Logical systems with left-sequential versions of NAND and XOR

MSc Thesis (*Afstudeerscriptie*)

written by

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Abstract

In this thesis, we define two new left-sequential connectives ℓNAND and ℓXOR that prescribe a short-circuit evaluation strategy, in which the second argument is evaluated only if the first argument does not suffice to determine the value of an expression, and a full evaluation strategy, in which each argument is evaluated, respectively. First, we define and axiomatize free left-sequential nand logic (FLNL) to investigate which logical laws axiomatize short-circuit evaluation of terms with ℓNAND in a setting where repeated occurrences of the same atomic proposition can yield different Boolean values, that is, modulo free valuation congruence. Then, we define and axiomatize free left-sequential xor logic (FLXL) to investigate which logical laws axiomatize full evaluation of terms with ℓXOR in the same setting. Finally, we investigate expressiveness modulo free valuation congruence of terms with (combinations of) ℓNAND, ℓXOR, left-sequential short-circuit conjunction and disjunction (the primitive connectives of short-circuit logic) and Hoare's conditional (the primitive connective of proposition algebra).

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Contents

Chapter 1

Introduction

Most programming languages have logical and bitwise conjunction and disjunction operators, whose operands are evaluated from left to right. These different types of operators prescribe different left-sequential evaluation strategies, namely *short-circuit (left-sequential) evaluation* and *full (left-sequential) evaluation*. Using the following block of Python code, we discuss these evaluation strategies.

```
def print_and_return(value):
    print(str(value) + " is evaluated")
    return value

print("The outcome of True and False is: " +
    str(print_and_return(True) and print_and_return(False)))

print("The outcome of False and True is: " +
    str(print_and_return(False) and print_and_return(True)))

print("The outcome of False & True is: " +
    str(print_and_return(False) & print_and_return(True)))
```

The function `print_and_return()` simply prints and returns its input. It allows us to check which operands of the three expressions at stake are evaluated when computing their outcomes. The expression `True and False` has outcome `False` and both Boolean values are evaluated. The expression `False and True` has outcome `False` and only the first Boolean value `False` is evaluated. Irrespective of the value of the second operand, this expression will always evaluate to `False` because the first operand evaluates to `False`. The evaluation strategy prescribed by the `and`-operator, in which the second operand is evaluated only if the first operand does not suffice to determine the value of the expression, is called *short-circuit evaluation*. Note that the evaluation of `True and False` cannot be short-circuited. Finally, `False & True` also has outcome `False`, but now both operands are evaluated. The evaluation strategy prescribed by `&`, in which each operand is evaluated, is called *full evaluation*.

Furthermore, the `and`-operator is not commutative. When we initialize the variable `var = 2`, the expression

`var += 1 and var == 2`, (1)

in which `var` is first incremented by one and then tested for equality to 2, evaluates to `False`, whereas the expression

`var == 2 and var += 1`, (2)

in which the order of evaluation is switched, evaluates to `True`.

We say that the atomic proposition `var += 1` yields an *atomic side effect* because it affects the state of the execution environment. In (1) this side effect affects the evaluation result, while it does not in (2). In the following we will refer to atomic propositions as *atoms*.

In [BPS13] the question is raised which logical laws axiomatize short-circuit evaluation. This question can be answered using short-circuit logics (SCL). A generic definition of SCL is given in [BP12] in terms of Proposition Algebra [BP11], using the connectives $\neg$, ${}_{\circ}\!\!\wedge$ and ${}^{\circ}\!\!\vee$. The connectives ${}_{\circ}\!\!\wedge$ and ${}^{\circ}\!\!\vee$ represent short-circuit left-sequential versions of conjunction and disjunction respectively. The notation stems from [BBR95]. The position of the circle indicates which argument is evaluated first. Furthermore, an empty circle prescribes a short-circuit evaluation strategy and a filled circle prescribes a full evaluation strategy, e.g. ${}_{\bullet}\!\!\wedge$ is the full left-sequential version of $\wedge$. Because negation is a unary connective it prescribes only one evaluation strategy, so $\neg$ has no circle.

An answer to the above-mentioned question depends on assumptions about (the possiblity of) atomic side effects and commutativity of the connectives. In [BP11] several valuation congruences are defined by means of these assumptions. A valuation congruence can be interpreted as a congruence relation between closed terms that yield the same evaluation result. Different valuation congruences lead to different short-circuit logics on which we will elaborate next.

Free SCL (FSCL) is the least identifying SCL. It characterizes *free valuation congruence*, where each atom can yield an atomic side effect during the sequential evaluation of a propositional statement, so different Boolean values may be returned for repeated occurrences of the same atom. In [PS18] an equational axiomatization of FSCL is given for closed terms. All the results in this thesis will be modulo free valuation congruence.

Memorizing SCL (MSCL) characterizes the setting where no atomic side effects are allowed. Static SCL, the variant of MSCL with commutative connectives, defines a left-sequential version of propositional logic and is the most identifying SCL. Both logics are axiomatized for closed terms in [BPS18]. In repetition-proof SCL subsequent equal atoms yield the same atomic evaluation result. Some axioms are given in [BP12] and a semantics is defined in [BP15]. Lastly, axiomatizations for contractive SCL, which characterizes the setting where subsequent equal atoms are contracted, are discussed in [vW16].

Taking the above concepts as a point of departure, we will consider the new connectives ℓNAND and ℓXOR, short-circuit left-sequential NAND and full left-sequential XOR respectively.

NAND, written as $|$, designates the truth-functional operator 'not and'. Other names for NAND are the Sheffer stroke (first mentioned in [She13] and first called thus in [Nic17]) and alternative denial. In propositional logic we can write

$$x \mid y = \neg(x \wedge y).$$

XOR, written as $\oplus$, designates the truth-functional operator 'either ... or'. An alternative name for XOR is exclusive disjunction. In propositional logic we can write

$$x \oplus y = (x \wedge \neg y) \vee (\neg x \wedge y).$$

Observe that the evaluation of XOR cannot be short-circuited. To evaluate $x \oplus y$, we must always evaluate y.

We will define ℓNAND and ℓXOR using Hoare's conditional [Hoa85], the ternary connective $y \triangleleft x \triangleright z$ that can be read as "if x then y else z". We will investigate which logical laws axiomatize short-circuit evaluation of terms with ℓNAND and which logical laws axiomatize full evaluation of terms with ℓXOR, under the assumption that each atom can yield an atomic side effect during the sequential evaluation of a propositional statement. To this end, we will define free left-sequential nand logic (FLNL) and free left-sequential xor logic (FLXL). In Chapters 3 and 4 we will provide equational axiomatizations of these logics for closed terms.

In Chapter 5 we will investigate the expressive power modulo free valuation congruence of closed terms over signatures with the SCL-connectives, with ℓXOR, with Hoare's conditional and with the SCL-connectives and ℓXOR.

The further content of this thesis is structured as follows: In Chapter 2 we lay the groundwork for this thesis by introducing results mainly from [PS18]. In Chapter 3 we define ℓNAND and FLNL and we provide an equational axiomatization of FLNL for closed terms. In Chapter 4 we define ℓXOR and FLXL and we provide an equational axiomatization of FLXL for closed terms. In Chapter 5 we investigate the expressive power modulo free valuation congruence of terms over the signatures that were previously mentioned. In Chapter 6 we make some concluding remarks. Finally, in Appendices A, B and C we provide some proofs that were omitted in Chapters 3, 4 and 5 respectively to enhance readability.

Chapter 2

Preliminaries

In this chapter we lay out the framework that will serve as the foundation of this thesis. The vast part of this chapter's content is a summary of the relevant results from [PS18]. In Section 2.1 we recall the definition of evaluation trees and leaf replacements. In Section 2.2 we introduce the ternary connective $y \triangleleft x \triangleright z$. Most connectives that will be considered in this thesis are defined using this ternary connective. Furthermore, we provide the most basic set of equational axioms (CP) for Proposition Algebra, and axioms and inferential rules of equational logic. In Section 2.3 we describe the syntax and semantics for short-circuit logics (SCL). In Section 2.4 we recall the definition of free short-circuit logic (FSCL) in terms of SCL, provide an equational axiomatization (EqFSCL) of this logic and state the result that EqFSCL axiomatizes FSCL for closed terms, which will be used in Chapter 3. In Section 2.5 we introduce the SCL Normal Form (*SNF*) and in Section 2.6 we recall a function that inverts evaluation trees to terms in *SNF*, using unique decompositions of evaluation trees. The last two sections will be used in Chapter 5.

2.1 Evaluation trees

Except for the last remark, the content in this section about evaluation trees and leaf replacements comes from [PS18]. The concepts introduced in this section are used to define the semantics of propositional statements.

From now on we let A be a non-empty countable set of atoms (atomic propositions that return a Boolean value upon evaluation) and we use constants T and F for the truth values *true* and *false* respectively.

Definition 2.1.1. *Let A be a set of atoms. The set $\mathcal{T}_A$ of **evaluation trees** over A with leaves in $\{\mathsf{T}, \mathsf{F}\}$ is defined inductively by*

$$\mathsf{T} \in \mathcal{T}_A, \quad \mathsf{F} \in \mathcal{T}_A, \quad (Y \trianglelefteq a \trianglerighteq Z) \in \mathcal{T}_A \text{ for any } Y, Z \in \mathcal{T}_A \text{ and } a \in A.$$

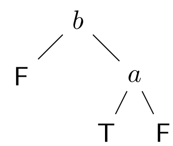
*The operator $\_ \trianglelefteq a \trianglerighteq \_$ is called **tree composition over** a. In the evaluation tree $X = (Y \trianglelefteq a \trianglerighteq Z)$, the root is represented by a, the left branch by Y and the right branch by Z. The **height** $h(X)$ is defined by*

$$h(\mathsf{T}) = h(\mathsf{F}) = 0 \text{ and } h((Y \trianglelefteq a \trianglerighteq Z)) = 1 + \max(h(Y), h(Z)).$$

The leaves of an evaluation tree represent evaluation results. Next to the formal notation for evaluation trees we also use a more pictorial representation. For example, the tree

$$(\mathsf{F} \trianglelefteq b \trianglerighteq (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F}))$$

can be depicted as Tree 1, where $\trianglelefteq$ yields a left branch, and $\trianglerighteq$ a right branch.



(Tree 1)

We now define the *leaf replacement*-operator on trees in $\mathcal{T}_A$.

Definition 2.1.2. *For $X, Y, Z \in \mathcal{T}_A$, the **leaf replacement** on X of T with Y and F with Z, denoted by*

$$X[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z]$$

is defined recursively by

$$\mathsf{T}[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] = Y, \tag{3}$$

$$\mathsf{F}[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] = Z, \tag{4}$$

$$(X_1 \trianglelefteq a \trianglerighteq X_2)[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] = (X_1[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] \trianglelefteq a \trianglerighteq X_2[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z]). \tag{5}$$

We note that the order in which the replacements of leaves of X is listed is irrelevant and adopt the convention of not listing identities inside the brackets, e.g.,

$$X[\mathsf{F} \mapsto Z] = X[\mathsf{T} \mapsto \mathsf{T}, \mathsf{F} \mapsto Z]. \tag{6}$$

For $X, Y_1, Y_2, Z_1, Z_2 \in \mathcal{T}_A$, repeated replacements on X satisfy

$$\begin{aligned} &X[\mathsf{T} \mapsto Y_1, \mathsf{F} \mapsto Z_1][\mathsf{T} \mapsto Y_2, \mathsf{F} \mapsto Z_2] = \\ &\qquad X\big[\mathsf{T} \mapsto Y_1[\mathsf{T} \mapsto Y_2, \mathsf{F} \mapsto Z_2],\ \mathsf{F} \mapsto Z_1[\mathsf{T} \mapsto Y_2, \mathsf{F} \mapsto Z_2]\big]. \end{aligned} \tag{7}$$

Note that an evaluation tree is in fact a rooted, labeled, full binary tree. That is, a rooted binary tree in which each internal node is labeled from A, has two child nodes and in which each leaf node is labeled from $\{\mathsf{T}, \mathsf{F}\}$. So we can use terminology on binary trees to reason about evaluation trees, see [CLRS01, Section B.5.3]. We call a leaf node with label T (or F) a T-leaf (or a F-leaf) and we will often refer to nodes by their labels. Finally, we define the depth of a node in an evaluation tree.

Definition 2.1.3. *Let $X = (Y \trianglelefteq a \trianglerighteq Z)$ in $\mathcal{T}_A$ and let Px be the parent node of a node x in X that is not the root. The **depth** $d_x(X)$ of a node x in X is defined by*

$$d_x(X) = \begin{cases} 0 & \text{if } x \text{ is the root of } X, \\ 1 + d_{Px}(X) & \text{otherwise.} \end{cases}$$

2.2 CP and equational logic

The ternary connective *conditional disjunction*, notation $[y, x, z]$, is defined in 1956 by Church [Chu56, pp.129-132]. Church proposes an oral reading of this disjunction by "y or z, according as x or not x". In 1985, Hoare introduces the *conditional* [Hoa85], notation $y \triangleleft x \triangleright z$, which is interpreted as follows:

$$y \triangleleft x \triangleright z = \begin{cases} y & \text{if } x \text{ is } \textit{true}, \\ z & \text{otherwise.} \end{cases}$$

Because we only consider truth values *true* and *false* in this thesis, the second clause can be read as "if x is *false*".

In [BP11] Proposition Algebra is introduced as a general setting in which expressions over the signature $\Sigma_{\text{CP}}(A) = \{a, \mathsf{T}, \mathsf{F}, \_ \triangleleft \_ \triangleright \_ \mid a \in A\}$ can be studied. There is a constant a in $\Sigma_{\text{CP}}(A)$ for each atom $a \in A$ and there are constants T and F for *true* and *false*. The abbreviation CP stands for conditional propositions.

Definition 2.2.1. *The set T^A_{CP} of closed terms over A is generated by the following grammar, where $a \in A$:*

$$t ::= a \mid \mathsf{T} \mid \mathsf{F} \mid t \triangleleft t \triangleright t.$$

We interpret propositional statements in T^A_{CP} as evaluation trees by a function *ce* (abbreviating conditional evaluation) [Sta12].

Definition 2.2.2. *The unary* ***conditional evaluation function*** $ce : T^A_{\text{CP}} \to \mathcal{T}_A$ *is defined as follows, where $a \in A$:*

$$\begin{aligned} &ce(\mathsf{T}) = \mathsf{T}, && ce(a) = (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F}), \\ &ce(\mathsf{F}) = \mathsf{F}, && ce(s \triangleleft t \triangleright u) = ce(t)[\mathsf{T} \mapsto se(s), \mathsf{F} \mapsto se(u)]. \end{aligned}$$

The overloading of the symbol T in $ce(\mathsf{T}) = \mathsf{T}$ is harmless (and similarly for F).

$$x \triangleleft \mathsf{T} \triangleright y = x \tag{CP1}$$
$$x \triangleleft \mathsf{F} \triangleright y = y \tag{CP2}$$
$$\mathsf{T} \triangleleft x \triangleright \mathsf{F} = x \tag{CP3}$$
$$x \triangleleft (y \triangleleft z \triangleright u) \triangleright v = (x \triangleleft y \triangleright v) \triangleleft z \triangleright (x \triangleleft u \triangleright v) \tag{CP4}$$

Table 2.1: CP, a set of equational axioms for Proposition Algebra

The most basic set of equational axioms for Proposition Algebra is given in Table 2.1 [BP11] and is called CP. New equations can be derived from these equational axioms using equational logic. Equational logic is a first-order logic in which only quantifier-free terms are considered and in which the only predicate symbol is '$=$'. The axioms and inferential rules of equational logic for axiom set E are listed in Table 2.2 (cf. [BS12, pp.99-108]). We write $\sigma(t)$ for the application of substitution σ to term t.

Axioms:	$t = u$	for all equations $t = u$ in E	(E)
	$t = t$	for every term t	(Reflexivity)
Rules:	$\dfrac{t = u}{u = t}$		(Symmetry)
	$\dfrac{t = s, s = u}{t = u}$		(Transitivity)
	$\dfrac{t_1 = u_1, \ldots, t_n = u_n}{f(t_1, \ldots, t_n) = f(u_1, \ldots u_n)}$	for every n-ary f	(Congruence)
	$\dfrac{t = u}{\sigma(t) = \sigma(u)}$	for σ a substitution	(Substitution)

Table 2.2: Axioms and inferential rules of the equational logic E

We now state how the derivation system in Table 2.2 [BS12, Def.14.18] is used. If t, u are terms over a signature Σ, we say that $t = u$ is an equation over Σ.

Definition 2.2.3. *Let E be a set of equational axioms over a signature Σ. The equation $t = u$ over Σ is **derivable from** E, notation*

$$E \vdash t = u,$$

if there is a sequence of equations

$$t_1 = u_1, \ldots, t_n = u_n$$

*over Σ such that each $t_i = u_i$ belongs to E, is of the form $t = t$ or is a result of applying one of the inferential rules in Table 2.2, and that the last equation $t_n = u_n$ is equal to $t = u$. We refer to n as the **length of the derivation**. Lastly, we say that t and u are **derivably equal**.*

In [BP11] it is proved that CP establishes a complete axiomatization for closed-term equations over $\Sigma_{\text{CP}}(A)$ with respect to *free valuation congruence*. In [BP15] it is shown that free valuation congruence can be defined as equality of *ce*-evaluation trees.

Definition 2.2.4. ***Free valuation congruence (FVC)**, notation $=_{ce}$, is defined on T^A_{CP} as follows:*

$$t =_{ce} u \iff ce(t) = ce(u).$$

We recall that FVC is a congruence relation on T^A_{CP} and repeat the completeness result of CP from [BP15].

Theorem 2.2.5. *For all $t, u \in T^A_{\text{CP}}$,*

$$\text{CP} \vdash t = u \iff t =_{ce} u.$$

2.3 SCL-connectives, syntax and *se*-evaluation trees

Negation and short-circuit left-sequential conjunction (notation $\wedge^{\circ}$) are defined using Hoare's conditional and the constants T and F [BPS13]. Short-circuit left-sequential disjunction (notation $\vee^{\circ}$) is defined in terms of $\neg$ and $\wedge^{\circ}$ [BBR95].

Definition 2.3.1. *The connectives $\neg$, $\wedge^{\circ}$ and $\vee^{\circ}$ are defined by*

$$\neg x = \mathsf{F} \triangleleft x \triangleright \mathsf{T}, \tag{8}$$

$$x \wedge^{\circ} y = y \triangleleft x \triangleright \mathsf{F}, \tag{9}$$

$$x \vee^{\circ} y = \neg(\neg x \wedge^{\circ} \neg y). \tag{10}$$

In [PS18] it is proved that the equation

$$x \vee^{\circ} y = \mathsf{T} \triangleleft x \triangleright y \tag{11}$$

is derivable from $\mathrm{CP} \cup \{(8), (9)\}$.

Definition 2.3.2. *The set T^A_{SCL} of closed terms over A is generated by the following grammar, where $a \in A$:*

$$t ::= a \mid \mathsf{T} \mid \mathsf{F} \mid \neg t \mid t \wedge^{\circ} t \mid t \vee^{\circ} t.$$

Its underlying signature is

$$\Sigma_{\mathrm{SCL}}(A) = \{\mathsf{T}, \mathsf{F}, a, \neg, \wedge^{\circ}, \vee^{\circ} \mid a \in A\}.$$

The abbreviation SCL stands for short-circuit logic. This notion will be defined in the following section. We interpret terms in T^A_{SCL} as evaluation trees using the function *se* (abbreviating short-circuit evaluation) [PS18].

Definition 2.3.3. *The unary **short-circuit evaluation function** $se : T^A_{\mathrm{SCL}} \to \mathcal{T}_A$ is defined as follows, where $a \in A$:*

$$\begin{aligned} se(\mathsf{T}) &= \mathsf{T}, & se(\neg t) &= se(t)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}],\\ se(\mathsf{F}) &= \mathsf{F}, & se(t \wedge^{\circ} u) &= se(t)[\mathsf{T} \mapsto se(u)],\\ se(a) &= (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F}), & se(t \vee^{\circ} u) &= se(t)[\mathsf{F} \mapsto se(u)]. \end{aligned}$$

The evaluation tree $se(t)$ of a term $t \in T^A_{\mathrm{SCL}}$ represents short-circuit evaluation in a way that can be compared to the notion of a truth table for propositional logic because it represents each possible evaluation of t. However, there are some important differences with truth tables. In $se(t)$, the sequentiality of t's evaluation is represented, and the same atom may occur multiple times in $se(t)$.

Example 2.3.4. We derive the evaluation tree of $\neg b \wedge^{\circ} a$.

$$se(\neg b \wedge^{\circ} a) = se(\neg b)[\mathsf{T} \mapsto se(a)] = (\mathsf{F} \trianglelefteq b \trianglerighteq \mathsf{T})[\mathsf{T} \mapsto se(a)] = (\mathsf{F} \trianglelefteq b \trianglerighteq (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F})),$$

which can be visualized as Tree 1 on page 10. Also, $se(\neg(b \vee^{\circ} \neg a)) = (\mathsf{F} \trianglelefteq b \trianglerighteq (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F}))$.

Definition 2.3.5. *The binary relation **se-congruence**, notation $=_{se}$, is defined on T^A_{SCL} as follows:*

$$t =_{se} u \iff se(t) = se(u).$$

It easily follows that $=_{se}$ is a congruence relation on T^A_{SCL}.

2.4 FSCL and EqFSCL-axioms

In this section we recall the definition of free short-circuit logic (FSCL) and we provide a set of equational axioms (EqFSCL) for this logic. First we need to recall the definition of a short-circuit logic [BP12]. This definition uses CP, equations

$$\neg x = \mathsf{F} \triangleleft x \triangleright \mathsf{T}, \tag{8}$$

$$x \mathbin{\&\!\wedge} y = y \triangleleft x \triangleright \mathsf{F}, \tag{9}$$

and the export-operator $\Box$ of *module algebra* [BHK90]. In module algebra $\Sigma \Box X$ is the operation that exports the signature Σ from module X, while declaring other signature elements hidden.

Definition 2.4.1. *A* ***short-circuit logic*** *is a logic that implies the consequences of the module expression*

$$\text{SCL} = \{\mathsf{T}, \neg, \mathbin{\&\!\wedge}\} \Box (\text{CP} \cup \{(8), (9)\}).$$

The constant F and the connective $\mathbin{\&\!\vee}$ do not occur in the exported signature of SCL, but can easily be added because $\text{CP} \cup \{(8), (9)\} \vdash \neg\mathsf{T} \overset{(8)}{=} \mathsf{F} \triangleleft \mathsf{T} \triangleright \mathsf{T} \overset{(\text{CP1})}{=} \mathsf{F}$ and by equation (11).

FSCL is defined as the least identifying short-circuit logic in [BP12].

Definition 2.4.2. ***Free short-circuit logic*** *(**FSCL**) is the short-circuit logic that implies no other consequences than those of the module expression* SCL.

For all terms t, u over $\Sigma_{\text{SCL}}(A)$ we write

$$\text{FSCL} \vdash t = u$$

if $\text{CP} \cup \{(8), (9)\} \vdash t = u$.

We recall two results on FSCL from [PS18] for closed terms over $\Sigma_{\text{SCL}}(A)$. The first result states that FSCL is characterized by *se*-congruence, i.e. for all closed terms $t, u \in T^A_{\text{SCL}}$,

$$\text{FSCL} \vdash t = u \iff t =_{se} u. \tag{12}$$

This characterization is used in [PS18] to show that EqFSCL, the set of equations listed in Table 2.3, constitutes an independent, equational axiomatization of FSCL for closed terms.

Theorem 2.4.3. *For all* $t, u \in T^A_{\text{SCL}}$,

$$\text{EqFSCL} \vdash t = u \iff \text{FSCL} \vdash t = u.$$

In [PS18] a more extensive, non-independent set of axioms is given, that also contains the axioms $\mathsf{F} = \neg\mathsf{T}$ and $\neg\neg x = x$ (in the paper these are axioms (F1) and (F3) respectively). In [PS18, Prop.2.1.8] it is shown that $\{(\text{A1}), \dots, (\text{A5})\} \vdash \mathsf{F} = \neg\mathsf{T}$ and $\{(\text{A1}), \dots, (\text{A5}), (\text{A7})\} \vdash \neg\neg x = x$. Therefore, we will refer to these equations as auxiliary results.

$$\mathsf{F} = \neg\mathsf{T}, \tag{Aux1}$$

$$\neg\neg x = x. \tag{Aux2}$$

$$x \mathbin{\overset{\circ}{\vee}} y = \neg(\neg x \mathbin{\underset{\circ}{\wedge}} \neg y) \tag{A1}$$
$$\mathsf{T} \mathbin{\underset{\circ}{\wedge}} x = x \tag{A2}$$
$$x \mathbin{\overset{\circ}{\vee}} \mathsf{F} = x \tag{A3}$$
$$\mathsf{F} \mathbin{\underset{\circ}{\wedge}} x = \mathsf{F} \tag{A4}$$
$$(x \mathbin{\underset{\circ}{\wedge}} y) \mathbin{\underset{\circ}{\wedge}} z = x \mathbin{\underset{\circ}{\wedge}} (y \mathbin{\underset{\circ}{\wedge}} z) \tag{A5}$$
$$\neg x \mathbin{\underset{\circ}{\wedge}} \mathsf{F} = x \mathbin{\underset{\circ}{\wedge}} \mathsf{F} \tag{A6}$$
$$(x \mathbin{\underset{\circ}{\wedge}} \mathsf{F}) \mathbin{\overset{\circ}{\vee}} y = (x \mathbin{\overset{\circ}{\vee}} \mathsf{T}) \mathbin{\underset{\circ}{\wedge}} y \tag{A7}$$
$$(x \mathbin{\underset{\circ}{\wedge}} y) \mathbin{\overset{\circ}{\vee}} (z \mathbin{\underset{\circ}{\wedge}} \mathsf{F}) = (x \mathbin{\overset{\circ}{\vee}} (z \mathbin{\underset{\circ}{\wedge}} \mathsf{F})) \mathbin{\underset{\circ}{\wedge}} (y \mathbin{\overset{\circ}{\vee}} (z \mathbin{\underset{\circ}{\wedge}} \mathsf{F})) \tag{A8}$$

Table 2.3: EqFSCL, a set of equational axioms for FSCL

In Section 5.1 we will give an alternative derivation of (Aux2), without using axiom (A7). We recall three more equations that are also derivable from $\{(\text{A1}), \dots, (\text{A5})\}$ by [PS18, Prop.2.1.8]. These auxiliary results will be used in Section 3.3 and in Section 5.1.

$$\neg \mathsf{F} = \mathsf{T}, \tag{Aux3}$$
$$\neg(\neg x \mathbin{\underset{\circ}{\wedge}} \neg \mathsf{F}) = x, \tag{Aux4}$$
$$\neg\neg x \mathbin{\underset{\circ}{\wedge}} \neg \mathsf{F} = x. \tag{Aux5}$$

We finish this section by recalling the definition of the dual of terms in T^A_{SCL} and stating the duality principle for equations over $\Sigma_{\text{SCL}}(A)$ [PS18].

Definition 2.4.4. *The **dual** of a closed term $t \in T^A_{\text{SCL}}$, notation t^{dl}, is defined by*

$$\begin{aligned} &\mathsf{T}^{dl} = \mathsf{F}, && a^{dl} = a, && (t \mathbin{\underset{\circ}{\wedge}} u)^{dl} = t^{dl} \mathbin{\overset{\circ}{\vee}} u^{dl}, \\ &\mathsf{F}^{dl} = \mathsf{T}, && (\neg t)^{dl} = \neg t, && (t \mathbin{\overset{\circ}{\vee}} u)^{dl} = t^{dl} \mathbin{\underset{\circ}{\wedge}} u^{dl}. \end{aligned}$$

In [PS18] is argued that (A1), (Aux1) and (Aux2) imply left-sequential versions of De Morgan's laws. Since these equations are derivable from EqFSCL, we find that EqFSCL satisfies the duality principle, i.e. setting the dual of a variable by $x^{dl} = x$ we find that for all terms s, t over $\Sigma_{\text{SCL}}(A)$,

$$\text{EqFSCL} \vdash s = t \iff \text{EqFSCL} \vdash s^{dl} = t^{dl}.$$

2.5 SCL Normal Form

We recall the definition of the SCL Normal Form [PS18], a normal form for terms in T^A_{SCL}.

Definition 2.5.1. *A term $P \in T^A_{\text{SCL}}$ is said to be in **SCL Normal Form (SNF)** if it is generated by the following grammar:*

$$
\begin{aligned}
P &::= P^{\mathsf{T}} \mid P^{\mathsf{F}} \mid P^{\mathsf{T}} \leftand P^* && (\textit{SNF-terms})\\
P^{\mathsf{T}} &::= \mathsf{T} \mid (a \leftand P^{\mathsf{T}}) \leftor P^{\mathsf{T}} && (\mathsf{T}\textit{-terms})\\
P^{\mathsf{F}} &::= \mathsf{F} \mid (a \leftand P^{\mathsf{F}}) \leftor P^{\mathsf{F}} && (\mathsf{F}\textit{-terms})\\
P^* &::= P^c \mid P^d && (*\textit{-terms})\\
P^c &::= P^\ell \mid P^* \leftand P^d\\
P^d &::= P^\ell \mid P^* \leftor P^c\\
P^\ell &:= (a \leftand P^{\mathsf{T}}) \leftor P^{\mathsf{F}} \mid (\neg a \leftand P^{\mathsf{T}}) \leftor P^{\mathsf{F}} && (\ell\textit{-terms})
\end{aligned}
$$

for $a \in A$. We refer to P^{T}-forms as T-terms, to P^{F}-forms as F-terms, to P^ℓ-forms as ℓ-terms (the name refers to literal terms) and to P^*-forms as $*$-terms. Finally, a term of the form $P^{\mathsf{T}} \leftand P^*$ is referred to as a T-$*$-term.

Although we usually use lower case letters $s, t, u, \ldots$ to denote closed terms in this thesis, we chose to stick to the original notation presented in [PS18]. This particularly enhances readability of the proof of Theorem 5.6.3.

Note that the evaluation trees of T-terms (F-terms) have only T-leaves (F-leaves), that $*$-terms are left-associative and that ℓ-terms (literal terms) are considered 'basic' in such terms. Moreover, ℓ-terms are the smallest grammatical units that generate *se*-images that have both T- and F-leaves. In the image of an ℓ-term, there is always a branch with only T-leaves, and a branch with only F-leaves.

A normalisation function $f : T^A_{\text{SCL}} \to \textit{SNF}$ is defined in [PS18]. We omit the exact definition because it is not necessary for the purpose of this thesis. However, we will mention one important theorem about this normalisation function. The function f maps any term in T^A_{SCL} to a derivably equal term in *SNF*.

Theorem 2.5.2. *For any $t \in T^A_{\text{SCL}}$, $f(t) \in \textit{SNF}$ and*

$$\text{EqFSCL} \vdash f(t) = t.$$

Corollary 2.5.3. *For any $t \in T^A_{\text{SCL}}$,*

$$\textit{se}(t) = \textit{se}(f(t)).$$

Proof. By Theorem 2.5.2 we have EqFSCL $\vdash f(t) = t$. By Theorem 2.4.3 we have FSCL $\vdash f(t) = t$. We find that $\textit{se}(f(t)) = \textit{se}(t)$ by equation (12) and by definition of *se*-congruence. $\square$

2.6 Tree decompositions and inverse function

The following section is a summary of Section 3 in [PS18]. Evaluation trees in $se[T^A_{\mathrm{SCL}}]$ can be uniquely decomposed in such a way that the constituents of each decomposition correspond to evaluation trees in $se[SNF]$. First we elaborate on such decompositions, then we define an inverse function g of se.

In addition to $se[T^A_{\mathrm{SCL}}] \subseteq \mathcal{T}_A$ we will also consider the set $\mathcal{T}_{A,\Delta}$, the set of evaluation trees over A with leaves in $\{\mathsf{T}, \mathsf{F}, \Delta\}$. The triangle is used as a placeholder when composing or decomposing trees. Replacement of the leaves of trees in $\mathcal{T}_{A,\Delta}$ by trees in $\mathcal{T}_A$ or $\mathcal{T}_{A,\Delta}$ is defined analogous to replacement for trees in $\mathcal{T}_A$, adopting the same notational conventions.

We start with a simple example of decomposing evaluation trees and we recall the lemma of non-decomposition.

Example 2.6.1. For a $*$-term $P \wedge\!\!\circ\, Q$, the evaluation tree $X = se(P \wedge\!\!\circ\, Q)$ can be decomposed as

$$X = se(P)[\mathsf{T} \mapsto \Delta][\Delta \mapsto Q].$$

Lemma 2.6.2. *There is no $*$-term P^* such that $X = se(P^*)$ can be decomposed as $X = Y[\Delta \mapsto Z]$ with $Y \in \mathcal{T}_{A,\Delta}$ and $Z \in \mathcal{T}_A$, where $Y \neq \Delta$ and Y contains Δ, but not T or F.*

Next, we recall the definition of candidate conjunction decompositions and candidate disjunction decompositions for evaluation trees of $*$-terms.

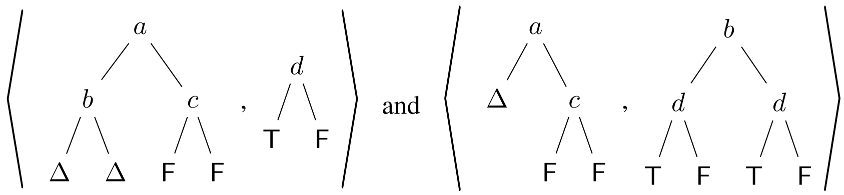
Definition 2.6.3.

1. *The pair $\langle Y, Z\rangle \in \mathcal{T}_{A,\Delta} \times \mathcal{T}_A$ is a **candidate conjunction decomposition (ccd)** of $X \in \mathcal{T}_A$, if*
   (a) $X = Y[\Delta \mapsto Z]$,
   (b) *Y contains Δ,*
   (c) *Y contains F, but not T, and*
   (d) *Z contains both T and F.*
2. *The pair $\langle Y, Z\rangle$ is a **candidate disjunction decomposition (cdd)** of $X \in \mathcal{T}_A$, if*
   (a) $X = Y[\Delta \mapsto Z]$,
   (b) *Y contains Δ,*
   (c) *Y contains T, but not F, and*
   (d) *Z contains both T and F.*

By definition of *SNF* we know that the constituents of $*$-terms must be $*$-terms themselves. Because ccd's and cdd's exist whose constituents are no evaluation trees corresponding to $*$-terms, ccd's and cdd's are not necessarily the decompositions we are looking for. We now give an example of a 'good' and a 'wrong' decomposition.

Example 2.6.4. The two pairs $\langle Y, Z\rangle$ depicted below are both ccd's of the same evaluation tree $se(P^\ell \wedge\!\!\circ\, Q^\ell)$, with $P^\ell = (a \wedge\!\!\circ\, ((b \wedge\!\!\circ\, \mathsf{T}) \vee\!\!\circ\, \mathsf{T})) \vee\!\!\circ\, ((c \vee\!\!\circ\, \mathsf{F}) \wedge\!\!\circ\, \mathsf{F})$ and $Q^\ell = (d \wedge\!\!\circ\, \mathsf{T}) \vee\!\!\circ\, \mathsf{F}$. The constituents of the first ccd are both evaluation trees corresponding to $*$-terms (upon replacing the Δ-leaves by T). The second constituent of the second ccd is not. It is an evaluation tree of the

T-$*$-term $((b \mathbin{{}_{\circ}\!\wedge} \mathsf{T}) \mathbin{{}_{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} ((d \mathbin{{}_{\circ}\!\wedge} \mathsf{T}) \mathbin{{}_{\circ}\!\vee} \mathsf{F})$. So the first decomposition yields the appropriate constituents of $se(P^\ell \mathbin{{}_{\circ}\!\wedge} Q^\ell)$ and the second does not.



The conjunction decomposition and the disjunction decomposition are defined in such a way that the constituents of the decomposition are always evaluation trees corresponding to $*$-terms (upon replacing the Δ-leaves by T and F respectively).

Definition 2.6.5.

1. *The pair $\langle Y, Z\rangle \in \mathcal{T}_{A,\Delta} \times \mathcal{T}_A$ is a* ***conjunction decomposition (cd)*** *of $X \in \mathcal{T}_A$, if it is a ccd of X and there is no other ccd $\langle Y', Z'\rangle$ of X where the height of Z' is smaller than that of Z.*

2. *The pair $\langle Y, Z\rangle \in \mathcal{T}_{A,\Delta} \times \mathcal{T}_A$ is a* ***disjunction decomposition (dd)*** *of $X \in \mathcal{T}_A$, if it is a cdd of X and there is no other cdd $\langle Y', Z'\rangle$ of X where the height of Z' is smaller than that of Z.*

Theorem 2.6.6.

