Locally finite varieties of Heyting algebras of width 2

MSc Thesis *(Afstudeerscriptie)*

written by

Thijs Benjamins
(born 09 July 1990 in Hilversum, Netherlands)

under the supervision of **dr. Nick Bezhanishvili** and **dr. Tommaso Moraschini**, and submitted to the Examinations Board in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:	Members of the Thesis Committee:
10 July 2020	dr. Benno van den Berg
	dr. Nick Bezhanishvili (co-supervisor)
	prof. dr. Dick de Jongh
	dr. Tommaso Moraschini (co-supervisor)
	prof. dr. Yde Venema (Chair)

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Abstract

In this thesis we investigate locally finite varieties of Heyting algebras of width 2. We show that a variety of width 2 is locally finite if and only if its 2-generated members are finite. This confirms a conjecture of G. Bezhanishvili and R. Grigolia (2005) for varieties of width 2. We prove this result by showing that non-locally finite varieties of width 2 contain the Rieger-Nishimura lattice with a new bottom element, which is a 2-generated infinite Heyting algebra. We also prove that this characterisation does not carry through to the case of varieties of width 3.

Using this characterisation we show that the variety generated by the Rieger-Nishimura lattice with a new bottom element is the only pre-locally finite variety of Heyting algebras of width 2. As a consequence, we obtain that local finiteness is decidable for finitely axiomatisable varieties of width 2. Finally, we show that there are continua of both locally finite and non-locally finite varieties of width 2.

Contents

Chapter 1

Introduction

Intuitionistic logic is generally considered as the logical foundation of constructive mathematics [11]. Formally speaking, the intuitionistic propositional calculus IPC is obtained from the classical propositional calculus CPC by removing the 'law of the excluded middle' $(p \vee \neg p)$ from its axiomatisation. The axiomatic extensions of IPC are called *superintuitionistic logics* (*si-logics* for short). It is well known that IPC is algebraised by the variety of Heyting algebras. As a consequence the lattice of si-logics is dually isomorphic to the lattice of varieties of Heyting algebras. Because of this, si-logics can be studied through the lenses of varieties of Heyting algebras, which are in turn amenable to the powerful methods of duality theory, universal algebra, and model theory (see, e.g., [10, 14]).

In this thesis we investigate the property of *local finiteness* for varieties of Heyting algebras. In doing so, we also study the logical counterpart of this property called *local tabularity*. Our main tool is the well-known *Esakia duality* between the category of Heyting algebras and that of Esakia spaces.

Recall that a variety is called locally finite if all its finitely generated members are finite. Local finiteness plays an important role in universal algebra [3, 9]. For instance, it implies the (hereditary) finite model property. It is well known that the variety of Boolean algebras, the algebraic dual of CPC, is locally finite. In general, varieties generated by a finite algebra are locally finite, these are called *finitely generated* varieties. There exist, however, locally finite varieties, for example the variety V_{LC} of Heyting algebras generated by chains, that are not finitely generated. Despite the fact that the problem of characterising locally finite varieties of Heyting algebras has received a lot of attention in the literature, it still remains a notoriously difficult open problem [6, 10, 14, 16].

In the case of modal algebras this property has proved relatively easy to characterise [10, Section 12.4]. In particular, a variety of $\mathsf{K4}$-algebras is locally finite if and only if it is of bounded depth or, equivalently, its free 1-generated algebra is finite [10, 18, 19, 22]. For the related property of *pre-local-finiteness* (call a variety pre-locally finite if it is not locally finite but all its proper subvarieties are) it was shown that the only pre-locally finite variety of $\mathsf{S4}$-algebras is the variety corresponding to the modal logic $\mathsf{Grz.3}$. This is the variety of modal algebras generated by the $\mathsf{S4}$-algebras dual to linear reflexive transitive frames.

For varieties of Heyting algebras the situation is more complicated. For example, it follows from [20] that there are continuum many pre-locally-finite varieties of Heyting algebras. Furthermore, while every variety of bounded depth is locally finite, there exist locally finite varieties that are of unbounded depth, such as V_{LC}. Additionally, unlike the case of $\mathsf{K4}$-algebras, the variety of Heyting algebras axiomatised by the 'weak law of the excluded middle' ($\neg p \vee \neg\neg p = 1$) fails to be locally finite despite having only finite 1-generated algebras. However, it *does* contain an infinite 2-generated algebra, a feature shared with all known non-locally finite varieties of Heyting algebras. This led to the problem posed in [6, Problem 2.4.(6)] of whether a variety of Heyting algebras is locally finite if and only if its 2-generated members are finite.

A positive solution to this problem was given for subvarieties of the variety V_{KG} corresponding to the Kuznetsov-Gerčiu logic KG [5, 7, 16]. This is the variety whose finitely generated subdirectly irreducible members are linear sums of 1-generated Heyting algebras. Notably, the variety V_{KG} is *of width* 2, in the sense that the Esakia spaces dual to its subdirectly irreducible members have no anti-chains of size more than 2. Motivated by this, it is natural to conjecture that this positive result extends to all varieties of width 2. This generalisation is non-trivial because the structure theory of the variety of *all* Heyting algebras of width 2 is much more complex than that of V_{KG}. For example, it does not admit a similarly transparent characterisation of its finitely generated subdirectly irreducible algebras.

Our main contribution is to confirm this conjecture. This means we prove that a variety of Heyting algebras of width 2 is locally finite if and only if its 2-generated members are finite. We do this by showing that a variety K of width 2 is not locally finite if and only if K contains a specific 2-generated infinite Heyting algebra, namely the Rieger-Nishimura lattice with a new bottom element, denoted by $\mathcal{L}_+$ (Theorem 4.17). However, our method does not extend to the case of varieties of width 3. We provide an example of a variety of Heyting algebras of width 3 that is not locally finite but fails to contain $\mathcal{L}_+$.

As a consequence of our main results we obtain that the variety generated by the algebra $\mathcal{L}_+$ is the only pre-locally finite variety of Heyting algebras of width 2 (Theorem 4.27). This gives a positive solution to [10, Problem 12.1] in the case of varieties of width 2. This problem asks whether every non-locally finite variety of Heyting algebras has a pre-locally finite subvariety. Furthermore, we show that local finiteness is decidable for varieties of Heyting algebras of width 2 that are presented by finite sets of equations (Theorem 4.30). This gives a positive solution to Maksimova's problem (see [10, Problem 17.4]) for varieties of width 2. However, these problems remain open in the general case. We also prove that there are continua of non-locally finite and locally finite varieties of width 2 (Corollaries 4.23 and 4.26).

This thesis is organised as follows: In Chapter 2 we set out the background material the reader needs to tackle the rest of the thesis, as well as our notation and conventions. In Chapter 3 we review some basic facts about local finiteness, as well as a dual characterisation of it in terms of colorability of Esakia spaces. In Chapter 4 we turn to our original contribution. After developing the necessary technical machinery, we establish our characterisation of non-locally finite varieties of Heyting algebras of width 2. We then provide an example of a non-locally finite variety of Heyting algebras of width 3 that does not contain $\mathcal{L}_+$. After this we prove our results on pre-local finiteness, decidability, and the cardinality of the classes of locally finite and non-locally finite varieties of Heyting algebras of width 2. Finally, in Chapter 5 we summarise our results and point the way to possible future research.

Chapter 2

Preliminaries

2.1 Order and lattices

Throughout this thesis we will be primarily interested in structures that are based in partially ordered sets. In this section we set out the basic terminology and notation. A survey of the subject can be found in [12].

For an arbitrary binary relation $R \subseteq X^2$ on a set X, we write $R[Y] := \{x \in X \mid \exists y \in Y((y, x) \in R)\}$.

Definition 2.1. A *partially ordered set*, generally referred to as a *poset*, is a pair $\mathbf{X} = (X, \leq)$ with X a set and $\leq \subseteq X^2$ a binary relation which is:

- *Reflexive*: $x \leq x$,
- *Transitive*: $x \leq y$ and $y \leq z$ implies $x \leq z$,
- *Anti-symmetric*: $x \leq y$ and $y \leq x$ implies $x = y$,

for all $x, y, z \in X$.

Given a poset $\mathbf{X} = (X, \leq)$ and $x, y \in X$ we say x and y are *comparable* if $x \leq y$ or $y \leq x$. If they are not comparable we call them *incomparable* and write $x \parallel y$. A subset $Y \subseteq X$ with $|Y| \geq 2$ and pairwise incomparable elements is called an *antichain*.

A poset in which every pair of elements is comparable is called *total*, *linear*, or a *chain*.

Now fix some poset $\mathbf{X} = (X, \leq)$ with elements $x, y \in X$ such that $x \leq y$. We say that x is *less than*, a *predecessor* of, or *below* y and conversely that y is *greater than*, a *successor* of, or *above* x. If further $x \neq y$ we call the relation *strict* and write $x < y$. The notations $y \geq x$ and $y > x$ express the same situations. If for all z with $x \leq z \leq y$ it is the case that $z = x$ or $z = y$ and $x < y$ we say x and y are *direct* predecessors respectively successors of each other and denote this by $x \prec y$. The relation $\prec$ is also called the *covering* relation and such a pair $x \prec y$ is called a *covering pair*.

An *upset* in the poset $\mathbf{X}$ is a subset $Y \subseteq X$ such that for all $x \in Y$ and $y \in X$, if $x \leq y$ then $y \in Y$. Given an arbitrary subset $Y \subseteq X$, the *upset generated by* Y is the set

$$\uparrow Y := \{x \in X \mid \exists y \in Y(y \leq x)\}.$$

This is the smallest upset in $\mathbf{X}$ containing Y. If Y is a singleton $\{y\}$ we will write $\uparrow y$ instead of $\uparrow\{y\}$. The notion of a *downset* is defined dually in the obvious way. The downset generated by $Y \subseteq X$ is denoted $\downarrow Y$. The collection of upsets of a poset $\mathbf{X}$ is denoted by $\mathrm{Up}(\mathbf{X})$.

An element $x \in X$ is called *maximal* if $x \leq y$ implies $x = y$. It is called a maximum if for all $y \leq x$ for all $y \in X$. We similarly have x *minimal* if for all $y \leq x$ implies $y = x$ and call x a *minimum* if $x \leq y$ for all y. A poset with a minimum is called *rooted* and this minimal element is called the *root* of the poset. For a subset $Y \subseteq X$ we let $\max(Y)$ and $\min(Y)$ denote the sets of maximal and minimal elements of Y under the order of $\mathbf{X}$.

For an arbitrary poset $\mathbf{X}$ we write $\mathbf{X}^+$ and $\mathbf{X}_+$ for the posets obtained from $\mathbf{X}$ by adding respectively a new maximum and minimum element.

Given a subset $Y \subseteq X$ we define

$$Y^u := \{z \in X \mid \forall y \in Y(y \leq z)\} \quad \text{and} \quad Y^l := \{z \in X \mid \forall y \in Y(z \leq y)\}$$

to be respectively the sets of *upper bounds* and *lower bounds* of Y.

The *order dual* of a poset $\mathbf{X} = (X, \leq)$ is the poset $(X, \leq')$ with $x \leq' y$ iff $y \leq x$ for all $x \in X$.

Definition 2.2. A *lattice* is a poset $\mathbf{X} = (X, \leq)$ such that for all $x, y \in X$ the set $\{x, y\}^u$ has a minimum (denoted $x \vee y$ and called the *join* of x and y) and the set $\{x, y\}^l$ has a maximum (denoted $x \wedge y$ and called the *meet* of x and y). If $\mathbf{X}$ has a minimum and maximum we say it is *bounded* and write $\perp$ and $\top$ for these elements.

We may consider the join and meet as binary operations $\vee, \wedge : X \times X \to X$. If these distribute over each other, i.e.

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \;\text{ and }\; x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

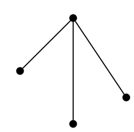
for all $x, y, z \in X$, we say the lattice $\mathbf{X}$ is *distributive*.

If $\mathbf{X} = (X, \leq)$ is a lattice it is not hard to see that its order dual is a lattice as well, with the roles of join and meet swapped.

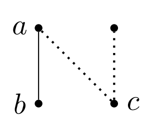
2.1.1 On diagrams

Here we establish some conventions on drawing diagrams representing posets. For a more formal treatment see e.g. [12, Section 1.15].

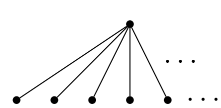
Finite posets are conventionally represented using so-called *Hasse diagrams*. These are fairly intuitive representations of a poset $\mathbf{X} = (X, \leq)$ by points and lines where if we wish to represent a relation $x < y$ of elements in $\mathbf{X}$, we ensure the point representing y is literally higher than the point representing x (in the sense of distance from the bottom of the page) and that the lines of the diagram trace a path moving strictly upward from the point representing x to that representing y. For example, the picture below to the left represents a three-element chain, while the picture below to the right is an acceptable (if somewhat wonky) representation of a three-element antichain below a single maximal element.



At a later stage we will need to be able to express a certain agnosticism as to whether a pair $x < y$ is in fact a covering pair $x \prec y$. To do this we adopt the convention in our diagrams that we connect two points $x < y$ with a solid line only if we *know* that $x \prec y$, and in all other situations we connect them with a dotted line. Points connected by dotted lines thus represent pairs $x < y$ that we either *know* not to be a covering pair, or for which the status as such is unknown. In the picture below we then have $b, c < a$ and we know $b \prec a$, but we express no knowledge of $c \prec a$.



Finally, we will extend these diagrams to representing infinite posets by employing the usual 'triple dots' implying continued structure as displayed before. As an example, the picture below represents an infinite antichain under a single maximal element.



2.2 Intuitionistic logic

In this section we discuss the syntax and semantics of the basic intuitionistic logic, as well as its extensions. For a thorough treatment the reader is referred to [10].

2.2.1 Syntax

We let our language consist of the symbols $\bot, \vee, \wedge$ and $\to$, and a fixed countable set of propositional variables $P = \{p_0, p_1, p_2, \ldots\}$. Formulas φ are defined by the following BNF:

$$\varphi ::= \bot \mid p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi$$

with p ranging over P. We write $\neg\varphi$ as shorthand for $\varphi \to \bot$, $\varphi \leftrightarrow \psi$ as shorthand for $(\varphi \to \psi) \wedge (\psi \to \varphi)$, and $\top$ as shorthand for $\neg\bot$. We denote the fact that the propositional variables that occur in a formula φ are among a set $\{p_0, \ldots, p_n\}$ by writing $\varphi(p_0, \ldots, p_n)$. Given a formula $\varphi(p_0, \ldots, p_n)$ and a set of formulas $\{\psi_0, \ldots, \psi_n\}$, we denote by $\varphi(\psi_0, \ldots, \psi_n)$ the formula obtained by substituting the formula ψ_i for each occurrence of the variable p_i.

We will make use of the following two derivation rules, named *modus ponens* and *uniform substitution* respectively:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ \mathbf{MP} \qquad\qquad \frac{\varphi(p_0, \ldots, p_n)}{\varphi(\psi_0, \ldots, \psi_n)}\ \mathbf{SUB}$$

Definition 2.3. The smallest set of formulas closed under **MP** and **SUB** and containing the following axioms

1. $p \to (q \to p)$,
2. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$,
3. $p \wedge q \to p$,
4. $p \wedge q \to q$,
5. $p \to p \vee q$,
6. $q \to p \vee q$,
7. $(p \to q) \to ((q \to r) \to ((p \vee q) \to r)))$,

8. $\bot \to p$,

is known as *intuitionistic propositional calculus*, abbreviated as IPC.

Definition 2.4. *Classical propositional calculus*, denoted CPC, is the smallest set of formulas closed under **MP** and **SUB** that contains IPC and the formula $p \vee \neg p$.

Definition 2.5. A set of formulas L closed under **MP** and **SUB** such that **IPC** $\subseteq \mathsf{L}$ is called a *superintuitionistic logic*. For the sake of brevity we usually write *si-logic* instead.

An si-logic L such that $\mathsf{L} \subseteq \mathsf{CPC}$ is called an *intermediate logic*. In fact, the only si-logic that is not intermediate is the *inconsistent logic* consisting of all formulas.

Definition 2.6. An si-logic L is said to be *axiomatised by* a set of formulas Φ if it is the smallest si-logic containing Φ. We denote this by $\mathsf{L} = \mathsf{IPC} + \Phi$, writing just $\mathsf{IPC} + \varphi$ if Φ is a singleton $\{\varphi\}$.

Throughout this thesis we will frequently refer to si-logics as just *logics*. Note that when ordered by inclusion the set of si-logics forms a lattice.

Definition 2.7. Let L be an intermediate logic. We say a formula φ is *derivable* in L from a set of formulas Γ, notation $\Gamma \vdash_{\mathsf{L}} \varphi$, if there exists a derivation of φ using premises in $\Gamma \cup \mathsf{L}$ and the rules **MP** and **SUB**.

2.2.2 Semantics

The semantics for intuitionistic logic traditionally involve so-called *intuitionistic Kripke frames*. These are really just partially ordered sets[1] and so we choose to present them as such.

Definition 2.8. An *intuitionistic Kripke model* (IKM) is a triple $\mathfrak{M} = (X, \leq, V)$ where $(X, \leq)$ is a poset and V is a *valuation function* $V : X \to \mathcal{P}(P)$ such that for all $x, y \in X$, if $x \leq y$ then $V(x) \subseteq V(y)$.

As all our models are intuitionistic we will refer to IKM's as just *models*.

Let $\mathfrak{M} = (X, \leq, V)$ be model and φ a formula. The truth of φ in $\mathfrak{M}$ at a point $x \in X$, notation $\mathfrak{M}, x \Vdash \varphi$ is defined inductively:

$$\begin{array}{lll}
\mathfrak{M}, x \Vdash \bot & & \text{Never} \\
\mathfrak{M}, x \Vdash p & \text{iff} & p \in V(x) \\
\mathfrak{M}, x \Vdash \varphi \vee \psi & \text{iff} & \mathfrak{M}, x \Vdash \varphi \text{ or } \mathfrak{M}, x \Vdash \psi \\
\mathfrak{M}, x \Vdash \varphi \wedge \psi & \text{iff} & \mathfrak{M}, x \Vdash \varphi \text{ and } \mathfrak{M}, x \Vdash \psi \\
\mathfrak{M}, x \Vdash \varphi \to \psi & \text{iff} & \text{For all } y \geq x, \text{ if } \mathfrak{M}, y \Vdash \varphi \text{ then } \mathfrak{M}, y \Vdash \psi
\end{array}$$

[1] Or pre-ordered sets, this makes no difference.

If $\mathfrak{M}, x \Vdash \varphi$ for all $x \in X$, we say that φ is *valid* on $\mathfrak{M}$. If for some poset $\mathbf{X} = (X, \leq)$ it is the case that for all valuations V, $(\mathbf{X}, V) \Vdash \varphi$, we likewise say φ is *valid* on $\mathbf{X}$. We denote the set of formulas valid on a poset $\mathbf{X}$ by $\mathrm{Log}(\mathbf{X})$ and call this the *logic of* $\mathbf{X}$. Given a class C of posets we write $\mathrm{Log}(C) := \{\varphi \mid \forall \mathbf{X} \in C(\mathbf{X} \Vdash \varphi)\}$. Likewise, given a set of formulas Φ we write $\mathrm{Fr}(\Phi)$ for the class of posets $\mathbf{X}$ such that $\mathbf{X} \Vdash \varphi$ for all $\varphi \in \Phi$.

The proof of the following is a routine check.

Proposition 2.9. *Let $\mathfrak{M} = (X, \leq, V)$ be a model and φ a formula. Then for all $x \in X$ we have $\mathfrak{M}, x \Vdash \varphi$ iff $\mathfrak{M}, y \Vdash \varphi$ for all $y \geq x$.*

Definition 2.10. Let $\mathbf{X} = (X, \leq)$ be a poset. A *generated subposet* of $\mathbf{X}$ is a poset $(X', \leq')$ where $X' \subseteq X$ and $\leq' = \leq \cap X \times X$ with the property that for all $x \in X'$, if $x \leq y$ then $y \in X'$.

Definition 2.11. A *bounded morphism*[2] from a poset $\mathbf{X} = (X, \leq)$ to a poset $\mathbf{Y} = (Y, \leq')$ is a map $f : X \to Y$ satisfying the following conditions:

(forth) If $x, y \in X$ are such that $x \leq y$, then $f(x) \leq' f(y)$.
(back) If $x \in X$ and $y' \in Y$ are such that $f(x) \leq' y'$, then there is a $y \in X$ with $x \leq y$ and $f(y) = y'$.

