An Algebraic Approach to Inquisitive and DNA-Logics*

Nick Bezhanishvili, Gianluca Grilletti, Davide Emilio Quadrellaro

Abstract

This article provides an algebraic study of the propositional system InqB of inquisitive logic. We also investigate the wider class of DNA-logics, which are negative variants of intermediate logics, and the corresponding algebraic structures, DNA-varieties. We prove that the lattice of DNA-logics is dually isomorphic to the lattice of DNA-varieties. We characterise maximal and minimal intermediate logics with the same negative variant, and we prove a suitable version of Tarski's and Birkhoff's classic variety theorems. We also introduce finite DNA-varieties and show that these varieties are axiomatised by the analogues of Jankov formulas. Finally, we prove that the lattice of extensions of InqB is dually isomorphic to the ordinal $\omega + 1$ and give an axiomatisation of these logics via Jankov DNA-formulas. This shows that these extensions coincide with the so-called inquisitive hierarchy of Ciardelli (2009).[1]

1 Introduction

Inquisitive logic was introduced a decade ago as a formal framework to analyse questions. More specifically, inquisitive semantics originates from the so-called "partition semantics" of Groenendijk and Stokhof [24, 26] and was formally developed by Ciardelli, Groenendijk and Roelofsen in [10–12, 14, 25]. In the last decade inquisitive semantics has been widely studied both from the linguistics point of view as well as from the perspective of logic. In particular, inquisitive propositional logic InqB has been thoroughly investigated in [10, 20, 37–39]. The recent textbook [13] gives the state of the art in the field.

It is only recently, however, that an algebraic approach to inquisitive logic has been developed. In [5] algebraic and topological semantics for InqB are introduced and investigated (see also [41] for a different algebraic approach to inquisitive logic). Algebraic semantics plays a crucial role in the study of intermediate, modal and other non-classical logics [6, 8, 19, 21]. A development of an algebraic semantics for inquisitive logic is thus an important milestone for better understanding the mathematics behind inquisitive semantics.

In this article we continue the study started in [5] and develop a full algebraic apparatus for inquisitive logic InqB and related systems. Using this machinery we give, among other things, a full description of the lattice of extensions of InqB. Also using this algebraic semantics for InqB we study the relation between inquisitive logic and intermediate logics.

While inquisitive logic is now widely known and recognized, the related class of DNA-logics has not been well investigated yet. In this article we introduce DNA-logics as negative variants of intermediate logics. A DNA-logic Λ is thus a set of propositional formulas such that, for some intermediate logic L, $\varphi \in \Lambda$ if and only if $\varphi[\overline{\neg p}/\overline{p}] \in L$. The name DNA stands for *double negation atoms*, since every DNA-logic Λ proves the formula $\neg\neg p \to p$ for every atomic formula

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[1] This article is based on [40].

$p \in \mathtt{AT}$. The relation between $\mathtt{InqB}$ and negative variants of intermediate logics was already pointed out in [10]. Also [36] establishes several properties of these systems. In this article we provide a systematic study of $\mathtt{DNA}$-logics and we investigate the corresponding classes of Heyting algebras, which we call $\mathtt{DNA}$-varieties.

The original contributions of this article are therefore twofold. On the one hand, we develop a general algebraic semantics for $\mathtt{DNA}$-logics and we prove some fundamental results concerning $\mathtt{DNA}$-logics and $\mathtt{DNA}$-varieties. In particular, we show that the lattice of $\mathtt{DNA}$-logics is dually isomorphic to the lattice of $\mathtt{DNA}$-varieties. We characterise maximal and minimal intermediate logics with the same negative variant, and we prove a suitable version of Tarski's and Birkhoff's classic variety theorems for $\mathtt{DNA}$-varieties. We also introduce locally finite $\mathtt{DNA}$-varieties and show that these varieties are axiomatised by the analogues of Jankov formulas.

On the other hand, we apply this general algebraic setting to inquisitive logic: we study the lattice of extensions of $\mathtt{InqB}$ and show that it forms a countable descending chain with an extra bottom element dually isomorphic to the ordinal $\omega + 1$. We also give an axiomatisation of these logics via the analogues of Jankov formulas. This shows that these extensions coincide with the so-called inquisitive hierarchy considered in [10]. It thus follows from our results that the inquisitive hierarchy comprises all the possible ways in which $\mathtt{InqB}$ can be extended to other $\mathtt{DNA}$-logics.

This article is structured as follows. In Section 2 we recall the preliminary notions about varieties, Heyting algebras, intermediate logics and inquisitive semantics which we will make use of in the course of this article. In Section 3 we introduce $\mathtt{DNA}$-logics and their algebraic semantics, and we prove that the lattice of $\mathtt{DNA}$-logics is dually isomorphic to the lattice of $\mathtt{DNA}$-varieties. In Section 4 we employ this duality result to make the first steps in the study of $\mathtt{DNA}$-logics and $\mathtt{DNA}$-varieties: we characterise maximal and minimal intermediate logics with the same negative variant, we prove a suitable version of Tarski's and Birkhoff's theorems about varieties and we introduce Jankov formulas to axiomatise locally finite $\mathtt{DNA}$-varieties. Finally, in Section 5, we continue the work done in [5] and we use the methods developed in this article to show that the extensions of $\mathtt{InqB}$ form a countable descending chain with an extra bottom element. Finally, we provide an axiomatisation of each of these logics and we show that they coincide with the so-called inquisitive hierarchy considered in [10].

2 Preliminaries

In this section we briefly discuss some of the basic facts that will be used throughout the article. We use [8] as our main references for the basic theory of intermediate logics. We also use [7] for universal algebra, and [17] and [43] for Heyting and Boolean algebras, respectively. Finally, we refer the reader to Ciardelli's original presentation in [10] and [13] for more details about inquisitive semantics and its applications in linguistics.

2.1 Universal Algebra

We write $f : A \twoheadrightarrow B$ if f is a surjective homomorphism between A and B and we say that B is *homomorphic image* of A. We denote by $B \preceq A$ that B is a subalgebra of A and by $\prod_{i\in I} A_i$ the product of the family of algebras $\{A_i\}_{i\in I}$. If I is finite, we also write $A_0 \times ... \times A_n$ for the product $\prod_{i\in I} A_i$. For every $i \in I$ we also have a *projection function* $\pi_i : \prod_{i\in I} A_i \to A_i$ such that $\pi_i : \alpha \mapsto \alpha(i)$. It is easy to show that every such projection function is a surjective homomorphism. We introduce the following closure maps.

Definition 2.1. Let $\mathcal{K}$ be a set of algebras of the same similarity-type, we then define the

following:

$A \in I(\mathcal{K})$ iff A is isomorphic to some algebra in $\mathcal{K}$
$A \in S(\mathcal{K})$ iff A is a subalgebra of some algebra in $\mathcal{K}$
$A \in H(\mathcal{K})$ iff A is homomorphic image of some algebra in $\mathcal{K}$
$A \in P(\mathcal{K})$ iff A is product of a nonempty family of algebras in $\mathcal{K}$.

The following proposition provides a characterisation of how the previous maps interact with one another.

Proposition 2.2. *Let $\mathcal{K}$ be an arbitrary class of algebras, we then have that $SH(\mathcal{K}) \subseteq HS(\mathcal{K})$, $PS(\mathcal{K}) \subseteq SP(\mathcal{K})$ and $PH(\mathcal{K}) \subseteq HP(\mathcal{K})$. Moreover, the operators I, S, H, P are all idempotent, i.e. $I^2(\mathcal{K}) = I(\mathcal{K})$, $S^2(\mathcal{K}) = S(\mathcal{K})$, $H^2(\mathcal{K}) = H(\mathcal{K})$ and $P^2(\mathcal{K}) = P(\mathcal{K})$.*

A *variety* is defined as a class of algebras $\mathcal{V}$ of the same similarity type which is closed under homomorphic images, subalgebras and products. If $\mathcal{K}$ is an arbitrary class of algebras of the same similarity type, then we write $\mathcal{V}(K)$ for the variety generated by $\mathcal{K}$, i.e. for the smallest class of algebras containing $\mathcal{K}$ which is closed under subalgebras, homomorphic images and products. An important theorem by Birkhoff establishes that varieties are exactly the classes of algebras which are definable by equations [7, Thm. 11.9].

Finally, we recall the following important theorems, which provide an internal characterisation of algebraic varieties. The first theorem, due to Tarski, characterizes the variety generated by a set of algebras in terms of the closure maps defined above. The second theorem is an important result by Birkhoff which shows that subdirectly irreducible algebras play an important role as generators of varieties. We use $\mathcal{V}_{SI}$ to denote the collection of subdirectly irreducible algebras of a variety $\mathcal{V}$. For a proof of these results see [7, Thms 9.5 and 9.7].

Theorem 2.3 (Tarski)**.** *Let $\mathcal{K}$ be a class of algebras of some similarity type, we then have that $\mathcal{V}(\mathcal{K}) = HSP(\mathcal{K})$.*

Theorem 2.4 (Birkhoff)**.** *Varieties are generated by their subdirectly irreducible members, i.e. for every variety $\mathcal{V}$, we have $\mathcal{V} = V(\mathcal{V}_{SI})$.*

2.2 Heyting Algebras

A *Heyting algebra* is a bounded lattice H such that for every $a, b \in H$ there is some element $a \to b \in H$ such that for all $c \in H$ we have that $c \leq a \to b \Leftrightarrow c \wedge a \leq b$. Given an element $a \in H$ of a Heyting algebra, we define its *pseudocomplement* $\neg a$ as $\neg a = a \to 0$. In case for all $a \in H$ it is the case that $a \wedge \neg a = 0$ and $a \vee \neg a = 1$ we say that H is a *Boolean algebra*. It is well known that a Heyting algebra H is subdirectly irreducible iff H has a second greatest element s_H.

A *power-set algebra* is a Boolean algebra $B = (\wp(X), \cup, \cap, \setminus, \emptyset, X)$, where the universe is a power-set, the algebraic operations of join and meet are the set-theoretic operations of union and intersection and complementation is the set-theoretic complement. We recall that every finite Boolean algebra B is isomorphic to a power-set algebra, i.e. $B \cong \wp(X)$ for some finite set X, see e.g., [16, Ch. 5]. Thus it follows that finite Boolean algebras are always equivalent up to isomorphism to $\wp(n)$ for some $n \in \mathbb{N}$. Therefore, it is easy to show that if $n \leq m$, then $\wp(n) \preceq \wp(m)$. Then, by identifying every $\wp(n)$ by 2^n, it follows that finite Boolean algebras form an ordered chain of subalgebras:

$$2^0 \preceq 2^1 \preceq 2^2 \preceq 2^3 \preceq 2^4 \preceq \dots$$

2.3 Intermediate Logics

Intermediate logics are a well-studied class of logics with many applications in mathematics and computer science. Fix a countable set $\mathtt{AT}$ of atomic propositional formulas, we define the set of propositional formulas $\mathcal{L}_P$ inductively as follows.

Definition 2.5. The language $\mathcal{L}_P$ is defined as follows, where $p \in \mathtt{AT}$:

$$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi$$

Negation can be defined as $\neg\varphi := \varphi \to \bot$. If φ is a formula, then we write $\varphi(\overline{x})$ or $\varphi(x_0, ..., x_n)$ to specify that the atomic formulas in φ are among those of $\overline{x}$ or respectively of $x_0, ..., x_n$. A *substitution* is a function $\eta : \mathtt{AT} \to \mathcal{L}_P$ which naturally lifts by induction to formulas by setting, for every connective $\odot$, the map $(\psi \odot \chi) \mapsto \eta(\psi) \odot \eta(\chi)$. If φ is a formula and q occurs in φ, we write $\varphi[^p/_q]$ for the formula obtained by the substitution $\eta : q \mapsto p$. Similarly, if $\overline{q} = q_0, ..., q_n$ are variables in φ, then we write $\varphi[^{\overline{p}}/_{\overline{q}}]$ for the formula obtained by the substitution $\eta : q_i \mapsto p_i$ for all $i \leq n$.

We denote by $\mathtt{IPC}$ the *intuitionistic propositional calculus* and by $\mathtt{CPC}$ the *classical propositional calculus*. Now, given a propositional language $\mathcal{L}_P$, we say that a set of formulas $L \subseteq \mathcal{L}_P$ is a *superintuitionistic logic* if $\mathtt{IPC} \subseteq L$ and in addition L is closed under modus ponens and uniform substitution. An *intermediate logic* is a superintuitionistic logic L which is also consistent, namely $\bot \notin L$.

It can be proven that $\mathtt{CPC}$ is the maximal intermediate logic and that intermediate logics are all the logics L such that $\mathtt{IPC} \subseteq L \subseteq \mathtt{CPC}$. We denote by $L + \varphi$ the closure under substitution and modus ponens of the set of formulas $L \cup \{\varphi\}$ and by $L + \Gamma$ the closure under substitution and modus ponens of the set of formulas $L \cup \Gamma$. If L is an intermediate logic and $\varphi \in L$ then we write $\vdash_L \varphi$ or $L \vdash \varphi$. Moreover, if φ can be obtained by closing the set $L \cup \Gamma$ under modus ponens, then we write $\Gamma \vdash_L \varphi$ and we say that φ is *derivable* from Γ in L. It is a well-know fact [8, Ch. 4.1] that intermediate logics form a *bounded lattice* **IL** with $\mathtt{IPC} = \bot$ and $\mathtt{CPC} = \top$ and where meet and join are defined as follows

$$L_0 \wedge L_1 = L_0 \cap L_1$$
$$L_0 \vee L_1 = L_0 + L_1.$$

We list here some intermediate logics that will be useful for us in this article:

$$\begin{aligned}
&\mathtt{KC} = \mathtt{IPC} + \neg p \vee \neg\neg p \\
&\mathtt{KP} = \mathtt{IPC} + (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r) \\
&\mathtt{ND} = \mathtt{IPC} + \{(\neg p \to \textstyle\bigvee_{i \leq k} \neg q_i) \to \bigvee_{i \leq k} (\neg p \to \neg q_i) : k \geq 2\}.
\end{aligned}$$

The logic $\mathtt{ND}$ was introduced by Maksimova in [34]. The logic $\mathtt{KP}$ was introduced by Kreisel and Putnam in [33]. The logic $\mathtt{KC}$ is also know as the *logic of the weak excluded middle* and was introduced by Jankov in [29]. While the previous logics are defined in axiomatic terms, one can also define logics by specifying the class of structures they correspond to. The logic $\mathtt{ML}$ is the logic of so-called Medvedev frames and it was introduced by Medvedev in [35]. A relational structure $\mathcal{F}$ is a *Medvedev frame* if $\mathcal{F} \cong (\wp_0(W), \supseteq)$, where W is a finite set and $\wp_0(W) = \{X \subseteq W : X \neq \emptyset\}$. A *Medvedev model* is then defined as a relational model over a Medvedev frame. Let $\mathcal{C}$ be the class of all Medvedev frames, then we have that $\mathtt{ML} = \{\varphi \in \mathcal{L}_P : \mathcal{C} \Vdash \varphi\}$, i.e. $\mathtt{ML}$ is the set of formulas valid in all Medvedev frames (here we assume the reader's familiarity with the standard Kripke semantics of intuitionistic logic).

We will now briefly recall the algebraic semantics of intermediate logics.

Definition 2.6 (Algebraic Model)**.** An *algebraic model* is a pair $M = (H, V)$ where H is a Heyting algebra and $V : \mathtt{AT} \to H$ is a valuation of propositional atoms over the elements of H.

Given an algebraic model $M = (H, V)$, we define by induction the interpretation of any formula $\varphi \in \mathcal{L}_P$.

Definition 2.7 (Interpretation of Arbitrary Formulas)**.** Given an algebraic model M and a formula $\varphi \in \mathcal{L}$, its *interpretation* $[\![\varphi]\!]^M$ is defined as follows:

$$[\![p]\!]^M = V(p) \qquad [\![\top]\!]^M = 1_H \qquad [\![\bot]\!]^M = 0_H$$
$$[\![\varphi \wedge \psi]\!]^M = [\![\varphi]\!]^M \wedge_H [\![\psi]\!]^M \qquad [\![\varphi \vee \psi]\!]^M = [\![\varphi]\!]^M \vee_H [\![\psi]\!]^M \qquad [\![\varphi \to \psi]\!]^M = [\![\varphi]\!]^M \to_H [\![\psi]\!]^M$$

When the valuation V is clear from the context, we simply write $[\![\varphi]\!]^H$ for the interpretation of φ in H under V. We say that a formula φ is *true under V in H* or *true in the model $M = (H, V)$* and write $M \vDash \varphi$ if $[\![\varphi]\!]^M = 1$. We say that φ is *valid in H* and write $H \vDash \varphi$ if φ is true in every algebraic model $M = (H, V)$ over H. Given a class of Heyting algebras $\mathcal{C}$, we say that φ is *valid in $\mathcal{C}$* and write $\mathcal{C} \vDash \varphi$ if φ is valid in every Heyting algebra $H \in \mathcal{C}$. Finally, we say that φ is a *validity* if φ is valid in any Heyting algebra H.

Let **HA** be the lattice of Heyting algebras and **IL** the lattice of intermediate logics, we then define the two maps $Var : \mathbf{IL} \to \mathbf{HA}$ and $Log : \mathbf{HA} \to \mathbf{IL}$ as follows:

$$Var : L \mapsto \{H \in \mathbf{HA} : H \vDash L\};$$
$$Log : \mathcal{V} \mapsto \{\varphi \in \mathcal{L}_P : \mathcal{V} \vDash \varphi\}.$$

That the two former functions are well defined follows from $Var(L)$ being a variety of Heyting algebras and $Log(\mathcal{V})$ being an intermediate logic. Also, one can prove that both these maps are order-reversing homomorphisms. We say that a variety of Heyting algebras $\mathcal{V}$ is *defined* by a set of formulas Γ if $\mathcal{V} = Var(\Gamma)$ and we say $\mathcal{V}$ is definable if there exists one such Γ. We say that an intermediate logic L is *algebraically complete* with respect to a class of Heyting algebras $\mathcal{C}$ if $L = Log(\mathcal{C})$.

Theorem 2.8 (Definability Theorem)**.** *Every variety of Heyting algebras $\mathcal{V}$ is defined by its validities, i.e. for every Heyting algebra H,*

$$H \in \mathcal{V} \Leftrightarrow H \vDash Log(\mathcal{V}).$$

Theorem 2.9 (Algebraic Completeness)**.** *Every intermediate logic L is complete with respect to its corresponding variety of Heyting algebras, i.e. for every $\varphi \in \mathcal{L}_P$,*

$$\varphi \in L \Leftrightarrow Var(L) \vDash \varphi.$$

We refer the reader to [8, Sec. 7] for a full proof of the aforementioned two results and the related constructions. Here let us only remark that the first of these two results is an immediate application of the fact that varieties and equational classes coincide. The second result relies essentially on the free-algebra construction, namely on the Lindenbaum-Tarski algebra of intermediate logics. These results together give us the following theorem.

Theorem 2.10 (Dual Isomorphism)**.** *The lattice of intermediate logics is dually isomorphic to the lattice of varieties of Heyting algebras, i.e.* $\mathbf{IL} \cong^{op} \mathbf{HA}$.

Here, the isomorphisms between **IL** and **HA** are the two maps Log and Var. We sometimes refer to the previous theorem as a duality result concerning **IL** and **HA**. Notice that we are implicitly excluding from the lattice **HA** the trivial variety generated by a singleton set, for it dually corresponds to the inconsistent logic containing all the formulas of $\mathcal{L}_P$.