1. *For any $*$-term $P \mathbin{{}_{\circ}\!\wedge} Q$, i.e. with $P \in P^*$ and $Q \in P^d$, $se(P \mathbin{{}_{\circ}\!\wedge} Q)$ has no dd and its unique cd is*
$$\langle se(P)[\mathsf{T} \mapsto \Delta], se(Q)\rangle.$$

2. *For any $*$-term $P \mathbin{{}_{\circ}\!\vee} Q$, i.e. with $P \in P^*$ and $Q \in P^c$, $se(P \mathbin{{}_{\circ}\!\vee} Q)$ has no cd and its unique dd is*
$$\langle se(P)[\mathsf{F} \mapsto \Delta], se(Q)\rangle.$$

Using this theorem we find that the decomposition in Example 2.6.1 and the first decomposition in Example 2.6.4 are the unique conjunction decompositions of the evaluation trees that were under consideration.

To be able to decompose evaluation trees of T-$*$-terms, we now recall the definition of candidate T-$*$-decompositions and the definition of the T-$*$-decomposition.

Definition 2.6.7. *The pair $\langle Y, Z\rangle \in \mathcal{T}_{A,\Delta} \times \mathcal{T}_A$ is a* ***candidate*** T***-$*$-decomposition (ctsd)*** *of $X \in \mathcal{T}_A$ if*

(a) $X = Y[\Delta \mapsto Z]$,

(b) Y *does not contain* T *or* F*, and*

(c) Z *contains both* T *and* F*,*

and there is no decomposition $\langle U, V\rangle \in \mathcal{T}_{A,\Delta} \times \mathcal{T}_A$ of Z such that

(a) $Z = U[\Delta \mapsto V]$,

(b) U *contains* Δ*,*

(c) $U \neq \Delta$, *and*

(d) U *contains neither* T *nor* F.

Definition 2.6.8. *The pair* $\langle Y, Z\rangle \in \mathcal{T}_{A,\Delta} \times \mathcal{T}_A$ *is a* T-$*$***-decomposition (tsd)*** *of* $X \in \mathcal{T}_A$, *if it is a ctsd of* X *and there is no other ctsd* $\langle Y', Z'\rangle$ *of* X *where the height of* Z' *is smaller than that of* Z.

Theorem 2.6.9. *For any* T*-term* P *and* $*$*-term* Q*, the unique tsd of* $se(P \mathbin{\&\!\wedge} Q)$ *is*

$$\langle se(P)[\mathsf{T} \mapsto \Delta], se(Q)\rangle.$$

Using Theorem 2.6.6 and Theorem 2.6.9, we know that the evaluation tree of a T-$*$-term is uniquely decomposable into a tsd $\langle Y, Z\rangle$ such that Z is uniquely decomposable into cd's and dd's.

In the following we recall the definition of an inverse function g of *se*. We need three auxiliary functions that aid in defining g.

Definition 2.6.10. *The function* $cd : \mathcal{T}_A \longrightarrow \mathcal{T}_{A,\Delta} \times \mathcal{T}_A$ *returns the conjunction decomposition of its argument and is defined by*

$$X \longmapsto \langle cd_1(X), cd_2(X)\rangle,$$

where $\langle cd_1(X), cd_2(X)\rangle$ *is the pair* $\langle Y, Z\rangle$ *from Theorem 2.6.6.1.* cd *is undefined when its argument does not have a conjunction decomposition. The functions* dd *(returning the disjunction decomposition of its argument) and* tsd *(returning the* T-$*$*-decomposition of its argument) are defined similarly.*

Definition 2.6.11. *We define* $g : \mathcal{T}_A \to$ *SNF using the functions* $g^{\mathsf{T}}, g^{\mathsf{F}}, g^{\ell}$ *and* g^* *that each have the same domain and codomain for inverting trees in the image of* T*-terms,* F*-terms,*ℓ*-terms and* $*$*-terms respectively. These functions are defined by*

$$g^{\mathsf{T}}(X) = \begin{cases} \mathsf{T} & \textit{if } X = \mathsf{T}, \\ (a \mathbin{\&\!\wedge} g^{\mathsf{T}}(Y)) \mathbin{\vee\!\!\&} g^{\mathsf{T}}(Z) & \textit{if } X = (Y \trianglelefteq a \trianglerighteq Z). \end{cases}$$

$$g^{\mathsf{F}}(X) = \begin{cases} \mathsf{F} & \textit{if } X = \mathsf{F}, \\ (a \mathbin{\&\!\wedge} g^{\mathsf{F}}(Y)) \mathbin{\vee\!\!\&} g^{\mathsf{F}}(Z) & \textit{if } X = (Y \trianglelefteq a \trianglerighteq Z). \end{cases}$$

$$g^{\ell}(X) = \begin{cases} (a \mathbin{\&\!\wedge} g^{\mathsf{T}}(Y)) \mathbin{\vee\!\!\&} g^{\mathsf{F}}(Z) & \textit{if } X = (Y \trianglelefteq a \trianglerighteq Z) \textit{ and } Y \textit{ only has } \mathsf{T}\textit{-leaves}, \\ (\neg a \mathbin{\&\!\wedge} g^{\mathsf{T}}(Z)) \mathbin{\vee\!\!\&} g^{\mathsf{F}}(Y) & \textit{if } X = (Y \trianglelefteq a \trianglerighteq Z) \textit{ and } Z \textit{ only has } \mathsf{T}\textit{-leaves}. \end{cases}$$

$$g^{*}(X) = \begin{cases} g^*(cd_1(X)[\Delta \mapsto \mathsf{T}]) \mathbin{\&\!\wedge} g^*(cd_2(X)) & \textit{if } X \textit{ has a cd}, \\ g^*(dd_1(X)[\Delta \mapsto \mathsf{F}]) \mathbin{\&\!\wedge} g^*(dd_2(X)) & \textit{if } X \textit{ has a dd}, \\ g^{\ell}(X) & \textit{otherwise}. \end{cases}$$

$$g(X) = \begin{cases} g^{\mathsf{T}}(X) & \textit{if } X \textit{ has only } \mathsf{T}\textit{-leaves}, \\ g^{\mathsf{F}}(X) & \textit{if } X \textit{ has only } \mathsf{F}\textit{-leaves}, \\ g^{\mathsf{T}}(tsd_1(X)[\Delta \mapsto \mathsf{T}]) \mathbin{\&\!\wedge} g^*(tsd_2(X)) & \textit{otherwise}. \end{cases}$$

We use the symbol $\equiv$ to denote ‘syntactic equivalence’. Finally, in [PS18] the following theorem about g is proved.

Theorem 2.6.12. *For any* P *in SNF,*

$$g(se(P)) \equiv P.$$

Chapter 3

ℓNAND and FLNL

In this chapter we consider ℓNAND, short-circuit left-sequential NAND, and provide an equational axiomatization of free left-sequential nand logic (FLNL). In Section 3.1 we define ℓNAND, notation $\circ\!|$, define a semantics for closed terms with $\circ\!|$ using a function *nse* that assigns evaluation trees, and we define *nse*-congruence that characterizes this semantics. In Section 3.2 we define FLNL and provide a set of equational axioms (EqFLNL) for this logic. In Section 3.3 we prove that EqFLNL and EqFSCL are 'translationally equivalent'. Finally, in Section 3.4 we show that EqFLNL axiomatizes FLNL for closed terms.

3.1 ℓNAND, syntax and *nse*-trees

We define the connective ℓNAND, short-circuit left-sequential NAND, using Hoare's conditional and the constants T and F.

Definition 3.1.1. *The connective ℓNAND, written as $\circ\!|$, is defined by*

$$x \circ\!| \; y = (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleleft x \triangleright \mathsf{T}. \tag{13}$$

Proposition 3.1.2. *The connective $\circ\!|$ is expressible in terms of $\neg$ and $\circ\!\wedge$:*

$$x \circ\!| \; y = \neg(x \circ\!\wedge\, y). \tag{14}$$

Proof.

$$\begin{aligned}
x \circ\!| \; y &= (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleleft x \triangleright \mathsf{T} && \text{by (13)}\\
&= (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleleft x \triangleright (\mathsf{F} \triangleleft \mathsf{F} \triangleright \mathsf{T}) && \text{by (CP2)}\\
&= \mathsf{F} \triangleleft (y \triangleleft x \triangleright \mathsf{F}) \triangleright \mathsf{T} && \text{by (CP4)}\\
&= \neg(x \circ\!\wedge\, y) && \text{by (8) and (9)}
\end{aligned}$$

□

Proposition 3.1.3. *The connectives* $\neg$*,* $\mathbin{^{\circ}\!\wedge}$ *and* $\mathbin{^{\circ}\!\vee}$ *are expressible in terms of* $\mathbin{^{\circ}\!|}$ *and* T*:*

$$\neg x = x \mathbin{^{\circ}\!|} \mathsf{T} = \mathsf{T} \mathbin{^{\circ}\!|} x, \tag{15}$$
$$x \mathbin{^{\circ}\!\wedge} y = (x \mathbin{^{\circ}\!|} y) \mathbin{^{\circ}\!|} \mathsf{T}, \tag{16}$$
$$x \mathbin{^{\circ}\!\vee} y = (x \mathbin{^{\circ}\!|} \mathsf{T}) \mathbin{^{\circ}\!|} (y \mathbin{^{\circ}\!|} \mathsf{T}). \tag{17}$$

Proof.
Proof of (15):

$$\begin{aligned} x \mathbin{^{\circ}\!|} \mathsf{T} &= \neg(x \mathbin{^{\circ}\!\wedge} \mathsf{T}) && \text{by (14)} \\ &= \neg x && \text{by (A3)}^{dl} \end{aligned}$$

and

$$\begin{aligned} \mathsf{T} \mathbin{^{\circ}\!|} x &= \neg(\mathsf{T} \mathbin{^{\circ}\!\wedge} x) && \text{by (14)} \\ &= \neg x && \text{by (A2)} \end{aligned}$$

Proof of (16):

$$\begin{aligned} (x \mathbin{^{\circ}\!|} y) \mathbin{^{\circ}\!|} \mathsf{T} &= \neg(\neg(x \mathbin{^{\circ}\!\wedge} y) \mathbin{^{\circ}\!\wedge} \mathsf{T}) && \text{by (14)} \\ &= \neg\neg(x \mathbin{^{\circ}\!\wedge} y) && \text{by (A3)}^{dl} \\ &= x \mathbin{^{\circ}\!\wedge} y && \text{by (Aux2)} \end{aligned}$$

Proof of (17):

$$\begin{aligned} (x \mathbin{^{\circ}\!|} \mathsf{T}) \mathbin{^{\circ}\!|} (y \mathbin{^{\circ}\!|} \mathsf{T}) &= \neg(\neg(x \mathbin{^{\circ}\!\wedge} \mathsf{T}) \mathbin{^{\circ}\!\wedge} \neg(y \mathbin{^{\circ}\!\wedge} \mathsf{T})) && \text{by (14)} \\ &= \neg(\neg x \mathbin{^{\circ}\!\wedge} \neg y) && \text{by (A3)}^{dl} \\ &= x \mathbin{^{\circ}\!\vee} y && \text{by (10)} \end{aligned}$$

□

Definition 3.1.4. *The set* T^A_{LNL} *of closed terms over* A *is generated by the following grammar, where* $a \in A$*:*

$$t ::= a \mid \mathsf{T} \mid \mathsf{F} \mid t \mathbin{^{\circ}\!|} t.$$

Its underlying signature is

$$\Sigma_{\text{LNL}}(A) = \{\mathsf{T}, \mathsf{F}, a, \mathbin{^{\circ}\!|} \mid a \in A\}.$$

The abbreviation LNL stands for left-sequential nand logic. We will elaborate on LNL in the next section. We interpret propositional statements in T^A_{LNL} as evaluation trees using the function *nse* (abbreviating nand short-circuit evaluation).

Definition 3.1.5. *The unary* ***nand short-circuit evaluation function*** $nse : T^A_{\text{LNL}} \to \mathcal{T}_A$ *is defined as follows, where* $a \in A$*:*

$$nse(\mathsf{T}) = \mathsf{T}, \qquad nse(a) = (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F}),$$
$$nse(\mathsf{F}) = \mathsf{F}, \qquad nse(t \mathbin{^{\circ}\!|} u) = nse(t)\big[\mathsf{T} \mapsto nse(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \mathsf{T}\big].$$

Note that *nse* indeed prescribes a short-circuit evaluation strategy because in $nse(t \mathbin{\triangleleft\!\!\mid} u)$, $nse(u)$ is not evaluated if $nse(t)$ yields F. We will shortly discuss full left-sequential NAND (notation $\mathbin{\blacktriangleleft\!\!\mid}$) in Chapter 6.

We now argue that equality of *nse*-trees defines a congruence relation on T^A_{LNL}.

Definition 3.1.6. *The binary relation* ***nse-congruence****, notation* $=_{nse}$*, is defined on* T^A_{LNL} *as follows:*

$$t =_{nse} u \iff nse(t) = nse(u).$$

Lemma 3.1.7. *The relation* $=_{nse}$ *is a congruence relation on* T^A_{LNL}.

Proof. Because reflexivity, symmetry and transitivity hold, $=_{nse}$ is an equivalence relation. Furthermore, it follows from Definition 3.1.5 that if $nse(t_1) = nse(u_1)$ and $nse(t_2) = nse(u_2)$, then $nse(t_1 \mathbin{\triangleleft\!\!\mid} t_2) = nse(u_1 \mathbin{\triangleleft\!\!\mid} u_2)$. $\square$

3.2 FLNL and equational axioms

In this section we define free left-sequential nand logic (FLNL) and we provide a set of equational axioms (EqFLNL) for this logic.

In Section 2.4 we recalled the generic definition of a short-circuit logic and the definition of free short-circuit logic from [BP12]. Following this line of reasoning, we will give a generic definition of a left-sequential nand logic (LNL) using equation

$$x \mathbin{\triangleleft\!\!\mid} y = (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleleft x \triangleright \mathsf{T}, \tag{13}$$

and we define free left-sequential nand logic (FLNL) as the least identifying LNL.

Definition 3.2.1. *A* ***left-sequential nand logic*** *is a logic that implies the consequences of the module expression*

$$\text{LNL} = \{\mathsf{T}, \mathbin{\triangleleft\!\!\mid}\} \,\square\, (\text{CP} \cup \{(13)\}).$$

Although the constant F does not occur in the exported signature of LNL, it can easily be added because $\text{CP} \cup \{(13)\} \vdash \mathsf{T} \mathbin{\triangleleft\!\!\mid} \mathsf{T} \overset{(13)}{=} (\mathsf{F} \triangleleft \mathsf{T} \triangleright \mathsf{T}) \triangleleft \mathsf{T} \triangleright \mathsf{T} \overset{(\text{CP1})}{=} \mathsf{F}$.

Definition 3.2.2. ***Free left-sequential nand logic (*****FLNL*****)*** *is the left-sequential nand logic that implies no other consequences than those of the module expression* LNL.

If $\text{CP} \cup \{(13)\} \vdash t = u$ for terms t, u over $\Sigma_{\text{LNL}}(A)$, we write

$$\text{FLNL} \vdash t = u.$$

We will show that FLNL is characterized by *nse*-congruence for closed terms in T^A_{LNL}. To this end, we define the combined signature $\Sigma_{\text{CP+LNL}}(A) = \Sigma_{\text{CP}}(A) \cup \{\mathbin{\triangleleft\!\!\mid}\}$ and $T_{\text{CP+LNL}}(A)$, the set of closed terms over $\Sigma_{\text{CP+LNL}}(A)$. We now extend the evaluation function *ce*.

Definition 3.2.3. *The evaluation function* $ce_{nse} : T_{\text{CP+LNL}}(A) \mapsto \mathcal{T}_A$ *extends ce as follows:*

$$ce_{nse}(t \mathbin{\triangleleft\!\!\mid} u) = ce_{nse}(t)\big[\mathsf{T} \mapsto ce_{nse}(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \mathsf{T}\big].$$

Note that ce_{nse} satisfies equation (13), i.e. $ce_{nse}(t \mathbin{\lhd\!\mid} u) = ce_{nse}(t)\big[\mathsf{T} \mapsto ce_{nse}(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \mathsf{T}\big] = ce_{nse}((\mathsf{F} \lhd u \rhd \mathsf{T}) \lhd t \rhd \mathsf{T})$. Using Definition 2.2.4 and the completeness result from Theorem 2.2.5, we find for all $t, u \in T^A_{\text{LNL}}$,

$$\text{CP} \cup \{(13)\} \vdash t = u \iff ce_{nse}(t) = ce_{nse}(u). \tag{18}$$

Lemma 3.2.4. *The evaluation function ce_{nse} is equal to nse when restricted to terms in T^A_{LNL}, i.e. for any $t \in T^A_{\text{LNL}}$,*

$$ce_{nse}(t) = nse(t).$$

Proof. The proof follows by induction on the complexity of closed terms. The base case is trivial. For the inductive step, assume that $s, u \in T^A_{\text{LNL}}$ are such that $ce_{nse}(s) = nse(s)$ and $ce_{nse}(u) = nse(u)$ (IH) and that t is of the form $s \mathbin{\lhd\!\mid} u$. We find that $ce_{nse}(s \mathbin{\lhd\!\mid} u) = ce_{nse}(s)\big[\mathsf{T} \mapsto ce_{nse}(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \mathsf{T}\big] \overset{\text{IH}}{=} nse(s)\big[\mathsf{T} \mapsto nse(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \mathsf{T}\big] = nse(s \mathbin{\lhd\!\mid} u)$. □

Using equation (18), we find by the previous lemma and by definition of *nse*-congruence that for all closed terms $t, u \in T^A_{\text{LNL}}$,

$$\text{FLNL} \vdash t = u \iff t =_{nse} u. \tag{19}$$

In Table 3.1 we list a set of equational axioms for FLNL, called EqFLNL. We will show that EqFLNL constitutes an independent, equational axiomatization of FLNL for closed terms.

$$(\mathsf{T} \mathbin{\lhd\!\mid} x) \mathbin{\lhd\!\mid} \mathsf{T} = x \tag{N1}$$
$$\mathsf{F} \mathbin{\lhd\!\mid} x = \mathsf{T} \tag{N2}$$
$$((x \mathbin{\lhd\!\mid} y) \mathbin{\lhd\!\mid} \mathsf{T}) \mathbin{\lhd\!\mid} z = x \mathbin{\lhd\!\mid} ((y \mathbin{\lhd\!\mid} z) \mathbin{\lhd\!\mid} \mathsf{T}) \tag{N3}$$
$$(x \mathbin{\lhd\!\mid} \mathsf{T}) \mathbin{\lhd\!\mid} \mathsf{F} = x \mathbin{\lhd\!\mid} \mathsf{F} \tag{N4}$$
$$(x \mathbin{\lhd\!\mid} \mathsf{F}) \mathbin{\lhd\!\mid} (y \mathbin{\lhd\!\mid} \mathsf{T}) = ((x \mathbin{\lhd\!\mid} \mathsf{F}) \mathbin{\lhd\!\mid} y) \mathbin{\lhd\!\mid} \mathsf{T} \tag{N5}$$
$$((x \mathbin{\lhd\!\mid} y) \mathbin{\lhd\!\mid} (z \mathbin{\lhd\!\mid} \mathsf{F})) \mathbin{\lhd\!\mid} \mathsf{T} = ((x \mathbin{\lhd\!\mid} \mathsf{T}) \mathbin{\lhd\!\mid} (z \mathbin{\lhd\!\mid} \mathsf{F})) \mathbin{\lhd\!\mid} ((y \mathbin{\lhd\!\mid} \mathsf{T}) \mathbin{\lhd\!\mid} (z \mathbin{\lhd\!\mid} \mathsf{F})) \tag{N6}$$

Table 3.1: EqFLNL, a set of equational axioms for FLNL

We first elaborate on the axioms of EqFLNL. Axiom (N1) is the EqFLNL-version of double negation. Axiom (N2) is a simple identity. Axiom (N3) defines an adapted form of associativity. Axiom (N4) always evaluates to T, and it does not matter for the evaluation result whether t or $t \mathbin{\lhd\!\mid} \mathsf{T}$ is evaluated before the evaluation of F. In axiom (N5) the evaluation of $(t \mathbin{\lhd\!\mid} \mathsf{F})$ always yields T. So u will always be evaluated. For the evaluation result it does not matter whether $(u \mathbin{\lhd\!\mid} \mathsf{T})$, or first u and then T is evaluated next. In axiom (N6) we see a variation of right-distributivity of $\mathbin{\lhd\!\mid}$ over itself. It only holds for $z \mathbin{\lhd\!\mid} \mathsf{F}$ instead of any z and the negation of the 'non-distributed' expression (by $\mathbin{\lhd\!\mid} \mathsf{T}$) transfers to x and y in the 'distributed' expression.

We derive two auxiliary results that will be used in the proofs of Lemma 3.3.3 and Theorem 3.3.6.

Proposition 3.2.5. *The following equations are derivable from* EqFLNL*:*

$$\mathsf{T} \mathbin{\lhd\!\mid} \mathsf{T} = \mathsf{F}, \tag{Aux6}$$
$$(x \mathbin{\lhd\!\mid} \mathsf{T}) \mathbin{\lhd\!\mid} \mathsf{T} = x. \tag{Aux7}$$

Proof. All proofs are distilled from output of the theorem prover *Prover9* [McC08].

Proof of (Aux6):

$$\begin{aligned}
\mathsf{T} \lhd \mathsf{T} &= (\mathsf{F} \lhd \mathsf{F}) \lhd \mathsf{T} && \text{by (N2)}\\
&= ((\mathsf{F} \lhd \mathsf{T}) \lhd \mathsf{F}) \lhd \mathsf{T} && \text{by (N4)}\\
&= (\mathsf{T} \lhd \mathsf{F}) \lhd \mathsf{T} && \text{by (N2)}\\
&= \mathsf{F} && \text{by (N1)}
\end{aligned}$$

Proof of (Aux7):

$$\begin{aligned}
(x \lhd \mathsf{T}) \lhd \mathsf{T} &= (((\mathsf{T} \lhd x) \lhd \mathsf{T}) \lhd \mathsf{T}) \lhd \mathsf{T} && \text{by (N1)}\\
&= (\mathsf{T} \lhd x) \lhd ((\mathsf{T} \lhd \mathsf{T}) \lhd \mathsf{T}) && \text{by (N3)}\\
&= (\mathsf{T} \lhd x) \lhd \mathsf{T} && \text{by (N1)}\\
&= x && \text{by (N1)}
\end{aligned}$$

□

To conclude this section we will show that the axioms of EqFLNL are independent when $|A| \geq 2$. We will show for each axiom (Ni) in EqFLNL that EqFLNL$\setminus\{$(Ni)$\}$ $\nvdash$ (Ni) by giving an independence model for each axiom (Ni), a model that satisfies the axioms of EqFLNL$\setminus\{$(Ni)$\}$ while it does not satisfy (Ni).

First we state when a model satisfies a set of equations [BS12, Def.11.1].

Definition 3.2.6. *A Σ-algebra $\mathcal{A}$ **satisfies** an equation $t(x_1, \ldots, x_n) = u(x_1, \ldots, x_n)$ over Σ, where $x_1, \ldots x_n$ are variables, notation*

$$\mathcal{A} \vDash t = u,$$

if for every choice of elements $a_1, \ldots, a_n$ in the domain of $\mathcal{A}$

$$[\![t(a_1, \ldots, a_n)]\!]^{\mathcal{A}} = [\![u(a_1, \ldots, a_n)]\!]^{\mathcal{A}}.$$

*Furthermore, $\mathcal{A}$ satisfies a set of equations E over Σ (notation $\mathcal{A} \vDash E$) if $\mathcal{A}$ satisfies every equation in E. We say that $\mathcal{A}$ is a **model** for $t = u$ and for E respectively.*

Theorem 3.2.7. *The axioms of* EqFLNL *are independent if A contains at least two atoms.*

Proof. All independence models are $\Sigma_{\text{LNL}}(A)$-algebras and were found with the tool *Mace4* [McC08]. We show independence of axiom (N6) here and for the remaining cases we refer to Appendix A.

Assume that $\{a, b\} \subseteq A$ and consider the model $\mathbb{M}$ for EqFLNL $\setminus$ $\{$(N6)$\}$ with domain $D = \{0, 1, 2, 3\}$, where the constants are interpreted by

$$[\![\mathsf{T}]\!]^{\mathbb{M}} = 1, \qquad [\![\mathsf{F}]\!]^{\mathbb{M}} = 0, \qquad [\![a]\!]^{\mathbb{M}} = 2, \qquad [\![b]\!]^{\mathbb{M}} = 3,$$

and where $\lhd$ is interpreted as follows:

$\lhd$	0	1	2	3
0	1	1	1	1
1	1	0	2	3
2	1	2	0	3
3	3	3	3	3

We find that $[\![((a \lhd a) \lhd (b \lhd \mathsf{F})) \lhd \mathsf{T}]\!]^{\mathbb{M}} = 0$ and $[\![((a \lhd \mathsf{T}) \lhd (b \lhd \mathsf{F})) \lhd ((a \lhd \mathsf{T}) \lhd (b \lhd \mathsf{F}))]\!]^{\mathbb{M}} = 3$, so axiom (N6) is not satisfied by $\mathbb{M}$. □

3.3 Translational equivalence

In this section we will show that EqFLNL and EqFSCL are *translationally equivalent* [Pel84]. In [PU03] it is shown that the notion of translationally equivalent logics is equivalent to the notion of *synonymous* logics from [Len79]. Intuitively, two logics are translationally equivalent if it is possible to translate one logic into the other, preserving theoremhood. That is, theorems are translated to theorems and non-theorems to non-theorems. We repeat the formal definition of translational equivalence that is given in [PU03], adapted to our framework.

Definition 3.3.1. *Two equational logics S_1 and S_2 with languages L_1 and L_2 are* ***translationally equivalent*** *if there are translation functions f, g such that*

1. *both f and g are* ***sound*** *(for equations $t = u \in L_1$ the translation $f(t) = f(u)$ is a theorem of S_2 whenever $t = u$ is a theorem of S_1, and vice-versa), and*
2. *for any term $t \in L_1$, $g(f(t)) = t$ is a theorem of S_1, and for any term $t \in L_2$, $f(g(t)) = t$ is a theorem of L_2.*

Let $\mathbb{T}^{A,X}_{\mathrm{SCL}}$ and $\mathbb{T}^{A,X}_{\mathrm{LNL}}$ denote the sets of all terms over $\Sigma_{\mathrm{SCL}}(A)$ and $\Sigma_{\mathrm{LNL}}(A)$ with variables in X respectively. In the following we take S_1 to be EqFSCL, we take S_2 to be EqFLNL, we take L_1 to be the union of $\mathbb{T}^{A,X}_{\mathrm{SCL}}$ and the set of all equations between terms in $\mathbb{T}^{A,X}_{\mathrm{SCL}}$ and we take L_2 to be the union of $\mathbb{T}^{A,X}_{\mathrm{LNL}}$ and the set of all equations between terms in $\mathbb{T}^{A,X}_{\mathrm{LNL}}$.

We now define two translation functions between $\mathbb{T}^{A,X}_{\mathrm{SCL}}$ and $\mathbb{T}^{A,X}_{\mathrm{LNL}}$ and we will show that these translation functions satisfy the conditions for translational equivalence from Definition 3.3.1.

Definition 3.3.2.
For terms $t, u \in \mathbb{T}^{A,X}_{\mathrm{SCL}}$, we define $f : \mathbb{T}^{A,X}_{\mathrm{SCL}} \to \mathbb{T}^{A,X}_{\mathrm{LNL}}$ by

$$f(t) = t \text{ for } t \in \{\mathsf{T}, \mathsf{F}\} \cup A \cup X, \qquad f(t \mathbin{{}^{\circ}\!\wedge} u) = (f(t) \mathbin{{}^{\circ}\!\mid} f(u)) \mathbin{{}^{\circ}\!\mid} \mathsf{T},$$
$$f(\neg t) = f(t) \mathbin{{}^{\circ}\!\mid} \mathsf{T}, \qquad f(t \mathbin{{}^{\circ}\!\vee} u) = (f(t) \mathbin{{}^{\circ}\!\mid} \mathsf{T}) \mathbin{{}^{\circ}\!\mid} (f(u) \mathbin{{}^{\circ}\!\mid} \mathsf{T}).$$

For terms $t, u \in \mathbb{T}^{A,X}_{\mathrm{LNL}}$, we define $g : \mathbb{T}^{A,X}_{\mathrm{LNL}} \to \mathbb{T}^{A,X}_{\mathrm{SCL}}$ by

$$g(t) = t \text{ for } t \in \{\mathsf{T}, \mathsf{F}\} \cup A \cup X, \qquad g(t \mathbin{{}^{\circ}\!\mid} u) = \neg(g(t) \mathbin{{}^{\circ}\!\wedge} g(u)).$$

The definitions of the translation functions f and g are direct consequences of Proposition 3.1.3 and Proposition 3.1.2 respectively.

We will show that the translation functions f and g are sound. Before we do this, we need to show that the translated axioms of EqFSCL, listed in Table 3.2, are derivable from EqFLNL and that the translated axioms of EqFLNL, listed in Table 3.3, are derivable from EqFSCL. We write $f(S)$ for $\{f(s) = f(t) \mid s = t \in S\}$, and similar for g.

Lemma 3.3.3. EqFLNL$\vdash$ *f(*EqFSCL*).*

Proof. All proofs are distilled from output of the theorem prover *Prover9* [McC08].

f(A1) is an instance of (Aux7). f(A2) is equal to axiom (N1). f(A3) is derived by replacing the last occurrence of T in (Aux7) by $(\mathsf{F} \mathbin{{}^{\circ}\!\mid} \mathsf{T})$. This inference is valid by axiom (N2). f(A4) is derived from axiom (N2) by adding $\mathbin{{}^{\circ}\!\mid} \mathsf{T}$ to both sides and by (Aux6). Note that adding $\mathbin{{}^{\circ}\!\mid} \mathsf{T}$ to both sides is a valid inference by congruence. f(A5) is derived from axiom (N3) by adding $\triangleleft\, \mathsf{T}$ to both sides. f(A6) is derived from axiom (N4) by adding $\triangleleft\, \mathsf{T}$ to both sides. f(A7) is derived as follows:

$$\begin{aligned}
(((x \triangleleft \mathsf{F}) \triangleleft \mathsf{T}) \triangleleft \mathsf{T}) \triangleleft (y \triangleleft \mathsf{T}) &= (x \triangleleft \mathsf{F}) \triangleleft (y \triangleleft \mathsf{T}) && \text{by (Aux7)} \\
&= ((x \triangleleft \mathsf{F}) \triangleleft y) \triangleleft \mathsf{T} && \text{by (N5)} \\
&= (((x \triangleleft \mathsf{T}) \triangleleft \mathsf{F}) \triangleleft y) \triangleleft \mathsf{T} && \text{by (N4)} \\
&= (((x \triangleleft \mathsf{T}) \triangleleft (\mathsf{T} \triangleleft \mathsf{T}) \triangleleft y) \triangleleft \mathsf{T} && \text{by (Aux6)}
\end{aligned}$$

Finally, f(A8) is derived from axiom (N6) by adding $\triangleleft\, \mathsf{T}$ to both sides and using (Aux7) five times. □

$$\begin{aligned}
(x \triangleleft \mathsf{T}) \triangleleft (y \triangleleft \mathsf{T}) &= (((x \triangleleft \mathsf{T}) \triangleleft (y \triangleleft \mathsf{T})) \triangleleft \mathsf{T}) \triangleleft \mathsf{T} && f(\text{A1}) \\
(\mathsf{T} \triangleleft x) \triangleleft \mathsf{T} &= x && f(\text{A2}) \\
(x \triangleleft \mathsf{T}) \triangleleft (\mathsf{F} \triangleleft \mathsf{T}) &= x && f(\text{A3}) \\
(\mathsf{F} \triangleleft x) \triangleleft \mathsf{T} &= \mathsf{F} && f(\text{A4}) \\
(((x \triangleleft y) \triangleleft \mathsf{T}) \triangleleft z) \triangleleft \mathsf{T} &= (x \triangleleft ((y \triangleleft z) \triangleleft \mathsf{T})) \triangleleft \mathsf{T} && f(\text{A5}) \\
((x \triangleleft \mathsf{T}) \triangleleft \mathsf{F}) \triangleleft \mathsf{T} &= (x \triangleleft \mathsf{F}) \triangleleft \mathsf{T} && f(\text{A6}) \\
(((x \triangleleft \mathsf{F}) \triangleleft \mathsf{T}) \triangleleft \mathsf{T}) \triangleleft (y \triangleleft \mathsf{T}) &= (((x \triangleleft \mathsf{T}) \triangleleft (\mathsf{T} \triangleleft \mathsf{T})) \triangleleft y) \triangleleft \mathsf{T} && f(\text{A7}) \\
(((x \triangleleft y) \triangleleft \mathsf{T}) \triangleleft \mathsf{T}) \triangleleft (((z \triangleleft \mathsf{F}) \triangleleft \mathsf{T}) \triangleleft \mathsf{T}) &= && f(\text{A8}) \\
&\big[((x \triangleleft \mathsf{T}) \triangleleft (((z \triangleleft \mathsf{F}) \triangleleft \mathsf{T}) \triangleleft \mathsf{T})) \triangleleft ((y \triangleleft \mathsf{T}) \triangleleft (((z \triangleleft \mathsf{F}) \triangleleft \mathsf{T}) \triangleleft \mathsf{T}))\big] \triangleleft \mathsf{T}
\end{aligned}$$

Table 3.2: f(EqFSCL), the translated axioms of EqFSCL

Lemma 3.3.4. EqFSCL $\vdash g$(EqFLNL).

Proof. All proofs are distilled from output of the theorem prover *Prover9* [McC08]. Here we show that EqFSCL $\vdash g$(N1). The proofs for the other translated axioms can be found in Appendix A.

$$\begin{aligned}
\neg(\neg(\mathsf{T} \mathbin{{}^{\circ}\!\wedge} x) \mathbin{{}^{\circ}\!\wedge} \mathsf{T}) &= \neg(\neg x \mathbin{{}^{\circ}\!\wedge} \mathsf{T}) && \text{by (A2)} \\
&= \neg(\neg x \mathbin{{}^{\circ}\!\wedge} \neg\mathsf{F}) && \text{by (Aux3)} \\
&= x && \text{by (Aux4)}
\end{aligned}$$

□

Using Lemma 3.3.3 and Lemma 3.3.4, we can show that f and g are sound.

$$\begin{aligned}
\neg(\neg(\mathsf{T} \mathbin{{}^{\circ}\!\wedge} x) \mathbin{{}^{\circ}\!\wedge} \mathsf{T}) &= x && g(\text{N1}) \\
\neg(\mathsf{F} \mathbin{{}^{\circ}\!\wedge} x) &= \mathsf{T} && g(\text{N2}) \\
\neg(\neg(\neg(x \mathbin{{}^{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\wedge} \mathsf{T}) \mathbin{{}^{\circ}\!\wedge} z) &= \neg(x \mathbin{{}^{\circ}\!\wedge} \neg(\neg(y \mathbin{{}^{\circ}\!\wedge} z) \mathbin{{}^{\circ}\!\wedge} \mathsf{T})) && g(\text{N3}) \\
\neg(\neg(x \mathbin{{}^{\circ}\!\wedge} \mathsf{T}) \mathbin{{}^{\circ}\!\wedge} \mathsf{F}) &= \neg(x \mathbin{{}^{\circ}\!\wedge} \mathsf{F}) && g(\text{N4}) \\
\neg(\neg(x \mathbin{{}^{\circ}\!\wedge} \mathsf{F}) \mathbin{{}^{\circ}\!\wedge} \neg(y \mathbin{{}^{\circ}\!\wedge} \mathsf{T})) &= \neg(\neg(\neg(x \mathbin{{}^{\circ}\!\wedge} \mathsf{F}) \mathbin{{}^{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\wedge} \mathsf{T}) && g(\text{N5}) \\
\neg(\neg(\neg(x \mathbin{{}^{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\wedge} \neg(z \mathbin{{}^{\circ}\!\wedge} \mathsf{F})) \mathbin{{}^{\circ}\!\wedge} \mathsf{T}) &= \neg(\neg(\neg(x \mathbin{{}^{\circ}\!\wedge} \mathsf{T}) \mathbin{{}^{\circ}\!\wedge} \neg(z \mathbin{{}^{\circ}\!\wedge} \mathsf{F})) \mathbin{{}^{\circ}\!\wedge} \neg(\neg(y \mathbin{{}^{\circ}\!\wedge} \mathsf{T}) \mathbin{{}^{\circ}\!\wedge} \neg(z \mathbin{{}^{\circ}\!\wedge} \mathsf{F}))) && g(\text{N6})
\end{aligned}$$

Table 3.3: g(EqFLNL), the translated axioms of EqFLNL

Theorem 3.3.5.

1. *The translation function f is sound, i.e. for all terms $t, u \in \mathbb{T}_{\text{SCL}}^{A,\mathcal{X}}$,*

$$\text{EqFSCL} \vdash t = u \implies \text{EqFLNL} \vdash f(t) = f(u).$$

2. *The translation function g is sound, i.e. for all terms $t, u \in \mathbb{T}_{\text{LNL}}^{A,\mathcal{X}}$,*

$$\text{EqFLNL} \vdash t = u \implies \text{EqFSCL} \vdash g(t) = g(u).$$

Proof.

Statement 1:

Proof by induction on n, the length of the derivation.

