What Structural Objects Could Be
– Mathematical Structuralism and its Prospects –

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis covers structuralism in the philosophy of mathematics, focusing on non-eliminative versions thereof and zooming in on three fresh and promising contemporary articulations. After introducing the topic and essential piece of terminology, we follow a quasi-historical route to modern mathematical structuralism: starting with Paul Benacerraf's seminal articles and after drawing a taxonomy of views playing out in the contemporary field, we discuss eliminative structuralism alongside introducing useful ideology, and we formulate eliminativist discontents which feed a line of reasoning which is crucially invoked by non-eliminativists to motivate their view. Moving thus on to non-eliminativism, we introduce Stewart Shapiro's early articulation thereof: *Sui Generis* Structuralism, followed by an extensive discussion of many of the the problems and ensuing objections leveraged against it. Gathered together, all these concerns constitute the canon we use to assess the three newly emerging articulations of positionalist non-eliminativist structuralism. After taking a motivated detour through non-positionalist non-eliminativism, we introduce in some detail Øystein Linnebo and Richard Pettigrew's Fregean Abstractionist Structuralism, Edward Zalta and Uri Nodelman's Object Theoretic Structuralism and Hannes Leitgeb's Graph Theoretic Structuralism. Assessing each of these views against our canon, we find that, for the most part, each of these is successfully replied. Our thesis is that in spite of sustained criticism, there is still fuel in the realist's tank, meaning that each of the three views is left standing following their assessment against the canon, albeit this claim will be qualified in Conclusion.

Acknowledgements

There are many people who deserve to be acknowledged for their direct or indirect contribution to this thesis; feeling as if committing an injustice, I'll only mention a few. This essay would not have been possible without the supervision of Bahram Assadian, whose professional expertise, careful guiding and own work in the field of mathematical structuralism, as well as his sustained moral support, kind encouragement and everlasting patience made possible the eventual virtues and depth of the present work. Katrin Schulz deserves special mention for her active mentoring, joyful planning and thorough understanding; our conversations always turned complicated situations into highways to the desired result. I owe to Tanja Kasenaar the taming of the bureaucratic beast through swift practical support and solutions to all things administrative, even in the worst of times. Hrafn Valtyr Oddson, my fellow master logician, passionate mathematician, flatmate and friend proved an invaluable source of meaning for complex proofs and mathematical results; our long discussions about anything mathematical – always turned set theoretical – have always been a source of light in my early pilgrimage through the formal jungle. The Master of Logic has been a roller coaster. The ILLC is ultimately an organism and the people mentioned above may also stand as representatives of the organs making it up and running it. The ILLC collective – researchers and teachers, TA's, students and administration – all deserve a heartful round of applause. I am grateful to Daniele Garancini for the many short hours – those past as well as those to come – spent discussing all things philosophical; moreover, for his unexpectedly fruitful *intervention*, for the DnD game taking us through the eventful last months, and so on and so forth. I owe too much to tell to Cătălina Frâncu, for being everything, from maecenas to the ultimate conversation partner to brilliant designer of worlds and much more.

Needless to say, any shortcomings of the present essay are of my own making; all its virtues are a collective fruit.

Contents

1 Introduction

The last five decades brought structuralism into the spotlight of philosophy, especially mathematical structuralism (MS) in the philosophy of mathematics. Ever since Paul Benacerraf's seminal article,[1] the philosophical community engaged closer and closer with structuralist themes, leading to the emergence of several different structuralist views to the point that no currently available taxonomy may comfortably accommodate them all. John Burgess sketches a history of the evolution of structuralist views in the last century.[2] *The following essay is concerned with central topics in the contemporary debates around* MS.

Coming of age against the background of a mathematical practice which emerged radically transformed at the end of more than a century of foundational disputes, modern *structuralism in the philosophy of mathematics* is a paradigm whose core claim is that structures constitute the subject matter of mathematical theories: such theories, the structuralist holds, are *about structures*.[3] The main motivation behind mathematical structuralism rests on peculiar phaenomena of *indifference* in the modern mathematical practice: the number theorist, for instance, doesn't care whether natural numbers are set theoretic systems, a sequence of Roman emperors or another of watermelons, at least as long as there are enough of them.

As much is a common to all sorts of currently trending structuralist views, but the agreement stops here. On the metaphysical side, divisions appear with respect to the status and nature of structures. On the ontological side, similar divisions emerge about the existence, nature and identity of mathematical objects understood as positions in structures; we are going to indiscriminately use 'metaphysical' and 'ontological' when referring to either cluster of issues in what follows. Such issues are paralleled by deep disagreements concerning the proper construal of ordinary mathematical discourse; such concerns are labeled 'semantic'. Further, yet mostly neglected topics are epistemological,[4] but these will be almost entirely bracketed in what follows. *In what follows we are chiefly concerned with metaphysical and semantic aspects of MS.*

Some of these disagreements are resolved along two main separation lines well known in philosophical circles: *eliminativism* and *non-eliminativism*, a.k.a.

[1] Benacerraf [1965].

[2] Burgess [2015, §3]. See also Hellman and Shapiro [2019, §2].

[3] Alongside *mathematical* structuralism, *scientific* structuralism is a boiling hot topic in the philosophy of science. See Ladyman et al. [2007] introducing *ontic structural realism* and Ladyman [2020] for a review of the contemporary field. We will not discuss scientific structuralism *per se*. Many problems and potential solutions discussed below have correspondents in scientific structuralism and a joint assessment would certainly prove most interesting. However, this is a topic for further work.

[4] See Shapiro [1997, §4] for an epistemology of structures in terms of pattern recognition. See MacBride [2008] for a discussion of the epistemological debts of non-eliminativism.

anti-realism and *realism*, respectively.[5] Eliminativists hold that structures do not exist and analyze discourse about *structures* as discourse about their *systems*. The distinction between structures and systems will become clear shortly; roughly, systems are understood as entities containing a domain of entities with relations and functions on them (like model theoretic structures), while structures are, if anything, something over and above systems that isomorphic systems have in common. Non-eliminativists, unlike eliminativists, hold that structures do exist and aim to provide an account of their nature. *Our main focus in what follows is non-eliminativist MS*. However, since an assessment of non-eliminativism is inevitably against the background of its main contender, a self-contained presentation of eliminativism will precede our core discussion. Non-eliminativism has been recently split into *positionalism* and *non-positionalism*:[6] the former pictures structures as endowed with a domain of positions perforating them, while the latter has no appetite for that. *Our focus is positionalism*, but non-positionalism will be shortly discussed nonetheless.

The main question of the present essay is the following (with a wink at Benacerraf's famous title):

Question 1: What could mathematical structures be?

In particular, what could non-eliminativists' mathematical structures be? For one, structures should be the kind of entities whose isomorphism suffices for identity, unlike other abstract objects such as set theoretic models, for instance. We review four metaphysical accounts of structure, aiming to showcase the contemporary non-eliminativist's options based on an uniform methodology (coming shortly). Since we focus on positionalism, the following needs attending:

Question 2? What could positions be?

However, as we shall shortly see, this question is not interesting as it stands once Question 1 has been answered. However, coming up with satisfactory identity criteria for positions in structures will prove tricky. So the following replaces it:

Question 2: What sort of facts govern the identity of positions?

One of the major temptations of positionalism is the promise of a simple semantic picture: mathematical theories are about structures, mathematical terms refer to positions thereof (and relations and functions on them). However, there is no such thing as a free-lunch in philosophy[7] and positionalists owe us an account of reference to structures and their positions:

[5] The distinction is first made explicit by Charles Parsons [1990]. A Hegelian remark on labeling philosophical views: historical priority is sharply marked by positive terms, even when the content of the corresponding view is rather *negative* (i.e. it negates some thesis). One can read historical order off labels.

[6] This distinction has been introduced by Bahram Assadian [2016]; see e.g. p. 29ff.

[7] Although there are a few used 'till abused catchphrases.

Question 3: What sort of reference do mathematical terms perform?

These questions correspond to the three topics mentioned above: metaphysical, ontological and semantic. They are the main focus of the present essay and will prove instrumental while presenting the views. Each of the three clusters of problems corresponds to one topic.

Our *thesis* is humble: in spite of sustained criticism, there is still fuel in the realist's tank. We focus on assessing three recent non-eliminativist articulations against an arsenal of problems and objections leveraged against Stewart Shapiro's and Michael Resnik early versions of positionalism.[8] In this order, after introducing Shapiro's *Sui Generis Strucutralism* (§3.1)[9] alongside its problems, we review *Fregean Abstractionist Structuralism* (§3.4.1),[10] Object Theoretic Structuralism (§3.4.2),[11], and Unlabeled Graph-theoretic Structuralism (§3.4.3).[12] We show that each of these is left standing following the assault, albeit we will qualify this thesis in conclusion (§4).

Concerning our methodology, we engage in score keeping with respect to an extensive collection of problems and objections. We label this collection the 'canon' and each view mentioned above will be assessed *against the canon*. Passing the canonical test, at least largely, is necessary for a view's worth of further theoretical interest. However, a full comparative assessment is a task for further work.

These contents are structured as follows. This Introduction (§1) continues with a short quasi-historical outline of the emergence of MS as a modern philosophy of mathematics in Benacerraf [1965], presenting his main arguments, conclusion, and the dynamics that played out between the exponents of its main versions (§1.1); we use this opportunity to introduce essential piece of vocabulary to be used throughout the essay. Concluding the Introduction, we present a taxonomy of the views playing out in the field (§1.2).

The second section (§2) provides an outline of eliminativism, considering the three most common variants thereof: *relativism* (§2.1), *universalism* (§2.2) - using Reck and Price [2000]'s jargon - and *modal structuralism* (§2.3), each presentation concluding with the main objections leveraged against the view just presented. Finally, we show how these serve as the chief motivation leading to non-eliminativism (§2.4).

The third section (§3) constitutes the core of the essay, bringing non-eliminativism into focus. We present Stewart Shapiro [1997]'s now classic positionalist account, indicating highlighting the crucial theses and time bombs. Afterwards we review the problems and objections raised against Shapiro's

[8] Mainly in Shapiro [1997] and Resnik [1997].

[9] Original in Shapiro [1997].

[10] Original in Linnebo and Pettigrew [2014] emended in Schiemer and Wigglesworth [2017] and Wigglesworth [2018a]. Reviewed in §3.4.1.

[11] Original in Nodelman and Zalta [2014].

[12] Leitgeb [forthcoming,a] and Leitgeb [forthcoming,b].

structuralism and build our canon off them (§3.2). Before going full on discussing positionalism, we consider non-positionalist accounts motivated by the observation that most of the canonical problems concern positions themselves or their roles in structures; a short assessment of this view will highlight some of its weaknesses (§3.3). We now turn to discussing the positionalist views mentioned above (§3.4), where each dedicated section has three parts: (1) the presentation of the theory is followed by (2) an assessment against the canonical problems and (3) concludes with a simplified picture and relevant remarks. Finally, the fourth and last section (§4) concludes the essay taking stock and highlighting further work.

It is certainly wise to inform the reader concerning those structuralist topics we are utterly silent about; to avoid repetition, we advise those interested in forming an accurate picture of the considerable gaps figuring in the essay at hand to take a glance at the last paragraph of §1.2.

1.1 Historical note

Modern MS begins with Paul Benacerraf's seminal article "What numbers could not be." At that time and under the influence of the Nicolas Bourbaki group, working mathematicians already subscribed to a version of structuralism in mathematics.[13] In Bourbaki's sense, a *structure* is a set together with a collection of relations between its elements. Such sets were taken to be those posited by the most mature set theory of the day (ZFC) and mathematical theories, understood as collections of axioms, were thought of as about those structures satisfying them, while mathematical objects were elements in such structures. This notion of structure is later in the century philosophically recovered under the label '*system*:' structures in the Bourbakist sense just are a particular sort of system, namely systems of sets:

> **(System)** A system is a collection of objects with relations on them.

In what follows, we reserve the terms 'structure' *simpliciter* for that which is common to isomorphic systems. We can now introduce our first distinction. Call a theory *categorical* if all the systems satisfying it are isomorphic, where isomorphisms are structure-preserving maps between systems. Bourbaki distinguishes between *univalent* and *multivalent* mathematical theories, a distinction to be later recovered under the labels *assertoric* and *algebraic* theories, respectively:

> **(Assertoric theories)** A theory T is assertoric if and only if it is categorical.

[13] Burgess [2015, p. 106-113] offers a neat presentation of their ideas; we draw on Burgess remarks in these paragraphs.

> **(Algebraic theories)** A theory T is algebraic if and only if it is not assertoric.

All mathematical theories are construed as being about structures (understood as set theoretic systems), but some of them, the assertoric ones, have the additional property that only systems of a certain isomorphism type satisfy them and, in *this* sense, they are held to describe their common structure. This distinction is only rough and for some purposes its characterization in terms of the *intended* interpretation of the theory might fare better; however, this is enough for our purposes in what follows.

This was and probably still is the mindset of the working mathematician; we call this view '*methodological structuralism*.'[14] This was made possible by developments in mathematics and logic in the 19$^{\text{th}}$ and 20$^{\text{th}}$ century, with the development of axiomatic systems and novel mathematical disciplines such as set theory and later on model theory. What is peculiar to this incipient form of structuralism is that it is free of any philosophical commitments concerning mathematical objects or structures: methodological structuralism was a faithful description of mathematical practice itself and, as such, unavoidably patchy and question begging from a philosophical viewpoint.

The middle of the 20$^{\text{th}}$ century brought about an "ontological turn" in analytic philosophy, notably in W.V.O. Quine's work.[15] Disputes between realism and anti-realism returned in focus transformed by half a century of analytic philosophy and engagement with formal means in the philosophical enquiry. Quinean dictums such as " "[t]o be is to be the value of a variable" " or "no entity without identity"[16] legitimized enquiries based on while at the same time going beyond the naive contents of methodological structuralism.

While practicing mathematicians were content to state that mathematical theories were about set theoretic structures, this much appeared now incomplete to those more philosophically inclined and well-informed concerning the latest philosophies of the day. Let's call versions of structuralism *about* mathematics which build more substantial philosophical conceptions upon the methodological structuralist scaffolding '*philosophical structuralism*.'[17] Philosophical structuralism takes seriously questions concerning the existence and nature of structures, mathematical objects, identity, reference or epistemology, supplementing the thin approach of methodological structuralism with substantial philosophical theses.

Against this background, Paul Benacerraf [1965][18] argues that numbers could not be objects, against the naive, implicit *Weltanschaung* of the working mathematician.

[14] See Reck and Price [2000, p. 346] and Reck and Schiemer [2020, §2.3].

[15] Burgess [2015, p. 119-120].

[16] Quine [1948, p. 34] and Quine [1969, p. 23], respectively.

[17] Following Reck and Schiemer [2020, §2.3].

[18] Reck and Schiemer [2020, §1.1] mention Hillary Putnam as another early proponent of philosophical structuralism; see Putnam [1975].

Benacerraf's first argument relies on simple set-theoretic observations while taking notions such as 'object' seriously. Using number theory as a case study, Benacerraf points out that there are multiple set-theoretic reductions of the natural numbers, as witnessed by the Zermelo and the von Neumann ordinals. Zermelo suggests an interpretation which assigns $\varnothing$ to 0, while the successor function is $s : x \mapsto \{x\}$. Von Neumann, instead, keeps the interpretation of 0, but takes the successor function to be rather be $s' : x \mapsto x \cup \{x\}$. The associated domain for each interpretation is the closure of $\{\varnothing\}$ under their respective successor functions. Both interpretations define set-theoretic structures satisfying PA^2. However, since the sets involved are distinct, the structure of natural numbers, $\mathbb{N}$, cannot be both at once and mathematical objects cannot be both Zermelo and von Neumann ordinals. So which one are they? Benacerraf's answer is uncompromising: none, since $\mathbb{N}$ could as well be any of them. If natural numbers are sets, thus objects, they should be *particular* ones with certain identity criteria distinguishing them from all other sets, so they should be *certain* sets. So they cannot be both Zermelo and von Neumann ordinals, hence they are neither.[19]

This is Benacerraf's first embarrassment of riches for set-theoretic reductions of number theory. However, one can go on to notice that *as well* is certainly not *best*: both systems attribute to natural numbers extra-arithmetical properties such as $1 \in 2$ or $1 \notin 2$.[20] Corresponding to the methodological structuralist's *indifference* concerning the choice of set-theoretic structures, is indifference concerning mathematical entities: mathematical entities appear to have exclusively structural properties:

> **Structural property**: Let S be a system, a be an element in the domain of S and φ be a property such that $\varphi(a)$. φ is a 'structural property' of a if and only if for all systems S$'$ and $f : \mathrm{S} \cong \mathrm{S}'$, $\varphi(a) \equiv \varphi(f(a))$.

Structural properties are isomorphism invariant properties, i.e. properties which are preserved along isomorphisms. Structural relations are those relations which are shared by all those systems which share the same structure.[21] If mathematical entities are objects at all, then they are a peculiar, *incomplete* sort thereof. This kind of indifference transferred from the level of systems to that of objects will

[19] Benacerraf [1965, p. 63].

[20] PA^2 is categorical – see Shapiro [1991] for a reconstruction of a proof traced back to Dedekind and Behman [trans]. However, we should notice that even though Zermelo and von Neumann ordinals - ordered by their appropriate successor functions s and s', respectively - are *isomorphic in the signature of arithmetic* ($\mathcal{L}^2 \cup \{0, s\}$), they are not set-theoretically isomorphic. As such, the fact that they are not set-theoretically elementarily equivalent doesn't conflict their being isomorphic in the relevant sense.

[21] See Korbmacher and Schiemer [2018] which distinguish another characterization found in the literature: structural properties are those expressible solely in terms of the primitive relations of the mathematical theory characterizing the structure concerned. The authors compare these two notions and find them extensionally distinct.

prove a cornerstone for non-eliminativism. Benacerraf sees reason to strengthen his conclusion that numbers are not sets: sets have the kind of properties numbers should better lack. This argument applies over the board to every assertoric mathematical theory such as integer, real and complex analysis, geometry or theories describing any finite unlabeled graph: if mathematical theories are about structures, then neither structures, nor their positions are sets.[22]

Benacerraf's second argument reinforces the former with a full on anti-realist sentence. Suppose that we somehow managed to pick out a unique system which we deem to be *the* structure described by a certain theory. It is a simple model theoretic result that (non-trivial) permutations of set-theoretic systems yield distinct albeit isomorphic set-theoretic systems:

> **Permutation**: Let $\mathcal{L}$ be any signature, let $\mathcal{M}$ be any $\mathcal{L}$-structure with underlying domain $\mathcal{M}$, and let $\pi : \mathcal{M} \to \mathcal{N}$ be any bijection. One can use π to induce another $\mathcal{L}$-structure $\mathcal{N}$ with underlying domain $\mathcal{N}$, just by "pushing through" the assignments in $\mathcal{M}$, i.e., by stipulating that $s^{\mathcal{N}} = \pi(s^{\mathcal{M}})$ for each $\mathcal{L}$-symbol s. Having done this, one can then check that $\pi : \mathcal{M} \to \mathcal{N}$ is an isomorphism.[23]

We follow the practice and abuse language by using $\mathcal{M}$ and $\mathcal{N}$ to refer to set-theoretic systems as well as their domains; in this case, we call f a 'permutation' of $\mathcal{M}$, and $\mathcal{N}$ a 'permuted copy' of $\mathcal{M}$. This much again concludes that structures are not set-theoretic systems; however, permutation appears simple enough to assume that whatever structures might be, they will afford some kind of permutation operation resulting in further distinct but isomorphic structures. Assuming (plausibly) that the permuted copy is just as good (and just as *little* bad) as the original for fixing the reference of mathematical terms, then we end up with the second embarrassment of riches, this time a more damning one. If something like the model theoretic permutation construction can be performed on the domain of the structure, then, given any of them, we end up with plenty of good choices. Taking arithmetic and the natural number structure as case study throughout his essay, Benacerraf [1965] makes the point in terms of progressions rather than permutations of given collections:

> *It was pointed out above that any system of objects, whether sets or not, that forms a recursive progression must be adequate. But this is odd, for any recursive set can be arranged in a recursive progression. So what matters, really, is not any condition on the objects (that is, on the set) but*

[22]The easiest way too apply this argument to integer, real and complex analysis is noticing that set theoretic structures thereof can be defined starting with a set theoretic structure of the natural numbers and it can be seen that different choices of structures for the latter will end up with different structures for the former ones.

[23]See e.g. Button and Walsh [2016, p. 284].

> *rather a condition on the relation under which they form a progression. To put the point differently – and this is the crux of the matter – that any recursive sequence whatever would do suggests that what is important is not the individuality of each element but the structure which they jointly exhibit. This is an extremely striking feature* (Benacerraf [1965, p. 69])

Since any choice would be arbitrary, Benacerraf concludes that none is *the* structure and hence that mathematical entities are not objects at all. This is what we will later present as the Permutation problem (§3.2.2) and, as one might expect, there are versions of it threatening mainstream non-eliminativist positions.

The constructive part of Benacerraf's article is far from being as well articulated as its destructive input. Benacerraf states that mathematical theories are about "*abstract structure*" such as the natural number structure[24] or the real number structure, that mathematical entities should be conceived of as " "elements" of the structure", that positions in the natural number structure are fully characterized by what "stem[s] from the relations they bear to one another in virtue of being arranged in a progression" (Benacerraf [1965, p. 70]); however, nothing is uttered concerning the nature of structures or whether discourse about them should be understood at face value (suggesting realism) or rather paraphrased away (suggesting a reduction of structures to another kind of entities). Straining it (arguably a bit too much), Benacerraf [1965]'s positive characterization leaves enough space for both eliminativist as well an non-eliminativist articulations of philosophical structuralism.

Modern structuralists[25] have recovered historical statements hinting in the direction of their views. Most notably, Richard Dedekind's views have been quoted by eliminativists and non-eliminativists alike to motivate their take on the matter. It turned out that the many of the important figures engaged with the foundations of mathematics at the turn of last century provide the means for a structuralist interpretation of sorts.[26] Some scholars have tried to resolve well-known disputes such as that recorded in the correspondence between Gottlob Frege and David Hilbert in terms of disputes on structuralism.[27] Such historical enquiries also brought to light early objections to modern day influential versions of structuralism, such as the Circularity objection (§3.2.2) which can be traced back

[24]Benacerraf states:

> *Arithmetic is therefore the science that elaborates the abstract structure that all progressions have in common merely in virtue of being progressions.* (Benacerraf [1965, p. 70])

[25]From now on, the terms 'structuralism', 'structuralist', 'MS' etc. *simpliciter* will be meant to refer to *philosophical structuralism* etc. unless otherwise stated.

[26]See e.g .Reck [2018] and Hellman and Shapiro [2019, §2].

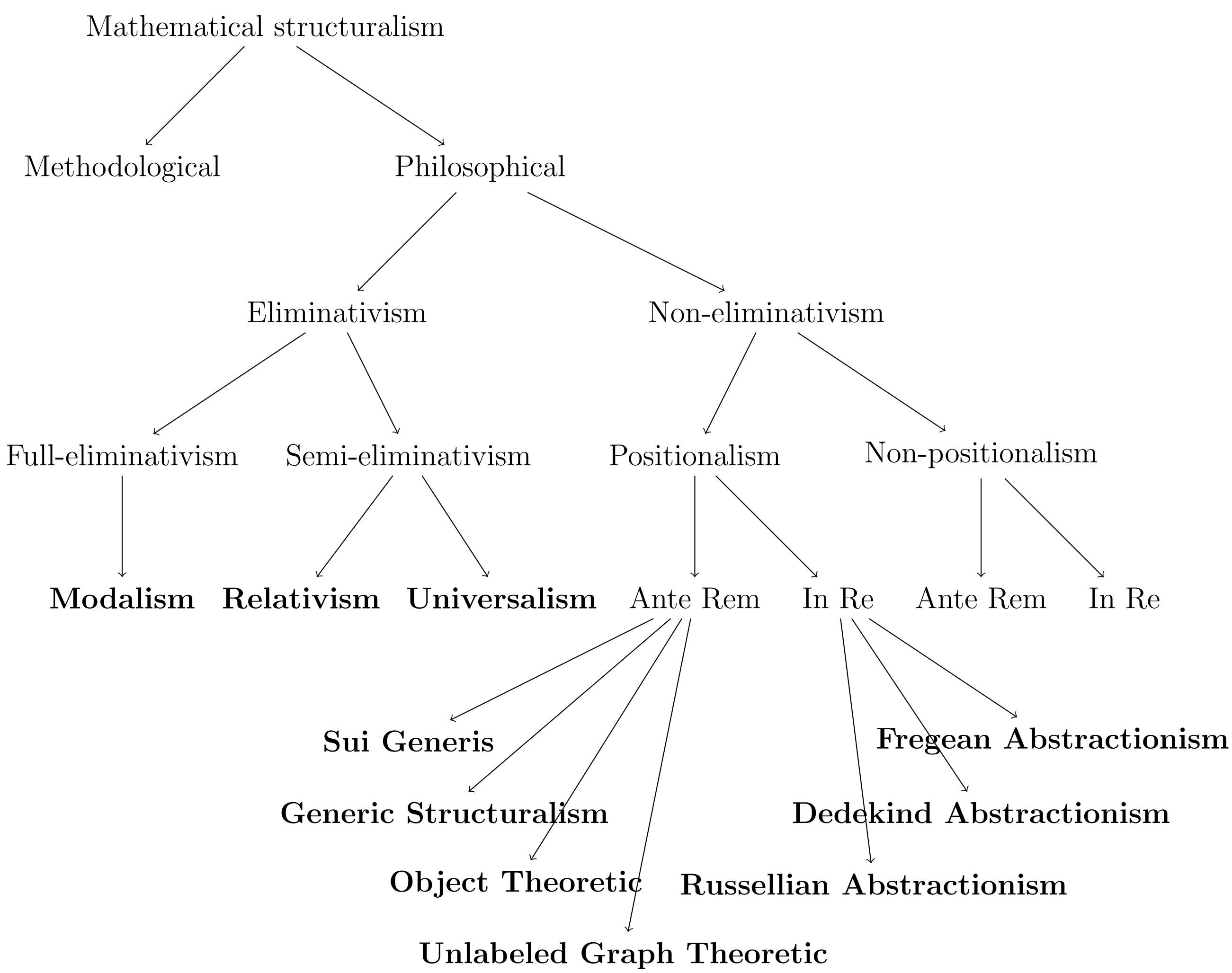
[27]Doherty [2019].

to Bertrand Russell's remarks against Dedekind's view.[28]

This section should have provided the necessary historical background for the discussion to follow. Before concluding our introduction, we provide a taxonomy of modern structuralist views.