A bounded morphism f from one *model* $(X, \leq, V)$ to another $(X', \leq', V')$ is a bounded morphism of the underlying posets satisfying the further condition that for all $x \in X$ and $p \in P$:

$$p \in V(x) \text{ iff } p \in V'(f(x)).$$

This is sometimes referred to as the *atom* condition.

If a bounded morphism $f : \mathbf{X} \to \mathbf{Y}$ is surjective we say $\mathbf{Y}$ is a *bounded morphic image* of $\mathbf{X}$. Similarly for models.

Definition 2.12. Let $\{\mathbf{X}_i\}_{i \in I}$ be family of posets with $\mathbf{X}_i = (X_i, \leq_i)$. The *disjoint union* of this family is the poset $\biguplus_{i \in I} \mathbf{X}_i := (X, \leq)$ with

$$X := \bigcup_{i \in I} (X_i \times \{i\})$$

and $(x, i) \leq (y, j)$ iff $i = j$ and $x \leq_i y$. If $V_i : X_i \to \mathcal{P}(P)$ is a valuation function for each $i \in I$ we set

$$V : X \to \mathcal{P}(P) : (x, i) \mapsto V_i(x),$$

and call the model $(\mathbf{X}, V)$ the disjoint union of the family $\{(\mathbf{X}_i, V_i)\}_{i \in I}$.

[2] Some authors call these maps *p-morphisms*, see e.g.[7].

These three methods of producing new posets and models from old ones all preserve validity. The proof of the following is a routine check.

Theorem 2.13. *Let φ be a formula, $\mathbf{X}$ a poset, and $\{\mathbf{X}_i\}_{i\in I}$ a family of posets. We have the following:*

1. *If $\mathbf{X} \Vdash \varphi$ and $\mathbf{Y}$ is a generated subposet of $\mathbf{X}$ then $\mathbf{Y} \Vdash \varphi$.*
2. *If $\mathbf{X} \Vdash \varphi$ and $\mathbf{Y}$ is a bounded morphic image of $\mathbf{X}$ then $\mathbf{Y} \Vdash \varphi$.*
3. *If $\mathbf{X}_i \Vdash \varphi$ for each $i \in I$ then $\biguplus_{i\in I} \mathbf{X}_i \Vdash \varphi$.*

2.3 Examples of logics

We have already met the logics IPC and CPC. In this section we introduce a few others, including some that will play an important role in the rest of this thesis.

We say a class of posets C is *characterised by* a set of formulas Φ if for every poset $\mathbf{X}$ it is the case that $\mathbf{X} \in C$ iff $\mathbf{X} \Vdash \varphi$ for all $\varphi \in \Phi$.

Definition 2.14. Let $\mathbf{X} = (X, \leq)$ be a poset. The *width* of $\mathbf{X}$ is the size of the largest antichain in $\mathbf{X}$.

Definition 2.15. Let $\mathbf{X} = (X, \leq)$ be a poset. The *depth* of a point $x \in X$, notation $d(x)$ is the largest $n \in \omega$ for which a chain

$$x = x_0 \prec x_1 \ldots \prec x_n,$$

exists in $\mathbf{X}$. If no such n exists we say x is of *infinite depth* and write $d(x) = \omega$. The depth of $\mathbf{X}$ is given by $d(\mathbf{X}) = \sup_{x\in X} d(x)$.

If $\mathbf{X} = (X <\leq)$ is a poset and $x, y \in X$ are distinct points such that $d(x) = d(y) < \omega$ we say x and y are *siblings*.

For each $n \in \omega$ we can characterise the class of posets of *bounded depth* n (i.e. the class of posets where every point has depth at most n) with a single formula. We define these formulas inductively as follows:

$$\mathbf{bd_1} = p_1 \vee \neg p_1$$

and

$$\mathbf{bd_{n+1}} = p_{n+1} \vee (p_{n+1} \to \mathbf{bd_n}).$$

We then have the following:

Proposition 2.16. *For any poset* $\mathbf{X}$*:*

$$\mathbf{X} \Vdash \mathbf{bd_n} \ \textit{iff} \ d(\mathbf{X}) \leq n.$$

Proof. By induction. For the base case consider $n = 1$ and let $\mathbf{X} = (X, \leq)$ be a poset. First suppose $d(\mathbf{X}) \leq 1$ so that no point in X has a successor. Then for every valuation $V : X \to \mathcal{P}(P)$ and $x \in W$ we have either $p_1 \in V(x)$ or $p_1 \notin V(x)$. The former implies $(\mathbf{X}, V), x \Vdash p_1$ and the latter $(\mathbf{X}, V), x \Vdash p_1 \to \bot$, so that in any case $(\mathbf{X}, V), x \Vdash \mathbf{bd_1}$. Since V and x were arbitrary we get $\mathbf{X} \Vdash \mathbf{bd_1}$. Conversely, suppose $d(\mathbf{X}) > 1$. Then there are $x, y \in W$ with $x < y$. Now set $V(y) = \{p_1\}$ and $V(z) = \varnothing$ for all $z \neq y$. Then $(\mathbf{X}, V), x \nVdash p_1$ and, since $(\mathbf{X}, V), y \Vdash p_1$, $(\mathbf{X}, V), y \nVdash p_1 \to \bot$. But then $(\mathbf{X}, V), x \nVdash p_1 \to \bot$ as well, so that $(\mathbf{X}, V), y \nVdash p_1 \vee \neg p_1$.

Now for the inductive step suppose the statement holds for some $n \in \omega$ and consider the case for $n + 1$. First, suppose for some poset $\mathbf{X} = (X, \leq)$ we have $\mathbf{X} \nVdash \mathbf{bd_{n+1}}$. Then there are $x \in X$ and a valuation $V : X \to \mathcal{P}(P)$ such that $(\mathbf{X}, V), x \nVdash \mathbf{bd_{n+1}}$. That is:

$$(\mathbf{X}, V), x \nVdash p_{n+1} \vee (p_{n+1} \to \mathbf{bd_n}).$$

This means $(\mathbf{X}, V), x \nVdash p_{n+1}$ and there is a successor $v > w$ such that $(\mathbf{X}, V), y \Vdash p_{n+1}$ but $(\mathbf{X}, V), y \nVdash \mathbf{bd_n}$. Then $\mathbf{bd_n}$ is not valid on the poset $({\uparrow}y, \leq \cap ({\uparrow}y)^2)$ so that the depth of this poset is strictly greater than n. Since y is a root, this implies that $d(y) > n$. But since $x < y$, this gives $d(x) > n+1$, showing that $d(\mathbf{X}) > n+1$. Conversely, suppose $\mathbf{X} = (X, \leq)$ is such that $d(\mathbf{X}) > n + 1$. Then there is a $x \in X$ with $d(x) > n + 1$, meaning there exists a chain

$$x < y_{n+1} < y_n < \ldots < y_2 < y_1.$$

Now define a valuation function $V : X \to \mathcal{P}(P)$ by setting $p_i \in V(z)$ iff $z = y_i$. We shall show that under this valuation $(\mathbf{X}, V), x \nVdash \mathbf{bd_{n+1}}$. Suppose toward a contradiction that $(\mathbf{X}, V), x \Vdash \mathbf{bd_{n+1}}$. Then as $(\mathbf{X}, V), x \nVdash p_n + 1$ it must be the case that $p_n + 1 \to \mathbf{bd_n}$ holds at all successors of x, and so in particular $(\mathbf{X}, V), y_{n+1} \Vdash p_n + 1 \to \mathbf{bd_n}$. This gives $(\mathbf{X}, V), y_{n+1} \Vdash \mathbf{bd_n}$, and so, as $(\mathbf{X}, V), y_{n+1} \nVdash p_n$, that $p_n \to \mathbf{bd_n}$ holds at all successors of y_{n+1} and so in particular at y_n. Continuing on like this we finally obtain that $(\mathbf{X}, V), y_2 \Vdash \mathbf{bd_1} = p_1 \vee \neg p_1$. But this cannot be the case, as $p_1 \notin V(y_2)$ and y_2 has a successor with p_1, giving the desired contradiction. We may conclude that $\mathbf{X} \nVdash \mathbf{bd_{n+1}}$. $\square$

We cannot provide a similar sequence of formulas that bound the width of a poset, as the validity of such a formula would be preserved under disjoint unions which increase width. We *can* however provide for each $n \in \omega$ with $n \geq 1$ a formula $\mathbf{bw_n}$ that characterises bounded width n in rooted posets:

$$\mathbf{bw_n} := \bigvee_{i=0}^{n} (p_i \to \bigvee_{j \neq i} p_j).$$

Then we have the following:

Proposition 2.17. *For each poset* $\mathbf{X}$ *and* $n \in \omega$ *we have* $\mathbf{X} \Vdash \mathbf{bw_n}$ *iff every rooted subposet of* $\mathbf{X}$ *has width at most* n*.*

Proof. Fix some $n \in \omega$ with $n \geq 1$.

First, suppose $\mathbf{X} = (X, \leq)$ is such that $\mathbf{X} \nVdash \mathbf{bw_n}$. Then there is a point $x \in X$ and a valuation $V : X \to P$ such that $(\mathbf{X}, V), x \nVdash \mathbf{bw_n}$. Then for each $i \leq n$ we have

$$(\mathbf{X}, V), x \nVdash p_i \to \bigvee_{j \neq 1} p_j.$$

That is, for each $i \leq n$ there is a successor y_i of x with $(\mathbf{X}, V), y_i \Vdash p_i$ but $(\mathbf{X}, V), y_i \nVdash p_j$ for $i \neq j$. But then we have that these y_i must be pairwise incomparable and so from an anti-chain, so that $\uparrow w$ is a rooted subposet of $\mathbf{X}$ that has width greater than n.

Conversely, suppose without loss of generality that $\mathbf{X} = (X, \leq)$ is a poset with root x and $y_0, \ldots, y_n \in X$ such that $i \neq j$ implies $y_i \parallel y_j$. Then define a valuation $V : X \to P$ by setting $p_i \in V(z)$ iff $i \leq n$ and $z = y_i$ for all $z \in W$. Then clearly $(\mathbf{X}, V), y_i \Vdash p_1$ but $(\mathbf{X}, V), y_i \nVdash \bigvee_{j \neq i} p_j$, so that

$$(\mathbf{X}, V), y_i \nVdash p_i \to \bigvee_{j \neq i} p_j.$$

Now since x is a root of $\mathbf{X}$, this implies

$$(\mathbf{X}, V), x \nVdash p_i \to \bigvee_{j \neq i} p_j,$$

for each i as well, so that $(\mathbf{X}, V), x \nVdash \mathbf{bw_n}$. This gives $\mathbf{X} \nVdash \mathbf{bw_n}$ as desired. □

The smallest si-logic containing $\mathbf{bw_1}$ is called the *linear calculus* or *Dummett's logic* and is denoted LC.

Let L be an intermediate logic. If $\mathbf{bd_n} \in \mathsf{L}$ we say it is of *bounded depth* n. Similarly, if $\mathbf{bw_n} \in \mathsf{L}$ we say L is of *bounded width* n or just *width* n.

2.4 Universal algebra

Universal algebra presents a framework developed to study algebraic structure in a very general sense. We are primarily interested in the algebraic presentation of a certain type of lattice know as *Heyting algebras* which provide an algebraic interpretation of si-logics. We nevertheless present the material in its most general from. A good introduction to the subject can be found in [9].

Definition 2.18. A *type* $\mathcal{F}$ consists of a set F of function symbols together with a function $a_{\mathcal{F}} : F \to \omega$ assigning to each symbol a finite *arity*. Types are often also referred to as *signatures*.

Definition 2.19. Given a type $\mathcal{F}$, an *algebra* $\mathcal{A}$ *of type* $\mathcal{F}$ consists of a non-empty set A together with an n-ary operation $s^{\mathcal{A}} : A^{a_{\mathcal{F}}(s)} \to A$ for each $s \in F$. The set A is called the *carrier* of $\mathcal{A}$.

We will always write algebras of any type in calligraphic font. Whenever we discuss an algebra denoted e.g. $\mathcal{A}$ or $\mathcal{B}$ we will let the plain font A and B denote their carriers without further comment. We often suppress the superscript of operations $s^{\mathcal{A}}$ when this causes no confusion.

Definition 2.20. Given a type $\mathcal{F}$ and a set of variables X we define the set of $\mathcal{F}$-*terms* over X, denoted $T_{\mathcal{F}}(X)$, inductively:

- $x \in T_{\mathcal{F}}(X)$ for every variable $x \in X$
- $s(t_1, \ldots, t_n) \in T_{\mathcal{F}}(X)$ for each n-ary $s \in F$ and $t_1, \ldots, t_n \in T_{\mathcal{F}}(X)$

Terms with binary symbols s will be written with infix notation.

Definition 2.21. An *equation* over a set of variables X in a type $\mathcal{F}$ is a pair of terms $(t_1, t_2) \in T_{\mathcal{F}}(X)$, conventionally presented as $t_1 \doteq t_2$. Given an $\mathcal{F}$-algebra $\mathcal{A}$ we say the equation $t_1 \doteq t_2$ *holds* in $\mathcal{A}$ or is *satisfied* by $\mathcal{A}$ if for every possible assignment of elements of A to the variables in t_1, t_2 the resulting expression is true in $\mathcal{A}$.

From this point on we fix a set of variables $\{x, y, z, x_1, x_2, \ldots\}$ and assume terms are over this set unless specified otherwise.

Example 2.22. We have the following examples of types and algebras:

- Bounded latices are algebras of the type $\{\vee, \wedge, \bot, \top\}$ with $\vee$ and $\wedge$ binary, and $\bot$ and $\top$ nullary, satisfying the following equations:

$$\begin{array}{ll} x \vee x \doteq x & x \wedge x \doteq x \\ x \vee y \doteq y \vee x & x \wedge y \doteq y \wedge x \\ x \vee (y \vee z) \doteq (x \vee y) \vee z & x \wedge (y \wedge z) \doteq (x \wedge y) \wedge z \\ x \vee \bot \doteq x & x \wedge \top \doteq x \\ x \vee (y \wedge x) \doteq x & x \wedge (y \vee x) \doteq x \end{array}$$

  In addition, distributive bounded lattices satisfy:

$$\begin{array}{l} x \vee (y \wedge z) \doteq (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) \doteq (x \wedge y) \vee (x \wedge z) \end{array}$$

- Groups are algebras of type $\{\circ, (-)^{-1}, e\}$ with $\circ$ binary, $(-)^{-1}$ unary, and e nullary, satisfying the equations:

$$\begin{array}{ll} x \circ e \doteq x & e \circ x \doteq x \\ x \circ x^{-1} \doteq e & x^{-1} \circ x \doteq e \\ x \circ (y \circ z) \doteq (x \circ y) \circ z & \end{array}$$

Definition 2.23. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of the same type $\mathcal{F}$. A *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a map $f : A \to B$ such that for all function symbols s in $\mathcal{F}$ and $x_1, \ldots, x_{a_{\mathcal{F}}(s)}$:

$$f(s^{\mathcal{A}}(x_1, \ldots, x_{a_{\mathcal{F}}(s)})) = s^{\mathcal{B}}(f(x_1), \ldots, f(x_{a_{\mathcal{F}}(s)})).$$

If such a map is surjective, we say that $\mathcal{B}$ is a *homomorphic image* of $\mathcal{A}$.

If f is bijective we call it an *isomorphism* and say $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*. We denote this by $\mathcal{A} \cong \mathcal{B}$.

Definition 2.24. Let $\mathcal{A}$ be an algebra of type $\mathcal{F}$. A *congruence* θ on $\mathcal{A}$ is an equivalence relation $\theta \subseteq A \times A$ such that for all $s \in F$ and $x_1, \ldots, x_{a_{\mathcal{F}}(s)}, y_1, \ldots, y_{a_{\mathcal{F}}(s)}$ with $(x_i, y_i) \in \theta$ for each $1 \leq i \leq a_{\mathcal{F}}(s)$ we have

$$(s^{\mathcal{A}}(x_1, \ldots, x_{a_{\mathcal{F}}(s)}), s^{\mathcal{A}}(y_1, \ldots, y_{a_{\mathcal{F}}(s)})) \in \theta.$$

The set of congruences of an algebra $\mathcal{A}$ is denoted by $\mathrm{Con}(\mathcal{A})$. When ordered by inclusion, $\mathrm{Con}(\mathcal{A})$ forms a lattice with meet of two congruences θ_1 and θ_2 being their intersection and their join given by $(x, y) \in \theta_1 \vee \theta_2$ iff there is a sequence of elements $z_1, \ldots, z_n \in A$ such that $x = z_1$, $y = z_n$ and for each $i \in \{1, \ldots, n-1\}$

$$(z_i, z_{i+1}) \in \theta_1 \quad \text{or } (z_i, z_{i+1}) \in \theta_2,$$

(see [9, Sections I.4 and II.5]).

Definition 2.25. Given an algebra $\mathcal{A}$ in some type $\mathcal{F}$ and a congruence θ on $\mathcal{A}$ we define the *quotient algebra* $\mathcal{A}/\theta$ with carrier the set A/θ of equivalence classes $\overline{x}$ under θ and for each $s \in F$:

$$s^{\mathcal{A}/\theta}(\overline{x_1}, \ldots, \overline{x_{a_{\mathcal{F}}(s)}}) = \overline{s^{\mathcal{A}}(x_1, \ldots, x_{a_{\mathcal{F}}(s)})}.$$

For any algebra $\mathcal{A}$ and $\theta \in \mathrm{Con}(\mathcal{A})$ the map from $\mathcal{A}$ to $\mathcal{A}/\theta$ that maps an element to its equivalence class is a surjective homomorphism (see [9, Theorem II.6.10]). Conversely, given a surjective homomorphism of algebras $h : \mathcal{A} \to \mathcal{B}$ the *kernel* of h:

$$\ker(h) := \{(x, y) \in A \times A \mid h(x) = h(y)\}$$

is a congruence on $\mathcal{A}$ (see [9, Theorem II.6.8]). We then have the following well-known *Homomorphism Theorem*. For a proof see [9, Theorem II.6.12].

Theorem 2.26. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of the same type and $h : \mathcal{A} \twoheadrightarrow \mathcal{B}$ a surjective homomorphism. Further let $g : \mathcal{A} \to \mathcal{A}/\ker(h)$ map an element to its equivalence class. Then there is an isomorphism $f : \mathcal{A}/\ker(h) \to \mathcal{B}$ such that $f \circ g = h$.*

In light of this we may freely interchange congruences on and homomorphic images of an algebra $\mathcal{A}$.

Definition 2.27. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of the same type such that the carrier of $\mathcal{A}$ is a subset of the carrier of $\mathcal{B}$. We say $\mathcal{A}$ is a *subalgebra* of $\mathcal{B}$ if the inclusion map $\iota : \mathcal{A} \to \mathcal{B} : x \mapsto x$ is a homomorphism.

Definition 2.28. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of the same type $\mathcal{F}$. The *(direct) product* of $\mathcal{A}$ and $\mathcal{B}$, denoted $\mathcal{A} \times \mathcal{B}$ is the $\mathcal{F}$-algebra with carrier $A \times B$ and for each $s \in \mathcal{F}$ an operation given by:

$$s^{\mathcal{A}\times\mathcal{B}}((x_1, y_1), \ldots, (x_{a_{\mathcal{F}}(s)}, y_{a_{\mathcal{F}}(s)})) := (s^{\mathcal{A}}(x_1, \ldots, x_{a_{\mathcal{F}}(s)}), s^{\mathcal{B}}(y_1, \ldots, y_{a_{\mathcal{F}}(s)})).$$

More generally, let $\{\mathcal{A}_i\}_{i\in I}$ be a family of algebras of type $\mathcal{F}$. The product of this family is then defined as having carrier $\prod_{i\in I} A_i$ and setting for each $s \in \mathcal{F}$ and $i \in I$:

$$s^{\prod_{i\in I}\mathcal{A}_i}(f_1, \ldots, f_{a_{\mathcal{F}}(s)})(i) := s^{\mathcal{A}_i}(f_1(i), \ldots, f_{a_{\mathcal{F}}(s)}(i)).$$

Given such a product we set for each $j \in I$:

$$\pi_j : \prod_{i\in I} \mathcal{A}_i \to \mathcal{A}_j : f \mapsto f(j)$$

the *projection to the j-th coordinate* which is always a homomorphism.