2.4 Inquisitive Semantics

In this section we recall the definition of inquisitive logic and its standard semantics. Though sometimes inquisitive logic is introduced in a signature consisting of two different disjunctions, here we follow [10] and present $\mathtt{InqB}$ in the same language $\mathcal{L}_P$ of intermediate logics. Inquisitive logic is defined as the logic of all evaluation states. Given a set of atomic formulas in $\mathcal{L}_P$, a *classical valuation* (or simply *valuation*) is a function $w : \mathtt{AT} \to \{0, 1\}$. When the set $\mathtt{AT}$ is fixed, we refer to the set $2^{\mathtt{AT}}$ of all classical valuations over $\mathtt{AT}$ as the *evaluation space* over $\mathtt{AT}$. An *evaluation state* (or simply *state*) is a set s of valuations $s \in \wp(2^{\mathtt{AT}})$. We introduce as follows the notion of support in a state.

Definition 2.11 (Support at a State)**.** Let φ be a formula of $\mathcal{L}_P$ and $s \in \wp(2^{\mathtt{AT}})$ a state. We say that s *supports* φ and we define $s \vDash \varphi$ inductively as follows:

$$\begin{array}{lcl} s \vDash p & \iff & \forall w \in s\ (w(p) = 1) \\ s \vDash \top & \iff & s \subseteq 2^{\mathtt{AT}} \\ s \vDash \bot & \iff & s = \emptyset \\ s \vDash \psi \wedge \chi & \iff & s \vDash \psi \text{ and } s \vDash \chi \\ s \vDash \psi \vee \chi & \iff & s \vDash \psi \text{ or } s \vDash \chi \\ s \vDash \psi \to \chi & \iff & \forall t\ (\text{if } t \subseteq s \text{ and } t \vDash \psi \text{ then } t \vDash \chi). \end{array}$$

For $p \in \mathtt{AT}$ and a state s, we introduce the notation $[\![p]\!]^s = \{w \in s : w(p) = 1\}$, that is, $[\![p]\!]^s$ is the set of classical valuations in s that make p true. Since $\neg\varphi = \varphi \to \bot$, the semantic clause of negation is then the following:

$$s \vDash \neg\varphi \text{ iff } \forall t\ (\text{if } t \subseteq s \text{ then } t \nvDash \varphi).$$

The system of inquisitive logic $\mathtt{InqB}$ is then defined semantically as follows.

Definition 2.12 (Inquisitive Logic)**.** The valid formulas of *inquisitive logic* $\mathtt{InqB}$ are the formulas $\varphi \in \mathcal{L}_P$ which are supported in every evaluation state:

$$\mathtt{InqB} = \{\varphi \in \mathcal{L}_P : \forall s \in \wp(2^{\mathtt{AT}}), s \vDash \varphi\}.$$

Inquisitive logic can thus be seen as the logic of all evaluation states.

3 DNA-Logics and their Algebraic Semantics

In this section we introduce the class of DNA-logics and we show that these logics are complete with respect to DNA-varieties, a suitably defined class of Heyting algebras.

3.1 DNA-Logics

We proceed by introducing the negative variant of an intermediate logic. Negative variants were first introduced by Miglioli et al. in [36] and later employed by Ciardelli in [10]. If $\varphi \in \mathcal{L}_P$ is an arbitrary formula, we often say that the formula $\varphi[\overline{\neg p}/\overline{p}]$ obtained by replacing all the atomic letters in φ with their negation is its *negative variant*.

Definition 3.1 (Negative Variant)**.** For every intermediate logic L, its *negative variant* $L^{\neg}$ is defined as follows:

$$L^{\neg} = \{\varphi \in \mathcal{L}_P : \varphi[\overline{\neg p}/\overline{p}] \in L\}.$$

A DNA-*logic* is then defined as the negative variant of some intermediate logic L. The name DNA stands for *double negation atoms*, which refers to the fact that, as we shall see, DNA-logics prove $\neg\neg p \to p$ for all atomic formulas $p \in \mathtt{AT}$. We will use the notation $L^\neg$ to refer to the negative variant of an intermediate logic L. If not specified otherwise, we reserve uppercase greek letters Γ and Δ to denote arbitrary sets of formulas and Λ and Π to denote DNA-logics. The following proposition provides us with an axiomatisation for every DNA-logic.

Proposition 3.2. *Let Λ be a* DNA*-logic and L an intermediate logic with $\Lambda = L^\neg$. Then Λ is the least set of formulas such that:*

1. *$L \subseteq \Lambda$;*
2. *For all atomic propositional formulas $p \in \mathtt{AT}$ we have that $\neg\neg p \to p \in \Lambda$;*
3. *Λ is closed under the modus ponens rule: if $\varphi \in \Lambda$ and $\varphi \to \psi \in \Lambda$, then $\psi \in \Lambda$.*

Proof. It is trivial to show that Λ satisfies the three conditions; what remains to prove is that Λ is the least such set. Suppose X also validates the three conditions above, we need to show that $\Lambda \subseteq X$. Consider any $\varphi \in \Lambda = L^\neg$, then by the definition of negative variant, $\varphi[\overline{\neg p}/\overline{p}] \in L$. Therefore, by uniform substitution, $\varphi[\overline{\neg\neg p}/\overline{p}] \in L$ and therefore since $L \subseteq X$ also $\varphi[\overline{\neg\neg p}/\overline{p}] \in X$. Finally, since for every $p \in \mathtt{AT}$, $\neg\neg p \to p \in X$, it follows that $\varphi[\overline{\neg\neg p}/\overline{p}] \to \varphi \in X$. Given that X is closed under modus ponens we obtain that $\varphi \in X$. □

DNA-logics give rise to a lattice structure ordered by the set-theoretic inclusion. The meet of two DNA-logics Λ_0, Λ_1 is just their intersection and their join is the closure of their union under modus ponens. We will thus write $\Lambda_0 \wedge \Lambda_1 := \Lambda_0 \cap \Lambda_1$ and $\Lambda_0 \vee \Lambda_1 := (\Lambda_0 \cup \Lambda_1)^{MP}$, where we denote by $(\Gamma)^{MP}$ the closure under modus ponens of a set Γ of formulas. If φ can be obtained by closing the set Γ of formulas under modus ponens, then we have $\Gamma \vdash \varphi$, i.e. φ is *derivable* from Γ. We prove the following proposition.

Proposition 3.3. *Let Λ_0 and Λ_1 be two* DNA*-logics, then: (i) $\Lambda_0 \wedge \Lambda_1$ is a* DNA*-logic and it is the infimum of Λ_0 and Λ_1; (ii) $\Lambda_0 \vee \Lambda_1$ is a* DNA*-logic and it is the supremum of Λ_0 and Λ_1.*

Proof. Assume without the loss of generality that $\Lambda_0 = L_0^\neg$ and $\Lambda_1 = L_1^\neg$. It is immediate that $\Lambda_0 \wedge \Lambda_1 = \Lambda_0 \cap \Lambda_1$ is the least set of formulas satisfying the conditions in Proposition 3.2 with respect to the intermediate logic $L_0 \cap L_1$; and that $\Lambda_0 \vee \Lambda_1 = (\Lambda_0 \cup \Lambda_1)^{MP}$ is the least set of formulas satisfying the conditions in Proposition 3.2 with respect to the intermediate logic $L_0 \vee L_1 := (L_0 \cup L_1)^{MP}$. □

We denote by **DNAL** the lattice of DNA-logics. Since intermediate logics also form a lattice **IL**, we can then show that the map $(-)^\neg : \mathbf{IL} \to \mathbf{DNAL}$ which assigns each intermediate logic to its negative variant is a lattice homomorphism.

Proposition 3.4. *The map $(-)^\neg : \mathbf{IL} \to \mathbf{DNAL}$ is a bounded lattice homomorphism.*

Proof. Obviously $(-)^\neg$ sends $\bot_{\mathbf{IL}}$ to $\bot_{\mathbf{DNAL}}$ and $\top_{\mathbf{IL}}$ to $\top_{\mathbf{DNAL}}$, so it suffices to check that $(-)^\neg$ preserves meet and join.

(i) Consider two intermediate logics L_0 and L_1, then it is straightforward that:

$$\begin{aligned}(L_0 \wedge L_1)^\neg &= (L_0 \cap L_1)^\neg \\ &= \{\varphi \in \mathcal{L}_P : \varphi[\overline{\neg p}/\overline{p}] \in L_0 \cap L_1\} \\ &= \{\varphi \in \mathcal{L}_P : \varphi[\overline{\neg p}/\overline{p}] \in L_0\} \cap \{\varphi \in \mathcal{L}_P : \varphi[\overline{\neg p}/\overline{p}] \in L_1\} \\ &= L_0^\neg \cap L_1^\neg\end{aligned}$$

$$= L_0^{\neg} \wedge L_1^{\neg}.$$

which shows that $(-)^{\neg}$ preserves the meet operator.

(ii) Consider two intermediate logics L_0 and L_1. We have by definition that $(L_0 \vee L_1)^{\neg} = ((L_0 \cup L_1)^{MP})^{\neg}$ and $L_0^{\neg} \vee L_1^{\neg} = (L_0^{\neg} \cup L_1^{\neg})^{MP}$. It suffices to show that $((L_0 \cup L_1)^{MP})^{\neg} = (L_0^{\neg} \cup L_1^{\neg})^{MP}$. ($\subseteq$) Suppose $\varphi \in ((L_0 \cup L_1)^{MP})^{\neg}$, then it follows that $\varphi[\overline{\neg p}/\overline{p}] \in (L_0 \cup L_1)^{MP}$, hence for some formulas $\psi_0, ..., \psi_n \in L_0 \cup L_1$ we have $\psi_0, ..., \psi_n \vdash \varphi[\overline{\neg p}/\overline{p}]$. We immediately obtain that $\psi_0[\overline{\neg p}/\overline{p}], ..., \psi_n[\overline{\neg p}/\overline{p}] \vdash \varphi[\overline{\neg\neg p}/\overline{p}]$. So, since for every $p \in \mathtt{AT}$ we have $\neg\neg p \to p \in L_0^{\neg}, L_1^{\neg}$ it follows that $\psi_0[\overline{\neg p}/\overline{p}], ..., \psi_n[\overline{\neg p}/\overline{p}] \vdash \varphi$ and hence $\varphi \in (L_0^{\neg} \cup L_1^{\neg})^{MP}$. ($\supseteq$) Suppose $\varphi \in (L_0^{\neg} \cup L_1^{\neg})^{MP}$, then it follows that for some formulas $\psi_0, ..., \psi_n \in L_0^{\neg} \cup L_1^{\neg}$ we have that $\psi_0, ..., \psi_n \vdash \varphi$, that is, there is a derivation of φ from $\psi_0, ..., \psi_n$. Therefore, $\psi_0[\overline{\neg p}/\overline{p}], ..., \psi_n[\overline{\neg p}/\overline{p}] \in L_0 \cup L_1$ and by substituting each ψ_i with $\psi_i[\overline{\neg p}/\overline{p}]$ in the previous derivation, we obtain $\psi_0[\overline{\neg p}/\overline{p}], ..., \psi_n[\overline{\neg p}/\overline{p}] \vdash \varphi[\overline{\neg p}/\overline{p}]$. So $\varphi[\overline{\neg p}/\overline{p}] \in (L_0 \cup L_1)^{MP}$ and consequently $\varphi \in ((L_0 \cup L_1)^{MP})^{\neg}$. Thus, $(-)^{\neg}$ also preserves the join operator and is a lattice homomorphism. □

3.2 Algebraic Semantics for DNA-Logics

In the existing literature, negative variants have been considered from a syntactic point of view [10, 36]. An algebraic semantics for inquisitive logic was introduced in [5]. Here we extend this algebraic approach to DNA-logics.

Recall that, if H is a Heyting algebra, then we say that an element $x \in H$ is *regular* if $x = \neg\neg x$. For any Heyting algebra H we then denote by $H_\neg$ the set:

$$H_\neg = \{x \in H : x = \neg\neg x\}.$$

So $H_\neg$ consists of all regular elements of the Heyting algebra H. Note that since in every Heyting algebras we have that $\neg x = \neg\neg\neg x$, the set of regular elements of H can also be specified as $H_\neg = \{y \in H : \exists x \in H(y = \neg x)\}$. We define DNA-models as follows.

Definition 3.5 (DNA-Model)**.** A DNA-*model* is a pair $M = (H, \mu)$ where H is a Heyting algebra and $\mu : \mathtt{AT} \to H_\neg$ is a valuation of propositional atoms over the regular elements of H.

We then say that μ is a DNA-valuation over the Heyting algebra H. Given a DNA-model $M = (H, \mu)$, we define by induction the interpretation of any formula $\varphi \in \mathcal{L}_P$.

Definition 3.6 (Interpretation of Arbitrary Formulas)**.** Given a DNA-model $M = (H, \mu)$ and a formula $\varphi \in \mathcal{L}_P$, its *interpretation* $[\![\varphi]\!]^M$ is defined by the following recursive clauses:

$$[\![p]\!]^M = \mu(p) \qquad [\![\top]\!]^M = 1_H \qquad [\![\bot]\!]^M = 0_H$$

$$[\![\varphi \wedge \psi]\!]^M = [\![\varphi]\!]^M \wedge_H [\![\psi]\!]^M \qquad [\![\varphi \vee \psi]\!]^M = [\![\varphi]\!]^M \vee_H [\![\psi]\!]^M \qquad [\![\varphi \to \psi]\!]^M = [\![\varphi]\!]^M \to_H [\![\psi]\!]^M$$

When the valuation μ is clear from the context, we simply write $[\![\varphi]\!]^H$ for the interpretation of φ in H under μ. From the former definitions it is straightforward to adapt the usual definitions of truth at a model and validity. We say that a formula φ is *true under* μ *in* H or *true in the model* $M = (H, \mu)$ and write $M \vDash^{\neg} \varphi$ if $[\![\varphi]\!]^M = 1$. We say that φ is DNA-*valid in* H and write $H \vDash^{\neg} \varphi$ if φ is true in every model $M = (H, \mu)$ over H. Given a class $\mathcal{C}$ of Heyting algebras, we say that φ is DNA-*valid in* $\mathcal{C}$ and write $\mathcal{C} \vDash^{\neg} \varphi$ if φ is DNA-valid in every Heyting algebra $H \in \mathcal{C}$. Finally, we say that φ is a DNA-*validity* if φ is valid in any Heyting algebra H. When the context is clear, we drop the qualification DNA from the definitions above and talk simply of *validity*.

DNA-validity and (standard) validity are closely intertwined. To see how, we first introduce the notion of negative variant of a valuation.

Definition 3.7 (Negative Variant of a Valuation)**.** Let H be a Heyting algebra and V an arbitrary valuation over H. Then we say that $V^\neg$ is the *negative variant* of V if for all $p \in \mathtt{AT}$ we have that $V^\neg(p) = \neg V(p)$.

The following lemma shows that the set of DNA-valuations and the set of negative variants of standard valuations coincide.

Lemma 3.8. *A valuation μ is a* DNA*-valuation if and only if it is the negative variant of some valuation V.*

By the previous lemma, a generic DNA-valuation is always of the form $V^\neg$ for some valuation V. We will henceforth write $V^\neg$ for an arbitrary DNA-valuation. We can now prove the following important lemma.

Lemma 3.9. *For every Heyting algebra H, for every valuation V and any formula φ, we have*

$$[\![\varphi]\!]^{(H,V^\neg)} = [\![\varphi[\overline{\neg p}/\overline{p}]]\!]^{(H,V)}.$$

Proof. The prove goes by induction on the complexity of φ. The only non-trivial case is $\varphi = p \in \mathtt{AT}$:

$$[\![p]\!]^{(H,V^\neg)} = V^\neg(p) = \neg V(p) = \neg [\![p]\!]^{(H,V)} = [\![\neg p]\!]^{(H,V)}.$$

□

From this we can derive the following result.

Proposition 3.10. *For any Heyting algebra H we have $H \vDash^\neg \varphi$ iff $H \vDash \varphi[\overline{\neg p}/\overline{p}]$.*

Proof. We prove both directions by contraposition. ($\Rightarrow$) Suppose $[\![\varphi[\overline{\neg p}/\overline{p}]]\!]^{(H,V)} \neq 1$ for some valuation V. Then, given $V^\neg$ the negative variant of V, it follows by Lemma 3.9 that $[\![\varphi]\!]^{(H,V^\neg)} \neq 1$. ($\Leftarrow$) Suppose $[\![\varphi]\!]^{(H,V^\neg)} \neq 1$ for some DNA-valuation $V^\neg$. By Lemma 3.8, there exists a valuation V whose negative variant is V. Then, by Lemma 3.9, we have that $[\![\varphi]\!]^{(H,V)} \neq 1$. □

Thus we end up with the following proposition: if a Heyting algebra validates an intermediate logic, then it also validates its negative variant.

Corollary 3.11. *Let H be a Heyting algebra and L an intermediate logic. Then we have that $H \vDash L$ entails $H \vDash^\neg L^\neg$*

Notice that the converse of the previous proposition does not hold in general. DNA-valuations form a subclass of all valuation and it might very well be that a formula is true in a Heyting algebra under all DNA-valuations but not under all valuations. However, the next proposition is a weaker version of it which we will need later. Let $\langle H_\neg \rangle$ be the subalgebra of H generated by $H_\neg$. First we prove the following lemma.

Lemma 3.12. *For any Heyting algebra H we have that $H \vDash^\neg \varphi$ iff $\langle H_\neg \rangle \vDash^\neg \varphi$.*

Proof. Clearly $H_\neg = \langle H_\neg \rangle_\neg$. So we have that $V^\neg$ is a DNA-valuation over H iff it is a DNA-valuation over $\langle H_\neg \rangle$. Since $\langle H_\neg \rangle$ is a subalgebra of H, it readily follows that $H \vDash^\neg \varphi$ iff $\langle H_\neg \rangle \vDash^\neg \varphi$. □

This allows us to prove the following result.

Proposition 3.13. *Let H be a Heyting algebra and L an intermediate logic. Then we have that $H \vDash^{\neg} L^{\neg}$ entails $\langle H_{\neg}\rangle \vDash L$.*

Proof. Consider any Heyting algebra H, and suppose that $\langle H_{\neg}\rangle \nvDash L$, then there is some formula $\varphi \in L$ and some valuation V such that $(\langle H_{\neg}\rangle, V) \nvDash \varphi$. Now, since $\langle H_{\neg}\rangle$ is the subalgebra generated by $H_{\neg}$, we can express every element $x \in \langle H_{\neg}\rangle$ as a polynomial δ^x_H of elements of $H_{\neg}$. We thus have $x = \delta^x_H(\overline{y})$, where for each y_i we have that $y_i \in H_{\neg}$. By writing $\overline{p} = p_1, ..., p_n$ for the variables contained in φ and $\overline{\delta^x_H(\overline{y})}$ for the polynomials of the elements $x_1 = V(p_1), ..., x_n = V(p_n)$, we get that $[\![\varphi(\overline{p})]\!]^{(\langle H_{\neg}\rangle, V)} = \varphi_H(\overline{\delta^x_H(\overline{y})})$. Since all the elements $\overline{y}$ in the polynomials δ^x_H are regular elements, we can define a DNA-valuation $U^{\neg} : \mathtt{AT} \to H_{\neg}$ such that $U^{\neg} : q_i \mapsto y_i$ for all $i \leq n$. Then it follows immediately that $[\![\varphi[\overline{\delta^x(\overline{q})}/\overline{p}]]\!]^{(\langle H_{\neg}\rangle, U^{\neg})} = \varphi_H(\overline{\delta^x_H(\overline{y})})$. But then, since we also had $[\![\varphi(\overline{p})]\!]^{(\langle H_{\neg}\rangle, V)} = \varphi_H(\overline{\delta^x_H(\overline{y})})$, it follows that $[\![\varphi[\overline{\delta^x(\overline{q})}/\overline{p}]]\!]^{(\langle H_{\neg}\rangle, U^{\neg})} = [\![\varphi(\overline{p})]\!]^{(\langle H_{\neg}\rangle, V)}$. So since $(\langle H_{\neg}\rangle, V) \nvDash \varphi$, we also get that $(\langle H_{\neg}\rangle, U^{\neg}) \nvDash^{\neg} \varphi[\overline{\delta(\overline{y})}/\overline{p}]$. So it then follows by Lemma 3.12 that $(H, U^{\neg}) \nvDash^{\neg} \varphi[\overline{\delta^x(\overline{q})}/\overline{p}]$, hence $H \nvDash^{\neg} \varphi[\overline{\delta^x(\overline{q})}/\overline{p}]$. Now, since L is an intermediate logic, it admits free substitution and so, since $\varphi \in L$, we also get that $\varphi[\overline{\delta^x(\overline{q})}/\overline{p}] \in L$ and therefore as $L \subseteq L^{\neg}$ also $\varphi[\overline{\delta^x(\overline{q})}/\overline{p}] \in L^{\neg}$. Finally, this means that $H \nvDash^{\neg} L^{\neg}$, thus proving our claim. □

3.3 DNA-Varieties

The algebraic semantics for DNA-logics that we have defined in the previous section motivates the introduction of DNA-varieties. We define DNA-varieties as negative closures of varieties of Heyting algebras.

