since pre is a Sahlqvist formula, we know $\mathcal{M}^{\mathsf{L}} \models \mathsf{pre}$ and thus $Id \cup R; R_* \subseteq R_*$. However, the rule min is not valid on the canonical frame. Take for example the formula $\varphi := p \vee \Diamond^*(\neg p \wedge \Diamond p)$. We have,

$$\mathsf{mK}^- \vdash p \vee \Diamond\varphi \to \varphi$$

and thus by min, we have

$$\mathsf{mK} \vdash \Diamond^* p \to \varphi. \qquad (*)$$

Now let Γ be a maximal consistent set extending the set given at the beginning of Section 3. By the Truth Lemma there must be a Δ_0 such that $\Gamma R^{\mathsf{mK}}_* \Delta_0$ but not $\Gamma (R^{\mathsf{mK}})^* \Delta_0$. Setting $V(p) := S^{\mathsf{mK}} \setminus \{\Delta \mid \Gamma (R^{\mathsf{mK}})^* \Delta\}$ we obtain a valuation on the canonical frame of mK that falsifies $\Diamond^* p \to \varphi$ at Γ.

5. Filtrations of the canonical model

We define the notion of filtration for models of a general similarity type ε. Here ε is simply a finite alphabet containing indices for modalities. Modal formulas containing only modalities in ε are called ε-formulas. For instance, modal m-formulas are $\{\Diamond, \Diamond^*\}$-formulas.

Definition 9 Let $\mathbb{S} = (S, (R_e)_{e\in\varepsilon}, V)$ be a Kripke model and Σ a finite subformula closed set of ε-formulas. Let $\sim$ be the equivalence relation given by:

$$s \sim s' \text{ if and only if } s \Vdash \varphi \Leftrightarrow s' \Vdash \varphi \text{ for all } \varphi \in \Sigma.$$

For $\delta \subseteq \epsilon$, a δ-*filtration* of $\mathbb{S}$ through Σ is a model $\mathbb{S}^f = (S^f, (R^f_e)_{e\in\delta}, V^f)$ such that

(i) $S^f = S/{\sim}$;
(ii) $R^{\mathsf{min}}_e \subseteq R^f_e \subseteq R^{\mathsf{max}}_e$ for all $e \in \delta$;
(iii) $V^f(p) = \{[s] \mid \mathbb{S}, s \Vdash p\}$ for all $p \in \Sigma$.

where for each $e \in \varepsilon$:

$$R^{\mathsf{min}}_e := \{([s],[t]) \mid \text{there are } s' \sim s \text{ and } t' \sim t \text{ such that } R_e s't'\},$$
$$R^{\mathsf{max}}_e := \{([s],[t]) \mid \text{for all } \Diamond_e\varphi \in \Sigma\text{: if } t \Vdash \varphi, \text{ then } s \Vdash \Diamond_e\varphi\}.$$

Note that the set of states of any filtration is finite. The following standard theorem can be proven by induction on formulas.

Theorem 10 *Let $\mathbb{S}^f$ be a δ-filtration of $\mathbb{S}$ trough Σ. Then for all δ-formulas $\varphi \in \Sigma$:*

$$\mathbb{S}, s \Vdash \varphi \Leftrightarrow \mathbb{S}^f, [s] \Vdash \varphi.$$

In this section we will in particular be concerned with filtrations of the canonical model through some Σ. We shall use the letters $A, B, C, \dots$ to denote equivalence classes of maximal consistent sets. For $A = [\Gamma]$ we shall write $\varphi \in A$ whenever $\varphi \in \Gamma \cap \Sigma$ (note that this is well-defined). Since such A is finite, we can take its conjunction $\bigwedge A$, which will be denoted $\widehat{A}$.

The following is an immediate consequence of the Truth Lemma.

Lemma 11 *For any filtration M^f of a canonical model $\mathcal{M}^{\mathsf{L}}$ it holds that*

$$\mathcal{M}^f, A \Vdash \widehat{B} \Leftrightarrow A = B.$$

As a consequence every point of $\mathcal{M}^f$ can be distinguished by a formula. Such a model is often said to be *named*. Filtrations of the canonical model have another remarkable property.