For an arbitrary class of algebras K of the same type $\mathcal{F}$ we let $\mathbf{H}(\mathsf{K})$ denote the operation mapping K to the class of $\mathcal{F}$-algebras that are the homomorphic image of an algebra in K. Likewise we write $\mathbf{S}$ and $\mathbf{P}$ for the operations closing K under subalgebras and products. When applying these operations in succession we often omit superfluous parentheses.

Definition 2.29. A class of algebras of the same type closed under the operations $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ is called a *variety.* Given an arbitrary class K we denote the smallest variety containing K by $\mathbf{V}(\mathsf{K})$ and call this the *variety generated by* K.

A proof of the following can be found in [9, Theorem II.9.5]

Theorem 2.30 (Tarski)**.** *For every class* K *of algebras of the same type we have* $\mathbf{V}(\mathsf{K}) = \mathbf{HSP}(\mathsf{K})$*.*

It turns out to be the case that $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ preserve validity of equations, something that readily be checked. The following result shows in addition that a class of algebras defined by some set of equations is closed under these operations. For a proof, see e.g. [9, Theorem II.11.9]

Theorem 2.31 (Birkhoff)**.** *A class of algebras* K *is a variety iff it is equationally definable.*

As a consequence of this result we obtain that the classes of lattices and groups from varieties.

2.4.1 Subdirectly irreducible algebras

Definition 2.32. An algebra $\mathcal{A}$ is a *subdirect product* of a family $\{\mathcal{A}_i\}_{i\in I}$ if it is isomorphic to a subalgebra $\mathcal{B} \overset{\iota}{\hookrightarrow} \prod_{i\in I} \mathcal{A}_i$ with the property that $\pi_j \circ \iota$ is surjective for each $j \in I$. We call the induced injective homomorphism $\mathcal{A} \hookrightarrow \prod_{i\in I} \mathcal{A}_i$ a *subdirect embedding.*

Definition 2.33. An algebra $\mathcal{A}$ is called *subdirectly irreducible* or *SI* if for every subdirect embedding $\mathcal{A} \overset{f}{\hookrightarrow} \prod_{i\in I} \mathcal{A}_i$ there is a $j \in I$ such that the composition $\pi_j \circ f$ is an isomorphism.

Given a class of algebras K we denote the class of subdirectly irreducible elements of K by K_{SI}. The following is essentially [9, Corollary II.9.7].

Theorem 2.34. *Let* K *be a variety. Then* $\mathsf{K} = \mathbf{V}(\mathsf{K}_{SI})$*.*

2.5 Heyting algebras

Definition 2.35. If $\mathcal{A}$ is a bounded distributive lattice such that for all $x, y \in X$ there exists an element $x \to y$ (called a *Heyting implication*) with the property that for all $z \in X$:

$$z \wedge x \leq y \text{ iff } z \leq x \to y,$$

we call $\mathcal{A}$ a *Heyting algebra*.

There is a purely algebraic presentation of Heyting algebras.

Definition 2.36. A *Heyting algebra* $\mathcal{A} = (A, \vee, \wedge, \to, \bot, \top)$ is an algebra of type $\{\vee, \wedge, \to, \bot, \top\}$ with $\vee, \wedge$ and $\to$ binary, and $\bot$ and $\top$ nullary such that $(\vee, \wedge, \bot, \top)$ is a bounded distributive lattice and $\mathcal{A}$ further satisfies the equations:

$$\begin{array}{ll} x \to x \doteq \top & x \wedge (x \to y) \doteq x \wedge y \\ y \wedge (x \to y) \doteq y & x \to (y \wedge z) \doteq (x \to y) \wedge (x \to z) \end{array}$$

Following the conventions for the notation of formulas we write $\neg x$ as shorthand for the element $x \to \bot$ and $x \leftrightarrow y$ as shorthand for the element $(x \to y) \wedge (y \to x)$.

It follows from this equational definition that the class of Heyting algebras forms a variety.

Definition 2.37. Let $\mathcal{A}$ be a Heyting algebra. A subset $F \subseteq X$ is called a *filter* if the following hold:

1. $F \neq \varnothing$,
2. if $x \leq y \in X$ and $x \in F$, then $y \in F$,
3. if $x, y \in F$ then $x \wedge y \in F$.

If F is such that $F \neq X$ then it is called *proper*. A *prime* filter is a proper filter F such that $x \vee y \in F$ implies $x \in F$ or $y \in F$. The set of prime filters of $\mathcal{A}$ is denoted by $\mathrm{PFilt}(\mathcal{A})$.

We have the following useful characterisation of subdirectly irreducible Heyting algebras. A proof can be found in [1, Theorem IX.5]

Theorem 2.38. *A Heyting algebra is subdirectly irreducible iff it has a second greatest element.*

2.5.1 Logic and algebras

There is a correspondence between formulas and equations in the signature of Heyting algebras that allows us to talk about validity of formulas in algebras.

Definition 2.39. Let $t_1 \doteq t_2$ be an equation in the type of Heyting algebras, and φ some formula. We define for an arbitrary term t the translation $(t)_*$ to be the term obtained from t by replacing each variable x in t by a fresh propositional variable. Then define the following translations:

- $(t_1 \doteq t_2)_* = (t_1)_* \leftrightarrow (t_2)_*$,
- $\varphi^* = (\varphi \doteq \top)$ seen as an equation over P.

For a set of equations Σ we denote by Σ_* the set of translations. Likewise the set of translations of a set of formulas Φ is denoted by Φ^*.

Definition 2.40. Let $\mathcal{A}$ be a Heyting algebra and φ a formula. We say that φ *holds in* or is *satisfied by* $\mathcal{A}$ if the equation φ^* is satisfied by $\mathcal{A}$ and denote this by $\mathcal{A} \models \varphi$.

This definition allows us to talk about varieties of Heyting algebras being axiomatised by sets of formulas. In fact, there is an isomorphism between the lattice of varieties of Heyting algebras and the order dual of the lattice of si-logics given by sending a variety K to the set of translations of those equations that are satisfied in every member of K. Given a variety K of Heyting algebras we write $\mathrm{Log}(\mathsf{K})$ for the si-logic associated with K. Likewise we write V_{L} for the variety of Heyting algebras axiomatised by the translations of formulas in L for any si-logic L.

2.6 Esakia spaces

Esakia spaces were introduced by Esakia in order to provide a topological counterpart to Heyting algebras.

Definition 2.41. Let X be a set, a *topology* on X is a collection $\mathcal{O} \subseteq \mathcal{P}(X)$ satisfying the following three conditions:

1. $\varnothing, X \in \mathcal{O}$,
2. If $U, V \in \mathcal{O}$, then $U \cap V \in \mathcal{O}$,

3. For any collection $\{U_i\}_{i\in I}$, if $U_i \in \mathcal{O}$ for each $i \in I$, then also $\bigcup_{i\in I} U_i \in \mathcal{O}$.

Such a pair $(X, \mathcal{O})$ is called a *topological space*. Provided it causes no confusion, we will refer to such a space by its underlying set. The elements of $\mathcal{O}$ are called *open* subsets of X. A subset $U \subseteq X$ is called *closed* if $X \setminus U$ is open, and will be referred to as *clopen* if it is both open and closed. The set of clopens of X is denoted by $\mathrm{Clop}(X)$.

One special topology that can be defined on any set X is the so-called *discrete* topology $\mathcal{O} = \mathcal{P}(X)$.

Definition 2.42. Let $(X, \mathcal{O})$ and $(X', \mathcal{O}')$ be topological spaces. A function $f : X \to X'$ is called *continuous* if $f^{-1}(U) \in \mathcal{O}$ for every $U \in \mathcal{O}'$.

If f is bijective and f^{-1} is continuous as well, we say f is a *homeomorphism* and call X and X' *homeomorphic*

Definition 2.43. Let $(X, \mathcal{O})$ be a topological space. We say a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is a *basis* for this space if the following hold:

1. $\bigcup \mathcal{B} = X$,

2. For every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

3. $U \in \mathcal{O}$ iff U is an arbitrary union of elements from $\mathcal{B}$.

Definition 2.44. A *subspace* of a topological space $(X, \mathcal{O})$ is a subset $Y \subseteq X$ endowed with the topology generated by the basis

$$\{Y \cap U \mid U \in \mathcal{O}\}.$$

This topology on Y is referred to as the *subspace topology*.

Definition 2.45. For any set X, a *subbasis* $\mathcal{S}$ is a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ such that $\bigcup \mathcal{S} = X$. The *topology generated by* $\mathcal{S}$ has as open sets all arbitrary unions of finite intersections of members of $\mathcal{S}$.

Definition 2.46. Let $(X, \mathcal{O})$ be a topological space. If X is:

- *Compact*, i.e. for every collection $\{U_i\}_{i\in I} \subseteq \mathcal{O}$ with $\cup_{i\in I} U_i = X$ there is a finite $I_0 \subseteq I$ with $\cup_{i\in I_0} U_i = X$,
- *Hausdorff*, i.e. for every pair of points $x \neq y \in X$ there are $U_x, U_y \in \mathcal{O}$ with $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \varnothing$,

- *Zero-dimensional*, i.e. X has a basis of clopens,

then X is called a *Stone space*.

Now let $(X, \leq)$ be a poset. If $\mathcal{O} \subseteq \mathcal{P}(X)$ is such that $(X, \mathcal{O})$ is a Stone space satisfying the following *Priestley separation axiom*:

if $x \not\leq y$ then there is a clopen upset $U \subseteq X$ such that $x \in U$ and $y \notin U$,

we say the triple $(X, \mathcal{O}, \leq)$ is a *Priestley space*. In fact, we do not need to demand that X is a Stone space as every compact ordered space satisfying the Priestley separation axiom is Hausdorff and zero-dimensional.

If $\mathbf{X} = (X, \mathcal{O}, \leq)$ is a Priestley space such that $\downarrow U \in \mathrm{Clop}(\mathbf{X})$ for all $U \in \mathrm{Clop}(\mathbf{X})$ we say it is an *Esakia space*.

Given an Esakia space $\mathbf{X} = (X, \mathcal{O}, \leq)$ we write $\mathrm{ClopUp}(\mathbf{X})$ for the set of clopen upsets of $\mathbf{X}$.

Remark 2.47. Every finite Hausdorff space is discrete. We can see this by noting that if X is a finite Hausdorff space with $x \in X$, we can take for each $y \in X$ distinct from x an open U_x^y that contains x but not y. The (finite) intersection over these U_x^y will then be open and equal to $\{x\}$. As arbitrary unions of opens are open, we see that every subset in $\mathcal{P}(X)$ is open, so the topology is discrete.

Further, if $(X, \mathcal{O})$ is finite, we note that any collection $\{U_i\}_{i \in I} \subseteq \mathcal{O}$ must be finite as well, so that X is compact. Similarly note that every topology is a basis for itself, and that in a discrete space every subset is clopen, so that every discrete space is zero-dimensional. Then we can conclude that every finite Hausdorff space X is a Stone space under the discrete topology, and that this is the unique topology on X that makes it so.

Now consider a poset $(X, \leq)$ endowed with the discrete topology $\mathcal{O} = \mathcal{P}(X)$. Then for any $x \not\leq y$ the set $\uparrow x$ is a clopen upset that contains x but omits y, showing that X is a Priestley space. The fact that all subsets of X are clopen shows that it is an Esakia space. We conclude that every finite poset can be uniquely associated with the Esakia space based on it given by the discrete topology.

Definition 2.48. An *E-subspace* of an Esakia space $(X, \mathcal{O}_X, \leq_X)$ is an Esakia space $(Y, \mathcal{O}_Y, \leq_Y)$ where $(Y, \mathcal{O}_Y)$ is a subspace of $(X, \mathcal{O}_X)$, Y is closed in X, and $(Y, \leq_Y)$ is an upset in $(X, \leq_X)$.

When ordered by inclusion, the E-subspaces of an Esakia space $\mathbf{X}$ form a lattice.

Definition 2.49. A map $f : \mathbf{X} \to \mathbf{Y}$ between two Esakia spaces $\mathbf{X} = (X, \mathcal{O}_X, \leq_X)$ and $\mathbf{Y} = (Y, \mathcal{O}_Y, \leq_Y)$ is called an *Esakia morphism* or sometimes just *bounded morphism* if it is a bounded morphism when viewed as a map of the posets $(X, \leq_X)$ and $(Y, \leq_Y)$, and continuous when viewed as a map of the spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$.

We can add a valuation to an Esakia space $\mathbf{X} = (X, \mathcal{O}, \leq)$ just as we did to simple posets to view them as models. This allows us to evaluate validity formulas as before.

Definition 2.50. Let $\mathbf{X} = (X, \mathcal{O}, \leq)$ be an Esakia space. A *valuation* on $\mathbf{X}$ is a function $V : X \to \mathcal{P}(P)$ such that $V^{-1}(p) \in \text{ClopUp}(\mathbf{X})$ for all $p \in P$.

We say a formula φ is *valid* in $\mathbf{X}$ if $(X, \leq, V) \Vdash \varphi$ for all valuations V satisfying the condition above.

Remark 2.51. We may extend the notions of generated subposets and bounded morphic images to the setting of Esakia spaces and morphisms in the natural way. We consider bounded morphisms between Esakia spaces to be continuous bounded morphisms of the underlying posets, and generated subposets of Esakia spaces are just E-subspaces. Likewise we can define the disjoint union $\biguplus_{i \in I} \mathbf{X}_i$ of a finite family of Esakia spaces as the disjoint union of the underlying posets with a topology generated by the basis:

$$\{U \times \{i\} \mid U \textit{ open in } \mathbf{X}_i\}.$$

Which we call the *topological sum* of these spaces.

2.7 Duality

In this section we describe the duality between Heyting algebras and Esakia spaces. This was first established by Esakia in [13] and is more fully developed in [14].

First, let $\mathcal{A}$ be a Heyting algebra. We define a map $\varphi : \mathcal{A} \to \mathcal{P}(\text{PFilt}(\mathcal{A}))$ by setting for each $x \in A$:

$$\varphi(x) := \{F \in \text{PFilt}(\mathcal{A}) \mid x \in F\}.$$

Then we define an ordered topological space $\mathcal{A}_* := (\text{PFilt}(\mathcal{A}), \mathcal{O}, \subseteq)$ where $\mathcal{O}$ is the topology on $\text{PFilt}(\mathcal{A})$ generated by the subbasis:

$$\{\varphi(x), A \setminus \varphi(x) \mid x \in A\}.$$

Then $\mathcal{A}_*$ is an Esakia space called the *dual space* of $\mathcal{A}$.

Conversely, let $\mathbf{X}$ be an Esakia space. Then we define a Heyting implication $\to$ on $\mathrm{ClopUp}(\mathbf{X})$ by setting for all $U, V \in \mathrm{ClopUp}(\mathbf{X})$:

$$U \to V = X \setminus {\downarrow}(U \setminus V).$$

The algebra $\mathbf{X}^* := (\mathrm{ClopUp}(\mathbf{X}), \cup, \cap, \to, \varnothing, X)$ is then a Heyting algebra called the *dual algebra* of $\mathbf{X}$.

We can likewise associate with every homomorphism between Heyting algebras $f : \mathcal{A} \to \mathcal{B}$ an Esakia morphism $f_* : \mathcal{B}_* \to \mathcal{A}_*$ defined by $f_*(F) := f^{-1}(F)$ for each $F \in \mathrm{PFilt}(\mathcal{B})$. Additionally we have for each Esakia morphism $g : \mathbf{X} \to \mathbf{Y}$ a homomorphism of Heyting algebras $g^* : \mathbf{Y}^* \to \mathbf{X}^*$ given by $g^*(U) = g^{-1}(U)$ for each $U \in \mathrm{ClopUp}(\mathbf{Y})$.

Now denote by HA and ESP the categories of Heyting algebras with homomorphisms and Esakia spaces with Esakia morphisms respectively. Then the maps $(-)_* : \mathsf{HA} \to \mathsf{ESP}$ and $(-)^* : \mathsf{ESP} \to \mathsf{HA}$ are contravariant functors and we have the following:

Theorem 2.52. *The functors $(-)_*$ and $(-)^*$ witness a dual equivalence between the categories HA and ESP.*

For a variety of Heyting algebras K we write K_* for the class of Esakia spaces dual to an algebra in K. With this duality in hand, we can make sense of an important piece of terminology:

Definition 2.53. A Heyting algebra $\mathcal{A}$ is called *width n* if the poset underlying its dual Esakia space $\mathcal{A}_*$ is width n, i.e. all rooted subposets of $\mathcal{A}_*$ have no anti-chains of size more than n.

A variety K of Heyting algebras is said to be *of width n* if all its algebras are.

Remark 2.54. Recalling the remark that every finite poset can be viewed as an Esakia space in exactly one way, namely by endowing it with the discrete topology, we note that if $\mathbf{X}$ is a finite Esakia space the dual algebra $\mathbf{X}^*$ has as carrier $\mathrm{Up}(\mathbf{X})$. This dual is called the *algebra of upsets* of $\mathbf{X}$ and is often denoted just by $\mathrm{Up}(\mathbf{X})$ where this causes no confusion.

The following theorem shows validity-preserving operations on Esakia spaces and Heyting algebras correspond. It is proved in [7, Theorem 2.3.7]

Theorem 2.55. *Let $\mathbf{X}, \mathbf{Y}$ be Esakia spaces and $\{\mathbf{X}_i\}_{i \leq n \in \omega}$ a finite family of Esakia spaces. Then:*

1. $\mathbf{X}$ *is homeomorphic to a generated subposet of* $\mathbf{Y}$ *iff* $\mathbf{X}^*$ *is a homomorphic image of* $\mathbf{Y}^*$*.*
2. $\mathbf{X}$ *is a bounded morphic image of* $\mathbf{Y}$ *iff* $\mathbf{X}^*$ *is isomorphic to a subalgebra of* $\mathbf{Y}^*$*.*
3. $(\biguplus_{i\leq n}\mathbf{X}_i)^*$ *is isomorphic to* $\prod_{i\leq n}\mathbf{X}_i^*$*.*

Now let $\mathcal{A},\mathcal{B}$ *be Esakia spaces and* $\{\mathcal{A}_i\}_{i\leq n\in\omega}$ *a finite family of Heyting algebras. Then:*

a. $\mathcal{A}$ *is a homomorphic image of* $\mathcal{B}$ *iff* $\mathcal{A}_*$ *is isomorphic to a generated subposet of* $\mathcal{B}_*$*.*
b. $\mathcal{B}$ *is a subalgebra of* $\mathcal{B}$ *iff* $\mathcal{A}_*$ *is homeomorphic to a bounded morphic image of* $\mathcal{B}_*$*.*
c. $(\prod_{i\leq n}\mathcal{A})_*$ *is homeomorphic to* $\biguplus_{i\leq n}\mathcal{A}_*$ *.*

For a class of Heyting algebras K and a class of Esakia spaces C we write K_* and C^* for the classes of dual spaces and algebras respectively. Then the above theorem gives us that K is a variety iff the class K_* is closed under generated subposets, bounded morphic images, and disjoint unions.

Lemma 2.56. *Let* $\mathcal{A}$ *be a Heyting algebra. Then there is a lattice isomorphism* f *between the lattice of congruences on* $\mathcal{A}$ *and the order dual of the lattice of E-subspaces of* $\mathcal{A}_*$ *with the property that for any* $\theta\in\mathrm{Con}(\mathcal{A})$ *the space* $(\mathcal{A}/\theta)_*$ *is homeomorphic to* $f(\theta)$ *and for any E-subspace* $\mathbf{Y}$ *of* $\mathcal{A}_*$ *the algebra* $\mathbf{Y}^*$ *is isomorphic to* $\mathcal{A}/f^{-1}(\mathbf{Y})$*.*

For a binary relation R on a set X and subset $Y\subseteq X$ write

$$R[Y]:=\{x\in X\mid \exists y\in Y \text{ s.t. } (y,x)\in R\}.$$

Definition 2.57. Let $(X,\mathcal{O},\leq)$ be an Esakia space. An equivalence relation E on X is called a *bisimulation equivalence* if

1. For every $x\in X$ we have $\uparrow E[x]\subseteq E[\uparrow x]$.
2. If $(x,y)\notin E$ then there is a $U\in\mathrm{ClopUp}(X)$ with $E[U]=U$ and one of $x\in U$ and $y\notin U$, or $y\in U$ and $x\notin U$.

The bisimulation equivalences of an Esakia space $\mathbf{X}$ form a lattice when ordered by inclusion.