Definition 3.14 (Negative Closure of a Variety)**.** For every variety of Heyting algebras $\mathcal{V}$, its *negative closure* $\mathcal{V}^{\uparrow}$ is defined as follows:

$$\mathcal{V}^{\uparrow} = \{H : \exists A \in \mathcal{V} \text{ such that } A_{\neg} = H_{\neg} \text{ and } A \preceq H\}.$$

A DNA-*variety* is the negative closure of some variety $\mathcal{V}$ of Heyting algebras. We use the notation $\mathcal{V}^{\uparrow}$ to refer to the negative variant of a variety $\mathcal{V}$ and we generally write $\mathcal{X}$ for DNA-varieties. If not specified otherwise, we reserve $\mathcal{C}$ to denote arbitrary classes of Heyting algebras, $\mathcal{V}$ or $\mathcal{U}$ to denote standard varieties and $\mathcal{X}$ or $\mathcal{Y}$ to denote DNA-varieties.

We will now show that DNA-varieties are also standard varieties, i.e., they are closed under the usual operations of taking subalgebras, homomorphic images and products. Moreover, we also show that they are closed under the following operation.

Definition 3.15. We say that a Heyting algebra K is a *core superalgebra* of H if $H_{\neg} = K_{\neg}$ and $H \preceq K$.

A core superalgebra K of a Heyting algebra H is thus a subalgebra of H such that K and H share the same regular elements. The following proposition provides us with a characterisation of DNA-varieties.

Proposition 3.16. *A class of Heyting algebras $\mathcal{C}$ is a* DNA*-variety if and only if it is closed under subalgebras, homomorphic images, products and core superalgebras.*

Proof. ($\Leftarrow$) If a set of algebras $\mathcal{C}$ is closed under subalgebras, homomorphic images and products then it is a variety. Moreover, since it is also closed under core superalgebras, it is straightforward to see that $\mathcal{C} = \mathcal{C}^{\uparrow}$, so that we can see $\mathcal{C}$ as the negative variant of itself and thus as a DNA-variety.

($\Rightarrow$) Consider now a DNA-variety $\mathcal{X}$. By definition it is the negative variant of some standard variety $\mathcal{V}$, so we have $\mathcal{X} = \mathcal{V}^{\uparrow}$. We need to check that $\mathcal{V}^{\uparrow}$ is closed under the above four operations.

(1) We check closure under subalgebras. Suppose $H \in \mathcal{V}^{\uparrow}$ and $K \preceq H$. Then by definition of DNA-variety it follows that there is some $H' \in \mathcal{V}$ such that $H'_{\neg} = H_{\neg}$ and $H' \preceq H$. Now consider $K' = H' \cap K$, since K' is the intersection of two subalgebras of H, it will also be closed under the Heyting algebra operations. Thus we have that K' is also a Heyting algebra and $K' \preceq H'$ and $K' \preceq K$. Therefore, by the fact that $K \in \mathcal{V}$ and $\mathcal{V}$ is closed under subalgebras, it then follows that $K' \in \mathcal{V}$. Moreover, since $H'_{\neg} = H_{\neg} \supseteq K_{\neg}$, we have that $K'_{\neg} = H'_{\neg} \cap K_{\neg} = K_{\neg}$. Finally, we showed that for $K' \in \mathcal{V}$ we have $K' \preceq K$ and $K'_{\neg} = K_{\neg}$, which entails $K \in \mathcal{V}^{\uparrow}$.

(2) We check closure under homomorphic images. Suppose $H \in \mathcal{V}^{\uparrow}$ and $f : H \twoheadrightarrow K$, then by the definition of DNA-variety we have that for some $H' \in V$ that $H'_{\neg} = H_{\neg}$ and $H' \preceq H$. Consider $K' = f[H']$. Since homomorphic images preserve subalgebras, we have $K' \preceq K$ and, by the closure of standard varieties under homomorphic images $K' \in \mathcal{V}$. Moreover, since by assumption $H_{\neg} = H'_{\neg}$, we have $K_{\neg} = f[H_{\neg}] = f[H'_{\neg}] = K'_{\neg}$. Thus for $K' \in \mathcal{V}$ we have $K' \preceq K$ and $K'_{\neg} = K_{\neg}$, which yields $K \in \mathcal{V}^{\uparrow}$.

(3) We check closure under products. Suppose $H^i \in \mathcal{V}^{\uparrow}$ for all $i \in I$ of some index-set I. Then we need to check that $\prod_{i\in I} H^i \in \mathcal{V}^{\uparrow}$. By the definition of DNA-variety it immediately follows that there is, for every $i \in I$, a Heyting algebra $K^i \in \mathcal{V}$ such that $H^i_{\neg} = K^i_{\neg}$, and $K^i \preceq H^i$. Then by the closure under products of $\mathcal{V}$, we have that $\prod_{i\in I} K^i \in \mathcal{V}$. Now, since $K^i \preceq H^i$ holds for every $i \in I$, it follows immediately that $\prod_{i\in I} K^i \preceq \prod_{i\in I} H^i$. Similarly, we have that:

$$(\prod_{i\in I} H^i)_{\neg} = \prod_{i\in I} H^i_{\neg} = \prod_{i\in I} K^i_{\neg} = (\prod_{i\in I} K^i)_{\neg}$$

Hence, by the fact that $\prod_{i\in I} K^i \in \mathcal{V}$ and the definition of DNA-variety, it follows that $\prod_{i\in I} H^i \in \mathcal{V}^{\uparrow}$.

(4) We check closure under core superalgebras. Suppose $H \in \mathcal{V}^{\uparrow}$ and for some K we have that $H_{\neg} = K_{\neg}$ and $H \preceq K$. By the definition of DNA-varieties we have that there is some $H' \preceq H$ such that $H' \in \mathcal{V}$ and $H'_{\neg} = H_{\neg}$. Since $H' \preceq H$ and $H \preceq K$ we then have $H' \preceq K$ by the transitivity of subalgebra relation. Moreover, since $H'_{\neg} = H_{\neg} = K_{\neg}$ and $H \in \mathcal{V}$, it finally follows that $K \in \mathcal{V}^{\uparrow}$. □

As in the case of standard varieties, DNA-varieties give rise to a lattice structure ordered by the set-theoretic inclusion. As it is customary doing, we implicitly exclude from the lattice of DNA-varieties the trivial DNA-variety of one-element algebras. The meet of two DNA-varieties $\mathcal{X}_0, \mathcal{X}_1$ is then just their intersection and their join is the smallest class containing their union and closed under the DNA-variety operations.

For any class $\mathcal{C}$ of Heyting algebras we say that $\mathcal{X}$ is *generated* by the class $\mathcal{C} \subseteq \mathcal{X}$ and we write $\mathcal{X} = \mathcal{X}(\mathcal{C})$ if $\mathcal{X}$ is the least class of Heyting algebras such that $\mathcal{C} \subseteq \mathcal{X}$ and $\mathcal{X}$ is closed under subalgebras, homomorphic images, products and core superalgebras. It is then clear that $\mathcal{X}(\mathcal{C})$ is the smallest DNA-variety containing $\mathcal{C}$ and that $\mathcal{X}(\mathcal{C}) = \mathcal{V}(\mathcal{C})^{\uparrow}$. We will thus define $\mathcal{X}_0 \wedge \mathcal{X}_1 := \mathcal{X}_0 \cap \mathcal{X}_1$ and $\mathcal{X}_0 \vee \mathcal{X}_1 := \mathcal{X}(\mathcal{X}_0 \cup \mathcal{X}_1)$. We proceed to prove the following proposition.

Proposition 3.17. *Let $\mathcal{X}_0$ and $\mathcal{X}_1$ be two* DNA*-varieties. Then: (i) $\mathcal{X}_0 \wedge \mathcal{X}_1$ is a* DNA*-variety and it is the infimum of $\mathcal{X}_0$ and $\mathcal{X}_1$; (ii) $\mathcal{X}_0 \vee \mathcal{X}_1$ is a* DNA*-variety and it is the supremum of $\mathcal{X}_0$ and $\mathcal{X}_1$.*

Proof. (i) By definition $\mathcal{X}_0 \wedge \mathcal{X}_1 := \mathcal{X}_0 \cap \mathcal{X}_1$. That this is a DNA-variety follows immediately from the fact that, since both $\mathcal{X}_0$ and $\mathcal{X}_1$ are closed under subalgebras, homomorphic images,

products and core superalgebras, then also their intersection is closed under these operations. Moreover, since $\mathcal{X}_0 \wedge \mathcal{X}_1 := \mathcal{X}_0 \cap \mathcal{X}_1$, it follows that $\mathcal{X}_0 \wedge \mathcal{X}_1$ is the infimum of $\mathcal{X}_0$ and $\mathcal{X}_1$.

(ii) By definition $\mathcal{X}_0 \vee \mathcal{X}_1 = \mathcal{X}(\mathcal{X}_0 \cup \mathcal{X}_1) = \mathcal{V}(\mathcal{X}_0 \cup \mathcal{X}_1)^{\uparrow}$, which is a DNA-variety. Now suppose $\mathcal{Y}$ is also a DNA-variety and $\mathcal{X}_0 \cup \mathcal{X}_1 \subseteq \mathcal{Y}$. Then since $\mathcal{Y}$ is also a variety it follows that $\mathcal{V}(\mathcal{X}_0 \cup \mathcal{X}_1) \subseteq \mathcal{Y}$ and since $\mathcal{Y}$ is also closed under core superalgebras it follows that $\mathcal{V}(\mathcal{X}_0 \cup \mathcal{X}_1)^{\uparrow} = \mathcal{X}(\mathcal{X}_0 \cup \mathcal{X}_1) \subseteq \mathcal{Y}$ and in turn gives us $\mathcal{X}(\mathcal{X}_0 \cup \mathcal{X}_1) = \mathcal{X}_0 \vee \mathcal{X}_1$ is the supremum of $\mathcal{X}_0$ and $\mathcal{X}_1$. □

We denote the lattice of DNA-varieties by **DNAV**. As varieties of Heyting algebras also form a lattice **HA**, one can then show that the map $(-)^{\uparrow} : \mathbf{HA} \to \mathbf{DNAV}$ which assigns every variety of Heyting algebras to its negative closure is a lattice homomorphism.

Proposition 3.18. *The map $(-)^{\uparrow} : \mathbf{HA} \to \mathbf{DNAV}$ is a bounded lattice homomorphism.*

Proof. Obviously $(-)^{\uparrow}$ sends $\bot_{\mathbf{HA}}$ to $\bot_{\mathbf{DNAV}}$ and $\top_{\mathbf{HA}}$ to $\top_{\mathbf{DNAV}}$, so it suffices to check that $\uparrow$ preserves meets and joins.

(i) Consider two standard varieties $\mathcal{V}_0$ and $\mathcal{V}_1$, then we have:

$$\begin{aligned}(\mathcal{V}_0 \wedge \mathcal{V}_1)^{\uparrow} &= \{H : \exists A \in \mathcal{V}_0 \cap \mathcal{V}_1 \text{ such that } A_{\neg} = H_{\neg} \text{ and } A \preceq H\} \\ &= \{H : \exists A \in \mathcal{V}_0 (A_{\neg} = H_{\neg}, A \preceq H)\} \cap \{H : \exists A \in \mathcal{V}_1 (A_{\neg} = H_{\neg}, A \preceq H)\} \\ &= \mathcal{V}_0^{\uparrow} \wedge \mathcal{V}_1^{\uparrow}.\end{aligned}$$

which shows that $(-)^{\uparrow}$ preserves the meet operator.

(ii) Consider two standard varieties $\mathcal{V}_0$ and $\mathcal{V}_1$, then we have by definition that $(\mathcal{V}_0 \vee \mathcal{V}_1)^{\uparrow} = (\mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1))^{\uparrow}$ and $\mathcal{V}_0^{\uparrow} \vee \mathcal{V}_1^{\uparrow} = \mathcal{X}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow}) = \mathcal{V}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow})^{\uparrow}$. It thus suffices to show that $\mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1)^{\uparrow} = \mathcal{V}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow})^{\uparrow}$.

($\subseteq$) Let us suppose $H \in (\mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1))^{\uparrow}$ which implies that there is some $K \in \mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1)$ such that $K_{\neg} = H_{\neg}$ and $K \preceq H$. Then clearly $K \in \mathcal{V}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow})$ and thus $H \in \mathcal{V}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow})^{\uparrow}$. ($\supseteq$) Suppose now $H \in \mathcal{V}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow})^{\uparrow}$, then for some $K \in \mathcal{V}(\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow})$ we have that $K_{\neg} = H_{\neg}$ and $K \preceq H$. In turn, this means that there exist a family of algebras $\{A_i : i \in I\} \subseteq \mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow}$ such that $K \in \mathcal{V}(\{A_i : i \in I\})$. Since each A_i is in $\mathcal{V}_0^{\uparrow} \cup \mathcal{V}_1^{\uparrow}$, there exist algebras $B_i \in \mathcal{V}_0 \cup \mathcal{V}_1$ such that $B^i \preceq A^i$ and $A^i_{\neg} = B^i_{\neg}$ for every $i \leq n$. Consequently $A_i \in \mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1)^{\uparrow}$ for every $i \leq n$; and $K \in \mathcal{V}(\{A_i : i \in I\}) \subseteq \mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1)^{\uparrow}$. Since $\mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1)^{\uparrow}$ is a DNA-variety and H is a core superalgebra of K, we conclude that $H \in \mathcal{V}(\mathcal{V}_0 \cup \mathcal{V}_1)^{\uparrow}$. □

Let us remark that we have given to both DNA-logics and DNA-varieties a twofold characterisation. On the one hand, we have introduced them in terms of negative variants of some intermediate logics or in terms of negative closure of some variety of Heyting algebras. On the other hand, we have also given an independent characterization of DNA-logics and DNA-varieties, as sets of formulas closed under some conditions or as set of algebras closed under some operations. We will often alternate between the two perspectives, i.e., consider DNA-logics and DNA-varieties in terms of negative variants or consider them as sets satisfying some closure properties.

3.4 The Maps $Log^{\neg}$ and $Var^{\neg}$

There are two obvious ways to relate formulas and algebras. We define the map $Var^{\neg}$ sending sets of formulas to the class of Heyting algebras in which they are DNA-valid and the map $Log^{\neg}$ sending classes of Heyting algebras to the set of their DNA-validities. We have:

$$Var^{\neg} : \Gamma \mapsto \{H \in HA : H \vDash^{\neg} \Gamma\};$$

$$Log^{\neg} : \mathcal{C} \mapsto \{\varphi \in \mathcal{L}_P : \mathcal{C} \vDash^{\neg} \varphi\}.$$

We say that a DNA-variety of Heyting algebras $\mathcal{X}$ is DNA-*defined* by a set of formulas Γ if $\mathcal{X} = Var^{\neg}(\Gamma)$. A class of Heyting algebras $\mathcal{C}$ is DNA-*definable* if there is a set Γ of formulas such that $\mathcal{C} = Var^{\neg}(\Gamma)$. When the context is clear, we often drop the qualification DNA and talk simply of definability. We say that a DNA-logic Λ is *algebraically complete* with respect to a class of Heyting algebras $\mathcal{C}$ if $\Lambda = Log^{\neg}(\mathcal{C})$. We shall prove in the next section a definability theorem and an algebraic completeness theorem for DNA-logics. We shall thus establish that every DNA-variety is defined by its validities and that every DNA-logic is complete with respect to its corresponding DNA-variety.

We will next show that $Var^{\neg}(\Gamma)$ is always a DNA-variety and $Log^{\neg}(\mathcal{C})$ is always a DNA-logic. First we prove the following important lemma showing that the DNA-validity of a formula is preserved by the key operations of a DNA-variety.

Lemma 3.19 (Preservation of DNA-Validity)**.** *The* DNA*-validity of a formula φ is preserved by the operations of subalgebras, homomorphic images, products and core superalgebras, i.e:*

(i) if $H \vDash^{\neg} \varphi$ and $K \preceq H$, then $K \vDash^{\neg} \varphi$;

(ii) if $H \vDash^{\neg} \varphi$ and $H \twoheadrightarrow K$, then $K \vDash^{\neg} \varphi$;

(iii) if $A_i \vDash^{\neg} \varphi$ for all $i \in I$ of a family $\{A_i\}_{i \in I}$ of algebras, then $\prod_{i \in I} A_i \vDash^{\neg} \varphi$;

(iv) if $H \vDash^{\neg} \varphi$ and for some K such that $K_{\neg} = H_{\neg}$ we have that $H \preceq K$, then $K \vDash^{\neg} \varphi$.

Proof. (i) By contraposition: If $K, V^{\neg} \nvDash^{\neg} \varphi$ for some DNA-valuation $V^{\neg}$, then $H, V^{\neg} \nvDash^{\neg} \varphi$.

(ii) Let $f : H \twoheadrightarrow K$ be a surjective morphism. By contraposition: If $K \nvDash^{\neg} \varphi$, then by Proposition 3.10 it follows that $K \nvDash \varphi[\overline{\neg p}/\overline{p}]$. Since validity is preserved by homomorphic images, it follows that $H \nvDash \varphi[\overline{\neg p}/\overline{p}]$ and therefore, by Proposition 3.10, $H \nvDash^{\neg} \varphi$.

(iii) The claim follows readily by noticing $\left(\prod_{i \in I} A_i\right)_{\neg} = \prod_{i \in I} (A_i)_{\neg}$, and so DNA-valuations over $\prod_{i \in I} A_i$ are all and only the functions of the form $V^{\neg}(p) = \langle V_i^{\neg}(p) : i \in I \rangle$ where every $V_i^{\neg}$ is some DNA-valuation over A_i.

(iv) Suppose by *reductio ad absurdum* that $K \nvDash^{\neg} \varphi$. Then for some valuation $V^{\neg}$ we have $(K, V^{\neg}) \nvDash^{\neg} \varphi$. Since $H_{\neg} = K_{\neg}$ and $H \preceq K$, $V^{\neg}$ is a valuation over H and $[\![\varphi]\!]^{(H,V^{\neg})} = [\![\varphi]\!]^{(K,V^{\neg})} \neq 1$. □

It follows immediately that for every set of formulas Γ the class of Heyting algebras $Var^{\neg}(\Gamma)$ is a DNA-variety.

