

Bounded Symbiosis and Upwards Reflection

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Abstract

In [2], Bagaria and Väänänen developed a framework for studying the large cardinal strength of *downwards* Löwenheim-Skolem theorems and related set theoretic reflection properties. The main tool was the notion of *symbiosis*, originally introduced by the third author in [12, 13].

Symbiosis provides a way of relating model theoretic properties of strong logics to definability in set theory. In this paper we continue the systematic investigation of *symbiosis* and apply it to *upwards* Löwenheim-Skolem theorems and reflection principles. To achieve this, we need to adapt the notion of *symbiosis* to a new form, called *bounded symbiosis*. As one easy application, we obtain upper and lower bounds for the large cardinal strength of upwards Löwenheim-Skolem-type principles for second order logic.

1 Introduction

Mathematicians have two ways of characterizing a class \mathcal{C} of mathematical structures: *defining* the class in set theory, or *axiomatizing* the class by sentences in logic. Symbolically:

1. $\Phi(\mathcal{A})$, where Φ is a formula in the language of set theory, vs.
2. $\mathcal{A} \models \varphi$, where φ is a sentence in some logic.

In general, set theory is much more powerful than first order logic.

However, by *restricting* the allowed complexity of Φ on one hand, while considering *extensions* of first-order logic on the other hand, one gets a more interesting picture. *Symbiosis* aims to capture an equivalence in strength between set-theoretic definability and model-theoretic axiomatisability. One application of this is connecting properties of some strong logic \mathcal{L}^* to specific set-theoretic principles (often expressed in terms of large cardinals). *Symbiosis* was first introduced by the third author in [13], and studied further in [9, 1, 2].

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If \mathcal{A} is a structure and ϕ a first-order formula, then the statement “ $\mathcal{A} \models \phi$ ” is Δ_1 in set theory. Therefore every first-order axiomatizable class of structures, i.e., every class of the form $\text{Mod}(\phi) = \{\mathcal{A} : \mathcal{A} \models \phi\}$, is Δ_1 -definable.

The converse does not hold: for example, the class of all well-ordered structures is easily seen to be Δ_1 -definable but not first-order axiomatizable. So it is natural to look for a logic \mathcal{L}^* extending first order logic, with the property that every Δ_1 -definable class would be axiomatizable by an \mathcal{L}^* -sentence

Consider the logic $\mathcal{L}_I = \mathcal{L}_{\omega\omega}(I)$ obtained from first order logic $\mathcal{L}_{\omega\omega}$ by adding the *Härtig quantifier* I , defined by

$$\mathcal{A} \models Ixy \phi(x)\psi(y) \text{ iff } |\{a \in A : \mathcal{A} \models \phi[a]\}| = |\{b \in A : \mathcal{A} \models \psi[b]\}|$$

and consider its closure under the so-called $\underline{\Delta}$ -operator (Definition 3.1).¹ We then obtain a logic, which we will call $\underline{\Delta}(\mathcal{L}_I)$, such that every Δ_1 -definable class, if closed under isomorphisms, is $\underline{\Delta}(\mathcal{L}_I)$ -axiomatisable (see Proposition 3.5 or [13, Example 2.3]).

However, $\underline{\Delta}(\mathcal{L}_I)$ -axiomatisability is now too strong to be “symbiotic” with Δ_1 -definability: the class

$$\{(A, P) : |\{x \in A : P(x)\}| = |\{x \in A : \neg P(x)\}|\}$$

is not Δ_1 (it is not absolute), but it is axiomatisable in \mathcal{L}_I by the sentence

$$Ixy(P(x))(\neg P(y)).$$

One can observe that all $\underline{\Delta}(\mathcal{L}_I)$ -axiomatisable classes are Δ_2 -definable, but once more, there are Δ_2 -definable classes that are not $\underline{\Delta}(\mathcal{L}_I)$ -axiomatisable (see Figure 1).

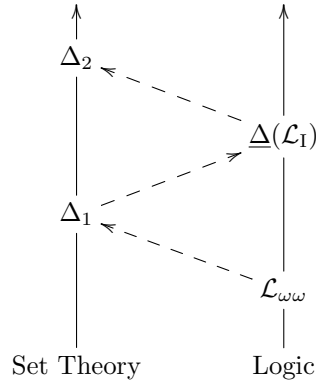


Figure 1: Set-theoretic definability vs. axiomatization in a logic

Interesting *symbiosis* relationships take place for complexity levels $\Delta_1(R)$, for fixed predicates R . In this paper, we will focus on Π_1 predicates R , so the complexity levels will lie below Δ_2 . Many such relations have been established

¹Usually the symbol used here is a simple Δ , but in this paper we choose the symbol $\underline{\Delta}$, and similarly $\underline{\Sigma}$, in order to easily distinguish the model-theoretic notions from the Lévy complexity of formulas in the language of set theory, i.e., Σ_n and Δ_n formulas.

in [13, 2]. To name some prominent examples, let \mathcal{L}^2 be second order logic with full semantics, and let \mathcal{L}_{WF} be the logic obtained from $\mathcal{L}_{\omega\omega}$ by adding the generalized quantifier WF defined by

$$\mathcal{A} \models \text{WF}xy \phi(x, y) \text{ iff } \{(x, y) \in A \times A : \mathcal{A} \models \phi(x, y)\} \text{ is well-founded.}$$

Furthermore, let $\text{Cd}(x)$ be the Π_1 predicate “ x is a cardinal”, and let $\text{PwSt}(x, y)$ be the Π_1 predicate “ $y = \wp(x)$ ”. Then we have the *symbiosis* relationships depicted in Figure 2 (see Propositions 3.4, 3.5 and 3.6).

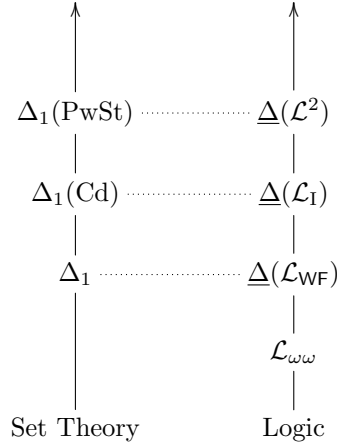


Figure 2: Symbiosis relations

As an application of *symbiosis*, Bagaria and Väänänen [2] considered the following principles:

Definition 1.1. The *downward Löwenheim-Skolem-Tarski number* $\text{LST}(\mathcal{L}^*)$ is the smallest cardinal κ such that for all $\phi \in \mathcal{L}^*$, if $\mathcal{A} \models_{\mathcal{L}^*} \phi$ then there exists a substructure $\mathcal{B} \subseteq \mathcal{A}$ such that $|B| \leq \kappa$ and $\mathcal{B} \models_{\mathcal{L}^*} \phi$. If such a κ does not exist, $\text{LST}(\mathcal{L}^*)$ is undefined.

Definition 1.2. Let R be a predicate in the language of set theory. The *structural reflection number* $\mathcal{SR}(R)$ is the smallest cardinal κ such that for every $\Sigma_1(R)$ -definable class \mathcal{K} of models in a fixed vocabulary, for every $\mathcal{A} \in \mathcal{K}$ there exists $\mathcal{B} \in \mathcal{K}$ with $|B| \leq \kappa$ and a first-order elementary embedding $e : \mathcal{B} \preceq \mathcal{A}$. If such a κ does not exist, $\mathcal{SR}(R)$ is undefined.

Theorem 1.3 (Bagaria & Väänänen [2]). *Suppose \mathcal{L}^* and R are symbiotic. Then $\text{LST}(\mathcal{L}^*) = \kappa$ if and only if $\mathcal{SR}(R) = \kappa$.*

Proof. See [2, Theorem 6]. □

Theorem 1.3 links a meta-logical property of a strong logic to a reflection principle in set theory. Depending on the predicate R , the principle $\mathcal{SR}(R)$ has a varying degree of large cardinal strength. In fact, Definition 1.2 may be regarded as a kind of Vopěnka principle, restricted to classes of limited complexity.²

²See [1, Sections 3, 4] for more on the connection between $\mathcal{SR}(R)$ and Vopěnka-type principles.

Indeed, in [2] Theorem 1.3 was used to compute the large cardinal strength of $\mathcal{SR}(R)$ and $\text{LST}(\mathcal{L}^*)$ for various symbiotic pairs.

In this paper we continue the work of Bagaria and Väänänen by developing a framework for the study of *upward* Löwenheim-Skolem and reflection principles. These principles are also interesting because they are closely related to the *compactness* of the strong logic, although the two notions are not equivalent, and in this paper we do not consider compactness explicitly. The main innovation of the current work is that, in order to deal with upwards rather than downwards reflection, we need to adapt the notion of *symbiosis*.

The paper is organized as follows: in Section 2 we introduce the necessary terminology and some background and in Section 3 we present the notion of *symbiosis*. In Section 4 we introduce *bounded symbiosis*. Section 5 is devoted to examples of *bounded symbiosis*, and in Section 6 we prove the main theorem, showing that under appropriate conditions the *upward* Löwenheim-Skolem number corresponds to a suitable upwards set-theoretic reflection principle. Finally, in Section 7 we apply our results to compute upper and lower bounds for the upward Löwenheim-Skolem number of second order logic and the corresponding reflection principle, noting that this also provides an upper bound for all other Π_1 predicates.

This paper contains research carried out by the first author as part of his PhD Dissertation. Some details have been left out of the paper for the sake of easier readability and navigation. The interested reader may find these details in [5, Chapter 6].

2 Abstract Logics

We assume that the reader is familiar with standard set theoretic and model theoretic notation and terminology. We will consider abstract logics \mathcal{L}^* , without providing a precise definition for what counts as a “logic”. Typical examples are infinitary logics $\mathcal{L}_{\kappa\lambda}$, full second-order logic \mathcal{L}^2 , and various extensions of first-order logic by *generalized quantifiers*. For a more detailed analysis we refer the reader to [5, Chapter 6] and [3]. Here we only want to stress two important points.

First, we will generally work with *many-sorted* languages, using the symbols s_0, s_1, \dots to denote *sorts*. In this setting, a *domain* may be a collection of domains (one for each sort), and all constant, relation and function symbols must have their sort specified in advance. This is a matter of convenience, since many-sorted logic can be simulated by standard single-sorted logic by introducing additional predicate symbols. The following definition is essential for what follows.

Definition 2.1. Suppose that $\tau \subseteq \tau'$ are many-sorted vocabularies and that \mathcal{M} is a τ' -structure. The *reduct* (or *projection*) of \mathcal{M} to τ , denoted by $\mathcal{M}|_\tau$, is the structure whose domains are those domains in the sorts available in τ , and the interpretations of all symbols not in τ are ignored.