Base case: Assume that the derivation EqFSCL $\vdash t = u$ has length one. If $t = u$ is derived using reflexivity, the result follows trivially. Otherwise $t = u$ is an axiom of EqFSCL and the result follows by Lemma 3.3.3.

Inductive step: Consider a derivation of length $n > 1$. We assume that the result holds for each derivation of lesser length (IH). If EqFSCL $\vdash t = u$ is derived using symmetry or transitivity, the result follows trivially by IH. Assume that

$$\frac{\text{EqFSCL} \vdash t_1 = u_1, \text{EqFSCL} \vdash t_2 = u_2}{\text{EqFSCL} \vdash t_1 \mathbin{{}_{\circ}\!\wedge} u_1 = t_2 \mathbin{{}_{\circ}\!\wedge} u_2}$$

is derived using congruence, with $t_1, t_2, u_1, u_2 \in \mathbb{T}_{\text{LNL}}^{A}$. By IH we have EqFLNL $\vdash f(t_1) = f(u_1)$ and EqFLNL $\vdash f(t_2) = f(u_2)$. So by congruence we also have EqFLNL $\vdash (f(t_1) \lhd f(u_1)) \lhd \mathsf{T} = (f(t_2) \lhd f(u_2)) \lhd \mathsf{T}$, which is equivalent to EqFLNL $\vdash f(t_1 \mathbin{{}_{\circ}\!\wedge} u_1) = f(t_2 \mathbin{{}_{\circ}\!\wedge} u_2)$ by Definition 3.3.2. The cases for $\neg$ and $\mathbin{{}_{\circ}\!\vee}$ follow similarly.

Finally, assume that we have a derivation

$$\frac{\text{EqFSCL} \vdash t = u}{\text{EqFSCL} \vdash \sigma(t) = \sigma(u)}$$

using substitution, for $t, u \in \mathbb{T}_{\text{SCL}}^{A}$. Assume WLOG that σ replaces every occurrence of the variable x in t and u by some expression v over $\Sigma_{\text{SCL}}(A)$. By IH we have EqFLNL $\vdash f(t) = f(u)$. By substitution it follows that EqFLNL $\vdash f(\sigma)f(t) = f(\sigma)f(u)$, where $f(\sigma)$ replaces every occurrence of x in $f(t)$ and $f(u)$ by $f(v)$. Because $f(x) = x$, we find that $f(\sigma)f(t) = f(\sigma(t))$ and $f(\sigma)f(u) = f(\sigma(u))$. So EqFLNL $\vdash f(\sigma(t)) = f(\sigma(u))$ follows.

The proof of Statement 2 follows similarly. □

We now show that translating a term t back and forth between $\mathbb{T}_{\text{SCL}}^{A,\mathcal{X}}$ and $\mathbb{T}_{\text{LNL}}^{A,\mathcal{X}}$ (or vice-versa) provides us with a derivably equal term.

Theorem 3.3.6.

1. *For any $t \in \mathbb{T}_{\text{LNL}}^{A,\mathcal{X}}$,*

$$\text{EqFLNL} \vdash f(g(t)) = t.$$

2. *For any $t \in \mathbb{T}_{\text{SCL}}^{A,\mathcal{X}}$,*

$$\text{EqFSCL} \vdash g(f(t)) = t.$$

Proof.

Statement 1.

Proof by induction on the complexity of terms in $\mathbb{T}^{A,X}_{\text{LNL}}$.

Base case: Let $t \in \{\mathsf{T}, \mathsf{F}\} \cup A \cup X$. Then $f(g(t)) = t$, by Definition 3.3.2. So also EqFLNL $\vdash$ $f(g(t)) = t$.

Inductive step: Assume that $s, u \in \mathbb{T}^{A,X}_{\text{LNL}}$ are such that EqFLNL $\vdash$ $f(g(s)) = s$ and EqFLNL $\vdash$ $f(g(u)) = u$ (IH). Furthermore assume that t is of the form $s \mathbin{{}^{\circ}\!\!\mid} u$. Using Definition 3.3.2, we compute

$$\begin{aligned}
f(g(s \mathbin{{}^{\circ}\!\!\mid} u)) &= f(\neg(g(s) \mathbin{{}^{\circ}\!\!\wedge} g(u))) \\
&= f(g(s) \mathbin{{}^{\circ}\!\!\wedge} g(u)) \mathbin{{}^{\circ}\!\!\mid} \mathsf{T} \\
&= ((f(g(s)) \mathbin{{}^{\circ}\!\!\mid} f(g(u))) \mathbin{{}^{\circ}\!\!\mid} \mathsf{T}) \mathbin{{}^{\circ}\!\!\mid} \mathsf{T}
\end{aligned}$$

Using the axioms of EqFLNL, we derive

$$\begin{aligned}
f(g(s \mathbin{{}^{\circ}\!\!\mid} u)) &= ((f(g(s)) \mathbin{{}^{\circ}\!\!\mid} f(g(u))) \mathbin{{}^{\circ}\!\!\mid} \mathsf{T}) \mathbin{{}^{\circ}\!\!\mid} \mathsf{T} && \text{by the above} \\
&= ((s \mathbin{{}^{\circ}\!\!\mid} u) \mathbin{{}^{\circ}\!\!\mid} \mathsf{T}) \mathbin{{}^{\circ}\!\!\mid} \mathsf{T} && \text{by IH} \\
&= s \mathbin{{}^{\circ}\!\!\mid} u && \text{by (Aux7)}
\end{aligned}$$

For the proof of Statement 2 we refer to Appendix A. □

By Theorem 3.3.5 and Theorem 3.3.6 we find that EqFLNL and EqFSCL satisfy the requirements of being translationally equivalent.

3.4 An equational axiomatization of FLNL

Before we show that EqFLNL axiomatizes FLNL for closed terms, we need two more lemmas, the first of which concerns evaluation trees.

Lemma 3.4.1. *For any closed term $t \in T^{A}_{\text{LNL}}$,*

$$nse(t) = se(g(t)).$$

Proof. Proof by induction on the complexity of closed terms.

Base case: Let $t \in \{\mathsf{T}, \mathsf{F}\} \cup A$. We find that $nse(t) = se(g(t))$ by Definition 3.3.2.

Inductive step: Assume that $s, u \in T^{A}_{\text{LNL}}$ are such that $nse(s) = se(g(s))$ and $nse(u) = se(g(u))$ (IH). If t is of the form $s \mathbin{{}^{\circ}\!\!\mid} u$, we find that

$$\begin{aligned}
nse(s \mathbin{{}^{\circ}\!\!\mid} u) &= nse(s)\big[\mathsf{T} \mapsto nse(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \mathsf{T}\big] && \text{by Definition 3.1.5} \\
&= nse(s)\big[\mathsf{T} \mapsto nse(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \mathsf{F}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]\big] && \text{by (4)} \\
&= nse(s)\big[\mathsf{T} \mapsto nse(u)\big]\big[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}\big] && \text{by (6) and (7)} \\
&= se(g(s))\big[\mathsf{T} \mapsto se(g(u))\big]\big[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}\big] && \text{by IH} \\
&= se(g(s) \mathbin{{}^{\circ}\!\!\wedge} g(u))\big[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}\big] && \text{by Definition 2.3.3}
\end{aligned}$$

$$
\begin{aligned}
&= se(\neg(g(s) \mathbin{{}^{\circ}\!\!\wedge}\ g(u))) && \text{by Definition 2.3.3}\\
&= se(g(s \mathbin{{}^{\circ}\!\!\mid}\ u)) && \text{by Definition 3.3.2}
\end{aligned}
$$

□

As a consequence of Theorem 3.3.6 and Lemma 3.4.1 we find that $se[T^A_{\mathrm{SCL}}]$ is equal to $nse[T^A_{\mathrm{LNL}}]$.

Corollary 3.4.2. $se[T^A_{\mathrm{SCL}}] = nse[T^A_{\mathrm{LNL}}]$.

Proof. By Lemma 3.4.1 it suffices to show that $se(t) = nse(f(t))$ for any closed term $t \in T^A_{\mathrm{SCL}}$. Let $t \in T^A_{\mathrm{SCL}}$. By Theorem 3.3.6.2 we know that $\mathrm{EqFSCL} \vdash g(f(t)) = t$. Because EqFSCL axiomatizes FSCL for closed terms and by equation (12) we find that $g(f(t)) =_{se} t$. By definition of *se*-congruence we have $se(t) = se(g(f(t)))$. Since $f(t) \in T^A_{\mathrm{LNL}}$ we know that $se(g(f(t))) = nse(f(t))$ by Lemma 3.4.1 and the result follows. □

Lemma 3.4.3. *For any $t, u \in \mathbb{T}^{A,\mathcal{X}}_{\mathrm{LNL}}$,*

$$\mathrm{EqFSCL} \vdash g(t) = g(u) \implies \mathrm{EqFLNL} \vdash t = u.$$

Proof.

$$
\begin{aligned}
\mathrm{EqFSCL} \vdash g(t) = g(u) &\implies \mathrm{EqFLNL} \vdash f(g(t)) = f(g(u)) && \text{by Theorem 3.3.5.1}\\
&\implies \mathrm{EqFLNL} \vdash t = u && \text{by Theorem 3.3.6.1}
\end{aligned}
$$

□

We now show that EqFLNL constitutes an equational axiomatization of FLNL for closed terms, which by equation (19) immediately yields a completeness result.

Theorem 3.4.4. *For all $t, u \in T^A_{\mathrm{LNL}}$,*

$$\mathrm{EqFLNL} \vdash t = u \iff \mathrm{FLNL} \vdash t = u.$$

Proof.

$$
\begin{aligned}
\mathrm{EqFLNL} \vdash t = u &\iff \mathrm{EqFSCL} \vdash g(t) = g(u) && \text{by Theorem 3.3.5.2 and Lemma 3.4.3}\\
&\iff \mathrm{FSCL} \vdash g(t) = g(u) && \text{by Theorem 2.4.3}\\
&\iff g(t) =_{se} g(u) && \text{by (12)}\\
&\iff se(g(t)) = se(g(u)) && \text{by Definition 2.3.5}\\
&\iff nse(t) = nse(u) && \text{by Lemma 3.4.1}\\
&\iff t =_{nse} u && \text{by Definition 3.1.6}\\
&\iff \mathrm{FLNL} \vdash t = u && \text{by (19)}
\end{aligned}
$$

□

Corollary 3.4.5. *For all $t, u \in T^A_{\mathrm{LNL}}$,*

$$\mathrm{EqFLNL} \vdash t = u \iff t =_{nse} u.$$

Chapter 4

ℓXOR and FLXL

In this chapter we consider ℓXOR, full left-sequential XOR, and provide an equational axiomatization of free left-sequential xor logic (FLXL) for closed terms. In Section 4.1 we define ℓXOR, notation $\mathbin{\bullet\!\!\oplus}$, define a semantics for closed terms with $\mathbin{\bullet\!\!\oplus}$ using a function *xe* that assigns evaluation trees, and we define *xe*-congruence that characterizes this semantics. In Section 4.2 we define FLXL, provide a set of equational axioms (EqFLXL) for this logic and show that any closed-term equation that is derivable from EqFLXL is also derivable from FLXL. In Section 4.3 we define ℓIFF, full left-sequential biconditional, notation $\mathbin{\bullet\!\!\leftrightarrow}$, and show that the duality principle holds for equations with $\mathbin{\bullet\!\!\oplus}$ and $\mathbin{\bullet\!\!\leftrightarrow}$. In Section 4.4 we discuss basic forms as a preferred notation for closed terms with $\mathbin{\bullet\!\!\oplus}$ and in Section 4.5 we discuss properties of *xe*-trees. Finally, in Section 4.6 we define a function that inverts *xe*-trees and show that EqFLXL axiomatizes FLXL for closed terms.

4.1 ℓXOR, syntax and *xe*-trees

We define the connective ℓXOR, full left-sequential XOR, using Hoare's conditional and the constants T and F.

Definition 4.1.1. *The connective ℓXOR, written as $\mathbin{\bullet\!\!\oplus}$, is defined by*

$$x \mathbin{\bullet\!\!\oplus} y = (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleleft x \triangleright y. \tag{20}$$

Observe that it is not possible to define a short-circuit version of XOR. Irrespective of the value of the first argument, the second argument must always be evaluated.

Definition 4.1.2. *The set T^A_{LXL} of closed terms over A is generated by the following grammar, where $a \in A$:*

$$t ::= a \mid \mathsf{T} \mid \mathsf{F} \mid t \mathbin{\bullet\!\!\oplus} t.$$

Its underlying signature is

$$\Sigma_{\mathrm{LXL}}(A) = \{\mathsf{T}, \mathsf{F}, a, \mathbin{\bullet\!\!\oplus} \mid a \in A\}.$$

The abbreviation LXL stands for left-sequential xor logic. We will elaborate on LXL in the next section. We interpret closed terms over $\Sigma_{\mathrm{LXL}}(A)$ as evaluation trees by a function *xe* (abbreviating xor evaluation).

Definition 4.1.3. *The unary* ***xor evaluation function*** $xe : T^A_{\text{LXL}} \to \mathcal{T}_A$ *is defined a follows, where* $a \in A$*:*

$$\begin{aligned} xe(\mathsf{T}) &= \mathsf{T}, & xe(a) &= (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F}), \\ xe(\mathsf{F}) &= \mathsf{F}, & xe(t \mathbin{\bullet\!\oplus} u) &= xe(t)\big[\mathsf{T} \mapsto xe(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto xe(u)\big]. \end{aligned}$$

Like equality of *nse*-trees, equality of *xe*-trees defines a congruence relation on T^A_{LXL}.

Definition 4.1.4. *The binary relation* ***xe-congruence****, notation* $=_{xe}$*, is defined on* T^A_{LXL} *as follows:*

$$t =_{xe} u \iff xe(t) = xe(u).$$

Lemma 4.1.5. *The relation* $=_{xe}$ *is a congruence relation on* T^A_{LXL}*.*

Proof. Because reflexivity, symmetry and transitivity hold, $=_{xe}$ is an equivalence relation. Furthermore, if $xe(t_1) = xe(u_1)$ and $xe(t_2) = xe(u_2)$, then $xe(t_1 \mathbin{\bullet\!\oplus} t_2) = xe(u_1 \mathbin{\bullet\!\oplus} u_2)$. This follows immediately from Definition 4.1.3. □

4.2 FLXL and equational axioms

In this section we will define free left-sequential xor logic (FLXL) and we will provide a set of equational axioms (EqFLXL) for this logic.

As in the previous chapter, we will give a generic definition of a left-sequential xor logic (LXL) using equation

$$x \mathbin{\bullet\!\oplus} y = (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleleft x \triangleright y, \tag{20}$$

and we define free left-sequential xor logic (FLXL) as the least identifying LXL.

Definition 4.2.1. *A* ***left-sequential xor logic*** *is a logic that implies the consequences of the module expression*

$$\text{LXL} = \{\mathsf{T}, \bullet\!\oplus\} \mathbin{\square} (\text{CP} \cup \{(20)\}).$$

The constant F does not occur in the exported signature of LXL, but can easily be added because $\text{CP} \cup \{(20)\} \vdash \mathsf{T} \mathbin{\bullet\!\oplus} \mathsf{T} \overset{(20)}{=} (\mathsf{F} \triangleleft \mathsf{T} \triangleright \mathsf{T}) \triangleleft \mathsf{T} \triangleright \mathsf{T} \overset{(\text{CP1})}{=} \mathsf{F}$.

Definition 4.2.2. ***Free left-sequential xor logic*** *(***FLXL***) is the left-sequential xor logic that implies no other consequences than those of the module expression* LXL.

For all terms t, u over $\Sigma_{\text{LXL}}(A)$ we write

$$\text{FLXL} \vdash t = u$$

if $\text{CP} \cup \{(20)\} \vdash t = u$.

In the following we argue that FLXL is characterized by *xe*-congruence for closed terms over $\Sigma_{\text{LXL}}(A)$. First we define the combined signature $\Sigma_{\text{CP+LXL}}(A) = \Sigma_{\text{CP}}(A) \cup \{\bullet\!\oplus\}$ and $T_{\text{CP+LXL}}(A)$, the set of closed terms over $\Sigma_{\text{CP+LXL}}(A)$. We now extend the evaluation function *ce*.

Definition 4.2.3. *The evaluation function $ce_{xe} : T_{\text{CP+LXL}}(A) \mapsto \mathcal{T}_A$ extends ce as follows:*

$$ce_{xe}(t \mathbin{\bullet\!\!\oplus} u) = ce_{xe}(t)\big[\mathsf{T} \mapsto ce_{xe}(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto ce_{xe}(u)\big].$$

Note that equation (20) is satisfied by ce_{xe}, i.e. $ce_{xe}(t \mathbin{\bullet\!\!\oplus} u) = ce_{xe}(t)\big[\mathsf{T} \mapsto ce_{xe}(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto ce_{xe}(u)\big] = ce_{xe}((\mathsf{F} \triangleleft u \triangleright \mathsf{T}) \triangleleft t \triangleright u)$. Using Definition 2.2.4 and the completeness result from Theorem 2.2.5, we find for all $t, u \in T^A_{\text{LXL}}$,

$$\text{CP} \cup \{(20)\} \vdash t = u \iff ce_{xe}(t) = ce_{xe}(u). \tag{21}$$

Lemma 4.2.4. *The evaluation function ce_{xe} is equal to xe when restricted to terms in T^A_{LXL}, i.e. for any $t \in T^A_{\text{LXL}}$,*

$$ce_{xe}(t) = xe(t).$$

Proof. The proof follows by induction on the complexity of closed terms. The base case is trivial. For the inductive step, assume that $s, u \in T^A_{\text{LXL}}$ are such that $ce_{xe}(s) = xe(s)$ and $ce_{xe}(u) = xe(u)$ (IH) and that t is of the form $s \mathbin{\bullet\!\!\oplus} u$. We find that $ce_{xe}(s \mathbin{\bullet\!\!\oplus} u) = ce_{xe}(s)\big[\mathsf{T} \mapsto ce_{xe}(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto ce_{xe}(u)\big] \stackrel{\text{IH}}{=} xe(s)\big[\mathsf{T} \mapsto xe(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto xe(u)\big] = xe(s \mathbin{\bullet\!\!\oplus} u)$. $\square$

Using equation (21), we find by the previous lemma and by definition of *xe*-congruence that for all closed terms $t, u \in T^A_{\text{LXL}}$,

$$\text{FLXL} \vdash t = u \iff t =_{xe} u. \tag{22}$$

In Table 4.1 we provide a set of equational axioms for FLXL, called EqFLXL. We will show that EqFLXL axiomatizes FLXL for closed terms.

$$\mathsf{T} \mathbin{\bullet\!\!\oplus} \mathsf{T} = \mathsf{F} \tag{X1}$$
$$(x \mathbin{\bullet\!\!\oplus} y) \mathbin{\bullet\!\!\oplus} z = x \mathbin{\bullet\!\!\oplus} (y \mathbin{\bullet\!\!\oplus} z) \tag{X2}$$
$$x \mathbin{\bullet\!\!\oplus} \mathsf{T} = \mathsf{T} \mathbin{\bullet\!\!\oplus} x \tag{X3}$$
$$x \mathbin{\bullet\!\!\oplus} \mathsf{F} = x \tag{X4}$$

Table 4.1: EqFLXL, a set of equational axioms for FLXL

We first elaborate on this set of axioms. Axiom (X1) is a simple identity that describes a relation between the constants. Axiom (X2) defines associativity. Axiom (X3) tells us that it does not matter for the evaluation result whether T is evaluated first or second. Finally, in axiom (X4) we see that the evaluation result of the first argument is preserved if the second argument is F.

We now derive two auxiliary results that will be used in the remainder of this thesis.

Proposition 4.2.5. *The following equations are derivable from* EqFLXL*:*

$$(x \mathbin{\bullet\!\!\oplus} \mathsf{T}) \mathbin{\bullet\!\!\oplus} \mathsf{T} = x, \tag{Aux8}$$
$$\mathsf{F} \mathbin{\bullet\!\!\oplus} x = x \mathbin{\bullet\!\!\oplus} \mathsf{F}, \tag{Aux9}$$

Proof. The equations are checked by the theorem prover *Prover9* [McC08].

Proof of (Aux8):

$$
\begin{aligned}
(x \mathbin{\bullet\!\!\oplus} \mathsf{T}) \mathbin{\bullet\!\!\oplus} \mathsf{T} &= x \mathbin{\bullet\!\!\oplus} (\mathsf{T} \mathbin{\bullet\!\!\oplus} \mathsf{T}) && \text{by (X2)}\\
&= x \mathbin{\bullet\!\!\oplus} \mathsf{F} && \text{by (X1)}\\
&= x && \text{by (X4)}
\end{aligned}
$$

Proof of (Aux9):

$$
\begin{aligned}
\mathsf{F} \mathbin{\bullet\!\!\oplus} x &= (\mathsf{T} \mathbin{\bullet\!\!\oplus} \mathsf{T}) \mathbin{\bullet\!\!\oplus} x && \text{by (X1)}\\
&= \mathsf{T} \mathbin{\bullet\!\!\oplus} (\mathsf{T} \mathbin{\bullet\!\!\oplus} x) && \text{by (X2)}\\
&= \mathsf{T} \mathbin{\bullet\!\!\oplus} (x \mathbin{\bullet\!\!\oplus} \mathsf{T}) && \text{by (X3)}\\
&= (\mathsf{T} \mathbin{\bullet\!\!\oplus} x) \mathbin{\bullet\!\!\oplus} \mathsf{T} && \text{by (X2)}\\
&= (x \mathbin{\bullet\!\!\oplus} \mathsf{T}) \mathbin{\bullet\!\!\oplus} \mathsf{T}) && \text{by (X3)}\\
&= x \mathbin{\bullet\!\!\oplus} (\mathsf{T} \mathbin{\bullet\!\!\oplus} \mathsf{T}) && \text{by (X2)}\\
&= x \mathbin{\bullet\!\!\oplus} \mathsf{F} && \text{by (X1)}
\end{aligned}
$$

□

We will show that if the equation $t = u$ is derivable from EqFLXL for $t, u \in T^A_{\text{LXL}}$, then $t = u$ is also derivable from FLXL. To this end, we first establish a model $\mathbb{M}$ for EqFLXL and show that EqFLXL is sound for $\mathbb{M}$.

Definition 4.2.6. *Let $\mathbb{M}$ be the $\Sigma_{\text{LXL}}(A)$-algebra with domain $D = \{xe(t) \mid t \in T^A_{\text{LXL}}\}$ in which the interpretation of the constants is defined by*

$$[\![\mathsf{T}]\!]^{\mathbb{M}} = \mathsf{T}, \quad [\![\mathsf{F}]\!]^{\mathbb{M}} = \mathsf{F}, \quad [\![a]\!]^{\mathbb{M}} = (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F}),$$

where $a \in A$, and in which the interpretation of the connective $\mathbin{\bullet\!\!\oplus}$ is defined by

$$[\![t \mathbin{\bullet\!\!\oplus} u]\!]^{\mathbb{M}} = [\![t]\!]^{\mathbb{M}}\big[\mathsf{T} \mapsto [\![u]\!]^{\mathbb{M}}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto [\![u]\!]^{\mathbb{M}}\big].$$

Lemma 4.2.7. *For all terms t, u over $\Sigma_{\text{LXL}}(A)$,*

$$\text{EqFLXL} \vdash t = u \implies \mathbb{M} \vDash t = u.$$

Proof. Proof by induction on n, the length of the derivation.

Base case: Assume that the derivation EqFLXL $\vdash t = u$ has length one. If $t = u$ is derived using reflexivity, the result follows trivially. Otherwise $t = u$ is an axiom of EqFLXL. Here we show that $\mathbb{M}$ satisfies (X3) and for the proofs of (X1), (X2) and (X4) we refer to Appendix B. Fix an arbitrary interpretation i of variables. Then,

$$
\begin{aligned}
[\![x \mathbin{\bullet\!\!\oplus} \mathsf{T}]\!]^{\mathbb{M},i} &= [\![x]\!]^{\mathbb{M},i}\big[\mathsf{T} \mapsto [\![\mathsf{T}]\!]^{\mathbb{M}}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto [\![\mathsf{T}]\!]^{\mathbb{M}}\big] && \text{by 4.2.6}\\
&= [\![x]\!]^{\mathbb{M},i}\big[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}\big] && \text{by (3)}\\
&= [\![\mathsf{T}]\!]^{\mathbb{M}}\big[\mathsf{T} \mapsto [\![x]\!]^{\mathbb{M},i}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto [\![x]\!]^{\mathbb{M},i}\big] && \text{by (3)}\\
&= [\![\mathsf{T} \mathbin{\bullet\!\!\oplus} x]\!]^{\mathbb{M},i}. && \text{by 4.2.6}
\end{aligned}
$$

Because i was chosen arbitrarily, we have $\mathbb{M} \vDash$ (X3).

Inductive step: Consider a derivation of length $n > 1$. We assume that the result holds for each derivation of lesser length (IH). If EqFLXL $\vdash t = u$ is derived using symmetry or transitivity, the result follows trivially by IH. Assume that we have a derivation

$$\frac{\text{EqFLXL} \vdash t_1 = u_1, \text{EqFLXL} \vdash t_2 = u_2}{\text{EqFLXL} \vdash t_1 \mathbin{\bullet\!\!\oplus} t_2 = u_1 \mathbin{\bullet\!\!\oplus} u_2}$$

using congruence, with t_1, t_2, u_1, u_2 over $\Sigma_{\text{LXL}}(A)$. By IH we have $\mathbb{M} \vDash t_1 = u_1$ and $\mathbb{M} \vDash t_2 = u_2$, so we also have $[\![t_1]\!]^{\mathbb{M},i} = [\![u_1]\!]^{\mathbb{M},i}$ and $[\![t_2]\!]^{\mathbb{M},i} = [\![u_2]\!]^{\mathbb{M},i}$ for each interpretation i of variables. Fix such an interpretation i arbitrarily. We find that

$$\begin{aligned}
[\![t_1 \mathbin{\bullet\!\!\oplus} t_2]\!]^{\mathbb{M},i} &= [\![t_1]\!]^{\mathbb{M},i}[\mathsf{T} \mapsto [\![t_2]\!]^{\mathbb{M},i}, \mathsf{F} \mapsto [\![t_2]\!]^{\mathbb{M},i}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]] && \text{by 4.2.6} \\
&= [\![u_1]\!]^{\mathbb{M},i}[\mathsf{T} \mapsto [\![u_2]\!]^{\mathbb{M},i}, \mathsf{F} \mapsto [\![u_2]\!]^{\mathbb{M},i}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]] && \text{by IH} \\
&= [\![u_1 \mathbin{\bullet\!\!\oplus} u_2]\!]^{\mathbb{M},i}, && \text{by 4.2.6}
\end{aligned}$$

so $\mathbb{M} \vDash t_1 \mathbin{\bullet\!\!\oplus} t_2 = u_1 \mathbin{\bullet\!\!\oplus} u_2$ because i was chosen arbitrarily.

Finally, assume we have a derivation

$$\frac{\text{EqFLXL} \vdash t = u}{\text{EqFLXL} \vdash \sigma(t) = \sigma(u)}$$

using substitution, for t, u over $\Sigma_{\text{LXL}}(A)$. Assume WLOG that σ replaces every occurrence of the variable x in t and u by some expression v over $\Sigma_{\text{LXL}}(A)$. Let i be an arbitrary interpretation of the variables in $\sigma(t)$ and $\sigma(u)$. Observe that there is an interpretation i' of variables in t and u such that $[\![\sigma(t)]\!]^{\mathbb{M},i} = [\![t]\!]^{\mathbb{M},i'}$ and $[\![\sigma(u)]\!]^{\mathbb{M},i} = [\![u]\!]^{\mathbb{M},i'}$. By IH we have $\mathbb{M} \vDash t = u$, hence also $\mathbb{M}, i' \vDash t = u$. Then $\mathbb{M}, i \vDash \sigma(t) = \sigma(u)$ holds as well, and the result follows. ☐

Theorem 4.2.8. *For $t, u \in T^A_{\text{LXL}}$,*

$$\text{EqFLXL} \vdash t = u \implies \text{FLXL} \vdash t = u.$$

Proof. By Lemma 4.2.7 and equation (22) it suffices to show that $\mathbb{M} \vDash t = u \iff t =_{xe} u$. Note that we have $[\![t]\!]^{\mathbb{M}} = xe(t)$ for any $t \in T^A_{\text{LXL}}$ (this follows easily by structural induction). Therefore, $\mathbb{M} \vDash t = u$ holds if and only if $[\![t]\!]^{\mathbb{M}} = [\![u]\!]^{\mathbb{M}}$, if and only if $xe(t) = xe(u)$, if and only if $t =_{xe} u$ by Definition 4.1.4. ☐

4.3 ℓIFF and duality

In this section we extend $\Sigma_{\text{LXL}}(A)$ to $\Sigma_{\text{LXL}^+}(A)$ with ℓIFF, full left-sequential biconditional, and EqFLXL to EqFLXL$^+$ with an extra axiom. We define the dual of closed terms over $\Sigma_{\text{LXL}^+}(A)$ and show that EqFLXL$^+$ satisfies the duality principle for all equations over $\Sigma_{\text{LXL}^+}(A)$.

First we define the connective ℓIFF, full left-sequential biconditional, using Hoare's conditional and the constants T and F.

Definition 4.3.1. *The connective ℓIFF, written as $\mathbin{\bullet\!\!\leftrightarrow}$, is defined by*

$$x \mathbin{\bullet\!\!\leftrightarrow} y = y \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{T}). \tag{23}$$

As with XOR, it is not possible to define a short-circuit version of the biconditional.

Proposition 4.3.2. *The connective* $\mathbin{\bullet\!\!\leftrightarrow}$ *is expressible in terms of* $\mathbin{\bullet\!\!\oplus}$ *and* T*:*

$$x \mathbin{\bullet\!\!\leftrightarrow} y = (x \mathbin{\bullet\!\!\oplus} y) \mathbin{\bullet\!\!\oplus} \mathsf{T}. \tag{24}$$

Proof.

$$\begin{aligned}
(x \mathbin{\bullet\!\!\oplus} y) \mathbin{\bullet\!\!\oplus} \mathsf{T} &= (\mathsf{F} \triangleleft \mathsf{T} \triangleright \mathsf{T}) \triangleleft ((\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleleft x \triangleright y) \triangleright \mathsf{T} && \text{by (20)}\\
&= (\mathsf{F} \triangleleft (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleright \mathsf{T}) \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) && \text{by (CP1) and (CP4)}\\
&= ((\mathsf{F} \triangleleft \mathsf{F} \triangleright \mathsf{T}) \triangleleft y \triangleright (\mathsf{F} \triangleleft \mathsf{T} \triangleright \mathsf{T})) \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) && \text{by (CP4)}\\
&= (\mathsf{T} \triangleleft y \triangleright \mathsf{F}) \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) && \text{by (CP1) and (CP2)}\\
&= y \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) && \text{by (CP3)}\\
&= x \mathbin{\bullet\!\!\leftrightarrow} y && \text{by (23)}
\end{aligned}$$

□

In the remainder of this section we will consider the set of closed terms $T^A_{\text{LXL}^+}$ over the extended signature

$$\Sigma_{\text{LXL}^+}(A) = \Sigma_{\text{LXL}}(A) \cup \{\mathbin{\bullet\!\!\leftrightarrow}\}.$$

We also extend EqFLXL to

$$\text{EqFLXL}^+ = \text{EqFLXL} \cup \{(24)\}.$$

We now define the dual of a term $t \in T^A_{\text{LXL}^+}$ and we prove that the duality mapping is an involution.

Definition 4.3.3. *The* ***dual*** *of a term* $t \in T^A_{\text{LXL}^+}$*, notation* t^{dl}*, is defined by*

$$\begin{aligned}
\mathsf{T}^{dl} &= \mathsf{F}, & a^{dl} &= a, & (t \mathbin{\bullet\!\!\leftrightarrow} u)^{dl} &= t^{dl} \mathbin{\bullet\!\!\oplus} u^{dl}.\\
\mathsf{F}^{dl} &= \mathsf{T}, & (t \mathbin{\bullet\!\!\oplus} u)^{dl} &= t^{dl} \mathbin{\bullet\!\!\leftrightarrow} u^{dl}, & &
\end{aligned}$$

Lemma 4.3.4. $(\ )^{dl} : T^A_{\text{LXL}^+} \to T^A_{\text{LXL}^+}$ *is an involution, i.e.* $(t^{dl})^{dl} = t$*.*

Proof. Proof by induction on the complexity of closed terms.

Base case: The result for $t \in \{\mathsf{T}, \mathsf{F}\} \cup A$ follows trivially.

Inductive step: Assume that $t, u \in T^A_{\text{LXL}^+}$ are such that $(t^{dl})^{dl} = t$ and $(u^{dl})^{dl} = u$ (IH). Then $((t \mathbin{\bullet\!\!\oplus} u)^{dl})^{dl} = (t^{dl} \mathbin{\bullet\!\!\leftrightarrow} u^{dl})^{dl} = (t^{dl})^{dl} \mathbin{\bullet\!\!\oplus} (u^{dl})^{dl} = t \mathbin{\bullet\!\!\oplus} u$, where the last equality holds by IH and the others by Definition 4.3.3. The case for $((t \mathbin{\bullet\!\!\leftrightarrow} u)^{dl})^{dl}$ follows similarly. □

Setting the dual of a variable by $x^{dl} = x$, we show that EqFSCL satisfies the duality principle for equations over $\Sigma_{\text{LXL}^+}(A)$.

Theorem 4.3.5. *For all* t, u *over* $\Sigma_{\text{LXL}^+}(A)$,

$$\text{EqFLXL}^+ \vdash t = u \iff \text{EqFLXL}^+ \vdash t^{dl} = u^{dl}.$$

Proof.
'$\Longrightarrow$'
Proof by induction on n, the length of the derivation.

Base case: Assume that the derivation EqFLXL$^+ \vdash t = u$ has length one. If $t = u$ is derived using reflexivity, the result follows trivially. Otherwise $t = u$ is an axiom of EqFLXL$^+$. We show that the dual of each axiom is also derivable from EqFLXL$^+$. The dual axioms of EqFLXL$^+$ can be found in Table 4.2. Below we show that EqFLXL$^+ \vdash$ (X1)^{dl}. The proofs for the remaining dual axioms can be found in Appendix B.

$$\begin{aligned}
\mathsf{F} \bullet\!\!\leftrightarrow \mathsf{F} &= (\mathsf{F} \bullet\!\!\oplus \mathsf{F}) \bullet\!\!\oplus \mathsf{T} && \text{by (24)}\\
&= \mathsf{T} \bullet\!\!\oplus (\mathsf{F} \bullet\!\!\oplus \mathsf{F}) && \text{by (X3)}\\
&= (\mathsf{T} \bullet\!\!\oplus \mathsf{F}) \bullet\!\!\oplus \mathsf{F} && \text{by (X2)}\\
&= \mathsf{T} && \text{by (X4)}
\end{aligned}$$

Inductive step: Consider a derivation of length $n > 1$. We assume that the result holds for each derivation of lesser length (IH). If EqFLXL$^+ \vdash t = u$ is derived using symmetry or transitivity, the result follows trivially by IH. Assume that we have a derivation

$$\frac{\text{EqFLXL}^+ \vdash t_1 = u_1, \text{EqFLXL}^+ \vdash t_2 = u_2}{\text{EqFLXL}^+ \vdash t_1 \bullet\!\!\oplus t_2 = u_1 \bullet\!\!\oplus u_2}$$

using congruence, with t_1, t_2, u_1, u_2 over $\Sigma_{\text{LXL}^+}(A)$. The case for $\bullet\!\!\leftrightarrow$ follows similarly. By IH we find that EqFLXL$^+ \vdash t_1^{dl} = u_1^{dl}$ and that EqFLXL$^+ \vdash t_2^{dl} = u_2^{dl}$. So by congruence we have EqFLXL$^+ \vdash t_1^{dl} \bullet\!\!\leftrightarrow t_2^{dl} = u_1^{dl} \bullet\!\!\leftrightarrow u_2^{dl}$. Then EqFLXL$^+ \vdash (t_1 \bullet\!\!\oplus t_2)^{dl} = (u_1 \bullet\!\!\oplus u_2)^{dl}$ follows by Definition 4.3.3.

Finally, assume we have a derivation

$$\frac{\text{EqFLXL}^+ \vdash t = u}{\text{EqFLXL}^+ \vdash \sigma(t) = \sigma(u)}$$

using substitution, for t, u over $\Sigma_{\text{LXL}^+}(A)$. Assume WLOG that σ replaces every occurrence of the variable x in t and u by some expression v over $\Sigma_{\text{LXL}^+}(A)$. By IH we have EqFLXL$^+ \vdash t^{dl} = u^{dl}$. By substitution it follows that EqFLXL$^+ \vdash \sigma^{dl}(t^{dl}) = \sigma^{dl}(u^{dl})$, where σ^{dl} replaces every occurrence of x in t^{dl} and u^{dl} by v^{dl}. Because $x^{dl} = x$, we find that $\sigma^{dl}(t^{dl}) = (\sigma(t))^{dl}$ and $\sigma^{dl}(u^{dl}) = (\sigma(u))^{dl}$. So EqFLXL$^+ \vdash (\sigma(t))^{dl} = (\sigma(u))^{dl}$ follows.