Proposition 3.20. *The class of Heyting algebras $Var^{\neg}(\Gamma)$ is a* DNA*-variety.*

Proof. Consider any set of formulas Γ, then by the previous Lemma 3.19 it follows that the corresponding set $Var^{\neg}(\Gamma)$ is closed under the operations of taking subalgebras, homomorphic images, products and core superalgebras. Therefore, it follows by Proposition 3.16 that it is a DNA-variety. □

It is a straightforward consequence of Proposition 3.20 that every DNA-definable class of Heyting algebras is also a DNA-variety. The next proposition shows that for every class $\mathcal{C}$ of Heyting algebras its set of validities $Log^{\neg}(\mathcal{C})$ is a DNA-logic.

Proposition 3.21. *The class of formulas $Log^{\neg}(\mathcal{C})$ is a* DNA*-logic.*

Proof. We check that for any class $\mathcal{C}$ of Heyting algebras the corresponding set of formulas $Log^{\neg}(\mathcal{C})$ is a DNA-logic. In particular, we show that $Log^{\neg}(\mathcal{C}) = Log(\mathcal{C})^{\neg}$. We have:

$$\begin{aligned}
\varphi \notin Log^{\neg}(\mathcal{C}) &\Leftrightarrow \exists H \in \mathcal{C} \text{ such that } H \not\models^{\neg} \varphi \\
&\Leftrightarrow \exists H \in \mathcal{C} \text{ such that } H \not\models \varphi[\overline{\neg p}/\overline{p}] \qquad \text{(by Proposition 3.10)} \\
&\Leftrightarrow \varphi[\overline{\neg p}/\overline{p}] \notin Log(\mathcal{C}) \\
&\Leftrightarrow \varphi \notin Log(\mathcal{C})^{\neg}.
\end{aligned}$$

This shows that $Log^{\neg}(\mathcal{C})$ is the negative variant of $Log(\mathcal{C})$. □

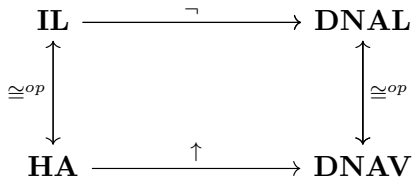
3.5 Duality between DNA-Logics and DNA-Varieties

We shall now prove our main result about DNA-logics and DNA-varieties, showing that their lattices are dually isomorphic. Notice that, so far, we have considered $Var^{\neg}$ as a map defined over arbitrary classes of Heyting algebras and $Log^{\neg}$ as a map defined over arbitrary sets of propositional formulas. Now we restrict our attention to the case in which the domain of $Var^{\neg}$ is the lattice of DNA-logics **DNAL** and the domain of $Log^{\neg}$ is the lattice of DNA-varieties **DNAV**.

Since we have shown above that $Var^{\neg}(\Gamma)$ is always a DNA-variety and $Log^{\neg}(\mathcal{C})$ is always a DNA-logic it follows that we have two maps:

$$\begin{aligned}
Var^{\neg} &: \mathbf{DNAL} \to \mathbf{DNAV}; \\
Log^{\neg} &: \mathbf{DNAV} \to \mathbf{DNAL}.
\end{aligned}$$

We shall now prove that these two maps describe a dual isomorphism between the lattice of DNA-logics and the lattice of DNA-varieties. Our proof essentially relies on the standard isomorphism between the lattice of intermediate logics and the lattice of varieties of Heyting algebras. An alternative proof, making use of Lindenbaum-Tarski algebras for DNA-logics, was given in [40]. Let us introduce the following diagram:



where the four objects in the diagram are the following:

IL is the lattice of intermediate logics;
HA is the lattice of varieties of Heyting algebras;
DNAL is the lattice of DNA-logics;
DNAV is the lattice of DNA-varieties.

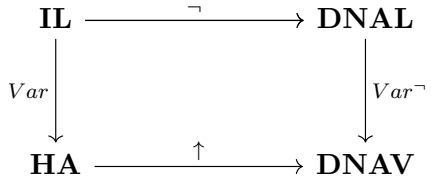
And the arrows are the following. Firstly, $(-)^{\neg} : \mathbf{IL} \to \mathbf{DNAL}$ is the map we introduced above that assigns to every intermediate logic L its negative variant $L^{\neg}$. Secondly, $(-)^{\uparrow} : \mathbf{HA} \to \mathbf{DNAV}$ is the map that assigns to each variety of Heyting algebras $\mathcal{V}$ its negative closure $\mathcal{V}^{\uparrow}$. The isomorphism $\mathbf{IL} \cong^{op} \mathbf{HA}$ is given by the standard duality for intermediate logics and varieties of Heyting algebras. The two maps of this bijection are $Log : \mathbf{HA} \to \mathbf{IL}$ and $Var : \mathbf{IL} \to \mathbf{HA}$, which we have defined in the preliminaries. By using the fact that $\mathbf{IL} \cong^{op} \mathbf{HA}$ we show now that also $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$ holds. We

proceed as follows. First we show that the diagram that we have described commutes, then we show that $Var^{\neg}$ and $Log^{\neg}$ are inverse maps of each other and finally we prove they are order-reversing homomorphisms between **DNAL** and **DNAV**. Thus we will obtain a dual isomorphism $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$.

3.5.1 Commutativity of the Diagram

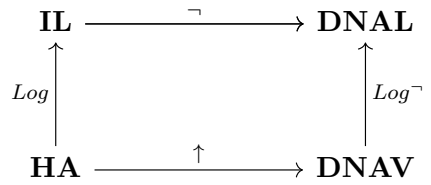
We first prove the two following propositions, thereby establishing that our diagram commutes.

Proposition 3.22. *For every intermediate logic L we have $Var^{\neg}(L^{\neg}) = Var(L)^{\uparrow}$.*



Proof. ($\subseteq$) Consider any Heyting algebra $H \in Var^{\neg}(L^{\neg})$. We have $H \vDash^{\neg} L^{\neg}$ and so by Proposition 3.13 that $\langle H_{\neg} \rangle \vDash L$. So we have $\langle H_{\neg} \rangle \in Var(L)$ and since $\langle H_{\neg} \rangle_{\neg} = H_{\neg}$ and $\langle H_{\neg} \rangle \preceq H$ also $H \in Var(L)^{\uparrow}$. ($\supseteq$) Consider any Heyting algebra $H \in Var(L)^{\uparrow}$, then there is some $K \in Var(L)$ such that $K \preceq H$ and $H_{\neg} = K_{\neg}$. Then $K \vDash L$ and by Lemma 3.11 we obtain $K \vDash^{\neg} L^{\neg}$, which entails $K \in Var^{\neg}(L^{\neg})$. Finally, since DNA-varieties are closed under core superalgebra, it follows that $H \in Var^{\neg}(L^{\neg})$. □

Proposition 3.23. *For every variety $\mathcal{V}$ of Heyting algebras $Log^{\neg}(\mathcal{V}^{\uparrow}) = Log(\mathcal{V})^{\neg}$.*



Proof. We prove both directions by contraposition. ($\subseteq$) Suppose $\varphi \notin Log(\mathcal{V})^{\neg}$, then $\varphi[\overline{\neg p}/\overline{p}] \notin Log(\mathcal{V})$, so there is some Heyting algebra $H \in \mathcal{V}$ such that $H \nvDash \varphi[\overline{\neg p}/\overline{p}]$. By Proposition 3.10 this means that $H \nvDash^{\neg} \varphi$ and so, since $H \in \mathcal{V} \subseteq \mathcal{V}^{\uparrow}$, we also have $\varphi \notin Log^{\neg}(\mathcal{V}^{\uparrow})$. ($\supseteq$) Suppose $\varphi \notin Log^{\neg}(\mathcal{V}^{\uparrow})$. It follows that there is some Heyting algebra $H \in \mathcal{V}^{\uparrow}$ such that $H \nvDash^{\neg} \varphi$, hence by Lemma 3.12 we have that $\langle H_{\neg} \rangle \nvDash^{\neg} \varphi$. It follows by Proposition 3.10 that $\langle H_{\neg} \rangle \nvDash \varphi[\overline{\neg p}/\overline{p}]$. Now, since $H \in \mathcal{V}^{\uparrow}$, we have for some $K \in \mathcal{V}$ that $K \preceq H$ and $K_{\neg} = H_{\neg}$. Therefore $\langle H_{\neg} \rangle \preceq K$ and $\langle H_{\neg} \rangle \in \mathcal{V}$. Finally, since $\langle H_{\neg} \rangle \nvDash \varphi[\overline{\neg p}/\overline{p}]$ we get that $\varphi[\overline{\neg p}/\overline{p}] \notin Log(\mathcal{V})$ and hence $\varphi \notin Log(\mathcal{V})^{\neg}$. □

In particular, when $\mathcal{V}$ is itself a DNA-variety we obtain the following corollary.

Corollary 3.24. *For every DNA-variety $\mathcal{X}$ we have $Log^{\neg}(\mathcal{X}) = Log(\mathcal{X})^{\neg}$.*

3.5.2 Definability Theorem and Algebraic Completeness

By relying on the commutativity result described above, we can now prove that the two maps $Var^{\neg}$ and $Log^{\neg}$ are inverse of one another. It is then easy to see that suitable versions of the definability theorem and algebraic completeness follow from this result.

Proposition 3.25. $Var^{\neg} \circ Log^{\neg} = 1_{\boldsymbol{DNAV}}$.

Proof. For any DNA-variety $\mathcal{X}$ we have:

$$\begin{aligned} Var^{\neg}(Log^{\neg}(\mathcal{X})) &= Var^{\neg}(Log(\mathcal{X})^{\neg}) && \text{(by Corollary 3.24)} \\ &= Var(Log(\mathcal{X}))^{\uparrow} && \text{(by Proposition 3.22)} \\ &= \mathcal{X}^{\uparrow} && \text{(by standard duality)} \\ &= \mathcal{X}. \end{aligned}$$

And thus $Var^{\neg} \circ Log^{\neg} = 1_{\mathbf{DNAV}}$. □

Theorem 3.26 (Definability Theorem)**.** *Every* DNA*-variety $\mathcal{X}$ is defined by its* DNA*-validities, i.e. for every Heyting algebra H,*

$$H \in \mathcal{X} \Leftrightarrow H \vDash^{\neg} Log^{\neg}(\mathcal{X}).$$

We then have that every DNA-variety is DNA-definable. Moreover, by Proposition 3.20 we have that every DNA-definable class is also a DNA-variety, the following corollary also follows.

Corollary 3.27 (Birkhoff Theorem for DNA-Varieties)**.** *A class of Heyting algebras $\mathcal{C}$ is a* DNA*-variety if and only if it is* DNA*-definable by some set of formulas.*

The algebraic completeness of DNA-logics is proved as follows.

Proposition 3.28. $Log^{\neg} \circ Var^{\neg} = 1_{\mathbf{DNAL}}$.

Proof. For any DNA-logic Λ such that $\Lambda = L^{\neg}$ we have:

$$\begin{aligned} Log^{\neg}(Var^{\neg}(\Lambda)) &= Log^{\neg}(Var^{\neg}(L^{\neg})) \\ &= Log^{\neg}(Var(L)^{\uparrow}) && \text{(by Proposition 3.22)} \\ &= Log(Var(L))^{\neg} && \text{(by Proposition 3.23)} \\ &= L^{\neg} && \text{(by standard duality)} \\ &= \Lambda. \end{aligned}$$

And thus $Log^{\neg} \circ Var^{\neg} = 1_{\mathbf{DNAL}}$. □

Theorem 3.29 (Algebraic Completeness)**.** *Every* DNA*-logic Λ is complete with respect to its corresponding* DNA*-variety, i.e. for every $\varphi \in \mathcal{L}_P$,*

$$\varphi \in \Lambda \Leftrightarrow Var^{\neg}(\Lambda) \vDash^{\neg} \varphi.$$

3.5.3 Dual Isomorphism

Finally, by relying on the standard dual isomorphism $\mathbf{HA} \cong^{op} \mathbf{IL}$ and the commutative square above, it is easy to show that $Var^{\neg}$ and $Log^{\neg}$ are order-reversing homomorphisms that invert the lattice structure of **DNAL** and **DNAV**.

Proposition 3.30. *$Var^{\neg}$ is an order-reversing homomorphism.*

Proof. It suffices to check that $Var^{\neg}$ inverts meet and join. Let Λ_0, Λ_1 be two DNA-logics such that $\Lambda_0 = L_0^{\neg}$ and $\Lambda_1 = L_1^{\neg}$. The case for $\wedge$ is as follows:

$$\begin{aligned} Var^{\neg}(\Lambda_0 \wedge \Lambda_1) &= Var^{\neg}(L_0^{\neg} \wedge L_1^{\neg}) \\ &= Var^{\neg}((L_0 \wedge L_1)^{\neg}) && \text{(by Proposition 3.4)} \end{aligned}$$

$$
\begin{aligned}
&= Var(L_0 \wedge L_1)^{\uparrow} && \text{(by Proposition 3.22)}\\
&= (Var(L_0) \vee Var(L_1))^{\uparrow} && \text{(by standard duality)}\\
&= Var(L_0)^{\uparrow} \vee Var(L_1)^{\uparrow} && \text{(by Proposition 3.18)}\\
&= Var^{\neg}(L_0^{\neg}) \vee Var^{\neg}(L_1^{\neg}) && \text{(by Proposition 3.22)}\\
&= Var^{\neg}(\Lambda_0) \vee Var^{\neg}(\Lambda_1).
\end{aligned}
$$

The case for $\vee$ is analogous. □

Proposition 3.31. *$Log^{\neg}$ is an order-reversing homomorphism.*

Proof. It suffices to check that $Log^{\neg}$ inverts meet and join. Let $\mathcal{X}_0, \mathcal{X}_1$ be two DNA-varieties such that $\mathcal{X}_0 = \mathcal{V}_0^{\uparrow}$ and $\mathcal{X}_1 = \mathcal{V}_1^{\uparrow}$. The case for $\wedge$ is as follows:

$$
\begin{aligned}
Log^{\neg}(\mathcal{X}_0 \wedge \mathcal{X}_0) &= Log^{\neg}(\mathcal{V}_0^{\uparrow} \wedge \mathcal{V}_1^{\uparrow})\\
&= Log^{\neg}((\mathcal{V}_0 \wedge \mathcal{V}_1)^{\uparrow}) && \text{(by Proposition 3.18)}\\
&= Log(\mathcal{V}_0 \wedge \mathcal{V}_1)^{\neg} && \text{(by Proposition 3.23)}\\
&= (Log(\mathcal{V}_0) \vee Log(\mathcal{V}_1))^{\neg} && \text{(by standard duality)}\\
&= Log(\mathcal{V}_0)^{\neg} \vee Log(\mathcal{V}_1)^{\neg} && \text{(by Proposition 3.4)}\\
&= Log^{\neg}(\mathcal{V}_0^{\uparrow}) \vee Log^{\neg}(\mathcal{V}_1^{\uparrow}) && \text{(by Proposition 3.23)}\\
&= Log^{\neg}(\mathcal{X}_0) \vee Log^{\neg}(\mathcal{X}_1).
\end{aligned}
$$

The case for $\vee$ is analogous. □

It is a consequence of the previous results that $Var^{\neg}$ and $Log^{\neg}$ are two order-reversing homomorphisms between **DNAL** and **DNAV** which are inverse of one another. The following duality theorem follows.

Theorem 3.32 (Duality)**.** *The lattice of* DNA*-logics is dually isomorphic to the lattice of* DNA*-varieties of Heyting algebras, i.e.* **DNAL** $\cong^{op}$ **DNAV**.

4 DNA-Varieties

In this section we prove some further results on DNA-varieties. Firstly, we investigate the relation between DNA-logics and the intermediate logics they are a negative variant of, and we characterize maximal and minimal elements in the sublattice of intermediate logics which have the same negative variant. We introduce regularly generated Heyting algebras and we use them to characterize the maximal logics with a negative variant. We prove for DNA-varieties a suitable version of two key results of universal algebra, namely the Tarski and Birkhoff variety theorems. We introduce a suitable notion of local finiteness for DNA-varieties and of local tabularity for DNA-logics. Finally, we introduce Jankov DNA-formulas and we prove a version of Jankov theorem for our setting.

4.1 Connections to Intermediate Logics

In the previous section we have introduced DNA-logics as negative variants of intermediate logics under the map $(-)^{\neg} : \mathbf{IL} \to \mathbf{DNAL}$. Now we shall investigate the relation between intermediate logics and DNA-logics in more detail. We will first show that the map $(-)^{\neg}$ which sends every intermediate logic to its negative variant is not injective. The following proposition was proved by Ciardelli in [10, Lemma 5.2.20] and exemplifies how different

intermediate logics can share the same negative variant. We recall that KC is the logic of the weak excluded middle, i.e., $\mathtt{KC} = \mathtt{IPC} + \neg\varphi \vee \neg\neg\varphi$.

Lemma 4.1. *Let L be any intermediate logic such that $\mathtt{KC} \subseteq L$, then $L^{\neg} = \mathtt{CPC}$.*

Proof. Suppose L is an intermediate logic such that $\mathtt{KC} \subseteq L$. One can show that for every formula φ we have $\varphi \vee \neg\varphi \in L^{\neg}$. We prove this by induction on the complexity of φ. For the base case, suppose that $p \in \mathtt{AT}$, then since $\mathtt{KC} \subseteq L$ we have for all $p \in \mathtt{AT}$ that $\neg p \vee \neg\neg p \in L$ and therefore that $p \vee \neg p \in L^{\neg}$. The induction steps follow easily by observing that for every formulas ψ and χ we have

$$(\psi \vee \neg\psi) \wedge (\chi \vee \neg\chi) \to ((\psi \odot \chi) \vee \neg(\psi \odot \chi)) \in \mathtt{IPC} \qquad \text{for } \odot \in \{\wedge, \vee, \to\}.$$

This shows that $L^{\neg} = \mathtt{IPC} + \varphi \vee \neg\varphi = \mathtt{CPC}$. □

Therefore, for intermediate logics L_0, L_1 such that $\mathtt{KC} \subseteq L_0, L_1$ and $L_0 \neq L_1$ we have that $L_0^{\neg} = L_1^{\neg} = \mathtt{CPC}$, hence $(-)^{\neg}$ is clearly not injective. Every DNA-logic Λ thus determines a subset of the lattice **IL** of those logics which have Λ as their negative variant. It is easy to see that this subset is also a sublattice, since the map $(-)^{\neg}$ is a homomorphism. Similarly, since also $(-)^{\uparrow}$ is a homomorphism, we can also consider the sublattice of all varieties $\mathcal{V}$ in **HA** whose negative closure is $\mathcal{X}$. We then define the preimage of a DNA-logic and the preimage of a DNA-variety as follows.

Definition 4.2. Let Λ be a DNA-logic and $\mathcal{X}$ be a DNA-variety. The *preimage* of Λ is the sublattice $\mathcal{I}(\Lambda)$ of all intermediate logics L such that $L^{\neg} = \Lambda$. The *preimage* of $\mathcal{X}$ is the sublattice $\mathcal{I}(\mathcal{X})$ of all varieties $\mathcal{V}$ such that $\mathcal{V}^{\uparrow} = \mathcal{X}$.

By the duality $\mathbf{IL} \cong^{op} \mathbf{HA}$ and the fact that the square introduced in Section 3.5 commutes, we then immediately have the following proposition.