So a reduct $\mathcal{M} \upharpoonright \tau$ can have a smaller domain, and a smaller cardinality, than the original model \mathcal{M} .

Secondly, we should note that one needs to be careful with the syntax of a given logic, because an unrestricted use of syntax may give rise to some undesirable effects. Consider, for example, an arbitrary set $X \subseteq \omega$, and a vocabulary τ which has ω -many relation symbols $\{R_i : i < \omega\}$, such that the arity of R_i is 1 if $i \in X$ and 2 if $i \notin X$. The information about arities of relation symbols must be encoded in the vocabulary τ . Therefore, the set X can be computed from τ .

In infinitary logic, we can encode a set $X \subseteq \omega$ even with finite vocabularies. Let ϕ_0, ϕ_1, \dots be a recursive enumeration of $\mathcal{L}_{\omega\omega}$ -sentences in some fixed τ , and consider the $\mathcal{L}_{\omega_1\omega}$ -sentence $\Phi := \bigwedge_{n \in X} \phi_n$. Then X can be computed from Φ .

Definition 2.2. Let \mathcal{L}^* be a logic. The *dependence number* of \mathcal{L}^* , $\text{dep}(\mathcal{L}^*)$, is the least λ such that for any vocabulary τ and any \mathcal{L}^* -formula ϕ in τ , there is a sub-vocabulary $\sigma \subseteq \tau$ such that $|\sigma| < \lambda$ and ϕ only uses symbols in σ . If such a number does not exist, $\text{dep}(\mathcal{L}^*)$ is undefined.

Definition 2.3. We say that a logic \mathcal{L}^* has *Δ_0 -definable syntax* if every \mathcal{L}^* -formula is Δ_0 -definable in set theory (as a syntactic object), possibly with the vocabulary of ϕ as parameter.

In our main Theorem 6.3, we will restrict attention to logics with a Δ_0 -definable syntax and $\text{dep}(\mathcal{L}^*) = \omega$. Note that this includes all finitary logics obtained by adding finitely many generalized quantifiers to first- or second-order logic.

We end this section by defining a version of the upward Löwenheim-Skolem number for abstract logics.

Definition 2.4 (Upward Löwenheim-Skolem number). Let \mathcal{L}^* be a logic.

1. The *upward Löwenheim-Skolem number of \mathcal{L}^* for $<\lambda$ -vocabularies*, denoted by $\text{ULST}_\lambda(\mathcal{L}^*)$, is the smallest cardinal κ such that

for every vocabulary τ with $|\tau| < \lambda$ and every ϕ in $\mathcal{L}^*[\tau]$, if there is a model $\mathcal{A} \models \phi$ with $|\mathcal{A}| \geq \kappa$, then for every $\kappa' > \kappa$, there is a model $\mathcal{B} \models \phi$ such that $|\mathcal{B}| \geq \kappa'$ and \mathcal{A} is a substructure of \mathcal{B} .

As usual, if there is no such cardinal then $\text{ULST}_\lambda(\mathcal{L}^*)$ is undefined.

2. The *upward Löwenheim-Skolem number of \mathcal{L}^** , denoted by $\text{ULST}_\infty(\mathcal{L}^*)$ is the smallest cardinal κ such that $\text{ULST}_\lambda(\mathcal{L}^*) \leq \kappa$ for all cardinals λ . Again, if there is no such cardinal then $\text{ULST}_\infty(\mathcal{L}^*)$ is undefined.

Notice that when $\text{dep}(\mathcal{L}^*) = \lambda$, then $\text{ULST}_\lambda(\mathcal{L}^*) = \kappa$ implies $\text{ULST}_\infty(\mathcal{L}^*) = \kappa$. In general, $\text{ULST}_\infty(\mathcal{L}^*)$ may fail to be defined even if all $\text{ULST}_\lambda(\mathcal{L}^*) = \kappa$ are defined.

Recall also that the *Hanf-number* of a logic is defined analogously to Definition 2.4 but without the assumption that \mathcal{A} is a substructure of \mathcal{B} . This additional assumption is rather crucial: it is easy to see that if the dependence number of a logic is defined, then the Hanf number is also defined (see [3, Theorem 6.4.1]). However, as we shall see in Section 7, the existence of upward Löwenheim-Skolem numbers in the sense of Definition 2.4, even for logics with dependence number ω , implies the existence of large cardinals.

3 Symbiosis

Symbiosis was introduced by the third author in [13]. To motivate its definition, let \mathcal{L}^* be a logic and R a predicate in set theory. The aim is to establish an equality in strength between \mathcal{L}^* -axiomatizability and $\Delta_1(R)$ -definability. One direction should be the statement “the satisfaction relation $\models_{\mathcal{L}^*}$ is $\Delta_1(R)$ -definable”, or, equivalently, “every \mathcal{L}^* -axiomatizable class of structures is $\Delta_1(R)$ -definable.”

The converse direction should say, roughly speaking, that every $\Delta_1(R)$ -definable class is \mathcal{L}^* -axiomatizable. This cannot literally work, because \mathcal{L}^* -model classes are closed under isomorphisms whereas this is not necessarily true for arbitrary $\Delta_1(R)$ classes. Therefore we try the approach “every $\Delta_1(R)$ -definable class closed under isomorphisms is \mathcal{L}^* -axiomatizable.”

Unfortunately, this does not always work: *symbiosis* can only be established for logics that are closed under the $\underline{\Delta}$ -operation. This operation closes the logic under operations which are in a sense “simple” but not as simple as mere conjunction, negation, and whatever operations the logic has. In order to define the property of a structure \mathcal{A} being in a model class based merely on the knowledge that the class is $\Delta_1(R)$, the only way seems to be to use the means of the logic to build a piece of the set theoretic universe around \mathcal{A} , and work in the small universe. The $\underline{\Delta}$ -operation is then used to eliminate the extra symbols used to build the small universe. See [2, 13, 11] for more details on the $\underline{\Delta}$ -operation and its use.

Definition 3.1. Let \mathcal{L}^* be a logic and let τ a fixed vocabulary. A class \mathcal{K} of τ -structures is $\underline{\Sigma}(\mathcal{L}^*)$ -axiomatisable if there exists ϕ in some extended vocabulary $\tau' \supseteq \tau$ such that

$$\mathcal{K} = \{\mathcal{A} : \exists \mathcal{B} (\mathcal{B} \models \phi \text{ and } \mathcal{A} = \mathcal{B} \upharpoonright \tau)\}.$$

We say that \mathcal{K} is the *projection* of the class $\text{Mod}(\phi)$ to τ .

A class \mathcal{K} is $\underline{\Pi}(\mathcal{L}^*)$ -axiomatisable if the complement of \mathcal{K} (i.e., the class of τ -structures not in \mathcal{K}) is $\underline{\Sigma}(\mathcal{L}^*)$ -axiomatizable, and $\underline{\Delta}(\mathcal{L}^*)$ -axiomatizable if it is both $\underline{\Sigma}(\mathcal{L}^*)$ and $\underline{\Pi}(\mathcal{L}^*)$ -axiomatizable.

Note that, if τ' has more sorts than τ , then the structures \mathcal{B} can be larger than their reducts $\mathcal{A} = \mathcal{B} \upharpoonright \tau$.

Since $\underline{\Delta}(\mathcal{L}^*)$ -axiomatizable classes are closed under unions, intersections, complements and projections, one could consider $\underline{\Delta}(\mathcal{L}^*)$ itself as an abstract logic, whose model classes are exactly the $\underline{\Delta}(\mathcal{L}^*)$ -axiomatizable classes. In general, $\underline{\Delta}(\mathcal{L}^*)$ is a non-trivial extension of \mathcal{L}^* . However, for first-order logic, and in general any logic satisfying the *Craig Interpolation Theorem*, two notions coincide, see [11, Lemma 2.7].

Definition 3.2 (Symbiosis). Let \mathcal{L}^* be a logic and R a predicate in the language of set theory. Then we say that \mathcal{L}^* and R are *symbiotic* if:

- (1) the relation $\models_{\mathcal{L}^*}$ is $\Delta_1(R)$ -definable, and
- (2) for every finite vocabulary τ , every $\Delta_1(R)$ -definable class of τ -structures closed under isomorphisms is $\underline{\Delta}(\mathcal{L}^*)$ -axiomatisable.

In [2], *symbiosis* was established for many logic-predicate pairs, among them the ones mentioned in the introduction.

In practice, there is an equivalent condition to (2) which is easier to both verify and to apply. Let R be an n -ary predicate in the language of set theory. We say that a transitive model of set theory M is R -correct if for all $m_1, \dots, m_n \in M$ we have $M \models R(m_1, \dots, m_n)$ iff $R(m_1, \dots, m_n)$.

Lemma 3.3 (Väänänen [13]). *For any predicate R and logic \mathcal{L}^* , the following are equivalent:*

- (a) *For every finite τ , every $\Delta_1(R)$ class of τ -structures closed under isomorphisms is $\underline{\Delta}(\mathcal{L}^*)$ -axiomatisable.*
- (b) *The class $\mathcal{Q}_R := \{\mathcal{A} : \mathcal{A} \text{ is isomorphic to a transitive } R\text{-correct } \in\text{-model}\}$ is $\underline{\Delta}(\mathcal{L}^*)$ -axiomatisable.*

Proof. See [13], or a simpler version of Theorem 4.9. □

For completeness, and to illustrate how proofs of *symbiosis* typically work in view of the results in the next section, we will now sketch proofs of some paradigmatic examples (see Section 1 for the definitions). ZFC^{-*} refers to a sufficiently large fragment of ZFC –Power Set.

Proposition 3.4 (Väänänen [13]). *\mathcal{L}_{WF} and \emptyset (no predicates) are symbiotic.*

Proof. (1) Since the statement “ (A, E) is well-founded” for sets A is Δ_1 and therefore absolute for transitive models, “ $\mathcal{A} \models_{\mathcal{L}_{\text{WF}}} \phi$ ” is also absolute for transitive models. Then $\mathcal{A} \models_{\mathcal{L}_{\text{WF}}} \phi$

$$\begin{aligned} & \text{iff } \exists M (M \text{ transitive} \wedge M \models \text{ZFC}^{-*} \wedge \mathcal{A} \in M \wedge M \models (\mathcal{A} \models_{\mathcal{L}_{\text{WF}}} \phi)) \\ & \text{iff } \forall M ((M \text{ transitive} \wedge M \models \text{ZFC}^{-*} \wedge \mathcal{A} \in M) \rightarrow M \models (\mathcal{A} \models_{\mathcal{L}_{\text{WF}}} \phi)). \end{aligned}$$

This gives a Δ_1 -definition.