'$\Longleftarrow$'
By the above we find that EqFLXL$^+ \vdash t^{dl} = u^{dl} \implies$ EqFLXL$^+ \vdash (t^{dl})^{dl} = (u^{dl})^{dl}$. The result follows by Lemma 4.3.4. $\square$

$$\mathsf{F} \mathbin{{\scriptstyle\bullet}\!\!\leftrightarrow} \mathsf{F} = \mathsf{T} \tag{X1^{dl}}$$
$$(x \mathbin{{\scriptstyle\bullet}\!\!\leftrightarrow} y) \mathbin{{\scriptstyle\bullet}\!\!\leftrightarrow} z = x \mathbin{{\scriptstyle\bullet}\!\!\leftrightarrow} (y \mathbin{{\scriptstyle\bullet}\!\!\leftrightarrow} z) \tag{X2^{dl}}$$
$$x \mathbin{{\scriptstyle\bullet}\!\!\leftrightarrow} \mathsf{F} = \mathsf{F} \mathbin{{\scriptstyle\bullet}\!\!\leftrightarrow} x \tag{X3^{dl}}$$
$$x \mathbin{{\scriptstyle\bullet}\!\!\leftrightarrow} \mathsf{T} = x \tag{X4^{dl}}$$
$$x \mathbin{{\scriptstyle\bullet}\!\oplus} y = (x \mathbin{{\scriptstyle\bullet}\!\!\leftrightarrow} y) \mathbin{{\scriptstyle\bullet}\!\!\leftrightarrow} \mathsf{F} \tag{24^{dl}}$$

Table 4.2: $(\text{EqFLXL}^{+})^{dl}$, the dual axioms of EqFLXL^{+}

4.4 Basic Form

In this section we define a basic form for terms in T^A_{LXL} and a function that maps any term in T^A_{LXL} to a derivably equal term in basic form.

Definition 4.4.1. *A term $t \in T^A_{\text{LXL}}$ is said to be in **Basic Form (BF)** if it is generated by the following grammar:*

$$\begin{aligned} t &::= t^B \mid t^\star \mathbin{{\scriptstyle\bullet}\!\oplus} t^B && \textit{(BF-terms)} \\ t^B &::= \mathsf{T} \mid \mathsf{F} && \textit{(Boolean constants)} \\ t^\star &::= a \mid t^\star \mathbin{{\scriptstyle\bullet}\!\oplus} a && \textit{($\star$-terms)} \end{aligned}$$

for $a \in A$. We refer to t^B-forms as Boolean constants, to $t^\star$-forms as $\star$-terms and to $t^\star \mathbin{{\scriptstyle\bullet}\!\oplus} t^B$-forms as $\star$-B-terms.

Note that each *BF*-term is of the form $(((a_1 \mathbin{{\scriptstyle\bullet}\!\oplus} a_2) \mathbin{{\scriptstyle\bullet}\!\oplus} \ldots) \mathbin{{\scriptstyle\bullet}\!\oplus} a_n) \mathbin{{\scriptstyle\bullet}\!\oplus} t^B$ for some $n \geq 0$.

Definition 4.4.2. *We define $f : T^A_{\text{LXL}} \to BF$ recursively using the auxiliary function f^x by*

$$\begin{aligned} f(\mathsf{T}) &= \mathsf{T}, & f(a) &= a \mathbin{{\scriptstyle\bullet}\!\oplus} \mathsf{F}, \\ f(\mathsf{F}) &= \mathsf{F}, & f(t \mathbin{{\scriptstyle\bullet}\!\oplus} u) &= f^x(f(t), f(u)). \end{aligned}$$

We define $f^x : BF \times BF \to BF$ recursively by a case distinction on the first argument. We start with the Boolean constants. If the first argument is T, we make a case distinction on the second argument.

$$\begin{aligned} f^x(\mathsf{T},\mathsf{T}) &= \mathsf{F}, & f^x(\mathsf{T}, t^\star \mathbin{{\scriptstyle\bullet}\!\oplus} t^B) &= t^\star \mathbin{{\scriptstyle\bullet}\!\oplus} f^x(\mathsf{T}, t^B), \\ f^x(\mathsf{T},\mathsf{F}) &= \mathsf{T}, & f^x(\mathsf{F}, t) &= t. \end{aligned}$$

If the first argument is of the form $t^\star \mathbin{{\scriptstyle\bullet}\!\oplus} t^B$, we make a case distinction on the second argument again. If the second argument is of the form $u^\star \mathbin{{\scriptstyle\bullet}\!\oplus} u^B$, we need the auxiliary function f^x_1.

$$\begin{aligned} f^x(t^\star \mathbin{{\scriptstyle\bullet}\!\oplus} t^B, u^B) &= t^\star \mathbin{{\scriptstyle\bullet}\!\oplus} f^x(t^B, u^B), \\ f^x(t^\star \mathbin{{\scriptstyle\bullet}\!\oplus} t^B, u^\star \mathbin{{\scriptstyle\bullet}\!\oplus} u^B) &= f^x_1(t^\star, u^\star) \mathbin{{\scriptstyle\bullet}\!\oplus} f^x(t^B, u^B). \end{aligned}$$

Finally, we define the function f^x_1 that takes two $\star$-terms as arguments and returns another $\star$-term by a case distinction on the second argument.

$$\begin{aligned} f^x_1(t^\star, a) &= t^\star \mathbin{{\scriptstyle\bullet}\!\oplus} a, \\ f^x_1(t^\star, u^\star \mathbin{{\scriptstyle\bullet}\!\oplus} a) &= f^x_1(t^\star, u^\star) \mathbin{{\scriptstyle\bullet}\!\oplus} a. \end{aligned}$$

Before we can show that the function f maps any term in T^{A}_{LXL} to a derivably equal term in *BF*, we need two results on the auxiliary functions that were used to define f.

Lemma 4.4.3. *For all $\star$-terms $t^\star, u^\star$, $f^x_1(t^\star, u^\star)$ is a $\star$-term and*

$$\text{EqFLXL} \vdash f^x_1(t^\star, u^\star) = t^\star \mathbin{\bullet\!\!\oplus} u^\star.$$

Proof. Proof by induction on n, the number of atoms in the second argument of f^x_1.

Base case: If $u^\star$ is an atom the result follows trivially.

Inductive step: Assume that the result holds if the second argument of f^x_1 has n atoms (IH). Consider the $\star$-term $u^\star$ having $n+1$ atoms. $u^\star$ is of the form $s^\star \mathbin{\bullet\!\!\oplus} a_{n+1}$ for some $\star$-term $s^\star$ with n atoms. We derive that

$$\begin{aligned}
f^x_1(t^\star, u^\star) &= f^x_1(t^\star, s^\star \mathbin{\bullet\!\!\oplus} a_{n+1}) && \text{by Definition 4.4.2}\\
&= f^x_1(t^\star, s^\star) \mathbin{\bullet\!\!\oplus} a_{n+1} && \text{by Definition 4.4.2}\\
&= (t^\star \mathbin{\bullet\!\!\oplus} s^\star) \mathbin{\bullet\!\!\oplus} a_{n+1}, && \text{by IH}
\end{aligned}$$

By IH we know that $f^x_1(t^\star, s^\star)$ is a $\star$-term, so also $f^x_1(t^\star, u^\star) = f^x_1(t^\star, s^\star) \mathbin{\bullet\!\!\oplus} a_{n+1}$ is a $\star$-term. □

Lemma 4.4.4. *For all t, u in BF, $f^x(t, u)$ is a BF-term and*

$$\text{EqFLXL} \vdash f^x(t, u) = t \mathbin{\bullet\!\!\oplus} u.$$

Proof. If t is F and u is a *BF*-term, we find that $f^x(\mathsf{F}, u) = u$ is a *BF*-term as well and we derive that $f^x(\mathsf{F}, u) = u \overset{\text{(X4)}}{=} u \mathbin{\bullet\!\!\oplus} \mathsf{F} \overset{\text{(Aux9)}}{=} \mathsf{F} \mathbin{\bullet\!\!\oplus} u$.

If t is T and $u \in \{\mathsf{T}, \mathsf{F}\}$, we find that $f^x(\mathsf{T}, u) \in \{\mathsf{T}, \mathsf{F}\}$ is a *BF*-term. By (X1) and (X4) it follows easily that $\text{EqFLXL} \vdash f^x(\mathsf{T}, u) = \mathsf{T} \mathbin{\bullet\!\!\oplus} u$. If u is of the form $u^\star \mathbin{\bullet\!\!\oplus} u^B$, we find that $f^x(\mathsf{T}, u^\star \mathbin{\bullet\!\!\oplus} u^B) = u^* \mathbin{\bullet\!\!\oplus} f^x(\mathsf{T}, u^B)$ is a *BF*-term because $u^\star$ is a $\star$-term and because $f^x(\mathsf{T}, u^B) \in \{\mathsf{T}, \mathsf{F}\}$. We derive

$$\begin{aligned}
f^x(\mathsf{T}, u^\star \mathbin{\bullet\!\!\oplus} u^B) &= u^\star \mathbin{\bullet\!\!\oplus} f^x(\mathsf{T}, u^B)\\
&= u^\star \mathbin{\bullet\!\!\oplus} (\mathsf{T} \mathbin{\bullet\!\!\oplus} u^B) && \text{by the above}\\
&= (u^\star \mathbin{\bullet\!\!\oplus} \mathsf{T}) \mathbin{\bullet\!\!\oplus} u^B && \text{by (X2)}\\
&= (\mathsf{T} \mathbin{\bullet\!\!\oplus} u^\star) \mathbin{\bullet\!\!\oplus} u^B && \text{by (X3)}\\
&= \mathsf{T} \mathbin{\bullet\!\!\oplus} (u^\star \mathbin{\bullet\!\!\oplus} u^B) && \text{by (X2)}
\end{aligned}$$

If t is of the form $t^\star \mathbin{\bullet\!\!\oplus} t^B$ and u is a Boolean constant, the result is derived similarly. If u is of the form $u^\star \mathbin{\bullet\!\!\oplus} u$, we find that $f^x(t^\star \mathbin{\bullet\!\!\oplus} t^B, u^\star \mathbin{\bullet\!\!\oplus} u^B) = f^x_1(t^\star, u^\star) \mathbin{\bullet\!\!\oplus} f^x(t^B, u^B)$ is a *BF*-term by Lemma 4.4.3 and because $f^x(t^B, u^B) \in \{\mathsf{T}, \mathsf{F}\}$. We derive that

$$\begin{aligned}
f^x(t^\star \mathbin{\bullet\!\!\oplus} t^B, u^\star \mathbin{\bullet\!\!\oplus} u^B) &= f^x_1(t^\star, u^\star) \mathbin{\bullet\!\!\oplus} f^x(t^B, u^B)\\
&= (t^\star \mathbin{\bullet\!\!\oplus} u^\star) \mathbin{\bullet\!\!\oplus} (t^B \mathbin{\bullet\!\!\oplus} u^B) && \text{by Lemma 4.4.3 and the above}\\
&= (t^\star \mathbin{\bullet\!\!\oplus} (u^\star \mathbin{\bullet\!\!\oplus} t^B)) \mathbin{\bullet\!\!\oplus} u^B && \text{by (X2)}\\
&= (t^\star \mathbin{\bullet\!\!\oplus} (t^B \mathbin{\bullet\!\!\oplus} u^\star)) \mathbin{\bullet\!\!\oplus} u^B && \text{by (X3) or (Aux9)}\\
&= (t^\star \mathbin{\bullet\!\!\oplus} t^B) \mathbin{\bullet\!\!\oplus} (u^\star \mathbin{\bullet\!\!\oplus} u^B) && \text{by (X2)}
\end{aligned}$$

□

Theorem 4.4.5. *For any $t \in T^A_{\mathrm{LXL}}$, $f(t)$ is a BF-term and*

$$\mathrm{EqFLXL} \vdash f(t) = t.$$

Proof. Proof by induction on the complexity of closed terms.

Base case: If $t \in \{\mathsf{T}, \mathsf{F}\}$ the result is trivial. If t is an atom the result follows by axiom (X4).

Inductive step: Assume that $s, u \in T^A_{\mathrm{LXL}}$ are such that $\mathrm{EqFLXL} \vdash f(s) = s$ and $\mathrm{EqFLXL} \vdash f(u) = u$ and that $f(s)$ and $f(u)$ are *BF*-terms (IH). If t is of the form $s \mathbin{\bullet\!\!\oplus} u$, $f(s \mathbin{\bullet\!\!\oplus} u) = f^x(f(s), f(u))$ is a *BF*-term by Lemma 4.4.4 and we derive that

$$\begin{aligned} f^x(f(s), f(u)) &= f(s) \mathbin{\bullet\!\!\oplus} f(u) && \text{by Lemma 4.4.4} \\ &= s \mathbin{\bullet\!\!\oplus} u && \text{by IH} \end{aligned}$$

□

Corollary 4.4.6. *For any $t \in T^A_{\mathrm{LXL}}$,*

$$\mathit{xe}(f(t)) = \mathit{xe}(t).$$

Proof. By Theorem 4.4.5 we have $\mathrm{EqFLXL} \vdash f(t) = t$. By Theorem 4.2.8 and equation (22) we find that $f(t) =_{\mathit{xe}} t$. The result follows by definition of *xe*-congruence. □

4.5 Properties of *xe*-trees

In this section we will prove some properties of *xe*-trees. By Corollary 4.4.6 it suffices to only consider *xe*-trees of *BF*-terms. Consider a *BF*-term t with n atoms. We will show that $\mathit{xe}(t)$ is a complete binary tree of height n, that is, every internal node in $\mathit{xe}(t)$ has two child nodes and all leaves occur at the same depth. Furthermore, we will show that the internal nodes at the same depth are labeled with the same atom and that there exactly two ways to label the leaves with T and F.

Lemma 4.5.1. *Let $t = ((((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} a_{n+1}) \mathbin{\bullet\!\!\oplus} t^B$ be a BF-term with $n+1$ atoms. Then*

$$\mathit{xe}(t) = \mathit{xe}(t')[\mathsf{T} \mapsto (\mathsf{F} \trianglelefteq a_{n+1} \trianglerighteq \mathsf{T}), \mathsf{F} \mapsto (\mathsf{T} \trianglelefteq a_{n+1} \trianglerighteq \mathsf{F})], \tag{25}$$

with $t' = (((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} t^B$.

Proof. We show that $\mathrm{EqFLXL} \vdash t = t' \mathbin{\bullet\!\!\oplus} a_{n+1}$. The result follows by Theorem 4.2.8, equation (22) and the definition of *xe*-congruence. We derive that

$$\begin{aligned} t &= ((((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} a_{n+1}) \mathbin{\bullet\!\!\oplus} t^B \\ &= ((((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} (a_{n+1} \mathbin{\bullet\!\!\oplus} t^B) && \text{by (X2)} \\ &= ((((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} (t^B \mathbin{\bullet\!\!\oplus} a_{n+1}) && \text{by (X3) or (Aux9)} \\ &= ((((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} t^B) \mathbin{\bullet\!\!\oplus} a_{n+1} && \text{by (X2)} \\ &= t' \mathbin{\bullet\!\!\oplus} a_{n+1} \end{aligned}$$

□

Lemma 4.5.2. *Let $t = (((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} t^B$ be a BF-term with n atoms. The height of xe(t) is n. In xe(t) each internal node has two child nodes, there are 2^i nodes at depth $i \in \{0, 1, \ldots, n-1\}$ that are all labeled with atom a_{i+1} and there are 2^n leaves at depth n labeled with* T *and* F.

Proof. Proof by induction on n, the number of atoms in t.

Base case: If $n = 1$, the evaluation tree of t is either $(\mathsf{T} \trianglelefteq a_1 \trianglerighteq \mathsf{F})$ or $(\mathsf{F} \trianglelefteq a_1 \trianglerighteq \mathsf{T})$. The result follows trivially.

Inductive step: Assume that the result holds for the *xe*-tree of any *BF*-term with n atoms (IH). Consider a *BF*-term $t = ((((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} a_{n+1}) \mathbin{\bullet\!\!\oplus} t^B$ with $n+1$ atoms. By Lemma 4.5.1 we can use equation (25), with $t' = (((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} t^B$.

Observe that *xe*(t') is a tree of height n by IH. So *xe*(t) is a tree of height $n + 1$ by equation (25). Let $i \in \{0, 1, \ldots, n-1\}$. By IH we know that the result holds for the nodes at depth i in *xe*(t'). Because level i of *xe*(t) is exactly level i of *xe*(t'), the result follows for the nodes at depth i in *xe*(t). We know by IH that there are 2^n leaves at depth n in *xe*(t'), with labels T or F. By equation (25) we know that all these leaves are replaced by $(\mathsf{F} \trianglelefteq a_{n+1} \trianglerighteq \mathsf{T})$ or $(\mathsf{T} \trianglelefteq a_{n+1} \trianglerighteq \mathsf{F})$ in *xe*(t). So there are 2^n nodes with two child nodes and label a_{n+1} at depth n in *xe*(t) and $2 \cdot 2^n = 2^{n+1}$ leaves with values T and F at depth $n + 1$ in *xe*(t). □

The following corollary will be useful in Section 5.7.

Corollary 4.5.3. *Let t be a BF-term with n atoms. Each internal node at depth $n-1$ in xe(t) has a child node with label* T *and a child node with label* F. *No other internal node in xe(t) has child nodes with labels* T *or* F.

Let t be a *BF*-term with n atoms. By the previous lemma, all the 2^n leaves of *xe*(t) are at depth n in *xe*(t) and can therefore be unambiguously ordered from left to right. The following lemma is about the labels of the leaves of *xe*(t).

Lemma 4.5.4. *Let $t = (((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} t^B$ be a BF-term with n atoms and let L_k be the k^{th} leaf from the left in xe(t), for $k \in \{1, 2, \ldots, 2^n\}$. For every i, j with $i \in \{0, 1, \ldots, n-1\}$ and $j \in \{1, 2, \ldots, 2^i\}$, the labels of L_j and L_{j+2^i} have opposite values.*

Proof. Proof by induction on n, the number of atoms in t.

Base case: If $n = 1$, the evaluation tree of t is either $(\mathsf{T} \trianglelefteq a_1 \trianglerighteq \mathsf{F})$ or $(\mathsf{F} \trianglelefteq a_1 \trianglerighteq \mathsf{T})$. We easily find that the result holds for $i = 0$ and $j = 1$, because the labels of L_1 and L_2 have opposite values.

Inductive step: Assume that the result holds for any *BF*-term with n atoms (IH). Consider a *BF*-term $t = ((((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} a_{n+1}) \mathbin{\bullet\!\!\oplus} t^B$ with $n + 1$ atoms. Again we use equation (25), with $t' = (((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} t^B$.

Let M_{k_1} be the k_1^{th} leaf from the left in *xe*(t'), for $k_1 \in \{1, 2, \ldots, 2^n\}$, and let L_{k_2} be the k_2^{th} leaf from the left in *xe*(t), for $k_2 \in \{1, 2, \ldots, 2^{n+1}\}$. The result is trivial if $i = 0$ and $j = 1$, because each node with label a_{n+1} branches to a T-leaf and to a F-leaf.

Now pick $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, 2^i\}$ arbitrarily. The node that branches to L_j replaced leaf $M_{\lceil \frac{j}{2} \rceil}$ and the node that branches to L_{j+2^i} replaced leaf $M_{\lceil \frac{2^i + j}{2} \rceil} = M_{(\lceil \frac{j}{2} \rceil + 2^{i-1})}$ by equation (25). Because $i - 1 \leq n - 1$ and $\lceil \frac{j}{2} \rceil \leq 2^{i-1}$, we know by IH that the labels of $M_{\lceil \frac{j}{2} \rceil}$ and $M_{(\lceil \frac{j}{2} \rceil + 2^{i-1})}$ in *xe*(t') have opposite values. So by equation (25) one of these leaves is replaced by $(\mathsf{T} \trianglelefteq a_{n+1} \trianglerighteq \mathsf{F})$ and the other one by $(\mathsf{F} \trianglelefteq a_{n+1} \trianglerighteq \mathsf{T})$. Because L_j and $L_{j + 2^i}$ are either both a left branch (if j is odd) or a right branch (if j is even), we conclude that the labels of L_j and $L_{j + 2^i}$ in *xe*(t) have opposite values. $\square$

By the previous lemma we find that the label of every leaf is known if the label of one of the leaves is known. In the next lemma we show that we can easily determine the label of the leftmost leaf in *xe*(t).

Lemma 4.5.5. *Let $t = (((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \dots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} t^B$ be a BF-term with n atoms and let lef(xe(t)) be the label of the leftmost leaf in xe(t). Then,*

$$\textit{lef}(\textit{xe}(t)) = \begin{cases} \mathsf{F} & \textit{if } n \textit{ is odd and } t^B = \mathsf{T}, \\ \mathsf{T} & \textit{if } n \textit{ is odd and } t^B = \mathsf{F}, \\ t^B & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Proof by induction on n, the number of atoms in t.

Base case: Let $n = 1$. If $t^B = \mathsf{T}$, we find that $\textit{lef}(\textit{xe}(a_1 \mathbin{\bullet\!\!\oplus} \mathsf{T})) = \textit{lef}((\mathsf{F} \trianglelefteq a_1 \trianglerighteq \mathsf{T})) = \mathsf{F}$. The result for $t^B = \mathsf{F}$ follows similarly.

Inductive step: Assume that the result holds for any *BF*-term with n atoms (IH). Let $t = ((((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \dots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} a_{n+1}) \mathbin{\bullet\!\!\oplus} t^B$ be a *BF*-term with $n+1$ atoms. Again we use equation (25), with $t' = (((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \dots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} t^B$.

If n is odd (so $n + 1$ is even) and $t^B = \mathsf{T}$, we find that $\textit{lef}(\textit{xe}(t')) = \mathsf{F}$ by IH. By equation (25) we know that each F-leaf of *xe*(t') is replaced by $(\mathsf{T} \trianglelefteq a_{n+1} \trianglerighteq \mathsf{F})$ in *xe*(t), so $\textit{lef}(\textit{xe}(t)) = \mathsf{T} = t^B$. The cases where $t^B = \mathsf{F}$ and n is even follow similarly. $\square$

4.6 An equational axiomatization of FLXL

Before we show that EqFLXL axiomatizes FLXL for closed terms, we define a function that maps any tree in $\textit{xe}[T^A_{\mathrm{LXL}}]$ to a *BF*-term.

Definition 4.6.1. *We define $g : \mathcal{T}_A \to BF$ by*

$$g(X) = \begin{cases} \mathsf{T} & \textit{if } X = \mathsf{T}, \\ \mathsf{F} & \textit{if } X = \mathsf{F}, \\ (((b_1 \mathbin{\bullet\!\!\oplus} b_2) \mathbin{\bullet\!\!\oplus} \dots) \mathbin{\bullet\!\!\oplus} b_n) \mathbin{\bullet\!\!\oplus} g(\textit{xe}(\textit{lef}(X) \mathbin{\bullet\!\!\oplus} \mathsf{T})) & \textit{if } n \textit{ is odd,} \\ (((b_1 \mathbin{\bullet\!\!\oplus} b_2) \mathbin{\bullet\!\!\oplus} \dots) \mathbin{\bullet\!\!\oplus} b_n) \mathbin{\bullet\!\!\oplus} g(\textit{xe}(\textit{lef}(X))) & \textit{if } n \textit{ is even,} \end{cases}$$

where $n \geq 1$ is the height of X and b_i is the label of the 2^{i-1} nodes at depth $i - 1$ in X.

We now show that g inverts evaluation trees of terms in *BF* correctly.

Theorem 4.6.2. *For all t in BF,*

$$g(xe(t)) \equiv t.$$

Proof. If $t \in \{\mathsf{T}, \mathsf{F}\}$, then $xe(t) = t$ and $g(xe(t)) \equiv t$ follows trivially. Assume that t is of the form $(((a_1 \bullet\!\!\oplus a_2) \bullet\!\!\oplus \ldots) \bullet\!\!\oplus a_n) \bullet\!\!\oplus t^B$ for some odd $n \geq 1$. The case where n is even follows similarly.

First notice that the atom a_i in t is the label of all the 2^{i-1} nodes at depth $i-1$ in $xe(t)$ by Lemma 4.5.2, for $i \in \{1, 2, \ldots, n\}$. So by definition of g we find that the atom b_i in $g(xe(t))$ is equal to the atom a_i in t.

We now argue that $xe(\mathit{lef}(xe(t)) \bullet\!\!\oplus \mathsf{T}) = xe(t^B)$. If $t^B = \mathsf{T}$, we find by Lemma 4.5.5 that $\mathit{lef}(xe(t)) = \mathsf{F}$. Then $xe(\mathsf{F} \bullet\!\!\oplus \mathsf{T}) = xe(\mathsf{T})$ follows because $\mathsf{F} \bullet\!\!\oplus \mathsf{T} \overset{\text{(X3)}}{=_{xe}} \mathsf{T} \bullet\!\!\oplus \mathsf{F} \overset{\text{(X4)}}{=_{xe}} \mathsf{T}$ and by definition of *xe*-congruence. The case for $t^B = \mathsf{F}$ follows similarly by axiom (X1).

Finally, we find that

$$\begin{aligned}
g(xe(t)) &\equiv (((b_1 \bullet\!\!\oplus b_2) \bullet\!\!\oplus \ldots) \bullet\!\!\oplus b_n) \bullet\!\!\oplus g(xe(\mathit{lef}(xe(t)) \bullet\!\!\oplus \mathsf{T})) \\
&\equiv (((a_1 \bullet\!\!\oplus a_2) \bullet\!\!\oplus \ldots) \bullet\!\!\oplus a_n) \bullet\!\!\oplus g(xe(t^B)) && \text{by the above} \\
&\equiv (((a_1 \bullet\!\!\oplus a_2) \bullet\!\!\oplus \ldots) \bullet\!\!\oplus a_n) \bullet\!\!\oplus t^B
\end{aligned}$$

□

Using the results from Theorem 4.4.5 and Theorem 4.6.2, we show that EqFLXL constitutes an equational axiomatization of FLXL for closed terms.

Theorem 4.6.3. *For all $t, u \in T^A_{\text{LXL}}$,*

$$\text{EqFLXL} \vdash t = u \iff \text{FLXL} \vdash t = u.$$

Proof.
'$\Longrightarrow$':
Follows by Theorem 4.2.8.

'$\Longleftarrow$':
Assume that FLXL $\vdash t = u$. Then $t =_{xe} u$ by equation (22). By Theorem 4.4.5 we find that EqFLXL $\vdash f(t) = t$ and EqFLXL $\vdash u = f(u)$. By '$\Longrightarrow$' and equation (22) we have $f(t) =_{xe} t$ and $f(u) =_{xe} u$, so also $f(t) =_{xe} f(u)$ by transitivity. So $xe(f(t)) = xe(f(u))$ by definition of *xe*-congruence. By Theorem 4.6.2 it follows that $f(t) \equiv g(xe(f(t)) = g(xe(f(u)) \equiv f(u)$. So $f(t)$ and $f(u)$ are syntactically equivalent and thus EqFLXL $\vdash f(t) = f(u)$. By transitivity it follows that EqFLXL $\vdash t = u$. □

By equation (22) this yields the following completeness result.

Corollary 4.6.4. *For all $t, u \in T^A_{\text{LXL}}$,*

$$\text{EqFLXL} \vdash t = u \iff t =_{xe} u.$$

Chapter 5

Expressiveness of ℓXOR and SCL-connectives modulo FVC

In this chapter we investigate the expressiveness modulo free valuation congruence (FVC) of terms over the signatures $\Sigma_i(A)$ we considered so far (for $i \in \{\text{CP}, \text{SCL}, \text{LNL}, \text{LXL}\}$) and the new combined signature $\Sigma_{\text{LXSCL}}(A) = \{\mathsf{T}, \mathsf{F}, a, \neg, {}^{\circ}\!\!\wedge, {}^{\circ}\!\!\vee, {\bullet}\!\oplus \mid a \in A\}$. We will use the shorter phrase "expressible over $\Sigma_i(A)$" to abbreviate "expressible over $\Sigma_i(A)$ modulo FVC".
In Section 5.1 we provide a set of equational axioms for terms over $\Sigma_{\text{LXSCL}}(A)$. In Section 5.2 we discuss evaluation-unanimous terms. In Section 5.3 and Section 5.4 we discuss expressibility of subtrees in $se[T^A_{\text{SCL}}]$ and other matters. In Section 5.5 we characterize which terms over $\Sigma_{\text{LXL}}(A)$ are expressible over $\Sigma_{\text{SCL}}(A)$, in Section 5.6 which terms over $\Sigma_{\text{LXSCL}}(A)$ are expressible over $\Sigma_{\text{SCL}}(A)$ and in Section 5.7 which terms over $\Sigma_{\text{LXSCL}}(A)$ are expressible over $\Sigma_{\text{LXL}}(A)$. Finally, in Section 5.8 we show that the term $b \triangleleft a \triangleright a \in T^A_{\text{CP}}$ is not expressible over $\Sigma_{\text{LXSCL}}(A)$.

The sets of evaluation trees corresponding to closed terms over the different signatures are related as depicted in the Venn diagram in Figure 5.1.

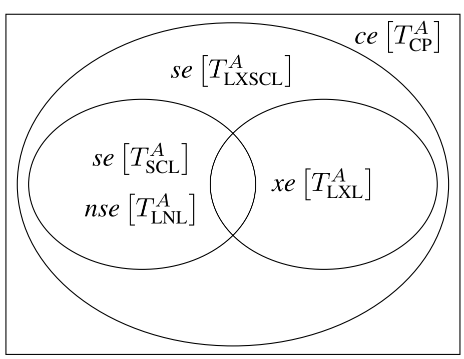


Figure 5.1: Venn diagram of sets of evaluation trees

5.1 FLXSCL and equational axioms

In the following we will not distinguish between the different evaluation functions *ce*, *se*, *nse* and *xe*. Instead, we refer to the evaluation tree of a closed term t over any signature by $se(t)$. Let T^A_{LXSCL} be the set of closed terms over $\Sigma_{\text{LXSCL}}(A)$. Terms in T^A_{LXSCL} are interpreted as *se*-trees by combining the original definition of *se* with the definition of *xe*. The abbreviation LXSCL stands for left-sequential xor and short-circuit logic, and FLXSCL stands for free LXSCL.

We redefine *se*-congruence on T^A_{LXSCL} and use this congruence relation to define expressibility modulo FVC.

Definition 5.1.1. *The binary relation* ***se-congruence****, notation* $=_{se}$*, is defined on* T^A_{LXSCL} *as follows:*

$$t =_{se} u \iff se(t) = se(u)$$

Definition 5.1.2. *A closed term* t *is* ***expressible*** *modulo free valuation congruence over a signature* $\Sigma_i(A)$ *if there is a closed term* u *over* $\Sigma_i(A)$ *such that* $t =_{se} u$*. We say that* t ***is expressible by*** u*.*

Sometimes we use the equivalent fact that $se(t) = se(u)$ to argue that t is expressible by u.

To investigate expressiveness of terms over $\Sigma_{\text{LXSCL}}(A)$, we need to be able to show that a closed term over $\Sigma_{\text{LXSCL}}(A)$ is *se*-congruent to a closed term over $\Sigma_{\text{SCL}}(A)$ or $\Sigma_{\text{LXL}}(A)$, or not. To this end, we first provide a basic set of axioms, named EqFLXSCL, for equations over $\Sigma_{\text{LXSCL}}(A)$ in Table 5.1. Then we list a set of equations over $\Sigma_{\text{LXSCL}}(A)$ in Table 5.2 and give their EqFLXSCL-derivations. Finally, we show that EqFLXSCL is sound with respect to *se*-congruence for closed-term equations over $\Sigma_{\text{LXSCL}}(A)$.

$$x \mathbin{{}^{\circ}\!\vee} y = \neg(\neg x \mathbin{{}_{\circ}\!\wedge} \neg y) \tag{A1}$$
$$\mathsf{T} \mathbin{{}_{\circ}\!\wedge} x = x \tag{A2}$$
$$x \mathbin{{}^{\circ}\!\vee} \mathsf{F} = x \tag{A3}$$
$$\mathsf{F} \mathbin{{}_{\circ}\!\wedge} x = \mathsf{F} \tag{A4}$$
$$(x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}_{\circ}\!\wedge} z = x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}_{\circ}\!\wedge} z) \tag{A5}$$
$$(x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\vee} (z \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) = (x \mathbin{{}^{\circ}\!\vee} (z \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}^{\circ}\!\vee} (z \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) \tag{A8}$$

$$(x \mathbin{\bullet\!\!\oplus} y) \mathbin{\bullet\!\!\oplus} z = x \mathbin{\bullet\!\!\oplus} (y \mathbin{\bullet\!\!\oplus} z) \tag{X2}$$
$$x \mathbin{\bullet\!\!\oplus} \mathsf{T} = \mathsf{T} \mathbin{\bullet\!\!\oplus} x \tag{X3}$$

$$\neg x = x \mathbin{\bullet\!\!\oplus} \mathsf{T} \tag{AX1}$$
$$((x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} y) \mathbin{\bullet\!\!\oplus} z = (x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} (y \mathbin{\bullet\!\!\oplus} z) \tag{AX2}$$
$$x \mathbin{\bullet\!\!\oplus} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) = (x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}^{\circ}\!\vee} \mathsf{T})) \mathbin{{}^{\circ}\!\vee} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \tag{AX3}$$

Table 5.1: EqFLXSCL, a set of equational axioms over $\Sigma_{\text{LXSCL}}(A)$

Note that the name of EqFLXSCL suggests that it is also a set of axioms for FLXSCL, the equational theory that is axiomatized by the union of the axioms of FSCL and FLXL, thus by CP $\cup \{\neg x = \mathsf{F} \triangleleft x \triangleright \mathsf{T},\ x \mathbin{{}_{\circ}\!\wedge} y = y \triangleleft x \triangleright \mathsf{F},\ x \mathbin{\bullet\!\!\oplus} y = (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleleft x \triangleright y\}$.

We now elaborate on the axioms of EqFLXSCL. This set of axioms is a combination of EqFSCL-axioms, EqFLXL-axioms and new axioms for equations with connectives in each of $\Sigma_{\mathrm{SCL}}(A)$ and $\Sigma_{\mathrm{LXL}}(A)$. Axiom (AX1) characterizes negation over $\Sigma_{\mathrm{LXL}}(A)$. In axiom (AX2) the left-hand side requires evaluation of $(s \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} t$ and the evaluation result is determined by u, while in the right-hand side the evaluation result is determined by $t \mathbin{\bullet\!\oplus} u$ after evaluating $s \mathbin{{}^{\circ}\!\vee} \mathsf{T}$. Axiom (AX3) states that if the second argument of $\mathbin{\bullet\!\oplus}$ always evaluates to F, then the expression $s \mathbin{\bullet\!\oplus} (u \mathbin{{}_{\circ}\!\wedge} \mathsf{F})$ is expressible in terms of s, u and constants and connectives in $\Sigma_{\mathrm{SCL}}(A)$.

Before we show that the equations in Table 5.2 are derivable from EqFLXSCL, we argue that EqFLXSCL satisfies the duality principle for equations over $\Sigma_{\mathrm{SCL}}(A)$. Following the line of reasoning from Section 2.4, it suffices to show that (A1), (Aux1) and (Aux2) are derivable from EqFLXSCL.

In Section 2.4 we mentioned the result from [PS18, Prop.2.1.8] that (Aux1), (Aux3), (Aux4) and (Aux5) are derivable from $\{$(A1), . . . , (A5)$\}$ and that (Aux2) is derivable from $\{$(A1), . . . , (A5), (A7)$\}$. We now sharpen this result by showing that (Aux2) is derivable from $\{$(A1), . . . , (A5)$\}$.

Proposition 5.1.3. $\{$(A1), . . . , (A5)$\} \vdash$ (Aux2).

Proof. The proof is distilled from output of the theorem prover *Prover9* [McC08].

$$\begin{aligned}
\neg\neg x &= \neg(\neg\neg\neg x \mathbin{{}_{\circ}\!\wedge} \neg\mathsf{F}) && \text{by (Aux4)}\\
&= \neg(\neg\neg\neg x \mathbin{{}_{\circ}\!\wedge} \mathsf{T}) && \text{by (Aux3)}\\
&= \neg(\neg\neg\neg x \mathbin{{}_{\circ}\!\wedge} (\mathsf{T} \mathbin{{}_{\circ}\!\wedge} \mathsf{T})) && \text{by (A2)}\\
&= \neg((\neg\neg\neg x \mathbin{{}_{\circ}\!\wedge} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} \mathsf{T}) && \text{by (A5)}\\
&= \neg((\neg\neg\neg x \mathbin{{}_{\circ}\!\wedge} \neg\mathsf{F}) \mathbin{{}_{\circ}\!\wedge} \neg\mathsf{F}) && \text{by (Aux3)}\\
&= \neg(\neg x \mathbin{{}_{\circ}\!\wedge} \neg\mathsf{F}) && \text{by (Aux5)}\\
&= x && \text{by (Aux4)}
\end{aligned}$$

□

By the above and because $\{$(A1), . . . , (A5)$\} \subseteq$ EqFLXSCL, we find that (A1), (Aux1) and (Aux2) are derivable from EqFLXSCL. So EqFLXSCL satisfies the duality principle for equations over $\Sigma_{\mathrm{SCL}}(A)$.