Proposition 12 ([2, Proposition 5.27]) *If $\mathcal{M}^f$ is a filtration of the canonical model $\mathcal{M}^{\mathsf{L}}$ of L such that $\mathsf{M}^f \models \mathsf{L}$, then $\mathcal{M}^f$ is the minimal filtration.*

Proof. First note that for any world $[\Delta]$ of $\mathcal{M}^f$, the *theory* of $[\Delta]$, *i.e.* the set

$$Th([\Delta]) := \{\varphi \mid \mathcal{M}^f, [\Delta] \Vdash \varphi\}$$

is maximally L-consistent. Indeed, suppose not, then $\mathsf{L} \vdash (\varphi_0 \wedge \ldots \wedge \varphi_n) \to \bot$ for some finite sequence of φ_i's in $Th([\Delta])$. But since $\mathsf{L} \subseteq Th([\Delta])$ it follows by the semantics of $\to$ that $\mathcal{M}^f, [\Delta] \Vdash \bot$, a contradiction.

Now suppose that $[\Gamma] R^f_e [\Delta]$. It holds by the Filtration Theorem that $Th([\Gamma]) \sim \Gamma$ and $Th([\Delta]) \sim \Delta$. Moreover, for all $\varphi \in Th([\Delta])$, we have that $\Diamond_e\varphi \in Th([\Gamma])$. It follows that by definition $Th([\Gamma]) R^{\mathsf{L}} Th([\Delta])$, as required. □

The relation R^{min}_e has a useful characterisation.

Proposition 13 *For any filtration of $\mathcal{M}^{\mathsf{L}}$:*

$$A R^{\mathsf{min}}_e B \text{ if and only if } \widehat{A} \wedge \Diamond_e \widehat{B} \text{ is consistent.}$$

Proof. $\Rightarrow$. If $AR_e^{\mathsf{min}}B$, then there are $[\Gamma] = A$ and $[\Delta] = B$ such that $\Gamma R_e^{\mathsf{L}}\Delta$. It follows that $\widehat{A} \wedge \Diamond_e \widehat{B} \in \Gamma$ and thus is consistent.

$\Leftarrow$. Conversely, if $\widehat{A} \wedge \Diamond_e \widehat{B}$ is consistent, then we can extend it to a maximal consistent set Γ. By the Truth Lemma, there is a maximal consistent set Δ such that $\Gamma R^{\mathsf{L}} \Delta$ and $\widehat{B} \in \Delta$. The result follows from the fact that $A = [\Gamma]$ and $B = [\Delta]$. □

We will also use the following property of filtrations of $\mathcal{M}^{\mathsf{L}}$.

Proposition 14 $\mathsf{L} \vdash \bigvee_{A \in S^\Sigma} \widehat{A}$.

Proof. Suppose not, then $\neg \bigvee_{A \in S^\Sigma} \widehat{A}$ is consistent and as such contained in a maximal consistent set Γ. But then for $B = [\Gamma]$, we have $\widehat{B} \in \Gamma$ and $\neg\widehat{B} \in \Gamma$, a contradiction. □

Definition 15 The *FL-closure* of a set Φ of formulas is the least $\Psi \supseteq \Phi$ such that:

(i) If $\neg\varphi \in \Psi$, then $\varphi \in \Psi$;
(ii) If $\neg\varphi \vee \psi \in \Psi$, then $\varphi, \psi \in \Psi$;
(iii) If $\Diamond\varphi \in \Psi$, then $\varphi \in \Psi$;
(iv) If $\diamond\!\!\!\!*\,\varphi \in \Psi$, then $\varphi \vee \Diamond\diamond\!\!\!\!*\,\varphi \in \Psi$.

Let $\sim : \mathsf{mML} \to \mathsf{mML}$ be the function that sends each φ to $\neg\varphi$ and $\neg\varphi$ to φ.

Definition 16 The *$\neg$FL-closure* of a set is the $\sim$-closure of its FL-closure.

We write FL(Φ) for the FL-closure of Φ and $\neg$FL(Φ) for its $\neg$FL-closure. Note that FL($\neg$FL(Φ)) = $\neg$FL(Φ), *i.e.* $\neg$FL(Φ) is *FL-closed.* It is a well-known fact that the $\neg$FL-closure of a finite set of formulas is finite.

The following theorem repackages the key elements of a common completeness proof for PDL (originally from [3]) into a statement about filtrations of the canonical model.