- (2) There are no predicates so $\mathcal{Q}_{\emptyset} = \{(A, E) : (A, E) \text{ is isomorphic to a transitive } \in\text{-model}\}$. But (A, E) is isomorphic to a transitive \in -model iff E is well-founded and extensional. Therefore $(A, E) \in \mathcal{Q}_{\emptyset}$ iff

$$(A, E) \models \text{EXT} \wedge \text{WF}xy(xEy)$$

which is an \mathcal{L}_{WF} -sentence. Thus \mathcal{Q}_{\emptyset} is \mathcal{L}_{WF} -axiomatisable and therefore also $\underline{\Delta}(\mathcal{L}_{\text{WF}})$ -axiomatisable. □

Proposition 3.5 (Väänänen [13]). *\mathcal{L}_1 and Cd are symbiotic.*

Proof.

- (1) It is easy to see that “ $\mathcal{A} \models_{\mathcal{L}_1} \phi$ ” is absolute for models of set theory which are Cd-correct. Therefore $\mathcal{A} \models_{\mathcal{L}_1} \phi$

$$\begin{aligned} & \text{iff } \exists M (M \text{ trans. and Cd-correct} \wedge M \models \text{ZFC}^{-*} \wedge \mathcal{A} \in M \wedge M \models (\mathcal{A} \models_{\mathcal{L}_1} \phi)) \\ & \text{iff } \forall M ((M \text{ trans. and Cd-correct} \wedge M \models \text{ZFC}^{-*} \wedge \mathcal{A} \in M) \rightarrow \\ & M \models (\mathcal{A} \models_{\mathcal{L}_1} \phi)) \end{aligned}$$

Note that “ M is Cd-correct” is the statement $\forall x \in M ((M \models \text{Cd}(x)) \leftrightarrow \text{Cd}(x))$ which is $\Delta_1(\text{Cd})$. Thus the above is a $\Delta_1(\text{Cd})$ formula.

(2) We need to check that $\mathcal{Q}_{\text{Cd}} = \{(A, E) : (A, E) \text{ is isomorphic to a transitive Cd-correct } \in\text{-model}\}$ is $\underline{\Delta}(\mathcal{L}_I)$. We have:

$(A, E) \in \mathcal{Q}_{\text{Cd}}$ iff

- (a) E is wellfounded
- (b) $(A, E) \models \text{EXT}$
- (c) $(A, E) \models_{\mathcal{L}_I} \text{“}\forall\alpha (\text{Cd}(\alpha) \rightarrow \forall x \in \alpha \neg \text{Iyz}(y \in x)(y \in \alpha))\text{”}$ (written in E instead of \in).

Conditions (b) and (c) are \mathcal{L}_I -sentences, so it remains to show that (a) is $\underline{\Delta}(\mathcal{L}_I)$.

First, we add a new unary predicate symbol P and consider the sentence “ P has no E -least element”, i.e.,

$$\phi \equiv \forall x(P(x) \rightarrow \exists y(P(y) \wedge yEx)).$$

Clearly the class of all models (A, E) such that E is *not* well-founded is the projection of $\text{Mod}(\phi)$ to $\{E\}$. Therefore (a) is $\underline{\Pi}(\mathcal{L}_I)$.

To show that (a) is also $\underline{\Sigma}(\mathcal{L}_I)$ we use a trick due to Per Lindström [7]: (A, E) is well-founded if and only if we can associate sets X_a to every $a \in A$ in such a way that $aEb \rightarrow |X_a| < |X_b|$. Let the original sort be called s_0 , extend the language with a second sort s_1 , add a new binary relation symbol R from s_0 to s_1 , and consider the sentence

$$\phi \equiv \forall^0 a \forall^0 b (aEb \rightarrow (\forall^1 x(R(a, x) \rightarrow R(b, x)) \wedge \neg \text{Iyz}R(a, y)R(b, z)))$$

where we have used \forall^0 and \forall^1 to denote quantification over the two sorts.

Now we can easily see that if $\mathcal{A} = (A, X, E^{\mathcal{A}}, R^{\mathcal{A}}) \models \phi$, then the sets $X_a := \{x \in X : R^{\mathcal{A}}(a, x)\}$ are exactly as required, hence $E^{\mathcal{A}}$ is well-founded. Conversely, if $(A, E^{\mathcal{A}})$ is well-founded, let $\text{rk}_E : A \rightarrow \text{Ord}$ be the rank function induced by $E^{\mathcal{A}}$, let $X := \aleph_{\sup_{a \in A} \{\text{rk}_E(a)+1\}}$, and define $R^{\mathcal{A}} \subseteq A \times X$ by $R^{\mathcal{A}}(a, \alpha) \Leftrightarrow \alpha < \aleph_{\text{rk}_E(a)}$. Then $(A, X, E^{\mathcal{A}}, R^{\mathcal{A}}) \models \phi$.

Thus we conclude that the class of well-founded structures is the projection of $\text{Mod}(\phi)$ to s_0 , completing the proof. \square

Proposition 3.6 (Väänänen [13]). \mathcal{L}^2 and PwSt are symbiotic.

Proof. (1) As before, note that the statement “ $\mathcal{A} \models_{\mathcal{L}^2} \phi$ ” is absolute for transitive models which are PwSt-correct, so we have $\mathcal{A} \models_{\mathcal{L}^2} \phi$

$$\text{iff } \exists M (M \text{ trans. and PwSt-correct} \wedge M \models \text{ZFC}^{-*} \wedge \mathcal{A} \in M \wedge M \models (\mathcal{A} \models_{\mathcal{L}_I} \phi))$$

$$\text{iff } \forall M ((M \text{ trans. and PwSt-correct} \wedge M \models \text{ZFC}^{-*} \wedge \mathcal{A} \in M) \rightarrow M \models (\mathcal{A} \models_{\mathcal{L}_I} \phi))$$

- (2) Consider $\mathcal{Q}_{\text{PwSt}} = \{(A, E) : (A, E) \text{ is isomorphic to a PwSt-correct } \in\text{-model}\}$. We have $(A, E) \in \mathcal{Q}_{\text{PwSt}}$ iff E is wellfounded and extensional and $(A, E) \models_{\mathcal{L}^2} (y = \wp(x) \leftrightarrow \Phi(x, y))$ where $\Phi(x, y)$ is the \mathcal{L}^2 -formula expressing that y is the *true* power set of x , written using E instead of \in .³ All of this is expressible in \mathcal{L}^2 . \square

4 Bounded Symbiosis

Although *symbiosis* is stated as a property of \mathcal{L}^* , it is really a property of $\underline{\Delta}(\mathcal{L}^*)$. For many applications, this is irrelevant: for example, the *downwards* Löwenheim-Skolem principles are all preserved by the $\underline{\Delta}$ -operation. However, in [14, Theorem 4.1] it was shown that the Hanf-number may not be preserved, and the *bounded* $\underline{\Delta}$ -operation was introduced as a closely related operation which still fulfills most of the properties but, in addition, preserves Hanf-numbers. The bounded $\underline{\Delta}$ coincides with the original $\underline{\Delta}$ in many but not all cases, see [14].

If we want to apply *symbiosis* to upwards Löwenheim-Skolem principles, we need to accommodate this bounded version of the $\underline{\Delta}$ -operation. Unfortunately, this also requires adapting the set-theoretic complexity classes to bounded versions. This section is devoted to the definition of these concepts. We start with the model-theoretic side. The following definition generalizes Definition 3.1 and was first introduced in [14, p. 45].

Definition 4.1. A class \mathcal{K} of τ -structures is $\underline{\Sigma}^{\text{B}}(\mathcal{L}^*)$ -*axiomatisable* if there exists ϕ in an extended vocabulary $\tau' \supseteq \tau$ such that

$$\mathcal{K} = \{\mathcal{A} : \exists \mathcal{B} (\mathcal{B} \models \phi \text{ and } \mathcal{A} = \mathcal{B} \upharpoonright \tau)\},$$

and for all \mathcal{A} there exists a cardinal $\lambda_{\mathcal{A}}$, such that for any τ' -structure \mathcal{B} : if $\mathcal{B} \models \phi$ and $\mathcal{A} = \mathcal{B} \upharpoonright \tau$ then $|\mathcal{B}| \leq \lambda_{\mathcal{A}}$.

We say that \mathcal{K} is a *bounded projection* of $\text{Mod}(\phi)$. \mathcal{K} is $\underline{\Delta}^{\text{B}}(\mathcal{L}^*)$ -*axiomatisable* if both \mathcal{K} and its complement are $\underline{\Sigma}^{\text{B}}(\mathcal{L}^*)$ -axiomatisable.

In other words, \mathcal{K} is a bounded projection of $\text{Mod}(\phi)$ if it is a projection and, in addition, every structure $\mathcal{B} \in \text{Mod}(\phi)$ is bounded in its cardinality by a function that depends on the respective reduct $\mathcal{B} \upharpoonright \tau$. Note that this definition really only plays a role when the extended vocabulary has additional sorts, since otherwise the cardinalities of $\mathcal{B} \upharpoonright \tau$ and \mathcal{B} are the same.

Typical examples of bounded projection will be seen, e.g., in Propositions 5.1, 5.2 and 5.6.

It will be useful to define a bound given by a function from ordinals to ordinals rather than models to ordinals.

³If we use superscripts 0 and 1 to denote first- and second-order quantification, and the relation symbols \in^{00} and \in^{01} to denote sets-in-sets membership and sets-in-classes membership, respectively, the sentence $\Phi(x, y)$ can be written as follows:

$$\forall^1 Z (\exists^0 v (v \in^{00} y \wedge \forall^0 w (w \in^{00} v \leftrightarrow w \in^{01} Z)) \leftrightarrow \forall^0 v (v \in^{01} Z \rightarrow v \in^{00} y))$$

Lemma 4.2. *Suppose \mathcal{K} is $\underline{\Sigma}^B(\mathcal{L}^*)$ -axiomatisable. Then there exists ϕ in an extended vocabulary $\tau' \supseteq \tau$, and a non-decreasing function $h : \text{Ord} \rightarrow \text{Ord}$ such that*

$$\mathcal{K} = \{\mathcal{A} : \exists \mathcal{B} (\mathcal{B} \models \phi \text{ and } \mathcal{A} = \mathcal{B} \upharpoonright \tau)\}$$

and

$$\forall \mathcal{B} (\mathcal{B} \models \phi \rightarrow |\mathcal{B}| \leq h(|\mathcal{B} \upharpoonright \tau|)).$$

Proof. Define h by $h(\lambda) := \sup\{\lambda_{\mathcal{A}} : |\mathcal{A}| \leq \lambda\}$ where each $\lambda_{\mathcal{A}}$ is as in Definitions 4.1. Since there are only set-many non-isomorphic models of any cardinality, h is well-defined. \square

Now we move to the set-theoretic side of things, which is more involved. In particular, we may no longer refer to arbitrary Σ_1 -formulas, since the witness in such formula may be unbounded, making it impossible to establish a *symbiotic* relationship for bounded projective classes. So we would like to restrict attention to formulas $\phi(x)$ of the form $\exists y \psi(x, y)$ but where, in addition, the (hereditary) size of at least one witness y is bounded by a function of the (hereditary) size of x , and this function itself can be “captured” by first-order logic. Note that a similar concept was introduced by the third author in [13, Definition 3.1].