Proposition 5.1.4. *The equations in Table 5.2 are derivable from* EqFLXSCL.

Proof. All proofs are distilled from output of the theorem prover *Prover9* [McC08]. Here we show that axiom (A6) is derivable from EqFLXSCL, for the remaining proofs we refer to Appendix C.

$$\begin{aligned}
\neg x \mathbin{{}_{\circ}\!\wedge} \mathsf{F} &= \neg\neg(\neg x \mathbin{{}_{\circ}\!\wedge} \neg\mathsf{T}) && \text{by (Aux1) and (Aux2)}\\
&= \neg(x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) && \text{by (A1)}\\
&= \neg(x \mathbin{{}^{\circ}\!\vee} (\mathsf{T} \mathbin{{}^{\circ}\!\vee} (x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}))) && \text{by (A4)}^{dl}\\
&= \neg((x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}^{\circ}\!\vee} (x \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) && \text{by (A5)}^{dl}\\
&= \neg((\mathsf{T} \mathbin{{}_{\circ}\!\wedge} (x \mathbin{{}^{\circ}\!\vee} \mathsf{T})) \mathbin{{}^{\circ}\!\vee} (x \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) && \text{by (A2)}\\
&= \neg(\mathsf{T} \mathbin{\bullet\!\oplus} (x \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) && \text{by (AX3)}\\
&= \neg((x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \mathbin{\bullet\!\oplus} \mathsf{T}) && \text{by (X3)}\\
&= \neg\neg(x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) && \text{by (AX1)}\\
&= x \mathbin{{}_{\circ}\!\wedge} \mathsf{F} && \text{by (Aux2)}
\end{aligned}$$

□

$$\neg x \mathbin{{}_{\circ}\!\wedge} \mathsf{F} = x \mathbin{{}_{\circ}\!\wedge} \mathsf{F} \tag{A6}$$
$$(x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \mathbin{{}^{\circ}\!\vee} y = (x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} y \tag{A7}$$

$$\mathsf{T} \mathbin{\bullet\!\oplus} \mathsf{T} = \mathsf{F} \tag{X1}$$
$$x \mathbin{\bullet\!\oplus} \mathsf{F} = x \tag{X4}$$

$$(x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{\bullet\!\oplus} y = (x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} \neg y \tag{Eq1}$$
$$(x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \mathbin{\bullet\!\oplus} y = (x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \mathbin{{}^{\circ}\!\vee} y \tag{Eq2}$$
$$x \mathbin{\bullet\!\oplus} (y \mathbin{{}^{\circ}\!\vee} \mathsf{T}) = (\neg x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}^{\circ}\!\vee} \mathsf{T})) \mathbin{{}^{\circ}\!\vee} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \tag{Eq3}$$
$$x \mathbin{\bullet\!\oplus} (y \mathbin{{}^{\circ}\!\vee} \mathsf{T}) = \neg(x \mathbin{\bullet\!\oplus} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) \tag{Eq4}$$

Table 5.2: Equations that are derivable from EqFLXSCL

We turn to the last result of this section. We will show that EqFLXSCL is sound with respect to *se*-congruence for closed terms. First we provide a model $\mathbb{M}'$ for EqFLXSCL that expands the model $\mathbb{M}$ for EqFLXL from Definition 4.2.6 and show that EqFLXSCL is sound for $\mathbb{M}'$.

Definition 5.1.5. *Let $\mathbb{M}'$ be the $\Sigma_{\text{LXSCL}}(A)$-algebra with domain $D' = \{se(t) \mid t \in T^A_{\text{LXSCL}}\}$, expanding the model $\mathbb{M}$ from Definition 4.2.6, in which the connectives $\neg$, $\mathbin{{}_{\circ}\!\wedge}$ and $\mathbin{{}^{\circ}\!\vee}$ are interpreted by*

$$\begin{aligned}
[\![\neg t]\!]^{\mathbb{M}'} &= [\![t]\!]^{\mathbb{M}'}\big[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}\big],\\
[\![t \mathbin{{}_{\circ}\!\wedge} u]\!]^{\mathbb{M}'} &= [\![t]\!]^{\mathbb{M}'}\big[\mathsf{T} \mapsto [\![u]\!]^{\mathbb{M}'}\big],\\
[\![t \mathbin{{}^{\circ}\!\vee} u]\!]^{\mathbb{M}'} &= [\![t]\!]^{\mathbb{M}'}\big[\mathsf{F} \mapsto [\![u]\!]^{\mathbb{M}'}\big].
\end{aligned}$$

Lemma 5.1.6. *For all terms t, u over $\Sigma_{\text{LXSCL}}(A)$,*

$$\text{EqFLXSCL} \vdash t = u \implies \mathbb{M}' \models t = u.$$

Proof. Proof by induction on n, the length of the derivation.

Base case: Assume that we have a derivation EqFLXSCL $\vdash t = u$ of length one. If $t = u$ is derived using reflexivity, the result follows trivially. Otherwise $t = u$ is an axiom of EqFLXSCL. We now argue that $\mathbb{M}'$ satisfies all the axioms of EqFLXSCL. For the axioms that are also in EqFSCL we refer to [PS18, Thm.2.1.4]. For the axioms that are also in EqFLXL we refer to the proof of Lemma 4.2.7. Below we show that $\mathbb{M}'$ satisfies (AX1). The proofs for (AX2) and (AX3) can be found in Appendix C. Fix an interpretation i of variables. Then,

$$\begin{aligned}
[\![\neg x]\!]^{\mathbb{M}',i} &= [\![x]\!]^{\mathbb{M}',i}\big[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}\big] && \text{by 5.1.5}\\
&= [\![x]\!]^{\mathbb{M}',i}\big[\mathsf{T} \mapsto [\![\mathsf{T}]\!]^{\mathbb{M}'}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto [\![\mathsf{T}]\!]^{\mathbb{M}'}\big] && \text{by (3)}\\
&= [\![x \mathbin{\bullet\!\oplus} \mathsf{T}]\!]^{\mathbb{M}',i}. && \text{by 4.2.6}
\end{aligned}$$

Because i was chosen arbitrarily, we have $\mathbb{M}' \models$ (AX1).

Inductive step: Consider a derivation of length $n > 1$. If EqFLXSCL $\vdash t = u$ is derived using symmetry, transitivity, congruence with $\mathbin{{}_{\bullet}\!\oplus}$ or substitution, we refer to the proof of Lemma 4.2.7. For derivations using congruence with $\neg$, $\mathbin{{}_{\circ}\!\wedge}$ and $\mathbin{{}^{\circ}\!\vee}$, we refer to [PS18, Thm.2.1.4]. $\square$

Using the previous lemma, soundness of EqFLXSCL with respect to *se*-congruence follows easily.

Theorem 5.1.7. *For any $t, u \in T^A_{\text{LXSCL}}$,*

$$\text{EqFLXSCL} \vdash t = u \implies t =_{se} u.$$

Proof. By Lemma 5.1.6 it suffices to show that $\mathbb{M}' \vDash t = u \iff t =_{se} u$. Note that we have $[\![t]\!]^{\mathbb{M}'} = se(t)$ for any $t \in T^A_{\text{LXSCL}}$ (this follows easily by structural induction). Therefore, $\mathbb{M}' \vDash t = u$ holds if and only if $[\![t]\!]^{\mathbb{M}'} = [\![u]\!]^{\mathbb{M}'}$, if and only if $se(t) = se(u)$, if and only if $t =_{se} u$ by Definition 5.1.1. $\square$

5.2 Evaluation-unanimous terms

Many expressibility results that will be proved in this chapter deal with closed terms whose evaluation trees have only T-leaves or only F-leaves. We now introduce some terminology for this property.

Definition 5.2.1. *Consider a closed term t over any signature.*

1. *If $se(t)$ has only T-leaves, then t is called T-**unanimous**.*
2. *If $se(t)$ has only F-leaves, then t is called F-**unanimous**.*
3. *If t is T-unanimous or F-unanimous, then t is called **evaluation-unanimous**.*

Lemma 5.2.2. *If a closed term t is T-unanimous, then $t =_{se} t \mathbin{{}^{\circ}\!\vee} \mathsf{T}$. If t is F-unanimous, then $t =_{se} t \mathbin{{}_{\circ}\!\wedge} \mathsf{F}$.*

Proof. If t is T-unanimous, then $se(t)$ has only T-leaves. So $se(t \mathbin{{}^{\circ}\!\vee} \mathsf{T}) = se(t)[\mathsf{F} \mapsto \mathsf{T}] = se(t)$ because there are no F-leaves in $se(t)$. The case where t is F-unanimous follows similarly. $\square$

Lemma 5.2.3. *A closed term $t = s \mathbin{{}_{\bullet}\!\oplus} u$ in T^A_{LXSCL} is evaluation-unanimous if and only if s and u are evaluation-unanimous.*

Proof.
'$\Longrightarrow$'
Assume that s and u are evaluation-unanimous. Recall from Definition 4.1.3 that

$$se(s \mathbin{{}_{\bullet}\!\oplus} u) = se(s)[\mathsf{T} \mapsto se(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto se(u)]. \tag{26}$$

If the leaves of $se(s)$ are all labeled with t^B_1 and the leaves of $se(u)$ with t^B_2, for $t^B_1, t^B_2 \in \{\mathsf{T}, \mathsf{F}\}$, then the leaves of $se(s \mathbin{{}_{\bullet}\!\oplus} u)$ are all labeled with F if $t^B_1 = t^B_2$ and with T otherwise. So $s \mathbin{{}_{\bullet}\!\oplus} u$ is also evaluation-unanimous.

'$\Longleftarrow$'
If s is not evaluation-unanimous, $se(s)$ has T- and F-leaves. By equation (26) both $se(u)$ and $se(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ are subtrees of $se(s \,\bullet\!\!\oplus\, u)$. Irrespective of whether u is evaluation-unanimous or not, $se(s \,\bullet\!\!\oplus\, u)$ has T- and F-leaves, so $s \,\bullet\!\!\oplus\, u$ is not evaluation-unanimous.

If u is not evaluation-unanimous, both $se(u)$ and $se(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ have T- and F-leaves. Because at least one of $se(u)$ or $se(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ is a subtree of $se(s \,\bullet\!\!\oplus\, u)$, also $se(s \,\bullet\!\!\oplus\, u)$ has T- and F-leaves. So $se(s \,\bullet\!\!\oplus\, u)$ is not evaluation-unanimous. □

We now show that every evaluation-unanimous term is expressible over $\Sigma_{\text{SCL}}(A)$. We show this using the SCL Normal Form that was defined in Definition 2.5.1.

Lemma 5.2.4.

1. *For an evaluation tree $X \in \mathcal{T}_A$ that has only T-leaves, there exists a T-term P^{T} in SNF such that $se(P^{\mathsf{T}}) = X$.*

2. *For an evaluation tree $X \in \mathcal{T}_A$ that has only F-leaves, there exists a F-term P^{F} in SNF such that $se(P^{\mathsf{F}}) = X$.*

Proof.

Statement 1:

Proof by induction on n, the height of X.

Base case: If $n = 0$, then $X = \mathsf{T}$. The result follows because T is a T-term in *SNF* by Definition 2.5.1.

Inductive step: Assume that the result holds for any evaluation tree of height $\leq n$ that has only T-leaves (IH). Further assume that $X = (Y \trianglelefteq a \trianglerighteq Z)$ is an evaluation tree of height $n+1$ that has only T-leaves, for some $a \in A$. Clearly Y and Z are of height $\leq n$ and have only T-leaves. So by IH we have $Y = se(P^{\mathsf{T}})$ and $Z = se(Q^{\mathsf{T}})$ for T-terms $P^{\mathsf{T}}, Q^{\mathsf{T}}$ in *SNF*. The result follows because $(a \curlywedge P^{\mathsf{T}}) \curlyvee Q^{\mathsf{T}}$ is a T-term in *SNF* by Definition 2.5.1 and because $se((a \curlywedge P^{\mathsf{T}}) \curlyvee Q^{\mathsf{T}}) = se(a)[\mathsf{T} \mapsto se(P^{\mathsf{T}})][\mathsf{F} \mapsto se(Q^{\mathsf{T}})] = se(a)[\mathsf{T} \mapsto Y][\mathsf{F} \mapsto Z] = X$, where the last equality holds because Y has no F-leaves.

The proof of Statement 2 follows similarly. □

5.3 Expressibility of SCL-subtrees

We will show that each subtree of an evaluation tree in $se[T^A_{\text{SCL}}]$ is also in $se[T^A_{\text{SCL}}]$. To this end, we first define a function f_r that takes a term $t \in T^A_{\text{SCL}}$ as input and returns a term $u \in T^A_{\text{SCL}}$ such that $se(u)$ is equal to the subtree of $se(t)$ whose root is labeled with r, for some $r \in A$. If r occurs multiple times in t, we can number each occurrence of r to establish uniqueness. So we assume WLOG that the atom r is unique in t.

Definition 5.3.1. *Let $r \in A$. We define $f_r : T_{\mathrm{SCL}}^A \to T_{\mathrm{SCL}}^A$ by*

$$f_r(t) = t \qquad \text{if } r \text{ is the root of } se(t),$$

otherwise

$$f_r(\neg t) = \neg f_r(t).$$
$$f_r(t \mathbin{⩓} u) = \begin{cases} f_r(t) \mathbin{⩓} u & \text{if } r \text{ in } t, \\ f_r(u) & \text{if } r \text{ in } u. \end{cases}$$
$$f_r(t \mathbin{⩔} u) = \begin{cases} f_r(t) \mathbin{⩔} u & \text{if } r \text{ in } t, \\ f_r(u) & \text{if } r \text{ in } u. \end{cases}$$

We now show that the function f_r works correctly.

Lemma 5.3.2. *Let $X = se(t)$ for some $t \in T_{\mathrm{SCL}}^A$. For every subtree X' of X with root $r \in A$,*

$$se(f_r(t)) = X'.$$

Proof. Proof by induction on the complexity of evaluation trees in $se[T_{\mathrm{SCL}}^A]$.

Base case: If $se(t) = (\mathsf{T} \trianglelefteq r \trianglerighteq \mathsf{F})$ we find that $se(t)$ is the only subtree of $se(t)$ of which the root is an atom. Because r is the root of $se(t)$, we have $se(f_r(t)) = se(t)$.

Inductive step: Let $Y = se(s)$ and $Z = se(u)$ be evaluation trees of $s, u \in T_{\mathrm{SCL}}^A$ and assume that the result holds for every subtree Y' of Y and every subtree Z' of Z (IH).

Let $X = se(\neg s) = Y[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ and consider an arbitrary subtree X' of X with root r. Note that $X' = Y'[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$, where Y' is the subtree of Y with root r. We find that

$$\begin{aligned} X' &= Y'[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}] \\ &= se(f_r(s))[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}] && \text{by IH} \\ &= se(\neg f_r(s)) && \text{by Definition 2.3.3} \\ &= se(f_r(\neg s)) && \text{by Definition 5.3.1} \end{aligned}$$

Let $X = se(s \mathbin{⩓} u) = Y[\mathsf{T} \mapsto Z]$ and consider an arbitrary subtree X' of X with root r. There are two options. If X' is a subtree of Z, we find that

$$\begin{aligned} X' &= se(f_r(u)) && \text{by IH} \\ &= se(f_r(s \mathbin{⩓} u)) && \text{by Definition 5.3.1} \end{aligned}$$

Otherwise the root of X is a node in Y. Then $X' = Y'[\mathsf{T} \mapsto Z]$, where Y' is the subtree of Y with root r, and

$$\begin{aligned} X' &= Y'[\mathsf{T} \mapsto Z] \\ &= se(f_r(s))[\mathsf{T} \mapsto se(u)] && \text{by IH} \\ &= se(f_r(s) \mathbin{⩓} u) && \text{by Definition 2.3.3} \\ &= se(f_r(s \mathbin{⩓} u)) && \text{by Definition 5.3.1} \end{aligned}$$

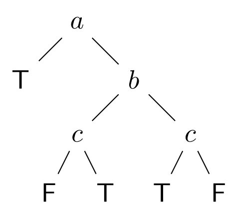
The case where $X = se(s \mathbin{⩔} u) = Y[\mathsf{F} \mapsto Z]$ follows similarly. $\square$

Theorem 5.3.3. *Let $t, u \in T^A_{\text{LXSCL}}$ such that $se(u)$ is a subtree of $se(t)$. If t is expressible over $\Sigma_{\text{SCL}}(A)$, then u is also expressible over $\Sigma_{\text{SCL}}(A)$.*

Proof. Because t is expressible over $\Sigma_{\text{SCL}}(A)$ there is a term $t' \in T^A_{\text{SCL}}$ such that $se(t) = se(t')$. Let r be the label of the root of $se(u)$. By Lemma 5.3.2 we have $se(u) = se(f_r(t'))$. Because $f_r(t') \in T^A_{\text{SCL}}$ we find that u is expressible over $\Sigma_{\text{SCL}}(A)$. $\square$

The contrapositive of this theorem is very important in the remainder of this chapter. We will illustrate it with a simple example.

Example 5.3.4. Consider the terms $a \mathbin{{}^{\circ}\!\vee} (b \mathbin{\bullet\!\!\oplus} c)$ and $b \mathbin{\bullet\!\!\oplus} c$ in T^A_{LXSCL} and assume that $b \mathbin{\bullet\!\!\oplus} c$ is not expressible over $\Sigma_{\text{SCL}}(A)$. Because $se(b \mathbin{\bullet\!\!\oplus} c)$ is a subtree of $se(a \mathbin{{}^{\circ}\!\vee} (b \mathbin{\bullet\!\!\oplus} c))$ (depicted by Tree 2), we find that $a \mathbin{{}^{\circ}\!\vee} (b \mathbin{\bullet\!\!\oplus} c)$ is not expressible over $\Sigma_{\text{SCL}}(A)$ by the contrapositive of Theorem 5.3.3.



(Tree 2)

5.4 Special atom

If a closed term t is not evaluation-unanimous, the evaluation tree $se(t)$ has T- and F-leaves. Clearly there is an atom in t such that there are T- and F-leaves below a node in $se(t)$ that is labeled with this atom (take for instance the label of the root of $se(t)$). There is one such atom in t that will be very useful in the remainder of this chapter, which we will call the *special atom* of t.

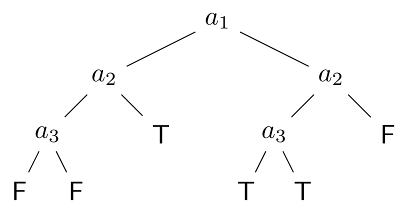
Definition 5.4.1. *Let $t \in T^A_{\text{LXSCL}}$ such that t is not evaluation-unanimous. The* ***special atom*** *of t is the rightmost atom in t that is the label of the root of a subtree of $se(t)$ with* T*- and* F*-leaves.*

Note that the special atom is unique by definition. Let a be the special atom of a term $t \in T^A_{\text{LXSCL}}$ that is not evaluation-unanimous. If a occurs multiple times in t, we can number each occurrence of a to establish uniqueness. We will reason about the special atom of a term t and the special node of an evaluation tree $se(t)$. We refer to the special atom by the unique occurrence of the atom in t and we refer to the special node by any node in $se(t)$ that is labeled with the special atom.

In what follows, we will give an example to illustrate these notions and we show two properties of evaluation trees that are rooted at a special node.

Example 5.4.2. Consider the term $t = a \mathbin{{}^{\circ}\!\vee} (b \mathbin{\bullet\!\!\oplus} c)$ in T^A_{LXSCL}. The special atom of t is c and the special node occurs twice in $se(t)$ (depicted as Tree 2 on page 50). Let $u = a \mathbin{\bullet\!\!\oplus} (a \mathbin{{}^{\circ}\!\wedge} (a \mathbin{{}^{\circ}\!\vee} \mathsf{T}))$ in T^A_{LXSCL}. The special atom of u is (the second occurrence of) a. Because a occurs multiple times in u, we number each occurrence of a to establish uniqueness. We find that a_2 is the special atom of

$u = a_1 \mathbin{\bullet\!\!\oplus} (a_2 \mathbin{\circ\!\!\!\wedge} (a_3 \mathbin{\circ\!\!\!\vee} \mathsf{T}))$ and that the special node occurs twice in *se*(u) (depicted as Tree 3).



(Tree 3)

Lemma 5.4.3. *Let $t \in T^A_{\text{LXSCL}}$ and let X be a subtree of se(t) that is rooted at a special node of se(t). Then X has a branch with only T-leaves and a branch with only F-leaves.*

Proof. Proof by contraposition. Assume that the root of X is labeled with a. If X has only T-leaves or only F-leaves, a is not the special atom of t by Definition 5.4.1. If X has a branch with T- and F-leaves, then X has a proper subtree with T- and F-leaves of which the root is labeled with $a' \neq a$. Because the node with label a' is a descendant of the node with label a in *se*(t), we know that a' occurs more to the right in t than a. Since there are T- and F-leaves below the node with label a' in *se*(t), a is not the special atom of t by Definition 5.4.1. In both cases we find that the node with label a is not a special node of *se*(t). □

Lemma 5.4.4. *Let $t \in T^A_{\text{LXSCL}}$ and let $X = (Y \trianglelefteq a \trianglerighteq Z)$ be a subtree of se(t) that is rooted at a special node of se(t). Then there is a literal term P^ℓ in SNF such that $X = $ se(P^ℓ).*

Proof. By Lemma 5.4.3 we find that X has a branch with only T-leaves and a branch with only F-leaves. Assume that Y is the branch of X that has only T-leaves. The proof for Z follows similarly. By Lemma 5.2.4.1 we know that there is a T-term P^{T} in *SNF* such that *se*$(P^{\mathsf{T}}) = Y$ and by Lemma 5.2.4.2 we know that there is a F-term P^{F} in *SNF* such that *se*$(P^{\mathsf{F}}) = Z$. The result follows because $(a \mathbin{\circ\!\!\!\wedge} P^{\mathsf{T}}) \mathbin{\circ\!\!\!\vee} P^{\mathsf{F}}$ is a literal term in *SNF* by Definition 2.5.1 and because *se*$((a \mathbin{\circ\!\!\!\wedge} P^{\mathsf{T}}) \mathbin{\circ\!\!\!\vee} P^{\mathsf{F}}) = $ *se*$(a)[\mathsf{T} \mapsto$ *se*$(P^{\mathsf{T}})][\mathsf{F} \mapsto$ *se*$(P^{\mathsf{F}})] = $ *se*$(a)[\mathsf{T} \mapsto Y][\mathsf{F} \mapsto Z] = X$, where the last equality holds because Y has no F-leaves. □

In the following we prove three more results that will be used in the remainder of this chapter.

Lemma 5.4.5. *Let P^ℓ be a literal term in SNF, let $P^{\mathsf{T}} \mathbin{\circ\!\!\!\wedge} P^*$ be a T-$*$-term in SNF and let $X = $ se(P^ℓ) be a subtree of se$(P^{\mathsf{T}} \mathbin{\circ\!\!\!\wedge} P^*)$. Then the root of X is not a node in se(P^{T}).*

Proof. Suppose for contradiction that the root of X is a node in *se*(P^{T}) and let Y be the subtree of *se*(P^{T}) that is rooted at this node. Observe that Y has height ≥ 1 and that Y has only T-leaves. If we replace the T-leaves of Y by Δ, we find that $X = Y[\Delta \mapsto$ *se*$(P^*)]$, where $Y \neq \Delta$ and Y has only Δ-leaves. Because a literal term is also a $*$-term, the existence of this decomposition of X contradicts Lemma 2.6.2. □

Lemma 5.4.6. *Let $t \in T^A_{\text{SCL}}$ be a term that is not evaluation-unanimous and let X be a subtree of se(t) that is rooted at a special node. Then X is a subtree of se(t) and $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ is not.*

Proof. By Corollary 2.5.3 and because t is not evaluation-unanimous we find that *se*$(t) = $ *se*$(P^{\mathsf{T}} \mathbin{\circ\!\!\!\wedge} P^*)$ for a T-$*$-term in *SNF*. We argue that it suffices to show the result for *se*(P^*). By Lemma 5.4.4 we find that $X = $ *se*(P^ℓ) for a literal term P^ℓ in *SNF*. Then also $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ is the evaluation tree of a literal term. By Lemma 5.4.5 it follows that X and $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ are subtrees of $\mathit{se}(P^*)$ if they are subtrees of $\mathit{se}(t)$.

We prove the result by induction on the number of ℓ-terms in P^*.

Base case: If $P^* = P^\ell$, the special node of $\mathit{se}(P^*)$ is its root. So $X = \mathit{se}(P^*)$ and the result follows trivially.

Inductive step: Assume that the result holds for a P^d-form (IH) and consider the $*$-term $P^* \mathbin{{}_{\circ}\!\wedge} P^d$. The case for a P^c-form and the $*$-term $P^* \mathbin{{}^{\circ}\!\vee} P^c$ follows similarly. By IH, X is a subtree of $\mathit{se}(P^d)$ and $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ is not. Clearly X is a subtree of $\mathit{se}(P^* \mathbin{{}_{\circ}\!\wedge} P^d)$. We now argue that $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ is not a subtree of $\mathit{se}(P^* \mathbin{{}_{\circ}\!\wedge} P^d)$. Suppose it is. Then its root must be a node in $\mathit{se}(P^*)$. Either only F-leaves occur below the root of $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ in $\mathit{se}(P^* \mathbin{{}_{\circ}\!\wedge} P^*)$, or $\mathit{se}(P^d)$, hence also X, is a subtree of $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$. Both cases lead to a contradiction. □

Corollary 5.4.7. *Let $t = s \mathbin{\bullet\!\oplus} u$ in T^A_{LXSCL} such that s and u are not evaluation-unanimous and $u \in T^A_{\text{SCL}}$. Let X, Y be subtrees of $\mathit{se}(t)$ such that X is rooted at a special node of $\mathit{se}(u)$ and Y is rooted at a special node of $\mathit{se}(s)$. Then X is only a subtree of the left branch of Y or only of the right branch of Y.*

Proof. Let Y' be the subtree of $\mathit{se}(s)$ that has the same root as Y in $\mathit{se}(s \mathbin{\bullet\!\oplus} u)$. We know that Y' has a branch with only T-leaves and a branch with only F-leaves by Lemma 5.4.3. By equation (26) we find that Y is obtained from Y' by replacing the F-leaves of Y' with $\mathit{se}(u)$, and the T-leaves of $\mathit{se}(s)$ with $\mathit{se}(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$. By Lemma 5.4.6 we know that X is a subtree of $\mathit{se}(u)$ and that $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ is not. By Lemma 5.4.6 we also know that $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ is a subtree of $\mathit{se}(u)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ and that $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}][\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}] = X$ is not. So X is a subtree of the branch of Y in which F-leaves were replaced and not of the branch of Y in which T-leaves were replaced. □

5.5 FLXL and FSCL

In this section we will characterize which closed terms t over $\Sigma_{\text{LXL}}(A)$ are expressible over $\Sigma_{\text{SCL}}(A)$. In Lemma 5.5.6 we will first show exactly which closed terms of the form $s \mathbin{\bullet\!\oplus} u$ in T^A_{LXSCL} are expressible over $\Sigma_{\text{SCL}}(A)$. The proof of Lemma 5.5.6 has several cases that will be covered in Lemma 5.5.5. For the proof of Lemma 5.5.5.2 we need to know when the evaluation tree $X = \mathit{se}(P^{\mathsf{T}})$ of a T-term P^{T} in *SNF* is a subtree of the evaluation tree $Y = \mathit{se}(P^*)$ of a $*$-term P^* in *SNF*. In Corollary 5.5.3 we will show when this is the case. This corollary combines the results of the following two lemmas.

Lemma 5.5.1. *Let $X = \mathit{se}(P^{\mathsf{T}})$ be the evaluation tree of a T-term P^{T} in SNF and let $Y = \mathit{se}(P^\ell)$ be the evaluation tree of a literal term P^ℓ in SNF. Then X is a subtree of Y if $g^{\mathsf{T}}(X)$ is a subterm of $g^\ell(Y)$.*

Proof. Let $Y = (Z \trianglelefteq a \trianglerighteq Z')$ for some $a \in A$ and assume that Z has only T-leaves. The case where Z' has only T-leaves follows similarly. Note that X is a subtree of Y if and only if X is a subtree of Z and that $g^{\mathsf{T}}(X)$ is a subterm of $g^\ell(Y) = (a \mathbin{{}_{\circ}\!\wedge} g^{\mathsf{T}}(Z)) \mathbin{{}^{\circ}\!\vee} g^{\mathsf{F}}(Z')$ if and only if $g^{\mathsf{T}}(X)$ is a subterm of $g^{\mathsf{T}}(Z)$. So it suffices to show that X is a subtree of Z if $g^{\mathsf{T}}(X)$ is a subterm of $g^{\mathsf{T}}(Z)$.

Proof by induction on the complexity of $g^{\mathsf{T}}(Z)$.

Base case: If $g^{\mathsf{T}}(Z) = \mathsf{T}$, the result follows trivially.

Inductive step: Let $g^{\mathsf{T}}(Z) = (b \mathbin{\curlywedge} g^{\mathsf{T}}(Z_1)) \mathbin{\curlyvee} g^{\mathsf{T}}(Z_2)$ and assume that the result holds for $g^{\mathsf{T}}(Z_1)$ and $g^{\mathsf{T}}(Z_2)$ (IH). Consider a subterm $g^{\mathsf{T}}(X)$ of $g^{\mathsf{T}}(Z)$. If $g^{\mathsf{T}}(X) = g^{\mathsf{T}}(Z)$, the result is trivial. If $g^{\mathsf{T}}(X)$ is a subterm of $g^{\mathsf{T}}(Z_1)$ or $g^{\mathsf{T}}(Z_2)$, we find by IH that X is a subtree of Z_1 or of Z_2, hence of $Z = (Z_1 \trianglelefteq b \trianglerighteq Z_2)$. □

Lemma 5.5.2. *Let $X = se(P^\ell)$ be the evaluation tree of a literal term P^ℓ in SNF and let $Y = se(P^*)$ be the evaluation tree of a $*$-term P^* in SNF. The branch of X with T-leaves is a subtree of Y if*

1. *$g^\ell(X)$ is a subterm of $g^*(Y)$, and*
2. *there is no evaluation tree X' such that*
   (a) *$g^*(cd_1(X')[\Delta \mapsto \mathsf{T}]) \mathbin{\curlywedge} g^*(cd_2(X'))$ is a subterm of $g^*(Y)$, and*
   (b) *$g^\ell(X)$ is a subterm of $g^*(cd_1(X')[\Delta \mapsto \mathsf{T}])$.*

Proof. Proof by induction on the complexity of $g^*(Y)$.

Base case: Assume that $g^*(Y) = g^\ell(Y)$. If $g^\ell(X)$ is a subterm of $g^*(Y)$ we find that $g^\ell(X) = g^\ell(Y)$. So X is equal to Y and the branch of X with T-leaves is a subtree of Y.

Inductive step: Assume that the result holds for $g^*(dd_1(Y)[\Delta \mapsto \mathsf{F}])$, $g^*(dd_2(Y))$ and $g^*(cd_2(Y))$ (IH). Assume that $g^*(Y) = g^*(dd_1(Y)[\Delta \mapsto \mathsf{F}]) \mathbin{\curlyvee} g^*(dd_2(Y))$ and that clauses 1 and 2 hold for $g^*(dd_1(Y)[\Delta \mapsto \mathsf{F}])$. The case for $g^*(dd_2(Y))$ follows similarly. By IH, the branch of X with T-leaves is a subtree of $dd_1(Y)[\Delta \mapsto \mathsf{F}]$. Observe that $Y = dd_1(Y)[\Delta \mapsto dd_2(Y)] = dd_1(Y)[\Delta \mapsto \mathsf{F}][\mathsf{F} \mapsto dd_2(Y)]$, where the first equality holds by Theorem 2.6.6.2 and by Definition 2.6.10 and where the second equality holds because $dd_1(Y)$ has no F-leaves. So the branch of X with T-leaves in $dd_1(Y)[\Delta \mapsto \mathsf{F}]$ is preserved in Y, so it is also a subtree of Y.

Assume that $g^*(Y) = g^*(cd_1(Y)[\Delta \mapsto \mathsf{T}]) \mathbin{\curlywedge} g^*(cd_2(Y))$. We find that clauses 1 and 2 cannot hold for $g^*(cd_1(Y)[\Delta \mapsto \mathsf{T}])$ simultaneously. The proof where clauses 1 and 2 hold for $g^*(cd_2(Y))$ follows similar to the proof for $g^*(dd_2(Y))$. □

Corollary 5.5.3. *Let $X = se(P^{\mathsf{T}})$ be the evaluation tree of a T-term P^{T} in SNF and let $Y = se(P^*)$ be the evaluation tree of a $*$-term P^* in SNF. Then X is a subtree of Y if*

1. *$g^{\mathsf{T}}(X)$ is a subterm of $g^*(Y)$, and*
2. *there is no evaluation tree X' such that*
   (a) *$g^*(cd_1(X')[\Delta \mapsto \mathsf{T}]) \mathbin{\curlywedge} g^*(cd_2(X'))$ is a subterm of $g^*(Y)$, and*
   (b) *$g^{\mathsf{T}}(X)$ is a subterm of $g^*(cd_1(X')[\Delta \mapsto \mathsf{T}])$.*

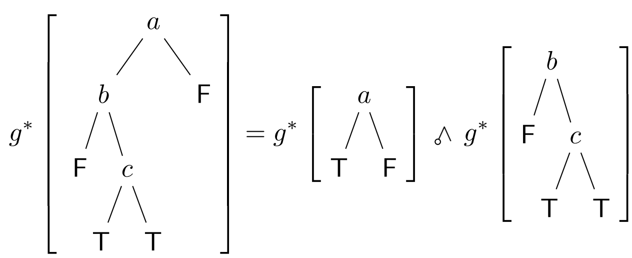
Proof. Note that X is a subtree of Y if there is an evaluation tree $Z = se(P^\ell)$ for some literal term P^ℓ in *SNF* such that X is a subtree of Z and the branch of Z with T-leaves is a subtree of Y. The result follows by Lemma 5.5.1 and Lemma 5.5.2. □

Next, we give an example of Corollary 5.5.3.

Example 5.5.4. Consider the T-term $P^{\mathsf{T}} = (c \mathbin{\circ\!\!\wedge} \mathsf{T}) \mathbin{\circ\!\!\vee} \mathsf{T}$ in *SNF* and the $*$-term $P^* = \big((a \mathbin{\circ\!\!\wedge} \mathsf{T}) \mathbin{\circ\!\!\vee} \mathsf{F}\big) \mathbin{\circ\!\!\wedge} \big((\neg b \mathbin{\circ\!\!\wedge} ((c \mathbin{\circ\!\!\wedge} \mathsf{T}) \mathbin{\circ\!\!\vee} \mathsf{T})) \mathbin{\circ\!\!\vee} \mathsf{F}\big)$ in *SNF*. Let $X = \mathit{se}(P^{\mathsf{T}})$ and $Y = \mathit{se}(P^*)$. Observe that

$$\begin{aligned} g^*(Y) &= g^*(cd_1(Y)[\Delta \mapsto \mathsf{T}]) \mathbin{\circ\!\!\wedge} g^*(cd_2(Y)) \\ &= \big((a \mathbin{\circ\!\!\wedge} \mathsf{T}) \mathbin{\circ\!\!\vee} \mathsf{F}\big) \mathbin{\circ\!\!\wedge} \big((\neg b \mathbin{\circ\!\!\wedge} ((c \mathbin{\circ\!\!\wedge} \mathsf{T}) \mathbin{\circ\!\!\vee} \mathsf{T})) \mathbin{\circ\!\!\vee} \mathsf{F}\big). \end{aligned}$$

As we can see, $g^{\mathsf{T}}(X) = P^{\mathsf{T}}$ is a subterm of $g^*(Y)$ and not a subterm of $g^*(cd_1(Y)[\Delta \mapsto \mathsf{T}])$. So X is a subtree of Y by Corollary 5.5.3.



Lemma 5.5.5. *Consider a closed term $t = s \mathbin{\bullet\!\!\oplus} u$ in T^A_{LXSCL}.*

1. *If s and u are not evaluation-unanimous and u is expressible over $\Sigma_{\mathrm{SCL}}(A)$, then $t = s \mathbin{\bullet\!\!\oplus} u$ is not expressible over $\Sigma_{\mathrm{SCL}}(A)$.*

2. *If s is not expressible over $\Sigma_{\mathrm{SCL}}(A)$ and u is T-unanimous, then $t = s \mathbin{\circ\!\!\wedge} u$ is not expressible over $\Sigma_{\mathrm{SCL}}(A)$.*

3. *If s is not expressible over $\Sigma_{\mathrm{SCL}}(A)$ and u is F-unanimous, then $t = s \mathbin{\bullet\!\!\oplus} u$ is not expressible over $\Sigma_{\mathrm{SCL}}(A)$.*

4. *If s is not expressible over $\Sigma_{\mathrm{SCL}}(A)$ and u is T-unanimous, then $t = s \mathbin{\bullet\!\!\oplus} u$ is not expressible over $\Sigma_{\mathrm{SCL}}(A)$.*

Proof.