1.2 A Taxonomy of MS

We are drawing upon Reck and Schiemer [2020, §2.3]'s "broader" taxonomy of structuralism, understood here generally to include not only its philosophical variants, but also methodological structuralism itself.



[28] Russell [2009, p. 251]

Following Bahram Assadian [2016], we enriched Reck and Schiemer [2020]'s suggestion by adding a further split within non-eliminativist views between positionalism and non-positionalism. Let us shortly characterize the views just mentioned.

Methodological structuralism corresponds to the characterization provided in the previous section: it is a depiction of the mathematical practice as it emerged at the end of the first quarter of the last century, lacking any concessions to satisfy the curiosity of the more philosophically inclined. It is however in principle possible to conceive of such a view as philosophically loaded, enriching it with theses holding ontological questions to be meaningless.[29] But this is a pure theoretical possibility in the contemporary field.

Philosophical structuralism has been also sketched in the previous section: it is the philosophically mature endorsement of methodological structuralism. The taxonomy of non-eliminativist views goes along metaphysical separation lines. On the one hand, *eliminativists* hold that discourse about structures has to analyse structures away, or otherwise reduce structures to another kind of entity; *non-eliminativists*, on the other hand, hold that structures are *sui generis* inhabitants of our ontology.

Among the eliminativists, some aim to fully do away with abstract objects, holding to have reduced structures in ways that do not rely on the existence of any kind of abstracta; these are the *full-eliminativists*. By way of contrast, *semi-eliminativists* allow the existence of some, 'more concrete' mathematical objects, most notably sets, and propose ways to reduce structures to set theoretic systems. *Naive set theoretic structuralism*, *relativist structuralism*[30] and *universalist structuralism*[31] are of the latter sort, while *modal structuralism*[32] is of the former. We will briefly discuss these in §2, alongside those objections leveraged against them chiefly employed by non-eliminativists to motivate their views.

Among *non-eliminativists*, *positionalists* hold that mathematical structures are perforated by positions which are themselves objects. In minority and motivated by problems surrounding positionalism, *non-positionalists* do not ontologically commit to positions, deeming mathematical objects a sort of shadowy artefact of our discourse concerning structures. Both views afford *ante rem* as well as *in re* articulations, depending on whether structures are taken to be ontologically independent of, or rather abstracted from, the systems having it them common.[33]

Ante rem positionalism is usually identified with Stewart Shapiro's *sui generis*

[29] Reck and Price [2000] mention Ludwig Wittgenstein and Rudolf Carnap in this respect.

[30] E.g. Reck and Price [2000], Schiemer and Gratzl [2016].

[31] E.g. Pettigrew [2008], Reck and Price [2000], probably Putnam [1975]. Charles Parsons [2008]'s *Conceptualist Structuralism* is probably another version of semi-eliminativism.

[32] E.g. Hellman [1989].

[33] For instance, Ketland [2015] and Isaacson [2011] arguably provide *ante rem* versions of non-positionalism; Assadian [2016, §6.4] articulates a version of *in re* non-positionalism.

structuralism,[34] but at least three other versions of this kind can be identified in the field; we will discuss two of these below (§3.4). Research into the full potential of *in re* positionalism is still ongoing, but the option is already crowded by views assuming different abstraction principles; among these, we will extensively discuss a version based on Fregean abstractionist principles holding that structures are logical objects corresponding to isomorphism classes (§3.4.1).[35]

Notable omissions from the above taxonomy are category theoretic structuralism (Awodey [1996]), homotopy type theoretic (with the Univalence axiom) based structuralism (Awodey [2014]), Charles Chihara's own version of modal structuralism (Burgess [2005]), Modal Set-Theoretic Structuralism (Hellman and Shapiro [2019, §7]), and probably others. Mathematical structuralism is a rapidly evolving field and a complete taxonomy is still awaiting historical sedimentation. Beside these, Russellian and Dedekind structuralisms, Charles Parsons' Conceptualist structuralism, Generic structuralism,[36] as well as scientific structuralism and epistemological aspects of non-eliminativism will be utterly absent but for further work.

2 Eliminativist MS

Eliminative structuralism denies that structures, understood as that various isomorphic systems have in common, *really* exist. What is probably the first version of eliminativism is only a bit more than methodological structuralism. What we call *naive set-theoretic structuralism*[37] barely enriches methodological structuralism with a conventional assignment of a set-theoretic system as the structure of interest, not unlike the way we sometimes talk about isomorphism classes through a choice of their representatives. Model theory provides the means to talk about the systems being described by theories through a recursively defined notion of satisfaction. If T is a theory with primitive non-logical vocabulary in t, then an interpretation of T is a $\mathcal{L}^t$-(set-theoretic-)-structure $\mathcal{I} = \langle \mathrm{D}, \mathrm{I} \rangle$ where D is a domain of objects and I is a function assigning elements in D to constants in t, subsets of D - i.e. properties - to predicates in t, sets of n-tuples of elements form D to n-ary relation symbols from t and functions from D to D to functional symbols in t. Notice that most of these are not necessarily 'in' D in a set theoretical sense, but they have D as basis. $\mathcal{I}$ is sometimes called a $\mathcal{L}^t$-structure or

[34] Shapiro [1997].

[35] Linnebo and Pettigrew [2014], Schiemer and Wigglesworth [2017]). For Russellian abstractionism, see Reck [2018]); for Dedekind abstractionism, see Reck [2018], Reck [2003] and Linnebo [2007], all presenting slightly different reconstructions of Dedekind's thought on the matter.

[36] Original in Horsten [2019], drawing on Finean topics from Fine and Tennant [1983] and Fine [1998].

[37] Reck and Schiemer [2020, §1.1] call it 'set-theoretic foundationalism'.

model or interpretation, keeping track of the language it interprets, here the language of T.

If we disregard the linguistic component, we end up with a set of sets, a set-theoretic model. Consider for instance $PA^2(0, s)$, where 0 and s constitute its sole non-logical vocabulary. The von Neumann $\{0, s\}$-interpretation of $PA^2(0, s)$ is $\mathcal{I} = \langle \omega, \mathrm{I}(0) = \varnothing, \mathrm{I}(s) = f : x \mapsto x \cup \{x\} \rangle$. From a set-theoretic perspective, $\langle \omega, \varnothing, f \rangle$ - often called simply ω - is a model of PA^2 and she might go about - as it is by no means unusual - identifying the natural number "structure" $\mathbb{N}$ with ω itself. "Structures" emerge as conventionally privileged set theoretic systems, where mathematical objects are sets in such systems and mathematical (arithmetical etc.) language should be interpreted as referring to such structures and their elements.

Naive set theoretic structuralism is the main target of Benacerraf's first argument. What makes it *structuralist* is the conventionalism behind the choice of system: the thought is that any relevantly isomorphic choice would have been just as good as the actual one and therefore, say, if another individual or community makes a different but relevantly isomorphic choice, the naivist would not go about holding that she is mistaken, but rather adapt her discourse to fit the circumstances. Conversely, the structuralist would not hold of say, arithmetic, anything not following from all systems which are isomorphic to the conventionally chosen one.

We discuss several more sophisticated variants of eliminativism which emerged as enlightened versions of naive structuralism in the face of Benacerraf's objections. None of the views is tied to a set theoretic background ontology; however, for reasons to be fully explained when discussing their problems at the end of each section), relativism and universalism are customarily carried out against such a background, thus bearing commitment to at least some abstract objects (sets), which in turn justifies their posting under the label of 'semi-eliminativism', rather than 'full-eliminativism'.

2.1 Relativist MS

Relativist structuralism[38] resembles naive structuralism in that it takes structures to be particular, typically set-theoretical systems. On the semantic side, just like naive structuralism, relativists hold that mathematical vocabulary is relative to a certain system.[39] As such, just like before, they can, for the most part, hold onto a *grammatically accurate*[40] interpretation of the mathematical language, matching

[38]See Reck and Price [2000, §2].

[39]This is what justifies the label.

[40]Some might say that we could as well have used here 'face value' instead of 'grammatical accuracy'; however, talking of a *face value interpretation of the mathematical discourse* in the context of eliminativism might might be confusing to those strongly associating the literal construal

the logical .

However, there are significant differences when it comes to the choice of system. Unlike naivists, relativists hold that this choice is *arbitrary*, in such a way that we can neither know, nor semantically determine which system is the one involved. This account is structuralist in that given the complete arbitrariness of the choice of system, its elements can only be characterized up to those relations holding in all the relevantly isomorphic systems; after all, given arbitrariness, the system concerned could be any of those isomorphic ones. This is the relativist explanation of mathematical entities' exclusively structural properties. Relativism is arguably superior to naivism in that conventionalism is essentially avoided; however, this comes at the cost of having to deal with matters semantic.

The burden of relativism is on the semantics of mathematical discourse: we are owed an account of reference which allows for the sort of arbitrariness being advertised, namely *arbitrary reference*.[41] Involvement with arbitrary reference is not the exclusive trade of eliminativism: contemporary non-eliminativist conceptions invoke arbitrary reference as well, and we will engage with such views in §3.4.[42] Schiemer and Gratzl's relativist account, which will be presented in some detail here, contains a semantics of arbitrarily referring terms; the reader will be later reminded to revisit the current section for details concerning an understanding of such terms.

The articulation of relativism introduced in Schiemer and Gratzl [2016] borrows ideas from Rudolf Carnap's mature reconstruction of scientific theories and employs David Hilbert's ϵ-calculus alongside an associated choice-theoretic semantics to account for arbitrarily referring terms.

We assume set-theory alongside second-order logic with identity in the background. Let T be an assertoric mathematical theory, and let $t = \langle t_1, ..., t_n \rangle$ be the non-logical vocabulary of T. Then T can be fully characterized by a single formula of $\mathcal{L}^t$ (the language of T):

of ordinary mathematical discourse to non-eliminativism. We refer the reader to the next section (§2.2) for a paragraph on this issue in connection to Pettigrew [2008]'s thesis.

[41] For a presentation and defense of arbitrary reference as an account of instantial terms – terms such as those used in mathematical reasoning naturally construed as employing Existential Elimination and Universal Introduction – see Breckenridge and Magidor [2012]. The general thesis concerning arbitrary reference is stated as follows:

> **(AR)** *It is possible to fix the reference of an expression arbitrarily. When we do so, the expression receives its ordinary kind of semantic value, though we do not and cannot know which value in particular it receives.* (Breckenridge and Magidor [2012, p. 378])

Connections between instantial terms and mathematical terms generally have been drawn in Shapiro [2008], Shapiro [2012] and Breckenridge and Magidor [2012].

[42] For instance when discussing Leitgeb [forthcoming,b]'s Unlabeled Graph-theoretic Structuralism.

($\Phi_{\mathbf{T}}$) $\Phi(t_1, ..., t_n)$

where Φ_{T} can be taken to be the conjunctions of all the axioms of T. The Ramsey sentence corresponding to T is then:

($\mathbf{RS_T}$) $\exists X_1, ..., \exists X_n \Phi(X_1, ..., X_n)$

Notice that all the terms of T have been eliminated in $\mathrm{RS_T}$; however, every system in which $\mathrm{RS_T}$ holds is (or can under the right interpretation function be turned into) a model of T. $\mathrm{RS_T}$ can be seen as capturing the *structural content* of T.

In order to recover the structural content of the terms in t, we need to introduce Hilbert's ϵ-calculus. ϵ is a term forming operator governed by the following two axioms:

(Critical Formulas) $A(t) \rightarrow A(\epsilon x A(x))$;

(Extensionality) $\forall x(A(x) \rightarrow B(x)) \rightarrow \epsilon x A(x) = \epsilon x B(x)$

Upon close inspection of the axioms, one can see that the intended meaning of the ϵ-operator is to pick out objects satisfying certain conditions, but only *arbitrarily*.[43]

We can now recover through explicit definitions the (structural content of the) vocabulary of T in two steps. First, we define t:

(ϵ-Def) $t := \epsilon z \exists X_1, ..., \exists X_n [z = \langle X_1, ..., X_n \rangle \wedge \Phi(X_1, ..., X_n)]$

If the Ramsey Sentence of T is true, that is, if the theory is satisfiable, then the sequence of T's theoretical terms is defined by referring to an arbitrary tuple of relations which is a model of T. The use of ϵ in ϵ-Def is the first and the essential occurrence of ϵ in this reconstruction; before clarifying this, let us formulate the explicit definition of each term from t:

(ϵ-Def*) $t_i := \epsilon Y \exists X_1, ..., \exists X_n [t = \langle X_1, ..., X_n \rangle \wedge Y = X_i]$

The intended meaning of ϵ-Def is that t picks out an arbitrary system satisfying $\mathrm{RS_T}$. However, the use of ϵ in ϵ-Def* is redundant: once a system of terms t has been picked out, arbitrary reference is not called upon in defining each term in t.[44]

The relativist can thus explicitly define mathematical terms which refer arbitrarily, and she can account for their inferential role; but we still lack a semantic understanding of such arbitrarily referring ϵ-terms. Schiemer and Gratzl [2016] present us with a choice-theoretic semantics for ϵ-terms.[45]

[43] See Schiemer and Gratzl [2016, §4] for a comparison to the definite description operator ι, which picks out the only object satisfying a certain property.

[44] See Schiemer and Gratzl [2016, p. 412-413].

[45] The authors refer Zach [2014] for formal details. The semantics for the rest of the language is Tarskian.

(Choice-semantics for ϵ-terms) Let $\mathcal{M}$ be a model and D be its domain. Let $X \subseteq D$ and $\delta : \mathcal{P}(D) \to D$ be a choice function, as follows: $\delta(X) = \begin{cases} x \in X, & if X \neq \varnothing \\ x \in D, & otherwise \end{cases}$. Let s be an assignment and A be a formula with at most x free. Then: $(\epsilon x A(x))^{\mathcal{M},s,\delta} = \delta(A(x))^{\mathcal{M},s} = \delta(\{d \in D | \mathcal{M}, s[x/d] \models A(x)\})$.

ϵ-terms are evaluated on models alongside assignments and a given choice function: given a formula A with only x free, A induces a subset of the domain of the model; an arbitrarily picked A-element form the domain of the model (i.e. $\epsilon x A(x)$) is the A-element picked out by the given choice function. The interpretation of ϵ terms is given in terms of a choice function: ϵ terms pick out elements in the domain, if any, satisfying the embedded formula in accordance to a given choice function. The semantics of sentences containing ϵ terms can then be given as follows:

(Evaluation of sentences containing ϵ-terms) Let A be a sentence in $\mathcal{L}^t$ and A^* be its correspondent in $\mathcal{L}_\epsilon$. Let $\mathcal{M}$ be a $\mathcal{L}_\epsilon$-model. Then:

- A is *true* in $\mathcal{M}$ iff there is a choice function δ such that $\mathcal{M}, \delta \models A^*$;
- A is *universally true* in $\mathcal{M}$ iff for all choice functions δ, $\mathcal{M}, \delta \models A^*$.

Mathematical truth would then correspond to universal truth. Consider the arithmetical formula $2 + 3 = 5$. The relativist interprets it on the background of some arbitrarily chosen set-theoretic structure:

$$\mathbb{N} := (\epsilon z)(\exists X)(\exists x)(\exists f)(\exists \circ)[z = \langle X, x, f, \circ \rangle \wedge \mathrm{PA}^2(X, x, f, \circ)]$$

where each individual term in $\mathbb{N} = \langle N_\mathbb{N}, 0_\mathbb{N}, s_\mathbb{N}, +_\mathbb{N} \rangle$ is defined as in ϵ-Def*.[46] Therefore, the relativist interpretation of $2 + 3 = 5$ would be $2_\mathbb{N} +_\mathbb{N} 3_\mathbb{N} = 5_\mathbb{N}$ (where, of course, $2_\mathbb{N} = s_\mathbb{N} s_\mathbb{N}(0_\mathbb{N})$ and so on), where the ϵ-term $\mathbb{N}$ figures in the ϵ-Def* definition of each other individual term. Which tuple of terms $\mathbb{N}$ actually is depends on the choice function involved in its interpretation; however, what this choice function is is entirely opaque.

A final remark before listing objections. Relativism requires that some system satisfying the Dedekind-Peano axioms *actually* exists:

(Exist) $\exists X \exists y \exists f\ \mathrm{PA}^2(X, y, f)$

Relativism is committed to the existence of a domain of entities interpreting '$\mathbb{N}$', a distinguished entity in this domain interpreting '0' and a function on X interpreting 'successor' such that PA^2 comes up true under these assignments. Intuitively, if no

[46] Preferably replacing the ϵ-operator in front of ϵ-Def^2 with a definite ι-operator defined as $s = \iota x P(x) :\equiv \exists! y(P(y) \wedge y = s)$.

such entities exist, then the arithmetical discourse is strictly speaking meaningless. In terms of the choice interpretation of ϵ-terms, such a discourse would be about an arbitrarily picked out system which doesn't satisfy PA^2, deeming the arithmetical discourse nothing more than a random display of truths and falsehoods.

Several objections have been leveraged against relativism, some of them also applying to other versions of eliminativism discussed below:

1. Relativism is committed to the existence of an actual system satisfying the theory.[47] The system is either one of abstract objects, commonly sets, or otherwise one of concrete ones, sometimes space-time regions. In the latter case, even assuming that it is possible to define the right relations on the concrete system, mathematical truth and facts would be deemed contingent, since such a system could have failed to exist altogether; but this goes against the orthodoxy holding that mathematical truth is necessary. Therefore one customarily concludes towards preserving the necessity of mathematical truth, that relativism is committed to the existence of a system of abstract objects, most commonly a set-theoretic one. However, commitment to set theory implies that set theory itself cannot be provided with a structuralist interpretation on pain of vicious circularity, deeming relativism incoherent at its root. Moreover, the particular set theory assumed in the background would exclude other set theories from the structuralist picture since they conflict with the chosen one on matters concerning sets. Finally, quantification over sets has set theory committed to a (class like) totality of sets which cannot be extended, which clashes with an Extendibility principle for structures.[48] In short, neither concrete systems, nor a set theoretic background offer the needed ambient for structuralist views.

2. Relativism, just like naivism, attributes too much structure to mathematical entities.[49] For instance, $1 \in 2$ would obtain if the choice of system would actually be the von Neumann ordinals on some choice, even if concluding to this effect from within arithmetical discourse would be semantically blocked; arguably, numbers have no such properties, which makes them unlike sets. This objection certainly applies to naivism, but it is doubtful that it has the same force against the version of relativism we presented above. For one, the system arbitrarily chosen to provide us with the semantic contents of a piece of mathematical discourse is not, strictly speaking, identified with the natural numbers: in this strict sense, the natural numbers do not exist and so there is no question concerning the amount of structure imposed on *them*.

[47] See Reck and Price [2000], Hellman [2005].
[48] See Hellman [2005] for a systematic presentation of such concerns.
[49] See Reck and Price [2000], Leitgeb [forthcoming,a, p. 10].

Things would change if relativism would be understood as providing a reduction of mathematical entities to sets; however, relativism is an eliminative position, holding that there are no structures over and above systems, and thus no mathematical objects such as natural numbers. Since the relevant mathematical ordinary discourse is rendered just right through the arbitrariness of the choice of system, this objection appears to be unmotivated against relativism as conceived of here. We only mention this objection here because recent defenders of non-eliminativism such as Leitgeb [forthcoming,a, p. 10] consider this to be the main complaint motivating relativism's rejection. However, these remarks is the farthest we go in this essay in the direction of a reply on behalf of relativism.

3. Relativism is committed to arbitrary reference, which brings about primitive semantic facts, i.e. semantic facts which do not supervene on use facts broadly construed, against the contemporary philosophical orthodoxy.[50] The likely picture painted by semantic non-supervenientionism is that of some semantic facts figuring in a presumed description of fundamental reality alongside quark color and charge. Although this might be only metaphorical, alternative metaphysical pictures of a world in which semantic non-supervenientionism obtains are yet to be provided. Meanwhile, the intuitive picture is seemingly unacceptable.

2.2 Universalist MS

Universalism[51] holds that structure-talk is to be paraphrased away as talk about relevantly isomorphic systems. Unlike the former views, universalism doesn't recommend systems serving as surrogates for structures: mathematical discourse is paraphrased such as to do away with any purported reference to structures, ordinary mathematical terms, the universalist holds, are not singular terms.

Resembling relativism, the burden of universalism is on semantics. The universalist construes mathematical discourse concerning structures as discourse

[50]See Kearns and Magidor [2012] for a general defense of 'Semantic Sovereignty', their label for the thesis that semantic facts do not (necessarily) supervene on use facts, broadly construed.

[51]Reck and Schiemer [2020, §1.1] mention Hilary Putnam - especially Putnam [1975] - its probably first defender under the label of 'if-then-isms'; another usually mentioned defender of this view is Mayberry [2000]. This version of eliminativism is sometimes called 'set-theoretic structuralism' ('STS') (see Hellman and Shapiro [2019, §3], Hellman [2001], Hellman [2005]). However, there at least two reasons to prefer the label 'universalist'. First, set theoretic structuralism would potentially generate confusion when it comes to distinguishing between what we here called universalist and relativist structuralisms. Second, it isn't strictly speaking necessary to assume a background universe of sets for universalism to hold: provided that we have enough objects organized appropriately, any kind of objects would do, so the 'set-theoretic' label would be voided. Be that as it is, relying on other sorts of objects raises potentially intractable problems and a set theoretic background is usually assumed; we will make no exception in this respect.

about all systems satisfying a certain categorical condition. Consider arithmetic $\mathrm{PA}^2(N, 0, s)$ and let φ be a sentence in its language. First, replace all terms in φ with their analyses such as to end up with a formula $\varphi(N, 0, s)$ only containing $N, 0, s$ as non-logical vocabulary. The structuralist would then construe the arithmetical meaning of φ as follows:

(Univ) $\forall X, y, f(PA^2(X, y, f) \to \varphi(X, y, f))$[52]

For instance, the mathematician's assertion that $2 + 3 = 5$ is construed as

$$\forall X, y, f(PA^2(X, y, f) \to ff(y) + fff(y) = fffff(y))$$

(where + is recursively defined as usual). If there are no systems (X, y, f) satisfying $\mathrm{PA}^2(X, y, f)$, then every arithmetical statement comes out vacuously true. So the universalist has to make the Exist assumption stated when discussing relativism.

This semantic account construes all mathematical assertions as universal statements quantifying over all systems satisfying certain conditions; this is a far cry from the face value grammar of an ordinary mathematical statement. However, Richard Pettigrew [2008] adds a machinery of (pragmatically) *dedicated variables* and argues that ordinary mathematical discourse can be recovered in an essentially universalist setting. Mathematical terms such as '$\mathbb{N}$', 's', '0', '1', '+', '$\times$' etc. as employed in ordinary mathematical discourse are *dedicated free variables*, i.e. free variables introduced into the discourse by stipulations such as:

(Stip) "Let $\mathbb{N}$, s, 0 satisfy the Peano Axioms" (or "$\mathrm{PA}^2(\mathbb{N}, s, 0)$")

An ordinary mathematical assertion $\varphi(\mathbb{N}, s, 0)$ can be recovered as a formula rather than a sentence. However, this formula follows the *surface grammar* of the asserted statement, just like relativism and naivism recommend; unlike these, mathematical terms are free variables rather than singular terms and, as such, there is nothing they refer to. The semantic content of the mathematical assertion is essentially captured by Univ, but ordinary mathematical discourse is not construed as trading in *explicit* generalities anymore; rather, generalities are concealed in the generality of the open formulas.