We first need to introduce the concept of “being captured by first-order logic”.

Definition 4.3. A non-decreasing class function $F : \text{Card} \rightarrow \text{Card}$ is called *definably bounding* if the class of structures

$$\mathcal{K} := \{(A, B) : |B| \leq F(|A|)\}$$

(in the vocabulary with two sorts and no symbols) is $\underline{\Sigma}^B(\mathcal{L}_{\omega\omega})$ -axiomatisable.

The intuition here is that the size of $|B|$ (the “witness”) may be larger than $|A|$, but not by too much — and by exactly how much is determined by F . For example, the identity function $F = \text{id}$ is definably bounding since we can always extend the language with a new function symbol f between the two sorts and express “ f is a surjection” in $\mathcal{L}_{\omega\omega}$. A more interesting example is the following:

Example 4.4. The function $F(\kappa) = 2^\kappa$ is definably bounding.

Proof. Consider the class $\mathcal{K} := \{(A, B) : |B| \leq 2^{|A|}\}$. Extend the vocabulary with a new relation symbol E between the two sorts in reverse order, and consider the first-order formula

$$\phi \equiv \forall^B b, b' (\forall^A a (aEb \leftrightarrow aEb') \rightarrow b = b'),$$

where we used the notation \forall^A and \forall^B to informally refer to quantification over the sorts. It is easy to see that if $\mathcal{M} = (A, B, E^{\mathcal{M}}) \models \phi$ then the map $i : B \rightarrow \wp(A)$ given by $i(b) := \{a \in A : aE^{\mathcal{M}}b\}$ is injective, so $|B| \leq |\wp(A)|$. It follows that \mathcal{K} is the projection of $\text{Mod}(\phi)$. The “bounded” part is immediate since we have not added new sorts. \square

By an additional argument (see [5, Lemma 6.23]), it is not hard to prove that if F is definably bounding, then so is any iteration F^n . In particular, if we define the cardinal function \beth_n for infinite cardinals λ by setting $\beth_0(\lambda) := 2^\lambda$ and $\beth_{n+1}(\lambda) := 2^{\beth_n(\lambda)}$, then each such \beth_n is also definably bounding. This is typically strong enough for most interesting applications.

Definition 4.5.

1. For a set x , the \mathbf{H} -rank of x , denoted by $\rho_{\mathbf{H}}(x)$, is the least infinite κ such that $x \in \mathbf{H}_{\kappa+}$ (i.e., $\rho_{\mathbf{H}}(\kappa) = \min(\aleph_0, |\text{trcl}(x)|)$).
2. Let F be a definably bounding function. A set-theoretic formula $\phi(x)$ is Σ_1^F if there exists a Δ_0 formula $\psi(x, y)$ such that
 - (a) $\forall x (\phi(x) \leftrightarrow \exists y \psi(x, y))$, and
 - (b) $\forall x (\phi(x) \rightarrow \exists y' (\rho_{\mathbf{H}}(y') \leq F(\rho_{\mathbf{H}}(x)) \wedge \psi(x, y'))$

A formula is Π_1^F if its negation is Σ_1^F , and Δ_1^F if it is equivalent to both a Σ_1^F - and a Π_1^F -formula.

3. Let R be a predicate in the language of set theory. All of the above can be generalized to $\Sigma_1^F(R)$, $\Pi_1^F(R)$ and $\Delta_1^F(R)$ in the obvious way.

So, a Σ_1^F formula is a Σ_1 formula such that, in addition, at least one “witness” y is not too far up in terms of \mathbf{H} -rank in relation to x itself, where by “not too far up” we mean “bounded by the definably bounding function F ”. An important example is the satisfaction relation of first-order logic:

Remark 4.6. The satisfaction relation $\models_{\mathcal{L}_{\omega\omega}}$ is Δ_1^{id} (see [4]).

This leads us to introduce a new notion of symbiosis. A similar idea already appeared in [13, Definition 3.3].

Definition 4.7 (Bounded Symbiosis). Let \mathcal{L}^* be a logic and R a set theoretic predicate. We say that \mathcal{L}^* and R are *boundedly symbiotic* if

- (1) The relation $\models_{\mathcal{L}^*}$ is $\Delta_1^F(R)$ -definable for some definably bounding F .
- (2) Every $\Delta_1^F(R)$ -definable class of τ -structures closed under isomorphisms is $\underline{\Delta}^B(\mathcal{L}^*)$ -axiomatizable (for every definably bounding F).

Just as before, condition (2) of *bounded symbiosis* has an equivalent form which is usually easier to verify and to apply. A side effect will be that in (2), we may assume that $F = \text{id}$ without loss of generality. First, we need the following:

Lemma 4.8. *Let R be a Π_1 predicate in set theory.*

1. *Every \mathbf{H}_{κ} is R -correct.*
2. *Let ϕ be a $\Sigma_1^F(R)$ -formula. Then for every x and every $\kappa > F(\rho_{\mathbf{H}}(x))$, $\phi(x)$ is absolute (upwards and downwards) for \mathbf{H}_{κ} .*

Proof. 1 is a classical result of of Lèvy [6] (see also [5, Lemma 6.27]). From this, it follows that $\Delta_0(R)$ -formulas are absolute for \mathbf{H}_{κ} .

For 2, it suffices to prove downwards absoluteness. Let x be arbitrary and suppose $\phi(x)$ holds. Then there exists y such that $\rho_{\mathbf{H}}(y) \leq F(\rho_{\mathbf{H}}(x)) < \kappa$ and $\psi(x, y)$ holds, where $\psi(x, y)$ is the corresponding $\Delta_0(R)$ -formula. But then $y \in \mathbf{H}_{\kappa}$ and $\mathbf{H}_{\kappa} \models \psi(x, y)$ by the above. It follows that $\mathbf{H}_{\kappa} \models \phi(x)$. \square

Lemma 4.9. *Let \mathcal{L}^* be a logic and R be a Π_1 predicate. Then the following are equivalent:*

- (a) Every $\Delta_1^F(R)$ class of τ -structures closed under isomorphisms is $\underline{\Delta}^B(\mathcal{L}^*)$ -axiomatisable (for every definably bounding F).
- (b) Every $\Delta_1^{\text{id}}(R)$ class of τ -structures closed under isomorphisms is $\underline{\Delta}^B(\mathcal{L}^*)$ -axiomatisable.
- (c) The class $\mathcal{Q}_R := \{\mathcal{A} : \mathcal{A} \text{ is isomorphic to a transitive } R\text{-correct } \in\text{-model}\}$ is $\underline{\Delta}^B(\mathcal{L}^*)$ -axiomatisable.

Proof. (a) \Rightarrow (b) is immediate. For (b) \Rightarrow (c), it is enough to prove that \mathcal{Q}_R itself is $\Delta_1^{\text{id}}(R)$ -definable. We have $\mathcal{A} \in \mathcal{Q}_R$ iff $\exists M \exists f$ such that

1. $\rho_{\mathbf{H}}(M) \leq \rho_{\mathbf{H}}(\mathcal{A})$
2. $\rho_{\mathbf{H}}(f) \leq \rho_{\mathbf{H}}(\mathcal{A})$
3. M is transitive
4. $f : \mathcal{A} = (A, E) \cong (M, \in)$ is an isomorphism
5. $\forall x_1 \dots x_n \in M (M \models R(x_1, \dots, x_n) \leftrightarrow R(x_1, \dots, x_n))$

Since clauses 3–5 are $\Delta_1(R)$, this gives a $\Sigma_1^{\text{id}}(R)$ statement. Similarly, $\mathcal{A} \notin \mathcal{Q}_R$ iff (A, E) is not well-founded or $\exists M \exists f$ such that

1. $\rho_{\mathbf{H}}(M) \leq \rho_{\mathbf{H}}(\mathcal{A})$
2. $\rho_{\mathbf{H}}(f) \leq \rho_{\mathbf{H}}(\mathcal{A})$
3. M is transitive
4. $f : \mathcal{A} = (A, E) \cong (M, \in)$ is an isomorphism
5. $\neg \forall x_1 \dots x_n \in M (M \models R(x_1, \dots, x_n) \leftrightarrow R(x_1, \dots, x_n))$

It is easy to see that being ill-founded is Σ_1^{id} , so the conjunction is again a $\Sigma_1^{\text{id}}(R)$ statement.

Now we look at (c) \Rightarrow (a). Let \mathcal{K} be a $\Delta_1^F(R)$ -definable class over a fixed vocabulary τ which is closed under isomorphisms. Let $\Phi(x)$ be the $\Sigma_1^F(R)$ formula defining \mathcal{K} . For simplicity, assume that τ consists only of one binary predicate P and only one sort.

Let τ' be a language in two sorts s_0 and s_1 , with E a binary relation symbol of sort s_0 , G a function symbol from s_1 to s_0 , c a constant symbol of sort s_0 , and P a unary predicate symbol in s_1 (i.e., s_1 is the original sort of τ , while s_0 adds a “model of set theory” on the side).

Let \mathcal{K}' be the class of all τ' -structures

$$\mathcal{M} := (M, A, E^{\mathcal{M}}, c^{\mathcal{M}}, G^{\mathcal{M}}, P^{\mathcal{M}})$$

satisfying the following conditions:

1. $(M, E^{\mathcal{M}}) \in \mathcal{Q}_R$, i.e., $(M, E^{\mathcal{M}})$ is isomorphic to a transitive R -correct model

2. $(M, E^{\mathcal{M}}) \models \text{ZFC}^{-*}$
3. $\mathcal{M} \models \Phi(c)$
4. $|M| \leq 2^{2^{F(|A|)}}$
5. $\mathcal{M} \models "c = (a, b) \text{ and } b \subseteq a \times a"$ (written using E instead of \in)
6. $\mathcal{M} \models "G \text{ is an isomorphism between } (A, P) \text{ and } (a, b)^{(M, E)}"$. In this sentence, $(a, b)^{(M, E)}$ refers to the domain and binary relation on it which is described by a and b when interpreting \in by $E^{\mathcal{M}}$ (e.g., the domain is really $\{x \in M : xE^{\mathcal{M}}a\}$ etc.)