Statement 1:

Suppose for contradiction that $t = s \mathbin{\bullet\!\!\oplus} u$ is expressible over $\Sigma_{\mathrm{SCL}}(A)$. Then $\mathit{se}(s \mathbin{\bullet\!\!\oplus} u) = \mathit{se}(P)$ for some P in *SNF* by Corollary 2.5.3. By Lemma 5.2.3 we know that $s \mathbin{\bullet\!\!\oplus} u$ is not evaluation-unanimous. Therefore, P is a T-$*$-term of the form $P^{\mathsf{T}} \mathbin{\circ\!\!\wedge} P^*$. So $\mathit{se}(s \mathbin{\bullet\!\!\oplus} u) = \mathit{se}(P^{\mathsf{T}} \mathbin{\circ\!\!\wedge} P^*)$ has a T-$*$-decomposition $\langle tsd_1(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u)), tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))\rangle$ by Theorem 2.6.9, and

$$g(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u)) = g^{\mathsf{T}}(tsd_1(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))[\Delta \mapsto \mathsf{T}]) \mathbin{\circ\!\!\wedge} g^*(tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u)))$$

by Definition 2.6.11. We will reach a contradiction by showing that $g^*(tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u)))$, hence also $g(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$, is undefined.

We will argue that $tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$ has no cd. The proof that $tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$ has no dd follows similarly. Suppose for contradiction that $tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$ has a ccd $\langle Y, Z\rangle$ such that $tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u)) = Y[\Delta \mapsto Z]$, where Y has Δ- and F-leaves but no T-leaves, and where Z has both T- and F-leaves. Before we reach a contradiction, we will prove three facts.

First of all, we show that there is no special node of $\mathit{se}(s)$ in $tsd_1(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$. Suppose there is. Then there are Δ-leaves in both the left branch and the right branch below this special node in

$tsd_1(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$. Let X be a subtree of $\mathit{se}(u)$ that is rooted at a special node of $\mathit{se}(u)$, so X is a subtree of $\mathit{se}(s \mathbin{\bullet\!\!\oplus} u)$. Then $X = \mathit{se}(P^\ell)$ for a literal term P^ℓ in *SNF* by Lemma 5.4.4. By Lemma 5.4.5 we know that the root of X cannot be a node in $tsd_1(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$, so X is a subtree of $tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$. Then X is a subtree of both the left branch and the right branch below the special node of $\mathit{se}(s)$ in $\mathit{se}(s \mathbin{\bullet\!\!\oplus} u) = tsd_1(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))[\Delta \mapsto tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))]$, which violates Corollary 5.4.7.

Secondly, there must be a root of $\mathit{se}(u)$ in Y. Because no special node of $\mathit{se}(s)$ occurs in $tsd_1(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$ there is no root of $\mathit{se}(u)$ in $tsd_1(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$. Suppose there is neither a root of $\mathit{se}(u)$ in Y. Then there is a F-leaf that is not below a root of $\mathit{se}(u)$ in $tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u)) = Y[\Delta \mapsto Z]$ by definition of Y, hence in $\mathit{se}(s \mathbin{\bullet\!\!\oplus} u) = tsd_1(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))[\Delta \mapsto tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))]$. Because all the leaves in $\mathit{se}(s \mathbin{\bullet\!\!\oplus} u)$ are below a root of $\mathit{se}(u)$ this cannot be the case.

Thirdly, special nodes of $\mathit{se}(u)$ occur only in Z. Because no special node of $\mathit{se}(s)$ occurs in $tsd_1(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$, there is also no special node of $\mathit{se}(u)$ in $tsd_1(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$. Suppose there is a special node of $\mathit{se}(u)$ in Y, and consider the subtree Y' of $Y[\Delta \mapsto Z]$ that is rooted at this special node. Then Y' either has Z as a proper subtree or it has only F-leaves. Both cases violate the result from Lemma 5.4.3 that Y' has a branch with only T-leaves and a branch with only F-leaves.

We now have enough information to reach the first contradiction. Consider a root of $\mathit{se}(u)$ in Y. By Lemma 5.4.6 and equation (26) we know that only one of X and $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ is a subtree below this particular root in $\mathit{se}(s \mathbin{\bullet\!\!\oplus} u)$, hence in $tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u)) = Y[\Delta \mapsto Z]$. Because a special node of $\mathit{se}(u)$ occurs only in Z, this implies that only one of X and $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ is a subtree of Z. So we find that X and $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ are not both a subtree of $tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$, hence not of $\mathit{se}(s \mathbin{\bullet\!\!\oplus} u) = tsd_1(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))[\Delta \mapsto tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))]$. This contradicts the fact that X and $X[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}]$ are both a subtree of $\mathit{se}(s \mathbin{\bullet\!\!\oplus} u)$ by Lemma 5.4.6 and equation (26). So $tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$ has no cdd, hence no cd.

Since $tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$ has no cd and no dd, we compute $g^*(tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))) = g^\ell(tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u)))$ by Definition 2.6.11. The function g^ℓ is only defined for evaluation trees with a branch that has only T-leaves and a branch that has only F-leaves. Because $tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$ has a special node of $\mathit{se}(s)$ and because u is not evaluation-unanimous, we find that $tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$ is not such a tree. So $g^*(tsd_2(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$, hence also $g(\mathit{se}(s \mathbin{\bullet\!\!\oplus} u))$, is undefined. This contradicts our assumption that $s \mathbin{\bullet\!\!\oplus} u$ is expressible over $\Sigma_{\mathrm{SCL}}(A)$.

Statement 2:

Because u is T-unanimous, $u =_{\mathit{se}} u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T}$ by Lemma 5.2.2. Because $=_{\mathit{se}}$ is a congruence relation on T^A_{LXSCL}, we have $s \mathbin{{}^{\circ}\!\!\wedge} u =_{\mathit{se}} s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})$. Suppose for contradiction that $s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})$ is expressible over $\Sigma_{\mathrm{SCL}}(A)$. We will reach a contradiction by showing that s is expressible over $\Sigma_{\mathrm{SCL}}(A)$.

Since $s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})$ is expressible over $\Sigma_{\mathrm{SCL}}(A)$, we find that $\mathit{se}(s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})) = \mathit{se}(P)$ for some P in *SNF* by Corollary 2.5.3. Because s is not expressible over $\Sigma_{\mathrm{SCL}}(A)$, it is not evaluation-unanimous by Lemma 5.2.4. So neither $s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})$ is evaluation-unanimous. Therefore, P is a T-$*$-term of the form $P^{\mathsf{T}} \mathbin{{}^{\circ}\!\!\wedge} P^*$. By Theorem 2.6.9 we find that $\mathit{se}(s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})) = \mathit{se}(P^{\mathsf{T}} \mathbin{{}^{\circ}\!\!\wedge} P^*)$ has a T-$*$-decomposition $\langle tsd_1(\mathit{se}(s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T}))), tsd_2(\mathit{se}(s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})))\rangle$, and by Definition 2.6.11 we have

$$g(\mathit{se}(s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T}))) = g^{\mathsf{T}}(tsd_1(\mathit{se}(s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})))[\Delta \mapsto \mathsf{T}]) \mathbin{{}^{\circ}\!\!\wedge} g^*(tsd_2(\mathit{se}(s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})))).$$

We argue that no root of $\mathit{se}(u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})$ is in $tsd_1(\mathit{se}(s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})))$. Suppose it is. Because $tsd_2(\mathit{se}(s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})))$ has T- and F-leaves by Definition 2.6.8, there are T- and F-leaves below this root in $\mathit{se}(s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})) = tsd_1(\mathit{se}(s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})))[\Delta \mapsto tsd_2(\mathit{se}(s \mathbin{{}^{\circ}\!\!\wedge} (u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T})))]$, which contradicts the fact that $u \mathbin{{}^{\circ}\!\!\vee} \mathsf{T}$ is T-unanimous.

By the above we find that $se(u \mathbin{\circ\!\!\vee} \mathsf{T})$ is a subtree of $se(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T}))) = tsd_1(se(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})))[\Delta \mapsto tsd_2(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})))]$ if and only if it is a subtree of $tsd_2(se(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})))$. Note that $se(u \mathbin{\circ\!\!\vee} \mathsf{T}) = se(Q^{\mathsf{T}})$ for a T-term Q^{T} in *SNF* by Lemma 5.2.4 and that $tsd_2(se(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T}))) = se(P^*)$ by Theorem 2.6.9. So by Corollary 5.5.3 we know when $se(u \mathbin{\circ\!\!\vee} \mathsf{T})$ is a subtree of $tsd_2(se(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})))$, hence of $se(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T}))$. If we replace all the subterms $g^{\mathsf{T}}(se(u \mathbin{\circ\!\!\vee} \mathsf{T}))$ of $g^*(tsd_2(se(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T}))))$ that meet the demands of Corollary 5.5.3 by T, we get a term $s' \in T^A_{\text{SCL}}$ such that $se(s) = se(s')$. This contradicts the fact that s is not expressible over $\Sigma_{\text{SCL}}(A)$. So $s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})$ is not expressible over $\Sigma_{\text{SCL}}(A)$.

Statement 3:

Because u is F-unanimous, $u =_{se} u \mathbin{\circ\!\!\wedge} \mathsf{F}$ by Lemma 5.2.2. Because $=_{se}$ is a congruence relation on T^A_{LXSCL}, we have $s \mathbin{\circ\!\!\oplus} u =_{se} s \mathbin{\circ\!\!\oplus} (u \mathbin{\circ\!\!\wedge} \mathsf{F})$. Finally, by axiom (AX3) and Theorem 5.1.7 we find that $s \mathbin{\circ\!\!\oplus} (u \mathbin{\circ\!\!\wedge} \mathsf{F}) =_{se} (s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})) \mathbin{\circ\!\!\vee} (u \mathbin{\circ\!\!\wedge} \mathsf{F})$. So it suffices to show that $(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})) \mathbin{\circ\!\!\vee} (u \mathbin{\circ\!\!\wedge} \mathsf{F})$ is not expressible over $\Sigma_{\text{SCL}}(A)$.

Suppose for contradiction that $(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})) \mathbin{\circ\!\!\vee} (u \mathbin{\circ\!\!\wedge} \mathsf{F})$ is expressible over $\Sigma_{\text{SCL}}(A)$. Then its negation $\neg((s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})) \mathbin{\circ\!\!\vee} (u \mathbin{\circ\!\!\wedge} \mathsf{F})) \overset{\text{(A1)}}{=_{se}} \neg\neg(\neg(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})) \mathbin{\circ\!\!\wedge} \neg\neg(\neg u \mathbin{\circ\!\!\vee} \neg\mathsf{F})) \overset{\text{(Aux2),(Aux3)}}{=_{se}} \neg(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})) \mathbin{\circ\!\!\wedge} (\neg u \mathbin{\circ\!\!\vee} \mathsf{T})$ is also expressible over $\Sigma_{\text{SCL}}(A)$. By Lemma 5.5.5.2 we know that $s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})$ is not expressible over $\Sigma_{\text{SCL}}(A)$, so neither $\neg(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T}))$ is. Then it follows by Lemma 5.5.5.2 that also $\neg(s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})) \mathbin{\circ\!\!\wedge} (\neg u \mathbin{\circ\!\!\vee} \mathsf{T})$ is not expressible over $\Sigma_{\text{SCL}}(A)$, which leads to a contradiction.

Statement 4:

Because u is T-unanimous, $u =_{se} u \mathbin{\circ\!\!\vee} \mathsf{T}$ and $s \mathbin{\circ\!\!\oplus} u =_{se} s \mathbin{\circ\!\!\oplus} (u \mathbin{\circ\!\!\vee} \mathsf{T})$. Suppose that $s \mathbin{\circ\!\!\oplus} (u \mathbin{\circ\!\!\vee} \mathsf{T})$ is expressible over $\Sigma_{\text{SCL}}(A)$. Then also $\neg(s \mathbin{\circ\!\!\oplus} (u \mathbin{\circ\!\!\vee} \mathsf{T})) \overset{\text{(Eq4)}}{=_{se}} \neg\neg(s \mathbin{\circ\!\!\oplus} (u \mathbin{\circ\!\!\wedge} \mathsf{F})) \overset{\text{(Aux2)}}{=_{se}} s \mathbin{\circ\!\!\oplus} (u \mathbin{\circ\!\!\wedge} \mathsf{F})$ is expressible over $\Sigma_{\text{SCL}}(A)$, which violates the result from Lemma 5.5.5.3. □

We now show exactly which closed terms of the form $s \mathbin{\circ\!\!\oplus} u$ are expressible over $\Sigma_{\text{SCL}}(A)$. Then we will characterize which terms $t \in T^A_{\text{LXL}}$ are expressible over $\Sigma_{\text{SCL}}(A)$.

Lemma 5.5.6. *A closed term $t = s \mathbin{\circ\!\!\oplus} u$ in T^A_{LXSCL} is expressible over $\Sigma_{\text{SCL}}(A)$ if and only if s or u is evaluation-unanimous and s and u are expressible over $\Sigma_{\text{SCL}}(A)$.*

Proof.
'$\Longleftarrow$'
We will show that $t = s \mathbin{\circ\!\!\oplus} u$ is expressible by a term t' containing only s, u and SCL-connectives if s or u is evaluation-unanimous. The result follows by the assumption that s and u are expressible over $\Sigma_{\text{SCL}}(A)$.

If s is T-unanimous, then $s =_{se} s \mathbin{\circ\!\!\vee} \mathsf{T}$ by Lemma 5.2.2. We find that $s \mathbin{\circ\!\!\oplus} u =_{se} (s \mathbin{\circ\!\!\vee} \mathsf{T}) \mathbin{\circ\!\!\oplus} u \overset{\text{(Eq1)}}{=_{se}} (s \mathbin{\circ\!\!\vee} \mathsf{T}) \mathbin{\circ\!\!\wedge} \neg u$, where the first equality holds because $=_{se}$ is a congruence relation on T^A_{LXSCL} and the second by Theorem 5.1.7. If s is F-unanimous, then $s =_{se} s \mathbin{\circ\!\!\wedge} \mathsf{F}$ and $s \mathbin{\circ\!\!\oplus} u =_{se} (s \mathbin{\circ\!\!\wedge} \mathsf{F}) \mathbin{\circ\!\!\oplus} u \overset{\text{(Eq2)}}{=_{se}} (s \mathbin{\circ\!\!\wedge} \mathsf{F}) \mathbin{\circ\!\!\vee} u$.

Similarly, if u is T-unanimous, then $s \mathbin{\circ\!\!\oplus} u =_{se} s \mathbin{\circ\!\!\oplus} (u \mathbin{\circ\!\!\vee} \mathsf{T}) \overset{\text{(Eq3)}}{=_{se}} (\neg s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T}) \mathbin{\circ\!\!\vee} (u \mathbin{\circ\!\!\wedge} \mathsf{F}))$. If u is F-unanimous, then $s \mathbin{\circ\!\!\oplus} u =_{se} s \mathbin{\circ\!\!\oplus} (u \mathbin{\circ\!\!\wedge} \mathsf{F}) \overset{\text{(AX3)}}{=_{se}} (s \mathbin{\circ\!\!\wedge} (u \mathbin{\circ\!\!\vee} \mathsf{T})) \mathbin{\circ\!\!\vee} (u \mathbin{\circ\!\!\wedge} \mathsf{F})$.

'$\Longrightarrow$'
Proof by contraposition. First assume that s and u are not evaluation-unanimous. If u is expressible over $\Sigma_{\text{SCL}}(A)$, then $t = s \mathbin{\bullet\!\!\oplus} u$ is not expressible over $\Sigma_{\text{SCL}}(A)$ by Lemma 5.5.5.1. If u is not expressible over $\Sigma_{\text{SCL}}(A)$, the result follows by the contrapositive of Theorem 5.3.3 because *se*(u) is a subtree of *se*$(s \mathbin{\bullet\!\!\oplus} u)$ by equation (26).

Assume that s is not expressible over $\Sigma_{\text{SCL}}(A)$. The case where u is not evaluation-unanimous is already covered because s is not evaluation-unanimous by Lemma 5.2.4. The cases where u is F-unanimous and T-unanimous are covered by Lemma 5.5.5.3 and Lemma 5.5.5.4 respectively.

Finally, assume that u is not expressible over $\Sigma_{\text{SCL}}(A)$. The case where s is not evaluation-unanimous is already covered. By equation (26), *se*(u) is a subtree of *se*$(s \mathbin{\bullet\!\!\oplus} u)$ if s if F-unanimous and *se*$(\neg u)$ is a subtree of *se*$(s \mathbin{\bullet\!\!\oplus} u)$ if s is T-unanimous. Because u is not expressible over $\Sigma_{\text{SCL}}(A)$, neither $\neg u$ is. In both cases the result follows by the contrapositive of Theorem 5.3.3. □

Theorem 5.5.7. *A closed term $t \in T^A_{\text{LXL}}$ is expressible over $\Sigma_{\text{SCL}}(A)$ if and only if the BF-term $f(t)$ has at most one atom.*

Proof.
'$\Longleftarrow$'
If $f(t)$ has zero atoms, then $f(t) \in \{\mathsf{T}, \mathsf{F}\}$ and the result is trivial. If $f(t)$ has one atom, then $f(t)$ is expressible over $\Sigma_{\text{SCL}}(A)$ because $a \mathbin{\bullet\!\!\oplus} \mathsf{F} =_{se} a$ by axiom (X4) and because $a \mathbin{\bullet\!\!\oplus} \mathsf{T} =_{se} \neg a$ by axiom (AX1).

'$\Longrightarrow$'
Assume that $f(t) = (((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} t^B$ has $n \geq 2$ atoms. We show by induction on n that $((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n$ is not expressible over $\Sigma_{\text{SCL}}(A)$. The result follows by Lemma 5.5.6.

Base case: Because a_1 and a_2 are not evaluation-unanimous, $a_1 \mathbin{\bullet\!\!\oplus} a_2$ is not expressible over $\Sigma_{\text{SCL}}(A)$ by Lemma 5.5.6.

Inductive step: Assume that $((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n$ is not expressible over $\Sigma_{\text{SCL}}(A)$ for some $n \geq 2$ (IH). By Lemma 5.5.6 we find that neither $(((a_1 \mathbin{\bullet\!\!\oplus} a_2) \mathbin{\bullet\!\!\oplus} \ldots) \mathbin{\bullet\!\!\oplus} a_n) \mathbin{\bullet\!\!\oplus} a_{n+1}$ is expressible over $\Sigma_{\text{SCL}}(A)$. □

Using this result, we can easily determine which terms $t \in T^A_{\text{LXSCL}}$ are expressible over $\Sigma_{\text{SCL}}(A)$ and $\Sigma_{\text{LXL}}(A)$.

Corollary 5.5.8. *The intersection se*$[T^A_{\text{SCL}}] \cap$ *se*$[T^A_{\text{LXL}}]$ *is equal to* $\{\mathsf{T}, \mathsf{F}, (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F}), (\mathsf{F} \trianglelefteq a \trianglerighteq \mathsf{T}) \mid a \in A\}$.

Proof. Let $t \in T^A_{\text{LXL}}$, so *se*$(t) \in$ *se*$[T^A_{\text{LXL}}]$. By Theorem 5.5.7 we know that *se*$(t) \in$ *se*$[T^A_{\text{SCL}}]$ if and only if the *BF*-term $f(t)$ has at most one atom, if and only if $se(f(t)) \in \{\mathsf{T}, \mathsf{F}, (\mathsf{T} \trianglelefteq a \trianglerighteq \mathsf{F}), (\mathsf{F} \trianglelefteq a \trianglerighteq \mathsf{T}) \mid a \in A\}$. □

5.6 FLXSCL and FSCL

In this section we will characterize which closed terms t over $\Sigma_{\text{LXSCL}}(A)$ are expressible over $\Sigma_{\text{SCL}}(A)$. First we introduce notation for T-unanimous and F-unanimous terms.

Definition 5.6.1. *We refer to any* T*-unanimous term in* T^A_{LXSCL} *as* $\widetilde{\mathsf{T}}$. *Similarly, we refer to any* F*-unanimous term in* T^A_{LXSCL} *as* $\widetilde{\mathsf{F}}$.

To present the proof of Theorem 5.6.3 more clearly, we cover one case separately below.

Lemma 5.6.2. *Let* $z \in T^A_{\text{LXSCL}}$ *such that* z *is not expressible over* $\Sigma_{\text{SCL}}(A)$ *and let* v *be a* T-$*$*-term in SNF of the form* $P^{\mathsf{T}} \mathbin{{}_{\circ}\!\!\wedge} P^d$. *Then* $t = z \mathbin{{}_{\circ}\!\!\wedge} v$ *is not expressible over* $\Sigma_{\text{SCL}}(A)$.

Proof. Suppose for contradiction that $z \mathbin{{}_{\circ}\!\!\wedge} v =_{se} z \mathbin{{}_{\circ}\!\!\wedge} (P^{\mathsf{T}} \mathbin{{}_{\circ}\!\!\wedge} P^d)$ is expressible over $\Sigma_{\text{SCL}}(A)$. We will reach a contradiction by showing that $z \mathbin{{}_{\circ}\!\!\wedge} P^{\mathsf{T}}$ is expressible over $\Sigma_{\text{SCL}}(A)$. Because $z \mathbin{{}_{\circ}\!\!\wedge} v$ is expressible over $\Sigma_{\text{SCL}}(A)$, we find that $se(z \mathbin{{}_{\circ}\!\!\wedge} v) = se(Q)$ for some Q in *SNF* by Corollary 2.5.3. Because z and v are not evaluation-unanimous (because z is not expressible over $\Sigma_{\text{SCL}}(A)$, z is not evaluation-unanimous by Lemma 5.2.4), neither $z \mathbin{{}_{\circ}\!\!\wedge} v$ is evaluation-unanimous. So Q is a T-$*$-term of the form $Q^{\mathsf{T}} \mathbin{{}_{\circ}\!\!\wedge} Q^*$. So $se(z \mathbin{{}_{\circ}\!\!\wedge} v) = se(Q^{\mathsf{T}} \mathbin{{}_{\circ}\!\!\wedge} Q^*)$ has a T-$*$-decomposition $\langle tsd_1(se(z \mathbin{{}_{\circ}\!\!\wedge} v)), tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v)\rangle$ by Theorem 2.6.9.

We argue that there is no special node of $se(z)$ in $tsd_1(se(z \mathbin{{}_{\circ}\!\!\wedge} v))$. If there is a special node of $se(z)$ in $tsd_1(se(z \mathbin{{}_{\circ}\!\!\wedge} v))$, there will be T- and F-leaves in the left branch and the right branch below this node in $se(z \mathbin{{}_{\circ}\!\!\wedge} v) = tsd_1(se(z \mathbin{{}_{\circ}\!\!\wedge} v))[\Delta \mapsto tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v))]$ because $tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v))$ has T- and F-leaves. This cannot be the case because there is a branch with only F-leaves below each special node of $se(z)$ in $se(z \mathbin{{}_{\circ}\!\!\wedge} v)$.

Because no special node of $se(z)$ occurs in $tsd_1(se(z \mathbin{{}_{\circ}\!\!\wedge} v))$, $se(P^d)$ is a subtree of $tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v))$. We write $tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v))[se(P^d) \mapsto \Delta]$ for the evaluation tree $tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v))$ in which each subtree $se(P^d)$ is replaced by Δ. We will show that

$$\langle Y, Z\rangle = \langle tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v))[se(P^d) \mapsto \Delta], se(P^d)\rangle$$

is the conjunction decomposition of $tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v))$. First we argue that $\langle Y, Z\rangle$ is a candidate conjunction decomposition. Obviously, $tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v)) = Y[\Delta \mapsto Z]$. Furthermore, Y has Δ-leaves, F-leaves because there is a branch with F-leaves below the special node of $se(z)$, but no T-leaves because all the T-leaves in $tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v))$ are T-leaves of $se(P^d)$. Finally, $Z = se(P^d)$ has both T- and F-leaves.

We now show that there is no other ccd $\langle Y', Z'\rangle$ of $tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v))$, where the height of Z' is smaller than the height of $Z = se(P^d)$. We show this by a case distinction on the complexity of P^d.

Let P^d be a literal term. Then $Z = se(P^d)$ has a branch with only T-leaves and a branch with only F-leaves. Since each T-leaf of $tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v)) = tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} (P^{\mathsf{T}} \mathbin{{}_{\circ}\!\!\wedge} P^d)))$ is below a root of $se(P^d)$, we find that each subtree of $tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v))$ with T- and F-leaves is equal to $se(P^d)$ or has $se(P^d)$ as a proper subtree. So there is no Z' with T- and F-leaves with a smaller height than Z.

Let P^d be of the form $P^* \mathbin{{}^{\circ}\!\!\vee} P^c$. Suppose for contradiction that $\langle Y', Z'\rangle$ is a ccd of $tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v))$, where Z' has a smaller height than $se(P^d)$. Because Z' has a smaller height than $se(P^d)$, the root of $se(P^d)$ is in Y'. Let Y'' be the subtree of Y' that is rooted at the root of $se(P^d)$. Then $se(P^d) = Y''[\Delta \mapsto Z']$. By definition of Y' there is no T-leaf in Y''. There must be a Δ-leaf in Y'', otherwise there are only F-leaves below the root of $se(P^d)$ in $tsd_2(se(z \mathbin{{}_{\circ}\!\!\wedge} v)) = Y'[\Delta \mapsto Z']$. Finally, there must be a F-leaf in Y'', otherwise the existence of the decomposition $se(P^d) = Y''[\Delta \mapsto Z']$ contradicts Lemma 2.6.2. By the above, we find that $\langle Y'', Z' \rangle$ is a ccd of $se(P^d)$. Because $se(P^d)$ has no conjunction decomposition by Theorem 2.6.6.2, hence no ccd, we reached a contradiction.

So $\langle tsd_2(se(z \leftand v))[se(P^d) \mapsto \Delta], se(P^d) \rangle$ is the conjunction decomposition of $tsd_2(se(z \leftand v))$. By Definition 2.6.11 we find that

$$\begin{aligned} g(se(z \leftand v)) &= g^{\mathsf{T}}(tsd_1(se(z \leftand v))[\Delta \mapsto \mathsf{T}]) \leftand g^*(tsd_2(se(z \leftand v))) \\ &= g^{\mathsf{T}}(tsd_1(se(z \leftand v))[\Delta \mapsto \mathsf{T}]) \leftand \big(g^*(tsd_2(se(z \leftand v))[se(P^d) \mapsto \mathsf{T}]) \leftand g^*(se(P^d))\big) \\ &= \big(g^{\mathsf{T}}(tsd_1(se(z \leftand v))[\Delta \mapsto \mathsf{T}]) \leftand g^*(tsd_2(se(z \leftand v))[se(P^d) \mapsto \mathsf{T}])\big) \leftand g^*(se(P^d)), \end{aligned}$$

where the last equality holds by axiom (A5). Note that $g^{\mathsf{T}}(tsd_1(se(z \leftand v))[\Delta \mapsto \mathsf{T}]) \leftand g^*(tsd_2(se(z \leftand v))[se(P^d) \mapsto \mathsf{T}])$ is a term in T^A_{SCL} for $z \leftand P^{\mathsf{T}}$. Because z is not expressible over $\Sigma_{\text{SCL}}(A)$, we know by Lemma 5.5.5.2 that $z \leftand P^{\mathsf{T}}$ is not expressible over $\Sigma_{\text{SCL}}(A)$. We reached a contradiction, so $z \leftand v$ is not expressible over $\Sigma_{\text{SCL}}(A)$. □

Theorem 5.6.3. *A closed term $t \in T^A_{\text{LXSCL}}$ is expressible over $\Sigma_{\text{SCL}}(A)$ if and only if for each subterm $s \leftxor u$ of t either*

1. *$s \leftxor u$ is expressible over $\Sigma_{\text{SCL}}(A)$, or*
2. *$s \leftxor u$ is a subterm of some term z such that*
   - *(a) z is an evaluation-unanimous subterm of t, or*
   - *(b) $\widetilde{\mathsf{T}} \leftor z$ or $\widetilde{\mathsf{F}} \leftand z$ is a subterm of t.*

Proof.
'$\Longleftarrow$'
Assume that for any subterm $s \leftxor u$ of t at least one of clause 1, 2a or 2b holds. Consider an arbitrary such subterm. We will argue that we can replace $s \leftxor u$ or a subterm of t of which $s \leftxor u$ is a subterm by a term $v \in T^A_{\text{SCL}}$, while preserving the evaluation tree $se(t)$. Because this can be done for every subterm $s \leftxor u$ of t, we find that t is expressible over $\Sigma_{\text{SCL}}(A)$.

If $s \leftxor u$ is expressible over $\Sigma_{\text{SCL}}(A)$, there is a term $v \in T^A_{\text{SCL}}$ such that $se(v) = se(s \leftxor u)$ and we can replace $s \leftxor u$ by v in t, while preserving the evaluation tree $se(t)$.

If $s \leftxor u$ is a subterm of some term z such that z is an evaluation-unanimous subterm of t, then z is expressible over $\Sigma_{\text{SCL}}(A)$ by Lemma 5.2.4. So there is a term $v \in T^A_{\text{SCL}}$ such that $se(v) = se(z)$ and we can replace z by v in t.

Finally, consider the case where $s \leftxor u$ is a subterm of a term z such that $\widetilde{\mathsf{T}} \leftor z$ is a subterm of t. The case for $\widetilde{\mathsf{F}} \leftand z$ follows similarly. By Lemma 5.2.4 we know that there is a term $v \in T^A_{\text{SCL}}$ such that $se(v) = se(\widetilde{\mathsf{T}}) = se(\widetilde{\mathsf{T}} \leftor z)$, where the last equality holds because $se(\widetilde{\mathsf{T}})$ has only T-leaves. So we can replace $\widetilde{\mathsf{T}} \leftor z$ by v in t.

'$\Longrightarrow$'
Assume that there is a subterm $s \leftxor u$ of t that does not meet clauses 1, 2a and 2b. We pick the last such subterm of t, so we can assume that no $\leftxor$ occurs after $s \leftxor u$ in t. We show that t is not expressible over $\Sigma_{\text{SCL}}(A)$ by induction on the complexity of closed terms.

Base case: If $t = s \leftxor u$, we know that t does not satisfy the first clause, hence that t is not expressible over $\Sigma_{\text{SCL}}(A)$.

Inductive step: Consider an arbitrary term $z \in T^A_{\text{LXSCL}}$ of which $s \mathbin{\bullet\!\!\oplus} u$ is a subterm and assume that z is not expressible over $\Sigma_{\text{SCL}}(A)$ (IH). Furthermore, take $v \in T^A_{\text{LXSCL}}$ arbitrarily.

Let t be of the form $\neg z$. Suppose for contradiction that t is expressible over $\Sigma_{\text{SCL}}(A)$, so there is a term $t' \in T^A_{\text{SCL}}$ such that $t' =_{se} \neg z$. Then $\neg t' =_{se} \neg\neg z \overset{\text{(Aux2)}}{=}_{se} z$. Since $\neg t' \in T^A_{\text{SCL}}$ it follows that z is expressible over $\Sigma_{\text{SCL}}(A)$, which contradicts the IH.

Let t be of the form $z \leftand v$. We assume that v is a term in T^A_{SCL} because no $\mathbin{\bullet\!\!\oplus}$ occurs after z in t. Because $s \mathbin{\bullet\!\!\oplus} u$ does not satisfy clause 2a we know that v is not F-unanimous. The case where v is T-unanimous follows directly because $z \leftand v =_{se} z \leftand (v \leftor \mathsf{T})$ and by Lemma 5.5.5.2.

If v is not evaluation-unanimous, there is a T-$*$-term of the form $P^{\mathsf{T}} \leftand P^*$ in *SNF* such that $v =_{se} P^{\mathsf{T}} \leftand P^*$. Suppose that P^* is of the form $Q^* \leftand Q^d$. Then we can consider the shorter term $z \leftand (P^{\mathsf{T}} \leftand Q^*)$ because $z \leftand v =_{se} z \leftand (P^{\mathsf{T}} \leftand (Q^* \leftand Q^d)) \overset{\text{(A5)}}{=}_{se} (z \leftand (P^{\mathsf{T}} \leftand Q^*)) \leftand Q^d$. So we let v be a T-$*$-term in *SNF* of the form $P^{\mathsf{T}} \leftand P^d$. The result follows by Lemma 5.6.2

Let t be of the form $z \leftor v$. Like before, we assume that $v \in T^A_{\text{SCL}}$. Because $s \mathbin{\bullet\!\!\oplus} u$ does not satisfy clause 2a, we find that v is not T-unanimous. Note that $z \leftor v \overset{\text{(A1)}}{=}_{se} \neg(\neg z \leftand \neg v)$. Because $\neg v \in T^A_{\text{SCL}}$ is not F-unanimous, the result follows by the inductive steps for $\neg z$ and $z \leftand v$.

Let t be of the form $v \leftand z$. Because $s \mathbin{\bullet\!\!\oplus} u$ does not satisfy clause 2b, we know that v is not F-unanimous. So $se(z)$ is a subtree of $se(v \leftand z)$. Because z is not expressible over $\Sigma_{\text{SCL}}(A)$ by IH, the result follows by the contrapositive of Theorem 5.3.3. The case for $v \leftor z$ follows similarly.

Finally, bet t be of the form $v \mathbin{\bullet\!\!\oplus} z$. If v is not T-unanimous, then $se(z)$ is a subtree of $se(v \mathbin{\bullet\!\!\oplus} z)$. If v is T-unanimous, $se(z)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}] = se(\neg z)$ is a subtree of $se(v \mathbin{\bullet\!\!\oplus} z)$. Because z and $\neg z$ are not expressible over $\Sigma_{\text{SCL}}(A)$ by IH and the inductive step for $\neg z$, the result follows in both cases by the contrapositive of Theorem 5.3.3. Because we assumed that the connective $\mathbin{\bullet\!\!\oplus}$ does not come after $s \mathbin{\bullet\!\!\oplus} u$ in t, we do not consider the case where t is of the form $z \mathbin{\bullet\!\!\oplus} v$. $\square$

Example 5.6.4. Consider the term $t = (((a \mathbin{\bullet\!\!\oplus} b) \leftor \mathsf{T}) \mathbin{\bullet\!\!\oplus} c) \leftand d$. First note that $z = (a \mathbin{\bullet\!\!\oplus} b) \leftor \mathsf{T}$ is a T-unanimous subterm of t, so z is expressible over $\Sigma_{\text{SCL}}(A)$ by Lemma 5.2.4. Because z and c are expressible over $\Sigma_{\text{SCL}}(A)$ and z is T-unanimous, the subterm $z \mathbin{\bullet\!\!\oplus} c$ of t is expressible over $\Sigma_{\text{SCL}}(A)$ by Lemma 5.5.6. So t is expressible over $\Sigma_{\text{SCL}}(A)$ by Theorem 5.6.3.

5.7 FLXSCL and FLXL

In this section we will characterize which closed terms t over $\Sigma_{\text{LXSCL}}(A)$ are expressible over $\Sigma_{\text{LXL}}(A)$. First we recall some results from Chapter 4. By Corollary 4.4.6 we find that t is expressible over $\Sigma_{\text{LXL}}(A)$ if *se*$(t) =$ *se*(t') for some *BF*-term t'. By Lemma 4.5.2 we know that the evaluation tree of a *BF*-term t' with n atoms has height n, has 2^i nodes at depth $i \in \{0, 1, \ldots, n-1\}$ that are all labeled with the same atom and has 2^n leaves at depth n. By Corollary 4.5.3 we know that any node at depth $n-1$ in *se*(t') has a T-leaf and a F-leaf as child nodes. Lemma 4.5.4 specifies how the leaves of *se*(t') are labeled. Namely, if L_k is the k^{th} leaf from the left in *se*(t'), for $k \in \{1, 2, \ldots, 2^n\}$, then the labels of L_j and L_{j+2^i} have opposite values for every i, j with $i \in \{0, 1, \ldots, n-1\}$ and $j \in \{1, 2, \ldots, 2^i\}$. It follows that t is expressible over $\Sigma_{\text{LXL}}(A)$ if Lemma 4.5.2 holds for the nodes of *se*(t) and if Lemma 4.5.4 holds for the leaves of *se*(t).

We introduced notation $\widetilde{\mathsf{T}}$ and $\widetilde{\mathsf{F}}$ for terms that are T-unanimous and F-unanimous in the previous section. We now introduce notation for terms of which the evaluation tree is a single leaf.

Definition 5.7.1. *We refer to any term $t \in T^A_{\text{LXSCL}}$ of which se$(t) = \mathsf{T}$ as $\widehat{\mathsf{T}}$. Similarly, we refer to any term $t \in T^A_{\text{LXSCL}}$ of which se$(t) = \mathsf{F}$ as $\widehat{\mathsf{F}}$.*

Before we turn to the main result of this section, we prove two lemmas about expressibility of closed terms of the form $s \mathbin{{}_\circ\!\wedge} u$ and $s \mathbin{{}^\circ\!\vee} u$.