Proposition 4.3. *For every DNA-logic Λ and every DNA-variety $\mathcal{X}$, we have that if $\mathcal{X} = Var^{\neg}(\Lambda)$ and $\Lambda = Log^{\neg}(\mathcal{X})$ then $\mathcal{I}(\Lambda) \cong^{op} \mathcal{I}(\mathcal{X})$.*

The isomorphism $\mathcal{I}(\Lambda) \cong^{op} \mathcal{I}(\mathcal{X})$ above is the restriction of the dual isomorphism $\mathbf{IL} \cong \mathbf{HA}$. We will now use this duality to characterize the two lattices $\mathcal{I}(\Lambda)$ and $\mathcal{I}(\mathcal{X})$.

First, we prove that the preimage $\mathcal{I}(\Lambda)$ of some DNA-logic Λ has a greatest element and we provide a characterisation of it. The following notion of schematic fragment of a DNA-logic was first introduced under the name of *standardization* in [36, Sec. 3] and later considered by Ciardelli in [10, Sec. 3.4]. That this operation on DNA-logics provides a maximal intermediate logic in $\mathcal{I}(\Lambda)$ was first proved in [36].

Definition 4.4 (Schematic Fragment)**.** Let Λ be a DNA-logic, we define its schematic fragment $Schm(\Lambda)$ as:

$$Schm(\Lambda) = \{\varphi \in \Lambda : \forall \overline{\psi} \in \mathcal{L}_P, \varphi[\overline{\psi}/\overline{p}]\}.$$

$Schm(\Lambda)$ is the set of all schematic formulas in Λ, namely those formulas for which Λ is closed under uniform substitution. One can easily check that $Schm(\Lambda)$ is an intermediate logic and that $Schm(\Lambda)^{\neg} = \Lambda$. Moreover, the following proposition show that $Schm(\Lambda)$ is the maximal intermediate logic whose negative variant is Λ.

Proposition 4.5. *Let Λ be any DNA-logic. Then, for every intermediate logic L such that $L^{\neg} = \Lambda$ we have that $L \subseteq Schm(\Lambda)$.*

Proof. Suppose that $\varphi \in L$. We denote by $\overline{p} = p_0, ..., p_n$ the atomic letters in φ. We need to check that for any sequence of formulas $\overline{\chi} = \chi_0(\overline{q}), ..., \chi_n(\overline{q}) \in \mathcal{L}_P$ with atomic letters $\overline{q}$, it is the case that $\varphi^{\chi} = \varphi[\overline{\chi}/\overline{p}] \in \Lambda$. Now, since $\varphi \in L$, it follows by uniform substitution that $\varphi^{\chi} \in L$. Then, again by uniform substitution, we have that $\varphi^{\chi}[\overline{\neg q}/\overline{q}] \in L$ and therefore $\varphi^{\chi} \in \Lambda$, which means that $\varphi \in Schm(\Lambda)$ and thus proves our claim. □

The following theorem immediately follows by the previous propositions.

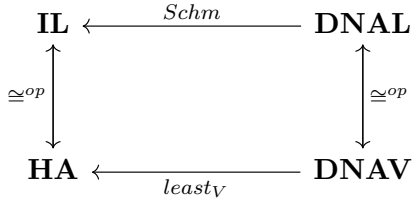
Theorem 4.6. *Let Λ be a* DNA*-logic. The schematic fragment $Schm(\Lambda)$ is the greatest intermediate logic whose negative variant is Λ.*

Therefore, the preimage $\mathcal{I}(\Lambda)$ of a DNA-logic Λ has always a greatest element. By Theorem 3.32 we also obtain a dual characterisation of the corresponding DNA-varieties. In fact, we have that $Var(Schm(\Lambda))$ is the least variety whose negative closure is $Var^{\neg}(\Lambda)$. We define the map $least_V : \mathbf{DNAV} \to \mathbf{HA}$ as follows:

$$least_V : \mathcal{X} \mapsto Var(Schm(Log^{\neg}(\mathcal{X}))).$$

The following proposition follows easily.

Proposition 4.7. *The following diagram commutes in both directions, i.e., $Var \circ Schm = least_V \circ Var^{\neg}$ and $Log \circ least_V = Schm \circ Log^{\neg}$.*



Proof. By the definition of $least_V$ and the dual isomorphism $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$ we have $least_V \circ Var^{\neg} = Var \circ Schm \circ Log^{\neg} \circ Var^{\neg} = Var \circ Schm$ and $Log \circ least_V = Log \circ Var \circ Schm \circ Log^{\neg} = Schm \circ Log^{\neg}$. □

Therefore, for every DNA-logic Λ we have that $Schm(\Lambda)$ is the greatest logic in $\mathcal{I}(\Lambda)$ and $least_V(Var^{\neg}(\Lambda))$ is the least variety in $\mathcal{I}(Var^{\neg}(\Lambda))$.

Similarly, one can show that the lattice $\mathcal{I}(\Lambda)$ has always a least element, which has so far been neglected in the literature. That this holds follows directly from the fact that for every DNA-variety $\mathcal{X}$, there is a greatest variety whose negative closure is exactly $\mathcal{X}$.

Proposition 4.8. *For every* DNA*-variety $\mathcal{X}$, there is a greatest variety $\mathcal{V}$ such that $\mathcal{V}^{\uparrow} = \mathcal{X}$.*

Proof. By Proposition 3.16 we have that DNA-varieties are also varieties and, moreover, $\mathcal{X}^{\uparrow} = \mathcal{X}$ for every DNA-variety $\mathcal{X}$. Hence $\mathcal{X}$ is clearly the greatest variety $\mathcal{V}$ such that $\mathcal{V}^{\uparrow} = \mathcal{X}$. □

The following theorem immediately follows by the previous propositions and DNA-duality.

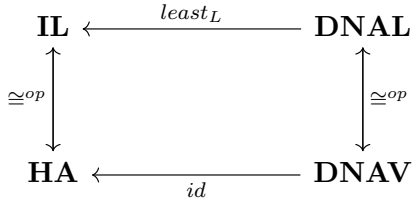
Theorem 4.9. *Let $\mathcal{X}$ be a* DNA*-variety. The logic $Log(\mathcal{X})$ is the least among the intermediate logics whose negative variant is $Log^{\neg}(\mathcal{X})$.*

We thus define a map $least_L : \mathbf{DNAL} \to \mathbf{IL}$ as follows:

$$least_L : \Lambda \mapsto Log(Var^{\neg}(\Lambda)).$$

The following proposition follows easily.

Proposition 4.10. *The following diagram commutes in both directions, i.e.,* $Var \circ least_L = id \circ Var^{\neg}$ *and* $Log \circ id = least_L \circ Log^{\neg}$.



Proof. By the definition of $least_L$ and the dual isomorphism $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$ we have $Var \circ least_L = Var \circ Log \circ Var^{\neg} = id \circ Var^{\neg}$ and $least_L \circ Log^{\neg} = Log \circ Var^{\neg} \circ Log^{\neg} = Log \circ id$. □

Therefore, it is the case that for every DNA-logic Λ we have that $least_L(\Lambda)$ is the smallest logic in $\mathcal{I}(\Lambda)$ and $Var^{\neg}(\Lambda)$ is the greatest variety in $\mathcal{I}(Var^{\neg}(\Lambda))$.

By the former results above it thus follows that the sublattices $\mathcal{I}(\Lambda)$ and $\mathcal{I}(\mathcal{X})$ are bounded sublattices of **IL** and **HA**. We introduce the following definitions.

Definition 4.11 (DNA-*maximality* and DNA-*minimality*)**.** Let L be an intermediate logic. (i) We say that L is DNA-*maximal* if it is the greatest logic in $\mathcal{I}(L^{\neg})$. (ii) We say that L is DNA-*minimal* if it is the least logic in $\mathcal{I}(L^{\neg})$.

In [36, Sec. 3] and [10, Sec. 5.2] intermediate logics L such that $L = Schm(L^{\neg})$ are called *stable*. The following proposition thus establishes that a logic is DNA-maximal iff it is stable. However, we will not use here this terminology, as the notion of stable logic has been employed e.g., in [27] with a rather different meaning. The following proposition is an immediate consequence of our definition and the previous results.

Proposition 4.12. *Let L be an intermediate logic, then:*

(i) L is DNA-*maximal iff $L = Schm(L^{\neg})$;*

(ii) L is DNA-*minimal iff $Var(L) = Var^{\neg}(L^{\neg})$.*

4.2 Regular Heyting Algebras

The previous characterisation of DNA-maximal and DNA-minimal logics is in a sense asymmetrical: we have a syntactic criterion for maximality and a semantic one for minimality. We are now after a semantic criterion for maximality. To this sake, we shall now define *regular* Heyting algebras, which also play a major role in the context of DNA-logics in general.

Definition 4.13 (Regular Heyting Algebras)**.** A Heyting algebra H is *regular* if $H = \langle H_{\neg} \rangle$.

These algebras have been introduced in [5] to provide an algebraic semantics to propositional inquisitive logic. A regular Heyting algebra is an algebra generated by its set $H_{\neg}$ of regular elements. For this reason we call regular Heyting algebras also *regularly generated*. Already in the previous section we have described some important properties of regular algebras in Lemma 3.12 and Proposition 3.13. Now we prove two further results showing that varieties $\mathcal{V}$ with the same negative closure $\mathcal{X}$ have the same collection of regular Heyting algebras. We first show the following proposition.

Proposition 4.14. *Let H be a regular Heyting algebra such that for some* DNA*-logic Λ we have that $H \models^{\neg} \Lambda$. Then, for every intermediate logic L such that $L^{\neg} = \Lambda$ we have that $H \models L$.*

Proof. Suppose that $H \vDash^{\neg} \Lambda$, then since $\Lambda = Schm(\Lambda)^{\neg}$ it follows that $H \vDash^{\neg} Schm(\Lambda)^{\neg}$ and so by Proposition 3.13 $H \vDash Schm(\Lambda)$. Finally, by Proposition 4.5 we have that $L \subseteq Schm(\Lambda)$ and thus $H \vDash L$. □

By the dual isomorphism $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$ we then obtain the following proposition.

Proposition 4.15. *Let H be a regular Heyting algebra. If $H \in \mathcal{X}$, then for every variety $\mathcal{V}$ such that $\mathcal{V}^{\uparrow} = \mathcal{X}$ we have that $H \in \mathcal{V}$.*

Proof. Suppose $H \in \mathcal{X}$, then $H \vDash^{\neg} Log^{\neg}(\mathcal{X})$. Then, since $\mathcal{V}^{\uparrow} = \mathcal{X}$, it follows by Proposition 3.23 that $Log(\mathcal{V})^{\neg} = Log^{\neg}(\mathcal{X})$. So $H \vDash^{\neg} Log(\mathcal{V})^{\neg}$ and by Proposition 4.14, $H \vDash Log(\mathcal{V})$, which entails $H \in \mathcal{V}$. □

We thereby have that all standard varieties whose negative closure is the same DNA-variety contain exactly the same regularly generated Heyting algebras. Interestingly, by Proposition 4.15 in order to check whether a regular Heyting algebra H validates an intermediate logic L, it is sufficient to check whether H DNA-validates $L^{\neg}$ (i.e., H validates $L^{\neg}$ under DNA-valuations).

Finally, we can strengthen the previous results and show that regular Heyting algebras provide a semantic characterisation of DNA-maximal logics. In [10, Sec. 5.2] a sufficient criterion for DNA-maximality was given in the context of Kripke frames: Ciardelli established that if L is the logic of a class of finite, everywhere branching trees, then it is DNA-maximal. We propose here a criterion in terms of regular algebras which is both sufficient and necessary.

Theorem 4.16. *An intermediate logic L is the logic of a class of regularly generated Heyting algebras if and only if it is* DNA*-maximal.*

Proof. ($\Rightarrow$) By the previous proposition this is equivalent to the statement that if an intermediate logic L is such that $L = Log(\mathcal{C})$, where $\mathcal{C}$ is a class of regularly generated Heyting algebras, then $L = Schm(L^{\neg})$. So, suppose that $\mathcal{C}$ is a class of regularly generated Heyting algebras, we need to show that $Log(\mathcal{C}) = Schm(Log(\mathcal{C})^{\neg})$. Since $Schm(Log(\mathcal{C})^{\neg})$ is DNA-maximal it follows that $Log(\mathcal{C}) \subseteq Schm(Log(\mathcal{C})^{\neg})$, so that we only need to show that $Schm(Log(\mathcal{C})^{\neg}) \subseteq Log(\mathcal{C})$. Now suppose by contraposition that $\varphi \notin Log(\mathcal{C})$, then we have that for some $H \in \mathcal{C}$ and for some valuation V, we have that $(H, V) \nvDash \varphi$. Now, since H is regularly generated, every element $x_i \in H$ can be written out as a polynomial $\delta_i(\overline{y_i^k})$ of regular elements of H. Then we define the DNA-valuation $V^{\neg} : p_i^k \mapsto y_i^k$ so that we then get $[\![\delta_i(\overline{p_i^k})]\!]^{(H,V^{\neg})} = \delta_i(\overline{y_i^k})$, so that we have, for some appropriate choice of polynomials, that $[\![\varphi]\!]^{(H,V)} = [\![\varphi[\overline{\delta_i(\overline{p_i^k})}/\overline{q}]]\!]^{(H,V^{\neg})}$. We then immediately get that $(H, V^{\neg}) \nvDash \varphi[\overline{\delta_i(\overline{p_i^k})}/\overline{q}]$ and, since $H \in \mathcal{C} \subseteq \mathcal{C}^{\uparrow}$, it follows $\varphi[\overline{\delta_i(\overline{p_i^k})}/\overline{q}] \notin Log^{\neg}(\mathcal{C}^{\uparrow}) = Log(\mathcal{C})^{\neg}$. Finally, since $\varphi[\overline{\delta_i(\overline{p_i^k})}/\overline{q}]$ is a substitution instance of φ, it follows that $\varphi \notin Schm(Log(\mathcal{C})^{\neg})$.

($\Leftarrow$) Suppose that L is a DNA-maximal logic and consider its corresponding variety $Var(L)$. Then let $Var_R(L)$ be the subclass of $Var(L)$ consisting of regular Heyting algebras only and let $L' = Log(Var_R(L))$, hence L' is clearly the logic of a class of regular Heyting algebras. Now, we have that $Var(L) \subseteq Var_R(L)^{\uparrow}$, since for every $H \in Var(L)$, we have that $\langle H_{\neg} \rangle \in Var_R(L)$, hence $H \in Var_R(L)^{\uparrow}$. As obviously $Var_R(L) \subseteq Var(L)$, it immediately follows that $Var(L)^{\uparrow} = Var_R(L)^{\uparrow}$. Thus, by duality $L = L' = Log(Var_R(L))$, implying that L is the logic of a class of regularly generated Heyting algebras. □

Hence we can restate Proposition 4.12 in purely semantical terms.

Proposition 4.17. *Let L be an intermediate logic, then:*

(i) L is DNA*-maximal iff $L = Log(\mathcal{C})$, for some class $\mathcal{C}$ of regular Heyting algebras;*

(ii) L is DNA*-minimal iff $Var(L) = Var^{\neg}(L^{\neg})$.*

4.3 DNA-Tarski and DNA-Birkhoff

The only characterisation that we have of DNA-varieties, so far, is that every DNA-variety $\mathcal{X}$ is the negative closure of some standard variety $\mathcal{V}$, i.e., $\mathcal{X} = \mathcal{V}^{\uparrow}$. Now we shall prove a version of Tarski's and Birkhoff's theorems for DNA-varieties to show that they can be generated by some suitable subclasses of Heyting algebras. Interestingly, another method to generate DNA-varieties is by constructing the Lindenbaum-Tarski algebra of their corresponding DNA-logic, see [40, Sec. 3.3.2].

Recall that if $\mathcal{X}$ is a DNA-variety, then we say that $\mathcal{X}$ is *generated* by the class $\mathcal{C} \subseteq \mathcal{X}$ if $\mathcal{X} = \mathcal{X}(\mathcal{C}) = \mathcal{V}(\mathcal{C})^{\uparrow}$. We can immediately adapt Tarski's variety theorem to the case of DNA-varieties.

Theorem 4.18 (DNA-Tarski)**.** *Let $\mathcal{C}$ be a class of Heyting algebras, then we have that $\mathcal{X}(\mathcal{C}) = HSP(\mathcal{C})^{\uparrow}$.*[2]

Proof. By definition we have that $\mathcal{X}(\mathcal{C}) = \mathcal{V}(\mathcal{C})^{\uparrow}$ and by Tarski's theorem 2.3 we have that $\mathcal{V}(\mathcal{C}) = HSP(\mathcal{C})$. Therefore $\mathcal{X}(\mathcal{C}) = HSP(\mathcal{C})^{\uparrow}$. □

We can then prove the following useful theorem.

Theorem 4.19. *Let $\mathcal{X}$ be a DNA-variety, then $\mathcal{X} = \mathcal{X}(\mathcal{C})$ iff $Log^{\neg}(\mathcal{X}) = Log^{\neg}(\mathcal{C})$.*

Proof. ($\Rightarrow$) Since $\mathcal{C} \subseteq \mathcal{X}$, the inclusion from right to left is straightforward. Suppose now that $\mathcal{X} \nvDash^{\neg} \varphi$ then there is some Heyting algebra $H \in \mathcal{X}$ such that $H \nvDash^{\neg} \varphi$. Then since $\mathcal{X} = \mathcal{X}(\mathcal{C})$, it follow by Theorem 4.18 that $H \in HSP(\mathcal{C})^{\uparrow}$. Thus, since DNA-validities are preserved under homomorphisms, subalgebras, products and core superalgebras, it follows that there is some Heyting algebra A such that $A \nvDash^{\neg} \varphi$. ($\Leftarrow$) Suppose now that $Log^{\neg}(\mathcal{X}) = Log^{\neg}(\mathcal{C})$. Then by the Duality Theorem 3.32 it follows $Var^{\neg}(Log^{\neg}(\mathcal{X})) = Var^{\neg}(Log^{\neg}(\mathcal{C}))$ and thus $\mathcal{X} = Var^{\neg}(Log^{\neg}(\mathcal{C}))$. Now, since $Log^{\neg}(\mathcal{X}(\mathcal{C})) = Log^{\neg}(\mathcal{C}))$ it follows by Proposition 3.19 and Duality that $Var^{\neg}(Log^{\neg}(\mathcal{C})) = Var^{\neg}(Log^{\neg}(\mathcal{X}(\mathcal{C})))$. Thus $Var^{\neg}(Log^{\neg}(\mathcal{X})) = Var^{\neg}(Log^{\neg}(\mathcal{X}(\mathcal{C})))$, which by duality means $\mathcal{X} = \mathcal{X}(\mathcal{C})$. □

Obtaining an analogue of Birkhoff's theorem is more involved. A first approximation is given by the following result, stating that every DNA-variety $\mathcal{X}$ is generated by its collection of regular Heyting algebras. If $\mathcal{X}$ is a DNA-variety, then we denote by $\mathcal{X}_R$ its subclass of regular Heyting algebras.

Proposition 4.20. *Every DNA-variety is generated by its collection of regular elements, i.e., $\mathcal{X} = \mathcal{X}(\mathcal{X}_R)$.*

Proof. Let $\mathcal{X}$ be a DNA-variety, then for any non-regular $H \in \mathcal{X}$ we have that $\langle H_{\neg} \rangle \preceq H$ and $H_{\neg} = \langle H_{\neg} \rangle_{\neg}$. So since $\langle H_{\neg} \rangle \in \mathcal{X}_R$ it follows $H \in \mathcal{X}(\mathcal{X}_R)$. □

We thus have, by the standard version of Birkhoff's theorem, that every DNA-variety is generated by its subdirectly irreducible elements and, by the previous proposition, that every DNA-variety is generated by its regular elements. We can actually prove more, namely that DNA-varieties are generated by their regular, subdirectly irreducible elements. Now if $\mathcal{X}$ is a DNA-variety, we denote by $\mathcal{X}_{RSI}$ its subset of regular subdirectly irreducible Heyting algebras. We will thus show that for every DNA-variety we have $\mathcal{X} = \mathcal{X}(\mathcal{X}_{RSI})$. Let us first recall the following result from the literature, originally due to Wronski [44].