Now we can see that conditions 2, 3, 5 and 6 are directly expressible in $\mathcal{L}_{\omega\omega}$, while 1 is $\underline{\Delta}^{\text{B}}(\mathcal{L}^*)$ -axiomatisable, and hence $\underline{\Sigma}^{\text{B}}(\mathcal{L}^*)$ -axiomatisable, by assumption. Moreover, 4 is also $\underline{\Sigma}^{\text{B}}(\mathcal{L}^*)$ -axiomatisable: this follows by the definition of “definably bounding”, by Example 4.4, and the discussion following it.

It remains to prove that \mathcal{K} is a bounded projection of \mathcal{K}' to τ . Note that the “bounded” part is immediate due to 4.

- First suppose $\mathcal{M} = (M, A, E^{\mathcal{M}}, c^{\mathcal{M}}, G^{\mathcal{M}}, P^{\mathcal{M}}) \in \mathcal{K}'$. We want to show that $(A, P^{\mathcal{M}}) \in \mathcal{K}$. By 1 $(M, E^{\mathcal{M}})$ is isomorphic to a transitive model (\bar{M}, \in) which is R -correct. Let $\bar{c}^{\mathcal{M}}$ be the image of $c^{\mathcal{M}}$ under this isomorphism. Then $(\bar{M}, \in) \models \Phi(\bar{c}^{\mathcal{M}})$. Moreover, since \bar{M} is R -correct and Φ is $\Sigma_1(R)$, by upwards absoluteness we have $\Phi(\bar{c}^{\mathcal{M}})$, i.e. $\bar{c}^{\mathcal{M}} \in \mathcal{K}$. By 6 we have $\bar{c}^{\mathcal{M}} \cong "(a, b)^{(M, E)}" \cong (A, P^{\mathcal{M}})$. Since \mathcal{K} is closed under isomorphism, it follows that $(A, P^{\mathcal{M}}) \in \mathcal{K}$.
- Conversely, let $\mathcal{A} = (A, P^{\mathcal{A}}) \in \mathcal{K}$, i.e., $\Phi(\mathcal{A})$ holds. We want to find a structure $\mathcal{M} \in \mathcal{K}'$ such that $\mathcal{A} = \mathcal{M} \upharpoonright \tau$.

The first idea would be to find an \mathbf{H}_{θ} which is sufficiently large to reflect $\Phi(\mathcal{A})$ while still being small enough to satisfy condition (4). In general, however, the transitive closure of A might be significantly larger than $|A|$. So we first find a model $\bar{\mathcal{A}}$ which is isomorphic to \mathcal{A} but whose domain is some cardinal μ . Since \mathcal{K} is closed under isomorphisms, $\bar{\mathcal{A}}$ is also in \mathcal{K} , i.e. $\Phi(\bar{\mathcal{A}})$ also holds.

Note that in this case $P^{\bar{\mathcal{A}}} \subseteq \mu \times \mu$, in particular, $\text{trcl}(\bar{\mathcal{A}}) = \text{trcl}((\mu, P^{\bar{\mathcal{A}}})) \subseteq \mu$, so $\rho_{\mathbf{H}}(\bar{\mathcal{A}}) = \mu$.⁴ Let $\theta := F(\mu)^+$. By Lemma 4.8 (2) $\mathbf{H}_{\theta} \models \Phi(\bar{\mathcal{A}})$.

Now let $\mathcal{M} = (\mathbf{H}_{\theta}, A, \in, \bar{\mathcal{A}}, g, P^{\mathcal{A}})$, where g is the isomorphism between \mathcal{A} and $\bar{\mathcal{A}}$. Now it is not hard to verify that all 6 conditions in the definition of \mathcal{K}' are satisfied. In particular, 1 holds because of Lemma 4.8 (1) and 4 because

$$|\mathbf{H}_{\theta}| \leq 2^{\theta} = 2^{F(\mu)^+} \leq 2^{2^{F(\mu)}} = 2^{2^{F(|A|)}}.$$

Thus $\mathcal{M} \in \mathcal{K}'$ as we wanted.

This shows that \mathcal{K} is a bounded projection of \mathcal{K}' and therefore \mathcal{K} is $\underline{\Sigma}^{\text{B}}(\mathcal{L}^*)$. Since \mathcal{K} is also $\Delta_1^F(R)$, the same proof works for the complement of \mathcal{K} , showing that \mathcal{K} is $\underline{\Delta}^{\text{B}}(\mathcal{L}^*)$. \square

⁴Even though we made an assumption to only consider the language τ with one binary relation symbol for the sake of clarity, the same holds for any number of predicate or function symbols on a model with domain μ .

5 Examples of Bounded Symbiosis

In general, all the pairs that are proved to be *symbiotic* in [2, Proposition 4] are also *bounded symbiotic*. For completeness, we now show how the proofs of Propositions 3.4, 3.5 and 3.6 can be strengthened to prove bounded symbiosis. In particular, Proposition 5.2 is a non-trivial result since by [14, §4] it is consistent that $\underline{\Delta}(\mathcal{L}_I) \neq \underline{\Delta}^B(\mathcal{L}_I)$.

Proposition 5.1. *The pairs \mathcal{L}_{WF} and \emptyset are bounded symbiotic.*

Proof. The same proof as Proposition 3.4 works. For (1), note that we may always use reflection to find M such that $|M| = |\text{trcl}(A)|$. This implies that $\models_{\mathcal{L}_{WF}}$ is Δ_1^{id} . For part (2), \mathcal{Q}_{WF} is actually \mathcal{L}_{WF} -definable, hence $\underline{\Delta}^B(\mathcal{L}_{WF})$ -definable. \square

Proposition 5.2. *The pairs \mathcal{L}_I and Cd are bounded symbiotic.*

Proof. Again we look at the proof of Proposition 3.5. For (1), we have the stronger equivalence: $\mathcal{A} \models_{\mathcal{L}_I} \phi$

$$\text{iff } \exists M (\rho_{\mathbf{H}}(M) \leq 2^{2^{\rho_{\mathbf{H}}(A)}} \wedge M \text{ transitive and Cd-correct} \wedge M \models \text{ZFC}^{-*} \\ \wedge \mathcal{A} \in M \wedge M \models (\mathcal{A} \models_{\mathcal{L}_I} \phi))$$

$$\text{iff } \forall M (((\rho_{\mathbf{H}}(M) \leq 2^{2^{\rho_{\mathbf{H}}(A)}} \wedge M \text{ transitive and Cd-correct} \\ \wedge M \models \text{ZFC}^{-*} \wedge \mathcal{A} \in M) \rightarrow M \models (\mathcal{A} \models_{\mathcal{L}_I} \phi))$$

As in the proof of Lemma 4.9, we know that for any \mathcal{A} we can let $\theta = |\text{trcl}(\mathcal{A})|^+$, so that

$$|\mathbf{H}_\theta| \leq 2^\theta = 2^{\rho_{\mathbf{H}}(A)^+} \leq 2^{2^{\rho_{\mathbf{H}}(A)}}$$

and \mathbf{H}_θ is Cd-correct by Lemma 4.8 (1). Thus, the relation $\models_{\mathcal{L}_I}$ is $\Delta_1^F(\text{Cd})$ for the definably bounding function $F(\alpha) = 2^{2^\alpha}$.

Now we check (2) of bounded symbiosis. Again, looking at the proof of Proposition 3.5, we see that clauses (b) and (c) are \mathcal{L}_I -sentences, and (a) is $\underline{\Pi}^B(\mathcal{L}_I)$ since we do not need to add new sorts. The only issue, then, is to prove that “ (A, E) is well-founded” is $\underline{\Sigma}^B(\mathcal{L}_I)$, which is less trivial because the method described previously does not yield an upper bound on the size of the second sort. So we need to adapt this method. The idea is to add a new linear ordering $(B, <)$ to the structure (A, E) , and a function $f : A \rightarrow B$, such that B plays the role of the appropriate cardinals \aleph_α .

Suppose $(B, <)$ is a linear order. For $b \in B$ let $b \downarrow = \{b' \in B : b' < b\}$ denote the set of $<$ -predecessors of b . We say that b is *cardinal-like* if for every $b' < b$ we have $|b' \downarrow| < |b \downarrow|$, and the set B itself is *cardinal-like* if for every $b \in B$ we have $|b \downarrow| < |B|$.

Consider the language with two sorts s_0 and s_1 , a binary relation symbol E in s_0 , a binary relation $<$ in s_1 , and a function symbol f from s_0 to s_1 . In the following proof, we will informally refer to the domains of the respective sorts by A and B .

First define the following abbreviations:

$$\text{Inf}(b) \equiv \forall^B x < b \text{I}yz(y < b \wedge y \neq x)(z < b)$$

i.e., b has infinitely many $<$ -predecessors.

$$\text{Like}(b) \equiv \forall^B b' < b \neg \text{I}yz(y < b)(z < b'),$$

i.e., b is cardinal-like. Let Φ be the conjunction of the following \mathcal{L}_I -formulas:

- (i) $<$ is a linear order;
- (ii) $\forall^B b \neg \text{I}xy(x < b)(y = y)$
i.e., “ B is cardinal-like”;
- (iii) $\forall^B b(\text{Inf}(b) \rightarrow \exists^B b' \leq b(\text{I}xy(x < b)(y < b') \wedge \text{Like}(b')))$
i.e., “no infinite cardinals are skipped”;
- (iv) $\forall^A a \forall^A a' (aEa' \rightarrow f(a) < f(a'))$
i.e., “ f is order-preserving”;
- (v) $\forall^A a(\text{Inf}(f(a)) \wedge \text{Like}(f(a)))$
i.e., “every $f(a)$ is infinite and cardinal-like”;
- (vi) $\forall^A a \forall^B b (b < f(a) \rightarrow \exists^A a' (a'Ea \wedge b \leq f(a')))$
i.e., “every $|f(a)\downarrow|$ is the least cardinal higher than $|f(a')\downarrow|$ for all $a'Ea$ ”;
- (vii) $\forall^B b \exists^A a (b \leq f(a))$
i.e., “the range of f is cofinal in B ”.

Now we prove several claims, which together imply that “ (A, E) is well-founded” is $\underline{\Sigma}^B(\mathcal{L}_I)$. For ease of notation we will identify the symbols $E, <$ and f with their respective interpretations.

Claim 5.3. (A, E) is well-founded iff $(A, B, E, <, f) \models_{\mathcal{L}_I} \Phi$ for some $B, <$ and f .