Lemma 5.7.2. *Let $s, u \in T^A_{\text{LXSCL}}$ such that se(s) and se(u) have depth ≥ 1. If Lemma 4.5.2 does not hold for the nodes of se(s) and se(u), then $t = s \mathbin{{}_\circ\!\wedge} u$ and $t = s \mathbin{{}^\circ\!\vee} u$ are not expressible over $\Sigma_{\text{LXL}}(A)$.*

Proof. We show the result for $t = s \mathbin{{}_\circ\!\wedge} u$. The proof for $t = s \mathbin{{}^\circ\!\vee} u$ follows similarly.

Assume that Lemma 4.5.2 does not hold for the nodes of *se*(s) because there are nodes at the same depth in *se*(s) with different labels. This will also be the case in *se*$(s \mathbin{{}_\circ\!\wedge} u)$, so the result follows by Lemma 4.5.2.

Assume that Lemma 4.5.2 does not hold for the nodes of *se*(s) because there are less than 2^i nodes at depth i in *se*(s), for some $i \in \{0, 1, \ldots, n-1\}$. Let n be the height of *se*(s) and let m be the height of *se*(u). If there are still less than 2^i nodes at depth i in *se*$(s \mathbin{{}_\circ\!\wedge} u)$, the result follows by Lemma 4.5.2. If there are 2^i nodes at depth i in *se*$(s \mathbin{{}_\circ\!\wedge} u)$ for every $i \in \{0, 1, \ldots, n-1\}$, there are two options. If the height of *se*$(s \mathbin{{}_\circ\!\wedge} u)$ is $n + m$, there are leaves at depth $i + m$. Because $i + m < n + m$, we find that there are leaves in *se*$(s \mathbin{{}_\circ\!\wedge} u)$ that are not at depth $n + m$ and the result follows by Lemma 4.5.2. If the height of *se*$(s \mathbin{{}_\circ\!\wedge} u)$ is $< n+m$, there are nodes in *se*$(s \mathbin{{}_\circ\!\wedge} u)$ at depth $n-1$ that have only F-leaves as children, which violates Corollary 4.5.3.

Finally, assume that Lemma 4.5.2 holds for the nodes of *se*(s) and not for the nodes of *se*(u). The result follows trivially. □

Lemma 5.7.3. *A closed term $t = s \mathbin{{}_\circ\!\wedge} u$ in T^A_{LXSCL} (or $t = s \mathbin{{}^\circ\!\vee} u$) is expressible over $\Sigma_{\text{LXL}}(A)$ if and only if either*

1. *s is expressible over $\Sigma_{\text{LXL}}(A)$ and $u = \widehat{\mathsf{T}}$ (or $u = \widehat{\mathsf{F}}$), or*
2. *$s = \widehat{\mathsf{T}}$ (or $s = \widehat{\mathsf{F}}$) and u is expressible over $\Sigma_{\text{LXL}}(A)$, or*
3. *$s = \widehat{\mathsf{F}}$ (or $s = \widehat{\mathsf{T}}$).*

Proof. We show the result for $t = s \,\triangleleft\!\!\wedge\, u$. The proof for $t = s \,\triangleleft\!\!\vee\, u$ follows similarly.
'$\Longleftarrow$'
If s is expressible over $\Sigma_{\text{LXL}}(A)$ and $u = \widehat{\mathsf{T}}$, then $s \,\triangleleft\!\!\wedge\, u$ is expressible over $\Sigma_{\text{LXL}}(A)$ because $s \,\triangleleft\!\!\wedge\, u =_{se} s \,\triangleleft\!\!\wedge\, \mathsf{T} =_{se} s$ by (A3)^{dl}. Similarly, if $s = \widehat{\mathsf{T}}$ and u is expressible over $\Sigma_{\text{LXL}}(A)$ the result follows because $s \,\triangleleft\!\!\wedge\, u =_{se} \mathsf{T} \,\triangleleft\!\!\wedge\, u \overset{\text{(A2)}}{=_{se}} u$ and if $s = \widehat{\mathsf{F}}$ because $s \,\triangleleft\!\!\wedge\, u =_{se} \mathsf{F} \,\triangleleft\!\!\wedge\, u \overset{\text{(A4)}}{=_{se}} \mathsf{F}$.

'$\Longrightarrow$'
We will first show that $s \,\triangleleft\!\!\wedge\, u$ is not expressible over $\Sigma_{\text{LXL}}(A)$ if *se*(s) has height $n \geq 1$, except for the case where s is not evaluation-unanimous and $u = \widehat{\mathsf{T}}$. Then we will show the result by contraposition.

Assume that s is T-unanimous. If u is evaluation-unanimous, there is a node in *se*$(s \,\triangleleft\!\!\wedge\, u)$ that has only T-leaves or only F-leaves. Then $s \,\triangleleft\!\!\wedge\, u$ is not expressible over $\Sigma_{\text{LXL}}(A)$ by Corollary 4.5.3. If u is not evaluation-unanimous, *se*(u) has height $m \geq 1$. We assume that Lemma 4.5.2 holds for the nodes of *se*(s) and *se*(u), otherwise the result follows by Lemma 5.7.2. So there are 2^m leaves at depth m in *se*(u) and 2^{m+n} leaves at depth $m+n$ in *se*$(s \,\triangleleft\!\!\wedge\, u)$. Let L_k be the k^{th} leaf from the left in *se*$(s \,\triangleleft\!\!\wedge\, u)$. Because *se*$(s)$ has only T-leaves, we know that the label of L_1 is equal to the label of L_{1+2^m}. The result follows because Lemma 4.5.4 does not hold for the leaves of *se*$(s \,\triangleleft\!\!\wedge\, u)$.

If s is F-unanimous, there are only F-leaves below every node in *se*(s). This implies that *se*$(s \,\triangleleft\!\!\wedge\, u) =$ *se*(s), so we find by Corollary 4.5.3 that $s \,\triangleleft\!\!\wedge\, u$ is not expressible over $\Sigma_{\text{LXL}}(A)$.

For the case where s is not evaluation-unanimous and $u \neq \widehat{\mathsf{T}}$, we consider a F-leaf of *se*(s) and the parent node of this leaf in *se*(s). The other child node of this parent node in *se*$(s \,\triangleleft\!\!\wedge\, u)$ cannot be a T-leaf because all the T-leaves of *se*(s) are replaced by *se*(u) in *se*$(s \,\triangleleft\!\!\wedge\, u)$ and because $u \neq \widehat{\mathsf{T}}$. The result follows by Corollary 4.5.3.

We now give the actual proof by contraposition. Assume that s is not expressible over $\Sigma_{\text{LXL}}(A)$. Then *se*(s) has height ≥ 1. If $u = \widehat{\mathsf{T}}$, then $s \,\triangleleft\!\!\wedge\, u$ is not expressible over $\Sigma_{\text{LXL}}(A)$ because $s \,\triangleleft\!\!\wedge\, u =_{se} s \,\triangleleft\!\!\wedge\, \mathsf{T} =_{se} s$ by (A3)^{dl}. The other cases are covered by the proof above. Assume that $u \neq \widehat{\mathsf{T}}$, $s \neq \widehat{\mathsf{T}}$ and $s \neq \widehat{\mathsf{F}}$. Then s has height ≥ 1 and the result follows by the proof above. Finally, assume that $u \neq \widehat{\mathsf{T}}$ is not expressible over $\Sigma_{\text{LXL}}(A)$ and that $s \neq \widehat{\mathsf{F}}$. If *se*$(s)$ has height ≥ 1, the result follows by the above. If $s = \widehat{\mathsf{T}}$, the result follows because $s \,\triangleleft\!\!\wedge\, u =_{se} \mathsf{T} \,\triangleleft\!\!\wedge\, u =_{se} u$ by axiom (A2). $\square$

Theorem 5.7.4. *A term $t \in T^A_{\text{LXSCL}}$ is expressible over $\Sigma_{\text{LXL}}(A)$ if and only if for each subterm $s \,\triangleleft\!\!\wedge\, u$ (or $s \,\triangleleft\!\!\vee\, u$) of t either*

1. *$s \,\triangleleft\!\!\wedge\, u$ (or $s \,\triangleleft\!\!\vee\, u$) is expressible over $\Sigma_{\text{LXL}}(A)$, or*

2. *$s \,\triangleleft\!\!\wedge\, u$ (or $s \,\triangleleft\!\!\vee\, u$) is a subterm of some term z such that $\widehat{\mathsf{T}} \,\triangleleft\!\!\vee\, z$ or $\widehat{\mathsf{F}} \,\triangleleft\!\!\wedge\, z$ is a subterm of t.*

Proof. First note that axiom (AX1) and soundness of EqFLXSCL with respect to *se*-congruence allow us to replace each subterm $\neg s$ of t by $s \bullet\!\!\oplus \mathsf{T}$, while preserving the evaluation tree *se*(t). So we assume that the connective $\neg$ does not occur in t.

'$\Longleftarrow$'
Assume that for any subterm $s \,\triangleleft\!\!\wedge\, u$ of t at least one of clause 1 or 2 holds. Consider an arbitrary such subterm. The case for $s \,\triangleleft\!\!\vee\, u$ follows similarly. We will argue that we can replace $s \,\triangleleft\!\!\wedge\, u$ or a subterm of t of which $s \,\triangleleft\!\!\wedge\, u$ is a subterm by a term $v \in T^A_{\text{LXL}}$, while preserving the evaluation tree *se*(t). Because this can be done for every subterm $s \,\triangleleft\!\!\wedge\, u$ and $s \,\triangleleft\!\!\vee\, u$ of t, we find that t is expressible over $\Sigma_{\text{LXL}}(A)$.

If $s \mathbin{\circ\!\wedge} u$ is expressible over $\Sigma_{\mathrm{LXL}}(A)$, there is a term $v \in T^{A}_{\mathrm{LXL}}$ such that $\mathit{se}(v) = \mathit{se}(s \mathbin{\circ\!\wedge} u)$. So we can replace $s \mathbin{\circ\!\wedge} u$ by v in t, while preserving the evaluation tree $\mathit{se}(t)$.

Consider the case where $s \mathbin{\circ\!\wedge} u$ is a subterm of a term z such that $\widehat{\mathsf{T}} \mathbin{\circ\!\vee} z$ is a subterm of t. The case for $\widehat{\mathsf{F}} \mathbin{\circ\!\wedge} z$ follows similarly. Then $\mathit{se}(\widehat{\mathsf{T}} \mathbin{\circ\!\vee} z) = \mathit{se}(\mathsf{T})$ and we can replace $\widehat{\mathsf{T}} \mathbin{\circ\!\vee} z$ by T in t.

'$\Longrightarrow$'
Assume that there is a subterm $s \mathbin{\circ\!\wedge} u$ of t that does not satisfy clauses 1 and 2. The case for $s \mathbin{\circ\!\vee} u$ follows similarly. We show that t is not expressible over $\Sigma_{\mathrm{LXL}}(A)$ by induction on the complexity of terms.

Base case: If $t = s \mathbin{\circ\!\wedge} u$, we know that t is not expressible over $\Sigma_{\mathrm{LXL}}(A)$ by the first clause.

Inductive step: Consider an arbitrary term $z \in T^{A}_{\mathrm{LXSCL}}$ of which $s \mathbin{\circ\!\wedge} u$ is a subterm and assume that z is not expressible over $\Sigma_{\mathrm{LXL}}(A)$ (IH). Furthermore, take $v \in T^{A}_{\mathrm{LXSCL}}$ arbitrarily.

Let t be of the form $z \mathbin{\circ\!\wedge} v$. Because z is not expressible over $\Sigma_{\mathrm{LXL}}(A)$, we know that $z \neq \widehat{\mathsf{T}}$ and $z \neq \widehat{\mathsf{F}}$. So $z \mathbin{\circ\!\wedge} v$ is not expressible over $\Sigma_{\mathrm{LXL}}(A)$ by Lemma 5.7.3. The case where t is of the form $z \mathbin{\circ\!\vee} v$ follows similarly.

Let t be of the form $v \mathbin{\circ\!\wedge} z$. Because z is not expressible over $\Sigma_{\mathrm{LXL}}(A)$, $z \neq \widehat{\mathsf{T}}$. Because $s \mathbin{\circ\!\wedge} u$ does not satisfy clause 2, $v \neq \widehat{\mathsf{F}}$. So we know that $v \mathbin{\circ\!\wedge} z$ is not expressible over $\Sigma_{\mathrm{SCL}}(A)$ by Lemma 5.7.3. The case where t is of the form $v \mathbin{\circ\!\vee} z$ follows similarly.

Let t be of the form $z \mathbin{\bullet\!\!\oplus} v$. We assume that Lemma 4.5.2 holds for the nodes of $\mathit{se}(z)$ and $\mathit{se}(v)$, otherwise the result follows by Lemma 5.7.2. Let n be the height of $\mathit{se}(z)$, let m be the height of $\mathit{se}(v)$, let M_{k_1} be the k_1^{th} leaf from the left in $\mathit{se}(z)$ for $k_1 \in \{1, 2, \ldots, 2^n\}$ and let L_{k_2} be the k_2^{th} leaf from the left in $\mathit{se}(z \mathbin{\bullet\!\!\oplus} v)$ for $k_2 \in \{1, 2, \ldots, 2^{m+n}\}$.
Because z is not expressible over $\Sigma_{\mathrm{LXL}}(A)$ by IH and because Lemma 4.5.2 holds for the nodes of $\mathit{se}(z)$, we find that Lemma 4.5.4 does not hold for the leaves of $\mathit{se}(z)$. So there are indices $i \in \{0, 1, \ldots, n-1\}$ and $j \in \{1, 2, \ldots, 2^i\}$ such that the labels of M_j and M_{j+2^i} are equal. Consider the leaves $L_{j \cdot 2^m}$ and $L_{(j+2^i)\cdot 2^m}$ in $\mathit{se}(z \mathbin{\bullet\!\!\oplus} v)$, which are the rightmost leaves of the subtrees of $\mathit{se}(z \mathbin{\bullet\!\!\oplus} v)$ that replaced the leaves M_j and M_{j+2^i} in $\mathit{se}(z)$ respectively. Because the labels of M_j and M_{j+2^i} are equal, also the labels of $L_{j \cdot 2^m}$ and $L_{(j+2^i)\cdot 2^m} = L_{(j \cdot 2^m + 2^{m+i})}$ are. Note that $m + i \in \{0, \ldots, m+n-1\}$ and that $j \cdot 2^m \in \{1, \ldots, 2^{m+i}\}$ (because $i \in \{0, \ldots, n-1\}$ and $j \in \{1, \ldots, 2^i\}$). Then the leaves of $\mathit{se}(z \mathbin{\bullet\!\!\oplus} v)$ are not labeled as described in Lemma 4.5.4. So $z \mathbin{\bullet\!\!\oplus} v$ is not expressible over $\Sigma_{\mathrm{LXL}}(A)$.

Finally, let t be of the form $v \mathbin{\bullet\!\!\oplus} z$. Again we assume that Lemma 4.5.2 holds for the nodes of $\mathit{se}(z)$ and $\mathit{se}(v)$ and that Lemma 4.5.4 does not hold for the leaves of $\mathit{se}(z)$. So there are indices $i \in \{0, \ldots, n-1\}$ and $j \in \{1, \ldots, 2^i\}$ such that the labels of M_j and M_{j+2^i} in $\mathit{se}(z)$ are equal. Because the labels of the first 2^n leftmost leaves of $\mathit{se}(v \mathbin{\bullet\!\!\oplus} z)$ are the same as the labels of the leaves of $\mathit{se}(z)$ or as the negation of these labels, we know that the labels of L_j and L_{j+2^i} in $\mathit{se}(v \mathbin{\bullet\!\!\oplus} z)$ are equal. So the leaves of $\mathit{se}(v \mathbin{\bullet\!\!\oplus} z)$ are not labeled as described in Lemma 4.5.4, so $v \mathbin{\bullet\!\!\oplus} z$ is not expressible over $\Sigma_{\mathrm{LXL}}(A)$. $\square$

5.8 CP and FLXSCL

In [BP11] it is shown that Hoare's conditional $\_ \triangleleft \_ \triangleright \_$ cannot be replaced by any collection of unary and binary connectives that are definable in $\Sigma_{\text{CP}}(A)$ modulo FVC if $|A| > 2$. We will show that terms over $\Sigma_{\text{LXSCL}}(A)$ are less expressive modulo FVC than terms over $\Sigma_{\text{CP}}(A)$ if $|A| > 1$. In fact, it is surprisingly easy to find terms in T^A_{CP} that are not expressible over $\Sigma_{\text{LXSCL}}(A)$. As an example, we show this for the term $b \triangleleft a \triangleright a$, with $a, b \in A$.

Proposition 5.8.1. *The term* $b \triangleleft a \triangleright a \in T^A_{\text{CP}}$ *is not expressible over* $\Sigma_{\text{LXSCL}}(A)$.

Proof. Suppose that $t \in T^A_{\text{LXSCL}}$ is a minimal expression of $b \triangleleft a \triangleright a$.

Suppose that $t \equiv s \mathbin{\bullet\!\!\oplus} u$. We first argue that $s, u \notin \{\widehat{\mathsf{T}}, \widehat{\mathsf{F}}\}$. If not, then t is not minimal because $\mathsf{T} \mathbin{\bullet\!\!\oplus} u \overset{\text{(X3)}}{=} u \mathbin{\bullet\!\!\oplus} \mathsf{T} \overset{\text{(AX1)}}{=} \neg u$ and because $\mathsf{F} \mathbin{\bullet\!\!\oplus} u \overset{\text{(Aux9)}}{=} u \mathbin{\bullet\!\!\oplus} \mathsf{F} \overset{\text{(X4)}}{=} u$. So s and u both evaluate atoms. If one of s and u evaluates more than two atoms, at least three atoms are evaluated in $t = s \mathbin{\bullet\!\!\oplus} u$ since s and u are always evaluated. This is not the case in $b \triangleleft a \triangleright a$. So s and u both evaluate exactly one atom. The second atom that is evaluated in $b \triangleleft a \triangleright a$ depends on the outcome of the first evaluation of a, whereas the second atom that is evaluated in $s \mathbin{\bullet\!\!\oplus} u$ is always the same (namely the atom that is evaluated in u). So $b \triangleleft a \triangleright a$ is not expressible by $t \equiv s \mathbin{\bullet\!\!\oplus} u$.

Now suppose that $t \equiv s \mathbin{\circ\!\!\wedge} u$. The case where $t \equiv s \mathbin{\circ\!\!\vee} u$ follows similarly. Again we argue that $s, u \notin \{\widehat{\mathsf{T}}, \widehat{\mathsf{F}}\}$. If s or u is $\widehat{\mathsf{F}}$, then t always yields F, which is not the case in $b \triangleleft a \triangleright a$. If s or u is $\widehat{\mathsf{T}}$, then t is not minimal by axiom (A2) or by (A3)^{dl}, respectively. So s and u both evaluate atoms.

We prove two facts about the evaluation of s. First, s can only yield T after exactly one evaluation of an atom. Note that s cannot yield T without evaluating an atom, because s must evaluate at least one atom by the above. If s cannot yield T, it is F-unanimous. Then t is not minimal because $(s \mathbin{\circ\!\!\wedge} \mathsf{F}) \mathbin{\circ\!\!\wedge} u \overset{\text{(A5)}}{=} s \mathbin{\circ\!\!\wedge} (\mathsf{F} \mathbin{\circ\!\!\wedge} u) \overset{\text{(A4)}}{=} s \mathbin{\circ\!\!\wedge} \mathsf{F}$. If s yields T after (more than) two evaluations of an atom, then (more than) three atoms are evaluated in $t = s \mathbin{\circ\!\!\wedge} u$, which is not the case in $b \triangleleft a \triangleright a$.

Secondly, s can only yield F after exactly two evaluations of an atom. If s cannot yield F, it will always yield T after evaluating exactly one atom in s. Then the second atom that is evaluated in t (the first atom that is evaluated in u) will always be the same, independent of the evaluation of the first atom. In $b \triangleleft a \triangleright a$ this is not the case. If s yields F after evaluating one atom or after evaluating (more than) three atoms, the evaluation of t is finished. This cannot be the case because in $b \triangleleft a \triangleright a$ always two atoms are evaluated.

Combining these two facts, we find that $s = a \mathbin{\circ\!\!\vee} (x \mathbin{\circ\!\!\wedge} \mathsf{F}) = \mathsf{T} \triangleleft a \triangleright (\mathsf{F} \triangleleft x \triangleright \mathsf{F})$ or $s = \neg a \mathbin{\circ\!\!\vee} (x \mathbin{\circ\!\!\wedge} \mathsf{F}) = (\mathsf{F} \triangleleft x \triangleright \mathsf{F}) \triangleleft a \triangleright \mathsf{T}$, for $x \in \{a, b\}$. This means that for a certain evaluation result of a (if a evaluates to F in the first case, and if a evaluates to T in the second case), the evaluation result of s is F. Then the evaluation result of t is also F, and it is independent of the evaluation result of the second atom. For $b \triangleleft a \triangleright a$ this is not the case, so $b \triangleleft a \triangleright a$ is not expressible by $t \equiv s \mathbin{\circ\!\!\wedge} u$.

So there is no term $t \in T^A_{\text{LXSCL}}$ such that $b \triangleleft a \triangleright a$ is expressible by t. $\square$

In [BP11, Prop.12.1] it is proved in a similar way that $a \triangleleft a \triangleright \neg a$ is not expressible over $\Sigma_{\text{SCL}}(A)$ modulo FVC. Note that this term is expressible over $\Sigma_{\text{LXSCL}}(A)$ by $\neg(a \mathbin{\bullet\!\!\oplus} a)$.

Chapter 6

Concluding remarks

In this chapter we first recall the main results of this thesis. Then we will discuss some topics that arose in the process of writing this thesis, but that were not included. Finally, we will provide some directions for future research.

6.1 Main results

We defined new connectives ℓNAND, ℓXOR and ℓIFF using Hoare's conditional [Hoa85] as a primitive, following the definitions of $\neg$ and $\wedge$ in [BPS13]. Defining the connectives this way allowed us to reason about the connectives using CP [BP11]. We found that Hoare's conditional is equal to Church's conditional disjunction that was defined three decades earlier [Chu56].

In Chapter 3 we defined ℓNAND, left-sequential short-circuit NAND, and we defined free left-sequential nand logic (FLNL) to investigate which logical laws axiomatize short-circuit evaluation of terms with ℓNAND modulo free valuation congruence. We showed that EqFLNL constitutes an independent, equational axiomatization of FLNL for closed terms. The proof of this result relies on the facts that EqFLNL and EqFSCL are translationally equivalent [PU03] and that EqFSCL axiomatizes FSCL for closed terms [PS18].

In Chapter 4 we defined ℓXOR, left-sequential full XOR, and we defined free left-sequential xor logic (FLXL) to investigate which laws axiomatize full evaluation of terms with ℓXOR modulo FVC. By defining a basic form for terms in T^A_{LXL} and fully describing how *xe*-trees of *BF*-terms are formed, we proved that EqFLXL constitutes an equational axiomatization of FLXL for closed terms. We also showed that ℓXOR and ℓIFF, left-sequential full biconditional, are dual.

Finally, in Chapter 5 we investigated the expressive power modulo FVC of terms over $\Sigma_i(A)$, for $i \in \{\text{CP}, \text{SCL}, \text{LNL}, \text{LXL}, \text{LXSCL}\}$. Proposition 5.1.3 sharpens the result from [PS18, Prop.2.1.8]. We proved that T^A_{SCL} and T^A_{LNL} are equally expressive in Corollary 3.4.2 and we showed which terms are expressible over both $\Sigma_{\text{SCL}}(A)$ and $\Sigma_{\text{LNL}}(A)$ in Corollary 5.5.8. In Theorem 5.6.3 and Theorem 5.7.4 we characterized which terms in T^A_{LXSCL} are expressible by terms over $\Sigma_{\text{SCL}}(A)$ and $\Sigma_{\text{LXL}}(A)$ respectively. Lastly, in Proposition 5.8.1 we showed that there are terms in T^A_{CP} that are not expressible over $\Sigma_{\text{LXSCL}}(A)$.

To illustrate the results from Chapter 5, we once more show the Venn diagram that depicts how the sets of evaluation trees corresponding to closed terms over the different signatures are related in Figure 6.1.

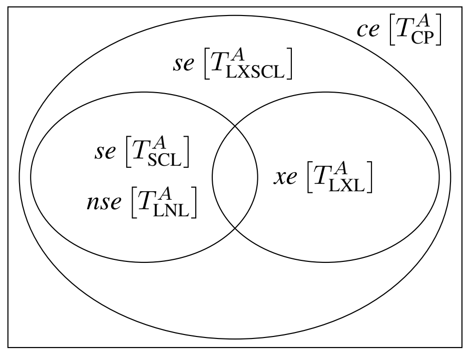


Figure 6.1: Venn diagram of sets of evaluation trees

6.2 Digression

We now discuss some topics that are outside the scope of the present work, but that are nevertheless quite interesting.

Functional completeness. In propositional logic we call a set of logical connectives functionally complete if every truth table can be expressed by a well-formed formula with connectives from this set. In [Chu56] it is shown that the signature $\{\_ \triangleleft \_ \triangleright \_, \mathsf{T}, \mathsf{F}\}$ is functionally complete. Also $\{|\}$ and $\{\downarrow\}$ are functionally complete, as well as $\{\neg, \mathbin{^{\circ}\!\vee}\}$, $\{\neg, \mathbin{{}_{\circ}\!\wedge}\}$ and less well known $\{\mathsf{T}, \mathbin{{}_{\circ}\!\wedge}, \oplus\}$ [Wer42].

Note that the functional completeness of $\{|\}$ and $\{\downarrow\}$ transfers to integrated circuit technology. NAND-gates and NOR-gates are universal gates with which every Boolean function can be implemented [MK14, pp.83-87]. For instance, the Apollo Guidance Computer guided the first lunar missions and was built using only NOR-gates [JPC14, p.2].

In the setting of left-sequential logics, the notion of a truth table is replaced by an evaluation tree. Each possible evaluation of a term t determines a path from the root of $se(t)$ to a leaf. So an evaluation tree covers both an evaluation *strategy* and all possible evaluation *results*. It might be interesting to develop a notion of functional completeness in the setting of left-sequential logics. This definition will be different when terms are evaluated modulo different valuation congruences.

It is reasonable to say that a signature $\Sigma_i(A)$ is functionally complete modulo FVC if for each $X \in \mathcal{T}_A$ there is a term $t \in T_i^A$ such that $X = se(t)$. Using this definition it is clear that $\Sigma_{\text{CP}}(A)$ is functionally complete modulo FVC and that none of $\Sigma_{\text{LNL}}(A)$, $\Sigma_{\text{SCL}}(A)$ and $\Sigma_{\text{LXSCL}}(A)$ are, although their propositional counterparts are in propositional logic.

We now discuss a possible definition of functional completeness modulo memorizing valuation congruence (MVC). In this setting the following axiom is added to CP [BP11]:

$$x \triangleleft y \triangleright (z \triangleleft u \triangleright (v \triangleleft y \triangleright w)) = x \triangleleft y \triangleright (z \triangleleft u \triangleright w) \tag{CPmem}$$

This axiom implies that once an atom has been evaluated, each subsequent evaluation of the same atom yields the same truth value. So we can say that $\Sigma_i(A)$ is functionally complete if for each $X \in \mathcal{T}_A$ in which each atom occurs at most once on a path from the root to a leaf there is a term $t \in T_i^A$ such that $X = se(t)$. Then we find that $\Sigma_{\mathrm{CP}}(A)$, $\Sigma_{\mathrm{SCL}}(A)$ and $\Sigma_{\mathrm{LNL}}(A)$ are functionally complete, because it is proved in [BP11] that

$$y \triangleleft x \triangleright z = (x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\vee} (\neg x \mathbin{{}_{\circ}\!\wedge} z).$$

is derivable from CP $\cup$ {(CPmem)}.

Left-sequential full NAND. We investigated left-sequential short-circuit NAND, but we could have also considered left-sequential full NAND, notation $\mathbin{\bullet\!\!\mid}$. This connective can be defined using Hoare's conditional, T and F:

$$x \mathbin{\bullet\!\!\mid} y = (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleleft x \triangleright (\mathsf{T} \triangleleft y \triangleright \mathsf{T}).$$

Using CP, we find that $\mathbin{\bullet\!\!\mid}$ is expressible in terms of $\mathbin{\circ\!\!\mid}$, T and F:

$$x \mathbin{\bullet\!\!\mid} y = ((x \mathbin{\circ\!\!\mid} \mathsf{T}) \mathbin{\circ\!\!\mid} (y \mathbin{\circ\!\!\mid} \mathsf{F})) \mathbin{\circ\!\!\mid} (y \mathbin{\circ\!\!\mid} \mathsf{T}).$$

So full evaluation of NAND can be seen as a special case of short-circuit evaluation of NAND.

In [Sta12], free fully evaluated logic (FFEL) is defined and axiomatized by EqFFEL for closed terms with $\neg$, $\mathbin{{}_{\bullet}\!\wedge}$ and $\mathbin{{}^{\bullet}\!\vee}$, where the latter two connectives are full versions of $\mathbin{{}_{\circ}\!\wedge}$ and $\mathbin{{}^{\circ}\!\vee}$ respectively. To investigate which logical laws axiomatize full evaluation of terms with $\mathbin{\bullet\!\!\mid}$ modulo FVC, a similar approach as in Chapter 3 is possible, using EqFFEL and the identities

$$\begin{aligned}
\neg x &= x \mathbin{\bullet\!\!\mid} \mathsf{T},\\
x \mathbin{{}_{\bullet}\!\wedge} y &= (x \mathbin{\bullet\!\!\mid} y) \mathbin{\bullet\!\!\mid} \mathsf{T},\\
x \mathbin{{}^{\bullet}\!\vee} y &= ((x \mathbin{\bullet\!\!\mid} \mathsf{T}) \mathbin{\bullet\!\!\mid} (y \mathbin{\bullet\!\!\mid} \mathsf{T})) \mathbin{\bullet\!\!\mid} \mathsf{T},
\end{aligned}$$

and

$$x \mathbin{\bullet\!\!\mid} y = \neg(x \mathbin{{}_{\bullet}\!\wedge} y).$$

ℓNOR, the dual of ℓNAND. In Section 4.3 we showed that ℓXOR and ℓIFF are dual. Similarly, we could have shown that ℓNOR, left-sequential short-circuit NOR, is the dual of ℓNAND. We can define ℓNOR, notation $\mathbin{\circ\!\!\downarrow}$, by

$$x \mathbin{\circ\!\!\downarrow} y = \mathsf{F} \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{T}).$$

Using CP, we find that $\mathbin{\circ\!\!\downarrow}$ is expressible in terms of $\mathbin{\circ\!\!\mid}$, T and F:

$$x \mathbin{\circ\!\!\downarrow} y = ((x \mathbin{\circ\!\!\mid} \mathsf{T}) \mathbin{\circ\!\!\mid} (y \mathbin{\circ\!\!\mid} \mathsf{T})) \mathbin{\circ\!\!\mid} \mathsf{T}.$$

Independence of EqFLXL and EqFLXSCL. We showed that EqFLNL is an independent axiomatization. Because EqFLNL $\vdash f(\text{EqFSCL})$ by Lemma 3.3.3, it follows that EqFSCL is derivable from EqFLNL$^+$ = EqFLNL $\cup \{\neg x = x \mathbin{\circ\!\!\mid} \mathsf{T}, x \mathbin{{}_{\circ}\!\wedge} y = (x \mathbin{\circ\!\!\mid} y) \mathbin{\circ\!\!\mid} \mathsf{T}, x \mathbin{{}^{\circ}\!\vee} y = \neg(\neg x \mathbin{{}_{\circ}\!\wedge} \neg y)\}$. Because EqFSCL is an independent axiomatization as well, independence of EqFLNL is relevant. By the tool *Mace4* [McC08] we know that the axioms of EqFLXL and EqFLXSCL are also independent, but we chose not to discuss this to keep the thesis compact. For EqFLXL the result is not crucial and for EqFLXSCL it does not make sense to prove independence because we only introduced this set of axioms as a means to show that closed terms over $\Sigma_{\mathrm{LXSCL}}(A)$ are *se*-congruent. Moreover, we know that EqFLXSCL is not complete with respect to *se*-congruence. For example, the equation

$$(x \mathbin{\bullet\!\!\oplus} y) \mathbin{{}^{\circ}\!\vee} (z \mathbin{{}^{\circ}\!\vee} \mathsf{T}) = (x \mathbin{{}_{\circ}\!\wedge} (\neg y \mathbin{{}^{\circ}\!\vee} (z \mathbin{{}^{\circ}\!\vee} \mathsf{T}))) \mathbin{{}^{\circ}\!\vee} (y \mathbin{{}^{\circ}\!\vee} (z \mathbin{{}^{\circ}\!\vee} \mathsf{T})) \qquad \text{(AX4)}$$

is sound but not derivable from EqFLXSCL by *Mace4* [McC08].

6.3 Future work

Finally, we pose some directions for future research.

FLNL as a way to solve open questions about FSCL from [PS18]. It is doubtful whether EqFLNL$^+$ is an easier axiomatization than EqFSCL. There is an extra connective, an extra axiom and reasoning with the SCL-connectives is more intuitive than reasoning with $\mathbin{\circ\!\!\mid}$. On the other hand, proofs by induction on the complexity of closed terms are shorter for terms in T^A_{LNL} than in T^A_{SCL}.

Also, the results from Chapter 3 might be used to give an answer to the open question from [PS18] whether there is a simpler proof of $\mathit{se}(P) = \mathit{se}(Q) \implies \text{EqFSCL} \vdash P = Q$ for $P, Q \in T^A_{\text{SCL}}$. The current proof of this result relies on the SCL Normal Form that has seven grammatical categories and on three decomposition theorems about evaluation trees in $\mathit{se}[T^A_{\text{SCL}}]$, as discussed in Section 2.5 and Section 2.6 respectively. Because there is only one connective in $\Sigma_{\text{LNL}}(A)$ instead of three, a similar approach might yield a more simple proof of $\mathit{nse}(f(P)) = \mathit{nse}(f(Q)) \implies \text{EqFLNL} \vdash f(P) = f(Q)$, hence of the mentioned result in [PS18].

Investigation of the research questions modulo other valuation congruences. It is interesting to investigate which logical laws axiomatize short-circuit evaluation of terms with ℓNAND and full evaluation of terms with ℓXOR modulo other valuation congruences. Variants of FLNL and FLXL can be defined, using the generic definitions of a left-sequential nand logic (LNL) and a left-sequential xor logic (LXL). Possible variants include memorizing, static, repetition-proof or contractive LNL or LXL.

Also expressiveness can be investigated modulo other valuation congruences. For the case modulo MVC we can show that

$$x \mathbin{\bullet\!\!\oplus} y = (x \leftand \neg y) \leftor (\neg x \leftand y)$$

is derivable from $\text{CP} \cup \{(\text{CPmem})\}$ and all defining equations of these connectives. Then each term in T^A_{LXL} and T^A_{LXSCL} is expressible over $\Sigma_{\text{SCL}}(A)$ modulo MVC, so the Venn diagram from Figure 6.1 will look different.

A complete axiomatization of *se*-congruence as defined on T^A_{LXSCL}. We already argued that EqFLXSCL is not complete with respect to *se*-congruence, because (AX4) is sound but not derivable from EqFLXSCL. As EqFLNL and EqFLXL are complete with respect to *nse*-congruence and *xe*-congruence by Corollary 3.4.5 and Corollary 4.6.4 respectively, it is interesting to find a complete axiomatization of *se*-congruence. We think that this will be tough because we did not find normal forms (or even basic forms) for terms in T^A_{LXSCL}. If we take the SCL Normal Form as starting point, we need at least two other grammatical categories. One for terms of the form $P \mathbin{\bullet\!\!\oplus} Q$ and one for terms of the form $(P \mathbin{\bullet\!\!\oplus} Q) \leftand (R \leftor \mathsf{T})$. Terms in the last grammatical category are not composed by ℓ-terms, but by ℓ-terms and T-terms.

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Appendices

A Appendix Chapter 3

We continue the proof of Theorem 3.2.7 on page 24.

Theorem 3.2.7. *The axioms of* EqFLNL *are independent if A contains at least two atoms.*

Proof. We present five independence models that are all $\Sigma_{\mathrm{LNL}}(A)$-algebras. All models were found with the tool *Mace4* [McC08]. In each model $\mathbb{M}$ that follows, $[\![\mathsf{T}]\!]^{\mathbb{M}} = 1$ and $[\![\mathsf{F}]\!]^{\mathbb{M}} = 0$.

Recall that $|A| \geq 2$. For the independence of axioms (N1) and (N2) no atoms are required. For the independence of axioms (N3), (N4) and (N5) one atom a is used. The independence of axiom (N6) is proved on page 24. Two atoms a, b were needed for this proof.

Independence of axiom (N1): Consider the model $\mathbb{M}$ for EqFLNL $\setminus$ $\{$(N1)$\}$ with domain $D = \{0, 1\}$, where $\lhd$ is interpreted as follows:

$\lhd$	0	1
0	1	1
1	1	1

We find that $[\![(\mathsf{T} \lhd \mathsf{F}) \lhd \mathsf{T}]\!]^{\mathbb{M}} = 1$ and $[\![\mathsf{F}]\!]^{\mathbb{M}} = 0$, so axiom (N1) is not satisfied by $\mathbb{M}$.