Theorem 2.58. *Let* $\mathbf{X}$ *be an Esakia space. Then there is a one-to-one correspondence between bisimulation equivalences on* $\mathbf{X}$ *and bounded morphic images of* $\mathbf{X}$*.*

For a proof see e.g. [7, Theorem 2.3.9]. The correspondence is given by defining for each surjective bounded morphism $f : \mathbf{X} \to \mathbf{Y}$ an equivalence relation $E_f \subseteq X \times X$ such that $(x, y) \in E_f$ iff $f(x) = f(y)$. Conversely define for each bisimulation equivalence E on an Esakia space $\mathbf{X} = (X, \mathcal{O}, \leq)$ the quotient space $\mathbf{X}/E$ as $(X/E, \mathcal{O}_E, \leq')$ where X/E consists of the equivalence classes $\overline{x}$ of elements in X,

$$\overline{x} \leq' \overline{y} \quad \text{iff there are } x' \in \overline{x} \text{ and } y' \in \overline{y} \text{ such that } x' \leq y',$$

and $\mathcal{O}_E := \{U \in \mathcal{O} \mid E[U] = U\}$. Then we get a surjective bounded morphism $f_E : \mathbf{X} \to \mathbf{X}/E$ by mapping each point of $\mathbf{X}$ to its equivalence class.

Given the known correspondence between subalgebras of a Heyting algebra and bounded morphic images of its dual space we obtain the following corollary.

Corollary 2.59. *Let $\mathcal{A}$ be a Heyting algebra. Then there is a one-to-one correspondence between subalgebras of $\mathcal{A}$ and bisimulation equivalences on $\mathcal{A}_*$.*

Although this follows immediately from Theorems 2.58 and 2.55 we sketch the correspondence for the sake of clarity. First, given a Heyting algebra $\mathcal{A}$ with a subalgebra $\mathcal{B}$ we define an equivalence relation $E_{\mathcal{B}}$ by setting for each $x, y \in \mathcal{A}_*$

$$(F, G) \in E_{\mathcal{B}} \quad \text{iff} \quad F \cap B = G \cap B.$$

For the other direction, let E be a bisimulation equivalence on $\mathcal{A}_* = (X, \mathcal{O}, \leq)$. Then the set $\mathcal{O}_E \subseteq \mathcal{O}$ given by

$$U \in \mathcal{O}_E \quad \text{iff} \quad E[U] = U$$

is the carrier of a subalgebra of $\mathcal{A}$.

Finally we present a method for easily checking whether there exists a surjective bounded morphism between finite Esakia spaces $\mathbf{X}$ and $\mathbf{Y}$. Then next two lemmas were first proven in [15], we borrow our presentation from [7]. The proof of the first is a routine check. For a proof of the second see [7, Lemma 3.1.7]

Lemma 2.60. *Let $\mathbf{X} = (X, \mathcal{O}, \leq)$ be a finite Esakia space and $x, y \in X$.*

1. *Suppose y is unique such that $x \prec y$. Then the smallest equivalence relation E with $(x, y) \in E$ is a bisimulation equivalence. We call the map $E_f : \mathbf{X} \to \mathbf{X}/E$ an* α-reduction.

2. *Suppose for all $z \in X$ that $x \prec z$ iff $y \prec z$. Then the smallest equivalence relation E with $(x, y) \in E$ is a bisimulation equivalence. We call the map $E_f : \mathbf{X} \to \mathbf{X}/E$ a β-*reduction.

We then have that every surjective bounded morphism between finite Esakia spaces can be seen as obtained from a finite sequence of α- and β-reductions.

Lemma 2.61. *Let $\mathbf{X}$ and $\mathbf{Y}$ be finite Esakia spaces. Then there exists a surjective bounded morphism $f : \mathbf{X} \to \mathbf{Y}$ iff there is a finite sequence of Esakia spaces $\mathbf{Z}_1, \ldots \mathbf{Z}_{n+1}$ such that $\mathbf{X} \cong \mathbf{Z}_1$, $\mathbf{Y} \cong \mathbf{Z}_{n+1}$, and for each $1 \leq i \leq n$ there is an α- or β-reduction $f_i : \mathbf{Z}_i \to \mathbf{Z}_{i+1}$.*

With this we conclude the preliminaries and move on to the specific properties that we will be investigating.

Chapter 3

Finitely generated algebras and local finiteness

In this chapter we discuss finitely generated algebras, and in particular *freely* generated algebras. We also introduce the equivalent notions of local finiteness of a variety of Heyting algebras and local tabularity of its associated si-logic, and review some results about such varieties. In conjunction with free Heyting algebras we discuss universal models.

3.1 Finitely generated algebras and coloring

We now introduce finitely generated Heyting algebras, as well as the dual notion of colorings of Esakia spaces. The latter will be what we use to prove our main results.

3.1.1 Finitely generated algebras

Definition 3.1. Let $\mathcal{F}$ be a type and $\mathcal{A}$ an $\mathcal{F}$-algebra. A *subuniverse* of $\mathcal{A}$ is a (possibly empty) subset $B \subseteq A$ such that for all $s \in F$ and $x_1, \dots, x_{a_{\mathcal{F}}(s)}$ we have

$$s^{\mathcal{A}}(x_1, \dots, x_{a_{\mathcal{F}}(s)}) \in B.$$

Given an arbitrary subset $X \subseteq A$, the *subuniverse generated by* X is defined as

$$\langle X \rangle := \bigcap \{B \mid X \subseteq B \text{ and } B \text{ is a subuniverse of } \mathcal{A}\}.$$

Definition 3.2. Given a type $\mathcal{F}$ and an $\mathcal{F}$-algebra $\mathcal{A}$ with a subset $X \subseteq A$, we say that X *generates* $\mathcal{A}$ if $\langle X \rangle = A$.

If an algebra $\mathcal{A}$ has a set of generators X with $|X| \leq n \in \omega$ we say $\mathcal{A}$ is *n-generated.* If $\mathcal{A}$ is n-generated for some $n \in \omega$ we say it is *finitely generated.*

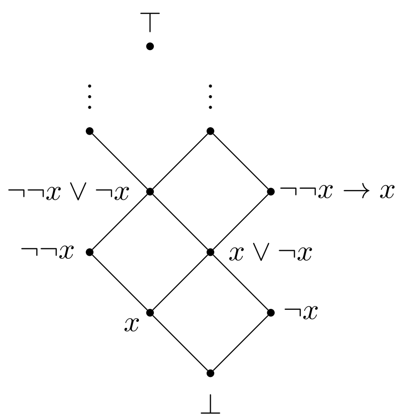
In fact, if $\mathcal{A}$ is an algebra generated by a set $X \subseteq A$, then every element in A can be obtained by a finite application of the operations of $\mathcal{A}$ to elements of X (see [9, Theorem II.3.2]). This motivates the terminology of X *generating* $\mathcal{A}$.

Definition 3.3. Let K be a variety and X a set. The *free X-generated algebra* for K is the unique (up to isomorphism) algebra $F(X)$ in K containing X and having the *universal mapping property* (UMP) over K. That is, for every $\mathcal{A} \in \mathsf{K}$ and function $f : X \to A$ there is a unique homomorphism $f' : F(X) \to \mathcal{A}$ extending f.

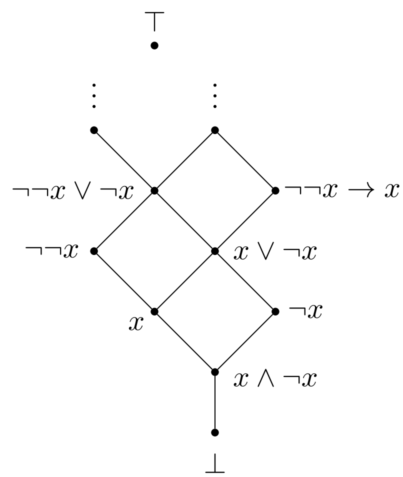
If $|X| = n \in \omega$ we call $F(X)$ the free n-generated algebra and often write $F(n)$.

Remark 3.4. We note that if an algebra $\mathcal{A}$ in a variety K is generated by a set X (or indeed a set Y such that $|X| = |Y|$), it is a homomorphic image of the free algebra $F(X)$ in K. This holds as the identity function $f : X \to X$ extends to a homomorphism $f' : F(X) \to \mathcal{A}$ and every element in $\mathcal{A}$ can be obtained from elements in X, so that they are images of elements in $F(X)$. This also ensures that if $|X| = |Y|$ then $F(X) \cong F(Y)$ so that we are justified in speaking of *the* n-generated free algebra.

Of particular interest is the free 1-generated Heyting algebra which is more commonly known as the *Rieger-Nishimura lattice* and is shown below. It serves as a kind of canonical example of a Heyting algebra that is finitely generated but infinite, and is generated by the singleton $\{x\}$. We denote it by $\mathcal{L}$.



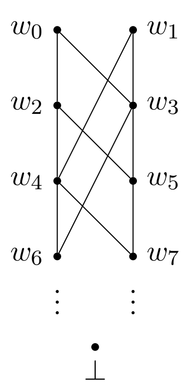
$\mathcal{L}$

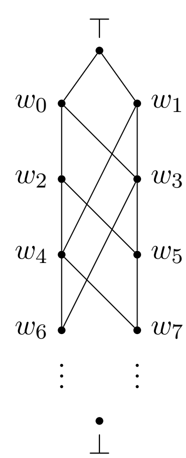


$\mathcal{L}_+$

To the right above we have what we call the Rieger-Nishimura lattice with a new bottom element and denote by $\mathcal{L}_+$, which will be of prime importance later. It is infinite as well, and is generated by the set $\{\perp, x\}$.

Below we have the posets underlying the respective dual Esakia spaces $\mathfrak{L} = \mathcal{L}_*$, which is called the *Rieger-Nishimura ladder*, and $\mathfrak{L}^+ = (\mathcal{L}_+)_*$. The naming of the points in the lower left poset should be considered standard from now on. Whenever a subscripted point w_i is mentioned it should be considered as a point in $\mathfrak{L}$. Likewise the notation $\uparrow w_i$ always refers to the corresponding upset in $\mathfrak{L}$, unless specifically stated otherwise.





3.1.2 Coloring

Definition 3.5. Let $\mathbf{X}$ be an Esakia space and $U_1, \dots, U_n \in \mathrm{ClopUp}(\mathbf{X})$. The *coloring* on $\mathbf{X}$ induced by $U_1, \dots, U_n$ is the map $\mathbf{c} : X \to \{0,1\}^n$ given by:

$$\mathbf{c}(x)(i) = \begin{cases} 1 & \text{if } x \in U_i, \\ 0 & \text{if } x \notin U_i. \end{cases}$$

For each $x \in X$ we call the sequence $\mathbf{c}(x)$ the *color* of x.

Theorem 3.6 ([7], Theorem 3.1.5)**.** *Let* $\mathbf{X}$ *be an Esakia space and* $U_1, \dots, U_n \in \mathrm{ClopUp}(\mathbf{X})$. *Then* $\mathbf{X}^*$ *is generated by* $U_1, \dots, U_n$ *iff every proper bisimulation equivalence on* $\mathbf{X}$ *identifies points of different color.*

Rephrasing this in the finite setting:

Definition 3.7. Given a finite poset $\mathbf{X}$, an *n-coloring* of $\mathbf{X}$ is a map $\mathbf{c} : X \to \{0,1\}^n$ such that:

(i) If $x \leq y$ and $\mathbf{c}(x)(i) = 1$ then $\mathbf{c}(y)(i) = 1$,

(ii) If $E \subseteq X^2$ is a bisimulation equivalence disregarding the topology, i.e. an equivalence relation not equal to the identity such that for all $(x, y) \in E$ and $z \geq x$ there is a $v \geq y$ such that $(z, v) \in E$, then E identifies points of different color.

Note that for a finite Esakia space $\mathbf{X}$ the n-colorings in definition 3.7 coincide with the colorings from 3.5 with the condition of Theorem 3.6. This is the case as in a finite Esakia space the topology is discrete, so that every upset is clopen.

We refer to the elements of $\{0,1\}^n$ as *n-colors*. We can impose a partial order on n-colors in a natural way by setting for all $c, d \in \{0,1\}^n$:

$$c \leq d \quad \text{iff} \quad c(i) \leq d(i) \text{ for all } i \in \{1, \dots, n\}.$$

Condition (i) in the above can then be restated by requiring the map $\mathbf{c}$ to be order-preserving. We then have a finitary version of Theorem 3.6:

Corollary 3.8. *Let* $\mathbf{X}$ *be a finite poset, then* $\mathrm{Up}(\mathbf{X})$ *is n-generated iff* $\mathbf{X}$ *admits an n-coloring.*

Proof. Follows from Theorem 3.6 when we recall that in finite Esakia spaces the topology is discrete. □

3.2 Universal models

Universal models can be seen as the image-finite parts of the duals of free Heyting algebras. Their theory has been developed in many sources, see e.g. [2, 17, 21, 23]. Our presentation closely resembles that of [7].

Definition 3.9. For each $n \in \omega$ we define the *n-universal model* $\mathcal{U}_n$ inductively by constructing an increasing chain of sets $W_0 \subseteq W_1 \subseteq \ldots$ together with a relation $R_i \subseteq W_i \times W_i$ for each set. The elements of these sets will be pairs consisting of an object and an n-color. We denote by π_1 and π_2 the standard projection functions for pairs.

First define $W_0 := \{w\} \times \{0,1\}^n$ where the object w can be chosen arbitrarily. Then note that we have for each n-color c exactly one world $(\{w\}, c) \in W_0$. Set $R_0 := \{(u,u) \mid u \in W_0\}$.

Now suppose W_m and R_m are defined. Then define W_{m+1} as the set containing W_m, for each $u \in W_m \setminus W_{m-1}$ and $c < \pi_2(u)$ a unique point $(\{u\}, c)$, and for every antichain $u_1, \ldots, u_k$ in the poset (W_m, R_m) with at least one point first defined in W_m and color $c \leq \pi_2(u_i)$ for $i \leq k$ a point $(\{u_1, \ldots, u_k\}, c)$. Then set R_{m+1} as the reflexive transitive closure of the set

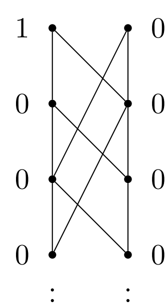
$$R_m \cup \{(u,v) \mid u \in W_{m+1}, v \in \pi_1(u)\}.$$

Finally, set $W := \bigcup_{m \in \omega} W_m$ and $\leq := \bigcup_{m \in \omega} R_m$. Then note that $(X, \leq)$ is a poset by construction. Now define a valuation function $V : W \to \mathcal{P}(P)$ by

$$p_i \in V(w) \text{ iff } i \leq n \text{ and } \pi_2(w)(i) = 1.$$

Then we define the *n-universal model as* $\mathcal{U}_m := (X, \leq, V)$.

Below is a picture of the 1-universal model $\mathcal{U}_1$. It is the only one we can really have a hope of depicting visually, as already complexity of even the top three layers of $\mathcal{U}_2$ is very great. As an example, a depiction of the first two layers of $\mathcal{U}_2$ can be found in [10, p.277].



The following theorem is [8, Theorem 4.1].

Theorem 3.10. *Let $n \in \omega$ and consider the free n-generated Heyting algebra $F(n)$.*

1. *The poset underlying $\mathcal{U}_n$ is isomorphic to the poset of finite depth elements in $F(n)_*$.*
2. *Let $x \in F(n)_*$. Then either x has finite depth or for every $m \in \omega$ with $1 \leq m$ there is a $y \in F(n)_*$ of depth m such that $x \leq y$.*
3. *For each $m \in \omega$ there are only finitely many points of depth m in $\mathcal{U}_n$.*

Corollary 3.11. *Let $n \in \omega$ and let $F(n)$ be the free n-generated Heyting algebra. If $\mathbf{X}$ is an infinite E-subspace of $F(X)_*$, then X contains an element of depth m for every $m \in \omega$ with $m \geq 1$.*

Proof. Toward a contradiction, suppose X is an infinite E-subspace of $F(X)_*$ that does not contain any elements of depth m for some $m \in \omega$ with $m \geq 1$. Then (2) in the theorem above implies that X does not have any points of infinite depth. On the other hand, as a point of depth $m+1$ must have a direct successor of depth m, and a point of depth $m+2$ must have a direct successor of depth $m+1$ and so on, we find that all points in $\mathbf{X}$ have depth strictly less than m. By (1) and (3) of the above theorem we then see that $\mathbf{X}$ must be finite, which gives the desired contradiction. □

3.3 Local finiteness

Definition 3.12. A variety K of algebras is called *locally finite* if every finitely generated algebra in K is finite.

We collect some characterisations of locally finite varieties.

Theorem 3.13. *Let K be a variety of algebras in some type $\mathcal{F}$. The following are equivalent:*

(i) K *is locally finite.*

(ii) For any $n \in \omega$ the free n-generated algebra in K is finite.

If in addition $\mathcal{F}$ is finite we may add:

(iv) For any $n \in \omega$ there is an $m \in \omega$ such that every n-generated algebra in K_{SI} has cardinality at most m.

The equivalence $(i) \Leftrightarrow (ii)$ is [9, Theorem 10.15] and $(i) \Leftrightarrow (iii)$ is item (4) in [4, Theorem 3.7].

The next lemma provides another useful characterisation and is proved in [9, Theorem 10.16].

Lemma 3.14. *Let K be a variety such that $\mathsf{K} = \mathbf{V}(C)$ for a finite set C of finite $\mathcal{F}$-algebras. Then K is locally finite.*

We also have an equivalent notion for si-logics. Given an si-logic L we say two formulas φ and ψ are L-*equivalent* if $\vdash_{\mathsf{L}} \varphi \leftrightarrow \psi$.

Definition 3.15. An si-logic L is called *locally tabular* if there are only finitely many non-L-equivalent formulas over any finite subset $P_0 \subseteq P$.

Theorem 3.16. *Let L be an si-logic and V_{L} its associated variety of Heyting algebras. Then V_{L} is locally finite iff L is locally tabular.*

We sketch a proof. The main tool is the so-called *Lindenbaum-Tarski* algebra $L_{\mathsf{L}}(P_0)$ for L over P_0. This algebra has as carrier the set of equivalence classes of formulas over P_0 under the equivalence relation $\varphi \sim \psi$ iff $\vdash_{\mathsf{L}} \varphi \leftrightarrow \psi$. The join of two equivalence classes $\overline{\varphi}$ and $\overline{\psi}$ is computed as $\overline{\varphi \vee \psi}$, the top element is $\overline{\top}$, etc.

Now suppose $|P_0| = n$. It turns out that $L_{\mathsf{L}}(P_0)$ is the n-generated free algebra in V_{L}. This means that if V_{L} is locally finite, $L_{\mathsf{L}}(P_0)$ will be finite for all finite P_0, and so L is locally tabular. Conversely, if V_{L} is not locally finite there is some infinite m-generated algebra in V_{L}. Then $|P_0| = m$ implies $L_{\mathsf{L}}(P_0)$ is infinite as well, as all m-generated algebras are homomorphic images of it. Then L is not locally tabular.

Example 3.17. We have some examples of locally finite varieties.

- The variety of all Heyting algebras is not locally finite as it contains $\mathcal{L}$ which is an infinite 1-generated algebra.

- Consider the *weak law of the excluded middle*: $\neg p_1 \vee \neg\neg p_1$. It can be shown (see e.g. [10, Proposition 2.37]) that this formula characterises the class of posets with a maximum. The logic that is axiomatised by this formula is known as $\mathsf{KC} = \mathsf{IPC} + \neg p_1 \vee \neg\neg p_1$.

  All 1-generated algebras in V_{KC} are finite. This can be seen by noting that they must all be homomorphic images of the free 1-generated Heyting algebra $\mathcal{L}$, and so their dual posets must be generated subposets of $\mathfrak{L}$. But since these dual posets must have a maximum, they can only equal $\uparrow w_2$,$\uparrow w_1$ or $\uparrow w_0$. Then these dual posets are finite and so are the original algebras.

However, since $\mathfrak{L}^+$ is a KC-poset, its 2-generated dual algebra $\mathcal{L}_+$ is in V_{KC}, so that this variety is not locally finite.

The second example leads to an interesting question. It shows that there are varieties of Heyting algebras where all 1-generated algebras are finite, but that nevertheless contain an infinite 2-generated algebra. It is natural to ask if this is a continuing pattern: does there exist a variety of Heyting algebras in which all 2-generated algebras are finite, but that contains an infinite 3-generated algebra? The following conjecture, which can be traced to [6, Problem 2.4.(6)], states that this is not possible.