We use this opportunity to highlight an issue that will be important later on (in particular in §3.2.3.2 when we discuss the Semantic objections against non-eliminativism). Interestingly, Pettigrew states the following:

> *I will argue that philosophers of elementary number theory—or arithmetic as philosophers and logicians tend to call it—have been wrong to assume that the platonist interpretation of that discourse is the only interpretation that takes its sentences at 'face value' or*

[52] This core component of universalism construing mathematical statements as universal ones is what backs its choice of label. See Reck and Price [2000, §3].

> *'literally'*. I will argue that the antirealist interpretation given by eliminative structuralists has at least as much claim to be the 'literal' or 'face-value' reading. (Pettigrew [2008, p. 310], our highlight)[53]

As glossed upon in §2.4, a face value or literal interpretation of ordinary mathematical discourse is regarded as a non-eliminativist stronghold against the eliminativist. Pettigrew argued against this claim. However, this raises a question concerning the precise meaning of the claim itself; in particular, this requires an understanding of what is customarily regarded as *the literal meaning of ordinary mathematical discourse*, and its tenability. Throughout Pettigrew [2008], notion employed appears to be the following:

> **(Literal Construal)** A construal of ordinary mathematical discourse is literal if and only if (i) it construes mathematical terms as syntactical *singular terms* and (ii) it is *grammatically accurate*, i.e. it matches the grammatical form of ordinary mathematical assertions.

We can already notice that (ii) is naturally achieved by naivism and relativism and, as we are on our way to find out, by universalism and modal structuralism as well: grammatical accuracy with respect to ordinary mathematical discourse is not the privilege of non-eliminativism. This leaves only (i) as a distinguished non-eliminativist dimension of Literal Construal. Stewart Shapiro, however, leads us to a seemingly richer notion:[54]

> **(Literal Construal$^+$)** In addition, (iii) syntactic singular terms have the semantic function of performing singular reference to mathematical objects, i.e. the semantic values of mathematical terms are appropriate mathematical objects.

[53] We'll say more about Pettigrew's argument in §3.2.3.2.

[54] The following should support our claim:

> *Because mathematics is a dignified and vitally important endeavor, one ought to try to take mathematical assertions literally,* ***"at face value."*** *This is just to hypothesize that mathematicians probably know what they are talking about, at least most of the time, and that they mean what they say. Another motivation for the desideratum comes from the fact that scientific language is thoroughly intertwined with mathematical language. It would be awkward and counterintuitive to provide separate semantic accounts for mathematical and scientific language, and yet another account of how various discourses interact* (Shapiro [1997, p. 3]) *In sum, the ante rem structuralist interprets statements of arithmetic, analysis, set theory, and the like, at* ***face value****. What appear to be singular terms are in fact singular terms that denote bona fide objects* (Shapiro [1997, p. 11]) *Moreover, if we take the language of mathematics, as reformulated in the idiom of mathematical logic, at* ***face value****, then we are committed to the existence of numbers, sets, and so forth, and have endorsed realism in ontology* (Shapiro [1997, p. 46], all highlights are ours)

The emerging notions are crucially distinct: Pettigrew's notion allows us to formulate an objection[55] against non-eliminativism that Shapiro's version would block. A proper inquiry into the notion of 'face value' is material for Further Work.

Returning to Pettigrew's dedicated free variables construal of ordinary mathematical terms, the number theorist's assertion that $2 + 3 = 5$ wears its logical form on the surface, provided that $2, 3, 5$ and $+$ are understood as $\mathbb{N}$-dedicated free variables;[56] in other words, the grammatical form of ordinary mathematical discourse can be recovered by the logical, real form suggested by the universalist construal. This brings the universalist semantics closer to the face value of mathematical discourse.[57] We will refer back to Pettigrew's construal of mathematical terms as free variables below when engaging with non-eliminativist accounts of reference; Pettigrew [2008]'s argument will then be used to add fuel to the fire set by the Semantic objection to non-eliminativism (§3.2.3.2).

It is clear what makes this view *structuralist*: it is what holds in all systems satisfying (say) arithmetic that counts, thus accounting for the indifference underlying methodological structuralism. Unlike relativist's bottom-up approach pealing off the non-structural properties of 'mathematical objects' of reference by the arbitrariness of choice, universalism takes a top-down approach by only building structural properties into (the discourse about) 'mathematical objects' to begin with.

Several objections have been raised against universalism.

1. The[58] same 'actuality-commitment' objection formulated regarding relativism applies *mutatis mutandis* to universalism, including the best case scenario commitment to sets;[59]

4. Universalism *mis*construes ordinary mathematical discourse:[60] what appear to be singular statements about certain entities, universalism construes as general statements about related elements in all systems of the same isomorphism-type. Pettigrew [2008]'s ameliorating strategy will may be employed to tackle

[55] The second strand of the Semantic objection (§3.2.3.2).

[56] This approach works under the assumption of a *quantificational account* of instantial terms; see Breckenridge and Magidor [2012, §2.1.2] for a critique. Shapiro [2008] suggests a similar approach similar in the context of the Automorphism problem (see §3.2.0).

[57] Again, we appeal to the distinction made above between a strong and a weak interpretation of a face value construal of mathematical discourse. In this sense, what Pettigrew's suggestion does for universalism is to provide it with the means to recapture a weak face value construal of ordinary mathematical discourse.

[58] We use the numbers mentioned in front to index particular objections across different eliminativist views. For instance, 1 here is essentially the same objection 1 leveraged against relativism, while objection 4 (following) has not been mentioned before, but it will be mentioned later on in connection with modal structuralism (§2.3).

[59] See Reck and Price [2000], Hellman [2005].

[60] See Shapiro [1997], Leitgeb [forthcoming,a, p. 10].

such concerns (§3.2.3).[61]

2.3 Modal MS

Geoffrey Hellman's modal structuralism[62] aims to be a full-eliminativist version of structuralism, in that it does away not only with structures as *sui generis* entities, but with commitment to abstract objects in general. In this sense it is sometimes characterized as a "structuralism without structures".[63]

Concerning the interpretation of mathematical discourse, modal structuralism resembles universalism. However, the universalist construal is endowed with a modal dimension. Given a arithmetical statement $\varphi(\mathbb{N}, s, 0)$, modal structuralism construes it as follows:

(Univ$_\Box$) $\Box\forall X, y, f(PA^2(X, y, f) \to \varphi(X, y, f))$

The background logic is second-order, while the modality is taken as primitive and is governed by S5 modal logic.[64] The modal construal doesn't need to rely on the *actual* existence of a system satisfying the Dedekin-Peano axioms to avoid vacuity. What she needs instead is the *possibility* of such a system:

(Exist$_\Box$) $\Diamond\exists X, y, f\ \mathrm{PA}^2(X, y, f)$

Just like in the case of universalism above, one can use dedicated variables as suggested by Pettigrew [2008] to tackle concerns related to a *mis*construal of ordinary mathematical discourse.

Relating this to matters metaphysical, the modalist proceeds as follows. Second-order comprehension is formulated such as to avoid commitment to cross-world relations:

(Comp$_\Box$) $\Box\exists R\forall x_1, ..., \forall x_n(R(x_1, ..., x_n) \leftrightarrow \varphi)$

where ϕ doesn't contain R free or modalities. Comp, however, carries commitment to classes as it stands, since we use second-order quantification over relations which are conceived of as classes. This is where the modalist deploys a complex machinery

[61] See Hellman and Shapiro [2019, p. 67] for mentioning Pettigrew's reply and suggesting that it is successful in defending modal structuralism.

[62] Introduced in Hellman [1989].

[63] E.g. Hellman and Shapiro [2019, p. 65].Of course, as we characterized it, eliminativists generally rule out structures as *sui generis* entities. However, semi-eliminativists require a background ontology of abstract objects, commonly sets (at their best), sometimes replacing structures with set theoretic representatives, be it pragmatically (naivists) or semantically (relativists); modal structuralism doesn't require any such background ontology, which makes it a true heaven for nominalists.

[64] Without the Barcan formula, such as to avoid inference from $\Diamond\exists x\varphi$ to $\exists x\Diamond\varphi$.

of plural quantification replacing second-order quantification[65] which, coupled with mereology, avoids quantification over abstracta entirely, assuming the possibility of a countably infinite system taken as an axiom, itself only using plural quantification and the language of mereology:

> **(Ax ∞)** There are some individuals, one of which is an atom, each of which combined with an atom not part of it is also one of them.[66]

This already provides us with a version of Exist$_\Box$ only using plural quantifiers and mereological notions, avoiding vacuity. Finally, mereological comprehension is given as follows:

> **(Σ Comp)** $\exists x\Phi(x) \rightarrow \exists y\forall z(y \circ z \leftrightarrow \exists u(z \circ u \wedge \Phi(u)))$

where $\circ$, intended to mean overlap in this context, is defined in terms of the primitive parthood relation. Summing up, modal structuralism is ontologically neutral in the sense of avoiding any sort of quantification over abstract objects: mathematical discourse is conceived of modally in an otherwise universalist fashion, second-order quantification is eliminated in favour of plural quantification so as to do away with classes and possibilia, and mereology is employed to deal with collections of objects as wholes.

Several objections have been raised against modal structuralism:

4. Just like universalism, modal structuralism has been also objected against on grounds of *mis*construing ordinary mathematical discourse, the same objection applying *mutatis mutandis* to its case.

5. The modality involved is primitive, which leaves us in the dark concerning the nature and the choice of states in the modal space, as well raising epistemological questions concerning access to the relevant modal knowledge.[67]

2.4 Concluding to non-eliminativism

Our focus in this essay is non-eliminativist structuralism; we conclude this section by highlighting the role played by the above objections in motivating the view. It is shown that a double metasemantic motivation assumes center stage in non-eliminativists' discourse; this aspect constitutes the core of some contemporary objections raised against non-eliminativism discussed later on (§3.2.3).

[65] So Comp$_\Box$ above should rather be rendered ass $\Box\exists xx\forall x_1, ..., \forall x_n((x_1, ..., x_n) \prec xx \leftrightarrow \varphi)$. This doesn't carry commitment to classes, but only to pluralities.

[66] Hellman and Shapiro [2019, p. 64].

[67] See Hellman [2001], Hellman [2005], Hellman and Shapiro [2019, p. 70].

Introducing *sui generis* non-eliminativist structuralism,[68] Stewart Shapiro appeals to Paul Benacerraf's second seminal contribution to the philosophy of mathematics. Benacerraf's "Mathematical Truth"[69] formulates the realism vs anti-realism dispute in the case of mathematics as a dilemma between semantic and epistemological desiderata. On the one hand, mathematical discourse appears to have the same face value structure as ordinary or scientific ones; since the latter are arguably best provided with Tarskian semantic understanding, semantic continuity suggests that mathematical discourse should better be itself understood in a similar manner, which seemingly leads to realism (both in ontology and truth value) about (presumably) abstract mathematical objects. However, a double faced problem emerges for the realist, who has troubles accounting for both the epistemology of abstracta, as well as their roles in understanding the empirical realm (what is usually called the 'applicability problem'). On the other hand, the anti-realist is in a much better position to account for the latter; however, her ways usually go through construing the 'real' or 'logical' form of the mathematical discourse in ways which depart from those provided in the ordinary and scientific cases, endangering semantic uniformity which in the least calls for an account of the schism. However, the anti-realist arguably fails to provide sufficient principled grounds for such facts. So goes Benacerraf's dilemma: on the one hand, realism satisfies semantic continuity, while it brings about seemingly intractable epistemological and metaphysical problems; anti-realism, on the other hand, could arguably better manage these, but it brings about seemingly intractable debts to explain semantic diversity, or otherwise provide an alternative semantic account for the ordinary and scientific discourses.

On the background of methodological structuralism in mathematics, the dispute between realists and anti-realists gets translated to one between non-eliminativists and eliminativists, respectively. One could complain that the horns of the dilemma misrepresent the situation in this case: we have, after all, considered two versions of eliminativism which take mathematical discourse at face-value, namely naivism and relativism. This is where Benacerraf [1965] comes to the fore pointing out that such reductions of mathematical ontology fall short of being satisfactory since they attribute too much structure to purported mathematical objects.

In this context, Shapiro argues that realist structuralism – i.e. non-eliminativism – could provide satisfactory answers to the realist challenges of the kind the traditional, non-structuralist realism could not appeal to; moreover, the non-eliminativist could better solve its debts than the eliminativist could pay hers. Naivism and relativism being arguably ruled out by the arguments of Benacerraf [1965], universalism and modal structuralism are ruled out by semantic considerations: unlike the realist, the eliminativist viciously *mis*construes

[68]See Shapiro [1997, p. 3].

[69]Benacerraf [1973].

mathematical discourse. Provided that non-eliminativism can pay its realist debts in a way non-eliminativism couldn't pay hers, non-eliminativism wins the day. The semantic motivation is central to the early non-eliminativist structuralist's justification against eliminativism.

Similar semantic concerns are also employed by recent non-eliminativists in defense of their view. For instance, Hannes Leitgeb argues similarly against universalism (and modal structuralism) upon introducing his non-eliminativist unlabeled-graph theoretic approach to mathematical structures:[70]

> *At least* prima facie*, the fact that the universalist reconstruction of arithmetic seems semantically to deviate more than necessary from mathematical practice should count against it.* (...) [O]*ne cannot help but wonder whether there might be a coherent way of combining the ontological benefits of universalist eliminative structuralism with the semantic benefits of relativist eliminative structuralism. This thought leads us to non-eliminative structuralism, the second large family of structuralist positions* (...) (Leitgeb [forthcoming,a, p. 11])

This constitutes what we will later on refer to as the central *metasemantic motivation*[71] for commitment to structures: structures provide us with the means to make the reference of mathematical terms scrutable, while delivering the best of both relativist and universalist worlds (as Leitgeb would have it):[72] mathematical terms are genuinely referential (as in relativism), albeit their reference would be crucially scrutable and would not commit us to predicating *foreign* properties of mathematical objects (as in universalism).

With these in mind, we can now approach the core of the present essay and start our discussion of non-eliminativism.

3 Non-eliminativist MS

We first present Stewart Shapiro early *Sui Generis* Structuralism (§3.1), followed by several of the most important problems and objections leveraged against it (§3.2). The latter will constitute the canon against which we shall assess the fresh approaches considered later on. We point out that most of the problems rely on

[70]Discussed below in §3.4.2.

[71]We borrow this expression from Assadian [2018], who stresses its centrality in the non-eliminativist discourse trying to induce commitment to structures as entities over and above systems. See also Linnebo and Pettigrew [2014, p. 277], seemingly implying that if structures cannot provide us with semantic benefits of the sort the eliminativist doesn't have at her disposal, then one of the "main advantages" of endorsing structures is lost.

[72]See Assadian [2018, p. 3201-2]'s discussion of the value of the Uniqueness thesis for non-eliminativism, drawing on comments made by Shapiro [1997, p. 141].

the positionalist component of Shapiro's view and go on to present non-positionalism – and its discontents (§3.3). Returning to positionalism, we showcase three recent approaches which aim to provide us with a satisfactory non-eliminativist account of mathematical structure (§3.4).

3.1 Non-eliminativism in 1997

Stewart Shapiro [1997][73] formulates an early articulation of non-eliminative positional structuralism, *Sui Generis* Structuralism (SGS), whose intuitive metaphysical outlook can be sketched as follows. *Ante rem* structures are understood as "abstract types exemplifying 'what all particular realizations thereof have in common' ";[74] in this sense, structures are like Platonic structural universals, structured entities shared by all their instances.[75] Like Platonic universals, structures are *ante rem* in that they are *freestanding*, their existence being independent of both their systems and the mind of the mathematician. Structures are *sui generis* in that they are not analyzed away in terms of another kind of entities, this also being what deems the account non-eliminativist. Places in structures are themselves objects in Shapiro's theory (thence 'positionalism'). On this background, mathematical objects are identified with positions in structures; mathematical theories in general are about structures, assertoric mathematical theories are about one structure in particular - the unique structure characterized by their second-order axiomatic scaffolding - while their objects are places in it.

On the semantic side, ordinary mathematical discourse is interpreted at face value: mathematical vocabulary comprises singular terms denoting properties, structures and positions therein. For instance, the arithmetical term '0' denotes, the natural number 0, that is, the first – predecessor-free – place in the natural number structure, the latter constituting the subject matter of arithmetic and the semantic value of 'ℕ' in mathematical discourse. Crucially, and as a hallmark of contemporary structuralism,[76] structures are unique up to isomorphism, which is

[73]Other notable early proponents of non-eliminativist structuralism are Michael Resnik [1997] (anticipated in Resnik [1981] and Resnik [1982]) and Charles Parsons [1990]. However, Shapiro's SGS was at the time arguably the most articulated view of the three and, as such, it has been the most widely engaged with and criticised. We will focus on it in this section. However, the publication of both Shapiro [1997] and Resnik [1997] propelled the discussion of non-eliminativism after *1997*, which motivates this section's title.

[74]Hellman and Shapiro [2019, p. 54]. Also, echoing Dedekind:

> *A structure is the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system.* (Shapiro [1997, p. 74])

[75]Shapiro says slightly more on this topic only later on in Shapiro [2008, §4].

[76]Steve Awodey [2014, p. 1]'s cherished "Principle of Structuralism: Isomorphic objects are

arguably the central tenant in the non-eliminativist semantic project. This feature allows us to employ a standard Tarskian semantic account for the ordinary mathematical discourse, fulfilling semantic continuity and providing for the metasemantic motivation recommending non-eliminativism. What makes this account structuralist is the presupposed aboutness of mathematical theories: quite literally, mathematical theories are about structures, i.e. entities whose isomorphism suffices for identity; mathematical objects, on the other hand, are purely structural objects, i.e. objects possessing only structural properties.

Concerning the formal scaffolding of SGS, Shapiro [1997] introduces mathematical structures through an axiomatic *ante rem* structure theory modeled upon ZFC, with the notable addition of the Coherence axiom. Coherence is the ”main principle behind structuralism”, holding that ”any coherent theory characterizes a structure, or a class of structures”:[77] in this setting, ‘coherent’ is a primitive predicate for formulas modeled upon ‘joint satisfiability’ in model theory.

In nuce, the background logic of *ante rem* structure theory is full second-order with identity. Structures and positions therein are entities in the first-order domain of the theory, with two sorts of corresponding variables (and quantifiers). Systems are defined as collections of positions from (one or more) structures, together with relations and functions on them; in this framework, structures are also systems, which are values of the second-order variables of structure theory. Intuitively, systems are ”concrete” collections of objects with relations on them such as set theoretic structures for instance.

Shapiro endorses the Quinean dictum ”no entity without identity”.[78] Concerning the identity criteria for the objects of structure theory – i.e. entities in its first order domain, structures and positions therein – Shapiro has notoriously little to say. Shapiro *concedes*[79] to postulate an identity criterion for structures in terms of

identical.” Although Shapiro [2006] gives up on uniqueness of structures, and with it on his previous semantic project, all the views presented below validate the ”Principle of Structuralism.”

[77] Shapiro [1997, p. 95].

[78] Quine [1969, p. 23]. The following is from a paragraph criticizing Resnik [1997] on grounds of floating this principle:

> [T]*he Quinean dictum “no entity without identity.” Quine’s thesis is that within a given theory, language, or framework, there should be definite criteria for identity among its objects. There is no reason for structuralism to be the single exception to this. If we are to have a theory of structures, we need an identity relation on them.* (Shapiro [1997, p. 92])

Given Shapiro’s maxims concerning positions in structures, we are entitled to assume that Quine’s principle should not be dropped in their case either.

[79] Three remarks. Shapiro concedes to postulate this criterion of identity for structures only after making the remark that their identity is to be taken as *primitive* (Shapiro [1997, p. 93]). It is unclear what the meaning of this remark is. Second, Shapiro points out that one could also replace isomorphism in Id-Struct with Resnik’s more coarse-grained notion of ’structure-equivalence’, which would identify some structures with different signatures, such as $\langle \mathbb{N}, 0, s, +, \times \rangle$

isomorphism:

(Id-Struct) $\mathrm{S} = \mathrm{S}' \equiv \mathrm{S} \cong \mathrm{S}'$

The identity relation on structures matches isomorphism between structures. Shapiro motivates Id-Struct invoking Ockham's razor: there is no use for multiple isomorphic structures in the ontology. Given Id-Struct, (coherent and) categorical theories characterize one and only one structure: existence comes by Coherence, while Id-Struct grants uniqueness. Let S be a structure, Y a system and [Y] be the structure of Y.[80] If S = [Y] we say that Y *instantiates* S and we have the following principle connecting structures with the systems instantiating them:

(Inst) $\mathrm{S} = [\mathrm{Y}] \equiv \mathrm{S} \cong \mathrm{Y}$

Instantiation is analyzed as isomorphism between systems and structures.[81]

Concerning identity between positions from within the same structure (*intra*-structural identities), one is driven by Shapiro's famous slogans[82] to conclude that their identity is governed by a version of Leibniz' Principle of Identity of Indiscernibles formulated in terms of structural properties:

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and $\langle \mathbb{N}, 0, s, +, \times, < \rangle$. This is rejected on grounds of technical inconvenience. Finally, postulating an identity criterion for structures diverges from Resnik's approach, who takes identity between structures, as well as identities between positions from distinct structures, to be indeterminate, holding that there is "no fact of the matter" concerning their obtaining or failure to do so; see e.g. Resnik [1997, p. 244].

[80] The use of the term forming operator $[\cdot]$ is justified given Inst. The intended meaning of the operator is that of a function such that, given a system X as argument, it returns an objects, the structure of X. As such, it has to be the case that given any system X, [X] *exists* and, moreover, it is *unique*. *Existence* comes by Coherence (assuming that the system is characterized by a coherent collection of formulas), while *uniqueness* is granted since $[\mathrm{X}] \neq [\mathrm{Y}] \to \mathrm{X} \neq \mathrm{Y}$, which follows by the counterpositive of the left to right direction of Convergence in the next footnote, which is a consequence of Inst.

[81] We note that Inst entails each of the following, assuming that structures are systems:

1. **Id-Struct**: $\mathrm{S} = \mathrm{S}' \equiv \mathrm{S} \cong \mathrm{S}'$
2. **Convergence**: $\mathrm{X} \cong \mathrm{Y} \equiv [\mathrm{X}] = [\mathrm{Y}]$
3. **Fixed-Pt**: $\mathrm{S} = [\mathrm{S}]$
4. **Isomorphism**: $\mathrm{X} \cong [\mathrm{X}]$

where S is a structure, X is a system and [X] is the structure of X; we use different sorts of variables to mark systems and structures apart given that (i) some views would draw a sharp distinction between them and (ii) we want to be able to talk about (*ante rem*) uninstantiated structures, as well as about structures corresponding to particular systems. Conversely, Isomorphism together with Id-Struct (sometimes called 'Uniqueness', e.g. Assadian [2018, p. 3201]) entail Inst (Isomorphism alone only entails the left to right direction of Inst). This shows that Inst is equivalent to Id-Struct together with Isomorphism. If structures are not systems, then neither Fixed-Pt, nor Id-Struct would follow from Inst, although Inst would still follow from their conjunction.

[82] Such as "[t]here is no more to the individual numbers "in themselves" than the relations they bear to each other" (Shapiro [1997, p. 73]).

(Id-Posit) $\forall x, y(x = y \equiv \forall \Phi_S(\Phi_S(x) \equiv \Phi_S(y)))$

where Φ_S's are structural properties of S. Id-Posit holds that instantiating the same structural properties is sufficient for identity within a structure. When it comes to identity between places in distinct structures (*cross*-structural identities), Shapiro [1997] advocates relativity in ontology, holding these to be a matter of contextual decision or convention. In this sense, statements of identity concerning places in distinct structures (say natural 0 and real 0, $0_{\mathbb{N}} = 0_{\mathbb{R}}$) are semantically indeterminate, while semantic indeterminacy is backed by ontological indeterminacy.[83]

SGS has been criticized extensively and alternative non-eliminative versions of structuralism have emerged focusing on satisfactory replies to one criticism or another. We review several problems and objections leveraged against SGS, while ultimately having a wider scope and concerning non-eliminativism in general.

3.2 Non-eliminativist troubles

We operate a pragmatic distinction between *problems* and *objections* to guide our discussion below. We characterize *problems* as collections of mutually conflicting statements. *Objections*, by contrast, are arguments meant to conclude to the rejection of a certain theory itself. Objections customarily go by showing how one or more problems concerning a certain theory cannot be solved without rejecting its solid core. The Individuation *objection* below (§3.2.1.3) is a good example, while what we will call the Automorphism *problem* (§3.2.1.2) is one of its central components.