[2] We consider the operator $(-)^{\uparrow}$ to have the least priority, that is, $HSP(\mathcal{C})^{\uparrow}$ stands for $(HSP(\mathcal{C}))^{\uparrow}$.

Lemma 4.21. *Let $B \in HA$. Then if $b \neq 1_B$ there is a subdirectly irreducible algebra C and a surjective homomorphism $h : B \twoheadrightarrow C$ such that $f(b) = s_C$, where s_C is the second greatest element of C.*

By using this fact we can prove Birkhoff theorem for DNA-varieties.

Theorem 4.22 (DNA-Birkhoff)**.** *Every DNA-variety is generated by its collection of regular subdirectly irreducible elements: $\mathcal{X} = \mathcal{X}(\mathcal{X}_{RSI})$.*

Proof. By the dual isomorphism between DNA-logics and DNA-varieties it suffices to show that $Log^{\neg}(\mathcal{X}) = Log^{\neg}(\mathcal{X}(\mathcal{X}_{RSI}))$, which is equivalent by Theorem 4.19 to $Log^{\neg}(\mathcal{X}) = Log^{\neg}(\mathcal{X}_{RSI})$. The direction $Log^{\neg}(\mathcal{X}) \subseteq Log^{\neg}(\mathcal{X}_{RSI})$ follows immediately from the inclusion $\mathcal{X}_{RSI} \subseteq \mathcal{X}$. It thus suffices to show that $Log^{\neg}(\mathcal{X}_{RSI}) \subseteq Log^{\neg}(\mathcal{X})$.

Suppose by contraposition that $\varphi \notin Log^{\neg}(\mathcal{X})$, then for some $H \in \mathcal{X}$ and some DNA-valuation $V^{\neg}$, we have that $(H, V^{\neg}) \not\models^{\neg} \varphi$ and so by Proposition 3.10 that $(\langle H_{\neg}\rangle, V^{\neg}) \not\models^{\neg} \varphi$. Then, since $x = [\![\varphi]\!]^{(\langle H_{\neg}\rangle, V^{\neg})} \neq 1_H$ it follows by Lemma 4.21 that there is a subdirectly irreducible algebra C such that there is surjective homomorphism $h : \langle H_{\neg}\rangle \twoheadrightarrow C$ with $h(x) = s_C$. Then, since homomorphisms preserve regular elements, the valuation $U^{\neg} = h \circ V^{\neg}$ is a DNA-valuation. Now let $p_0, .., p_n$ be the variables in φ, it follows by the properties of homomorphisms that:

$$\begin{aligned}
[\![\varphi(p_0, .., p_n)]\!]^{(C,U^{\neg})} &= \varphi_C[U^{\neg}(p_0), ..., U^{\neg}(p_n)] \\
&= \varphi_C[h(V^{\neg}(p_0)), ..., h(V^{\neg}(p_n))] \\
&= h[\![\varphi(p_0, .., p_n)]\!]^{(\langle H_{\neg}\rangle, V^{\neg})} \\
&= s_C.
\end{aligned}$$

From which it immediately follows that $(C, U^{\neg}) \not\models \varphi$ and so that $C \not\models \varphi$. Now, since $H \in \mathcal{X}$, we have that $\langle H_{\neg}\rangle \in \mathcal{X}$ and so since $h : \langle H_{\neg}\rangle \twoheadrightarrow C$ also that $C \in \mathcal{X}$. Moreover, we have that C is subdirectly irreducible and regular, as it is homomorphic image of $\langle H_{\neg}\rangle$. Finally, this means that $C \in \mathcal{X}_{RSI}$ and so that $\varphi \notin Log^{\neg}(\mathcal{X}_{RSI})$, which proves our claim. □

4.4 Locally Tabular DNA-Logics and DNA-Varieties

The notions of local tabularity and local finiteness play an important role in the theory of intermediate logics and in universal algebra at large. Here we introduce a suitable notion of local finiteness for DNA-varieties and DNA-logics, which we will later employ in our study of inquisitive logic.

We say that a Heyting algebra H is DNA-*finitely generated* if there are finitely many elements $x_0, ..., x_n \in H_{\neg}$ such that $\langle x_0, ..., x_n\rangle = H$. We then define locally finite DNA-varieties and locally tabular DNA-logics.

Definition 4.23. A DNA-variety $\mathcal{X}$ is DNA-*locally finite* if every DNA-finitely generated $H \in \mathcal{X}$ is also finite. A DNA-logic Λ is DNA-*locally tabular* if its corresponding DNA-variety $Var^{\neg}(\Lambda)$ is locally finite.

When the context makes it clear we then drop the prefix DNA and talk simply of local finiteness and local tabularity. If not specified otherwise, every time we talk of local finiteness of a DNA-variety or local tabularity of a DNA-logic we actually refer to the property of DNA-local finiteness and DNA-local tabularity . The following proposition follows straightforwardly and allows us to relate the local finiteness of intermediate logics to the local finiteness of DNA-logics.

Proposition 4.24. *Let L be any intermediate logic, if L is locally tabular, then $L^{\neg}$ is locally tabular.*

Proof. If L is locally tabular, then every finitely generated $H \in Var(L)$ is also finite. Now consider any $H \in Var^{\neg}(L^{\neg})$ and suppose for some $x_0, ..., x_n \in H_{\neg}$ we have $\langle x_0, ..., x_n \rangle = H$. Then it follows that $H = \langle H_{\neg} \rangle$ and so that H is regular. Then, we have by Proposition 4.15 that $H \in Var(L)$ and so since H is finitely generated by $x_0, ..., x_n$ it also follows that H is finite. This shows that $L^{\neg}$ is locally tabular. □

A property of DNA-logics which is closely connected to local finiteness is the finite model property (FMP). We introduce it as follows.

Definition 4.25 (Finite Model Property)**.** A DNA-variety $\mathcal{X}$ has the DNA-*finite model property* (FMP) if $\mathcal{X} = \mathcal{X}(\mathcal{C})$, where $\mathcal{C}$ is a collection of finite Heyting algebras. A DNA-logic Λ has the DNA-*finite model property* if its corresponding DNA-variety $Var^{\neg}(\Lambda)$ has the finite model property.

When the context makes it clear we drop the prefix DNA and simply talk of finite model property. If not specified otherwise, every time we talk of the finite model property of a DNA-variety or a DNA-logic we actually refer to the DNA-finite model property. The finite model property allows, for every formula $\varphi \notin \Lambda$, to find a finite algebra H which validates Λ and refutes φ. Similarly to the case of local finiteness, the finite model property of an intermediate logic entails the finite model property of its negative variant.

Proposition 4.26. *Let L be any intermediate logic, if L has the finite model property then $L^{\neg}$ has the finite model property.*

Proof. Suppose L has the finite model property, then $Var(L) = \mathcal{V}(\mathcal{C})$ for some class $\mathcal{C}$ of finite Heyting algebras. Then, we have that $Var^{\neg}(L^{\neg}) = Var^{\neg}(L^{\neg})^{\uparrow} = \mathcal{V}(\mathcal{C})^{\uparrow} = \mathcal{X}(\mathcal{C})$, which shows that $L^{\neg}$ also has the finite model property. □

If a DNA-variety has the finite model property we can further refine our version of Birkhoff theorem. We denote by $\mathcal{X}_{RFSI}$ the collection of finite, regular, subdirectly irreducible elements in $\mathcal{X}$.

Theorem 4.27. *If a DNA-variety $\mathcal{X}$ has the finite model property, then it is generated by its finite, regular subdirectly irreducible elements, i.e., $\mathcal{X} = \mathcal{X}(\mathcal{X}_{RFSI})$.*

Proof. By Theorem 4.19 it suffices to check that $Log^{\neg}(\mathcal{X}_{RFSI}) = Log^{\neg}(\mathcal{X}(\mathcal{X}_{RFSI}))$. The direction $Log^{\neg}(\mathcal{X}) \subseteq Log^{\neg}(\mathcal{X}_{RFSI})$ is obvious, for if φ is true in every algebra in $\mathcal{X}$ it is also true in $\mathcal{X}_{RFSI}$. Now, consider the direction $Log^{\neg}(\mathcal{X}_{RFSI}) \subseteq Log^{\neg}(\mathcal{X})$. First notice that if a DNA-variety $\mathcal{X}$ has the finite model property, then for some class $\mathcal{C}$ of finite Heyting algebras, we have that $\mathcal{X} = \mathcal{X}(\mathcal{C})$. Suppose now by contradiction that $\varphi \notin Log^{\neg}(\mathcal{X})$, then by Theorem 4.19 there is some finite $H \in \mathcal{C}$ such that $H \not\models^{\neg} \varphi$. Therefore, it follows immediately by Lemma 3.10 that $\langle H_{\neg} \rangle \not\models^{\neg} \varphi$. Then, by the argument of the proof of DNA-Birkhoff Theorem 4.22, we obtain a regular subdirectly irreducible algebra C such that $h : \langle H_{\neg} \rangle \twoheadrightarrow C$ and $C \not\models \varphi$. Moreover, by the fact that C is a homomorphic image of $\langle H_{\neg} \rangle$ it also follows that C is finite. We thus obtain that $C \in \mathcal{X}_{RFSI}$ and since $C \not\models^{\neg} \varphi$ that $\varphi \notin Log^{\neg}(\mathcal{X}_{RFSI})$, which finishes the proof of the theorem. □

Moreover, we can also show that if a DNA-variety $\mathcal{X}$ is locally finite, then it has the finite model property. We denote by $\mathcal{X}_F$ the subcollection of finite Heyting algebras in $\mathcal{X}$.

Theorem 4.28. *Let $\mathcal{X}$ be a DNA-variety. If $\mathcal{X}$ is locally finite, then it has the finite model property.*

Proof. By Theorem 4.19 it suffices to show that $Log^{\neg}(\mathcal{X}) = Log^{\neg}(\mathcal{X}_F)$. The inclusion $Log^{\neg}(\mathcal{X}) \subseteq Log^{\neg}(\mathcal{X}_F)$ is obvious, so we show that $Log^{\neg}(\mathcal{X}_F) \subseteq Log^{\neg}(\mathcal{X})$. Suppose $\varphi \notin Log^{\neg}(\mathcal{X})$, then there is some $H \in \mathcal{X}$ such that for some DNA-valuation $V^{\neg}$ we have that $(H, V^{\neg}) \not\models^{\neg} \varphi$. Now let Let $\overline{p}$ be the variables in φ and $V^{\neg}(\overline{p})$ their interpretation in H. Then, since $\mathcal{X}$ is locally finite we have that the generated subalgebra $\langle V^{\neg}(\overline{p})\rangle$ is also finite. Moreover, since $(H, V^{\neg}) \not\models \varphi$ and by the fact that the interpretation of φ lies inside $\langle V^{\neg}(\overline{p})\rangle$, it immediately follows that $(\langle V(\overline{p})\rangle, V^{\neg}) \not\models \varphi$. So, since $\langle V^{\neg}(\overline{p})\rangle \in \mathcal{X}_F$, it follows that $\varphi \notin Log^{\neg}(\mathcal{X}_F)$, which finishes the proof of the theorem. □

One may wonder whether our definition trivializes or if it captures an interesting property that DNA-varieties may, or may not, have. That this is the case follows from the fact that one can find DNA-logics which have the finite model property but that are not locally tabular. Hence, exactly as in the case of intermediate logics, the property of local tabularity is stronger than that of the finite model property. In particular, since IPC has the finite model property, it follows immediately from Proposition 4.26 that $\mathtt{IPC}^{\neg}$ has the finite model property as well. However, similarly to the case of IPC, we can show that $\mathtt{IPC}^{\neg}$ is not locally tabular. This is done by adapting the method of the Rieger-Nishimura ladder to the context of DNA-logics. A proof of this result can be found in [40, Sec. 4.2.2].

4.5 Jankov Formulas for DNA-Models

Jankov formulas (or Jankov-de Jongh formulas) play an important role in the study of intermediate logics [4, 8]. These formulas are a sort of counterpart in algebraic logic of what diagrams are in model theory, as they express in syntactic terms some key semantic properties of the corresponding algebra. Jankov introduced these formulas in [28, 30], where he used them to show that the lattice of intermediate logic has the cardinality of the continuum. Formulas having similar properties have also been introduced around the same time by de Jongh [32] (see also [4, Sec. 3.3]) and later by Fine in the context of modal logics [18]. We refer the interested reader to [2, 8, 15] for more information on Jankov formulas and their history.

We introduce a version of Jankov formulas which suits our setting of DNA-logics and we show how they can be used to axiomatise locally tabular DNA-logics. We adapt the approach originally presented by Wronski in [44]. First, we show how to decorate a Heyting algebra $H \in HA_{RFSI}$ with what we call Jankov representatives. Consider any $H \in HA_{RFSI}$, then we have that $H = \langle H_{\neg}\rangle$ and also that $H_{\neg}$ is finite. We can thus assume without loss of generality that H is generated by a finite set of elements $a_0, ..., a_n$ and that every element $x \in H$ can be expressed as a polynomial $\delta_H(a_0, ..., a_n)$ over the regular elements of H. We then associate every element $x \in H$ to a formula ψ_x called its *Jankov representative*.

Definition 4.29 (Jankov Representative)**.** Let $H \in HA_{RFSI}$ and $x \in H$, then the *Jankov representative* of x is a formula ψ_x defined as follows:

(i) If $x \in H_{\neg}$, then $\psi_x = p_x$, where $p_x \in \mathtt{AT}$;

(ii) If $x = \delta_X(a_0, ..., a_n)$ with $a_0, ..., a_n \in H_{\neg}$, then $\psi_x = \delta(p_{a_0}, ..., p_{a_n})$.

Notice that when we decorate a Heyting algebra H with Jankov representatives we are making a fundamental use of the fact that H is regular. Notice also that the Jankov representative of an element $x \in H$ is not unique, as there are different polynomials over regular elements characterizing the same element of a regular Heyting algebra. The Jankov representative is thus the formula corresponding to any of those polynomials. Once we have the notion of Jankov representative, we can define Jankov formulas for the setting of DNA-logics as follows.

Definition 4.30 (Jankov DNA-Formula)**.** Let $H \in HA_{RFSI}$, let 0 be the least element of H and s its second greatest element. Then the *Jankov* DNA*-Formula* $\chi^{\mathtt{DNA}}(H)$ is defined as follows:

$$\chi^{\mathtt{DNA}}(H) = \alpha \to \beta,$$

where α and β are the following formulas:

$$\begin{aligned}\alpha = (\psi_0 \leftrightarrow \bot) \wedge \bigwedge\{(\psi_a \wedge \psi_b) \leftrightarrow \psi_{a\wedge b} : a, b \in H\} \wedge \\ \bigwedge\{(\psi_a \vee \psi_b) \leftrightarrow \psi_{a\vee b} : a, b \in H\} \wedge \\ \bigwedge\{(\psi_a \to \psi_b) \leftrightarrow \psi_{a\to b} : a, b \in H\}\end{aligned}$$

$$\beta = \psi_s.$$

When its clear from the context that we are working with Jankov DNA-formulas and not with the standard Jankov formulas, we drop the superscript and write just $\chi(H)$ for the Jankov DNA-formula of H. We now prove a lemma which plays an important role in the proof of our Jankov theorem.

Lemma 4.31. *Let $H \in HA_{RFSI}$, then $H \nvDash^{\neg} \chi(H)$.*

Proof. Suppose $H \in HA_{RFSI}$ and $\chi(H)$ is its DNA-Jankov formula. Then we define the DNA-valuation $V^{\neg}$ such that for all atomic Jankov representative we have that $V : p_a \mapsto a$, for all $a \in H_{\neg}$. Moreover, if an element $x \in H \setminus H_{\neg}$ is described by a polynomial $\delta_H(a_0, ..., a_n)$ over regular element of H, it follows by the definition of Jankov representative that $[\![\delta(p_a, ..., p_a)]\!]^{(H,V^{\neg})} = \delta_H(a_0, ..., a_n)$. We then have that for every element $x \in H$ it is the case that $[\![\psi_x]\!]^{(H,V^{\neg})} = x$. But then it follows straightforwardly that for all $a, b \in H$ and for any connective $\odot$ we have $[\![\psi_a \odot \psi_b]\!]^{(H,V^{\neg})} = [\![\psi_{a\odot b}]\!]^{(H,V^{\neg})}$ so that the antecedent of the DNA-Jankov formula is $[\![\alpha]\!]^{(H,V^{\neg})} = 1_A$ and its consequent is $[\![\beta]\!]^{(H,V^{\neg})} = [\![\psi_x]\!]^{(H,V^{\neg})} = s_c$. Therefore, we have that:

$$[\![\chi(H)]\!]^{(H,V^{\neg})} = [\![\alpha \to \beta]\!]^{(H,V^{\neg})} = [\![\alpha]\!]^{(H,V^{\neg})} \to [\![\beta]\!]^{(H,V^{\neg})} = 1_A \to s_A = s_A \neq 1_A.$$

And, therefore, we have that $(H, V^{\neg}) \nvDash^{\neg} \chi(H)$ and so that $H \nvDash^{\neg} \chi(H)$. □

If A and B are two Heyting algebras, then we define $A \leq B$ iff $A \in HS(B)$. It is easy to show that this is indeed a partial order. We now prove a suitable version of Jankov theorem for our setting. We adapt to our setting a similar proof given in [2].

Theorem 4.32 (Jankov Theorem for DNA-Models)**.** *Let $A \in HA_{RFSI}$ and $B \in HA$ then:*

$$B \nvDash^{\neg} \chi(A) \text{ iff } A \leq B.$$

Proof. ($\Rightarrow$) Suppose that $B \nvDash^{\neg} \chi(A)$, then for some DNA-valuation $V^{\neg}$ we have $[\![\chi(A)]\!]^{(B,V^{\neg})} = b \neq 1_B$. It follows from Lemma 4.21 that there is a subdirectly irreducible Heyting algebra C and a surjective homomorphism $f : B \twoheadrightarrow C$ such that $f(b) = s_C$. Hence, since f is a homomorphism, it follows that $U^{\neg} = f \circ V^{\neg}$ is a DNA-valuation. It thus follows that $[\![\chi(A)]\!]^{(C,U^{\neg})} = [\![\alpha \to \psi_s]\!]^{(C,U^{\neg})} = f(b) = s_C$. In particular, since s_C is the second-greatest element, this implies that $[\![\alpha]\!]^{(C,U^{\neg})} = 1_C$ and $[\![\psi_s]\!]^{(C,U^{\neg})} = s_C$.