Theorem 17 *Let Σ be a finite $\neg$FL-closed set of formulas and $(S^\Sigma, R^\Sigma, V^\Sigma)$ a $\Diamond$-filtration of $\mathcal{M}^{\mathsf{L}}$. Then $(S^\Sigma, R^\Sigma, (R^\Sigma)^*, V^\Sigma)$ is a $\{\Diamond, \diamond\!\!\!\!*\}$-filtration of $\mathcal{M}^{\mathsf{L}}$.*

Proof. We must show that $R_*^{\mathsf{min}} \subseteq (R^\Sigma)^* \subseteq R_*^{\mathsf{max}}$.

For the first inclusion, suppose that $AR_*^{\mathsf{min}}B$. By Proposition 13, we have that $\widehat{A} \wedge \diamond\!\!\!\!*\,\widehat{B}$ is consistent. We claim that there is a chain

$$A = C_0 R^{\mathsf{min}} \cdots R^{\mathsf{min}} C_n = B,$$

from which the required inclusion follows.

Let $\mathcal{D}$ be the set of equivalence classes for which there is such a chain. We will show that $B \in \mathcal{D}$. To that end, let $\delta = \bigvee_{D \in \mathcal{D}} \widehat{D}$. We claim that $\delta \wedge \Diamond\neg\delta$ is inconsistent. Suppose not, then there must be some $E \notin \mathcal{D}$ such that $\delta \wedge \Diamond\widehat{E}$ is consistent. Indeed, otherwise we would have

$$\mathsf{L} \vdash (\delta \wedge \Diamond\neg\delta) \to \Diamond\neg \bigvee_{A \in S^\Sigma} \widehat{A},$$

which contradicts Proposition 14. From the consistency of $\delta \wedge \Diamond\widehat{E}$ it follows that also $\widehat{D} \wedge \Diamond\widehat{E}$ is consistent for some $D \in \mathcal{D}$. But then $DR^{\mathsf{min}}E$, which is again a contradiction by the fact that $E \notin \mathcal{D}$.

Thus $\mathsf{L} \vdash \delta \to \Box\delta$ and by pos it follows that $\mathsf{L} \vdash \delta \to \boxtimes\delta$. Since $\widehat{A}$ is one of the disjuncts of δ, we find $\mathsf{L} \vdash \widehat{A} \to \boxtimes\delta$. It follows that $\widehat{A} \wedge \diamond\!\!\!\!*\,(\widehat{B} \wedge \delta)$ is consistent.

Therefore we have that $\widehat{A} \wedge \diamondplus(\widehat{B} \wedge \widehat{D})$ is consistent for some $D \in \mathcal{D}$. Since Σ is $\neg$-closed this must mean $\widehat{B} = \widehat{D}$.

For the second inclusion, suppose that there is a chain

$$A = C_0 R^{\mathsf{max}} \cdots R^{\mathsf{max}} C_n = B,$$

and $\diamondplus\varphi \in \Sigma$. Pick maximal consistent sets Δ_i such that $[\Delta_i] = C_i$ for each $0 \leq i \leq n$. We must show that if $\Delta_n \Vdash \varphi$, then $\Delta_0 \Vdash \diamondplus\varphi$. Since $Id \subseteq R^{\mathsf{L}}$ we have $\Delta_n \Vdash \diamondplus\varphi$. By the FL-closure of Σ and the fact that $C_{n-1} R^{\mathsf{max}} C_n$ it follows that $\Delta_{n-1} \Vdash \Diamond\diamondplus\varphi$. But from the fact that $R^{\mathsf{L}}; R^{\mathsf{L}}_* \subseteq R^{\mathsf{L}}_*$ we can again derive that $\Delta_{n-1} \Vdash \diamondplus\varphi$. Continuing in this way we eventually find $\Delta_0 \Vdash \diamondplus\varphi$, as required. □

We can use the results of this section so far to prove some general properties of the canonical model of mK, *i.e.* for the logic L obtained by setting $\Theta = \emptyset$.