We divide the problems discussed in three distinct albeit related clusters: Problems of Identity, Problems of Objects and Problems of Reference.[84] The problems in each group contain the roots of one objection against non-eliminativism, and each of these objections concludes rejecting non-eliminativism. This list is not exhaustive: notable omissions include objections relying on floating a general principle of extendibility for structures, as

[83]The following quotes should back our inline statements:

> *But it makes no sense to pursue the identity between a place in the natural-number structure and some other object, expecting there to be a fact of the matter. Identity between natural numbers is determinate; identity between numbers and other sorts of objects is not, and neither is identity between numbers and the positions of other structures.* (Shapiro [1997, p. 79]) *We point toward a relativity of ontology, at least in mathematics. Roughly, mathematical objects are tied to the structures that constitute them.* (Shapiro [1997, p. 80]) *The point here is that cross-identifications like these are matters of decision, based on convenience, not matters of discovery.* (Shapiro [1997, p. 81])

[84]We owe the idea of such a division to Leitgeb [forthcoming,b].

well as worries related to SGS' Coherence Axiom.[85] Their omission changes nothing concerning our verdicts below.

3.2.1 Problems of Identity

This section introduces two problems concerning the identities of positions in structures: the problem of Cross-structural Identifications and the Automorphism problem. The latter plays a central role in the Individuation objection introduced right thereafter.

3.2.1.1 Cross-structural Identities Problem. Shapiro [1997] holds that cross-structural identity statements are indeterminate, their assertion relying on category mistakes. For instance, when the teacher writing a closed integral on the blackboard asks the class whether it's value is a natural number or not, she is seemingly cross-structurally identifying natural numbers with certain real numbers. Similarly, when the set theorist holds that the natural number 2 is $\{\varnothing, \{\varnothing\}\}$, she is seemingly identifying natural numbers with von Neumann ordinals.[86] Shapiro holds that in such cases mathematicians employ the 'is' of *fiat*, by which they are ruling in a semantically indeterminate case of identity for the purposes of the present context. Shapiro holds that semantic indeterminacy is underscored by ontological indeterminacy: ' $2_{\mathbb{R}} = 2_{\mathbb{N}}$ ' is indeterminate because it is indeterminate whether in reality $2_{\mathbb{R}} = 2_{\mathbb{N}}$. However, ontological indeterminacy conflicts with the realist outlook of non-eliminativism: if mathematical entities are places in structures and such places are objects, then their identity has to be completely determinate.[87]

[85] Both problems were first formulated in Hellman [2001], their last iteration probably being Hellman and Shapiro [2019]. *In nuce*, the extendibility principle is credited to Ernst Zermelo and it states that every mathematical domain of objects can be extended and, as such, there is no (completed) totality of mathematical objects. However, given its commitment to full second order comprehension, SGS is committed to a totality of positions in structures, violating extendibility in all its generality; see Hellman and Shapiro [2019, p. 58]. Concerning the primitive notion 'coherent', the issue is that if explained formally, then it would come close to Hilbert's notion of 'formal consistency' and the Coherence Axiom would be akin to his idea that consistency suffices for mathematical existence, which has been proven untenable by Gödel's Incompleteness Theorems; however, the alternative is 'coherent' should be identified with second order logical possibility, which turns the Coherence Axiom into a problematic criterion of actual existence; see Hellman and Shapiro [2019, p. 61].

[86] This problem is related to Gottlob Frege's Caesar Problem. Right after suggesting Hume's Principle as a contextual account of numbers and number denoting terms – the number of F's is the same as the number of G's if and only if there is a one-to-one correspondence between F and G – Frege criticizes it on the ground that it doesn't settle cross-structural identity statements: for instance, it doesn't say anything concerning the statement '2 = Caesar'. For a discussion in the context of structuralism, see Shapiro [1997, p. 79ff], Shapiro [2006, p. 122ff]. For the original problem, see Frege and Austin [trans, §56, §66].

[87] MacBride [2005, p. 577].

Shapiro [2006, p. 128] rejects the ontological indeterminacy: distinct structures are disjoint and, as such, *unambiguous* cross-structural identity statements are strictly speaking *false*. However, the falsity view conflicts with mathematical practice: ordinary mathematical discourse contains many statements of identity concerning entities from distinct structures, while many mathematically interesting questions concern conjectured cross-structural identities. On the face of it, the falsity view would deem such discourse false and such questions spurious; we are thereby owed an account of such identifications in mathematical practice.[88]

3.2.1.2 The Automorphism Problem. [89]

Consider the group of integers under addition: $\langle \mathbb{Z}, + \rangle$. Any element of $\mathbb{Z}$ is distinct from its inverse under addition, i.e. 1 is distinct from -1 , 2 is distinct from -2 and so on and so forth; this is a mathematical fact which would better not be opposed on philosophical grounds.

Arguably, another fact is that any inverse integers z and $-z$ are *structurally indiscernible by properties* in the given group structure:[90] there is no structural property had by one but not by the other. This is underlined by the non-rigidity of $\mathbb{Z}$:

> **Rigid system**: A system is rigid if and only if it has only one automorphism.[91]

In particular, a mapping $f : \mathbb{Z} \to \mathbb{Z}$ holding 0 in place and taking any other element to its inverse can be shown to be an isomorphism; we call such distinct elements

[88] Shapiro [2006, p. 128ff] credits ordinary uses involving apparent cases of indeterminacy such as those above to *semantic* indeterminacy on the part of mathematical terms: the numeral '2', for instance, is ambiguous between natural, rational, real, complex etc. readings.

[89] Originally formulated in Burgess [1999], Keränen [2001], independently. The Automorphism problem and the Individuation objection that follows it have close semantic correlates which will be discussed in §3.2.3. We are only concerned with their ontological aspects in this section.

[90] The status of this "fact" depends to some degree on the notion of structural property one is working with. As noticed before, Korbmacher and Schiemer [2018] distinguish between two distinct notions of structural property: isomorphism invariance and definability in terms of the primitive vocabulary of the theory characterizing the structure concerned. We opted for isomorphism invariance (§1.1) without discussing Schiemer and Wigglesworth; however, we point out that the alternative notion – definability in the primitive vocabulary – makes this fact even more pressing. It is a well-known model-theoretic fact that no non-0 element of $\langle \mathbb{Z}, + \rangle$ can be defined in its signature. Zero is defined as $0 := \iota x(x + x = x)$, while difference is defined as $x - y = z :\equiv y + z = x$. Finally, given a non-0 element z of $\mathbb{Z}$, one can actually define its additive inverse as $-z := \iota x(z + x = 0)$. $\langle \mathbb{Z}, + \rangle$ is the same as $\langle \mathbb{Z}, 0, +, - \rangle$.

[91] The envisaged unique automorphism is the identity mapping. An automorphism is an isomorphism where the domain and the codomain coincide. In model theory, an isomorphism is a structure preserving map between $\mathcal{L}$-models, i.e. a bijection preserving constants, relations and functions: for relations, for instance, given two $\mathcal{L} \cup \{P\}$-models $\langle D, I \rangle$ and $\langle D', I' \rangle$ with P a unary predicate symbol, if $f : D \cong D'$, then $d \in P^I$ if and only if $f(d) \in P^{I'}$.

which are connected by some non-trivial automorphism ‘symmetric’. Symmetric elements are structurally indiscernible by properties.[92]

Finally, we remember that SGS holds that positions are identical if and only if they have the same structural properties. However, given that any (non-0) integer has the same structural properties as its inverse, we'd be lead to conclude that SGS is committed to identifying every non-0 integer with its inverse, which is anathema for any attempted account of mathematics.

This is the Automorphism problem. Sticking with our illustrative $\langle \mathbb{Z}, + \rangle$, it can be sketchily summarized as follows:

1. $1 \neq -1$;
2. 1 and -1 are structurally indiscernible in $\langle \mathbb{Z}, + \rangle$;
3. ID-Posit: structurally indiscernible positions are identical.

Since the positionalist holds that mathematical objects are positions in structures, then 3 and 2 imply that $1 = -1$, blatantly contradicting 1.

There are many important mathematical structures which contain symmetric positions, the complex field (i and $-i$ are symmetric) and the Euclidean space (any two points are symmetric) being only two of the commonly mentioned cases. There are even more somewhat trivial mathematical structures such as those called by Shapiro 'simple cardinal structures' – represented by unlabeled graphs with no relations on them – which would also be ruled out of existence by SGS as presented above. All these cases pose a similar problem as that sketched above concerning the group of additive integers.

3.2.1.3 The Individuation Objection. Jukka Keränen [2001] turns the Automorphism problem into an objection aimed at rejecting non-eliminativism.[93]

In good Quinean fashion of the kind Shapiro seemed willing to endorse,[94] Keränen holds that every theory should provide an account of identity for the objects in its domain of quantification. This, Keränen holds, is a "metaphysical requirement": objects, after all, are properly individuated entities, i.e. entities having determinate identity criteria.[95]

Keränen identifies essentially two ways one can provide an identity account: (i) substantially, by *general properties* – roughly, properties possibly being multiply instantiated – or otherwise (ii) somewhat trivializing the matter, by (ii.1) invoking

[92] A model theorist would say that they have the same 1-types. See Ladyman et al. [2012] and Button and Walsh [2018] for detailed analyses of discernibility and distinctness in models.

[93] Also Keränen [2006] which is a reply to Shapiro [2006]'s reply to the Keränen's article quoted inline.

[94] See footnote 78 (Page 29).

[95] Keränen [2001, p. 313].

haecceities or, rejecting the need to formulate any identity principle, (ii.2) by taking identity as *primitive*.

Regarding the first option (i), Shapiro [1997][96] notoriously holds that mathematical entities only have isomorphism invariant structural properties. Keränen argues that, among the general properties, only *structural* properties which can be specified by formulae with *one free variable* and *not containing individual constants* are of this kind and he insists that the structuralist should provide identity criteria exclusively invoking such properties;[97] his argument can be sketched as follows. Structural properties essentially expressed by formulas containing *multiple free variables* – structural *relations* – fare no better in discerning positions in non-rigid structures than do their single-variable counterparts. Concerning the involvement of *individual constants* in the specification of the relevant properties, their use would eventually lead to vicious regress. For instance, 1 and -1 would be distinguished by a property such as $x + x = 2$, in turn requires that the identity of 2 has been already properly provided; but trying to provide for its identity would involve another entity in equal need of individuation, ending up in regress. The structuralist might alternatively attempt to employ *non-structural properties*. However, since such properties are notoriously not invariant under isomorphism, this option would arguably amount to a full blown rejection of non-eliminativist structuralism.

So, the argument concludes, structuralism would better only employ *structural properties* free of any constants in her account of identity for positions in structures. However, the *Automorphism problem* rules out this path.

It is so we are driven to consider trivial identity accounts, the second collection of options (ii).[98] *Haecceities* (ii.1) are properties which are extensionally equivalent to self-identity: given an object a, its haecceity is a property which a and only a satisfies, such as $\lambda x(x = a)$. The structuralist could employ such properties to account for the distinctness of, say, 1 and -1: the former, but not the latter, satisfies $\lambda x(x = 1)$. However, haecceities are a notoriously non-structural sort of property, giving rise to the same objection mentioned before in connection with invoking non-structural, albeit general properties: such an account, Keränen argues, amounts to a rejection of the ontological project of realist *structuralism*.

One is thus led to give up any attempt to formulate a principle of identity for positions in structures and rely on *primitive identity* facts (ii.2). However, Keränen holds that such a move would burden the non-eliminativist with the task of explaining why no such principle could be formulated. Moreover, it would be metaphysically dubious to hold that there are distinct but otherwise *utterly indiscernible* objects, whose distinctness is not backed by any fact of the matter –

[96] Shapiro [2006, p. 115ff] notoriously qualifies earlier "slogans", holding that only the *essential* properties of positions in structures are structural.

[97] Keränen [2001, p. 315-319].

[98] Keränen [2001, p. 327-328].

that is, holding onto metaphysically primitive identity facts. Therefore the structuralist would have to postulate other facts grounding identity facts, and such facts would most likely appeal to haecceities, collapsing this option into the previous one and suffering its defeat.

Option (ii) is thus seemingly ruled out as well. Keränen concludes that the non-eliminativist cannot pay her debts: although an account of identity is the core requirement placed upon any theory characterizing a domain of objects (more pressingly so when those object are introduced by the theory), the non-eliminativist is unable to accomplish this task with respect to positions in structures.

This is Keränen's *Individuation objection*. The non-eliminativist is seemingly bound to hold that positions in structures are individuated by their structural properties. However, once mathematical entities are identified with positions, the thesis misfires giving rise to the Identity problem, revealing a tension withing the structuralist view which, Keränen argues, should lead to the rejection of non-eliminative structuralism.

The Individuation objection has been discussed at length in the literature. All the options considered and shortly dismissed by Keränen's found their own advocates and have been further articulated. We sketch a map of the now classical views on the matter.

James Ladyman [2005] formulated a more fine-grained identity principle for positions in structures in terms of structural *relations*. Even though symmetric integers are indiscernible by monadic structural properties, Ladyman points out that they are nonetheless *discernible by structural relations*: $z + (-z) = 0$, while it is not the case that $z + z = 0$, showing that for any non-0 integer, the relation 'x is the additive inverse of y' – $A(x, y) := y + x = 0$ – is a structural relation which sets apart integers from their inverses: given an integer z, $A(z, -z)$ but not $A(z, z)$. In general, any two elements which are in a symmetric albeit irreflexive relation can be discerned by such means. This shows that Keränen is wrong in holding that structural relations are no more discerning than properties.[99] The suggestion is then to amend Id-Posit as follows:

(Id-Posit-Rel) $\forall$ S $\forall x, y(x = y \equiv \forall \mathrm{R} \forall z\ (\mathrm{R}(x, z) \equiv \mathrm{R}(y, z)))$

where R is a (dyadic) structural relation. Borrowing a piece of useful terminology from Ladyman et al. [2012], we define as follows:

> **(Absolute Discernability)** Elements a and b are absolutely discernible in a system X if and only if they are discernible by (monadic) structural *properties*;

[99] Keränen [2001, p. 324]. See MacBride [2006, p. 67] for an objection against weak discernibility through relations.

> **(Weak Discernability)** Elements a and b are weakly discernible in a system X if and only if they are discernible by a symmetric and irreflexive (dyadic) structural *relation*.[100]

Ladyman et al. [2012] show that weak discernability is more discerning than absolute discernability (regardless of the language concerned) and, in general, that weak discernability is the most discerning relation on systems short of outright distinctness: if a and b are not weakly discernible, then they are *utterly indiscernible* in the system considered, i.e. there is no structural (n-adic) relation holding of one but not of the other in that system.[101] If structures are systems, as Shapiro [1997] holds, then these results also hold of structures. In this sense, Id-Posit-Rel is the most discerning principle on structures: if it fails to discern a and b, then no principle does so, including Id-Posit.[102] Id-Posit-Rel solves the Automorphism problem as it appears for many important mathematical structures: the additive group of integers, the complex field and the euclidean space. Symmetric entities in these structures are weakly discernible, albeit not absolutely so.

However, the Automorphism problem keeps its ground for structures which contain elements which are not related by symmetric and irreflexive relations[103] – e.g. the simple cardinal structures (e.g. Figure 1) – which lead to further attempted solutions to cover these cases. Some symmetric positions in such structures are utterly indiscernible: the nodes of the dumbbell graph, for instance, are utterly indiscernible in the above sense and, since they are distinct, the Automorphism problem would show its face if such structures are allowed for.

Tim Button [2006] provides a non-trivial account of identity for positions in structures. Following a dim suggestion mentioned by Keränen himself in a footnote,[104] Button suggests to draw a distinction between *basic* and *constructed* structures and provide a realist account of the former, while endorsing

[100] Definition 3.1 of Ladyman et al. [2012] introduces five grades or notions of discernibility in systems: *intrinsic*, *absolute*, *relative*, *weak* and outright *distinctness*. Intrinsically discernible entities are discernible by an intrinsic property definable in the signature of the system, relatively discernible ones by a relation definable in the same signature, while distinctness is the meta-theoretic, language independent discernibility. Absolute and weak discernibility are as defined inline, the latter being the notion suggested in Ladyman [2005]. The authors study these notions and their comparative strengths in four first-order sort of languages: with or without identity, with or without constants, and combinations thereof. For the extended discussion, see Ladyman et al. [2012, p. 170ff]. For similar discussions on notions of discernability in systems and structures, see also Ladyman [2020], Ketland [2011], Caulton and Butterfield [2012] and Button and Walsh [2018, §15].

[101] Ladyman et al. [2012], Theorems 5.1 and 7.1.

[102] We gloss over the fact that the authors use a notion of structural properties as properties definable in the signature of the system. See Korbmacher and Schiemer [2018]. For the purposes at hand, their notion and ours (§1.1) overlap.

[103] E.g. Ketland [2006, p. 309]

[104] Keränen [2001, p. 328], footnote 27. It is swiftly rejected on grounds of (i) not being able to

Figure 1: The dumbbell graph, a simple cardinal structure.

eliminativist customs with respect to the latter:[105] ruling that non-rigid structures are *constructed* from rigid, basic ones, the Automorphism problem is thereby done with by dispelling with non-rigid structures from the ontology. 'Trivial' accounts of identity have found their own fair share of defenders. Shapiro [2006] endorses *primitive identity* and suggests dropping any Leibnizian type of identity principle in favour of *primitive* identity, fully embracing utterly indiscernible entities.[106] Leitgeb and Ladyman [2008] suggest that identity is a structural property of structures and identity facts concerning positions in structures are grounded in the identity of the very structure they belong to. Shapiro as well as Leitgeb and Ladyman are far from holding such facts as somewhat metaphysically dubious and feel no need to fall back on some kind of haecceitism.[107] Recent structuralist accounts embracing primitive identity are Unlabeled Graph-theoretic Structuralism (§3.4.3 below) and some defenders of Fregean Abstractionist *in re* Structuralism (discussed in §3.4.1 below). Finally, and although enjoying little success, haecceitism has found its own defenders, for instance Bermúdez [2007].[108]

3.2.2 Problems of Objects

This section discusses the problems of Permutation, Circularity and Structural Properties. The latter alongside with the Automorphism problem (introduced previously in §3.2.1.2) are employed at the end to formulate MacBride's "bad news/old news" objection.

3.2.2.1 The Permutation Problem. [109]

account for all non-rigid mathematical structures and (ii) it would apply eliminativist means to many important mathematical structures, essentially giving in too eliminativism. Button dispels both worries.

[105] The label 'hybrid' is also mentioned by Keränen [2001], whose second objection against this view states that "adopting a*hybrid position* of the sort envisaged here would in any case amount to rejecting realist structuralism proper" (our italics).

[106] Alongside Keränen [2001], Button [2006] contains a swift rejection of utter indiscernibles. But see Assadian [2019a] for a defense of utter indiscernibles.

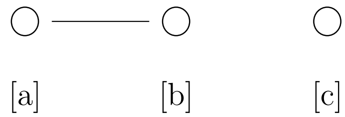
[107] See Assadian [2019a] for a defense of utter indiscernibles.

[108] See Menzel [2016] for a critique.

[109] As concerning non-eliminativist structuralism, the Permutation problem was originally raised in Hellman [2001, p. 195-196] and later in Hellman [2005, p. 546]; Ketland [2015, §2] elaborates on the matter. It has been recently expanded and given a semantic dimension in Assadian [2018, §4]; we draw on Bahram Assadian's exposition in this section. The roots of this problem can be

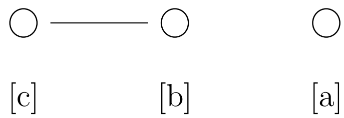
Suppose that the non-eliminativist notion of structure is coherent and consider the following structure:

S:



S can be categorically characterized by a purely second-order and coherent formula Φ_S.[110] The labels [a], [b] and [c] may be regarded as parameters – or free variables as in §2.2 or Skolem constants as in §3.4 – introduced by existential elimination performed on Φ_S and meant as an aid in distinguishing S's positions. Consider further the following π-permuted[111] copy of S:

S^π:



Clearly $\text{S} \cong \text{S}^\pi$ and, arguably, both are structures (if S is, as we assumed). However, following model theory and since the non-eliminativist holds that structures are systems we are driven to conclude that $\text{S} \neq \text{S}^\pi$. Therefore there are distinct albeit isomorphic structures, contradicting a core non-eliminativist principle holding that isomorphic structures are identical:

(Id-Struct) $\text{S} = \text{S}' \equiv \text{S} \cong \text{S}'$

We can summarize the Permutation problem as follows:

1. S is a structure;
2. $\text{S} \cong \text{S}^\pi$;
3. S^π is a structure;
4. $\text{S} \neq \text{S}^\pi$;
5. Id-Struct.

1 and 2 are seemingly non-negotiable. 1 lends credibility to 3, while the model theoretic construction suggests 4. 1 through 4 entail that there are distinct isomorphic structures, an embarrassment of riches contradicting 5.

traced back to Benacerraf [1965] objection against traditional mathematical Platonism (see §1.1).

110
$$\begin{aligned}\Phi_\text{S} := \; & \exists x_1, x_2, x_3[[x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3]\wedge \\ & [R(x_1, x_2) \wedge \neg R(x_1, x_3) \wedge \neg R(x_2, x_3) \wedge \neg R(x_1, x_1) \wedge \neg R(x_2, x_2) \wedge \neg R(x_3, x_3)]\wedge \\ & \forall y[y = x_1 \vee y = x_2 \vee y = x_3]]\end{aligned}$$

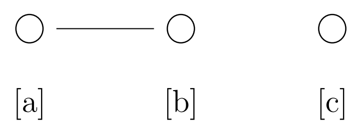
111 See §1.1 for the model-theoretic permutation construction.

Since no assumption concerning the nature of the structure S was made, the Permutation problem is a general threat to the non-eliminativist notion of structure, with overarching consequence for non-eliminativist semantics (see the Singular Reference problem in §3.2.3.1). We notice that the Permutation problem is more general than the Automorphism problem, in that the latter, unlike the former, only appears for non-rigid structures, whereas Permutation plagues all structures whatsoever.

3.2.2.2 The Circularity Problem. [112]

The non-eliminativist holds that structures are systems, i.e. entities consisting of a domain of positions together with distinguished relations on them. Structures are then individuated as follows: $S' = \langle D, R_1, ..., R_n \rangle$. For instance, consider again the structure:

S:



As a system, S may be characterized as $\langle D = \{[a], [b], [c]\}, R = \{([a], [b]), ([b], [a])\}\rangle$.

The structuralist also holds that positions are individuated by their relations with the other objects in their hosting structure. We can render non-eliminativist slogans concerning the identity of positions formally. Given a position [x] in a structure S', the identity of [x] is $\{ (y_1, ..., y_k)^{R_i^{S'}} \mid [x] = y_1 \vee ... \vee [x] = y_k\}$, where $R_i^{S'}$ is a structural relation on S'; in plain English, the identity of positions in structures is fully characterized by their relations with other positions in the very same structure.[113] For instance, the identity of [a] in S is $\{([a],[b])^R, ([b], [a])^R\} =$

[112] The original problem is formulated in Hellman [2001, p. 194-195]. Hellman traces this problem back to Bertrand Russell's remarks agaisnt Richard Dedekind's views:

> *Moreover it is impossible that the ordinals should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression. If they are to be anything at all, they must be intrinsically something; they must differ from other entities as points from instants, or colours from sounds.* (...) *And in any case, Dedekind does not show us what it is that all progressions have in common, nor give any reason for supposing it to be the ordinal numbers, except that all progressions obey the same laws as ordinals do, which would prove equally that any assigned progression is what all progressions have in common.* (Russell [2009, p. 251])

For modern formulations of the problem, see also Hellman [2005, p. 546] and MacBride [2006]. Leitgeb [forthcoming,b, p. 12-3] interprets the Circularity problem epistemically; Linnebo [2007, p. 69-70] and Nodelman and Zalta [2014, p. 64] interpret it ontologically. We think that both are partially right, and moreover that the problem has also a semantic side.

[113] A similar gloss of the structuralist slogans is proposed in Wigglesworth [2018b], discussed below in §3.4.3.

R. This is arguably an intuitive depiction of early non-eliminativists' statements on the matter.

We can easily notice that the identity of R depends on [a], since [a] appears in the identity of R. However, the identity of [a] itself depends on R, since R figures in the identity of [a]. In Hellman's own words:

> *Thus the notion of an* ante rem *structure seems to involve a vicious circularity: such a structure is supposed to consist of purely structural relations among* purely structural objects, but understanding either of these requires already understanding the other. *Whereas the Keränen–Burgess objection granted the relations and raised questions about how these alone could determine the objects (unless the structures are rigid), this objection questions such talk of relations in the first place, and thereby the very notion of "Dedekind abstraction," which is supposed to lead to them.* (Hellman [2005, p. 545], highlight in the original)

On the *ontological* side, we end up with a circular case of identity dependence: both positions and structures should be ontologically prior to each other and both have their identities seemingly depending on one another. This, it is held, is unacceptable.[114] This problem also has semantic and epistemic counterparts. *Semantically*, the problem is that reference to relations and thus to structures would proceed through reference to positions, while reference to positions itself presupposes reference to the relations on structures, seemingly vitiating the possibility of reference to either. On the *epistemic* side, it appears that understanding structures presupposes an understanding of their positions, while understanding positions relies on an understanding of structures themselves, seemingly tuning both pursuits impossible.