Conjecture 3.18. *Every non-locally finite variety of Heyting algebras contains an infinite 2-generated algebra.*

We will return to this conjecture in the next chapter, where we confirm it in the case of width 2 varieties.

An algebra $\mathcal{A}$ is called *finitely subdirectly irreducible* or *FSI* if for any pair of congruences θ, θ' on $\mathcal{A}$ with $\theta \cap \theta' = \Delta_A$ we have one of $\theta = \Delta_A$ or $\theta' = \Delta_A$, where Δ_A is the identity relation on $\mathcal{A}$. The following is item (i) in [8, Lemma 3.2].

Lemma 3.19. *Let $\mathcal{A}$ be a Heyting algebra. Then $\mathcal{A}$ is finitely subdirectly irreducible iff the poset underlying $\mathcal{A}_*$ is rooted.*

In [8, Theorem 4.3] we find the following characterisation of locally finite varieties of Heyting algebras:

Theorem 3.20. *Let K be a variety of Heyting algebras. Then K is locally finite iff K has, up to isomorphism only finitely many finite n-generated FSI members for each $n \in \omega$.*

Proof. We prove the contrapositive of the 'if' direction. So suppose K is not locally finite. Then for some $n \in \omega$ there is an infinite n-generated algebra $\mathcal{A}$ in K. Then $\mathcal{A}$ is a homomorphic image of the free n generated algebra $F(n)$ in K and so, by Lemma 2.56, $\mathcal{A}_*$ is (isomorphic to) an E-subspace of $F(X)_*$. Further, as $\mathcal{A}$ is infinite, so is $\mathcal{A}_*$. Then by corollary 3.11 we find for each $m \in \omega$ with $m \geq 1$ an element x_m of depth m in $\mathcal{A}_*$.

Now consider for each $m \in \omega$ with $m \geq 1$ the E-subspace $\uparrow x_m$ in $\mathcal{A}_*$. By Lemmas 2.56 and 3.19 we find that each of these spaces is isomorphic to an FSI homomorphic image $\mathcal{A}_m$ of $F(n)$, and so $\mathcal{A}_m \in \mathsf{K}$ for each m. Further, by items (1) and (3) of Theorem 3.10, the upsets $\uparrow x_m$ are finite, and so the algebras $\mathcal{A}_m$ are too. Then $\{\mathcal{A}_m \mid 1 \leq m \in \omega\}$ is a sequence of finite finitely generated FSI algebras in K.

Now note that as each $\uparrow x_m$ is of depth m and so size at least m, the size of the spaces $\{\uparrow x_m \mid 1 \leq m \in \omega\}$ is not bounded above by any natural number. Then the size of the algebras $\{\mathcal{A}_m \mid 1 \leq m \in \omega\}$ is not bounded either. But since they are finite, this means that there must be infinitely many pairwise non-isomorphic $\mathcal{A}_m$.

The 'only if' direction is straightforward. $\square$

With all requisite background material in hand, we are now able to move on to our original results.

Chapter 4

Local finiteness in width 2

In this chapter we finally turn toward our original contribution. The main result is Theorem 4.17 which shows that a width 2 variety of Heyting algebras will fail to be be locally finite if and only if it contains the Rieger-Nishimura lattice with a new bottom element $\mathcal{L}_+$. We establish this result by noting that this is omitted from a variety if and only if at least one of the algebras dual to a finite upset of the Rieger-Nishimura ladder with a new top element is omitted, and then characterising such classes.

First, we show that finding such finite upsets in a class of posets dual to a variety of Heyting algebras can be reduced to finding associated upsets in the Rieger-Nishimura ladder as subposets of posets in that class (Theorem 4.3). Next, we develop the technical machinery that will allow us to prove our main result. We follow this with a counterexample that shows our argument does not extend to varieties of width greater than 2. We close out the chapter by using our characterisation of local finiteness in width 2 to prove some additional results. First we prove that there are continuum many non-locally finite and locally finite width 2 varieties of Heyting algebras (corollaries 4.23 and 4.26). Then we show that $\mathrm{Log}(\mathfrak{L}^+)$ is the only pre-locally tabular width 2 logic (Theorem 4.27), and that every non-locally tabular width 2 logic is contained in a pre-locally tabular one (Corollary 4.28). Finally, we prove that the problem of local finiteness is decidable for width 2 varieties of Heyting algebras (Theorem 4.30). With these last two points we provide partial solutions to problems 12.1 and 17.4 from [10].

4.1 Finite upsets of the Rieger-Nishimura ladder

Given a variety K of Heyting algebras the question of whether an Esakia space $\mathbf{X}$ is in K_* is equivalent to the question of whether $\mathbf{X}$ is a bounded morphic image of some space in K_*. In this section we show that for $\mathfrak{L}^+$ and its finite principal upsets this question reduces to finding an associated generated subposet of $\mathfrak{L}$ as a subposet of a space in K_*.

For the sake of convenience we recall here the two conditions needed for a map $f : (X, \leq) \to (Y, \leq')$ to be a bounded morphism between posets $\mathbf{X}$ and $\mathbf{Y}$:

(i) If $x \leq y$ then $f(x) \leq' f(y)$.
(ii) If $f(x) \leq' y'$ then there is a $y \in X$ such that $x \leq y$ and $f(y) = y'$.

Proposition 4.1. *Let $\mathbf{X} = (X, \leq)$ be a poset. If a copy of $\mathfrak{L}$, denoted by $\mathfrak{L}_F$, is a sub-poset of $\mathbf{X}$, then $\mathfrak{L}^+$ is a bounded morphic image of $\mathbf{X}$.*

Proof. Given such a poset $\mathbf{X}$, write $w_0, w_1, \ldots$ for the elements of $\mathfrak{L}_F$ as well. We may assume that $|{\uparrow}w_0|, |{\uparrow}w_1| > 1$. If this is not the case, we can simply consider the copy of $\mathfrak{L}$ in $\mathbf{X}$ that has w_2 and w_3 as maximal elements instead. We need to define a map $f : \mathbf{X} \twoheadrightarrow \mathfrak{L}^+$ and show it is a bounded morphism.

For arbitrary $w \in X$ we define the image $f(w)$ based on the set of direct successors of w in $\mathfrak{L}_F$. That is, we consider the *minimal* elements among the successors of w in $\mathfrak{L}_F$. To this end, we define:

$$S_w := \min({\uparrow}w \cap \mathfrak{L}_F).$$

There are three cases:

C1. $S_w = \varnothing$
C2. $|S_w| = 1$
C3. $|S_w| = 2$

Note that a fourth possible case of $|S_w| > 2$ cannot occur, as this would imply the existence of an anti-chain of length 3 in $\mathfrak{L}$ which is width 2 and rooted.

C1: If $S_w = \varnothing$ there are two subcases. For subcase **1a**, suppose there is a $v \in \mathfrak{L}_F$ such that $w \leq v$. Then it must be the case that for all $v \in \mathfrak{L}_F \setminus \{\bot_{\mathfrak{L}_F}\}$ we have $w \leq v$. This can be seen from the fact that $\mathfrak{L}$ is conversely well-founded, and that no point $v \in \mathfrak{L}$ is incomparable with more than two others, so that if one $v \in \mathfrak{L}_F \setminus \{\bot_{\mathfrak{L}_F}\}$ is omitted from ${\uparrow}w$, all but finitely many are. In this subcase it is then natural to set $f(w) = \bot_{\mathfrak{L}}$. Subcase **1b** occurs when no $v \in \mathfrak{L}_F$ has $w \leq v$. Here we set $f(w) = \top_{\mathfrak{L}}$.

C2: If S_w is a singleton $\{v\}$, we note that for all $u \in \mathfrak{L}_F$ we have $w \leq u$ iff $v \leq u$. This motivates the definition: $f(w) = v$ (where v is now seen as an element of $\mathfrak{L}$).

C3: If $|S_w| = 2$ we note that, as S_w must be an anti-chain $\{v, u\}$ there is a unique common direct predecessor s of v and u in $\mathfrak{L}_F$. We then set $f(w) = s$.

With f now defined for all $w \in X$, that for $w \in \mathfrak{L}_F$ we have $S_w = \{w\}$, and so f maps each point in $\mathfrak{L}_F$ to its corresponding point in $\mathfrak{L}$. This, together with the fact that there exists something above w_0 mapped to $\top_{\mathfrak{L}}$, ensures that f is surjective. It remains to show the following:

Claim. *The map $f : \mathbf{X} \twoheadrightarrow \mathfrak{L}^+$ is a bounded morphism.*

Proof of claim. We will argue for each $w \in X$ that the conditions for bounded morphisms hold. So consider arbitrary $w, v \in X$ with $w \leq v$. We first verify condition (i). That is, we show f is order-preserving. Note that this condition holds trivially if $S_w = \varnothing$ so we may assume this is not the case. Let $v \in X$ be such that $w \leq v$, we need to show that $f(w) \leq f(v)$. There are several cases which correspond to the cases 1-3 of our definition of f:

C1: $S_v = \varnothing$. For subcase **1a** we have $\mathfrak{L}_F \setminus \bot_{\mathfrak{L}_F} \subseteq \uparrow v \subseteq \uparrow w$ and $f(v) = \bot_{\mathfrak{L}}$. Then by assumption we have $S_w = \{\bot_{\mathfrak{L}_F}\}$ so we conclude $f(w) = \bot_{\mathfrak{L}}$ and so $f(w) \leq f(v)$. For subcase **1b** we have $f(v) = \top_{\mathfrak{L}}$ and so clearly $f(w) \leq f(v)$.

C2: Here we have $S_v = \{u\}$ for some $u \in X$. Then either $u \in S_w$, in which case the condition holds as $f(w) \leq f(u) = f(v)$, or $u \notin S_w$. In the latter case we find that, as $u \in \uparrow w \cap \mathfrak{L}_F$, there is some $s \in S_w$ such that $s \leq u$ and we have:

$$f(w) \leq f(s) \leq f(u) = f(v).$$

So again the condition is satisfied.

C3: Analogous to case 2.

Next, we verify property (ii). For this we take an arbitrary $w \in X$ and suppose there is some $x \in \mathfrak{L}^+$ such that $f(w) \leq x$. We need to find a $v \in X$ such that $f(v) = x$. We again consider the three cases of our definition, this time as applied to S_w.

C1: In subcase **1a** we have $f(w) = \bot_{\mathfrak{L}}$. Now either $x = f(w)$, or the point corresponding to x in $\mathfrak{L}_F$ is in $\uparrow w$ and mapped to x. Either way the condition is satisfied. For subcase **1b** we note that $f(w) = \top_{\mathfrak{L}}$ so it must be the case that $x = f(w)$, this satisfies the condition.

C2: Say $S_w = \{v\}$. Then $f(w) = f(v)$ and so there is an element $v' \in \mathfrak{L}_F$ with $v \leq v'$ corresponding to x. Then clearly $w \leq v'$ and $f(v') = x$, so this case is satisfied.

C3: Here, we let $S_w = \{v, u\}$. Then we find that $f(w)$ is the unique common direct predecessor of $f(v)$ and $f(u)$, and so that there must be a point $s \in \mathfrak{L}_F$ corresponding to x such that at least one of:

$$w \leq s \quad \text{or} \quad v \leq s \quad \text{or} \quad u \leq s.$$

This satisfies the condition.

With both (i) and (ii) verified, we conclude that the claim holds. □

Now with this claim proven, we have that f is a surjective bounded morphism as desired. This completes the proof. □

Noting that $\mathcal{L}_+$ is 2-generated and infinite, we have the following corollary.

Corollary 4.2. *Let* L *be an si-logic that is characterised by a class of posets. If there is a poset of* L *that has* $\mathfrak{L}$ *as a sub-poset, then* L *is not locally tabular.*

We have a finitary version of Proposition 4.1 as well:

Theorem 4.3. *Let* $\mathbf{X} = (X, \leq)$ *be a poset such that for some* $n \in \omega$ *with* $n \geq 2$ *the set* $\uparrow w_n$ *is a sub-poset. Then* $(\uparrow w_{n-2})^+$ *is a bounded morphic image of* $\mathbf{X}$.

Proof. The argument is similar to that for Proposition 4.1, in particular the special case where "the" copy of $\mathfrak{L}$ is considered to have w_2 and w_3 as maximal elements. The map $f : \mathbf{X} \twoheadrightarrow (\uparrow w_{n-2})^+$ is then defined as before, with the remark that case **1a** can be disregarded. □

Finally, we obtain the characterisation that we will use in the sequel. The following is essentially [5, Theorem 8.49].

Theorem 4.4. *Let* K *be a variety of Heyting algebras and* K_* *be the class of Esakia spaces dual to some algebra in* K. *Then* $\mathfrak{L}^+ \in \mathsf{K}_*$ *iff* $\uparrow w_n \in \mathsf{K}_*$ *for all* $n \in \omega$.

Corollary 4.5. *Let* K *be a variety of Heyting algebras and* K_* *be the class of Esakia spaces dual to some algebra in* K. *Then* $\mathfrak{L}^+ \in \mathrm{Fr}(\mathsf{L})$ *iff for all* $n \in \omega$ *there is an Esakia space* $\mathbf{X} \in \mathsf{K}_*$ *that has* $\uparrow w_n$ *as a sub-poset.*

Proof. This follows directly from theorems 4.3 and 4.4. □

4.2 Local finiteness in bounded width varieties

We now turn toward a characterisation of locally finite varieties of Heyting algebras in terms of n-colorings on posets dual to their algebras.

Given a finite poset $\mathbf{X}$ with an n-coloring $\mathbf{c}$, we denote for each $c \in \{0,1\}^n$ by $c(\mathbf{X})$ the set of points in $\mathbf{X}$ colored by c. For the sake of brevity we write $0(\mathbf{X})$ for the set of points colored by the constant sequence of n zeros. Similarly, we will often just refer to this sequence by calling it the color 0.

Remark 4.6. Let $\mathbf{X}$ be a finite poset with a coloring. We remark the following:

1. If $x, y \in X$ are such that $x \prec y$ and $y \in 0(\mathbf{X})$, then there is a $z \in X$ distinct from y such that $x \prec z$. Consequently, if $x, y \in X$ are such that $x < y$ and $y \in 0(\mathbf{X})$, then x has at least two distinct direct successors.
2. Suppose $x_1, \dots, x_n \in 0(\mathbf{X})$ form an antichain, then there are no distinct $y, z \in 0(\mathbf{X})$ that have $x_1, \dots, x_n$ as their direct successors.

To see why 2 and the first part of 1 hold, we note that if this were not the case there would be α- and β-reductions identifying y, z resp. y, x which would violate (ii) in Definition 3.7. The latter part of 1 follows as $\mathbf{X}$ is finite.

The following is a restating of Theorem 3.20 to a form that we will use in our subsequent proofs.

Theorem 4.7. *A variety K of Heyting algebras fails to be locally finite iff there is a sequence $\{\mathbf{X}_m\}_{m\in\omega}$ of finite rooted posets and $n \in \omega$ such that: $\mathrm{Up}(\mathbf{X}_m) \in \mathsf{K}$ and $\mathbf{X}_m$ admits an n-coloring for each $m \in \omega$, and $\{|\mathbf{X}_m|\}_{m\in\omega}$ is unbounded.*

Proof. This follows from Theorem 3.20 and Corollary 3.8. $\square$

We can further refine this theorem by showing that we can not just find a sequence $\{\mathbf{X}_m\}_{m\in\omega}$ where the size of the $\mathbf{X}_m$ is unbounded, but that we can in fact find such a sequence where the size of the subposets that are colored 0 is unbounded and the size of the subposets colored with any other color is bounded.

Theorem 4.8. *Let K be a variety of Heyting algebras of bounded width. Then the following are equivalent:*

1. *K is not locally finite,*

2. there are $n \in \omega$ and a sequence $\{\mathbf{X}_m\}_{m\in\omega}$ of n-colorable finite rooted posets such that $\mathrm{Up}(\mathbf{X}_m) \in \mathsf{K}$ for each $m \in \omega$, and $\{|0(\mathbf{X}_m)|\}_{m\in\omega}$ is unbounded.

In fact, we may further assume in (2) that there is some $M \in \omega$ such that $|\mathbf{X}_m \setminus 0(\mathbf{X}_m)| \leq M$ for each $m \in \omega$.

Proof. "(2) ⇒ (1)": Follows immediately from Theorem 4.7.

"(1) ⇒ (2)": By Theorem 4.7 there are $n \in \omega$ and a sequence $\{\mathbf{X}_m\}_{m\in\omega}$ of finite rooted n-colored posets such that $\mathrm{Up}(\mathbf{X}_m) \in \mathsf{K}$ for each m and $\{|\mathbf{X}_m|\}_{m\in\omega}$ is unbounded. We shall construct out of this sequence a new one $\{\mathbf{X}'_m\}_{m\in\omega}$ of n-colored finite rooted posets with $\mathrm{Up}(\mathbf{X}'_m) \in \mathsf{K}$ for each m and such that $\{|0(\mathbf{X}'_m)|\}_{m\in\omega}$ is unbounded.

Now note that there are only 2^n colors as each of these can be seen as a sequence of n ones and zeroes, and that $\{|\mathbf{X}_m|\}_{m\in\omega}$ is unbounded, so there must be at least one color $d \in \{0,1\}^n$ such that $\{|d(\mathbf{X}_m)|\}_{m\in\omega}$ is unbounded as well. Let $C \subseteq \{0,1\}^n$ be the set of such colors d. Then note that C is finite and partially ordered by the order on colorings so that is has maximal elements. This allows us to select $c \in \max(C)$ arbitrarily. Now we are able to define our new sequence of posets. Take, for each $m \in \omega$ and $x \in {\uparrow}c(\mathbf{X}_m)$, the subposet $\mathbf{Y}^x_m := {\uparrow}x$. We note that each $\mathbf{Y}^x_m$ is finite and rooted. Further, recalling the correspondence between generated subposets and subalgebras, we note that $\mathrm{Up}(\mathbf{Y}^x_m) \in \mathsf{K}$. Finally we have that, as for each m there are finitely many $\mathbf{Y}^x_m$, the set $\{\mathbf{Y}^x_m \mid m \in \omega \text{ and } x \in {\uparrow}c(\mathbf{X}_m)\}$ is countable and so can form an ω-indexed sequence. It remains to show that our $\mathbf{Y}^x_m$ can be recolored so that $\{|0(\mathbf{Y}^x_m)| \mid m \in \omega, x \in {\uparrow}c(\mathbf{X}_m)\}$ is unbounded.