Proof. First, suppose (A, E) is well-founded. Let rk_E be the rank function induced by E , let $B = \aleph_{\text{rk}_E(A)}$, and let $f(a) = \aleph_{\text{rk}_E(a)}$. Then it is easy to verify that $(A, B, E, <, f) \models_{\mathcal{L}_I} \Phi$. Conversely, suppose $(A, B, E, <, f) \models \Phi$. Then for every aEa' we have $|f(a)\downarrow| < |f(a')\downarrow|$, as follows easily from the fact that $<$ is transitive, that f is order-preserving, and that every $f(a)$ is cardinal-like. But then E must be well-founded. \square (Claim 5.3)

Claim 5.4. Suppose $(A, B, E, <, f) \models_{\mathcal{L}_I} \Phi$. Then

1. For all $b \in B$ and all $\lambda < |b\downarrow|$, there exists $c < b$ such that $\lambda \leq |c\downarrow| < |b\downarrow|$.
2. For all $b \in B$ and all $\lambda < |b\downarrow|$, there exists $d < b$ such that $|d\downarrow| = \lambda$.
3. For all $\lambda < |B|$, there exists d such that $|d\downarrow| = \lambda$.

Proof.

1. Let $b \in B$ and $\lambda < |b\downarrow|$. By (iii) there is a $b' < b$ such $|b'\downarrow| = |b\downarrow|$ and b' is cardinal-like. We claim that there is $c < b'$ such that $\lambda \leq |c\downarrow|$. Towards contradiction, suppose this is false. Let $\{c_\alpha : \alpha < |b'\downarrow|\}$ enumerate $b'\downarrow$ and consider the initial λ -sequence of this enumeration, i.e., $\{c_\alpha : \alpha < \lambda\}$. This sequence cannot be $<$ -cofinal in b' , otherwise we would have $|b'\downarrow| = |\bigcup_{\alpha < \lambda} (c_\alpha\downarrow)| \leq \lambda \times \lambda = \lambda$, which is a contradiction. Therefore, there is $c < b'$ such that $\{c_\alpha : \alpha < \lambda\} \subseteq c\downarrow$. But then $\lambda \leq |c\downarrow|$, also a contradiction.
2. First apply (1) to find $c_0 < b$ such that $\lambda \leq |c_0\downarrow| < |b\downarrow|$. Apply again to find $c_1 < c_0$ such that $\lambda \leq |c_1\downarrow| < |c_0\downarrow|$, etc. By well-foundedness, this process will stop after finitely many steps and we will find $d < b$ such that $\lambda = |d\downarrow|$.
3. By an analogous argument as in (1) above, and using (ii), we first find $b \in B$ such that $\lambda \leq |b\downarrow| < |B|$. Then proceed as in (2). \square (Claim 5.4).

Claim 5.5. *Suppose $(A, B, E, <, f) \models_{\mathcal{L}_1} \Phi$. Then $|A \cup B| \leq \aleph_{\text{rk}_E(A)}$.*

Proof. We prove, by induction on E , that for all $a \in A$:

$$|f(a)\downarrow| \leq \aleph_{\text{rk}_E(a)}.$$

Suppose the above holds for all $a' \in A$. Towards contradiction suppose $|f(a)\downarrow| > \aleph_{\text{rk}_E(a)}$. By Claim 5.4 (2), we can find $d < f(a)$ such that $|d\downarrow| = \aleph_{\text{rk}_E(a)}$. By (vi), there exists $a' \in A$ such that $d \leq f(a')$. But this means that

$$\aleph_{\text{rk}_E(a)} = |d\downarrow| \leq |f(a')\downarrow| = \aleph_{\text{rk}_E(a')}$$

which is a contradiction since $\text{rk}_E(a') < \text{rk}_E(a)$. This completes the induction.

Completing the proof requires repeating the above argument once more: if $|B| > \aleph_{\text{rk}_E(A)}$, then by Claim 5.4 (3) there is $d \in B$ such that $|d\downarrow| = \aleph_{\text{rk}_E(A)}$, and by (vii) there is $a \in A$ with $d \leq f(a)$, implying

$$\aleph_{\text{rk}_E(A)} = |d\downarrow| \leq |f(a)\downarrow| = \aleph_{\text{rk}_E(a)}$$

which is a contradiction since by definition $\text{rk}_E(a) < \text{rk}_E(A)$. It follows that $|A \cup B| = |B| \leq \aleph_{\text{rk}_E(A)}$. \square (Claim 5.5)

\square

Proposition 5.6. *The pairs \mathcal{L}^2 and PwSt are bounded symbiotic.*

Proof. A straightforward adaptation of the proof of Proposition 3.6 works. Using the same trick as above, in (1) we see that $\models_{\mathcal{L}^2}$ is $\Delta_1^F(\text{PwSt})$ for $F(\alpha) = 2^{2^\alpha}$. For (2), we do not need to change anything since the class $\mathcal{Q}_{\text{PwSt}}$ is already \mathcal{L}^2 -axiomatisable. \square

6 The Upwards Structural Reflection principle

Now we consider a *reflection number* analogous to the one in Definition 1.2 which, as in [2], will allow us to connect the strength of existence of upward Löwenheim-Skolem numbers for strong logics to large cardinals.

Definition 6.1. Let R be a Π_1 predicate in the language of set theory. The *bounded upwards structural reflection number* $\mathcal{USR}(R)$ is the least κ such that:

For every definably bounding function F , and every $\Sigma_1^F(R)$ -definable class of τ -structures in a fixed vocabulary τ closed under isomorphisms:

If there is $\mathcal{A} \in \mathcal{K}$ with $|\mathcal{A}| \geq \kappa$, then for every $\kappa' > \kappa$ there is a $\mathcal{B} \in \mathcal{K}$ with $|\mathcal{B}| \geq \kappa'$ and an elementary embedding $e : \mathcal{A} \preceq_{\mathcal{L}_{\omega\omega}} \mathcal{B}$.

If there is no such cardinal, $\mathcal{USR}(R)$ is undefined.

Remark 6.2. In this definition, we are assuming that \mathcal{K} is definable by a $\Sigma_1^F(R)$ -formula *without parameters*. In particular, the definition presupposes that τ , as a vocabulary, is itself $\Sigma_1^F(R)$ -definable (e.g., finite). Notice that if arbitrary τ were allowed, $\mathcal{USR}(R)$ would never be defined: for any κ , we could take a vocabulary τ with κ -many constant symbols and let \mathcal{K} be the class of τ -structures such that every element is the interpretation of a constant symbol, which is Δ_1^{id} in τ . One could avoid this problem by considering classes defined with parameters of a limited **H**-rank; but then, to prove results like the ones in this section, one would need to extend the corresponding logic in such a way that the parameter can be defined. For the current paper, the parameter-free version will be sufficient.

Our main theorem below is proved for logics which have Δ_0 -definable syntax and dependence number ω . This is necessary if we want to avoid parameters—recall the discussion in Section 2. All logics obtained by adding finitely many quantifiers to first- or second-order logic, such as \mathcal{L}_{WF} , \mathcal{L}_1 , \mathcal{L}^2 and the examples in [2, Proposition 4], are covered by this theorem. For the ULST-principle, see Definition 2.4 and recall that since we are assuming $\text{dep}(\mathcal{L}^*) = \omega$, $\text{ULST}_\omega(\mathcal{L}^*) = \text{ULST}_\infty(\mathcal{L}^*)$.

Theorem 6.3 (Main Theorem). *Let \mathcal{L}^* be a logic with Δ_0 -definable syntax and $\text{dep}(\mathcal{L}^*) = \omega$, and let R be a Π_1 predicate. Assume that \mathcal{L}^* and R are boundedly symbiotic. Then the following are equivalent:*

- (1) $\text{ULST}_\infty(\mathcal{L}^*) = \kappa$,

- (2) $\mathcal{USR}(R) = \kappa$.

Proof. (2) \Rightarrow (1). Suppose $\mathcal{USR}(R) = \kappa$. Let ϕ be an \mathcal{L}^* -formula, and let $\mathcal{A} \models_{\mathcal{L}^*} \phi$ with $|\mathcal{A}| \geq \kappa$. Letting κ' be any cardinal above κ , the goal is to find a super-structure \mathcal{B} of \mathcal{A} such that $\mathcal{B} \models_{\mathcal{L}^*} \phi$ and $|\mathcal{B}| \geq \kappa'$.

Consider the class $\mathcal{K} = \text{Mod}(\phi)$. By condition (1) of *bounded symbiosis*, \mathcal{K} is $\Delta_1^F(R)$ -definable, hence $\Sigma_1^F(R)$, with parameter ϕ . However, since $\text{dep}(\mathcal{L}^*) = \omega$, we may assume that the vocabulary of ϕ is finite. Moreover, \mathcal{L}^* has a Δ_0 -definable syntax, so ϕ is Δ_0 -definable, therefore \mathcal{K} is in fact $\Sigma_1^F(R)$ -definable *without parameters*. It is also clearly closed under isomorphisms.

Applying $USR(R)$ we find a $\mathcal{B}' \in \mathcal{K}$, such that $|\mathcal{B}'| \geq \kappa'$ and there is $e : \mathcal{A} \preceq_{\mathcal{L}_{\omega\omega}} \mathcal{B}'$. We can also easily find $\mathcal{B} \cong \mathcal{B}'$ such that \mathcal{A} is a substructure of \mathcal{B} , and this is what we need.

(1) \Rightarrow (2). Now assume $ULST_{\infty}(\mathcal{L}^*) = \kappa$, and let \mathcal{K} be a $\Sigma_1^F(R)$ -definable transitive class of τ -structures, with $\Phi(x)$ the defining $\Sigma_1^F(R)$ -formula.

Since the USR -principle involves elementary embeddings whereas $ULST$ does not, the proof must proceed indirectly. The intuition is that we first embed a given structure $\mathcal{A} \in \mathcal{K}$ in a larger structure that includes a model of set theory \mathbf{H}_{θ} and includes Skolem functions for first-order existential sentences; then we apply $ULST$ to (a further extension of) this larger structure. To make sure that enlarging the set-theoretic structure also yields an enlargement of the original structure, we must carefully keep track of the relations between cardinalities given by the various bounds in the definition of *bounded symbiosis*. See Figure 3.

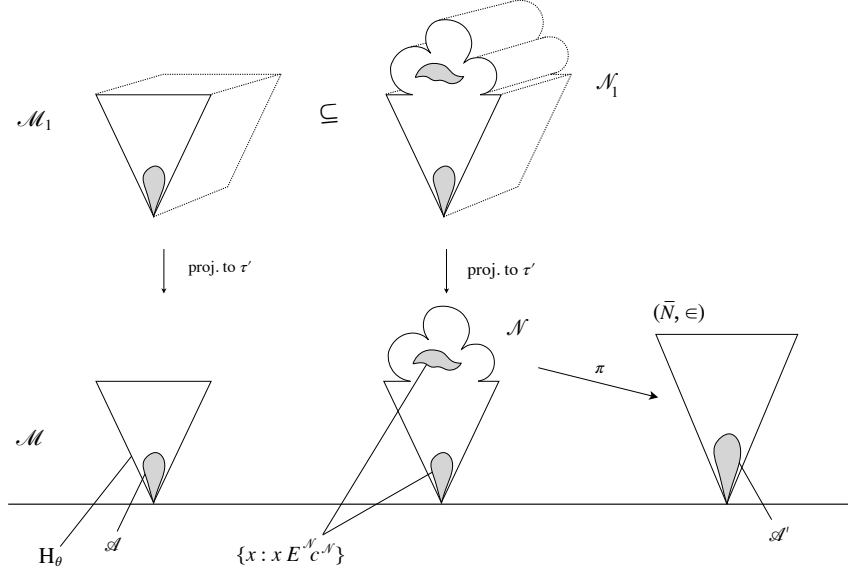


Figure 3: Structure of the proof.