3.2.2.3 The Problem of Structural Properties. [115]

Non-eliminativists hold that positions in structures only have structural properties, while mathematical objects are identified with such positions in mathematical structures.

First, this thesis faces counterexamples. Mathematical entities have non-structural properties such as 'being the number of the planets in the Solar system',[116] which as it happens holds true of the natural number 8. But then a certain position in the natural number structure has this property, against non-eliminativist claims.

[114] Linnebo [2007, p. 69] provides a slightly different interpretation of this objection.

[115] This problem has been first formulated by John Burgess [1999].

[116] In June 2020, and at least since 2015. See MacBride [2005, p. 584] for an argument that our counting practices commit us to accepting that mathematical objects have non-structural properties.

Second, the thesis is incoherent. If the non-eliminativist is right, then all positions in structures have the property 'having only structural properties'. However, this is not itself a structural property.[117]

3.2.2.4 MacBride's Objection. Fraser MacBride [2005, p. 582-586] employs the Automorphism problem (§3.2.1.2) and the previous problem of Structural Properties to build up a damning *dilemma* for the non-eliminativist: non-eliminativism either has absurd consequences such as '$i = -i$' (really *bad news*), or otherwise collapses into "good old-fashioned Platonism" (*old news* indeed). Structuralists such as Shapiro utter slogans such as the following:

> *There is no more to the individual numbers "in themselves" than the relations they bear to each other.* (Shapiro [1997, p. 73])

MacBride holds that renouncing such theses would be giving up any pretense that non-eliminativism is anything more than Platonism. The question is how to interpret such claims, and MacBride comes up with two allegedly exhaustive interpretations:

1. *Object Reductionism*: Mathematical objects – positions in structures – are *nothing more than* bundles of structural properties;

2. *Property Reductionism*: Mathematical properties are *fully* reducible to structural properties.

If we endorse an object reductionist reading of the slogans (horn number 1), then non-eliminativism is committed to a principle of Identity of Indiscernibles in terms of structural properties which entails that e.g. $i = -i$, facing the Automorphism problem, which is really *bad news*. The non-eliminativist, MacBride concludes, cannot reduce mathematical objects to bundles of mathematical properties and is thus committed to a dual ontology of *positions* and and *relations*.

Endorsing property reductionism (horn number 2) on the other hand, avoids the Automorphism problem since it allows for distinct albeit structurally indiscernible mathematical objects. However, this option falls prey to the problem of Structural Properties: mathematical objects, the problem shows, have properties which are not reducible to structural ones. But this reads like *old news*: the non-eliminativist is committed to irreducible mathematical objects which possess non-structural properties, which is essentially, MacBride holds, the traditional Platonist position.

What else could distinguish non-eliminativism from traditional Platonism? MacBride considers what is probably the one *big* difference between the two: structuralists, unlike Platonists, hold that mathematical objects are *positions* in

[117] Burgess [1999, p. 287].

structures. However, this thesis can be upheld only if positions are objects; MacBride argues that positions are better provided with a nominalist account.[118]

This is *MacBride's dilemma*. Endorsing Identity of Indiscernibles requires a non-eliminativist solution to the Automorphism problem; however, dropping it and endorsing property reductionism requires a non-eliminativist ontological articulation of positions in structures such as to lend credibility to the thesis that positions are *bona fide* objects. MacBride holds that none has been achieved.

MacBride's dilemma has been relatively little discussed in the literature. Maybe the most straightforward reply to this objection can be provided in a non-positionalist setting: after all, MacBride might be instructed, structuralism is not about mathematical objects, but about the subject matter of mathematical theories, namely mathematical structures, which are to be conceived as non-positional entities (e.g. as second order propositional functions as suggested in §3.3 below): nothing *bad*, nor *old* about that (at least not in the sense of this dilemma).

In a positionalist setting however, Øystein Linnebo [2007] provides an early discussion concerning possible articulations of the distinction between non-eliminative structuralism and traditional mathematical Platonism. Linnebo identifies two main non-eliminativist motifs or theses distinguishing positions in structures from other ordinary, including Platonistic, sorts of objects. First, the *Incompleteness claims*[119] hold that positions are crucially incomplete in some sense. *Intrinsic Incompleteness* holds that positions have no intrinsic properties, while *Non-Structural Incompleteness* holds that positions have no non-structural properties. However, both Incompleteness claims are found deeply wanting: the former since no meaningful notion of intrinsicness can be articulated in a mathematical setting, mathematical objects presumably having all their properties

[118]The argument runs as follows (MacBride [2005, p. 585-586]). If positions are objects, then they must play a role in an account of order; that is, they should play a role in explaining why aRb rather than bRa, where xRy is a asymmetric relation. The customary explanation of such facts holds that R obtains for (a, b) when a fits into the x place and b into the y place of the relation; bRa fails because b and a don't fit into the right slots. If places are objects, however, then further explanation is needed: why a's filling x and b's filling y suffices for aRb rather than bRa?

If we chose to answer we must seemingly follow the same logic: it suffices because x and y themselves have a certain order in xRy, which obtains in virtue of a further relation holding between the places x and y, say xR^*y, explaining it. However, we would need an even further relation $vR^{**}w$ explaining the obtaining of xR^*y rather than yR^*x; it can be easily seen that this leads to a *vicious infinite regress* undermining the purported explanation.

We can, on the other hand, break the explanatory chain and postulate as a primitive, brute fact that the places x and y are ordered as in xRy; however, in that case places play no role in the account of order, since the order itself (aRb and not bRa) could be taken as a brute fact to start with. So positions as objects would be the fifth wheel to the cart (a popular saying in Romania), better dropped than carried (this was presumably slightly more sensible in the older days when wheels were made of wood and heavy).

[119]Linnebo [2007, §2].

necessarily; the latter since it faces the problem of Structural Properties presented above. So Incompleteness presumably fails to set non-eliminativists free of MacBride's embrace.

However, crucially, Linnebo identifies a second sort of claims made by non-eliminativists towards delineating their view from Platonism, namely the *Dependence claims*.[120] Linnebo pins down two Dependence claims: non-eliminativists hold both that *positions depend on the other positions in their structure* (ODO), as well as that *positions depend on their very structure* (ODS), both unlike ordinary objects. ODO certainly faces the ontological flavour of the Circularity problem as presented above, but Linnebo argues that (i) it is unclear that non-circular dependence chains are problematic in general and (ii) the requirement that related objects have already had their identities grounded prior to their relating begs the question against the non-eliminativist if not supported by independent argument; none such has been provided in the literature. As such, even if ODO is presumably dismissed by (i), ODS is left standing and unaffected by either (i) or (ii); crucially, ODS itself, Linnebo argues, suffices for non-eliminativist purposes, holding mathematical entities to be subject to *upward* dependence which is unlike ordinary and Platonic objects.[121] Circularity is ultimately solved by dropping the requirement that relata should have their identities grounded prior to the relations involving them, so as to save at least ODS, if not also ODO.[122]

The notion of dependence involved in ODO and ODS is discussed[123] and Linnebo settles on *weak identity existence*: every individuation of a position makes use of entities which also suffice for individuation any of the other positions in the structure (ODO), while every individuation of any position in a structure makes use of entities which also suffice to individuate the structure itself (ODS). Given an *in re*, Fregean Abstractionist analysis of mathematical structures, Linnebo shows that upwards weak identity grounding claims hold for mathematical structures such as the Klein four-group and, in general, by structures corresponding to assertoric mathematical theories.[124] However, Linnebo argues that upwards grounding claim fails in some

[120] Linnebo [2007, §3].

[121] Linnebo [2007, p. 71].

[122] Wigglesworth [2018b] also provides an analysis of grounding in mathematical structuralism which endorses reflexive grounding relations.

[123] Linnebo [2007, §7]. Drawing on Kit Fine and E.J. Lowe, Linnebo operates a distinction between *existence* and *identity dependence*, while another distinction is introduced by Linnebo himself between *strong* and *weak* dependence. Linnebo provides an analysis mathematical structures along Fregean Abstractionist lines (similar to those we are about to present in §3.4.1) and shows that some (Linnebo [2007, §6]) albeit not all (Linnebo [2007, §5]) mathematical objects satisfy the non-eliminativist's Dependence claims.

[124] Understood as those which can be arrived at by a kind of Dedekind abstraction on systems; see Linnebo [2007, p. 76].

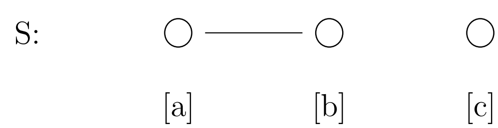
other mathematical structures such as the set theoretic universe.[125]

Invoking an articulation of non-eliminativism through the Dependence claims doesn't answer MacBride's second horn worries *right away*: as MacBride points out, we need a prior, independent reason to believe in positions as objects before identifying them as mathematical entities. However, Linnebo's Abstractionist account provides an antidote to this worry: if positions are conceived of as logical objects abstracted from systems and their elements, then there is seemingly no point in worrying about their existence. This provides *in re* non-eliminativism with the means to answer MacBride's dilemma. We will raise the question anew when discussing contemporary *ante rem* non-eliminativist and positionalist views in §3.4. However, we notice that even if positions are objects, the structuralist might still be in trouble if the kind of objects involved are, besides their label, essentially nothing but Platonic individuals; on a closer look, in this respect, both the Incompleteness and the Dependence claims are, even when considered apart from one another, key. However, if some account of positions validates some version of Dependence or Incompleteness and, on top of that, it provides us with an otherwise satisfactory and competitive philosophical account of mathematics, then the very theoretical virtues recommending this as an account of mathematics would provide us with as implicit motivation for believing in positions.

3.2.3 Problems of Reference

This section introduces the problem of Singular Reference and the related Semantic objections.

3.2.3.1 The Problem of Singular Reference. Non-eliminativists[126] hold that ordinary mathematical terms perform singular reference to structures and positions therein. For instance, '$\mathbb{Z}$' denotes *the* integer structure while '-1' denotes a particular position in this structure.[127] Consider the structure S once more:



[125] Linnebo [2007, §5].

[126] This problem is routed via semantic relatives of the Permutation and Automorphism problems discussed above. The route following Automorphism can be traced back to Shapiro [2008]; the Permutation route is explored in Assadian [2018, §4] as the Permutation plight.

[127] E.g. Hellman and Shapiro [2019, p. 55]. Remember the central metasemantic motivation Shapiro as well as recent non-eliminativists employ in defending their view (§2.4): unlike eliminativists, the realist holds that she can provide a face value reading of ordinary mathematical discourse, where the value on the face of the latter has it that mathematical terms are true singular terms denoting mathematical objects and structures.

The suggestion is that, in light of Id-Struct, mathematical terms such as 'ℕ' refer to *the* structure satisfying a certain categorical theory or condition.[128] Similarly concerning terms such as '0' in PA^2 aiming to denote mathematical entities, the suggestion is that, they pick out *the* entity uniquely instantiating a certain structural property.[129] So the non-eliminativist says: S is *the* Φ_S structure, for some categorical condition Φ_S, while [a] is *the* $\phi_{[a]}$ position of S, for some structural property defined by $\phi_{[a]}$ and uniquely instantiated by [a].

However, this account faces a problem on two fronts: concerning reference to structures and their contents, as well as to symmetric positions in non-rigid structures. First, concerning reference to structures and their contents,[130] even if structures are unique up to isomorphism – that is, if the Permutation problem is dispelled – it is not sufficient that the non-eliminativist simply asserts that mathematical vocabulary picks out structures and their positions, as opposed to systems and their elements: she also needs to provide us with an *account* explaining *why* structures and their positions are *more eligible* to attract the reference of mathematical terms than do systems and their elements. Lacking such an account, her claims that mathematical vocabulary performs singular reference to structures is hardly more than an ungrounded *fiat*: as far as the non-eliminativist's discourse goes, there is nothing substantial ruling out systems as referents of mathematical terms.

Second, concerning symmetric positions in non-rigid structures,[131] the non-eliminativist owes us an account of reference explaining *how* mathematical terms could perform singular reference to one and only one of the positions from a symmetric pair. Given the account sketched above and looking at the purported

[128] One can phrase it formally as follows (where S is a mathematical term denoting mathematical structures): $S := \iota\Sigma(\Phi_S(\Sigma) \wedge Structure(\Sigma))$, where ι is a term-forming definite description operator picking out the *unique* entity satisfying a certain formula, while Φ_S is a categorical condition characterizing the isomorphism type of S (see §3.2.2.1)

[129] Formally, if [a] is such a term: $[a] := \iota x(\phi_{[a]}(x) \wedge Position(x, S))$, where $\phi_{[a]}$ is formula in the language of S expressing a structural property only possessed by [a] from among the positions of S.

[130] Assadian [2018, §4] presents this route of the problem of singular reference as the "Permutation plight", formulated as an analysis of the replies the non-eliminativist might provide to the Permutation problem as presented above; Assadian's conclusion is that even if the Permutation problem is solved and uniqueness up to isomorphism of structures is granted, the non-eliminativist still owes us as account of reference magnetism explaining why mathematical vocabulary picks out structures and their positions, as opposed to systems and their elements. Other have also raised this objection, e.g. Button and Walsh [2016, p. 288] and the sources quoted there.

[131] Shapiro [2008] considers the problem of singular reference via the Automorphism problem: he endorses utter indiscernibles as a solution to the ontological problem raised by symmetric elements, but recognizes its semantic counterpart discussed in this section as independently problematic, aiming to provide solutions in Shapiro [2008] as well as in Shapiro [2012]. In the end, Shapiro drops a singular singular reference to indiscernibles and holds such terms seemingly referring to indiscernibles are free variables.

structure S, since S is non-rigid and [a] and [b] are symmetric positions thereof, there is no such formula $\phi_{[a]}$ that [a] and only [a] satisfies among the positions of S; any such formula would be as well satisfied by [b], and vice versa. Once again, lacking such an account, non-eliminativist's claims that mathematical terms perform singular reference are hardly more than ungrounded *fiat*s.

Either way, the non-eliminativist is faced with an embarrassment of riches giving rise to indeterminacy of reference which, if left unmitigated, undermines the prospects of the advertised singular reference account for mathematical terms.

3.2.3.2 The Semantic Objections. We present two related albeit mostly independent objections, united in their conclusion that non-eliminativism is ultimately unmotivated. First, we present an objection based on Bahram Assadian [2018, §5]'s recent critique;[132] second, to add fuel to the fire, we twist Richard Pettigrew [2008]'s argument to reach the same conclusion, as we will show, albeit driving in Assadian's opposite direction.

Given the problem of Singular Reference introduced above, a singular reference account for mathematical terms is still pending: Assadian argues that absent such an account, non-eliminativists fail to provide us with *any* distinctly non-eliminativist account of reference whatsoever, dealing severe damage to the metasemantic motivation brought about against eliminativism.

[132] Our presentation of Bahram Assadian [2018]'s arguments requires a caveat explaining both the way we understand their original formulation, as well as the way we distribute their content in this essay. Assadian presents what are meant to be two distinct objections against non-eliminativism – or, better, positionalism, as his footnote 3 points out – albeit the two are meant to complement each other.

The first objection appears in §4 and is labeled the 'permutation plight': starting from the Permutation problem (formulated close to §3.2.2.1 in the present essay), Assadian argues that the structuralist owes us an account of singular reference for the mathematical vocabulary yielding reference to structures rather then systems; absent this, non-eliminativists cannot deliver on their metasemantic promises, i.e. a singular reference account for the mathematical terms. We reformulated the Permutation plight as the problem of singular reference above (§3.2.3.1), largely independent from the Permutation problem; this choice is due to our impression that the problem raised is a general debt of the non-eliminativist, unrelated to Permutation related concerns.

The second objection appears in §5 and is labeled the 'reference plight': starting from the previous conclusion, Assadian identifies three alternative accounts of reference, of which two are pursued and shown that the eliminativist could just as well employ them; as such, Assadian concludes that the non-eliminativist – most likely, modulo the third option, which is left for further work – fails to provide any account of reference which makes up for the metasemantic motivation widely employed to support her view against the eliminativist (as shown in §2.4 in this essay and argued by Assadian in §3). This is the argument we detail in the present section as the first Semantic objection, starting from the problem of singular reference.

We can see how Assadian's objections are meant to supplement each other: the permutation plight conditionally rules out singular reference, while the reference plight aims to rule out the alternatives, undermining the metasemantic motivation. Given its centrality for non-eliminativism, as Assadian [2018, §3] argues, this arguably undermines non-eliminativism altogether.

Non-eliminativists seemingly have three options concerning reference to mathematical entities:

1. Simulated reference (related to §2.2);
2. Arbitrary reference (related to §2.1);
3. Singular reference to arbitrary objects.

The last option is not pursued: in general, an account of structures and positions therein as arbitrary objects is not covered by Assadian's comments.[133]

Simulated reference (1) is specific to free variables, Skolem constants or parameters:[134] such terms are meant as purely distinguishing terms, aiding in talking about the positions of some structure as purely distinct entities; since they could refer to any entity whatsoever, far from genuine, their sort of 'reference' is a simulation only preserving distinguishability. Simulated reference has been already employed by eliminativists, as Pettigrew suggests above. In this sense, besides its failure to be genuine reference, it fails to to be a distinctive non-eliminativist semantic account of ordinary mathematical discourse. *A fortiori*, it fails to provide any motivation for commitment to structures. Arbitrary reference (2) has also been employed by non-eliminativists, invoking a Carnapian structural relativist account (see §2.1). As such, just like the previous option, arbitrary reference fails to be a distinctly non-eliminativist account of reference.[135]

Assadian concludes that, modulo a successful account of non-eliminativist singular reference to structures as arbitrary objects (3) or otherwise, the non-eliminativist has nothing semantic to weigh in her favour against her eliminativist contender: that is, the metasemantic motivation for committing to structures and endorsing her view is entirely voided.[136]

[133] Attempting to fill in this gap is a task for further work. Employing Kit Fine and Tennant [1983]'s theory of arbitrary objects, Leon Horsten [2019]'s has recently formulated an account of structures as arbitrary objects (Generic Structuralism). Unfortunately, this view won't be discussed in the essay at hand.

[134] See Pettigrew [2008]'s suggestion in §2.2 for an instance, where such terms are called 'dedicated free variables'; Shapiro [2012] calls them 'parameters', while Ketland [2015] calls them 'Skolem constants'. Assadian calls this type of reference 'Skolemite reference'.

[135] Both Assadian [2018, p. 3212] and Pettigrew [2008, p. 320] raise a further objection against this view, holding that it would bring about semantic facts which do not supervene on use, broadly construed. Both authors find this *unacceptable*. The thesis that there are irreducible or fundamental semantic facts is labeled 'Semantic Sovereignty' by Kearns and Magidor [2012] and minutely defended. We point out that there are non-eliminativists who don't trouble over the failure of Semantic Sovereignty, for instance Hannes Leitgeb [forthcoming,b].

[136] Linnebo and Pettigrew [2014, p. 277] suggest a similar objection highlighting the centrality of the metasemantic benefits for endorsing structures as entities over and above systems. One might complain however that both Assadian's objection, as well as the one we derive from Pettigrew's argument in what follows, rely on some kind of semantic-concerns-aside sort of priority

Apart from showing that non-eliminativism cannot pay its debts, a similar conclusion can be reached by showing that eliminativism can do better than expected. Richard Pettigrew [2008] employs the palliative dedicated free variables strategy applied to the universalist interpretation of mathematical discourse (see §2.2) to conclude that "whatever philosophical reasons there are for rejecting the [non-eliminativist] interpretation, the [eliminativist] need not appeal to them to motivate her position" and, as such, "[h]er position begins the battle for philosophical plausibility with the same virtues as the [non-eliminativist] account."[137] Pettigrew's conclusion can be understood to provide additional fuel to the non-eliminativist's motivational collapse: given the centrality of the metasemantic motivation (see §2.4), such conclusions can be leveraged against non-eliminativism following Assadian's strategy.

Pettigrew [2008]'s argument runs as follows. As shown in §2.2, the eliminativist can construe ordinary mathematical discourse close to its surface grammatical form, by rendering mathematical terms as dedicated free variables. Non-eliminativists might nonetheless insist that mathematical terms are singular terms, not free variables, and that eliminativist construal is still *mis*construal. However, Pettigrew shows that there is no uniform agreement among mathematicians as to whether mathematical terms are proper names or rather free variables. Furthermore, Pettigrew argues that there is no syntactic test distinguishing between singular terms and free variables; moreover, any semantic

of eliminativism over non-eliminativism. This is true in a chronological order: eliminativist options came to the philosophical structuralist table earlier than their non-eliminativist counterparts. This is part of the reason why non-eliminativism has been marketed as a better version of philosophical structuralism, chiefly on metasemantic grounds. As such, eliminativism might have the flavour that it is the default option to fall back on whenever there is nothing weighing in another direction. However, temporal priority is hardly a valid ground in such matters. The objector might then focus on another aspect: eliminativism is ontologically lighter than non-eliminativism, the former, unlike the later, not bringing about an additional exotic layer of structures as entities over and above their systems. In this case the eliminativist would argue that Ockham's razor suggests erasing the structures and make do with systems, or nothing at all. This is how we interpret Assadian's thought on the matter.

[137] Pettigrew [2008, p. 330]; we replace Pettigrew's 'aristotelian' and 'platonist' terminology with 'eliminativist' and 'non-eliminativist', respectively. Right before these, Pettigrew states:

> *In conclusion, I submit that the discourse of arithmetic provides no evidence that tells in favour of the [non-eliminativist] interpretation of that discourse and against the [eliminativist] interpretation. Equally, there are no considerations that tell in favour of the [eliminativist] interpretation and against the [non-eliminativist]. On the question of whether there are metaphysical or epistemological considerations that favour one over the other, I will say nothing.* (Pettigrew [2008, p. 330])

As such, we don't mean to imply that Pettigrew himself endorses our argument concerning the overall worth or motivation of endorsing non-eliminativism. Anything to this effect is our own twist of his argument. Pettigrew's contemporary views on the matter – set-theoretic eliminativism coupled with a nominalist approach to sets – may be found in Pettigrew [2018].

test which could distinguish the two categories is essentially question begging.[138] We push close inspection of these arguments to further work: our claim is that Pettigrew's point can be employed towards Assadian's conclusion, only if these arguments are substantially correct.

On the one hand, Assadian shows that non-eliminativists cannot live up to their standards and cannot deliver on their promises. On the other hand, Pettigrew's argument can be employed to pull things in the opposite direction: in light of the indeterminacy of ordinary mathematical discourse, eliminativists can after all stand up reasonable semantic demands. Either way, lowering non-eliminativists or raising their counterparts, leads us to the same likely conclusion: the non-eliminativist faces motivational collapse, undermining justified belief in the existence of structures.

3.2.4 Summing Up

Figure 2 should provide the reader with a clear picture of the problems and objections discussed above, alongside the main relations holdintg among them:

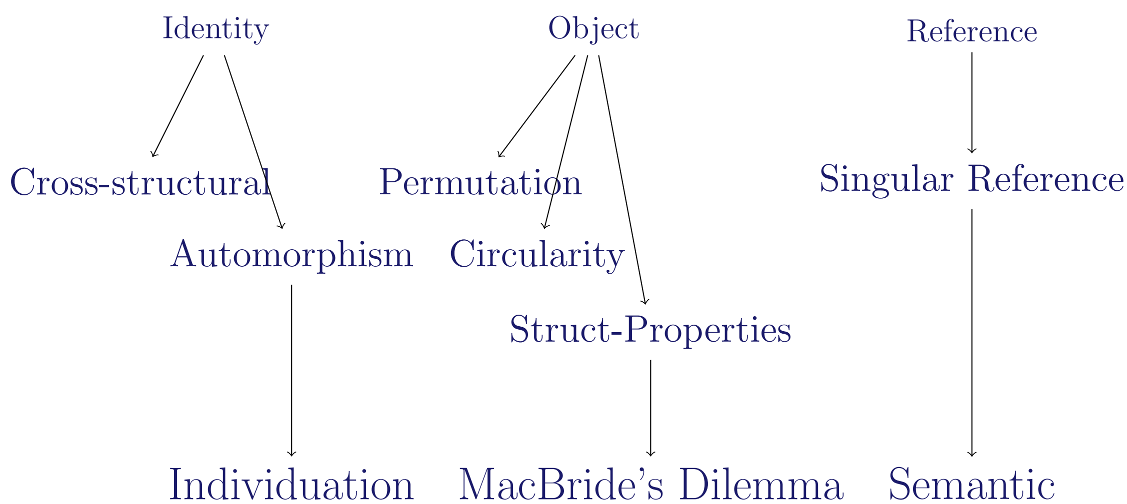


Figure 2: A canon of non-eliminativist concerns

The top row items represent the three clusters of problems, the items in the middle mention the problems in each cluster, while the bottom row names the three corresponding objections. Of course, as seen above, their relations are a tiny bit more complex than represented here: Automorphism plays a role in MacBride's dilemma, Permutation and Automorphism are closely related to two different branches of the Singular Reference problem and, as such, to the Semantic objection, while the Semantic objection itself is a composite of two, mostly separate arguments. However,

[138] Pettigrew [2008, §3].

we'll keep the picture clean for better comprehension. With the sole exception of the Reference concerns (§3.2.3), all canonical items crucially involve positions in structures conceived of as *bona fide* objects. As Bahram [Assadian, 2016, §6.2] argues with respect to several of the canonical items, we can see that whenever we invoke Identity or Objects's problems, instead of 'non-eliminativism' we should rather mention positionalism, the view that structures are endowed with a domain of positions (§1.1): positionalism, it appears, is the true root of most non-eliminativist evil. We take a look at an alternative in the following section.