For this, consider an arbitrary $\mathbf{Y}^x_m$. We construct a coloring for it by first restricting the coloring $\mathbf{c}$ of $\mathbf{X}_m$ to it and then recoloring some elements of $\mathbf{X}^x_m$. We write $\mathbf{c}' := \mathbf{c}|_{Y^x_m}$ for this restriction. To see that $\mathbf{c}'$ is an n-coloring of $\mathbf{Y}^x_m$ we first note that for each $y, z \in Y^x_m$ with $y \leq z$ we have

$$\mathbf{c}'(y) = \mathbf{c}(y) \leq \mathbf{c}(z) = \mathbf{c}'(z),$$

so that $\mathbf{c}'$ is order-preserving. Let E be a proper bisimulation equivalence on $\mathbf{Y}^x_m$. We can extend E to an equivalence relation E' on X_m by adding (y, y) to E for all $y \in X_m \setminus Y^x_m$. Now suppose $(y, z) \in E'$ and let $u \geq y$. Then either $(y, z) \in E$ so that there is a $v \geq z$ with $(u, v) \in E'$, or $y = z$ so that this condition is satisfied trivially. We conclude that any proper bisimulation equivalence on $\mathbf{Y}^x_m$ can be extended to one on $\mathbf{X}_m$, so that it must identify points of different color. This shows that $\mathbf{c}'$ is indeed a coloring of $\mathbf{Y}_m^x$. We now modify $\mathbf{c}'$ to another coloring $\mathbf{c}''$ on $\mathbf{Y}_m^x$ by setting for each $y \in Y_m^x$:

$$\mathbf{c}''(y)(i) = \begin{cases} 00\ldots 0 & \text{if } y \in c(\mathbf{X}_m) \\ \mathbf{c}'(y)(i) & \text{if } y \notin c(\mathbf{X}_m) \end{cases}$$

where $00\ldots0$ denotes the constant sequence with value 0 of length n. We claim the following:

Claim. *The map $\mathbf{c}''$ is a coloring of $\mathbf{Y}_m^x$.*

Proof of claim. For property (i) in Definition 3.7 this is clear. For property (ii) we consider an arbitrary proper bisimulation equivalence E on Y_m^x. Then $E \cup \mathrm{Id}_{\mathbf{X}_m}$ is a proper bisimulation equivalence on $\mathbf{X}_m$ so by property (ii) of n-colorings we have $y, z \in Y_m^x$ with $(y, z) \in E$ and $\mathbf{c}(y) \neq \mathbf{c}(z)$. We then need $\mathbf{c}''(y) \neq \mathbf{c}''(z)$ as well. For this, assume toward a contradiction that $\mathbf{c}''(y) = \mathbf{c}''(z)$. Then we must have one of y or z recolored, say without loss of generality $\mathbf{c}''(y) \neq \mathbf{c}(y)$. Then from the construction of $\mathbf{c}''$ we see that $\mathbf{c}(y) = c$ and $\mathbf{c}''(y) = 0$. As $\mathbf{c}''(y) = \mathbf{c}''(z)$ we get $\mathbf{c}''(z) = 0$ as well. From $\mathbf{c}(y) \neq \mathbf{c}(z)$ we obtain $\mathbf{c}(z) \neq c$ which, together with $\mathbf{c}''(z) = 0$, gives $\mathbf{c}(z) = 0$. Now since $z \in {\uparrow} c(\mathbf{X}_m)$ we must have $c \leq \mathbf{c}(z)$, so that $c = 0$. But then we get $\mathbf{c}(y) = c = 0 = \mathbf{c}(z)$, which is the desired contradiction. It follows that $\mathbf{c}''$ is a coloring on $\mathbf{Y}_m^x$. □

Finally, it remains to show that the sequence $\{|0(\mathbf{Y}_m^x)| \mid m \in \omega, x \in {\uparrow} c(\mathbf{X}_m)\}$, with each $\mathbf{Y}_m^x$ recolored as above, is indeed unbounded. This will follow from the fact that K is of bounded width, say by k. Now note that the sequence $\{|c(\mathbf{X}_m)|\}_{m\in\omega}$ is unbounded and that the size of antichains in each $\mathbf{X}_m$ is bounded by k. As a sequence of posets with bounded antichains and bounded principal upsets cannot be unbounded, it follows that the size of principal upsets in the posets $c(\mathbf{X}_m)$ is unbounded, so in particular the set $\{|0(\mathbf{Y}_m^x)| \mid m \in \omega, x \in {\uparrow} c(\mathbf{X}_m)\}$ is unbounded as desired.

Now note that by selecting c to be *maximal* among colors $d$such that $\{|d(\mathbf{X}_m)|\}_{m\in\omega}$ is unbounded, and subsequently defining our $\mathbf{Y}_m^x$ on principal upsets of points colored c, we have ensured that for each color $d \neq 0$ the set $\{d(\mathbf{Y}_m^x)\}_{m\in\omega, x\in\mathbf{X}_m}$ is bounded. Then as there are only finitely many colors, we indeed have that there is some $M \in \omega$ such that $|\mathbf{Y}_m^x \setminus 0(\mathbf{Y}_m^x)| \leq M$ for each $m \in \omega, x \in \mathbf{X}_m$. □

4.3 Technical toolbox

This section introduces a number of technical lemmas that will be used in the proof of Theorem 4.16.

Lemma 4.9. *Let $M \in \omega$. There exists a function $f : \omega \to \omega$ such that if $\mathbf{X}$ is a finite n-colored width 2 poset with root $\perp$ and $a, b \in 0(\mathbf{X})$ such that $x_1 \prec a, b$ and $M = |{\uparrow}\{a, b\}|$, then for all $k \in \omega$: if there are $\perp = x_k \prec x_{k-1} \prec \ldots \prec x_1$, then $|\mathbf{X}| \leq f(k)$.*

Proof. The proof proceeds by induction on k. For the base case we have $k = 1$, so that $x_1 = \perp$. We shall prove that $|\mathbf{X}| = M + 1$, for which it will suffice to show that $\mathbf{X} = {\uparrow}\{a, b\} \cup \{x_1\}$. Suppose toward a contradiction that there is a $c \in \mathbf{X} \setminus ({\uparrow}\{a, b\} \cup \{x_1\})$. As $\mathbf{X}$ is rooted and width 2, $a \parallel b$, and $c \notin {\uparrow}\{a, b\}$ we have without loss of generality that $c < a$. Now since $c \neq x_1 = \perp$ we have $\perp < c < a$, contradicting $x_1 \prec a$. This establishes the base case.

For the inductive step we assume the statement holds for some k, and suppose there exists a chain

$$\perp = x_{k+1} \prec x_k \prec \ldots \prec x_1.$$

Now note that ${\uparrow}x_k$ is a finite rooted n-colored poset of width 2 that contains x_1, so we can apply the inductive hypothesis to it, obtaining $|{\uparrow}x_k| \leq f(k)$. Further, since $\perp < a \in 0(\mathbf{X})$ and $\mathbf{X}$ is n-colored we see that $\perp$ has at least two distinct direct successors. It has no more because $\mathbf{X}$ is rooted and width 2. One of these direct successors is x_k and we shall call the other d. Then $\mathbf{X} = {\uparrow}x_k \cup {\uparrow}d \cup \{\perp\}$ and so:

$$|\mathbf{X}| \leq f(k) + 1 + |{\uparrow}d \setminus {\uparrow}x_k|.$$

To complete the inductive step it is now sufficient to provide a bound for $|{\uparrow}d \setminus {\uparrow}x_k|$. Now note that, as $\perp \prec x_k$ we must have for each $e \in {\uparrow}d \setminus {\uparrow}x_k$ that $e \parallel x_k$, since it is certainly not the case that $x_k < e$. By width 2 and the fact that $\mathbf{X}$ is rooted this implies that ${\uparrow}d \setminus {\uparrow}x_k$ is a chain, say ${\uparrow}d \setminus {\uparrow}x_k = \{c_1, \ldots c_l\}$ with $c_1 < \ldots < c_l$ for some $l \in \omega$. In fact, we have $c_1 \prec c_2 \prec \ldots \prec c_l$, as $\mathbf{X}$ is finite and each c_i is incomparable with x_k.

Then consider $c_{l-1} \prec c_l$. As $x_k \leq a, b$ and $c_{l-1} \parallel x_k$ we must have $c_{l-1} \not\geq a, b$. By width 2 and the fact that ${\uparrow}x_k$ is rooted we may assume without loss of generality that c_{l-1} is comparable with a, and so $c_{l-1} < a$. Then $a \in 0(\mathbf{X})$ implies that c_{l-1} has two distinct direct successors, one is c_l and the other we will call d_{l-1}. Now we have $d_{l-1} \parallel c_l$ and $c_l \parallel x_k$, and so we get that d_{l-1} is comparable with x_k. Since $\perp \prec x_k$ we obtain $x_k \leq d_{l-1}$. Next we consider c_{l-2}, and through similar reasoning see that is has exactly two direct successors: c_{l-1} and an element d_{l-2}. Again similarly we get by width 2 that $d_{l-2} \geq x_k$. Iterating like this we find $l-1$ elements:

$$d_{l-1}, d_{l-2}, \ldots, d_1 \geq x_k.$$

To see that these elements are distinct, suppose toward a contradiction that there are d_i, d_j with $d_i = d_j$ and $i < j$. Then $c_i < c_j < d_j = d_i$, and so $c_i \not\prec d_i$ which gives a contradiction. Then the d_i are pairwise distinct elements in $\uparrow x_k$, and so $l \leq f(k)$. Then $|\uparrow d \setminus \uparrow x_k| \leq f(k) + 1$, which gives the upper bound $|\mathbf{X}| \leq 2(f(k) + 1)$. $\square$

Lemma 4.10. *Let $\mathbf{X}$ be a finite rooted n-colored poset of width 2, and let $a, b \in 0(\mathbf{X})$ be such that $a \parallel b$ and $x \prec a, b$. If $|\downarrow a| \geq 3$, then there is a maximal $y \in \downarrow\{a, b\} \setminus \{a, b\}$ distinct from x. Moreover, one of the following holds:*

1. *$y \prec a$ and y has a direct successor $z > b$,*
2. *$y \prec b$ and y has a direct successor $z > a$.*

Proof. As $x \prec a, b$ we clearly have x maximal in $\downarrow\{a, b\} \setminus \{a, b\}$. Suppose toward a contradiction that x is the unique maximum of this set. Then, as $\mathbf{X}$ is finite, this implies $\downarrow\{a, b\} = \{a, b\} \cup \downarrow x$. Since $|\downarrow a| \geq 3$, there is some $c < a$ distinct from x. Now it cannot be the case that $b \leq c$, as $a \parallel b$, we must have $c < x$. As $\mathbf{X}$ is finite we may assume $c \prec x$. If this were not the case we could select a $d \prec x$ with $c < d$. Now $c < a$ and $a \in 0(\mathbf{X})$ implies that c has two direct successors, one is x and the other we shall call y. By width 2 and the fact that $\mathbf{X}$ is rooted we must have y comparable with one of a or b. Since $x \prec a, b$ and $x \parallel y$ we get $y \not\geq a, b$, and so either $y < a$ or $y < b$. Then x being the maximum of $\downarrow\{a, b\} \setminus \{a, b\}$ yields $y \leq x$ which contradicts $x \parallel y$.

Our contradiction found, we get that $\downarrow\{a, b\} \setminus \{a, b\}$ has a maximal element z distinct from x. We may assume without loss of generality that $z \prec a$. Now $z \not\prec b$, since if this were the case, the least equivalence relation on $\mathbf{X}$ identifying x and z would violate property (ii) of n-colorings. As z is maximal in $\downarrow\{a, b\} \setminus \{a, b\}$ and $a \parallel b$ we further get $z \not\leq b$. Now since $z < a \in 0(\mathbf{X})$, it must have a direct successor distinct from a, which we call v. Then $a \parallel v$ and $a \parallel b$, and so we get by width 2 that v must be comparable to b. Since $z \not\leq b$ this gives $v > b$ as desired. $\square$

Lemma 4.11. *Let $\mathbf{X}$ be a finite rooted n-colored poset of width 2. If $a \parallel b$ with $a \in 0(\mathbf{X})$ are such that all direct successors of b are above a, then all direct predecessors of a are below b. Further, at least one such predecessor exists.*

Proof. As $\mathbf{X}$ is finite and rooted, and $a \parallel b$, there must be a direct predecessor of a. It suffices to show all of these are below b. Select and arbitrary $c \prec a$. As $c < a \in 0(\mathbf{X})$, there is a direct successor of c distinct from a, say d. Now $a \parallel d$ and $a \parallel b$ implies d and b are comparable by width 2. It suffices to show that $d \leq b$. Toward a contradiction, assume $d > b$. Then as all direct successors of b are above a, and $\mathbf{X}$ is finite, we get $a \leq d$ which contradicts the fact that $c \prec d$. □

Remark 4.12. For a finite n-colored poset $\mathbf{X}$ we will denote by $\hat{0}(\mathbf{X})$ the set $\{x \in X \mid \exists y_1, y_2 \succ x \text{ distinct, with } y_1, y_2 \in 0(\mathbf{X})\}$ and observe that $\hat{0}(\mathbf{X}) \subseteq 0(\mathbf{X})$.

Lemma 4.13. *Let $\mathbf{X}$ be a finite rooted poset of width 2 with an n-coloring. Then $0(\mathbf{X})$ and $\hat{0}(\mathbf{X})$ are downsets in $\mathbf{X}$.*

Proof. That $0(\mathbf{X})$ is a downset follows from the fact that colorings are order-preserving maps.

Now let $y \in \hat{0}(\mathbf{X})$ and $x \in X$ such that $x < y$. Then by definition of $\hat{0}(\mathbf{X})$ there are $u, v \in 0(\mathbf{X})$ distinct such that $y \prec u, v$. As $x < y \in 0(\mathbf{X})$ and $\mathbf{X}$ is rooted and width 2, we find that x has exactly two distinct direct successors w, z. For one of these, say w, we have $x \prec w \leq y$ and so $w \in 0(\mathbf{X})$. It now suffices to show that $z \in 0(\mathbf{X})$ as well.

By width 2 and the fact that $\mathbf{X}$ is rooted we may assume without loss of generality that z and u are comparable, as $u \parallel v$. Since $x < u$ and $x \prec z$ it cannot be the case that $u \leq z$, so that we conclude $z < u$. Then as $0(\mathbf{X})$ is a downset we obtain $z \in 0(\mathbf{X})$ and so $x \in \hat{0}(\mathbf{X})$ as desired. □

Lemma 4.14. *Let K be a non-locally finite variety of width 2. Then there exist a sequence $\{\mathbf{X}_m\}_{m \in \omega}$ of finite rooted n-colored posets and a natural number M^+ such that $\mathrm{Up}(\mathbf{X}_m) \in \mathsf{K}$ for each m, and $|\mathbf{X}_m \setminus \hat{0}(\mathbf{X}_m)| \leq M^+$ for all $m \in \omega$. Consequently, the sequence $\{\hat{0}(\mathbf{X}_m)\}_{m \in \omega}$ is unbounded.*

Proof. We know from Theorem 4.8 that such a sequence $\{\mathbf{X}_m\}_{m \in \omega}$ exists such that $\{\mathbf{X}_m\}_{m \in \omega}$ and with the property that there is an $M \in \omega$ with $|\mathbf{X}_m \setminus 0(\mathbf{X}_m)| \leq M$ for each $m \in \omega$. As the second part of the statement follows from the first, we only need to provide the bound M^+. For this, fix some $\mathbf{X}_m$. Since $|\mathbf{X}_m \setminus 0(\mathbf{X}_m)| \leq M$, it suffices to provide a bound for

$|0(\mathbf{X}) \setminus \hat{0}(\mathbf{X})|$. Then take maximals $a_1, a_2 \in 0(\mathbf{X}) \setminus \hat{0}(\mathbf{X})$. Note that we may have $a_1 = a_2$ if there is a unique maximum in $0(\mathbf{X}) \setminus \hat{0}(\mathbf{X})$, but by width 2 there cannot be others. Now let $c_1 \in 0(\mathbf{X}) \setminus \hat{0}(\mathbf{X})$ be a direct predecessor of a_1, if any such exists. As $c_1 < a_1 \in 0(\mathbf{X})$, we must have another direct successor of c_1, say x_1. Because $c_1 \notin \hat{0}(\mathbf{X})$ we get $x_1 \in \mathbf{X} \setminus 0(\mathbf{X})$. Then select an direct predecessor of c_1 if any exist, call it c_2. With the same reasoning as above, we find another direct successor $x_2 \in \mathbf{X} \setminus 0(\mathbf{X})$ of c_2. Note that $x_1 \neq x_2$ as $c_2 \prec c_1$ and $c_1 \prec x_1$, so that $c_2 \not\prec x_1$. Proceeding like this we find that chains of the form $y_1 \prec y_2 \prec \ldots \prec y_k \prec a_1$ are bounded by $M + 1$. Now since $\hat{0}(\mathbf{X})$ and $0(\mathbf{X})$ are downsets by Lemma 4.13, we have $0(\mathbf{X}) \setminus \hat{0}(\mathbf{X})$ convex. Then every chain $y_1 < \ldots < y_k < a_1$ in $0(\mathbf{X}) \setminus \hat{0}(\mathbf{X})$ can be prolonged to one of the form $y_1 \prec \ldots \prec y_k \prec a_1$. As a result, we obtain that the size of chains in $0(\mathbf{X}) \setminus \hat{0}(\mathbf{X})$ is bounded by $M + 1$. As antichains in $0(\mathbf{X}) \setminus \hat{0}(\mathbf{X})$ are bounded by 2, we must have a bound M^+ on the size of $0(\mathbf{X}) \setminus \hat{0}(\mathbf{X})$. Since we only used M and 2 to arrive at this bound, it works for all $m \in \omega$. $\square$

Lemma 4.15. *Let K be a non-locally finite variety of width 2. Then there exist a sequence $\{\mathbf{X}_m\}_{m\in\omega}$ of finite rooted n-colored posets such that for every $m \in \omega$ there are $m^* \in \omega$ and $a \parallel b$ in $\hat{0}(\mathbf{X}_{m^*})$ with $x \prec a, b$ such that $|{\downarrow}\{a, b\}| \geq m$.*

Proof. Toward a contradiction, suppose the contrary. Then there is a $k \in \omega$ such that for all $m \in \omega$ and $a \parallel b$ in $\hat{0}(\mathbf{X}_m)$ with $x \prec a, b$, we have $|{\downarrow}\{a, b\}| \leq k$. We will bound the size of the sets $\hat{0}(\mathbf{X}_m)$, contradicting Lemma 4.14. Since $\hat{0}(\mathbf{X}_m)$ is a poset of width 2, it will suffice to bound the length of chains in it. First, observe that $\hat{0}(\mathbf{X}_m)$ is a downset, so that it suffices to consider chains of the form $c_1 \prec \ldots \prec c_l$ in $\hat{0}(\mathbf{X}_m)$. Our task now is to find an upper bound for l. For this, first note that if c_i has a direct successor in $\hat{0}(\mathbf{X}_m)$ other than c_{i+1}, we must have $i \leq k$ by assumption. Now assume i is the least index for which c_i has only c_{i+1} as a direct successor in $\hat{0}(\mathbf{X}_m)$. Then clearly $i \leq k$. Now as $c_i \in \hat{0}(\mathbf{X}_m)$ it has a successor in $0(\mathbf{X}_m)$, and so it has a direct successor $a_i \in \hat{0}(\mathbf{X}_m)^c$. Similarly for c_{i+1}, we find a direct successor $a_{i+1} \in \hat{0}(\mathbf{X}_m)^c$. Continuing like this, we construct a sequence $a_i, a_{i+1}, \ldots a_l \in \hat{0}(\mathbf{X}_m)^c$. These must all be distinct as we have for each j that $c_j \prec c_{j+1}, a_j$. This yields the bound

$$l \leq k + |\hat{0}(\mathbf{X}_m)^c| \leq k + M^+,$$

giving the desired contradiction. $\square$

4.4 Main result

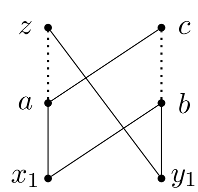
We are now in a position to prove a sufficient condition for an n-colored rooted width 2 poset to have a finite upset of the Rieger-Nishimura ladder as subposet. Following this, we give a full characterisation of locally finite width 2 varieties of Heyting algebras. The proof is rather involved, and so it is instructive to give a general overview. We will first state the condition and then discuss it informally, before moving on to the actual proof.

Theorem 4.16. *Let $\mathbf{X}$ be a finite n-colored width 2 poset with root $\perp$. Further, let $a, b, x_1 \in \hat{0}(\mathbf{X})$ be such that $a \parallel b$ and $x_1 \prec a, b$. Assume that $1 \leq m \in \omega$ and for all $m \geq k \in \omega$ there are no $x_2, \dots, x_k \in \mathbf{X}$ such that $\perp = x_k \prec x_{k-1} \prec \dots \prec x_2 \prec x_1$. Then $\uparrow\{w_{2(m+1)}, w_{2(m+1)+1}\}$ is a subposet of $\mathbf{X}$ in which $\uparrow\{w_{2(m+1)}, w_{2(m+1)+1}\} \setminus \{w_0, w_1\} \subseteq \downarrow\{a, b\}$ and:*

1. $w_{2(m+1)} \prec w_{2m}, w_{2(m-1)+1}$,
2. $w_{2(m+1)+1} \prec w_{2m+1}, w_{2m}$,
3. *there is a chain* $w_{2(m+1)+1} \prec \dots \prec x_1$ *of size less than* m.

The proof will run by induction on m. The inductive method will provide an explicit construction of the desired poset, and so we will describe this construction.

We start with the points a and b from the statement of the Theorem. These points will serve as the points w_3 and w_2 of the subposet. By our assumptions on the predecessors of these points, in particular the point x_1, we are able to apply Lemma 4.10 to find a point y_1 below a and b that is distinct from x_1, as well as a successor z of a and this new y_1. Using the assumption that $a \in \hat{0}(\mathbf{X})$ we find successor c of a and b that is distinct from z. Some argument then allows us to conclude that we have the following picture:



which is exactly the poset we were looking for in the case of $m = 1$. It would be natural to proceed with the construction in the same way as above, taking x_1 and y_1 in place of a and b, which is possible to some extent. The assumptions in the statement and the arguments made up to this point certainly allow us to apply Lemma 4.10 again to find predecessors and successors of x_1 and y_1 as we did for a and b. But here we run into trouble, because there is no guarantee that the successors of x_1 and y_1 that we find coincide with a and b.