Assume that τ is in one sort (a similar proof works in the general case). Similarly to the proof of Lemma 4.9, define a vocabulary τ' with two sorts: s_0 and s_1 , with all of the symbols occurring in τ written in sort s_1 . Let E be a binary relation symbol in sort s_0 , G a function symbol from s_1 to s_0 , and c a constant symbol in sort s_0 . In addition, for every quantifier-free first-order formula $\psi(x, y_1, \dots, y_n)$ in the language E, c , add an n -ary function symbol f_ψ of sort s_0 .

Let \mathcal{K}^* be the class of all structures $\mathcal{M} := (M, A, E^{\mathcal{M}}, G^{\mathcal{M}}, c^{\mathcal{M}}, \{f_\psi^{\mathcal{M}}\})$ such that

1. $(M, E^{\mathcal{M}}) \models \text{ZFC}^{-*}$,
2. $(M, E^{\mathcal{M}}) \in \mathcal{Q}_R$, i.e., it is isomorphic to a transitive R -correct model,
3. $|M| \leq 2^{2^{F(|\mathcal{A}|)}}$,
4. $\mathcal{M} \models \Phi(c)$, written with E instead of \in ,
5. $\mathcal{M} \models \forall \bar{z} (\exists x \psi(x, \bar{z}) \rightarrow \psi(f_\psi(\bar{z}), \bar{z}))$ for every quantifier-free ψ .
6. $\mathcal{M} \models G$ is a bijection between A and $\{x : x E c\}$.

Conditions 1, 4 and 6 are in first-order logic, whereas 2 is $\underline{\Delta}^{\text{B}}(\mathcal{L}^*)$ -axiomatizable by the equivalent condition (2) of *bounded symbiosis*. Moreover, 3 is $\underline{\Sigma}^{\text{B}}(\mathcal{L}^*)$ -axiomatisable by the definition of “definably bounding” (Definition 4.3), by Example 4.4, and the discussion following it. Finally, while 5 might look like an infinite set of sentences (and we are not assuming that \mathcal{L}^* is infinitary), it is still true that, since \models is Δ_1^{id} , the entire condition 5 can be expressed in a Δ_1^{id} way in set theory. By condition (2) of *bounded symbiosis*, the class of models satisfying 5 is $\underline{\Delta}^{\text{B}}(\mathcal{L}^*)$. Therefore, \mathcal{K}^* is $\underline{\Sigma}^{\text{B}}(\mathcal{L}^*)$.

Let \mathcal{A} be a structure in \mathcal{K} with $|\mathcal{A}| \geq \kappa$, and let $\kappa' > \kappa$ be any cardinal. Since \mathcal{K} is closed under isomorphisms, we may assume wlog. that \mathcal{A} is transitive. Our goal is to find $\mathcal{A}' \in \mathcal{K}$ such that $|\mathcal{A}'| \geq \kappa'$ and $\mathcal{A} \preceq_{\mathcal{L}^{\omega\omega}} \mathcal{A}'$.

Let $\theta := F(\rho_{\mathbf{H}}(\mathcal{A}))^+ = F(|\mathcal{A}|)^+$, choose Skolem functions $f_\psi^{\mathbf{H}_\theta} : \mathbf{H}_\theta^n \rightarrow \mathbf{H}_\theta$, and consider the structure

$$\mathcal{M} := (\mathbf{H}_\theta, A, \in, \text{id}, \mathcal{A}, \{f_\psi^{\mathbf{H}_\theta}\})$$

Clearly \mathcal{M} satisfies 1, 5 and 6 of the definition of \mathcal{K}^* . Moreover, due to Lemma 4.8 (1), \mathbf{H}_θ is R -correct and $\Phi(\mathcal{A})$ is absolute for \mathbf{H}_θ . Hence, 2 and 4 are satisfied as well. Finally, 3 holds because

$$|\mathbf{H}_\theta| \leq 2^\theta = 2^{F(|\mathcal{A}|)^+} \leq 2^{2^{F(|\mathcal{A}|)}}.$$

Therefore $\mathcal{M} \in \mathcal{K}^*$.

Let χ be an \mathcal{L}^* -sentence in an extended vocabulary τ'' such that \mathcal{K}^* is a “bounded projection” of $\text{Mod}(\chi)$. Let $h : \text{Ord} \rightarrow \text{Ord}$ be the function as in Lemma 4.2.

Let $\mathcal{M}_1 = (\mathcal{M}, \dots)$ be such that $\mathcal{M}_1 \models \chi$ and $\mathcal{M}_1 \upharpoonright \tau' = \mathcal{M}$. Since $|\mathcal{M}_1| \geq |\mathcal{M}| \geq |\mathcal{A}| \geq \kappa$, we can apply $\text{ULST}_\infty(\mathcal{L}^*) = \kappa$ to find \mathcal{N}_1 such that $\mathcal{N}_1 \models \chi$ and $|\mathcal{N}_1| > h(2^{2^{F(\kappa')}})$, and $\mathcal{M}_1 \subseteq \mathcal{N}_1$ (i.e., \mathcal{M}_1 is a substructure of \mathcal{N}_1). Let $\mathcal{N} = \mathcal{N}_1 \upharpoonright \tau'$. We write $\mathcal{N} = (N, B, E^{\mathcal{N}}, G^{\mathcal{N}}, c^{\mathcal{N}}, \{f_\psi^{\mathcal{N}}\})$ for this structure.

Let (\bar{N}, \in) be the transitive collapse of $(N, E^{\mathcal{N}})$, and $\bar{c}^{\mathcal{N}}$ be the image of $c^{\mathcal{N}}$ under this collapse. We claim that $\mathcal{A}' := \bar{c}^{\mathcal{N}}$ is the model we are looking for.

Claim 6.4. $\mathcal{A}' \in \mathcal{K}$.

Proof. Since $\mathcal{N} \in \mathcal{K}^*$, we know that $(N, E^{\mathcal{N}}) \models \Phi(c)$ (written in E), and therefore $\bar{N} \models \Phi(\bar{c})$ (written in \in). Also, since $(N, E^{\mathcal{N}}) \in \mathcal{Q}_R$, we know that \bar{N} is R -correct, in particular, $\Sigma_1(R)$ formulas are upwards absolute. Therefore $\Phi(\mathcal{A}')$ is true, so $\mathcal{A}' \in \mathcal{K}$. \square

Claim 6.5. $\kappa' < |\mathcal{A}'|$.

Proof. By the definition of the function h as in Lemma 4.2, we know that $|\mathcal{N}_1| \leq h(|\mathcal{N}|)$. Thus we have

$$h\left(2^{2^{F(\kappa')}}\right) < |\mathcal{N}_1| \leq h(|\mathcal{N}|)$$

and since h is order-preserving, $2^{2^{F(\kappa')}} < |\mathcal{N}|$. By condition 3 of the definition of \mathcal{K}^* , we have $|N| \leq 2^{2^{F(|B|)}}$. Therefore $2^{2^{F(\kappa')}} < |N| \leq 2^{2^{F(|B|)}}$, from which it follows that $\kappa' < |B|$. Finally, by condition 6 we get that $|B| = |\{x \in N : xE^{\mathcal{N}}c^{\mathcal{N}}\}| = |\mathcal{A}'|$. \square

Claim 6.6. *There is an $\mathcal{L}_{\omega\omega}$ -elementary embedding from \mathcal{A} to \mathcal{A}' .*

Proof. By condition 5, both models \mathcal{M}_1 and \mathcal{N}_1 satisfy the axioms for Skolem functions concerning first-order quantifier-free formulas in $\{E, c\}$. In addition, since \mathcal{M}_1 is a *substructure* of \mathcal{N}_1 , the interpretations of f_ψ coincide between the models, i.e., $f_\psi^{\mathcal{N}_1} \upharpoonright \mathbf{H}_\theta = f_\psi^{\mathbf{H}_\theta}$ for every ψ . Thus, if $\mathcal{N}_1 \models \exists x\psi(x, \bar{z})$ and $\bar{z} \in \mathbf{H}_\theta$, then $\mathcal{N}_1 \models \psi(f_\psi(\bar{z}), \bar{z})$, so $(\mathbf{H}_\theta, \in, \mathcal{A}) \models \psi(f_\psi(\bar{z}), \bar{z})$. It follows that \mathcal{N}_1 and $(\mathbf{H}_\theta, \in, \mathcal{A})$ satisfy the same Σ_1 -formulas in $\{E, c\}$.

Let $\pi : N \rightarrow \bar{N}$ be the collapsing map. Since the first-order satisfaction relation is Δ_1 , for every first-order ϕ and for every $\bar{a} = a_1, \dots, a_n \in \mathcal{A}$ we have

$$\begin{aligned} \mathcal{A} &\models \phi(\bar{a}) \\ &\Leftrightarrow \mathbf{H}_\theta \models (\mathcal{A} \models \phi(\bar{a})) \\ &\Leftrightarrow \mathcal{N}_1 \models (c \models \phi(\bar{a})) \\ &\Leftrightarrow (\bar{N}, \in, \mathcal{A}') \models (c \models \phi(\pi(\bar{a}))) \\ &\Leftrightarrow \mathcal{A}' \models \phi(\pi(\bar{a})). \end{aligned}$$

Hence $\pi \upharpoonright \mathcal{A} : \mathcal{A} \preceq_{\mathcal{L}_{\omega\omega}} \mathcal{A}'$ as required.⁵ \square

\square

7 The predicate PwSt and second order logic

In this section we apply our results to determine upper and lower bounds for the large cardinal strength of $\mathcal{USR}(\text{PwSt})$ and $\text{ULST}_\infty(\mathcal{L}^2)$, which will also yield upper bounds for other symbiotic pairs \mathcal{L}^* and R . The main point is that PwSt can be seen as an upper bound for all Π_1 predicates. The following is not hard to verify (see [5, Section 6.5] for the details).