3.3 A Detour Through Non-Positionalism

Following the train of thought concluding the previous section, a non-eliminativist might find it sensible to hold onto realism with respect to mathematical structures, while endorsing an eliminativist approach to positions therein. This is *non-positionalism*.[139]

The core idea is to provide an understanding of structures satisfying the following desiderata:

I. Structures are not eliminated;

II. Structures have no domain of positions;

III. Structures are what isomorphic systems have in common.

That is, the aim is to formulate a non-eliminativist version of structuralism (I), which is non-positionalist (II) and accounts for the methodological structuralist intuition that structures are what isomorphic systems have in common (III). The last item is captured by a (weaker) version of what we called Id-Struct in §3.1:

(Convergence) $[X] = [Y] \Leftrightarrow X \cong Y$[140]

Ketland [2015, §3] mentions four accounts of structures which arguably satisfy these conditions:

1. Structures are equivalence classes of isomorphic systems (or otherwise the property of being isomorphic to a given system);

2. Category-theoretic structures;

3. Structures are logical objects governed by a primitive abstraction principle (Convergence);

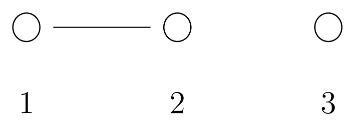
[139]In our terminology above, the non-positionalist implicitly holds that structures are not systems. Daniel Isaacson [2011] and Jeffrey Ketland [2015] have both articulated versions of non-positionalism. We are sketching Ketland [2015]'s suggestion in what follows.

[140]Following the notation used in §3.1: X and Y are systems, [X] and [Y] are their respective structures. Ketland [2015, §3] calls it Leibniz Abstraction.

4. Structures are categorical second-order propositional functions expressed by the a second-order categorical diagram formula.

Drawing on Carnapian ideas,[141] Ketland articulates the latter option. We consider a toy, finite case and afterwards mention the *infinite* intricacies. Consider the following labeled graph:

X:



We can characterize X up to isomorphism by a purely second-order logical formula Φ_X: for any model $\mathcal{M}$, $\mathcal{M}$ satisfies Φ_X if and only if $\mathcal{M}$ is isomorphic to X.[142] Although the third desiderata is thus in sight, the non-positionalist cannot quite identify the structure of X with Φ_X: since these are linguistic entities, doing so would allow for isomorphic systems Y whose Φ_Y, although equivalent to Φ_X, is a *notational variant* of Φ_X and thus distinct. In general, only this much would be granted:

$$\text{X} \cong \text{Y} \Leftrightarrow \Phi_X \equiv \Phi_Y$$

Φ characterizations of isomorphic systems are logically equivalent, albeit not necessarily identical. However, the propositional function expressed by Φ_X and Φ_Y is the same, that is, an entity to the effect that there are precisely three elements and only two distinct ones are related to one another. If we express such propositional functions as $\langle\Phi_X\rangle$ and $\langle\Phi_Y\rangle$, then the following is a plausible assumption concerning the relations holding between formulas and the propositional functions expressed:[143]

[141] We sketched some of these ideas in §2.1 when discussing relativism.

[142] Describe X in model theoretic terms as a σ-model $\text{X} = \langle D = \{1, 2, 3\}, \sigma = \{\text{‘1’,‘2’,‘3’}\}, I\rangle$, where D is X's domain, σ is its signature and I is an interpretation function. In general, in case σ doesn't contain a name for each element in D, then we can extend σ to σ^+ containing a constant for each element in D, and adapt I accordingly; we would then consider X as a σ^+-model. A *literal* is a closed σ-formula or the negation of such a formula; let Lit^σ be the collection of literals. The *diagram formula* of X is then is then $\text{diag}_X = \bigwedge\{\varphi \in \text{Lit}^\sigma | \text{X} \models \varphi\}$; since σ is finite, then Lit^σ is also finite (up to logical equivalence) and so diag_X is a finite σ-formula. It is a model theoretic result that $\mathcal{M} \models \text{diag}_\mathcal{N}$ if and only if there is an embedding $h : \mathcal{N} \to \mathcal{M}$. Furthermore, let $\text{dom}_X = \forall y \bigvee\{y = n | n \in D\}$, where n is a constant in σ such that $I(n) = n \in D$; again, since D is finite, dom_X is finite itself. Then $\phi_X = \text{diag}_X \wedge \text{dom}_X$ characterizes a surjective embedding (i.e. an isomorphism) and thus it is a formula such that for all σ-models $\mathcal{M}$, $\mathcal{M} \models \Phi_X$ if and only if $\mathcal{M} \cong X$. In general, the formula $\Phi_X = \exists x_1, x_2, x_3 \phi_X[x_1/1.x_2/2.x_3/3]$ is a purely second-order logical formula categorically characterizing X: for any model $\mathcal{M}$, $\mathcal{M} \models \Phi_X$ if and only if $\mathcal{M} \cong X$. However, in the case of an infinite model $\mathcal{M}$, $\text{diag}_\mathcal{M}$ and $\text{dom}_\mathcal{M}$ and thus $\Phi_\mathcal{M}$ may be infinite and thus we need to appeal to infinitary logics.

[143] Ketland [2015, p. 30].

$$\Phi_X \equiv \Phi_Y \Leftrightarrow \langle \Phi_X \rangle = \langle \Phi_Y \rangle$$

Plausibly, the propositional functions expressed by logically equivalent formulas are identical. Putting these together, we conclude that: $\langle \Phi_X \rangle = \langle \Phi_Y \rangle \Leftrightarrow \mathrm{X} \cong \mathrm{Y}$. The propositional functions expressed by formulas categorically characterizing isomorphic systems are identical. Ketland's suggestion is then to identify the structure of a system with the propositional function expressed by its diagram formula: $[X] := \langle \Phi_X \rangle$.[144] As such, Convergence holds as well: $[\mathrm{X}] = [\mathrm{Y}] \equiv \mathrm{X} \cong \mathrm{Y}$, satisfying the third desiderata. In this context, instantiation is understood as follows: X instantiates [A] if and only if $\mathrm{X} \models \Phi_A$.

From the viewpoint of someone who was already prepared to assent to mathematical structures, there is seemingly no obstacle in believing in such propositional functions and, therefore, the account may be held non-eliminativist. Furthermore, there is no domain of positions in [X] and therefore the account is non-positionalist, satisfying the second desiderata and avoiding the trouble. However, the non-eliminativist still has to provide a semantic account of ordinary mathematical terms: Ketland [2015, §7.2] suggests to understand these as *Skolem constants*, i.e. freshly added constants which substitute the quantified variables in Φ_X.[145]

The case of infinite structures brings about the intricacies mentioned above. The underlying idea is the same as in the finite case; however, there are systems whose characterizing formula Φ is infinite and, as such, appeal to infinitary logics (i.e. logics allowing for infinite formulas) is required. Ketland carries out a construction of diagram formulas in both finite and infinite cases in the fifth section.

Some might complain, for instance, that natural numbers should be elements or positions in the natural natural number structure: what else could they be? However, Ketland holds that natural numbers are *sui generis* entities characterized by Frege's Hume's Principle: $\mathrm{N}[S] = \mathrm{N}[P] \equiv S \sim P$, that is, the number of S's is the same as the number of P's if and only if there is a one-to-one correspondence between S and P. The collection of natural numbers would then be $\mathbb{N} := \{x | \exists X : x = \mathrm{N}[X]\}$; the natural numbers with the natural order, $\langle \mathbb{N}, < \rangle$ (alongside any set theoretic reduction thereof) would then instantiate the *natural number structure*, $\langle \Phi_{(\mathbb{N},<)} \rangle$. However, the natural numbers are not the natural number structure, nor positions therein: the former, unlike the latter, suffer from the permutation objection, whereas the latter, unlike the former, is a propositional function without any domain of distinguished elements, avoiding thus many of the pitfalls discussed in the previous section.

Regardless of its virtues, non-positionalism doesn't provide us with a unified

[144] This is likely carrying commitment to a version of *in re* non-eliminativism.

[145] The name is reminiscent of Skolemization in model theory: given a consistent theory, for every formula $\varphi(x)$ in its language, add a fresh constant c to its vocabulary such that $\exists x \varphi(x) \rightarrow \varphi(c)$ is a theorem. This is essentially similar to Pettigrew's strategy in §2.2.

account of ordinary mathematical practices, from everyday counting to theoretical endeavours: the former would be conceived in terms of cardinalities and natural numbers, while the latter is the study of certain propositional functions only related to (say) natural numbers in that the progression of numbers instantiates it, as do many other systems. If possible, we would better avoid such *discontinuities*. In general, this means that non-positionalism falls prey to the a particularly vicious form of the Singular Reference problem: since there are no positions in structures, the non-positionalist cannot provide us with a singular reference account for mathematical terms. In turn, this exposes non-positionalism to both fronts of the Semantic objection: it is not only the threat that the metasemantic motivation (§2.4) might not constitute reason enough for endorsing structures, but also that the metasemantic motivation itself it utterly absent. We are thus driven to consider contemporary positionalist reactions to the problems and objections mentioned in the previous section.

3.4 Positionalism

After introducing mathematical structuralism and discussing early versions of philosophical structuralism, notably eliminativist, we introduced Shapiro [1997]'s early articulation of non-eliminativism and considered extended criticisms against it. Noticing that many of these criticisms concern positions in structures, we briefly mentioned several versions of non-positional non-eliminativism and reviewed Ketland [2015]'s own articulation of it. However, several notable versions of positionalist non-eliminativism aim to solve the problems raised by positions in structures rather than avoid them altogether by doing away with positions to start with. These views articulate distinct notions of structural objects which aim to dispel the worries raised against SGS. We are now going to review three of the most influential such views in the literature: Fregean Abstractionist Structuralism (FAS), Object Theoretic Structuralism (OTS) and Unlabeled Graph-theoretic Structuralism (UGS).

3.4.1 Fregean Abstractionist Structuralism

This section introduces Øystein Linnebo and Richard Pettigrew's original *Fregean Abstractionist Structuralism* (FAS),[146] alongside Georg Schiemer and John Wigglesworth's updated version thereof.[147] Further on, we asses FAS against the canonical objections listed in §3.2 and conclude with a summary of FAS' performance against these.[148]

[146] Linnebo and Pettigrew [2014]. For an axiomatic account of mathematical structures based on Fregean abstraction principles which will not be discussed in this essay, see Leach-Krouse [2017].

[147] Schiemer and Wigglesworth [2017].

[148] The following remark in by Richard Dedekind seem to point in the Abstractionist direction:

3.4.1.1 Theory. Linnebo and Pettigrew provide a characterization of non-eliminativism resulting in three core theses:

> **(Instantiation)** Systems are isomorphic to their structures;
>
> **(Purity)** All fundamental properties of positions in structures are structural properties;
>
> **(Uniqueness)** [X] is unique in satisfying Instantiation and Purity. (Linnebo and Pettigrew [2014, p. 273])

Instantiation "is essential in order to give the semantics for mathematical language that the non-eliminative structuralist proposes."[149] Purity is a restricted form of the Non-Structural-Incompleteness claim (discussed at the end of §3.2.2) and it is a central tenant of the non-eliminative structuralist understanding of mathematical objects. Uniqueness is, alongside Instantiation, crucial for realizing the metasemantic motivation presented above (§2.4). These are in turn taken as desiderata of the Fregean Abstractionist account of structure: the aim is to show that they are substantial consequences of FAS.

Abstraction principles were introduced by Gottlob Frege and revived later on by neo-logicists. The idea is that of a relation (usually an equivalence)[150] on a given domain of objects abstracting away from these a new sort thereof corresponding to classes of appropriately related objects from the domain. For instance, neo-logicists hold that natural numbers are governed by Hume's Principle (HP):

> **(HP)** $N[P] = N[S] \Leftrightarrow P \sim S$

'N' is an abstraction operator on properties, 'S' and 'P' express properties and '$\sim$' is one-to-one correspondence between S and P; 'N[P]' should be read as 'the number of Ps'. HP introduces numbers into the discourse by providing their identity conditions and governing terms denoting them; to give an idea of the power of abstraction principles, PA^2 can be derived from HP in second order logic.[151]

> *If in the consideration of a simply infinite system N set in order by a transformation ϕ we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation ϕ, then are these elements called natural numbers or ordinal numbers or simply numbers, and the base-element 1 is called the base-number of the number series N.* (Dedekind and Behman [trans, Definition 73])

We point out that Linnebo and Pettigrew distance themselves from Dedekind Abstractionism.

[149] Linnebo and Pettigrew [2014, p. 272].

[150] This condition can be relaxed under some conditions; see Payne [2013].

[151] Abstraction principles are plagued by the so called Bad Company Objection: presumably coherent abstraction principles cannot be well delineated from inconsistent ones, such as Frege's infamous Basic Law V. Some authors suggest to tackle this problem by only allowing for predicative

According to FAS, structures are governed by the following Structure Abstraction (SA) principle:[152]

(SA) $[\mathrm{X}] = [\mathrm{Y}] \Leftrightarrow \mathrm{X} \cong \mathrm{Y}$

The structure of X (denoted by [X]) is the same as the structure of Y if and only if X is isomorphic to Y, where X and Y are systems and where $[\cdot]$ is an abstraction operator on systems yielding their structures, *sui generis* mathematical objects which correspond to the systems' isomorphism types. In order to avoid the Burali-Forti paradox,[153] systems are required to be set-sized. Furthermore, positions in structures are governed by Position Abstraction (PoA):

(PoA) $[x]_{\mathrm{X}} = [x']_{\mathrm{X}'} \Leftrightarrow \exists f(f : \mathrm{X} \cong \mathrm{X}' \wedge f(x) = x')$

The position corresponding to an element x from a system X (denoted by $[x]_{\mathrm{X}}$) is the same as the position corresponding to an element x' from a system X′ (denoted by $[x']_{\mathrm{X}'}$) if and only if there is an isomorphism f between X and X′ which takes x to x'. For instance, looking at the Zermelo and von Neumann ordinals with their respective orders, $\mathrm{Z} = \langle Z, <_Z \rangle$ and $\omega = \langle \omega, <_\omega \rangle$, Structure Abstraction yields that $[\mathrm{Z}] = [\omega]$; looking at their third elements according to their respective orders, Position Abstraction yields that $[\{\{\varnothing\}\}]_{\mathrm{Z}} = [\{\varnothing, \{\varnothing\}\}]_\omega$, since $f : \{x\} \mapsto \{x, \{x\}\}$ is an isomorphism between Z and ω which relates these elements appropriately.[154] Given a system $\mathrm{X} = \langle D, R_1, ..., R_n \rangle$, in light of PoA we can explicitly characterize [X] as a positional entity. The domain of [X] can be given as fallows:

(Domains of Structures) $[D]_{\mathrm{X}} = \{[x]_{\mathrm{X}} | x \in D\}$.

The domain of [X] contains all and only positions corresponding to elements of X. Furthermore, relations on $[D]_{\mathrm{X}}$ can be defined similarly. Given X and an k-ary relation R on D:

(Relations on Structures) $[R]_{\mathrm{X}}(x_1, ..., x_k)$ if and only if there are elements $u_1, ..., u_k$ in D such that for each i, $[u_i]_{\mathrm{X}} = x_i$ and $R(u_1, ..., u_k)$.

Positions are $[R]_{\mathrm{X}}$-related in [X] if and only if there are corresponding elements which are R-related in X; in the monadic case, a position $[x]_{\mathrm{X}}$ has property $[P]_{\mathrm{X}}$ if an only if $P(x)$ obtains in X. These yield a positional characterization of [X]:

abstraction principles, while recovering their strength through a dynamic approach; see Studd [2016] and Linnebo [2018]. Schiemer and Wigglesworth [2017] will employ dynamic abstraction in their amendments to FAS.

[152] Linnebo and Pettigrew [2014, p. 274]. We labeled this principle 'Convergence' in §3.3. However, the status of SA in the present context is radically different than that of Convergence in Ketland [2015].

[153] Hazen [1985, p. 253-254] points out this peril.

[154] With $f(\varnothing) = \varnothing$.

(Positional Structure) [X] = $\langle\, [D]_{\text{X}}, [R_1]_{\text{X}}, ..., [R_n]_{\text{X}} \,\rangle$

Finally, with an eye on the Purity thesis, given a system X and a relation R on [X]'s positions, *fundamental relations* are defined as follows:

(Fundamental Relations) R is a fundamental relation on [X] if and only if there is a relation R' on the elements of X such that R = $[R']_{\text{X}}$.

Fundamental relations on structures are those abstracted from a relation on some underlying system. In light of the above principles and definitions, Instantiation and Purity should be formally recaptured as follows:

(Instantiation) [X] ≅ X;

(Purity) $[R]_{\text{X}}([x_1]_{\text{X}}, ..., [x_n]_{\text{X}}) \Leftrightarrow R(x_1, ..., x_n)$, where $x_1, ..., x_n$ belong to the domain of X.[155]

We provisionally make the following two substantial assumptions:

(i) Every structure is the result of abstraction on some system (*in re* structuralism);

(ii) The systems on which abstraction is operated are all rigid.

Linnebo and Pettigrew argue that, under these assumptions, FAS satisfies all three theses: **Propositions 5.1** and **5.2**[156] state that Instantiation and Purity, respectively, are satisfied, while Uniqueness follows by Instantiation given SA.[157]

This is all neat looking, but Schiemer and Wigglesworth [2017] show that Purity fails even under the restrictive assumptions (i) and (ii) if relations on structures are *extensional*, that is, if relations are conceived of as fully characterized by their extensions. There are two related problems in this case. First, strictly speaking, relations on structures are *not structural relations*. Remember that we characterized (§1.1) structural relations as isomorphism invariant relations; however, Relations on Structures only yields relations obtaining on the domains of structures, and not on the domains of systems as well. Since structures are isomorphic to their systems, then relations on structures are not structural relations.[158]

[155] This formulation of Purity captures the original meaning only roughly: notice that it is not the structural property itself that holds in systems isomorphic to the structure, but a correspondent of it in the system. See the first of Schiemer and Wigglesworth's objections below.

[156] Linnebo and Pettigrew [2014, p. 276-7].

[157] If (i) fails, then Uniqueness doesn't fallow from FAS. If (ii) fails, then Instantiation is flouted: structures abstracted from non-rigid systems are not in general isomorphic to their systems. Given (ii) and in light of the seeming centrality of non-rigid structures in mathematics, Linnebo and Pettigrew eventually abandon FAS. John Wigglesworth [2018a] provides a reply to this objection, discussed below when engaging with the Automorphism problem. An alternative route was taken by Bahram Assadian [2016, §6.4], who presents non-positionalist Abstractionism.

[158] Schiemer and Wigglesworth [2017, p. 10].

Second, Schiemer and Wigglesworth show that under an *extensional* account of relations, Linnebo and Pettigrew's restricted version of Purity is plagued by the same problem of Structural Properties that it was originally meant to avoid by restricting the Incompleteness claim to fundamental properties. This is so because on an extensional account it follows that every relation on structures is fundamental,[159] entailing that positions in structures have no non-structural relations; however, as the problem of Structural Properties highlights, positions in structures do have non-structural relations.

Consequently, Schiemer and Wigglesworth amend FAS with an *intensional* account of relations carried out in the framework of Kripke models and resulting in a dynamic and predicative version of abstraction which allows for a natural alternative definition of fundamental relations avoiding the previous pitfalls and proving the restricted version of the Purity thesis. *In nuce*, the main idea is to conceive of systems as states in a Kripke model endowed with an accessibility relation relating all and only isomorphic systems; in this setting, relations are intensional: a relation is a function which given a system/state as input, it yields as output a class of appropriately sized collections (corresponding to the arity of the function) of entities from the domain of the input state; crucially, relations have local domains, and locally extensionally equivalent relations are not identical. Against this background, abstraction principles are conceived of as predicative and dynamic operators on Kripke models, giving rise to extensions of the original models containing in addition the positional structures corresponding to each isomorphism type represented by some system included in the model.

Systems are characterized as before. Variable domain Kripke (VDK) models are defined as follows:

(VDK model) $\mathcal{M} = \langle D, W, \sim, \nu \rangle$ is a variable domain Kripke model if and only if:

- D is a domain of objects;
- W is a collection of systems with domains subsets D;
- $\sim$ is an accessibility relation on W;
- For $w \in W$, $\nu(w) = D_w \subseteq D$.

[159] Schiemer and Wigglesworth [2017, p. 11]. The argument runs as follows: Let R be a relation on a structure [X]; this is without loss of generality since we consider an *in re* version of structuralism, by assumption (ii). By Instantiation, there is an isomorphism $f : [\mathrm{X}] \to \mathrm{X}$; f induces a relation Q on X such that $Q(x_1, ..., x_n)$ if and only if $\mathrm{R}([x_1]_\mathrm{X}, ..., [x_n]_\mathrm{X})$. By Relations on Structures, $[Q]_\mathrm{X}$ is a relation on [X] such that $[Q]_\mathrm{X}([x_1]_\mathrm{X}, ..., [x_n]_\mathrm{X})$ if and only if $Q(x_1, ..., x_n)$. Notice that it follows that $\mathrm{R}([x_1]_\mathrm{X}, ..., [x_n]_\mathrm{X})$ if and only if $[Q]_\mathrm{X}([x_1]_\mathrm{X}, ..., [x_n]_\mathrm{X})$. By Fundamental Relations, $[Q]_\mathrm{X}$ is a fundamental relation on [X]. However, if relations on structures are *extensional*, then $\mathrm{R} = [Q]_\mathrm{X}$. Hence R is a fundamental relation on [X]; generalizing, all relations on structures are fundamental and thus all relations of positions in structures are structural relations.

Such models have a domain of objects (D) and a domain of states which in our case are systems (W) such that, as expected, each w in W has itself a local domain which is a subset of D; intuitively, all sets will be in D and some local domain would be D_ω, i.e. the von Neumann ordinals, while ω itself would be an element in W. We will say more about the accessibility relation after introducing intensional relations on such models:

> **(Intensional Relation)** Given a VDK model $\mathcal{M} = \langle D, W, \sim, \nu \rangle$, an n-ary intensional relation R is a function $R : W \to \mathcal{P}(D^n)$ (where D^n is the collection of all n-tuples of elements from D).

The core idea behind intensional relations is that extensional equivalence is not sufficient for identity in the case of relations. This is precisely captured by the above definition, from which it follows that if R and Q are intensional relations, then $R = Q \Leftrightarrow \forall w \in W(R_w = Q_w)$, where $R_w = R(w)$ is the local extension of R at w. This as the identity criterion for intensional relations. A classical example of extensionally equivalent albeit intensionally distinct relations is provided by the relations 'creature with kidney' and 'creature with heart': it is true that, in actuality, any creature having one also possesses the other, but the properties are seemingly nonetheless distinct. Intuitively, intensional relations can avoid the argument leading from an arbitrary relation on structures to the conclusion that it is a fundamental relation: relations having the same local extension are not thereby the same relation, since their local extensions in another system might be distinct. Furthermore, intensional relations on structures can strictly speaking be structural relations, since they have local domains on the elements of the isomorphic systems. These two features of intensional relations show that they afford an answer to Schiemer and Wigglesworth's above objections to Linnebo and Pettigrew's abstractionist account. Given a relational language $\mathcal{L}$, we extend the definition above to $\mathcal{L}$-models:

> **(VDK $\mathcal{L}$-model)** $\mathcal{M} = \langle D, W, \sim, \nu \rangle$ is a VDK $\mathcal{L}$-model if and only if:
>
> - $\mathcal{M}$ is a VDK model;
> - for all 'R' $\in \mathcal{L}$: ν('R') is an intensional relation R on $\mathcal{M}$;
> - For $w, v \in W$: $w \sim v \Leftrightarrow w \cong v$.

In the definition of a VDK $\mathcal{L}$-model, isomorphism between systems, $w \cong v$, is defined as follows:

> **(Systems' Isomorphism)** For all $w, v \in W$, $w \cong v$ if an only if there is a bijection $f : D_w \to D_v$ such that for all n-ary intensional relations R and for all $x_1, ..., x_n \in D_w$: $R_w(x_1, ..., x_n) \Leftrightarrow R_v(f(x_1), ..., f(x_n))$.