The rest (and indeed majority) of the proof is devoted to showing that we can resolve this difficulty. We do this by proving a claim to the effect that if we have our subposet constructed up to this point and we can find such predecessors of x_1 and y_1, as well as a successor z' of x_1 analogous to z, we can either "fit" this z into our constructed poset or use it to find replacements of the points we have constructed so far.

Proof of Theorem 4.16. We proceed by induction on m. For the base case we have $m = 1$. Then by our assumptions on chains $\perp = x_k \prec \ldots \prec x_1$ we get $x_1 \neq \perp$. Then $|{\downarrow}a| \geq 3$ as $\perp < x_1 \prec a$, and so by Lemma 4.10 we obtain a maximal y_1 in ${\downarrow}\{a, b\} \setminus \{a, b\}$ distinct from x_1. From the same lemma we may assume without loss of generality that $y_1 \prec b$ and y_1 has a direct successor $z > a$. This leads us to picture (a) below. For our conventions on drawing dotted or solid lines in these diagrams see Subsection 2.1.1.

(a) (b)

Now as a is in $\hat{0}(\mathbf{X})$, it has two direct successors c and d such that $c \parallel d$. Since $\mathbf{X}$ is width 2 and rooted, it must be the case that without loss of generality c is comparable with b. As $a \parallel b$ this gives $b < c$. Taking into account all known covering relations between our points, we now have picture (b) above. To conclude the proof of the base case we need to show that (b) is faithful in the sense that no other relations exist between these six points. To show this we argue that for each point x depicted in (b) above as x not being below some y, it is in fact the case that $x \not\leq y$. We treat each case separately:

- As x_1, y_1 are distinct maximal points in ${\downarrow}\{a, b\} \setminus \{a, b\}$ we have $x_1 \not\leq y_1$ and $y_1 \not\leq x_1$. Further, as $y_1 \prec z$ and $a < z$ we have $y_1 \not\leq a$. This takes care of x_1 and y_1.

- For a, we have:

- $y_1 \leq b$ and $a \nleq b$, so that $a \nleq y_1$
- $a \parallel b$ so $a \nleq b$
- $x_1 \prec a$ so $a \nleq x_1$

- For b, we have:

  - $a \parallel b$ so $b \nleq a$
  - $y_1 \prec b, z$ and $z \neq b$, so $b \nleq z$
  - $x_1, y_1 \prec b$ so $b \nleq x_1, y_1$

- For z, we have:

  - $a < z$, so $z \nleq a$
  - $y_1 \prec b, z$ and $b \neq z$ and so $z \nleq b$
  - If it were the case that $z \leq c$, then $a \leq z \leq c$. This, together with $a < z$ and $a \prec c$, would give $z = c$. Then $b \leq c$ against $b \parallel z$. So we get $z \nleq c$.
  - $y_1 \prec z$ and $x_1 \prec a < z$ so $z \nleq x_1, y_1$.

- Finally, for c:

  - $b \parallel z$ and $b \leq c$ imply $c \nleq z$.
  - As $a, b < c$ we have $c \nleq a, b, x_1, y_1$.

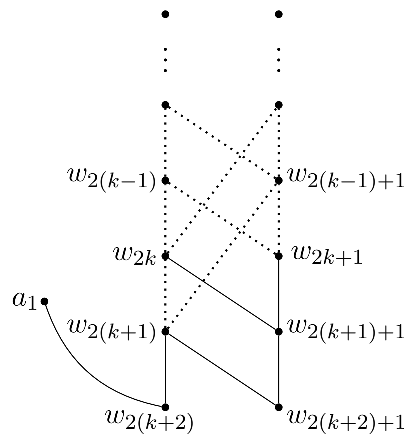
We can now conclude that picture (b) above is faithful, concluding the base case.

The inductive case will require significantly more work and makes extensive use of the lemmas in Section 4.3. Assume the statement holds for some $k \in \omega$ and consider $m = k + 1$. First we prove the following claim.

Claim. *To complete the inductive step, it is sufficient to show that* $\mathbf{X}$ *has a subposet isomorphic to the upset* $\uparrow\{w_{2(k+1)}, w_{2(k+1)+1}\}$ *such that all but* w_0, w_1 *are in* $\downarrow\{a, b\}$ *and the following hold:*

1. $w_{2(k+1)+1} \prec w_{2k+1}, w_{2k}$,
2. *there exists a point* $w_{2(k+2)+1} \in \mathbf{X}$ *such that* $w_{2(k+2)+1} \prec w_{2(k+1)}$ *and* $w_{2(k+2)+1} \prec w_{2(k+1)+1}$,
3. *there exists a point* $w_{2(k+2)} \in \mathbf{X}$ *such that* $w_{2(k+2)} \prec w_{2(k+1)}$ *and* $w_{2(k+2)}$ *has a direct successor* a_1 *with* $a_1 > w_{2(k+1)+1}$.

We recommend the reader keeps the picture below in mind while reading the proof. Note that the point a_1 will actually be strictly above $w_{2(k+1)+1}$, but we do not yet want to suggest an exact location.



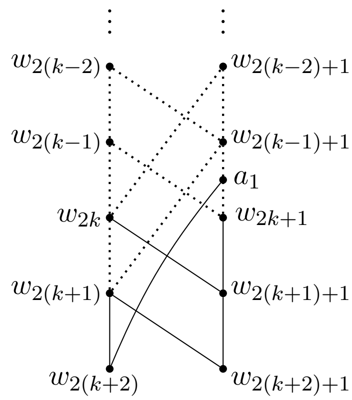
Proof of claim. Suppose the subposet with stated conditions as in the claim exists. We need to show that the inductive step goes through. First, note that $w_{2(k+1)+1}$ has two direct successors w_{2k} and w_{2k+1}.

First note that as both a_1 and $w_{2(k+1)}$ are direct successors of $w_2(k+2)$ and $w_{2(k+1)} \parallel w_{2(k+1)+1} < a_1$, we have $a_1 \parallel w_{2(k+1)}$. Since $w_{2(k+1)+1} < a_1$, it must be the case that a_1 is greater or equal than one of the immediate successors w_{2k} and w_{2k+1} of $w_{2(k+1)+1}$. As $a_1 \parallel w_{2(k+1)}$, this implies $a_1 \geq w_{2k+1}$. Now, as a_1 is a direct successor of $w_{2(k+1)}$, we have $w_{2k} \not\leq a_1$. Together with $a_1 \geq w_{2k+1}$ and $w_{2k+1} \parallel w_{2k}$, this implies $a_1 \parallel w_{2k}$. Finally, because $w_{2k} \parallel a_1$ and$w_{2k} \parallel w_2(k-1)+1$, we can apply the assumption that $\mathbf{X}$ is rooted and width 2 to obtain that a_1 is comparable with $w_2(k-1)+1$.

Together with $a_1 \parallel w_{2(k+1)}$ this implies

$$a_1 \in [w_{2k+1}, w_{2(k-1)+1}) = \{x \in X \mid w_{2k+1} \leq x < w_{2(k-1)+1}\}.$$

We can now provide an updated version of the picture above that shows the exact location of a_1:

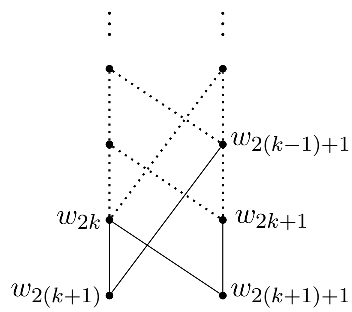


Now if $a_1 = w_{2k+1}$ we are done. Otherwise we consider the case where $w_{2k+1} < a_1 < w_{2(k-1)+1}$. Then, as $a_1 \in \downarrow\{a, b\}$ and $a, b \in 0(\mathbf{X})$, we see a_1 has two direct successors. By width 2, one of these is comparable with $w_{2(k-1)}$, call this point a_2. Now if $a_1 \leq w_{2(k-1)}$, we remove w_{2k+1} from the picture and replace it with a_1, which completes the construction. Otherwise we have $a_1 \not\leq w_{2(k-1)}$ which implies that $a_2 \not\leq w_{2(k-1)}$ as well. Since a_2 and $w_{2(k-1)}$ are comparable, we obtain $a_2 > w_{2(k-1)}$. Further, as $a_1 < w_{2(k-1)+1} < w_{2(k-2)}$ and $a_1 \prec a_2$, we get $a_2 \not\geq w_{2(k-2)}$. We shall see that $a_2 < w_{2(k-2)}$. Toward a contradiction, suppose $a_2 \not< w_{2(k-2)}$. Then $a_2 \parallel w_{2(k-2)}$. As we also have $w_{2(k-2)} \parallel w_{2(k-2)+1}$, width 2 gives that a_2 is comparable with $w_{2(k-2)+1}$. Since $a_1 \prec a_2$we obtain $a_2 < w_{2(k-2)+1}$, and so $w_{2(k-1)} \leq a_2 \leq w_{2(k-2)+1}$ which is a contradiction. We conclude that $a_2 < w_{2(k-2)}$, and so $a_2 \in (w_{2(k-1)}, w_{2(k-2)})$. We can then delete from the picture w_{2k+1} and $w_{2(k-1)}$, and replace them with a_1 and a_2 respectively. Next, we construct from a_2 a point a_3 in the same manner as we constructed a_2 from a_1. Continuing this way and recalling $m = k + 1$ we eventually get the desired copy of $\uparrow\{w_{2(m+1)}, w_{2(m+1)+1}\}$. $\square$

We now return to the proof of the theorem. From the induction hypothesis we get that there is a copy of $\uparrow\{w_{2(k+1)}, w_{2(k+1)+1}\}$ in $\mathbf{X}$ such that all its elements are in $\downarrow\{a, b\}$, except for w_0 and w_1 and:

1. $w_{2(k+1)} \prec w_{2k}, w_{2(k-1)+1}$,
2. $w_{2(k+1)+1} \prec w_{2k+1}, w_{2k}$,
3. there is a chain $w_{2(k+1)+1} \prec \ldots \prec x_1$ of size less than k.

The picture below shows the situation:



We can now conclude the proof of the theorem by constructing an element a_1 as in the statement of the claim. Note that there are two cases:

C1. There exists $z \prec w_{2(k+1)}, w_{2(k+1)+1}$.
C2. There is no common direct predecessor of $w_{2(k+1)}$ and $w_{2(k+1)}$.

We treat these cases in turn.

C1: By assumption there is a chain $w_{2(k+1)+1} \prec \ldots \prec x_1$ of at most k elements. Then there is a chain $z \prec w_{2(k+1)+1} \prec \ldots \prec x_1$ of at most $k+1 = m$ elements. By our assumptions on chains $\perp = x_k \prec \ldots \prec x_1$ this implies that $z \neq \perp$, and so $|{\downarrow} w_{2(k+1)}| \geq 3$. Consequently, we can apply Lemma 4.10 to conclude that the set

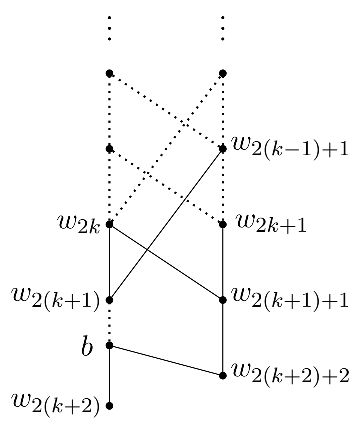
$$\downarrow\{w_{2(k+1)}, w_{2(k+1)+1}\} \setminus \{w_{2(k+1)}, w_{2(k+1)+1}\}$$

has a maximal element distinct from z, call this element a_1. Further considering Lemma 4.10, and examining the direct successors of $w_{2(k+1)}$ and $w_{2(k+1)+1}$, we find that a_1 has a direct successor strictly above $w_{2(k+1)+1}$, $a_1 \prec w_{2(k+1)}$ and $a_1 \not\leq w_{2(k+1)+1}$. But then we are in the situation described in the claim, and so we are done.

C2: In this case we find by Lemma 4.11 that there exists an element $w_{2(k+2)+1}$ strictly below $w_{2(k+1)}$ that is a direct predecessor of $w_{2(k+1)+1}$. Then $w_{2(k+2)+1}$ has a direct successor $b \leq w_{2(k+1)}$. We can choose $w_{2(k+2)+1}$ in such a way that for all other $c \prec w_{2(k+1)+1}$ with $c \leq w_{2(k+1)}$ the direct successor of c below $w_{2(k+1)}$ is smaller or equal to b. This property of be will be called the "*maximality*" of b. Now observe that $w_{2(k+2)}+1 \prec b, w_{2(k+1)+1}$. Further, there is a chain $w_{2(k+2)+1} \prec w_{2(k+1)+1} \prec \ldots x_1$ of at most $k+1 = m$ elements. From our assumptions we then get that $w_{2(k+2)+1} \neq \perp$, and so $|{\downarrow} w_{2(k+1)+1}| \geq 3$. Applying Lemma 4.10, we get that the set

$$\downarrow\{w_{2(k+1)+1}, b\} \setminus \{w_{2(k+1)+1}, b\}$$

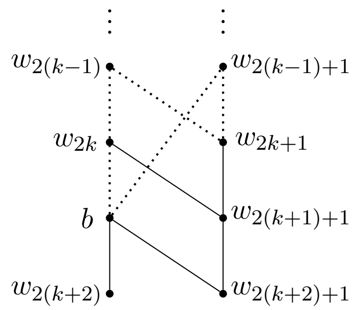
has a maximal element distinct from $w_{2(k+2)+1}$. We call this element $w_{2(k+2)}$. Again examining Lemma 4.10, we see $w_{2(k+2)}$ must satisfy exactly one of the two conditions numbered 1 and 2 in this lemma. Now suppose toward a contradiction that $w_{2(k+2)} \prec w_{2(k+1)+1}$. Then Lemma 4.11 indicates that $w_{2(k+2)}$ has an direct successor $c > b$. By the *maximality* of b we then get $c \not\leq w_{2(k+1)}$. As $w_{2(k+1)+1} \parallel c$ and $w_{2(k+1)+1} \parallel w_{2(k+1)}$ we get from width 2 that c is comparable to $w_{2(k+1)}$. Since $c \not\leq w_{2(k+1)}$, we have $w_{2(k+1)} < c$. Examining the picture above, we find that this implies $w_{2k} \leq c$ and so $w_{2(k+1)+1} < c$, which is not the case. The desired contradiction reached, we conclude $w_{2(k+2)} \not\prec w_{2(k+1)+1}$. Then the other condition in Lemma 4.11 must hold. That is, $w_{2(k+2)} \prec b, c$ for some $c > w_{2(k+1)+1}$. We now have the following update to our picture:



Not shown here is the direct successor c of $w_{2(k+2)}$ which is above $w_{2(k+1)+1}$. Considering the direct successors of $w_{2(k+1)+1}$ and recalling $c \parallel b$, we see that $c \geq w_{2(k+1)}$. Similarly we get from $c \parallel b$ that $c \not\geq w_{2k}$. Further, from $c \geq w_{2(k+1)}$ we obtain $c \not\leq w_{2k}$ and so conclude $c \parallel w_{2k}$. From this and $w_{2k} \parallel w_{2(k-1)+1}$ we get from width 2 that c is comparable of $w_{2(k-1)+1}$. As $c \parallel b$ this implies $c < w_{2(k-1)+1}$. Then:

$$w_{2(k+1)} \leq c < w_{2(k-1)+1}.$$

This, together with $b \parallel c$ implies that $b \not\leq w_{2(k+1)}$. We have shown that the following picture is faithful:



Further, we note that $c \prec w_{2(k+2)}$ and that $c \geq w_{2k+1}$ implies $c > w_{2(k+1)+1}$. We can now conclude that the conditions of the claim are satisfied and so that the inductive step goes through.

This conclude the proof. $\square$

We are now equipped to prove the following theorem, which fully characterises locally finite varieties of Heyting algebras of width 2:

Theorem 4.17. *A width* 2 *variety* K *of Heyting algebras fails to be locally finite iff it contains the Rieger-Nishimura lattice with a new bottom element* $\mathcal{L}_+$.

Proof. In both directions we prove the contrapositive.

"$\Leftarrow$": The Rieger-Nishimura lattice with a new bottom element is 2-generated and infinite, so a variety that contains it is not locally finite.

"$\Rightarrow$": Suppose K is a not locally finite variety of width 2. We need to show that the Rieger-Nishimura lattice with a new bottom element is in K. By Proposition 4.5, it is sufficient to show that every principal upset of the Rieger-Nishimura ladder can be embedded as a poset into the dual Esakia space of a finite member of K. For this, consider some arbitrary $m \in \omega$. From Lemmas 4.9 and 4.15 we know that there is a poset $\mathbf{X}_{m^*}$ with $\mathrm{Up}(\mathbf{X}_{m^*}) \in \mathsf{K}$ and elements $a, b, x_1 \in \hat{0}(\mathbf{X}_{m^*})$ such that:

1. $a \parallel b$,
2. $x_1 \prec a, b$,
3. For every chain $x_k \prec \ldots \prec x_1$ with $k \leq m$ we have $x_k \neq \bot$.

We are then in a position to apply theorem 4.16 and obtain that the upset $\uparrow\{w_{2(m+1),w_{2(m+1)+1}}\}$ of the Rieger-Nishimura ladder is a subposet of $\mathbf{X}_{m^*}$. Since m was arbitrary, we conclude that all upsets $\uparrow\{w_{2(k+1)}, w_{2(k+1)+1}\}$

with $k \in \omega$ can be obtained as a subposet of some Esakia space in K_*. Then every finite upset $\uparrow w_k$ is such a subposet of an Esakia space in K_* so that Corollary 4.5 implies that K contains a copy of the Rieger-Nishimura lattice with a new bottom element. $\square$

As a direct result of this theorem we obtain the following:

Corollary 4.18. *Conjecture 3.18 holds in width* 2 *varieties.*

4.5 Limitative result

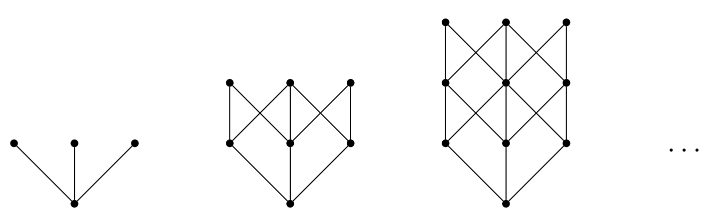
We now provide an example that shows the method applied above cannot be applied in width n varieties for $n > 2$. Note that this has no bearing on the truth or falsehood of Conjecture 3.18 in these varieties. More specifically, we will show that there exists a non-locally finite variety K of width 3 that omits the Rieger-Nishimura lattice with a new bottom element.

In order to show that our example works we will make use of *Jankov-de Jongh formulas.* In fact, the formulas introduced by Jankov and those introduced by de Jongh differ in their essential construction, but for our purposes this does not matter. What is important for us is that they both satisfy the following theorem. For a proof see e.g. [7, Theorem 3.3.3].

Theorem 4.19 (Jankov-de Jongh)**.** *For every finite rooted poset* $\mathbf{X}$ *there exists a formula* $\chi(\mathbf{X})$ *such that for all posets* $\mathbf{Y}$*:*

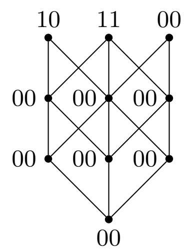
$\mathbf{Y} \not\Vdash \chi(\mathbf{X})$ *iff* $\mathbf{X}$ *is a bounded morphic image of a generated subposet of* $\mathbf{Y}$.

For each poset $\mathbf{X}$ the formula $\chi(\mathbf{X})$ will be called the *Jankov-de Jongh formula* of $\mathbf{X}$. We now turn to constructing our variety K. We will define this as the variety generated by a class of finite subdirectly irreducible Heyting algebras which we will specify as the duals of a certain class of finite rooted posets. Consider the posets in the picture below:



$\mathbf{X}_0$ $\mathbf{X}_1$ $\mathbf{X}_2$ . . .