We now prove that the map $h : A \to C$ defined as $h(x) = [\![\psi_x]\!]^{(C,U^{\neg})}$ is an embedding of A into C. First, we show that h is a homomorphism. Since $[\![\alpha]\!]^{(C,U^{\neg})} = 1_C$ it follows immediately that $[\![\psi_0 \leftrightarrow \bot]\!]^{(C,U^{\neg})} = 1_C$ and for every connective $\odot$ and every elements

$a, b \in C$, we have $[\![(\psi_a \odot \psi_b) \leftrightarrow \psi_{a \odot b}]\!]^{(C,U^\neg)} = 1_C$. From this we immediately get that $[\![\psi_0]\!]^{(C,U^\neg)} = 0_C$ and $[\![\psi_a \odot \psi_b]\!]^{(C,U^\neg)} = [\![\psi_{a \odot b}]\!]^{(C,U^\neg)}$. So we have the following two identities.

$$h(0_A) = [\![\psi_0]\!]^{(C,U^\neg)} = 0_C;$$
$$h(a \odot b) = [\![\psi_{a \odot b}]\!]^{(C,U^\neg)} = [\![\psi_a \odot \psi_b]\!]^{(C,U^\neg)} = [\![\psi_a]\!]^{(C,U^\neg)} \odot [\![\psi_b]\!]^{(C,U^\neg)} = h(a) \odot h(b).$$

We now want to prove that h is injective. It suffices to show that, for $a \neq b$, we have $h(a) \to h(b) \neq 1_C$. Since under this hypothesis $a \to b \leq s_A$—where s_A is the second greatest element of A—and since $[\![\psi_s]\!]^{(C,U^\neg)} = s_C$, we have that $h(a \to b) \leq h(s_A)$. Therefore:

$$h(a) \to h(b) \leq h([\![\psi_s]\!]^{(A,V^\neg)}) = [\![\psi_s]\!]^{(C,U^\neg)} = s_C$$

In particular $h(a) \not\leq h(b)$, thus proving the injectivity of h.

Therefore, we have that h is an embedding and thus $h[A] \preceq C$, showing that A is a subalgebra of C up to isomorphism. Since $B \twoheadrightarrow C$ it follows that $A \in SH(B)$ and, by Proposition 2.2, that $SH(B) \subseteq HS(B)$. Thus we obtain that $A \in HS(B)$, that is, $A \leq B$.

($\Leftarrow$) Suppose that $A \leq B$, namely that $A \in HS(B)$, then we know there is some subalgebra $B' \preceq B$ such that there is a surjective homomorphism $h : B' \twoheadrightarrow A$. Moreover, by the previous Lemma 4.31 we have that $A \not\models^\neg \chi(A)$. Then, since $h : B' \twoheadrightarrow A$ it follows immediately by the fact that the DNA-validity of a formula is preserved by homomorphic images that $B' \not\models^\neg \chi(A)$. Moreover, since $B' \preceq B$ it follows by the preservation of DNA-validity under subalgebra that $B \not\models^\neg \chi(A)$, which proves our claim. □

Once we have shown that Jankov theorem holds for our setting, we can use Jankov's machinery to characterize the lattice of subvarieties of locally finite DNA-varieties. We denote by $\Lambda^\neg(\mathcal{X})$ the lattice of subvarieties of some DNA-variety $\mathcal{X}$ and we first prove the following useful proposition.

Definition 4.33 (Hereditary FMP)**.** We say that a DNA-variety $\mathcal{X}$ has the *hereditary* DNA*-finite model property* if every DNA-variety $\mathcal{Y} \in \Lambda^\neg(\mathcal{X})$ has the finite model property.

As we always do, when the context is clear we drop the prefix DNA and talk simply of the hereditary finite model property.

Proposition 4.34. *If a* DNA*-variety $\mathcal{X}$ is locally finite, then $\mathcal{X}$ has the hereditary finite model property.*

Proof. Suppose that $\mathcal{X}$ is locally finite and consider any subvariety $\mathcal{Y} \in \Lambda^\neg(\mathcal{X})$. Since $\mathcal{X}$ is locally finite we have that every DNA-finitely generated $H \in \mathcal{X}$ is also finite and thus since $\mathcal{Y} \subseteq \mathcal{X}$ also that DNA-finitely generated $H \in \mathcal{X}_{RSI}$ is finite. Hence we have that $\mathcal{Y}$ is locally finite and therefore, by Proposition 4.28 above, it follows that $\mathcal{Y}$ also has the finite model property. □

We now prove the following theorem characterising the sublattice of locally finite DNA-varieties. We denote by $Dw(\mathcal{X}_{RFSI})$ the downsets of $\mathcal{X}_{RFSI}$ under the partial order $\leq$ defined above.

Theorem 4.35. *Let $\mathcal{X}$ be a* DNA*-variety which is locally finite. Then the lattice of negative subvarieties of $\mathcal{X}$ is isomorphic to the lattice of downsets over $\mathcal{X}_{RFSI}$, i.e:*

$$\Lambda^\neg(\mathcal{X}) \cong Dw(\mathcal{X}_{RFSI}).$$

Proof. Consider the map $\alpha : \mathcal{Y} \mapsto \mathcal{Y}_{RFSI}$ which sends every subvariety $\mathcal{Y} \subseteq \mathcal{X}$ to its subclass of finite regular subdirectly irreducible elements. We claim that α is welldefined and also it is an isomorphism between $\Lambda^{\neg}(\mathcal{V})$ and $Dw(\mathcal{X}_{RFSI})$. (i) First, we show that $\mathcal{Y}_{RFSI} \in Dw(\mathcal{X}_{RFSI})$. Suppose $B \in \mathcal{Y}_{RFSI}$, $A \in HS(B)$ and $A \in \mathcal{X}_{RFSI}$. As varieties are closed under homomorphic image and subalgebra, we have that $A \in \mathcal{Y}$ and so since $A \in \mathcal{X}_{RFSI}$ also that $A \in \mathcal{Y}_{RFSI}$. (ii) To show injectivity, consider two subvarieties $\mathcal{Y}, \mathcal{W} \in \Lambda(\mathcal{V})$ such that $\mathcal{Y} \neq \mathcal{W}$. By Proposition 4.34 we have that since $\mathcal{X}$ is DNA-locally finite then it has the hereditary finite model property. Therefore, it follows from Theorem 4.27 that every subvariety of $\mathcal{X}$ is generated by its finite regular subdirectly irreducible elements. So we have that $\mathcal{Y} = \mathcal{Y}_{RFSI}$ and $\mathcal{W} = \mathcal{W}_{RFSI}$ and so it follows that $\mathcal{Y}_{RFSI} \neq \mathcal{W}_{RFSI}$. (iii) For surjectivity, consider any downset $D \in Dw(\mathcal{X}_{RFSI})$. Then this defines a DNA-variety $\mathcal{Y} = \mathcal{X}(D)$. We now claim that $D = \mathcal{Y}_{RFSI}$. For the left-to-right inclusion suppose $A \in D$, then we also have that $A \in \mathcal{X}_{RFSI}$ and $A \in \mathcal{X}(D) = \mathcal{Y}$, which together imply $A \in \mathcal{Y}_{RFSI}$. For the other direction, suppose that $A \in \mathcal{Y}_{RFSI}$, then we have by Lemma 4.31 that $A \nvDash^{\neg} \chi^{\neg}(A)$. Then since $A \in \mathcal{Y} = \mathcal{X}(D)$ it follows that there is some $B \in D$ such that $B \nvDash^{\neg} \chi^{\neg}(A)$. Finally, it follows by the Jankov theorem for DNA-varieties 4.32 that $A \leq B$ and thus since D is a downset that $A \in D$. □

Moreover, we can also show how one can use Jankov formulas to axiomatise subvarieties of a DNA-variety $\mathcal{X}$ which is locally finite. To this end, we notice that for every proper subvariety $\mathcal{Y} \in \Lambda^{\neg}(\mathcal{X})$ we have that $\mathcal{Y}_{RFSI}$ is a downset and $\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}$ is a nonempty upset over $\mathcal{X}_{RFSI}$. Now, since every algebra in $H \in \mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}$ is finite, we cannot have infinite descending chains of the form $H_0 \geq H_1 \geq H_2 ...$, for $|H_n| \geq |H_{n+1}|$ and $|H_n|$ is finite. It follows that every set of the form $\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}$ has some minimal element. We thus define the following notion of minimal counterexamples of a subvariety of $\mathcal{X}$.

Definition 4.36 (Minimal Counterexample)**.** Let $\mathcal{Y} \in \Lambda^{\neg}(\mathcal{X})$ be a subvariety of $\mathcal{X}$ such that $\mathcal{Y} \neq \mathcal{X}$. A *minimal counterexample* to $\mathcal{Y}$ is a Heyting algebra $H \in \mathcal{X} \setminus \mathcal{Y}$ such that for all $K \leq H$, if $K \ncong H$ then $K \in \mathcal{Y}$.

For every $\mathcal{Y} \in \Lambda^{\neg}(\mathcal{X})$, we denote by $min(\mathcal{X} \setminus \mathcal{Y})$ its collection of minimal counterexamples in $\mathcal{X}$. It follows from our previous reasoning that this collection is always nonempty when $\mathcal{Y}$ is a proper subvariety of $\mathcal{X}$. We prove the following theorem.

Theorem 4.37. *Let $\mathcal{X}$ be a locally finite* DNA*-variety, then for every subvariety $\mathcal{Y} \in \Lambda^{\neg}(\mathcal{X})$ such that $\mathcal{Y} \neq \mathcal{X}$ we have that:*

$$\mathcal{Y} = \mathcal{X}\{H \in \mathcal{X}_{RFSI} : H \vDash^{\neg} \chi(A) \text{ for all } A \in min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})\}.$$

Proof. It suffices to show that $\mathcal{Y}_{RFSI} = \{H \in \mathcal{X}_{RFSI} : H \vDash^{\neg} \chi(A) \text{ for all } A \in min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})\}$. ($\subseteq$) Suppose $H \in \mathcal{Y}_{RFSI}$, then since $\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}$ is a nonempty upset it follows that $min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}) \neq \emptyset$. But then, for all $A \in min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})$ we have that $A \nleq H$. Therefore, it follows by Jankov theorem for DNA-varieties (Theorem 4.32) that $H \vDash^{\neg} \chi(A)$ and so $H \in \{H \in \mathcal{X}_{RFSI} : H \vDash^{\neg} \chi(A) \text{ for all } A \in min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})\}$. ($\supseteq$) Suppose now that $H \in \{H \in \mathcal{X}_{RFSI} : H \vDash^{\neg} \chi(A) \text{ for all } A \in min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})\}$, then for all $A \in min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})$ it follows that $H \vDash^{\neg} \chi(A)$, hence by Theorem 4.32 we have that $A \nleq H$. But then, since $min(\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI})$ is the set of minimal elements in $\mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}$, it follows that $H \notin \mathcal{X}_{RFSI} \setminus \mathcal{Y}_{RFSI}$ and so since $H \in \mathcal{X}_{RFSI}$ that $H \in \mathcal{Y}_{RFSI}$. □

The previous theorem provides a set of formulas which axiomatise the subvarieties of a locally finite variety. By the dual isomorphism $\mathbf{DNAL} \cong^{op} \mathbf{DNAV}$ we can extend the previous result to the corresponding DNA-logics. We say that a DNA-logic Π is an *extension* of a DNA-logic

Λ if $\Lambda \subseteq \Pi$. Theorem 4.37 thus immediately allows us to axiomatise the extensions of a logic Λ which is locally tabular. We denote by $Var^{\neg}_{RFSI}(\Lambda)$ the collection of finite, regular, subdirectly irreducible elements of the DNA-variety $Var^{\neg}(\Lambda)$ and by $\Lambda + \Gamma$ the closure under modus ponens of the set of formulas $\Lambda \cup \Gamma$.

Corollary 4.38. *Let Λ be a locally tabular* DNA*-logic. Then every* DNA*-logic Π such that $\Lambda \subseteq \Pi$ can be axiomatised as follows:*

$$\Pi = \Lambda + \{\chi(A) : A \in min(Var^{\neg}_{RFSI}(\Lambda) \setminus Var^{\neg}_{RFSI}(\Pi)\}.$$

Proof. Since Λ is locally tabular we have that $Var^{\neg}(\Lambda)$ is locally finite. Moreover, since $\Lambda \subseteq \Pi$ it follows by DNA-duality that $Var^{\neg}(\Pi) \subseteq Var^{\neg}(\Lambda)$. Let $K = \{H \in Var^{\neg}_{RFSI}(\Lambda) : H \vDash^{\neg} \chi(A)$ for all $A \in min(Var^{\neg}_{RFSI}(\Lambda) \setminus Var^{\neg}_{RFSI}(\Pi))\}$, then by Theorem 3.29 above it follows that $Var^{\neg}(\Pi) = \mathcal{X}(K)$. Moreover, we have by Theorem 4.19 that $Log^{\neg}(\mathcal{X}(K)) = Log^{\neg}(K)$. By DNA-duality we then have:

$$\Pi = Log^{\neg}(Var^{\neg}(\Pi)) = Log^{\neg}(\mathcal{X}(K)) = Log^{\neg}(K).$$

Hence, since it is easy to see that $Log^{\neg}(K) = \Lambda + \{\chi(A) : A \in min(Var^{\neg}_{RFSI}(\Lambda) \setminus Var^{\neg}_{RFSI}(\Pi)\}$, we finally obtain that $\Pi = \Lambda + \{\chi(A) : A \in min(Var^{\neg}_{RFSI}(\Lambda) \setminus Var^{\neg}_{RFSI}(\Pi)\}$, which proves our claim □

We will apply Corollary 4.38 and the method of Jankov formulas in next section to axiomatize the extensions of the system InqB of inquisitive logic.

5 Linearity of the Extensions of InqB

In this section we put to work the general theory of DNA-logics that we have developed in the previous sections and we provide a characterisation of the extensions of the system InqB of inquisitive logic. In particular, we use the algebraic semantics of DNA-logics to show that InqB is locally tabular and it can therefore be studied by using the method of Jankov formulas. We thus prove that the sublattice of DNA-logics which extend InqB is linearly ordered and that it coincides with the *inquisitive hierarchy* considered by Ciardelli in [10].

5.1 Axiomatisation of InqB

It is a well-known fact since the early days of inquisitive semantics that InqB can be considered as the negative variant of the intermediate logics ND, KP and ML. This result was originally proved by Ciardelli in [10, Thm. 3.4.9] and it immediately entails that InqB is a DNA-logic.

Theorem 5.1. $\mathtt{InqB} = \mathtt{ND}^{\neg} = \mathtt{KP}^{\neg} = \mathtt{ML}^{\neg}$.

Corollary 5.2. InqB *is a* DNA*-logic.*

This result allows us to apply the general setting described in this article to study InqB. Moreover, it allows to introduce an axiomatisation of InqB which makes use of the KP-axiom.

Theorem 5.3 (Axiomatisation of InqB)**.** *The following system of axioms and rules axiomatises* InqB*:*

Axioms *All tautologies of* IPC
$(\neg\varphi \to \psi \vee \chi) \to (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$ *for all* $\varphi, \psi, \chi \in \mathcal{L}_P$
$\neg\neg p \to p$ *for all* $p \in \mathtt{AT}$
Rule $\varphi, \varphi \to \psi \Rightarrow \psi$.

This proof system is presented in [5] and was first formulated in [10]. One may also wonder what are greatest and least elements in the sublattice $\mathcal{I}(\texttt{InqB})$. It was shown already in [10] that ND is the least intermediate logic having InqB as its negative variant and ML the greatest such logic. A different proof is given in [40], where it is shown that $Var(\texttt{ND}) = Var^{\neg}(\texttt{InqB})$.

5.2 The Method of ND-*extensions*

We introduce the ND-*extension* of a Boolean algebra, analogously as [5] defines it for KP-extensions. Let B be any Boolean algebra and consider the term algebra $T(B)$ over the signature $(\dot{\wedge}, \dot{\vee}, \dot{\rightarrow}, \dot{1}, \dot{0})$. The algebra $T(B)$ consists of all propositional formulas built over the set of atomic letters B, that is, using the elements of B as propositional formulas:

$$T(B) = \{\varphi(b_0, ..., b_n) : b_i \in B \text{ and } \varphi \text{ is a formula in } (\dot{\wedge}, \dot{\vee}, \dot{\rightarrow}, \dot{1}, \dot{0})\}.$$

Since $T(B)$ is a term algebra, we have that its algebraic operations are exactly the signature operations, i.e., we have that $\varphi \wedge_{T(B)} \psi = \varphi \dot{\wedge} \psi$ etc. We now quotient the term algebra $T(B)$ to obtain an ND-algebra. In order to do this, we define the congruence $\equiv^e_{\texttt{ND}}$.

Definition 5.4. Let B be an arbitrary Boolean algebra, then the congruence $\equiv^e_{\texttt{ND}}$ is the least congruence containing $\equiv_{\texttt{ND}}$ and such that for all $p, q \in B$ we have: $1_B \equiv^e_{\texttt{ND}} \dot{1}$, $0_B \equiv^e_{\texttt{ND}} \dot{0}$, $p \wedge_B q \equiv^e_{\texttt{ND}} p \dot{\wedge} q$, $p \rightarrow_B q \equiv^e_{\texttt{ND}} p \dot{\rightarrow} q$.

The ND-*extension* $H^{\texttt{ND}}(B)$ of B is then defined as the quotient algebra $T(B)/\equiv^e_{\texttt{ND}}$. Notice that since $\equiv_{\texttt{IPC}} \subseteq \equiv_{\texttt{ND}} \subseteq \equiv^e_{\texttt{ND}}$ we have that $H^{\texttt{ND}}(B)$ validates all the validities of IPC and thus is a Heyting algebra. KP-extensions are introduced analogously in [5] by using the equivalence relation $\equiv_{\texttt{KP}}$ instead of $\equiv_{\texttt{ND}}$.

Recall that a downset D over a poset $(P, \leq)$ is *finitely generated* if there is a nonempty, finite set of elements $x_0, ..., x_n$ such that $D = {\downarrow}\{x_0, ..., x_n\}$. We denote by $Dw_{fg}(B)$ the set of finitely generated downsets over B and we leave to the reader to verify that this form a Heyting algebra with the order induced by the set-theoretic inclusion. It was shown in [5] that $H^{\texttt{KP}}(B) \cong Dw_{fg}(B)$ and in [40] that $H^{\texttt{ND}}(B) \cong Dw_{fg}(B)$. Hence we have the following result.

Theorem 5.5. *Let B be a Boolean algebra, then $H(B)^{\texttt{ND}} \cong H(B)^{\texttt{KP}} \cong Dw_{fg}(B)$.*

We will henceforth drop the superscript and denote the ND-extension of B just by $H(B)$. We now recall some important facts about ND-extensions. The proof of the following claims was given in [5] for KP-extensions and in [40] for ND-extensions. The following proposition is an important universal mapping property of such constructions.

Proposition 5.6 (Universal Mapping Property)**.** *Let B be a Boolean algebra and $H(B)$ its ND-extension, then for every Heyting algebra K such that $K \vDash \texttt{ND}$ and $K_{\neg} = B$ there is a unique homomorphism $h : H(B) \to K$ such that $h \restriction B = id_B$. Moreover, if K is regular then h is also surjective.*

The following proposition gives us a description of the structure of the *ND-extension* $H(B)$ of a Boolean algebra B. In particular, we show that every element of $H(B)$ can be written in a unique way as a disjunction of elements of B. Following [5] we say that every $x \in H(B)$ has a *non-redundant representation*. With a slight abuse of notation we henceforth drop the square brackets and refer to elements of $H(B)$ as formulas rather than equivalence classes thereof. Also, since the algebra operations of $H(B)$ agree with the connectives in $(\dot{\wedge}, \dot{\vee}, \dot{\rightarrow}, \dot{1}, \dot{0})$, we drop the dots and use the same symbols both for connectives and operations. The proof of the following two propositions was first given in [5] and later adapted to the context of ND-extensions in [40].

Proposition 5.7. *For every $x \in H(B)$ we have that $x = \bigvee_{i \leq n} a_i$ where $a_i \in B$ for all $i \leq n$ and $a_i \not\leq a_j$ for $i \neq j$. Moreover $a_1, ..., a_n$ are uniquely determined.*

Moreover $H(B)$ is *well-connected*, i.e., we have that for any $x, y \in H(B)$ it is the case that $x \vee y = 1$ entails $x = 1$ or $y = 1$.