Fact 7.1. *The function $\alpha \mapsto \mathbf{V}_\alpha$ is $\Sigma_1^F(\text{PwSt})$ -definable (for suitable F). Also, the function \mathbf{H} that maps every infinite successor cardinal θ to \mathbf{H}_θ is $\Sigma_1^F(\text{PwSt})$ -definable.*

Lemma 7.2. *For every Π_1 predicate R , if ϕ is $\Sigma_1^F(R)$ then it is $\Sigma_1^F(\text{PwSt})$.*

⁵In this proof we have occasionally identified the syntax and semantics of first-order logic for ease of readability.

Proof. Suppose ϕ is $\Sigma_1^F(R)$. Then for every a we have

$$\phi(a) \Leftrightarrow \exists \mathbf{H}_\theta \left(\rho_{\mathbf{H}}(\mathbf{H}_\theta) < 2^{2^{F(\rho_{\mathbf{H}}(a))}} \wedge \mathbf{H}_\theta \models \phi(a) \right).$$

By the previous fact, “being \mathbf{H}_θ ” is $\Sigma_1^{F'}$ (PwSt)-definable (possibly another F'). In conjunction with the upper bound, the whole expression is also $\Sigma_1^{F''}$ (PwSt)-definable (for F'' being the maximum of F' and $\alpha \mapsto 2^{2^{F(\alpha)}}$). To see that the equivalence holds, let $\theta = F(\rho_{\mathbf{H}}(a))^+$. Then $\rho_{\mathbf{H}}(\mathbf{H}_\theta) \leq 2^\theta \leq 2^{2^{F(\rho_{\mathbf{H}}(a))}}$, and moreover $\phi(a)$ is absolute for \mathbf{H}_θ by Lemma 4.8 (2). \square

Corollary 7.3.

1. For every Π_1 predicate R we have $USR(R) \leq USR(\text{PwSt})$. In particular, if $USR(\text{PwSt})$ is defined then $USR(R)$ is also defined.
2. If \mathcal{L}^* is any logic which is boundedly symbiotic to some Π_1 -predicate R , has Δ_0 -definable syntax and $\text{dep}(\mathcal{L}^*) = \omega$, then $ULST_\infty(\mathcal{L}^*) \leq ULST_\infty(\mathcal{L}^2)$. In particular, if $ULST_\infty(\mathcal{L}^2)$ is defined, then so is $ULST_\infty(\mathcal{L}^*)$.

A famous result of Magidor [8] shows that the least cardinal κ for which \mathcal{L}^2 satisfies a κ -version of compactness, is the least extendible cardinal. One can show that this version of compactness implies $ULST_\infty(\mathcal{L}^2) = \kappa$. Therefore an extendible cardinal provides an upper bound for $ULST_\infty(\mathcal{L}^2)$ and $USR(\text{PwSt})$, as well as other pairs \mathcal{L}^* and R satisfying bounded symbiosis and the conditions of Theorem 6.3. For completeness, we include a short proof of this fact.

Theorem 7.4 (Magidor [8]). *If κ is an extendible cardinal, then*

$$USR(\text{PwSt}) = ULST(\mathcal{L}^2) \leq \kappa.$$

Moreover, $USR(R) \leq \kappa$ for every Π_1 predicate R , and $ULST_\infty(\mathcal{L}^) \leq \kappa$ for any \mathcal{L}^* which is boundedly symbiotic with some Π_1 predicate, and which has Δ_0 -definable syntax and $\text{dep}(\mathcal{L}^*) = \omega$.*

Proof. Let κ be extendible, and we will prove that $USR(\text{PwSt}) \leq \kappa$. The other statements follow by Theorem 6.3 and Corollary 7.3.

Let \mathcal{K} be a Σ_1^F (PwSt)-definable class of τ -structures closed under isomorphisms. Fix some $\mathcal{A} \in \mathcal{K}$ with $|\mathcal{A}| \geq \kappa$. Let $\kappa' > \kappa$ be arbitrary. Let $\eta > \kappa'$ be such that $\mathcal{A} \in \mathbf{V}_\eta$ and $\mathbf{V}_\eta \models \Phi(\mathcal{A}) \wedge (|\mathcal{A}| \geq \kappa)$. Then there is an elementary embedding $J : \mathbf{V}_\eta \preceq \mathbf{V}_\theta$ for some θ , and $J(\kappa) > \eta > \kappa'$. But then by elementarity we have $\mathbf{V}_\theta \models \Phi(J(\mathcal{A})) \wedge |J(\mathcal{A})| \geq J(\kappa)$. Since \mathbf{V}_θ is PwSt-correct, $\Phi(J(\mathcal{A}))$ holds, so $J(\mathcal{A}) \in \mathcal{K}$. Moreover, since θ is sufficiently large, we have $|\mathcal{A}| \geq J(\kappa) > \eta > \kappa'$. Finally, $\mathcal{A} \preceq_{\mathcal{L}^{\omega\omega}} J(\mathcal{A})$ holds by elementarity and first-order definability of “ $\mathcal{A} \models \phi$ ”. \square

Now we look at how much large cardinal strength we can obtain from $USR(\text{PwSt})$.

Theorem 7.5. *If $USR(\text{PwSt})$ is defined, then there exists an n -extendible cardinal for every natural number $n > 0$.*

Proof. Assume that $\mathcal{USR}(\text{PwSt}) = \kappa$. Let \mathcal{K} to be the class of all structures which are isomorphic to $(\mathbf{V}_{\alpha+n}, \in, \alpha)$, in the language $\{E, a\}$.

By Fact 7.1, being a structure of the form $(\mathbf{V}_{\alpha+n}, \in, \alpha)$ is $\Sigma_1^F(\text{PwSt})$ -definable. Then $(M, a, E) \in \mathcal{K}$ iff $\exists(\mathbf{V}_{\alpha+n}, \in, \alpha)$ and $\exists f : (M, a, E) \cong (\mathbf{V}_{\alpha+n}, \in, \alpha)$. Notice that also $\rho_{\mathbf{H}}(\mathbf{V}_{\alpha+n}) \leq \rho_{\mathbf{H}}(M)$. Thus, \mathcal{K} is $\Sigma_1^F(\text{PwSt})$ -definable.

Take any $\mu \geq \kappa$. Since $(\mathbf{V}_{\mu+n}, \mu, \in) \in \mathcal{K}$, by $\mathcal{USR}(\text{PwSt})$ there exists an elementary embedding

$$J : (\mathbf{V}_{\mu+n}, \mu, \in) \preceq_{\mathcal{L}_{\omega\omega}} (\mathbf{V}_{\beta+n}, \beta, \in)$$

for some $\beta > \mu$, which maps μ to β . Let λ be the critical point of J , which is $\leq \mu$. But then $J \upharpoonright \mathbf{V}_{\lambda+n} : \mathbf{V}_{\lambda+n} \preceq \mathbf{V}_{J(\lambda)+n}$ (this includes the case $\lambda = \mu$), since:

$$\begin{aligned} \mathbf{V}_{\lambda+n} \models \varphi(x_1, \dots, x_n) &\Leftrightarrow \mathbf{V}_{\mu+n} \models (\mathbf{V}_{\lambda+n} \models \varphi(x_1, \dots, x_n)) \\ &\Leftrightarrow \mathbf{V}_{\beta+n} \models (J(\mathbf{V}_{\lambda+n}) \models \varphi(J(x_1), \dots, J(x_n))) \\ &\Leftrightarrow \mathbf{V}_{\beta+n} \models (\mathbf{V}_{J(\lambda)+n} \models \varphi(J(x_1), \dots, J(x_n))) \\ &\Leftrightarrow \mathbf{V}_{J(\lambda)+n} \models \varphi(J(x_1), \dots, J(x_n)) \end{aligned}$$

Since $n < J(\lambda)$, it follows that λ is n -extendible. \square

Corollary 7.6. *If $\text{ULST}_{\infty}(\mathcal{L}^2)$ is defined then there exists an n -extendible cardinal for every n .*

Notice that the only reason that the proof works for $n < \omega$ and not arbitrary α , is because the class \mathcal{K} needs to be definable. It is easy to adapt the proof to show that there exists a γ -extendible cardinal for any $\Sigma_1^F(\text{PwSt})$ -definable ordinal γ . In fact, we conjecture that the consistency strength is exactly an extendible.

Conjecture 7.7. *$\mathcal{USR}(\text{PwSt})$ and $\text{ULST}_{\infty}(\mathcal{L}^2)$ are defined if and only if there exists an extendible cardinal.*

8 Questions and concluding remarks

The biggest question left open in this paper is the exact consistency strength of $\mathcal{USR}(\text{PwSt})$ and $\text{ULST}_{\infty}(\mathcal{L}^2)$, i.e., Conjecture 7.7.

Other questions that we have not investigated involve a similar analysis of the large cardinal strength for other symbiotic pairs \mathcal{L}^* and R .

Question 8.1. *What is the large cardinal strength (or, at least, lower and upper bounds), for the principles $\mathcal{USR}(R)$ and $\text{ULST}_{\infty}(\mathcal{L}^*)$, for other boundedly symbiotic pairs R and \mathcal{L}^* , such as the ones in [2, Proposition 4]?*

Another important issue, which we have not investigated in this paper, is the study of various *compactness* properties of strong logics.

Definition 8.2. A logic \mathcal{L}^* is (κ, γ) -compact if for every set T of sentences of size γ , if every $< \kappa$ -sized subset of T has a model, then T has a model. A logic \mathcal{L}^* is (κ, ∞) -compact if it is (κ, γ) -compact for every γ . Classical compactness is (ω, ∞) -compactness.

Most strong logics are not (ω, ∞) -compact but may be (κ, ∞) -compact for some κ . Often such a κ will exhibit large cardinal properties, e.g., Magidor’s result on \mathcal{L}^2 [8]. As we mentioned in the previous section, it is easy to see that:

If \mathcal{L}^* is (κ, ∞) -compact then $\text{ULST}_\infty(\mathcal{L}^*) \leq \kappa$.

We do not know whether the converse holds. The following questions seem interesting and worth investigating:

Question 8.3. *Assume that κ is a regular cardinal. For which logics does $\text{ULST}_\infty(\mathcal{L}^*) \leq \kappa$ imply (κ, ∞) -compactness?*

One can try to look for a set-theoretic principle involving $\Delta_1(R)$ definable classes of structures, which would correspond to (κ, ∞) -compactness in a similar way as in Theorem 6.3.

Question 8.4. *Is there a set-theoretic principle $P(R)$, for classes definable using a Π_1 -parameter R , such that if R and \mathcal{L}^* are (bounded) symbiotic, then $P(R) = \kappa$ if and only if \mathcal{L}^* is (κ, ∞) -compact?*

Answering the last question could involve extensions of partial orders within a fixed $\Delta_1(R)$ -class, using ideas from [10]. Notice, however, that when dealing with compactness, large vocabularies are essential, so the corresponding principles will require the use of parameters, which will restrict the class of logics \mathcal{L}^* to which it can apply.

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