Independence of axiom (N2): Consider the model $\mathbb{M}$ for EqFLNL $\setminus$ $\{$(N2)$\}$ with domain $D = \{0, 1\}$, where $\lhd$ is interpreted as follows:

$\lhd$	0	1
0	0	0
1	0	1

We find that $[\![\mathsf{F} \lhd \mathsf{F}]\!]^{\mathbb{M}} = 0$ and $[\![\mathsf{T}]\!]^{\mathbb{M}} = 1$, so axiom (N2) is not satisfied by $\mathbb{M}$.

Independence of axiom (N3): Consider the model $\mathbb{M}$ for EqFLNL $\setminus$ $\{$(N3)$\}$ with domain $D = \{0, 1, 2, 3\}$, in which $[\![a]\!]^{\mathbb{M}} = 2$ for some $a \in A$ and where $\lhd$ is interpreted as follows:

$\lhd$	0	1	2	3
0	1	1	1	1
1	1	0	2	3
2	3	2	1	3
3	3	3	2	3

We find that $[\![((a \mathbin{\triangleleft} \mathsf{F}) \mathbin{\triangleleft} \mathsf{T}) \mathbin{\triangleleft} a]\!]^{\mathbb{M}} = 2$ and $[\![a \mathbin{\triangleleft} ((\mathsf{F} \mathbin{\triangleleft} a) \mathbin{\triangleleft} \mathsf{T})]\!]^{\mathbb{M}} = 3$, so axiom (N3) is not satisfied by $\mathbb{M}$.

Independence of axiom (N4): Consider the model $\mathbb{M}$ for EqFLNL $\setminus$ $\{$(N4)$\}$ with domain $D = \{0, 1, 2, 3, 4\}$, in which $[\![a]\!]^{\mathbb{M}} = 2$ for some $a \in A$ and where $\triangleleft$ is interpreted as follows:

$\triangleleft$	0	1	2	3	4
0	1	1	1	1	1
1	1	0	3	2	4
2	1	3	1	1	1
3	4	2	4	2	4
4	4	4	4	4	4

We find that $[\![(a \mathbin{\triangleleft} \mathsf{T}) \mathbin{\triangleleft} \mathsf{F}]\!]^{\mathbb{M}} = 4$ and $[\![a \mathbin{\triangleleft} \mathsf{F}]\!]^{\mathbb{M}} = 1$, so axiom (N4) is not satisfied by $\mathbb{M}$.

Independence of axiom (N5): Consider the model $\mathbb{M}$ for EqFLNL $\setminus$ $\{$(N5)$\}$ with domain $D = \{0, 1, 2, 3, 4\}$, in which $[\![a]\!]^{\mathbb{M}} = 2$ for some $a \in A$ and where $\triangleleft$ is interpreted as follows:

$\triangleleft$	0	1	2	3	4
0	1	1	1	1	1
1	1	0	2	4	3
2	3	2	2	2	3
3	3	4	4	4	3
4	3	3	3	3	3

We find that $[\![(a \mathbin{\triangleleft} \mathsf{F}) \mathbin{\triangleleft} (a \mathbin{\triangleleft} \mathsf{T})]\!]^{\mathbb{M}} = 4$ and $[\![((a \mathbin{\triangleleft} \mathsf{F}) \mathbin{\triangleleft} a) \mathbin{\triangleleft} \mathsf{T}]\!]^{\mathbb{M}} = 3$, so axiom (N5) is not satisfied by $\mathbb{M}$. $\square$

We finish the proof of Lemma 3.3.4 on page 26.

Lemma 3.3.4. EqFSCL $\vdash$ g(EqFLNL).

Proof. All proofs are distilled from output of the theorem prover *Prover9* [McC08].

Proof of EqFSCL $\vdash$ g(N2):

$$\begin{aligned} \neg(\mathsf{F} \mathbin{{}_{\circ}\!\wedge} x) &= \neg\mathsf{F} && \text{by (A4)} \\ &= \mathsf{T} && \text{by (Aux3)} \end{aligned}$$

Proof of EqFSCL $\vdash$ g(N3):

$$\begin{aligned} \neg(\neg(\neg(x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}_{\circ}\!\wedge} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} z) &= \neg(\neg(\neg(x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}_{\circ}\!\wedge} \neg\mathsf{F}) \mathbin{{}_{\circ}\!\wedge} z) && \text{by (Aux3)} \\ &= \neg((x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}_{\circ}\!\wedge} z) && \text{by (Aux4)} \\ &= \neg(x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}_{\circ}\!\wedge} z)) && \text{by (A5)} \\ &= \neg(x \mathbin{{}_{\circ}\!\wedge} \neg(\neg(y \mathbin{{}_{\circ}\!\wedge} z) \mathbin{{}_{\circ}\!\wedge} \neg\mathsf{F})) && \text{by (Aux4)} \\ &= \neg(x \mathbin{{}_{\circ}\!\wedge} \neg(\neg(y \mathbin{{}_{\circ}\!\wedge} z) \mathbin{{}_{\circ}\!\wedge} \mathsf{T})) && \text{by (Aux3)} \end{aligned}$$

Proof of EqFSCL $\vdash$ g(N4):

$$\begin{aligned} \neg(\neg(x \mathbin{{}_{\circ}\!\wedge} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) &= \neg(\neg x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) && \text{by (A3)}^{dl} \\ &= \neg(x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) && \text{by (A6)} \end{aligned}$$

Proof of EqFSCL $\vdash g$(N5):

$$
\begin{aligned}
\neg(\neg(x \mathbin{{}_\circ\!\wedge} \mathsf{F}) \mathbin{{}_\circ\!\wedge} \neg(y \mathbin{{}_\circ\!\wedge} \mathsf{T})) &= \neg(\neg(x \mathbin{{}_\circ\!\wedge} \mathsf{F}) \mathbin{{}_\circ\!\wedge} \neg y) && \text{by (A3)}^{dl} \\
&= (x \mathbin{{}_\circ\!\wedge} \mathsf{F}) \mathbin{{}^\circ\!\vee} y && \text{by (A1)} \\
&= (x \mathbin{{}^\circ\!\vee} \mathsf{T}) \mathbin{{}_\circ\!\wedge} y && \text{by (A7)} \\
&= \neg(\neg x \mathbin{{}_\circ\!\wedge} \neg\mathsf{T}) \mathbin{{}_\circ\!\wedge} y && \text{by (A1)} \\
&= \neg(\neg x \mathbin{{}_\circ\!\wedge} \mathsf{F}) \mathbin{{}_\circ\!\wedge} y && \text{by (Aux1)} \\
&= \neg(x \mathbin{{}_\circ\!\wedge} \mathsf{F}) \mathbin{{}_\circ\!\wedge} y && \text{by (A6)} \\
&= \neg\neg(\neg(x \mathbin{{}_\circ\!\wedge} \mathsf{F}) \mathbin{{}_\circ\!\wedge} y) && \text{by (Aux2)} \\
&= \neg(\neg(\neg(x \mathbin{{}_\circ\!\wedge} \mathsf{F}) \mathbin{{}_\circ\!\wedge} y) \mathbin{{}_\circ\!\wedge} \mathsf{T}) && \text{by (A3)}^{dl}
\end{aligned}
$$

Finally, we show that EqFSCL $\vdash g$(N6):

$$
\begin{aligned}
\neg(\neg(\neg(x \mathbin{{}_\circ\!\wedge} y) \mathbin{{}_\circ\!\wedge} \neg(z \mathbin{{}_\circ\!\wedge} \mathsf{F})) \mathbin{{}_\circ\!\wedge} \mathsf{T}) &= \neg(((x \mathbin{{}_\circ\!\wedge} y) \mathbin{{}^\circ\!\vee} (z \mathbin{{}_\circ\!\wedge} \mathsf{F})) \mathbin{{}_\circ\!\wedge} \mathsf{T}) && \text{by (A1)} \\
&= \neg((x \mathbin{{}_\circ\!\wedge} y) \mathbin{{}^\circ\!\vee} (z \mathbin{{}_\circ\!\wedge} \mathsf{F})) && \text{by (A3)}^{dl} \\
&= \neg((x \mathbin{{}^\circ\!\vee} (z \mathbin{{}_\circ\!\wedge} \mathsf{F})) \mathbin{{}_\circ\!\wedge} (y \mathbin{{}^\circ\!\vee} (z \mathbin{{}_\circ\!\wedge} \mathsf{F}))) && \text{by (A8)} \\
&= \neg(((x \mathbin{{}_\circ\!\wedge} \mathsf{T}) \mathbin{{}^\circ\!\vee} (z \mathbin{{}_\circ\!\wedge} \mathsf{F})) \mathbin{{}_\circ\!\wedge} ((y \mathbin{{}_\circ\!\wedge} \mathsf{T}) \mathbin{{}^\circ\!\vee} (z \mathbin{{}_\circ\!\wedge} \mathsf{F}))) && \text{by (A3)}^{dl} \\
&= \neg(\neg(\neg(x \mathbin{{}_\circ\!\wedge} \mathsf{T}) \mathbin{{}_\circ\!\wedge} \neg(z \mathbin{{}_\circ\!\wedge} \mathsf{F})) \mathbin{{}_\circ\!\wedge} \neg(\neg(y \mathbin{{}_\circ\!\wedge} \mathsf{T}) \mathbin{{}_\circ\!\wedge} \neg(z \mathbin{{}_\circ\!\wedge} \mathsf{F}))) && \text{by (A1)}
\end{aligned}
$$

□

We now prove the second statement of Theorem 3.3.6 on page 27.

Theorem 3.3.6.

1. *For any* $t \in \mathbb{T}^{A,\mathcal{X}}_{\text{LNL}}$,
$$\text{EqFLNL} \vdash f(g(t)) = t.$$
2. *For any* $t \in \mathbb{T}^{A,\mathcal{X}}_{\text{SCL}}$,
$$\text{EqFSCL} \vdash g(f(t)) = t.$$

Proof. Recall the functions f and g that were defined in Definition 3.3.2.

Statement 2:

Proof by induction on the complexity of terms in $\mathbb{T}^{A,\mathcal{X}}_{\text{SCL}}$.

Base case: Let $t \in \{\mathsf{T}, \mathsf{F}\} \cup A \cup \mathcal{X}$. Then $g(f(t)) = t$ by Definition 3.3.2. So also EqFSCL $\vdash$ $g(f(t)) = t$.

Inductive step: Assume that $s, u \in \mathbb{T}^{A,\mathcal{X}}_{\text{SCL}}$ are such that EqFSCL $\vdash g(f(s)) = s$ and EqFSCL $\vdash$ $g(f(u)) = u$ (IH). Furthermore assume that t is of the form $\neg s$. Using Definition 3.3.2 we compute

$$
\begin{aligned}
g(f(\neg s)) &= g(f(s) \mathbin{\circ\!\mid} \mathsf{T}) \\
&= \neg(g(f(s)) \mathbin{{}_\circ\!\wedge} g(\mathsf{T})) \\
&= \neg(g(f(s)) \mathbin{{}_\circ\!\wedge} \mathsf{T})
\end{aligned}
$$

Using the axioms of EqFSCL, we derive that

$$\begin{aligned}
\text{EqFSCL} \vdash g(f(\neg s)) &= \neg(g(f(s)) \mathbin{{}_{\circ}\!\wedge} \mathsf{T}) && \text{by the above}\\
&= \neg(s \mathbin{{}_{\circ}\!\wedge} \mathsf{T}) && \text{by IH}\\
&= \neg s && \text{by (A3)}^{dl}
\end{aligned}$$

Assume that t is of the form $s \mathbin{{}_{\circ}\!\wedge} u$. Using Definition 3.3.2, we compute

$$\begin{aligned}
g(f(s \mathbin{{}_{\circ}\!\wedge} u)) &= g((f(s) \triangleleft f(u)) \triangleleft \mathsf{T})\\
&= \neg(g(f(s) \triangleleft f(u)) \mathbin{{}_{\circ}\!\wedge} g(\mathsf{T}))\\
&= \neg(\neg(g(f(s)) \mathbin{{}_{\circ}\!\wedge} g(f(u))) \mathbin{{}_{\circ}\!\wedge} \mathsf{T})
\end{aligned}$$

Using the axioms of EqFSCL, we derive that

$$\begin{aligned}
g(f(s \mathbin{{}_{\circ}\!\wedge} u)) &= \neg(\neg(g(f(s)) \mathbin{{}_{\circ}\!\wedge} g(f(u))) \mathbin{{}_{\circ}\!\wedge} \mathsf{T}) && \text{by the above}\\
&= \neg(\neg(s \mathbin{{}_{\circ}\!\wedge} u) \mathbin{{}_{\circ}\!\wedge} \mathsf{T}) && \text{by IH}\\
&= \neg\neg(s \mathbin{{}_{\circ}\!\wedge} u) && \text{by (A3)}^{dl}\\
&= s \mathbin{{}_{\circ}\!\wedge} u && \text{by (Aux2)}
\end{aligned}$$

If t is of the form $s \mathbin{{}^{\circ}\!\vee} u$ we know that $g(f(s \mathbin{{}^{\circ}\!\vee} u)) = g(f(\neg(\neg s \mathbin{{}_{\circ}\!\wedge} \neg u)))$ by axiom (A1). The result follows by the previous results for $\neg$ and $\mathbin{{}_{\circ}\!\wedge}$. □

B Appendix Chapter 4

We continue the proof of Lemma 4.2.7 on page 33.

Lemma 4.2.7. *For all terms t, u over $\Sigma_{\text{LXL}}(A)$,*

$$\text{EqFLXL} \vdash t = u \implies \mathbb{M} \models t = u.$$

Proof. We show that the model $\mathbb{M}$ that was defined in Definition 4.2.6 satisfies axioms (X1), (X2) and (X4). Fix an arbitrary interpretation i of variables.

Proof of $\mathbb{M}, i \models$ (X1):

$$\begin{aligned}
[\![\mathsf{T} \mathbin{\bullet\!\!\oplus} \mathsf{T}]\!]^{\mathbb{M}} &= [\![\mathsf{T}]\!]^{\mathbb{M}}\big[\mathsf{T} \mapsto [\![\mathsf{T}]\!]^{\mathbb{M}}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto [\![\mathsf{T}]\!]^{\mathbb{M}}\big] && \text{by 4.2.6}\\
&= [\![\mathsf{T}]\!]^{\mathbb{M}}\big[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}\big] && \text{by (3)}\\
&= [\![\mathsf{F}]\!]^{\mathbb{M}} && \text{by (3)}
\end{aligned}$$

Proof of $\mathbb{M}, i \models$ (X2):

$$\begin{aligned}
&[\![(x \mathbin{\bullet\!\!\oplus} y) \mathbin{\bullet\!\!\oplus} z]\!]^{\mathbb{M},i}\\
&\quad = [\![x]\!]^{\mathbb{M},i}\big[\mathsf{T} \mapsto [\![y]\!]^{\mathbb{M},i}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto [\![y]\!]^{\mathbb{M},i}\big]\\
&\qquad\qquad \big[\mathsf{T} \mapsto [\![z]\!]^{\mathbb{M},i}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto [\![z]\!]^{\mathbb{M},i}\big] && \text{by 4.2.6}
\end{aligned}$$

$$
\begin{aligned}
&= \llbracket x\rrbracket^{\mathbb{M},i}\Big[\mathsf{T}\mapsto \llbracket y\rrbracket^{\mathbb{M},i}\big[\mathsf{T}\mapsto\mathsf{F},\mathsf{F}\mapsto\mathsf{T}\big]\big[\mathsf{T}\mapsto \llbracket z\rrbracket^{\mathbb{M},i}[\mathsf{T}\mapsto\mathsf{F},\mathsf{F}\mapsto\mathsf{T}],\mathsf{F}\mapsto \\
&\qquad \llbracket z\rrbracket^{\mathbb{M},i}\big],\mathsf{F}\mapsto \llbracket y\rrbracket^{\mathbb{M},i}\big[\mathsf{T}\mapsto \llbracket z\rrbracket^{\mathbb{M},i}[\mathsf{T}\mapsto\mathsf{F},\mathsf{F}\mapsto\mathsf{T}],\mathsf{F}\mapsto \llbracket z\rrbracket^{\mathbb{M},i}\big]\Big] && \text{by (7)}\\
&= \llbracket x\rrbracket^{\mathbb{M},i}\Big[\mathsf{T}\mapsto \llbracket y\rrbracket^{\mathbb{M},i}\big[\mathsf{T}\mapsto\mathsf{F},\mathsf{F}\mapsto\mathsf{T}\big]\big[\mathsf{T}\mapsto \llbracket z\rrbracket^{\mathbb{M},i}[\mathsf{T}\mapsto\mathsf{F},\mathsf{F}\mapsto\mathsf{T}],\mathsf{F}\mapsto \\
&\qquad \llbracket z\rrbracket^{\mathbb{M},i}\big],\mathsf{F}\mapsto \llbracket y \mathbin{\bullet\!\!\oplus} z\rrbracket^{\mathbb{M},i}\Big] && \text{by 4.2.6}\\
&= \llbracket x\rrbracket^{\mathbb{M},i}\Big[\mathsf{T}\mapsto \llbracket y\rrbracket^{\mathbb{M},i}\big[\mathsf{T}\mapsto \llbracket z\rrbracket^{\mathbb{M},i},\mathsf{F}\mapsto \llbracket z\rrbracket^{\mathbb{M},i}[\mathsf{T}\mapsto\mathsf{F},\mathsf{F}\mapsto\mathsf{T}]\big], \\
&\qquad\quad \mathsf{F}\mapsto \llbracket y \mathbin{\bullet\!\!\oplus} z\rrbracket^{\mathbb{M},i}\Big] && \text{by (3), (4), (7)}\\
&= \llbracket x\rrbracket^{\mathbb{M},i}\Big[\mathsf{T}\mapsto \llbracket y\rrbracket^{\mathbb{M},i}\big[\mathsf{T}\mapsto \llbracket z\rrbracket^{\mathbb{M},i}[\mathsf{T}\mapsto\mathsf{F},\mathsf{F}\mapsto\mathsf{T}][\mathsf{T}\mapsto\mathsf{F},\mathsf{F}\mapsto\mathsf{T}], \\
&\qquad\quad \mathsf{F}\mapsto \llbracket z\rrbracket^{\mathbb{M},i}[\mathsf{T}\mapsto\mathsf{F},\mathsf{F}\mapsto\mathsf{T}]\big],\mathsf{F}\mapsto \llbracket y \mathbin{\bullet\!\!\oplus} z\rrbracket^{\mathbb{M},i}\Big] && \text{by (3), (4), (6), (7)}\\
&= \llbracket x\rrbracket^{\mathbb{M},i}\Big[\mathsf{T}\mapsto \llbracket y\rrbracket^{\mathbb{M},i}\big[\mathsf{T}\mapsto \llbracket z\rrbracket^{\mathbb{M},i}[\mathsf{T}\mapsto\mathsf{F},\mathsf{F}\mapsto\mathsf{T}],\mathsf{F}\mapsto \llbracket z\rrbracket^{\mathbb{M},i}\big]\big[\mathsf{T}\mapsto\mathsf{F}, \\
&\qquad\quad \mathsf{F}\mapsto\mathsf{T}\big],\mathsf{F}\mapsto \llbracket y \mathbin{\bullet\!\!\oplus} z\rrbracket^{\mathbb{M},i}\Big] && \text{by (7)}\\
&= \llbracket x\rrbracket^{\mathbb{M},i}\big[\mathsf{T}\mapsto \llbracket y \mathbin{\bullet\!\!\oplus} z\rrbracket^{\mathbb{M},i}[\mathsf{T}\mapsto\mathsf{F},\mathsf{F}\mapsto\mathsf{T}],\mathsf{F}\mapsto \llbracket y \mathbin{\bullet\!\!\oplus} z\rrbracket^{\mathbb{M},i}\big] && \text{by 4.2.6}\\
&= \llbracket x \mathbin{\bullet\!\!\oplus} (y \mathbin{\bullet\!\!\oplus} z)\rrbracket^{\mathbb{M},i} && \text{by 4.2.6}
\end{aligned}
$$

Proof of $\mathbb{M}, i \vDash$ (X4):

$$
\begin{aligned}
\llbracket x \mathbin{\bullet\!\!\oplus} \mathsf{F}\rrbracket^{\mathbb{M},i} &= \llbracket x\rrbracket^{\mathbb{M},i}\big[\mathsf{T}\mapsto \llbracket \mathsf{F}\rrbracket^{\mathbb{M}}[\mathsf{T}\mapsto\mathsf{F},\mathsf{F}\mapsto\mathsf{T}],\mathsf{F}\mapsto \llbracket \mathsf{F}\rrbracket^{\mathbb{M}}\big] && \text{by 4.2.6}\\
&= \llbracket x\rrbracket^{\mathbb{M},i}\big[\mathsf{T}\mapsto\mathsf{T},\mathsf{F}\mapsto\mathsf{F}\big] && \text{by (4)}\\
&= \llbracket x\rrbracket^{\mathbb{M},i} && \text{by (6)}
\end{aligned}
$$

Because i was chosen arbitrarily, it follows that (X1), (X2) and (X4) are satisfied by $\mathbb{M}$. $\square$

We finish the proof of Theorem 4.3.5 on page 36.

Theorem 4.3.5. *For all t, u over $\Sigma_{\text{LXL}^+}(A)$,*

$$\text{EqFLXL}^+ \vdash t = u \iff \text{EqFLXL}^+ \vdash t^{dl} = u^{dl}.$$

Proof.

Proof of $\text{EqFLXL}^+ \vdash (\text{X2})^{dl}$:

$$
\begin{aligned}
(x \mathbin{\bullet\!\!\leftrightarrow} y) \mathbin{\bullet\!\!\leftrightarrow} z &= (((x \mathbin{\bullet\!\!\oplus} y) \mathbin{\bullet\!\!\oplus} \mathsf{T}) \mathbin{\bullet\!\!\oplus} z) \mathbin{\bullet\!\!\oplus} \mathsf{T} && \text{by (24)}\\
&= (x \mathbin{\bullet\!\!\oplus} (y \mathbin{\bullet\!\!\oplus} (\mathsf{T} \mathbin{\bullet\!\!\oplus} z))) \mathbin{\bullet\!\!\oplus} \mathsf{T} && \text{by (X2)}\\
&= (x \mathbin{\bullet\!\!\oplus} (y \mathbin{\bullet\!\!\oplus} (z \mathbin{\bullet\!\!\oplus} \mathsf{T}))) \mathbin{\bullet\!\!\oplus} \mathsf{T} && \text{by (X3)}\\
&= (x \mathbin{\bullet\!\!\oplus} ((y \mathbin{\bullet\!\!\oplus} z) \mathbin{\bullet\!\!\oplus} \mathsf{T})) \mathbin{\bullet\!\!\oplus} \mathsf{T} && \text{by (X2)}\\
&= x \mathbin{\bullet\!\!\leftrightarrow} (y \mathbin{\bullet\!\!\leftrightarrow} z) && \text{by (24)}
\end{aligned}
$$

Proof of EqFLXL$^+$ $\vdash$ (X3)^{dl}:

$$
\begin{aligned}
x \mathbin{\bullet\!\!\leftrightarrow} \mathsf{F} &= (x \mathbin{\bullet\!\!\oplus} \mathsf{F}) \mathbin{\bullet\!\!\oplus} \mathsf{T} && \text{by (24)}\\
&= (\mathsf{F} \mathbin{\bullet\!\!\oplus} x) \mathbin{\bullet\!\!\oplus} \mathsf{T} && \text{by (Aux9)}\\
&= \mathsf{F} \mathbin{\bullet\!\!\leftrightarrow} x && \text{by (24)}
\end{aligned}
$$

Proof of EqFLXL$^+$ $\vdash$ (X4)^{dl}:

$$
\begin{aligned}
x \mathbin{\bullet\!\!\leftrightarrow} \mathsf{T} &= (x \mathbin{\bullet\!\!\oplus} \mathsf{T}) \mathbin{\bullet\!\!\oplus} \mathsf{T} && \text{by (24)}\\
&= x && \text{by (Aux8)}
\end{aligned}
$$

Proof of EqFLXL$^+$ $\vdash$ (24)^{dl}:

$$
\begin{aligned}
x \mathbin{\bullet\!\!\leftrightarrow} y &= (x \mathbin{\bullet\!\!\oplus} y) \mathbin{\bullet\!\!\oplus} \mathsf{T} && \text{by (24)}\\
&= ((x \mathbin{\bullet\!\!\oplus} y) \mathbin{\bullet\!\!\oplus} \mathsf{F}) \mathbin{\bullet\!\!\oplus} \mathsf{T} && \text{by (X4)}\\
&= (x \mathbin{\bullet\!\!\oplus} y) \mathbin{\bullet\!\!\leftrightarrow} \mathsf{F} && \text{by (24)}
\end{aligned}
$$

□

C Appendix Chapter 5

We continue the proof of Proposition 5.1.4 on page 45.

Proposition 5.1.4. *The equations in Table 5.2 are derivable from* EqFLXSCL*.*

Proof. All proofs are distilled from output of the theorem prover *Prover9* [McC08].

Proof of EqFLXSCL $\vdash$ (A7):

$$
\begin{aligned}
(x \mathbin{{}_{\circ}\!\!\wedge} \mathsf{F}) \mathbin{{}^{\circ}\!\!\vee} y &= (\neg x \mathbin{{}_{\circ}\!\!\wedge} \mathsf{F}) \mathbin{{}^{\circ}\!\!\vee} y && \text{by (A6)}\\
&= (\neg x \mathbin{{}_{\circ}\!\!\wedge} \neg\mathsf{T}) \mathbin{{}^{\circ}\!\!\vee} y) && \text{by (Aux1)}\\
&= \neg(\neg(\neg x \mathbin{{}_{\circ}\!\!\wedge} \neg\mathsf{T}) \mathbin{{}_{\circ}\!\!\wedge} \neg y) && \text{by (A1)}\\
&= \neg((x \mathbin{{}^{\circ}\!\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\!\wedge} \neg y) && \text{by (A1)}\\
&= \neg((x \mathbin{{}^{\circ}\!\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\!\wedge} (y \mathbin{\bullet\!\!\oplus} \mathsf{T})) && \text{by (AX1)}\\
&= \neg(((x \mathbin{{}^{\circ}\!\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\!\wedge} y) \mathbin{\bullet\!\!\oplus} \mathsf{T}) && \text{by (AX2)}\\
&= \neg\neg((x \mathbin{{}^{\circ}\!\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\!\wedge} y) && \text{by (AX1)}\\
&= (x \mathbin{{}^{\circ}\!\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\!\wedge} y && \text{by (Aux2)}
\end{aligned}
$$

Proof of EqFLXSCL $\vdash$ (X1):

$$
\begin{aligned}
\mathsf{T} \mathbin{\bullet\!\!\oplus} \mathsf{T} &= \neg\mathsf{T} && \text{by (AX1)}\\
&= \mathsf{F} && \text{by (Aux1)}
\end{aligned}
$$

Proof of EqFLXSCL $\vdash$ (X4):

$$\begin{aligned}
x \mathbin{\bullet\!\!\oplus} \mathsf{F} &= x \mathbin{\bullet\!\!\oplus} (\mathsf{T} \mathbin{\bullet\!\!\oplus} \mathsf{T}) && \text{by (X1)}\\
&= (x \mathbin{\bullet\!\!\oplus} \mathsf{T}) \mathbin{\bullet\!\!\oplus} \mathsf{T} && \text{by (X2)}\\
&= \neg\neg x && \text{by (AX1)}\\
&= x && \text{by (Aux2)}
\end{aligned}$$

We now have EqFLXSCL $\vdash$ EqFLXL. So we can use the auxiliary result (Aux9) which is derivable from EqFLXL. We continue with the proof of EqFLXSCL $\vdash$ (Eq1):

$$\begin{aligned}
(x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{\bullet\!\!\oplus} y &= ((x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} \mathsf{T}) \mathbin{\bullet\!\!\oplus} y && \text{by (A3)}^{dl}\\
&= (x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} (\mathsf{T} \mathbin{\bullet\!\!\oplus} y) && \text{by (AX2)}\\
&= (x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} (y \mathbin{\bullet\!\!\oplus} \mathsf{T}) && \text{by (X3)}\\
&= (x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} \neg y && \text{by (AX1)}
\end{aligned}$$

Proof of EqFLXSCL $\vdash$ (Eq2):

$$\begin{aligned}
(x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \mathbin{\bullet\!\!\oplus} y &= ((x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \mathbin{{}^{\circ}\!\vee} \mathsf{F}) \mathbin{\bullet\!\!\oplus} y && \text{by (A3)}\\
&= ((x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \mathbin{\bullet\!\!\oplus} y && \text{by (A7)}\\
&= (x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} (\mathsf{F} \mathbin{\bullet\!\!\oplus} y) && \text{by (AX2)}\\
&= (x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} (y \mathbin{\bullet\!\!\oplus} \mathsf{F}) && \text{by (Aux9)}\\
&= (x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} y && \text{by (X4)}\\
&= (x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \mathbin{{}^{\circ}\!\vee} y && \text{by (A7)}
\end{aligned}$$

Proof of EqFLXSCL $\vdash$ (Eq3):

$$\begin{aligned}
x \mathbin{\bullet\!\!\oplus} (y \mathbin{{}^{\circ}\!\vee} \mathsf{T}) &= x \mathbin{\bullet\!\!\oplus} \neg(\neg y \mathbin{{}_{\circ}\!\wedge} \neg\mathsf{T}) && \text{by (A1)}\\
&= x \mathbin{\bullet\!\!\oplus} \neg(\neg y \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) && \text{by (Aux1)}\\
&= x \mathbin{\bullet\!\!\oplus} \neg(y \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) && \text{by (A6)}\\
&= x \mathbin{\bullet\!\!\oplus} ((y \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \mathbin{\bullet\!\!\oplus} \mathsf{T}) && \text{by (AX1)}\\
&= x \mathbin{\bullet\!\!\oplus} (\mathsf{T} \mathbin{\bullet\!\!\oplus} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) && \text{by (X3)}\\
&= (x \mathbin{\bullet\!\!\oplus} \mathsf{T}) \mathbin{\bullet\!\!\oplus} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) && \text{by (X2)}\\
&= \neg x \mathbin{\bullet\!\!\oplus} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) && \text{by (AX1)}\\
&= (\neg x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}^{\circ}\!\vee} \mathsf{T})) \mathbin{{}^{\circ}\!\vee} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) && \text{by (AX3)}
\end{aligned}$$

Proof of EqFLXSCL $\vdash$ (Eq4):

$$\begin{aligned}
x \mathbin{\bullet\!\!\oplus} (y \mathbin{{}^{\circ}\!\vee} \mathsf{T}) &= x \mathbin{\bullet\!\!\oplus} ((y \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \mathbin{\bullet\!\!\oplus} \mathsf{T}) && \text{by the proof of (Eq3)}\\
&= (x \mathbin{\bullet\!\!\oplus} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) \mathbin{\bullet\!\!\oplus} \mathsf{T} && \text{by (X2)}\\
&= \neg(x \mathbin{\bullet\!\!\oplus} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) && \text{by (AX1)}
\end{aligned}$$

$\square$

Finally, we finish the proof of Lemma 5.1.6 on page 46.

Lemma 5.1.6. *For all terms t, u over $\Sigma_{\mathrm{LXSCL}}(A)$,*

$$\mathrm{EqFLXSCL} \vdash t = u \implies \mathbb{M}' \vDash t = u.$$

Proof. We show that the model $\mathbb{M}'$ that was defined in Definition 5.1.5 satisfies axioms (AX2) and (AX3). Fix an arbitrary interpretation i of variables.

Proof of $\mathbb{M}', i \vDash$ (AX2):

$$\begin{aligned}
&\llbracket ((x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} y) \mathbin{\bullet\!\oplus} z \rrbracket^{\mathbb{M}',i} \\
&\quad = \llbracket x \mathbin{{}^{\circ}\!\vee} \mathsf{T} \rrbracket^{\mathbb{M}',i}\big[\mathsf{T} \mapsto \llbracket y \rrbracket^{\mathbb{M}',i}\big]\big[\mathsf{T} \mapsto \llbracket z \rrbracket^{\mathbb{M}',i}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \llbracket z \rrbracket^{\mathbb{M}',i}\big] && \text{by 5.1.5} \\
&\quad = \llbracket x \mathbin{{}^{\circ}\!\vee} \mathsf{T} \rrbracket^{\mathbb{M}',i}\Big[\mathsf{T} \mapsto \llbracket y \rrbracket^{\mathbb{M}',i}\big[\mathsf{T} \mapsto \llbracket z \rrbracket^{\mathbb{M}',i}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \llbracket z \rrbracket^{\mathbb{M}',i}\big], \\
&\qquad\qquad \mathsf{F} \mapsto \mathsf{F}\big[\mathsf{T} \mapsto \llbracket z \rrbracket^{\mathbb{M}',i}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \llbracket z \rrbracket^{\mathbb{M}',i}\big]\Big] && \text{by (6), (7)} \\
&\quad = \llbracket x \mathbin{{}^{\circ}\!\vee} \mathsf{T} \rrbracket^{\mathbb{M}',i}\big[\mathsf{T} \mapsto \llbracket y \mathbin{\bullet\!\oplus} z \rrbracket^{\mathbb{M}',i}, \mathsf{F} \mapsto \llbracket z \rrbracket^{\mathbb{M}',i}\big] && \text{by 5.1.5, (4)} \\
&\quad = \llbracket x \mathbin{{}^{\circ}\!\vee} \mathsf{T} \rrbracket^{\mathbb{M}',i}\big[\mathsf{T} \mapsto \llbracket y \mathbin{\bullet\!\oplus} z \rrbracket^{\mathbb{M}',i}\big]
\end{aligned}$$

because $\llbracket x \mathbin{{}^{\circ}\!\vee} \mathsf{T} \rrbracket^{\mathbb{M}',i}$ has no F-leaves,

$$= \llbracket (x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} (y \mathbin{\bullet\!\oplus} z) \rrbracket^{\mathbb{M}',i} \qquad \text{by 5.1.5}$$

Proof of $\mathbb{M}', i \vDash$ (AX3):

$$\begin{aligned}
&\llbracket x \mathbin{\bullet\!\oplus} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \rrbracket^{\mathbb{M}',i} \\
&\quad = \llbracket x \rrbracket^{\mathbb{M}',i}\big[\mathsf{T} \mapsto \llbracket y \rrbracket^{\mathbb{M}',i}[\mathsf{T} \mapsto \llbracket \mathsf{F} \rrbracket^{\mathbb{M}'}][\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \llbracket y \mathbin{{}_{\circ}\!\wedge} \mathsf{F} \rrbracket^{\mathbb{M}',i}\big] && \text{by 5.1.5} \\
&\quad = \llbracket x \rrbracket^{\mathbb{M}',i}\big[\mathsf{T} \mapsto \llbracket y \rrbracket^{\mathbb{M}',i}[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{F}][\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \llbracket y \mathbin{{}_{\circ}\!\wedge} \mathsf{F} \rrbracket^{\mathbb{M}',i}\big] && \text{by (6)} \\
&\quad = \llbracket x \rrbracket^{\mathbb{M}',i}\big[\mathsf{T} \mapsto \llbracket y \rrbracket^{\mathbb{M}',i}[\mathsf{T} \mapsto \mathsf{T}, \mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \llbracket y \mathbin{{}_{\circ}\!\wedge} \mathsf{F} \rrbracket^{\mathbb{M}',i}\big] && \text{by (3), (4), (7)} \\
&\quad = \llbracket x \rrbracket^{\mathbb{M}',i}\big[\mathsf{T} \mapsto \llbracket y \rrbracket^{\mathbb{M}',i}[\mathsf{F} \mapsto \mathsf{T}][\mathsf{F} \mapsto \llbracket y \mathbin{{}_{\circ}\!\wedge} \mathsf{F} \rrbracket^{\mathbb{M}',i}], \mathsf{F} \mapsto \mathsf{F}[\mathsf{F} \mapsto \llbracket y \mathbin{{}_{\circ}\!\wedge} \mathsf{F} \rrbracket^{\mathbb{M}',i}]\big]
\end{aligned}$$

because $\llbracket y \rrbracket^{\mathbb{M}',i}[\mathsf{T} \mapsto \mathsf{T}, \mathsf{F} \mapsto \mathsf{T}]$ has only T-leaves and by (4) and (6),

$$\begin{aligned}
&= \llbracket x \rrbracket^{\mathbb{M}',i}\big[\mathsf{T} \mapsto \llbracket y \rrbracket^{\mathbb{M}',i}[\mathsf{F} \mapsto \mathsf{T}], \mathsf{F} \mapsto \mathsf{F}\big]\big[\mathsf{F} \mapsto \llbracket y \mathbin{{}_{\circ}\!\wedge} \mathsf{F} \rrbracket^{\mathbb{M}',i}\big] && \text{by (7)} \\
&= \llbracket (x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}^{\circ}\!\vee} \mathsf{T})) \mathbin{{}^{\circ}\!\vee} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \rrbracket^{\mathbb{M}',i} && \text{by 5.1.5, (6)}
\end{aligned}$$

Because i was chosen arbitrarily, it follows that $\mathbb{M}'$ satisfies (AX2) and (AX3). $\square$