Lemma 18 *For a given filtration $\mathcal{M}^f$ of the canonical model $\mathcal{M}^{\mathsf{mK}}$, we have:*

(i) $R^{\mathsf{min}} = R^{\mathsf{max}}$*;*
(ii) not generally $R^{\mathsf{min}}_* = R^{\mathsf{max}}_*$*;*
(iii) $\mathcal{M}^f$ *need not be regular.*

Proof. (i). Since $(S^\Sigma, R^{\mathsf{max}}, V^\Sigma)$ is a $\Diamond$-filtration, it follows from Theorem 17 that the model $(S^\Sigma, R^{\mathsf{max}}, (R^{\mathsf{max}})^*, V^\Sigma)$ is a $\{\Diamond, \diamondplus\}$-filtration of $\mathcal{M}^{\mathsf{mK}}$. But since this model is regular, it is a model of mK and thus, by Proposition 12, it is the minimal filtration. It follows that $R^{\mathsf{min}} = R^{\mathsf{max}}$.

(ii). Since always $R^{\mathsf{min}}_* \subseteq R^{\mathsf{max}}_*$, we must invalidate $R^{\mathsf{max}}_* \subseteq R^{\mathsf{min}}_*$. Suppose we filtrate through $\Sigma = \neg\mathrm{FL}(\{\diamondplus p\})$, *i.e.*

$$\Sigma = \{\diamondplus p, p \vee \Diamond\diamondplus p, p, \Diamond\diamondplus p, \neg\diamondplus p, \neg(p \vee \Diamond\diamondplus p), \neg p, \neg\Diamond\diamondplus p\}.$$

We consider maximal consistent sets $\Gamma \supseteq \{\boxplus p\}$ and $\Delta \supseteq \{\neg p\}$. Since $p \notin \Delta$, we vacuously have $[\Gamma] R^{\mathsf{max}}_* [\Delta]$. However, it does not hold that $[\Gamma] R^{\mathsf{min}}_* [\Delta]$ since $\boxplus p \wedge \diamondplus\neg p$ is inconsistent.

(iii). By the same argument as for (i) we have that $(R^{\mathsf{min}})^* = R^{\mathsf{min}}_*$. It then follows from (ii) that the filtration $(S^\Sigma, R^{\mathsf{min}}, R^{\mathsf{max}}_*, V^\Sigma)$ need not be regular. □

We end this section with a definition that will be useful in the following.

Definition 19 Let Γ be a set of ε-formulas. We say that Γ *admits filtration* if for every model $\mathbb{S} = (S, (R_e)_{e\in\varepsilon}, V)$ such that $(S, (R_e)_{e\in\varepsilon}) \models \Gamma$ and finite subformula closed set Σ, there is a filtration $(S^f, (R^f_e)_{e\in\varepsilon}, V^f)$ of $\mathbb{S}$ through Σ such that $(S^f, (R_e)_{e\in\varepsilon}) \models \Gamma$.

Example 20 Since any filtration of a reflexive model is reflexive, the axioms $p \to \Diamond p$ and $p \to \diamondplus p$ admit filtration. The following extensions of K are among those known to admit filtration: T, D, 4, B, S5, D4, S4, K4.2, K4.3, S4.2, S4.3.

6. Basic modal Θ

In this section we suppose that the set Θ of additional axioms for L consists of only basic modal formulas.

Theorem 21 *If Θ is basic modal, canonical and admits filtration, then* L *is complete wrt* RegFr(Θ).

Proof. Suppose $\mathsf{L} \not\vdash \varphi$. Then $\{\neg\varphi\}$ is consistent and can be extended to a maximal consistent set Δ. Let $\Sigma = \neg\mathrm{FL}(\{\neg\varphi\})$. Since Θ admits filtration, there is a $\Diamond$-filtration $\mathcal{M}^\Sigma$ of $\mathcal{M}^\mathsf{L}$ through Σ such that $\mathcal{M}^\Sigma$ is based on a Θ-frame. By assumption Θ does not contain $\Diamond\!\!\!\!\ast$, therefore it is also valid on the regular filtration $\mathcal{M}^r$ of Theorem 17. It follows from soundness that $\mathcal{M}^r \models \mathsf{L}$ and by the Filtration Theorem that $\mathcal{M}^r, [\Delta] \Vdash \neg\varphi$. Thus $\mathsf{L} \not\models \varphi$, as required. □

Remark 22 By Propositon 12 the filtration $\mathcal{M}^r$ in the proof of Theorem 21 is in fact the minimal filtration.