Continuing this sequence we obtain a class of posets $\{\mathbf{X}_i\}_{i\in\omega}$. We then set the variety: $\mathsf{K} := \mathbf{V}(\{\mathbf{X}_i^*\}_{i\in\omega})$. Now consider the following 2-coloring of $\mathbf{X}_2$ as a prototype of the colorings for each $\mathbf{X}_i$:



To see that this is a coloring of $\mathbf{X}_2$, consider an arbitrary proper bisimulation equivalence E. We can see E as the result of a finite sequence of α- and β-reductions. Now noting that no point in $\mathbf{X}_2$ has a unique direct successor, and that the only pairs of points that share sets of direct successors are the maximal ones, we conclude that this sequence must start with a β-reduction that identifies two maximal points. That is, it must identify points of different color. We readily see that this coloring is increasing, and so conclude that it is a coloring. Apply this same coloring (only the maximal points have distinct colors, the rest have color 00) to all $\mathbf{X}_i$. By Theorem 3.8 we then see that each $\mathbf{X}_i^*$ is 2-generated. As K then has arbitrarily large finite 2-generated algebras, we see that the free 2-generated algebra in K is infinite. Then K is not locally finite.

To complete the counterexample we need to show that the Heyting algebra $\mathcal{L}_+$ is not in K. We do this by deriving a contradiction from the contrary assumption. Supposing that $\mathcal{L}_+ \in \mathsf{K}$, it follows that for every principal upset $\uparrow w_n$ of the Rieger-Nishimura *ladder* with a new top element we have $(\uparrow w_n)^* \in \mathsf{K}$ as well. Now every finite rooted poset is trivially a bounded morphic image of a generated subposet of itself, and so refutes its own Jankov-de Jongh formula by Theorem 4.19. Then we have that for each $n \in \omega$ the formula $\chi(\uparrow w_n)$ is refuted in K. Since K is generated by the class $\{\mathbf{X}_i^*\}_{i\in\omega}$, we obtain that for each $m \in \omega$ there is a $k \in \omega$ such that

$$\mathbf{X}_k \not\Vdash \chi(\uparrow w_m),$$

and so by Theorem 4.19, $\uparrow w_m$ is without loss of generality a bounded morphic image of $\mathbf{X}_k$. We shall show that this is not possible.

Select some $m \in \omega$ sufficiently large, and let $k \in \omega$ be such that $\mathbf{X}_k \not\Vdash \chi(\uparrow w_m)$. Call a point x in some poset $\mathbf{X}$ *standard* if there exist $y, z \in \mathbf{X}$ such that $y \parallel x \parallel z$ and $y < z$. Note that in $\uparrow w_m$, all but w_0 and the top and bottom elements are standard. By contrast, in $\mathbf{X}_k$ every point in the 'center' (i.e. every point with exactly three direct successors) is not standard. We have the following:

Lemma 4.20. *Let $f : \mathbf{X} \twoheadrightarrow \mathbf{Y}$ be a surjective bounded morphism. For all $x \in \mathbf{X}$, if $f(x)$ is standard, then x is standard.*

Proof. Let x be arbitrary such that $f(x)$ is standard. Then there are $y_1 \parallel f(x) \parallel y_2$ in $\mathbf{Y}$ with $y_1 < y_2$. By surjectivity of f there must be $x_1 \in \mathbf{X}$ such that $f(x_1) = y_1$. By the forth condition on bounded morphisms, we must have $x_1 \parallel x$. By the back condition, there must be an $x_2 \in \mathbf{X}$ such that $x_1 < x_2$ and $f(x_2) = y_2$. Finally, by the forth condition we have $x_2 \parallel x$ as well. Then x is standard as desired. □

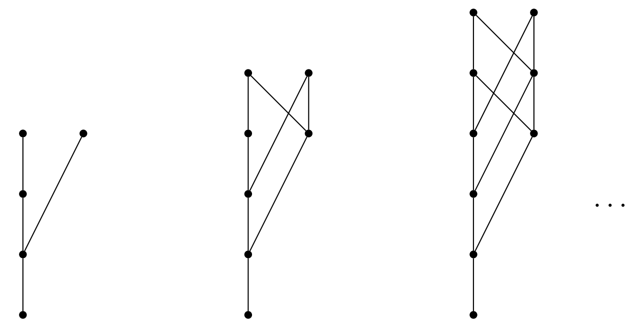
It follows that every non-standard point in $\mathbf{X}_k$ must be mapped to one of w_0 or the top or bottom elements of $\uparrow w_m$. Clearly we cannot have all such non-standard points x mapped to the bottom element, as then every y below x would be mapped to the bottom element as well, leaving too few points to cover the rest of $\uparrow w_m$. Similarly for mapping all such x to w_0 or the top element. In fact, if we let x_0 be the largest non-standard point mapped to the bottom, and x_1 its direct non-standard successor, we find that at most four points remain outside of $\downarrow x_0 \cup \uparrow x_1$ that can be mapped to points in $\uparrow w_0$ besides to w_0 or the top or bottom elements. Since we selected m to be large enough, this shows the impossibility of there being a bounded morphism, giving the desired contradiction.

4.6 Cardinality

In this section we examine the cardinality of the classes of locally finite and non-locally finite width 2 varieties of Heyting algebras.

We first show that there are continuum many non-locally finite varieties of width 2 Heyting algebras. We do this by considering the class of posets

$\Gamma = \{(\uparrow w_{2n})_+\}_{n\in\omega, n\geq 2}$ depicted below:



The following is item (1) in [7, Lemma 4.5.3]

Lemma 4.21. *Let $m, n \in \omega$ such that $2 \leq m < n$, then neither of $\mathbf{X} = (\uparrow w_{2m})_+$ and $\mathbf{Y} = (\uparrow w_{2n})_+$ is a bounded morphic image of a generated subposet of the other.*

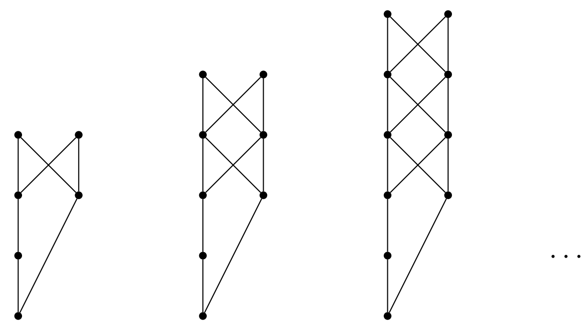
Theorem 4.22 ([7], Theorem 3.4.18)**.** *Let $\{\mathbf{Y}_i\}_{i\in\omega}$ be a sequence of Kripke posets such that for all $n \neq m \in \omega$ neither of $\mathbf{Y}_m, \mathbf{Y}_n$ is a bounded morphic image of a generated subposet of the other. Then for every $A, B \subseteq \omega$ with $A \neq B$ we have $\mathrm{Log}(\{\mathbf{Y}_i\}_{i\in A}) \neq \mathrm{Log}(\{\mathbf{Y}_i\}_{i\in B})$*

Proof. Assume $\{\mathbf{Y}_i\}_{i\in\omega}$ and A, B are as in the statement. Then without loss of generality we may assume that $A \not\subseteq B$, and so there is a $\mathbf{X} \in \{\mathbf{Y}_i\}_{i\in A}$ such that $\mathbf{X} \notin \{\mathbf{Y}_i\}_{i\in B}$. Then as $\mathbf{X}$ is a bounded morphic image of itself, we have by Theorem 4.19 that $\mathbf{X} \not\Vdash \chi(\mathbf{X})$. This gives $\chi(\mathbf{X}) \notin \mathrm{Log}(\{\mathbf{Y}_i\}_{i\in A})$. By assumption we have that $\mathbf{X}$ is not a bounded morphic image of any poset in $\{\mathbf{Y}_i\}_{i\in B}$, so again by Theorem 4.19 we get for all $\mathbf{X}' \in \{\mathbf{Y}_i\}_{i\in B}$ that $\mathbf{X}' \Vdash \chi(\mathbf{X})$. This gives $\chi(\mathbf{X}) \in \mathrm{Log}(\{\mathbf{Y}_i\}_{i\in B})$, establishing the result. □

Corollary 4.23. *There are continuum many non-locally finite width 2 varieties of Heyting algebras.*

Proof. Note that each principal upset $\uparrow w_n$ of the Rieger-Nishimura ladder is a subposet of some $(\uparrow w_{2n})_+$. Then as there are continuum many infinite subsets of ω we obtain continuum many subvarieties of $\mathbf{V}(\Gamma^*)$ that have arbitrarily large finite 2-generated algebras. Then the free 2-generated algebra in each of these will be infinite, and so these varieties will not be locally finite. By Lemma 4.21 and Theorem 4.22 they are all distinct. □

We can use similar reasoning to come to the complimentary result that there are continuum many locally finite varieties of Heyting algebras as well. For this, consider the following sequence of posets:

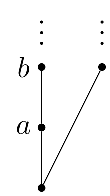


$\mathbf{Y}_0$ $\mathbf{Y}_1$ $\mathbf{Y}_2$

Again we get a sequence $\{\mathbf{Y}_i\}_{i\in\omega}$. Then we have the following lemma in parallel to Lemma 4.21.

Lemma 4.24. *Let $m \neq n \in \omega$, then neither of $\mathbf{Y}_m$ nor $\mathbf{Y}_n$ is a bounded morphic image of a generated subposet of the other.*

Proof. Assume $m < n$. Then clearly $\mathbf{Y}_n$ is not a bounded morphic image of a generated subposet of $\mathbf{Y}_m$. Conversely, suppose toward a contradiction that there is a generated subposet $\mathbf{Y}_n'$ of $\mathbf{Y}_n$ with a surjective bounded morphism $f : \mathbf{Y}_n \twoheadrightarrow \mathbf{Y}_m$. Then f can be seen as the result of a finite sequence of α- and β-reductions. Then the first of these reductions must either identify two points of the same depth, or identify a and b in the picture below. We see that neither can lead to a poset isomorphic to $\mathbf{Y}_m$, establishing the contradiction.



$\mathbf{Y}_n$

$\square$

Lemma 4.25. *For each $A \subseteq \omega$ the variety $\mathbf{V}(\{\mathbf{Y}_i^*\}_{i\in A})$ is locally finite.*

Proof. By contradiction. Assume that for some $A \subseteq \omega$ the variety $\mathsf{K} := \mathbf{V}(\{\mathbf{Y}_i\}_{i\in A})$ is not locally finite. Then as K is width 2, we get from Theorem 4.17 that the Rieger-Nishimura lattice with a new bottom element is in K. This implies that for each $n \in \omega$ the formula $\chi(\uparrow w_n)$ is refuted in K, and so by one of $\mathbf{Y}_m$. Theorem 4.19 then gives that for each $\uparrow w_n$ there is a $\mathbf{Y}_m$ with a generated subposet $\mathbf{Y}'_m$ and surjective bounded morphism $f : \mathbf{Y}'_m \twoheadrightarrow \uparrow w_n$. However, selecting n large enough, we find that this is impossible, as $\uparrow w_n$ has standard points and no point in the $\mathbf{Y}_i$ are standard. So such a morphism f contradicts Lemma 4.20. $\square$

Corollary 4.26. *There are continuum many locally finite varieties of width 2 Heyting algebras,*

Proof. Identical to that of Corollary 4.23 when using the above lemmas. $\square$

4.7 Pre-local tabularity and decidability

4.7.1 Pre-locally tabular si-logics

Call a logic L *pre-locally tabular* if it is not locally tabular, but every logic L' with $\mathsf{L} \subset \mathsf{L}'$ is. It is known that there is a continuum of pre-locally tabular logics (see [20]). We obtain the following from our main characterisation of locally finite width 2 varieties of Heyting algebras:

Theorem 4.27. *The logic* $\mathsf{L} = \mathrm{Log}(\mathfrak{L}^+)$ *is the only pre-locally tabular width 2 logic.*

Proof. That L is not locally tabular follows from Theorem 3.16 and the fact that V_{L} contains $\mathcal{L}_+$.

Now suppose L' is a logic such that $\mathsf{L} \subset \mathsf{L}'$. Then $\mathbf{bw_2} \in \mathsf{L}'$ and so it is width 2. As $\mathsf{L}' \neq \mathsf{L}$ there is a formula $\varphi \in \mathsf{L}'$ such that $\mathfrak{L}^+ \not\Vdash \varphi$, and so $\mathcal{L}_+ \notin V_{\mathsf{L}'}$. Then L' is locally tabular by Theorems 4.17 and 3.16.

By the same theorems we get that if L'' is width 2 and non-locally tabular, it must be the case that $\mathsf{L}'' \subseteq \mathsf{L}$. This establishes unicity. $\square$

In [10, p.429] we find the following problem:

Problem 12.1 *Is it true that every non-locally tabular si-logic is contained in a pre-locally tabular one?*

While we cannot provide a solution for all cases, the previous proposition together with Theorem 4.17 provides a positive one for the case of width 2

Corollary 4.28. *Every non-locally tabular width 2 logic* L *contained in a pre-locally tabular logic.*

Proof. Suppose L is a non-locally tabular width 2 logic. Then by Theorem 4.17 we have $\mathfrak{L}^+ \in V_{\mathsf{L}}$, and so $\mathsf{L} \subseteq \mathrm{Log}(\mathfrak{L}^+)$. $\square$

4.7.2 Decidability

In this subsection we show that it is decidable whether a width 2 variety of Heyting algebras K defined by a finite set of equations Σ is locally finite. We know from Theorem 4.17 that to decide this problem it is sufficient to decide whether $\mathcal{L}_+ \in \mathsf{K}$. In doing so we provide a partial solution to the following problem from [10], which is originally due to Maksimova.

Problem 17.4 *Is local tabularity decidable for si-logics?*

We note that deciding whether $\mathcal{L}_+ \in \mathsf{K}$ can be done by determining whether all equations in Σ hold on $\mathcal{L}_+$ which can in turn be decided by determining whether all formulas φ in Σ_* (the set of formula translations of equations in Σ) are in $\mathrm{Log}(\mathfrak{L}^+)$. Effectively, our desired result will follow if we can show that $\mathrm{Log}(\mathfrak{L}^+)$ is decidable.

Theorem 4.29. *The logic* $\mathrm{Log}(\mathfrak{L}^+)$ *is decidable.*

We will provide a commonly used decision procedure that relies on two properties of $\mathrm{Log}(\mathfrak{L}^+)$. First that it is finitely axiomatisable and secondly that it has the *finite model property* (fmp), i.e. that if $\varphi \notin \mathrm{Log}(\mathfrak{L}^+)$, then there is a finite model $\mathfrak{M}$ such that $\mathfrak{M} \Vdash \mathrm{Log}(\mathfrak{L}^+)$ and $\mathfrak{M} \nVdash \varphi$ for all formulas φ. The former follows immediately from [7, Theorem 4.6.4] while the latter is [7, Theorem 4.6.2].

Now let φ be a formula and $\mathsf{L} = \mathrm{Log}(\mathfrak{L}^+)$. To determine whether $\varphi \in \mathsf{L}$ we run two procedures in parallel.

First, we make use of the fact that L is finitely axiomatised to systematically enumerate all $\psi \in \mathsf{L}$ that have propositional variables that occur in φ. We do this by first writing down all one-step derivations using only variables that occur in φ and where all formulas have at most ten symbols, say. Then we write down all such derivations that have at most two steps and fifteen symbols. Continuing on like this, if it is the case that $\varphi \in \mathsf{L}$ then this procedure will produce it.

On the other hand, we enumerate systematically all models on a one-point set, verify whether they are a model of L (which we can do as L has finitely many axioms) and if this is the case, verify if φ is valid. Next we repeat this process with all models on a two-point set and so on. Since L has the finite model property, if it is the case that $\varphi \notin \mathsf{L}$ then there will be a finite model of L that refutes it, which our procedure will find.

We combine these procedures by running them alternately, then regardless of whether $\varphi \in \mathsf{L}$ or not, we will discover this in a finite number of steps, showing that L is decidable.

Theorem 4.30. *Let* K *be a variety of width* 2 *Heyting algebras defined by a finite set of equations. Then the problem of whether* K *is locally finite is decidable.*

Proof. This follows immediately from Theorem 4.29 and the preceding remarks. $\square$

Chapter 5

Conclusion

In this thesis we have proved that every non-locally finite variety of Heyting algebras of width 2 contains an infinite 2-generated algebra. We did this by providing a full characterisation of such varieties by showing that they must contain the Rieger-Nishimura lattice with a new bottom element $\mathcal{L}_+$. Using this characterisation we were able to prove the decidability of local finiteness for varieties of Heyting algebras of width 2 that are defined by finite sets of equations and show that the logic of $\mathcal{L}_+$ is the only pre-locally tabular si-logic of width 2. We also proved that there are continua of both locally finite and non-locally finite varieties of Heyting algebras of width 2.

We have also shown that this characterisation does not extend to the case of arbitrary width n by exhibiting a non-locally finite variety of Heyting algebras of width 3 that does not contain $\mathcal{L}_+$. This leads to the most natural problem for further research: can we find an equally transparent characterisation in arbitrary width n? If not, can we at least do this in width 3? Doing this would require finding an appropriate width 3 analogue of $\mathcal{L}_+$. However, even if this cannot be done, it might still be possible that every non-locally finite variety of Heyting algebras of width 3 (or more generally, of bounded width) contains *some* infinite 2-generated algebra.

Bibliography

[1] R. Balbes, P. Dwinger, *Distributive Lattices*, University of Missouri Press, 1974.

[2] F. Bellissima, "Finitely generated free Heyting algebras" *The Journal of Symbolic Logic, vol. 51(1), pp.152–165*, 1986.

[3] C. Bergman, *Universal Algebra: Fundamentals and Selected Topics*, CRC Press, 2011.

[4] G. Bezhanishvili, "Locally finite varieties", *Algebra Universalis, vol. 46(4), pp.532-548*, 2001.

[5] G. Bezhanishvili, N. Bezhanishvili, D. de Jong, "The Kuznetsov-Gerciu and Rieger-Nishimura logics: The boundaries of the finite model property", *Logic and Logical Philosophy, vol. 17, pp.73-110*, 2008.

[6] G. Bezhanishvili, R. Grigolia, "Locally finite varieties of Heyting algebras", *Algebra Universalis, vol. 54, pp.465–473*, 2005.

[7] N. Bezhanishvili, *Lattices of Intermediate and Cylindric Modal Logics*, PhD Thesis. ILLC, University of Amsterdam, 2006.

[8] N. Bezhanishvili, T. Moraschini, "Hereditarily structurally complete intermediate logics: Citkin's theorem via Esakia duality", submitted.

[9] S. Burris, H.P. Sankanappavar, *A Course in Universal Algebra*, Springer, 1981.

[10] A. Chagrov, M. Zakharyaschev, *Modal Logic*, Oxford University Press, 1997.

[11] D. van Dalen, A. Troelstra, *Constructivism in mathematics : an introduction*, Volumes I and II, North-Holland, 1988.

[12] B. A. Davey, H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, 2002.

[13] L. Esakia, "Topological Kripke models", *Soviet Mathematics Doklady, vol. 15, pp.147-151*, 1974.

[14] L. Esakia, *Heyting algebras. Duality Theory*, English translation of 1985 book, Springer, 2019.

[15] D. de Jongh, A. Troelstra, "On the connection of partially ordered sets with some pseudo-Boolean algebras", *Indagationes Mathematicae, vol. 69, C, pp.317-329*, 1966.

[16] V. Ja. Gerčiu, A. V. Kuznetsov, "The finitely axiomatizable superintuitionistic logics", *Soviet Mathematics Doklady, vol. 11, pp.1654–1658*, 1970.

[17] R. Grigolia, "Free Algebras of Non-Classical Logics", *Metsniereba*, Tbilisi, 1987 (in Russian).

[18] L. L. Maksimova, "Modal logics of finite slices", *Algebra and Logic, vol. 14, pp.188-197*, 1975.

[19] L. L. Maksimova, "Interpolation in the modal logics of the infinite slice containing the logic $\mathsf{K4}$", *Mathematical Logic and Algorithmic Problems, pp.73-91*, 1989.

[20] S. I. Mardaev, "Number of pre-locally tabular superintuitionistic propositional logics", *Algebra and Logic, vol. 23(1) pp.56-66*, 1984.

[21] V. V. Rybakov, "Admissibility of logical inference rules", volume 136 of *Studies in Logic*. Elsevier, Amsterdam, 1997.

[22] K. Segerberg, "An essay in classical modal logic", *Philosophical Studies, Uppsala, 13*, 1971.

[23] V. B. Shehtman. "Rieger-Nishimura lattices", *Soviet Mathematics Doklady, vol. 19, pp.1014–1018*, 1978.