Further on, an extension of a VDK $\mathcal{L}$-model is defined as usual:

(VDK $\mathcal{L}$-model extension) Let $\mathcal{M} = \langle D, W, \sim, \nu \rangle$ and $\mathcal{M}' = (D', W', \sim', \nu')$ be variable domain $\mathcal{L}$-Kripke models. $\mathcal{M}'$ is an extension of $\mathcal{M}$ if and only if $D \subseteq D'$, $W \subseteq W'$, $\sim \subseteq \sim'$ and $\nu \subseteq \nu'$.

Abstraction principles are now introduced as abstraction operators on the systems (W) or the objects (D) of a VDK $\mathcal{L}$-model $\mathcal{M}$. Structure abstraction is then expresses as follows:

(SA$_i$) Given a VDK $\mathcal{L}$-model $\mathcal{M} = \langle D, W, \sim, \nu \rangle$, $[\cdot]$: $W \to W_{\mathrm{S}}$ is a structure abstraction operator such that for all $w, v \in W$: $[w] = [v] \Leftrightarrow w \sim v$. Then $W_{\mathrm{S}} = \{[w] | w \in W\}$, with $W \cap W_{\mathrm{S}} = \varnothing$.

SA$_i$ is a predicative abstraction principle:[160] none of the $[w]$'s are in W itself. Given a system w in W, $[\cdot]$ introduces a fresh entity which corresponds to w's isomorphism type; all these newly abstracted objects are collected in W_S and, as a consequence of their mode of introduction, their identity matches isomorphism. Similarly, abstraction for positions is defined as follows:

(PoA$_i$) Given a VDK $\mathcal{L}$-model $\mathcal{M} = \langle D, W, \sim, \nu \rangle$, $[\cdot]$: $D \to D_{\mathrm{P}}$ is a position abstraction operator such that for all $w, v \in W$, for all $a \in D_w$ and $b \in D_v$: $[a]_w = [b]_v \Leftrightarrow \exists f : w \cong v \wedge f(a) = b$. Then $D_{\mathrm{P}} = \{ [a]_w | a \in D_w, w \in W\}$, with $D \cap D_{\mathrm{P}} = \varnothing$.[161]

As in the case of structures, PoA$_i$ is a predicative abstraction principle itself. We are now aiming to extend the original VDK $\mathcal{L}$-model $\mathcal{M}$ to an extended model containing W_S and D_P in its domains; but first we have to define the local domain of relations on the new abstracted domains:

(Intensional Relations on Structures) Given a VDK $\mathcal{L}$-model $\mathcal{M} = \langle D, W, \sim, \nu \rangle$ and an intensional relation R on $\mathcal{M}$, the expansion of R to W_S is a function R^* such that:

1. For all $w \in W$, $R^*_w = R_w$;
2. For all $u \in W_{\mathrm{S}}$, for all $d_1, ..., d_n \in D_u$: $(d_1, ..., d_n) \in R^*_u$ if and only if there are $v \in W$ and $b_1, ..., b_n \in D_v$ such that

[160] This is important since predicative abstraction principles are 'good': none of them is subject to the sort of issues plaguing some non-predicative principles. In this sense, predicative abstraction principles are a way of answering the Bad Company Objection. The reverse of the medal is that predicative principles are in many cases too weak to yield rich mathematical domains (e.g. the set theoretic universe). However, dynamic abstraction can make up for the lack in deductive power. See Linnebo [2018]. In what follows, Schiemer and Wigglesworth themselves employ dynamic abstraction.

[161] We hold onto the structure notation introduced before; we trust that no confusion will emerge. For instance, it is clear that the abstraction operator for positions is distinct than the abstraction operator for structures, defined in SA$_i$.

(a) $d_i = [b_i]_v$, for all $i \leqslant n$;
(b) $(b_1, ..., b_n) \in R_v$.

This follows Linnebo and Pettigrew's definition introduced above. We can now proceed to define the model extensions induced by the abstraction operators for structures and positions:

(Abstract Extension) Let $\mathcal{M} = \langle D, W, \sim, \nu \rangle$ be a VDK $\mathcal{L}$-model. $\mathcal{M}^* = \langle D^*, W^*, \sim^*, \nu^* \rangle$ is the abstract extension of $\mathcal{M}$, where:

- $D^* = D \cup D_{\mathrm{P}}$;
- $W^* = W \cup W_{\mathrm{S}}$;
- for all $w, v \in W^*$: $w \sim^* v \equiv (w \sim v \vee w = [v] \vee [w] = v)$;
- Concerning ν^*:
  - for $w \in W$: $\nu^*(w) = \nu(w)$;
  - for $w \in W_{\mathrm{S}}$: $\nu^*(w) = \{[a]_v | a \in D_v, v \in W, v \sim^* w\}$;
  - for 'R' $\in \mathcal{L}$: ν('R') $= R^*$.

This concludes the construction of $\mathcal{M}^*$, a model containing all systems as well as their corresponding structures abstracted from these systems along with positions in these structures. Having supposed that all systems in W are rigid (assumption (ii) above), it can be easily shown that the Instantiation thesis holds.[162] Moreover, we notice that in this framework all structures are the result of abstraction on some system.[163] Uniqueness can then be also shown to obtain: if $u, v \in W_S$ and $[w] = u \cong v = [w']$, then by Instantiation $w \cong u \cong v \cong w'$ and so $w \cong w'$ and hence by SA_i it fallows that $u = [w] = [w'] = v$, concluding that $u = v$. To verify that the Purity thesis also holds, we need to just a little more defining. Structural relations, can be precisely recaptured in this framework as follows:

(Structural Relations) A property P is a structural property of $[w]$ if and only if for all systems $v \in W$ and all isomorphisms $f : [w] \cong v$, if $a \in P_{[w]}$ then $f(a) \in P_v$.[164]

[162] We notice that each $u \in W_S$ can be provided with a positional characterization: $u = \langle \nu(u), R_1^*(u), ..., R_n^*(u) \rangle$. Assuming that all systems are rigid, it is then routine to show that if $u = [w]$, then $u \cong w$.

[163] Corresponding to assumption (i) above, we suppose that W only contains systems and no structures. We can also notice that in this framework structures are not systems, against Shapiro's view. Assadian [2016, §???] suggests this feature as another way to characterize the distinction between *in re* and *ante rem* non-eliminativism: structures are systems on an *ante rem* conception of structure, whereas on an *in re* conception they are not.

[164] This can be easily generalized to n-ary relations.

Structural relations are, once again, isomorphism invariant relations. We still need to formulate the crucial definition of fundamental relation in this framework. First, Schiemer and Wigglesworth introduce the notion of definable relation:

> **(Definable relation)** Let $\mathcal{M} = \langle D, W, \sim, \nu \rangle$ be a VDK $\mathcal{L}$-model, and let X= $\langle \nu(\mathrm{X}), R_{1_\mathrm{X}}, ..., R_{n_\mathrm{X}} \rangle$ be a $\mathcal{L}_\mathrm{X}$-system in W. An n-ary intensional relation R is $\mathcal{L}_\mathrm{X}$-definable if and only if there exists an $\mathcal{L}_\mathrm{X}$-formula $\varphi(x_1, ..., x_n)$ such that for all $v \in W$, for all $b_1, ..., b_n \in D_v$: $(d_1, ..., d_n) \in R_v \Leftrightarrow v \models \varphi(d_1, ..., d_n)$.

An intesional relation is $\mathcal{L}$-definable if and only if there is a $\mathcal{L}$-formula which locally holds of a tuple of entities when and only when their tuple is in the local domain of the relation. Given a system and a relation from among those holding on it, this notion is meant to yield the primitive relations of the system, or the relations which are definable in terms of the primitive ones. We finally define fundamental relations on structures:

> **(Fundamental relation)** An n-ary relation R on the positions of a pure structure $[w]$ is fundamental if and only if there exists an n-ary relation Q on the objects of an $\mathcal{L}$-system w and a formula $\varphi \in \mathcal{L}$ such that:
>
> 1. Q is defined by φ;
> 2. R is an extension of Q.

Fundamental relations on structures are extensions of relations which are definable in the systems of the structure concerned. Schiemer and Wigglesworth point out that binding fundamental relations to definability in languages introduces an essential linguistic component into our notion of mathematical object; in general, the stronger (e.g. higher order) the language, the more properties would turn out to be definable and, as such, the more properties would be fundamental. In light of Purity, this means that as we go up in language order, more and more properties would be fundamental and thus more and more properties would be[165] structural; the more expressive a language, the more 'concrete' the structures it describes, the more properties count as structural.

Schiemer and Wigglesworth mention three virtues of this notion of fundamental relation. First, all intuitively fundamental relations turn out fundamental on this definition. Second, if R is an n-ary relation definable by some formula φ in the language of w, then the extension of R at $\Sigma(w)$ contains an n-tuple of positions if and only if those positions are the abstracts of the elements in some n-tuple in the

[165] That is, if Purity holds, as we shortly show it does. For the potential perils the kind of language relativity arrived at poses in the realist context, see Schiemer and Wigglesworth [2017, p. 21]. Wigglesworth [2018a] will also defend the language relativity of mathematical structures.

extension of R at w.[166] Finally, and crucial in the proof below, if φ defines R, then φ also defines R^*, the extension of R.[167] We can now prove that the Purity thesis is a consequence of FAS.

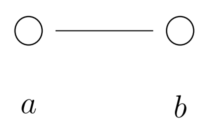
> **(Purity)** Let a be a position in $[w]$. If R is a fundamental property of a in $[w]$, then R is a structural property of a in $[w]$.[168]

The proof can be found in Schiemer and Wigglesworth [2017, p. 24].[169] This is *Fregean Abstractionist Structuralism.* We conclude by answering the three programmatic questions. First, structures and positions are *sui generis* logical objects abstracted from systems and their elements (**Question 1**). The identity of structures and positions is governed by the implicit definitions provided by abstraction principles (**Question 2**); however, as it stands, this will be shown unsuccessful in what follows. It is frequently held that the terms introduced by abstraction principles are singular terms performing singular reference; as such, terms introduced by FAS' abstraction principles are presumably performing singular reference to mathematical structures and positions therein (**Question 3**).

3.4.1.2 Against the Canon. The *problem of Cross-structural identities* is settled in the negative by FAS: since distinct structures are abstracted from non-isomorphic systems, none of their positions are related by isomorphisms and hence, by POA$_i$, their domains are disjoint. This is a theorem of FAS and, as such, non-negotiable. Ordinary mathematical practice seemingly identifying positions from distinct structures are explained away as cases of semantic indeterminacy.

We've been assuming that all systems are rigid (ii) precisely because the *Automorphism problem* looms in FAS, and it is this problem that eventually leads Linnebo and Pettigrew [2014, p. 278] to give up the view; Schiemer and Wigglesworth's amendments are no aid against Automorphism either. Consider the following unlabeled graph:

G_2:



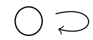
Since $f : G_2 \to G_2$ with $f(a) = b$ and $f(b) = a$ is an automorphism (and thus an isomorphism), then PoA yields that $[a]_{G_2} = [b]_{G_2}$. But then the positional characterization of structures yields that the structure $[G_2]$ is the following:

[166] Formally: $R^*_{\Sigma(w)} = \{\langle\sigma(x_1), ..., \sigma(x_n)\rangle | (x_1, ..., x_n) \in R_w\}$.

[167] Schiemer and Wigglesworth [2017, p. 22] provide a short proof.

[168] Schiemer and Wigglesworth [2017, p. 23], Proposition 1.

[169] We notice that Schiemer and Wigglesworth take $[w]$ to be a model theoretic $\mathcal{L}$-structure when carrying out this proof; however, if all systems are in our original $\mathcal{M}$, then W_S doesn't contain any set theoretic structures. In other words, abstract structures are not set theoretic entities, but *sui generis* mathematical entities. Therefore this is more of a plausible argument than a strict proof.

$[G_2]$: 

$[a]_{G_2}$

In Linnebo and Pettigrew's jargon, this is a case of Instantiation failure.[170] PoA collapses distinct but symmetric elements on the same positions in the corresponding structure and, as such, e.g. additive inverses in the group of integers, as well as i and $-i$ in the complex field would be identified. It is easy to see that a similar reasoning applies to PoA_i, leading to the same result: using the same isomorphism f form G_2 to G_2 (this time conceived of as elements of W), PoA_i yields that $[a]_{G_2} = [b]_{G_2}$. We do not hold that in general PoA and PoA_i validate the the same identities, although we find this likely; however, at least in the case at hand, they both validate $[a]_{G_2} = [b]_{G_2}$ and hence give rise to the Automorphism problem.

Drawing on Leitgeb and Ladyman [2008], John Wigglesworth [2018a] has recently presented a solution to the Automorphism problem on behalf of FAS. The formal idea is twofold. *First*, Wigglesworth insists on the model-theoretic understanding of systems as language dependent $\mathcal{L}$-models, i.e. models of the

[170]Linnebo and Pettigrew [2014, p. 277-8] propose two remedies to the Automorphism problem in FAS. First, it is suggested that PoA be entirely dropped and only structure abstraction left standing. The authors complain employing a form of the Semantic objection against this option: by endorsing it "we lose one of the main advantages of the [non-eliminative] structuralist position, namely, an account of the subject matter of mathematics that equips us with a straightforward semantics for mathematical language." (See Ketland [2015] for reiterating this suggestion and Assadian [2016, §6.4] for carrying it out in some detail as a version of Hybrid Structuralism.) The second option is to amend PoA as follows:

$$\textbf{(PoA}'\textbf{)}\ [x]_{\mathrm{X}} = [x']_{\mathrm{X}'} \Leftrightarrow \exists f(f : \mathrm{X} \cong \mathrm{X}') \wedge \forall f(f : \mathrm{X} \cong \mathrm{X}' \rightarrow f(x) = x')$$

However, the relation characterized on the right hand side is not an equivalence relation since it is not reflexive: given the graph G_2, for instance, PoA would yield that $[a]_{G_2} = [b]_{G_2}$, whereas now PoA$'$ yields $[a]_{G_2} \neq [a]_{G_2}$, since besides identity, $f : G_2 \rightarrow G_2$ with $f(a) = b$, $f(b) = a$ is an isomorphism which doesn't send a to a. Both consequences seem equally unacceptable. However, see Payne [2013] for a discussion of non-reflexive abstraction principles in the context of negative free logic. We leave this for further work.

language $\mathcal{L}$.[171] Given a relational language $\mathcal{L}$,[172] a system $\mathrm{X} = \langle D^{\mathrm{X}}, \nu^{\mathrm{X}} \rangle$ is a $\mathcal{L}$-intepretation or $\mathcal{L}$-model where D^{X} is a domain of objects and $\nu^{\mathrm{X}} : \mathcal{L} \to \mathcal{P}(D^{\mathrm{X}^n})$ is an interpretation function taking any n-ary relation symbol in $\mathcal{L}$ to a collection of n-tuples from D^{X}. An isomorphism between $\mathcal{L}$-systems X and Y is then a bijection $f : D^{\mathrm{X}} \to D^{\mathrm{Y}}$ such that for all n-ary relations $R \in \mathcal{L}$ and for all $x_1, ..., x_n \in D^{\mathrm{X}}$: $\langle x_1, ..., x_n \rangle \in \nu^{\mathrm{X}}(R)$ if and only if $\langle f(x_1), ..., f(x_n) \rangle \in \nu^{\mathrm{Y}}(R)$. This definition is meant to replace **Systems' Isomorphism** above. Both structure as well as positions abstraction is now performed on such systems.

Second, non-rigid systems should be rigidified before performing abstraction on them: following Leitgeb and Ladyman [2008] argument to the effect that mathematical practice regards identity as an intrinsic relation of structures,[173] Wigglesworth suggests to enrich systems with enough structure to achieve this. Given the language $\mathcal{L}$ and the $\mathcal{L}$-system X mentioned above, given an element $x \in D^{\mathrm{X}}$, an *identity predicate* P_x for x is a predicate such that $\nu^{\mathrm{X}}(P_x) = \{x\}$. Let $\mathcal{L}^*$ be the *identity expansion* of $\mathcal{L}$, that is, the expansion of $\mathcal{L}$ with predicates for every $x \in D^{\mathrm{X}}$: $\mathcal{L}^* = \mathcal{L} \cup \{P_x | x \in D^{\mathrm{X}}\}$. Then ν^{X} can be extended accordingly so as to obtain the $\mathcal{L}^*$-system X^*, the *identity expansion* of X. Consider again the graph G_2: its corresponding signature is $\mathcal{L} = \{R(x, y)\}$, and viewed as a $\mathcal{L}$-system $\mathcal{G}_2 = \langle D^{G_2} = \{a, b\}, \nu^{\mathcal{G}_2} = \{\langle R, \{\langle a, b \rangle, \langle b, a \rangle\}\rangle\}\rangle$. The identity expanded $\mathcal{L}^*$-system $\mathcal{G}_2^*$ is then rigid: since $a \in \nu^{\mathrm{X}}(P_a) = \{a\}$ while $b \notin \nu^{\mathrm{X}}(P_a)$, it follows that any function such that $f(a) = b$ is not an isomorphism on $\mathcal{G}_2^*$. And this holds in general: given a (rigid or not) $\mathcal{L}$-system X, its identity expansion X^* is rigid. The suggestion is to amend FAS[174] so as to perform abstraction on systems in two

[171] Wigglesworth [2018a, §4]. This insistence comes against what Wigglesworth calls the intuitive, language free notion of isomorphism, e.g. implicitly employed above in the definition of **Systems' Isomorphism** (§3.4.1). Wigglesworth quotes two arguments against the intuitive notion of a system/structure. First, consider

$$\mathrm{X} = \langle D^{\mathrm{X}} = \{a, b, \langle a, a \rangle\}, R^{\mathrm{X}} = \{\langle a, a \rangle\}\rangle$$
$$\mathrm{Y} = \langle D^{\mathrm{Y}} = \{a, b, \langle a, b \rangle\}, R^{\mathrm{Y}} = \{\langle a, b \rangle\}\rangle$$

Both X and Y are defined as language free sets of sets, endowed with a domain (D^{X} and D^{Y}) and a simple relation (R^{X} and R^{Y}, respectively). What set theoretic structures are X and Y? To see that their language free characterizations above are ambiguous, we can show that it is ambiguous whether X and Y are isomorphic or not. If R^{X} and R^{Y} are both monadic relations, then X and Y are isomorphic; however, if they are both binary relations (or at least one of them is) then they are *not* isomorphic (there is an element with a reflexive edge in X but none in Y). This argument is formulated in Halvorson [2016, p. 593] while discussing the semantic view of theories. A second quoted argument holds that the characterization of isomorphism associated to the language free notion of systems is incoherent, since it allows for elementarily non-equivalent systems which are nonetheless isomorphic (Halvorson [2013], Glymour [2013]).

[172] The restriction to relational languages is not substantial.

[173] Leitgeb and Ladyman [2008, p. 392].

[174] Notice that this strategy fits as well on Schiemer and Wigglesworth [2017]'s dynamic abstractionist account.

steps. Given an $\mathcal{L}$-system X:

1. Build up the $\mathcal{L}^*$-system X*;

2. Perform abstraction on X^* and its elements.

According to Wigglesworth suggestion, the structure of X is then [X*], while if $x \in D^{\mathrm{X}}$, then $[x]_{\mathrm{X}^*} \in [D]_{\mathrm{X}^*}$.

Wigglesworth discusses several potential problems arising for this account, among which its plausibility as a rational reconstruction of mathematical practice, the induced language relativity and the relation to haecceitism; we won't engage with any of these in this section, but would rather formulate a further worry ourselves. Wigglesworth says:

> *After the first step of moving to an expanded $\mathcal{L}^*$-system, the second step uses abstraction to obtain* [X*]*, which is the intended pure structure of* X. (Wigglesworth [2018a, §4], substituting in our notation)

We remember that Linnebo and Pettigrew insisted on the the Instantiation thesis as a central component of a non-eliminativist semantic account, and that Wigglesworth account is precisely meant to save FAS from Instantiation failure. However, strictly speaking, Instantiation fails in this context. [X*] is a $\mathcal{L}^*$-structure, whereas X is only a $\mathcal{L}$-structure and, therefore, $[\mathrm{X}^*] \ncong \mathrm{X}$.[175] However, we have two options. First, we can read Wigglesworth suggestion as that, in actuality, the systems considered in mathematical practice are never like X, but rather like its identity expansion X*, having all their elements absolutely discernible; in this case, we would actually only be interested in X* to start with. However, this is hardly defensible, since mathematicians are, as a matter of fact, ordinarily engaging with non-rigid systems. Alternatively, if [X*] is a $\mathcal{L}^*$-structure, we can consider its restriction to $\mathcal{L}$ and consider it as a $\mathcal{L}-$structure, in which case indeed $[\mathrm{X}^*] \cong \mathrm{X}$; however, in that case, as an $\mathcal{L}$-system, [X*] is not rigid anymore. We leave an assessment of this worry for further work. Concluding the cluster of identity problems, we notice that the *Individuation objection* is thereby answered by providing a trivial account of identity for mathematical objects: identity and distinctness are primitive relations of mathematical systems.

We can now consider the cluster of problems concerning objects. Drawing on Harold Hodes [1984]'s original argument against the ability of terms introduced by Fregean abstraction principles to perform singular reference, Bahram Assadian [2019b, p. 181] presents and expands upon a version of the *Permutation problem* plaguing singular reference to mathematical entities on the part of mathematical terms introduced by abstraction principles. Assadian doesn't explicitly consider

[175] Only systems with the same signature can be isomorphic. $\mathcal{L}$ and $\mathcal{L}^*$ could be identical; however, they are not so in general.

such principles in a structuralist context and, moreover, they don't consider dynamic abstraction principles of the kind employed by Schiemer and Wigglesworth in formulating FAS; however, although we do not aim to provide an analysis and full critical assessment of the Permutation argument in the dynamic and structuralist context at hand, we present the argument and list the problem for further work.

Assadian applies the Permutation argument to the case number theoretic terms as governed by Hume's Principle (see above). Let's call *numberer* a second order cardinality function from first-order concepts to objects, such that two concepts are mapped to the same object if and only if the concepts are one-to-one correlated. Let $\mathcal{A}$ be a numberer such that non-instantiated concepts are assigned 0, uniquely, doubly, triply concepts are assigned 1, 2, 3, respectively, and so on. Let $\mathcal{B}$ be another numberer just like $\mathcal{A}$, except that it assigns 5 to concepts fourthly instantiated, and 4 to concepts which are fifthly instantiated. Consider now a language, New English, just like common English except for the following:

- the numeral '4' denotes 5;
- the numeral '5' denotes 4;
- 'successor' denotes a function just like the one denoted in English, except that: $s(3) = 5$; $s(5) = 4$; $s(4) = 6$.

This and their consequences (e.g. $ss(3) = 4$) are the only differences between New English and English. The questions is now: why is English and the numberer $\mathcal{A}$ the case, rather than New English and the numberer $\mathcal{B}$?

	···	III	IV	V	VI	···
$\mathcal{A}$:	···	3	4	5	6	···
	···	'3'	'4'	'5'	'6'	···
$\mathcal{B}$:	···	3	**5**	**4**	6	···

I represents uniquely instantiated concepts, II represents doubly instantiated ones, III triply and so on. If IV_i is some fourthly instantiated concept, then $\mathcal{A}(\text{IV}_i) = 4$, while $\mathcal{B}(\text{IV}_i) = 5$. What makes numberer $\mathcal{A}$ with $\mathcal{L}_\mathcal{A}$ standard, while $\mathcal{B}$ with $\mathcal{L}_\mathcal{B}$ non-standard? Notice that linguistic use facts are the same in both cases: fourthly instantiated objects are said to be '4' in both languages, while fifthly instantiated ones said to be '5' in both. So use cannot help us distinguish between the two. The question for the abstractionist is then: what makes $\mathcal{A}$ rather than $\mathcal{B}$ standard or intended, if anything? The problem might be formulated as a dilemma. Either $\mathcal{B}$ is as eligible as the 'standard' numberer, or it is not. If it $\mathcal{B}$ is just as eligible as $\mathcal{A}$

is, then number words introduced by Fregean abstraction principles are referentially indeterminate and so they do not perform singular reference. If $\mathcal{B}$ is less eligible than $\mathcal{A}$, then the abstractionist owes us an account of why $\mathcal{A}$, rather than $\mathcal{B}$, is the intended numberer. However, the argument goes, there is no such account, since there is nothing making $\mathcal{A}$ intended as opposed to $\mathcal{B}$. Assadian concludes to referential indeterminacy.

We could seemingly apply the same argument against the referential determinacy of terms introduced by Structure Abstraction.[176] Call a *structurer* a first-order isomorphism-type function from systems to objects, particularly structures, such that two systems are mapped onto the same object if and only if they are isomorphic. Let $\mathcal{X}$ and $\mathcal{Y}$ be two distinct structurers of Isomorphism types II and III, respectively, only differing with respect the assignments they make to systems isomorphic to X $= \langle\{a, b\}\rangle$ and Y $= \langle\{a, b\}, \{\langle a, b\rangle, \langle b, a\rangle\}\rangle$ such that $[\mathrm{X}]^{\mathcal{X}} =$ • • and $[\mathrm{X}]^{\mathcal{Y}} =$ •-•, and the other way around.[177]

	I	II	III	IV	V	···
$\mathcal{X}$:	•	• •	•-•	• • •	•-• •	···
	'G_0'	'G_1'	'G_2'	'G_3'	'G_4'	···
$\mathcal{Y}$:	•	•-•	• •	• • •	•-• •	···

With similar linguistic potential readjustments, in this case the question for the abstractionist is what makes $\mathcal{X}$ intended rather than $\mathcal{Y}$ and, as in the case of HP, we could conclude to referential indeterminacy for terms meant to refer to structures and introduced through $\mathrm{SA}_{(i)}$.