In particular it follows that $\mathsf{mK} \oplus \mathsf{L_b}$ is complete with respect to $\mathsf{RegFr}(\mathsf{L_b})$ for all logics $\mathsf{L_b}$ mentioned in Example 20.

7. General Θ

In this exploratory section we consider a general $\Theta \subseteq \mathsf{mML}$ as set of extra axioms. We do not yet have a general completeness result in this setting, but make a few observations that could help towards that goal.

Lemma 23 *For any filtration of $\mathcal{M}^\mathsf{L}$ through a $\neg$FL-closed set, we have:*

$$(R^{\mathsf{min}})^* = (R_*^{\mathsf{min}})^*.$$

Proof. $\subseteq$. By the monotonicity of the reflexive-transitive closure operation, it suffices to show that $R^{\mathsf{min}} \subseteq R_*^{\mathsf{min}}$. This follows from Proposition 13 and the fact that $\mathsf{L} \vdash \Diamond\varphi \to \Diamond\!\!\!\!\ast\,\varphi$.

$\supseteq$. Again by general properties of the reflexive-transitive closure, it suffices to show that $R_*^{\mathsf{min}} \subseteq (R^{\mathsf{min}})^*$. This is simply one part of Theorem 17. □

Note that the direction of Theorem 17 used in the above proof of Lemma 23 does not require the filter to be FL-closed. This give us more room to experiment with different filters. Since a bigger filter makes the filtration inherit more properties from the original model, we could hope to always be able to find a filter as in part (iii) of the hypothesis of the following proposition.

Proposition 24 *Let Θ be a set of additional axioms such that:*

(i) Θ is canonical;
(ii) Θ is preserved by the minimal filtration of the canonical model;
(iii) any finite filter can be extended to a finite filter through which the minimal filtration R_*^{min} preserves the transitivity of R_*.

Then L is complete with respect to $\mathsf{RegFr}(\mathsf{L})$.

Proof. As in the proof of Theorem 21, we take some $\mathsf{L} \not\vdash \varphi$ and extend $\{\neg\varphi\}$ to a maximal consistent set Δ. Now let $\Sigma \supseteq \{\varphi\}$ as given by part (iii) of the hypothesis. By parts (i) and (ii), the minimal filtration through Σ:

$$\mathcal{M}^f := (S^\Sigma, R^{\mathsf{min}}, R_*^{\mathsf{min}}, V^\Sigma),$$

is based on a Θ frame. Since R_*^{min} is transitive and every filtration preserves reflexivity, we have $R_*^{\mathsf{min}} = (R_*^{\mathsf{min}})^*$. From Lemma 23 we deduce that $\mathcal{M}^f$ is based on a regular frame, which by soundness becomes an L-frame. The rest of the proof proceeds exactly as that of Theorem 21. □

Example 25 Let $\Theta = \{p \to \boxast\diamondast p\}$. This is canonical for the symmetry of R_*. Note that this does not in general mean that R is symmetric. Since this property is preserved by the minimal filtration, it satisfies part (i) and (ii) of the hypothesis of Theorem 24. It is unknown to the authors whether it also satisfies part (iii).

Remark 26 In a sense the hypothesis of Proposition 24 is not only sufficient but also necessary for proving completeness using filtration. Indeed, by Proposition 12 any filtration of $\mathcal{M}^{\mathsf{L}}$ based on an L-frame is the minimal filtration.

Remark 27 Possible options for extending a filter Σ in order to satisfy part (iii) are:

(i) closing under conjunction, such that each state of the filtration is named by a formula *that is in the filter*;
(ii) closing under (definable) $\diamondast$-modalities, of which there are up to equivalence only finitely many (since the $\diamondast$-fragment of L extend $\mathsf{S4}$).

Both of these extensions may very well result in an infinite filter, but it should always remain finitely based, *i.e.* finite up to L-equivalence.

8. Conclusion

We have provided a set of sufficient conditions for the completeness of extensions of modal logic with the master modality generated by additional basic modal axioms. When the additional axioms are allowed to contain the master modality the proof breaks down, but we have given some possible directions for enlarging the scope of our method.

Another avenue for further research is to try to extend these results to more complex fragments of the modal μ-calculus, such as PDL and CTL.

References

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