Proposition 5.8. *For any Boolean algebra B, its $\mathtt{ND}$-extension $H(B)$ is well-connected.*

We now make a short digression on Medvedev frames, that is, frames of the form $\mathcal{F} \cong (\wp_0(W), \supseteq)$, where W is a finite set. We recall the following two propositions on $\mathtt{ML}$-frames. The first connects the validity over a $\mathtt{ML}$-frame $\mathcal{F}$ to the validity of its corresponding downset algebra $Dw(\mathcal{F})$ (see e.g., [40, Sec. 5.1]), and it follows readily from the correspondence between Kripke semantics and algebraic semantics for finite frames [8, 17]. The second connects validity in $\mathtt{ML}$-frames and validity in state models, and it was already pointed out in [10]. A proof of both these results is contained in [40].

Proposition 5.9. *For every Medvedev frame $\mathcal{F}$ we have that $\mathcal{F} \Vdash \varphi$ iff $Dw(\mathcal{F}) \vDash \varphi$.*

Proposition 5.10. *Let $\mathcal{M} = (\wp_0(W), \supseteq, V)$ be a Medvedev model, then for any formula $\varphi \in \mathcal{L}_P$ and any $s \in \wp_0(W)$ we have that $X_s \vDash \varphi$ if and only if $(s, V) \Vdash \varphi[\overline{\neg p}/\overline{p}]$.*

5.3 Characterisation of $Var^{\neg}(\mathtt{InqB})$

We use the results that we recalled in the previous sections to characterise a set of generators of $Var^{\neg}(\mathtt{InqB})$. First, we use the method of the $\mathtt{ND}$-*extension* of a Boolean algebra to show that $\mathtt{InqB}$ is locally tabular. The following theorem also follows as an easy corollary of [5, Lemmas 4.1 and 4.3].

Theorem 5.11. $\mathtt{InqB}$ *is locally tabular.*

Proof. We need to show that every $\mathtt{DNA}$-finitely generated $\mathtt{InqB}$-algebra is finite. Consider any $H \in Var^{\neg}(\mathtt{InqB})$ and suppose H is $\mathtt{DNA}$-finitely generated, then there are elements $x_0, ..., x_n \in H_{\neg}$ such that $\langle x_0, ..., x_n \rangle = H$. In particular, H is regular. Moreover, by the fact that $\mathtt{ND} = Log(Var^{\neg}(\mathtt{InqB}))$ we also have that $Var^{\neg}(\mathtt{InqB}) = Var(\mathtt{ND})$ and so $H \in Var(\mathtt{ND})$.

Notice that $H_{\neg}$ is generated as a Boolean algebra by $x_0, \ldots, x_n$ and so in particular it is finite. Moreover, by Theorem 5.5, $H(H_{\neg}) \cong Dw_{fg}(H_{\neg})$ is also finite. By Proposition 5.6, $f : H(H_{\neg}) \twoheadrightarrow H$. So it follows that H is finite, as wanted. □

Since $\mathtt{InqB}$ is locally tabular, we have by Theorem 4.27 that it is generated by its collection of finite, regular, subdirectly irreducible elements. The next theorem provides a characterisation of this class of $\mathtt{InqB}$-algebras. Our proof adapts [5, Thm. 4.6].

Theorem 5.12. *Let H be an Heyting algebra. Then $H \in Var^{\neg}_{RFSI}(\mathtt{InqB})$ iff there is some finite Boolean algebra B such that $H \cong H(B)$.*

Proof. ($\Leftarrow$) Suppose $H \cong H(B)$ for some finite Boolean algebra B, then we need to show that H is finite, regular and subdirectly irreducible. First, since $H \cong H(B) \cong DW_{fg}(B)$, it follows that H is finite. Second, it by construction $H(B)$ is regular and so H is regular as well. Finally, by Proposition 5.8 $H(B)$ is well-connected and so—since it is finite—it has a second-greatest element, that is, it is a subdirectly irreducible algebra.

($\Rightarrow$) Let $H \in Var^{\neg}_{RFSI}(\mathtt{InqB})$, then since H is regular and $Var^{\neg}(\mathtt{InqB}) = Var(\mathtt{ND})^{\uparrow}$, it follows by Proposition 4.15 that $H \in Var(\mathtt{ND})$. From the universal mapping property of Proposition 5.6 there is a surjective homomorphism $h : H(H_{\neg}) \twoheadrightarrow H$, where $H_{\neg}$ is a finite Boolean algebra. We prove now that this homomorphism is also injective. Consider

$x, y \in H$ such that $h(x) = h(y)$, then it follows Proposition 5.7 that we have non-redundant representations $x = \bigvee_{i \leq n} a_i$ and $y = \bigvee_{j \leq m} b_j$. Since for all $i \leq n, j \leq m$ we have $a_i, b_j \in H_\neg$, it follows by Proposition 5.6 that $h \restriction H_\neg = id_{H_\neg}$ and so that $h(a_i) = a_i$ and $h(b_j) = b_j$, which means that $h(a_j), h(b_j) \in H_\neg$. Now, since $h(x) = h(y)$, we have that $h(\bigvee_{i \leq n} a_i) \leq h(\bigvee_{j \leq m} b_j)$ and $h(\bigvee_{j \leq m} b_j) \leq h(\bigvee_{i \leq n} a_i)$. From the former of these claims we have:

$$
\begin{aligned}
& h\left(\bigvee_{i \leq n} a_i\right) \leq h\left(\bigvee_{j \leq m} b_j\right) \\
\Longrightarrow & \bigwedge_{i \leq n}\left[h(a_i) \rightarrow \bigvee_{j \leq m} h(b_j)\right] = 1 && \text{(by properties of IPC)} \\
\Longrightarrow & \bigwedge_{i \leq n}\left[\neg\neg h(a_i) \rightarrow \bigvee_{j \leq m} h(b_j)\right] = 1 && \text{(since } h(a_i) \in H_\neg\text{)} \\
\Longrightarrow & \bigwedge_{i \leq n} \bigvee_{j \leq m} [\neg\neg h(a_i) \rightarrow h(b_j)] = 1 && \text{(using axioms of ND)} \\
\Longrightarrow & \bigwedge_{i \leq n} \bigvee_{j \leq m} [h(a_i) \rightarrow h(b_j)] = 1 && \text{(since } h(a_i) \in H_\neg\text{)} \\
\Longrightarrow & \forall i \leq n, \exists j \leq m \text{ such that } h(a_i) \leq h(b_i) && \text{(since } H \text{ is well-connected)} \\
\Longrightarrow & \forall i \leq n, \exists j \leq m \text{ such that } a_i \leq b_i && \text{(since } h \restriction H_\neg = id_{H_\neg}\text{)} \\
\Longrightarrow & x \leq y.
\end{aligned}
$$

Similarly, starting from $h(\bigvee_{j \leq m} b_j) \leq h(\bigvee_{i \leq n} a_i)$ we then get that $y \leq x$ and so that $x = y$. Finally, this means that the surjective homomorphism $h : H(H_\neg) \twoheadrightarrow H$ is also injective and so that $H \cong H(H_\neg)$. □

5.4 Extensions of InqB

Finally, we can prove our main result concerning extensions of InqB. From the former theorem it is easy to prove the following important lemma. We recall from Section 4.5 that if A and B are two Heyting algebras, the order $\leq$ between them is defined as $A \leq B$ iff $A \in HS(B)$. The next lemma shows that under this ordering the collection of regular, finite, subdirectly irreducible InqB-algebras is isomorphic to ω.

Lemma 5.13. *Let $Var^{\neg}_{RFSI}$(InqB) be the collection of finite, regular, subdirectly irreducible* InqB*-algebras. Then we have that*

$$(Var^{\neg}_{RFSI}(\texttt{InqB}), \leq) \cong \omega.$$

Proof. We show that $Var^{\neg}_{RFSI}$(InqB) is isomorphic to ω under the order $A \leq B$ iff $A \in HS(B)$. First, consider any algebra $H \in Var^{\neg}_{RFSI}$(InqB), then it follows by Theorem 5.12 that there is some finite Boolean algebra B such that $H = H(B)$. The representation theorem of the finite Boolean algebras entails that finite Boolean algebras form the following chain of length ω:

$$2^0 \preceq 2^1 \preceq 2^2 \preceq 2^3 \preceq 2^4 \preceq \dots$$

Now, we have by the definition of the ND*-extension* of a Boolean Algebra 2^n that $H(2^n)$ is regular and $H(2^n) = \langle 2^n \rangle$. Therefore, since we have that for all $n \in \mathbb{N}$, $2^n \preceq 2^{n+1}$, it follows

that $H(2^n) \preceq H(2^{n+1})$. Finally, since every $H \in Var^{\neg}_{RFSI}(\mathtt{InqB})$ is of the form $H(2^n)$ for some $n \in \mathbb{N}$, it follows that:

$$H(2^0) \preceq H(2^1) \preceq H(2^2) \preceq H(2^3) \preceq H(2^4) \preceq \dots$$

is a chain of length ω ordered by $A \leq B \Leftrightarrow A \in HS(B)$ which contains every element $H \in Var^{\neg}_{RFSI}(\mathtt{InqB})$. Finally, this means that the poset $(Var^{\neg}_{RFSI}(\mathtt{InqB}), \leq)$ is isomorphic to ω. □

Once we have the previous lemma, we can use use the method of Jankov formulas for DNA-logics developed in Section 4.5 to show that the lattice of extensions of the system of inquisitive logic InqB is linearly ordered and dually isomorphic to $\omega + 1$.

Theorem 5.14. *Let* $\Lambda^{\neg}(\mathtt{InqB})$ *be the lattice of extensions of* InqB. *Then there is a dual isomorphism* $\Lambda^{\neg}(\mathtt{InqB}) \cong^{op} \omega + 1$.

Proof. By the dual isomorphism **DNAL** $\cong^{op}$ **DNAV** we immediately have that $\Lambda^{\neg}(\mathtt{InqB}) \cong^{op} \Lambda^{\neg}(Var^{\neg}(\mathtt{InqB}))$, where $\Lambda^{\neg}(Var^{\neg}(\mathtt{InqB}))$ is the lattice of subvarieties of $Var^{\neg}(\mathtt{InqB})$. Therefore, to show that $\Lambda^{\neg}(\mathtt{InqB}) \cong^{op} \omega+1$ it suffices to show that $\Lambda^{\neg}(Var^{\neg}(\mathtt{InqB})) \cong \omega+1$. Now, by Proposition 5.11 we have that InqB is locally tabular and therefore it follows by Theorem 4.35 that $\Lambda^{\neg}(Var^{\neg}(\mathtt{InqB})) \cong Dw(Var^{\neg}_{RFSI}(\mathtt{InqB}))$. But then, we have by Lemma 5.13 that $Var^{\neg}_{RFSI}(\mathtt{InqB}) \cong \omega$ and therefore that $Dw(Var^{\neg}_{RFSI}(\mathtt{InqB})) \cong Dw(\omega) = \omega + 1$. To sum up, we have:

$$\Lambda^{\neg}(\mathtt{InqB}) \cong^{op} \Lambda^{\neg}(Var^{\neg}(\mathtt{InqB})) \cong Dw(Var^{\neg}_{RFSI}(\mathtt{InqB})) \cong Dw(\omega) = \omega + 1,$$

which proves our claim. □

The method of Jankov formulas allows us also to provide an axiomatisation for all the extensions Λ of InqB. Then by DNA-duality and Theorem 4.35 we have that $\Lambda^{\neg}(\mathtt{InqB}) \cong^{op} Dw(Var^{\neg}_{RFSI}(\mathtt{InqB}))$. Therefore we have that extensions Λ of InqB are uniquely identified by specifying a downset of elements of $Var^{\neg}_{RFSI}(\mathtt{InqB})$. For any $n \in \mathbb{N}$, we define by $\mathtt{InqB}_n$ the DNA-logic $\mathtt{InqB}_n = Log^{\neg}(\downarrow H(2^n))$. We now prove the following proposition.

Proposition 5.15. *Let* Λ *be a proper extension of* InqB, *i.e.,* Λ *is a* DNA*-logic and* $\mathtt{InqB} \subsetneq \Lambda$. *Then there is some* $n \in \mathbb{N}$ *such that*

$$\Lambda = \mathtt{InqB}_n = \mathtt{InqB} + \chi(H(2^{n+1})).$$

Proof. Suppose that Λ is a DNA-logic and $\mathtt{InqB} \subsetneq \Lambda$, then it follows by Theorem 4.35 that $Var^{\neg}(\Lambda) = \mathcal{X}(D)$, where $D \in Dw(Var^{\neg}_{RFSI}(\mathtt{InqB}))$. Now, since $\Lambda \neq \mathtt{InqB}$, it follows that $D \neq Var^{\neg}_{RFSI}(\mathtt{InqB})$. Therefore, it follows immediately from Lemma 5.13 that $D = \downarrow H(2^n)$ for some $n \in \mathbb{N}$ and hence $\Lambda = \mathtt{InqB}_n$. Moreover, it is easy to see that the only minimal counterexample in $Var^{\neg}(\mathtt{InqB}) \setminus Var^{\neg}(\mathtt{InqB}_n)$ is $H(2^{n+1})$. Therefore, we have by Theorem 4.38 that $\mathtt{InqB}_n$ is equivalent to $\mathtt{InqB} + \chi(H(2^{n+1}))$. □

The previous result allows us to introduce in an alternative way the inquisitive hierarchy originally introduced by Ciardelli [10, Ch. 4]. We define, for every $n \in \mathbb{N}$, the system $\mathtt{InqL}_n$ as follows:

$$\mathtt{InqL}_n = \{\varphi \in \mathcal{L}_P : \forall s \in \wp(2^{\mathtt{AT}}), \text{ such that } |s| \leq n, s \vDash \varphi\}.$$

We can now show that the inquisitive hierarchy is exactly the sublattice of all the proper extensions of InqB. Firstly, we say that a DNA-logic is *tabular* if it is the logic of a finite regular Heyting algebra. Then, since for all $H \in \downarrow H(2^n)$ we have that $H \preceq H(2^n)$, it follows immediately that $\mathtt{InqB}_n = Log^{\neg}(\downarrow H(2^n)) = Log^{\neg}(H(2^n))$, i.e. $\mathtt{InqB}_n$ is the logic of $H(2^n)$ and is thus tabular. Then we obtain the following theorem.

Theorem 5.16. *For any $n \in \mathbb{N}$, we have that* $\mathtt{InqB}_n = \mathtt{InqL}_n$.

Proof. For any $n \in \mathbb{N}$, we have the following equalities:

$$\begin{aligned}
\mathtt{InqB}_n &= Log^{\neg}(\downarrow H(2^n)) \\
&= Log^{\neg}(H(2^n)) \\
&= Log(H(2^n))^{\neg} && \text{(by Proposition 3.23)} \\
&= Log(Dw_{fg}(2^n))^{\neg} && \text{(by Theorem 5.5)} \\
&= \{\varphi \in \mathcal{L}_P : \wp_0(n) \Vdash \varphi[\overline{\neg p}/\overline{p}]\} && \text{(by Proposition 5.9)} \\
&= \{\varphi \in \mathcal{L}_P : n \vDash \varphi\} && \text{(by Proposition 5.10)} \\
&= \{\varphi \in \mathcal{L}_P : \forall s \in \wp(2^{\mathtt{AT}}), \text{ such that } |s| \leq n, s \vDash \varphi\} \\
&= \mathtt{InqL}_n.
\end{aligned}$$

Which proves our claim. □

Therefore, by defining for every $n \in \mathbb{N}$ the logic $\mathtt{ML}_n$ as the set of formulas valid in all Medvedev frames $\mathcal{F}$ whose cardinality is $|\mathcal{F}| \leq n$, it follows from the previous theorem that $(\mathtt{ML}_n)^{\neg} = \mathtt{InqB}_n = \mathtt{InqL}_n$. The following corollary follows directly from Theorem 5.14 and Theorem 5.16 and is an extension of [10, Cor. 4.1.6].

Corollary 5.17.

$$\mathtt{InqB} = \bigcap_{n\in\mathbb{N}} \mathtt{InqB}_n = \bigcap_{n\in\mathbb{N}} \mathtt{InqL}_n = \bigcap_{n\in\mathbb{N}} (\mathtt{ML}_n)^{\neg}.$$

The results in this section thus provide a characterisation of the extensions of InqB and show that they coincide precisely with the inquisitive hierarchy already studied in the literature.

6 Conclusion

In this article we developed algebraic semantics for DNA-logics and we applied this general setting to inquisitive logic. This semantics allows to apply methods of universal algebra to study DNA-logics and inquisitive logic from a novel perspective. Let us briefly summarize our main results. In Section 3 we introduced DNA-logics and their algebraic semantics and we proved the dual isomorphism **DNAL** $\cong^{op}$ **DNAV** between DNA-logics and DNA-varieties. In Section 4 we studied closer the relation between DNA-logics and intermediate logics and we proved a suitable version of some classical results for the setting of DNA-varieties. In particular, we showed that every DNA-variety is generated by its regular subdirectly irreducible members and we introduced a suitable version of Jankov formulas in order to axiomatise locally finite DNA-varieties. Finally, in Section 5 we used the algebraic semantics of DNA-logics to study the inquisitive logic InqB. In particular, we showed that the sublattice of its extensions is dually isomorphic to $\omega + 1$ and that it actually coincides with the inquisitive hierarchy studied in [10].

In addition to these results, in our view one of the main contributions of this article is that it provides a new setting for the study of inquisitive logic. The system InqB had so far been considered as the logic of the evaluation states or as the negative variant of the logics between ND and ML – here we showed that one can also consider InqB as the logic of a specific class of Heyting algebras, under a suitable semantics. This new perspective at the propositional system of inquisitive logic allows us to raise new questions and consider new issues.

One direction for future study is to consider what happens if, instead of the negative substitution $p \mapsto \neg p$, we consider the substitution $p \mapsto \chi(p)$ for an arbitrary polynomial $\chi \in \mathcal{L}_P$. In fact, it seems possible to extend at least part of the theory of DNA-logics to this extended framework. In the case of negative variants we rely on the fact that in intuitionistic logic $\neg\neg\neg p = \neg p$. This property however is shared in a more general form by every polynomial χ. Ruitenberg's Theorem [22, 42] states that for any polynomial χ we can find a number $n \in \mathbb{N}$ such that $\chi^n = \chi^{n+2}$. This allows to introduce the χ-variant of an intermediate logic L as $L^\chi = \{\varphi \in \mathcal{L}_P : \varphi[\overline{\chi^n(p)}/\overline{p}] \in L\}$ and to generalize our study of DNA-logics to arbitrary χ-variants. We refer the reader to the upcoming [23].

Similarly, the close connection between inquisitive logic and dependence logic that has been studied e.g., in [9, 11] suggests that a similar semantics might be developed for the system of propositional dependence logic [45]. This direction, originally hinted at in [1], also raises the issue of possible connections between the present framework and the setting of residuated lattices which is employed to give a semantics to separation logic and related formalisms [31].

Finally, a last direction of further research should go towards a deeper understanding of DNA-logics and DNA-varieties. For instance, a new topological semantics for InqB is introduced in [5]. This raises a question whether it is possible to translate our framework in topological terms and describe a suitable topological semantics for DNA-logics. Moreover, such semantics could be used to give a characterisation of finite regular subdirectly irreducible Heyting algebras. We know by Esakia duality that a finite subdirectly irreducible Heyting algebra is the upset algebra of a finite rooted frame. Can we obtain a similar characterisation for regular finite subdirectly irreducible Heyting algebras? What properties should a rooted frame satisfy in order for its dual Heyting algebra to be regular? Finally, it also seems natural to generalize Jankov formulas for DNA-models to canonical formulas, as it is the case both for intermediate [2] and modal logics [3]. These questions are for the moment open problems to be addressed in future investigations.

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