However, structurer $\mathcal{Y}$ is flouting the structuralist abstractionist endeared Instantiation thesis which is also a consequence of FAS: for all systems Z, the structure of Z is isomorphic to Z ($[\mathrm{Z}] \cong \mathrm{Z}$). Therefore we should have $[\mathrm{X}]^{\mathcal{X}} \cong \mathrm{X}$ and $[\mathrm{X}]^{\mathcal{Y}} \cong X$ as well, although we notice that

$$[\mathrm{X}]^{\mathcal{Y}} = \bullet\text{-}\bullet \ncong \{\{a, b\}\} = \mathrm{X}$$

plausibly against Instantiation.[178] $\mathcal{Y}$ is not an eligible structurer after all. Following a similar reasoning, since it would be found to be the only structurer satisfying Instantiation, $\mathcal{X}$ is the privileged, intended structurer in general. Therefore the Hodes-Assadian Permutation argument fails for Structure $\mathrm{Abstraction}_{(i)}$.

[176]Static as well as dynamic versions of SA would be seemingly affected in the same degree.

[177]For a uniform notation with the previous discussion on structuralist abstractionism, we use e.g. '$[\mathrm{X}]^{\mathcal{X}}$' instead of '$\mathcal{X}(\mathrm{X})$' to mean 'the structure corresponding to system X according to structurer $\mathcal{X}$'.

[178]The explicit positional characterization of $[\mathrm{X}]^{\mathcal{Y}}$ is $\langle\{[a]_\mathrm{X}, [b]_\mathrm{X}\}, \{\langle[a]_\mathrm{X}, [b]_\mathrm{X}\rangle, \langle[b]_\mathrm{X}, [a]_\mathrm{X}\rangle\}\rangle$.

A version of the Permutation argument plagues nonetheless terms introduced by Position Abstraction, in its static and dynamic versions as well. Call *positioner* a first-order function from elements-cum-systems to objects, such that two elements form their respective systems are mapped on the same object if and only if they are related by an isomorphism between their hosting systems. Let $\mathcal{R}$ and $\mathcal{Q}$ be two positioners, and let system X be $\langle\{a, b, c\}, \{\langle b, c\rangle\}\rangle$ such that $[a]^{\mathcal{R}}_{\mathrm{X}} = \alpha$, $[b]^{\mathcal{R}}_{\mathrm{X}} = \beta$, $[c]^{\mathcal{R}}_{\mathrm{X}} = \gamma$, $[a]^{\mathcal{Q}}_{\mathrm{X}} = \beta$, $[b]^{\mathcal{Q}}_{\mathrm{X}} = \alpha$ and $[c]^{\mathcal{Q}}_{\mathrm{X}} = \gamma$.[179]

X:	a	b ——— c
$\mathcal{R}$:	α	β ——— γ
$\mathcal{Q}$:	β	α ——— γ

The abstractionist seemingly cannot provide us with grounds for choosing one unique object as $[a]_{\mathrm{X}}$, presumably *the* position corresponding to the element a of X. As such, as far as the abstractionist structuralist *said* or *implied*, both $[a]^{\mathcal{R}}_{\mathrm{X}}$ and $[a]^{\mathcal{Q}}_{\mathrm{X}}$ would do, and hence both positional *structures* induced by $\mathcal{R}$ and $\mathcal{Q}$,[180] $[\mathrm{X}]^{\mathcal{R}}$ and $[\mathrm{X}]^{\mathcal{Q}}$, could as well be the the structure of X, conflicting with Uniqueness.

Drawing on Bahram Assadian [2018, §4]'s general discussion of the Permutation problem for non-eliminativism, the abstractionist has two options:[181] holding that one of $[\mathrm{X}]^{\mathcal{R}}$ and $[\mathrm{X}]^{\mathcal{Q}}$ is not a structure, but a system; alternatively, holding that $[\mathrm{X}]^{\mathcal{R}} = [\mathrm{X}]^{\mathcal{Q}}$. However, none of these is an option for the structuralist. Regarding the former, the abstractionist structuralist should provide an account concerning which of $[\mathrm{X}]^{\mathcal{R}}$ and $[\mathrm{X}]^{\mathcal{Q}}$ is the original and which the permuted copy; however, this task is as hopeless as that of explaining why any of them would be privileged over the other, that is, why one would be a structure while the other not. Concerning the second reply, $[\mathrm{X}]^{\mathcal{R}} = [\mathrm{X}]^{\mathcal{Q}}$, it implies that $\mathcal{R} = \mathcal{Q}$ and, in general, that positioners are unique. However, this seems highly implausible: after all, positioners are just functions mapping elements-cum-systems onto objects according to a certain rule and, as such, there is no constraint on the objects in the codomain of the positioner, nor any further constraint on the assignments.

A further and seemingly more natural reply for the abstractionist is to accept the conclusion, albeit hold that none of $[\mathrm{X}]^{\mathcal{R}}$ and $[\mathrm{X}]^{\mathcal{Q}}$ are structures, in particular that none is [X]: structures are fully characterized by isomorphism, while $[\mathrm{X}]^{\mathcal{R}}$ and

[179] Aiming at a uniform notation, as before, we write '$[a]^{\mathcal{R}}_{\mathrm{X}}$' for '$\mathcal{R}(\mathrm{X}, a)$', meaning 'the position associated to the element a from system X by the positioner $\mathcal{R}$'.

[180] For instance, the structure induced by $\mathcal{R}$, $[\mathrm{X}]^{\mathcal{R}}$ is $\langle\{[a]^{\mathcal{R}}_{\mathrm{X}}, [b]^{\mathcal{R}}_{\mathrm{X}}, [c]^{\mathcal{R}}_{\mathrm{X}}\}, \{\langle [b]^{\mathcal{R}}_{\mathrm{X}}, [c]^{\mathcal{R}}_{\mathrm{X}}\rangle\}\rangle = \langle\{\alpha, \beta, \gamma\}, \{\langle \beta, \gamma\rangle\}\rangle$.

[181] Assadian discusses a third option, credited to Shapiro [2006]: embracing the conclusion of permutations and holding that there might be many structures corresponding to some isomorphism type. However, regardless of other complaints, this option is not available to the endorser of FAS to start with, since Uniqueness is a consequence of FAS.

$[\mathrm{X}]^{\mathcal{Q}}$ are isomorphic albeit distinct and, as such, they are both only systems. In the static abstractionist case, this option implies that structures don't afford a positional characterization and, as such, they are not positional entities. However, we notice that Structure Abstraction is not directly affected by Permutation *only* because Instantiation holds, which in turn only holds because structures can be characterized as positional entities (Positional Structure above). This raises the spectrum of Permutation over non-positionalist abstractionist views taking Structure Abstraction as the sole abstraction principle and which, as such, don't allow for a positional characterization of structures.[182] This is, however, material for further work.

In the dynamic case, however, things are not so clear. Presumably, neither $[\mathrm{X}]^{\mathcal{R}}$, nor $[\mathrm{X}]^{\mathcal{Q}}$ are in the original domain of systems W, since they contain entities of the form $[x]^{\mathcal{R}}_{\mathrm{X}}$ which are not in D_P. Therefore they should be in W_S and thus be structures, against the supposition that both are systems. However, they cannot be structures on the dynamic account either, since by SA_i it follows that $[\mathrm{X}]^{\mathcal{R}} = [\mathrm{X}]^{\mathcal{Q}}$ if and only if $\mathrm{X} \sim \mathrm{X}$, which implies that $[\mathrm{X}]^{\mathcal{R}} = [\mathrm{X}]^{\mathcal{Q}}$, leading back to the issues introduced above. We leave his discussion open for further work and flag the Permutation problem as a potentially problematic case for FAS.

We focused on the Automorphism and the Permutation problems as FAS faces them; we will only be sketchy concerning FAS' performance against the remaining canonical problems and objections. Since the identities of structures and positions therein are grounded in systems and their elements, FAS faces none of the versions of the *Circularity problem* – metaphysical, semantic and epistemic.[183] The *problem of Structural Properties* is one of the main motivations leading to the formulation and later emendation of FAS and, as such, we discussed it above in some detail, highlighting the way it is dealt with within FAS. Consequently, *MacBride's dilemma* is seemingly toothless against FAS as well: unlike traditional Platonism, mathematical objects are positions in structures, which are unlike Platonic abstract objects in that both Incompleteness and Dependence holds of them (see the end of §3.2.2.4).

The *problem of Singular Reference* is at least as pressing for FAS as the Permutation problem. If the Permutation problem has a bite off FAS' Uniqueness thesis, then referential indeterminacy looms and there is no prospect of singular reference for mathematical terms introduced via structuralist abstraction principles.[184] The first branch of the *Semantic objection* is highly reliant on the problem of Singular Reference: if FAS cannot secure singular reference to the terms introduced by abstraction, then it seemingly fails to provide a strong face value semantics for ordinary mathematical discourse, undermining the original

[182] E.g. Assadian [2016, §6.4].
[183] See e.g. Wright [1983].
[184] Assadian [2019b, §3] formulates a further argument from semantic idleness concluding against singular reference for mathematical terms introduced through abstraction principles.

metasemantic motivation against eliminativism. FAS also fails to provide additional support for non-eliminativism and, as such, it doesn't have a say in managing the second breach of the objection.

3.4.1.3 Summing up. Figure 3 depicts in red the neuralgic canonical items FAS has left unsolved.

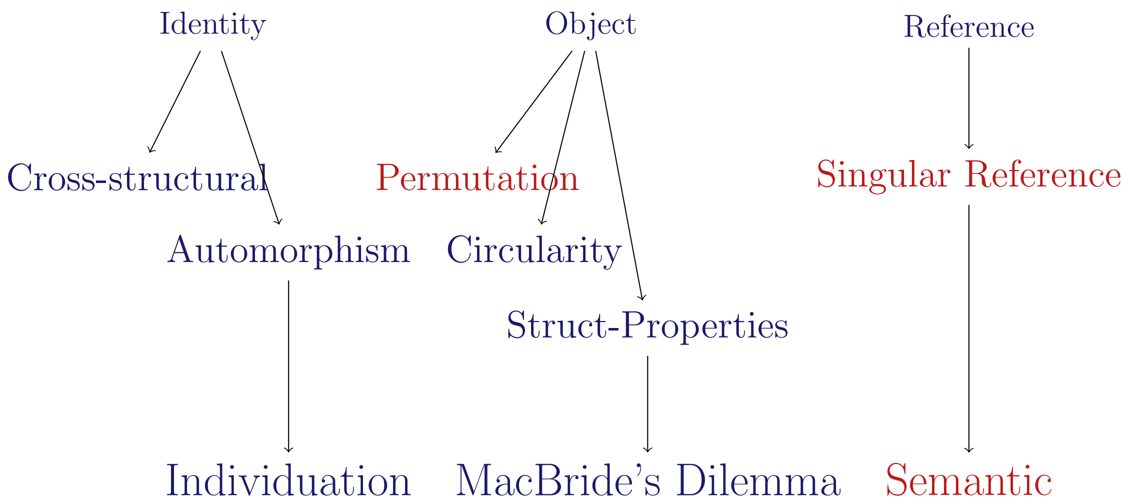


Figure 3: FAS against the canon

The conclusions formulated at the end of each of these three expository sections in this manner are by no means intended to be last words on the matter; rather, they are meant as directions for further enquiry.

3.4.2 Object Theoretic Structuralism

Edward Zalta and Uri Nodelman formulate Object Theoretic Structuralism (OTS) based on an axiomatic theory of abstract objects.[185] We begin with a sketch of Object Theory, followed by a presentation of its application as a foundation of non-eliminativist structuralism; finally, we test the theory against our canonical problems and objections and conclude with a map of the neuralgic spots.

3.4.2.1 Theory. *Object Theory* (OT) is an axiomatic theory of abstract objects formulated in a typed language in the framework of higher-order logic together with S5 modal logic (with first- and second-order Barcan formulas), alongside the machinery of λ-terms and (rigid) definite descriptions ι-terms. Due

[185] We draw on Nodelman and Zalta [2014]. Object Theory was introduced in Zalta [1983].

to space limitations, we will be brief and omit types. Most importantly, OT employs two sorts of predication: *exemplification* and *encoding*. Exemplification ($F(x)$, 'x exemplifies property F') is the classical sort of predication employed in predicate logic; objects are complete exemplification-wise, meaning that for every property, objects either exemplify it or its negation. Encoding (xF, 'x encodes property F') is the novel addition which, unlike the former, allows for objects being incomplete in the following sense: objects might encode neither a property, nor its complement.[186] The central idea is that abstract objects are *defined* by properties which are then understood to *constitute* these objects which, in turn, are therefore said to *encode* these properties. Encoding predication is governed by an axiom schema, $\Diamond xF \rightarrow \Box xF$: possibly encoding a property implies necessarily encoding it.

OT uses a primitive predicate $E!x$ with the intended meaning that 'x is concrete'. *Ordinary objects* are defined as possibly concrete objects ($O!(x) := \Diamond E!(x)$), while *abstract objects* are all other objects, those that are not possibly concrete ($A!(x) := \neg\Diamond E!(x)$).[187]

Identity between objects is explicitly defined in the theory: ordinary objects are identical if and only if, necessarily, they exemplify the same properties, while abstract objects are identical if and only if, necessarily, they encode the same properties.[188] Identity for properties is defined as necessary co-codification: F and G are identical if and only if, necessarily, they are encoded by the same objects; identity of propositions is defined in terms of identity of properties, using λ notation.[189] To Second-order comprehension for properties (easily adapted to relations) is formulated as follows:

> **(Comp)** $\exists F\Box\forall x(Fx \equiv \varphi)$, where φ is a formula not containing any encoding subformulas.[190]

The main difference between ordinary and abstract objects is that only the latter encode properties; this is formalized in the first of the two non-logical axioms of OT:

> **(No Encoding for $O!x$)** $O!x \rightarrow \Box\neg\exists F(xF)$;

The second axiom of OT provides us with the required field of abstract objects:

> **(Comprehension for $A!x$)** $\exists x(A!x \wedge \forall F(xF \equiv \varphi))$, where x is not free in φ.

[186] x is incomplete $:= \exists F(\neg xF \wedge \neg x\overline{F})$, where $\overline{F}$ denotes the negation of the property F (Nodelman and Zalta [2014, p. 53]).

[187] See Linsky and Zalta [1995] for more on encoding properties.

[188] $x = y := (O!x \wedge O!y \wedge \Box\forall F(F(x) \equiv F(y))) \vee (A!(x) \wedge A!(y) \wedge \Box\forall F(xF \equiv yF))$ (Nodelman and Zalta [2014, p. 43]).

[189] $F = G := \Box\forall x(xF \equiv xG)$; $p = q := [\lambda x.p] = [\lambda y.q]$ (Nodelman and Zalta [2014, p. 43]).

[190] The condition is required on pain of incosnsitency. We notice cross-worlds properties are allowed for.

Given identity as previously defined, it is a theorem that the abstract object in the second axiom is unique for each φ.[191]

Further on, Zalta and Nodelman provide an *analysis of mathematical discourse* in OT, which yields a natural structuralist construal of mathematics.[192] An abstract object x is a *situation* if and only if x only encodes propositional properties, where propositional properties are properties in which all terms are bound.[193] *Truth* of a proposition p in a situation x ($x \models p$) is then defined as x's encoding the propositional property corresponding to p.[194] For a theory T, the following is then a theorem of OT:

(Identity of T) $\mathrm{T} = \iota x(A!x \wedge \forall F(xF \equiv \exists p(\mathrm{T} \models p \wedge F = [\lambda x.p])))$

This serves as a *characterization*[195] of the theory: T is *the* abstract object encoding all and only those propositional properties corresponding to true propositions in T. Notice that talking of truth in T is justified since the left to right direction of the right hand side embedded equivalence shows that T is a situation. According to the interpretation provided here, a theory is constituted (in the same sense in which the properties encoded by an abstract object constitute it) by its truths, i.e. its theorems (including its axioms).

OT may be employed to provide a semantic account for theoretical discourse involving abstract objects. Zalta and Nodelman formulate rules for importing the truths of a theory into OT, extending OT and bringing the given theory against the ontological background provided by OT. Let T be a theory: if t and P are a singular term and a relation of T, respectively, let t_{T} and P_{T} be their T-indexed counterparts ('the t/P of T'). The following rule is then applied:

(Importation Rule) For each theorem φ of T, add the truth $\mathrm{T} \models \varphi^*$ to OT, where φ^* is the result of replacing all occurrences of all terms $t_1, ..., t_n, P_1, ..., P_m$ in φ with their respective T-indexed counterparts.

[191] The second axiom is really typed:

$$\exists x^t(A^{\langle t \rangle}!x \wedge \forall F^{\langle t \rangle}(xF \equiv \varphi))$$

This provides us with abstract objects encoding functions and relations of various orders. Similarly, we will implicitly assume a typed formulation for most of the principles to follow.

[192] The authors distinguish between *natural* and *theoretical* mathematics: the former is the kind of mathematics as employed in counting, measurements and other 'ordinary' applications, while the latter is the kind of mathematics done in the context of an explicit theory and mostly concerned with things such as proving theorems and relations between different theories. See Zalta [2000] for more on this distinction. The exercise in the foundations of non-eliminativist structuralism developed in Nodelman and Zalta [2014, §2.2 ff] is chiefly concerned with theoretical mathematics.

[193] If p is a sentence, thereby expressing a proposition, then $\lambda y.p$ is a propositional property, where y might substitute a term from p. Formally, situations are defined as follows: x is a *situation* $:= A!x \wedge \forall F(xF \rightarrow \exists p(F = [\lambda y.p]))$ (Nodelman and Zalta [2014, p. 45]).

[194] p is *true* in x, $x \models p$, if and only if $x[\lambda y.p]$ (Nodelman and Zalta [2014, p. 45]).

[195] The authors highlight that this is not a definition (Nodelman and Zalta [2014, p. 45]), since T appears on both sides of the identity sign.

This way one can import the theorems of T into object theory as truths under the scope of the theory operator.[196] The entities of a theory T can then be characterized as follows:

(Reduction of Individuals) $t_{\mathrm{T}} = \iota x(A!x \wedge \forall F(xF \equiv \mathrm{T} \models F(t_{\mathrm{T}})))$

This is a theorem of OT (when the truths of T have been imported) and it serves as a characterization of the entities denoted by the singular terms and the predicates making up the primitive vocabulary of the theory.[197] The 0 of PA^2 would then be characterized as the abstract object encoding all and only those properties that PA^2 holds to be exemplified by 0_{PA^2}.[198] Finally, the meaning of the mathematical statements is recovered through the following theorem:

(Equivalence Theorem) $t_{\mathrm{T}}F \equiv \mathrm{T} \models F(t_{\mathrm{T}})$

The meaning of an arithmetical (i.e. PA^2) statement such as '1 is a number', that is, '$\mathrm{PA}^2 \vdash N(1)$', is construed by the object theorist as $1_{\mathrm{T}}[\lambda x.N(x)]$. But a similar Equivalence theorem holds for all types of objects in OT, and therefore the proper analysis would rather be: $1_T[\lambda x.N_T(x)]$. For another example, '2 is less than π in $\mathbb{R}$', i.e. $\mathbb{R} \vdash 2 < \pi$ is construed as $2_{\mathbb{R}}[\lambda x.x <_{\mathbb{R}} \pi_{\mathbb{R}}]$.[199]

We can now explicitly formulate *Object Theoretic Structuralism* on these foundations, i.e. provide a structuralist interpretation to the OT rendering of theoretical mathematics. Given a theory T, its corresponding *structure* is defined as follows:

(Structure) The structure T := The theory T

Structures just are theories constituted by their truths. However, truths are not *elements* of structures; their elements are defined as follows:

(Elements of Structures): x is an element of a structure T :≡

$\mathrm{T} \models \forall y(y \neq_{\mathrm{T}} x \rightarrow \exists F(F(x) \wedge \neg F(y)))$

[196] The authors argue that truths such as $\mathrm{PA}^2 \models (1+3=4)^*$ are analytic in OT (Nodelman and Zalta [2014, p. 47]). Moreover, they show that the Importation Rule validates a Rule of Closure which ensures that reasoning under the theory operator is classical:

(Rule of Closure) If $p_1, ..., p_n \vdash q$ and $\mathrm{T} \models p_1$, ..., $\mathrm{T} \models p_n$, then $T \models q$

where $\vdash$ is logical consequence.

[197] An analog result holds for P_{T}'s; see Nodelman and Zalta [2014, p. 48].

[198] That is: $0_{\mathrm{PA}^2} = \iota x(A!x \wedge \forall F(xF \equiv \mathrm{PA}^2 \models F(0_{\mathrm{PA}^2})))$.

[199] The meaning of the sentence '2 is less than π in $\mathbb{R}$', as used in ordinary mathematical discourse would then be: $\pi_{\mathbb{R}}(<_{\mathbb{R}} (2_{\mathbb{R}}[\lambda x, Y, z.xYz]))$.

where $=_{\mathrm{T}}$ is the identity predicate of the theory T and the formula on the right hand side is one of the truths imported from T into OT.[200] Intuitively, elements of T are those abstract objects κ_{T} which are absolutely-discernible (see §3.2.1) in T, i.e. by T-properties, from any other entity in T.[201] *Relations* of structures are defined similarly:

> **(Relations of Structures)**: R is a relation of structure T $:\equiv$ T $\models$ $\forall S(S \neq_{\mathrm{T}} R \rightarrow \exists F(F(R) \wedge \neg F(S)))$

This is *Object Theoretic Structuralism*. Several core non-eliminativist structuralist claims can be readily shown to hold well in OTS. First, elements of mathematical structures are *incomplete* objects in the encoding sense of predication: for instance, since $E!$ is not a term of any mathematical theory, then no mathematical object, i.e. element of a mathematical structure, encodes $E!$ and none encodes $\overline{E!}$.[202]

Moreover, Zalta and Nodelman hold that mathematical objects are incomplete in a *right*, structuralist way.[203] More to the point, they argue that a restricted version of Linnebo [2007]'s *Non-Structural Incompleteness claim*[204] formulated in terms of essential properties as suggested in Shapiro [2006] also holds in OTS. Define the *essential properties* of abstract objects as the properties they encode;[205] then the following holds:

> **(E-Incompleteness Thesis)** All essential properties of mathematical objects are structural properties.[206]

[200] We notice that this understanding of elementhood goes in the direction of Gottlob Frege's conception of object as a properly individuated entity. It also gets divorced from one of the strands of the Quinean paradigm holding that objects are those entities in the first-order domain of quantification of theories. However, it fits well with another, more Fregean strand of the Quinean paradigm, usually formulated by mentioning the slogan "no entity without identity."

[201] We will say more concerning strongly-indiscernibles shortly when discussing the Automorphism problem. The notion of 'elementhood' assumed by OT might look *ad hoc*. OT is meant to capture set theory as well; one may ask how OT's notion of elementhood fits with the set theoretic notion; we won't pursue this question in these pages.

[202] That is: for all mathematical structures T and all their objects t_{T}, T$\not\models E!(t_{\mathrm{T}})$ and T$\not\models \overline{E!}(t_{\mathrm{T}})$. Therefore, by the Equivalence Theorem, $\neg t_{\mathrm{T}}E!$ and $\neg t_{\mathrm{T}}\overline{E!}$. However, one should notice that most abstract abstract objects are incomplete encoding wise. Only those abstract objects which are abstracted away from concrete objects might be held encoding complete (Nodelman and Zalta [2014, p. 45]), but these are relatively 'few'. In this sense, the kind of incompleteness exhibited by mathematical objects is not peculiarly mathematical.

[203] But see the previous footnote.

[204] See the discussion of Linnebo [2007] concluding our discussion of MacBride's dilemma in §3.2.2.

[205] That is: for an abstract object x, F is an essential property of $x :\equiv xF$; see Zalta [2006] for this notion of essence. They distinguish essential properties from *necessary exemplifications*, which are defined as $\Box F(x)$.

[206] Linnebo [2007, p. 65] distinguishes between *NS-Incompleteness* and *I-Incompleteness*, i.e. having no non structural properties and having all their intrinsic properties being structural,

Given OT we need to modify the notion of structural property suggested in §1.1. An alternative can be formulated:

> **(T-Structural Property)** F is a structural property of x_{T} :≡
>
> $\mathrm{T} \models F(x_{\mathrm{T}})$

To see that the E-Incompleteness thesis holds, consider an essential property F of t_{T}. Since t_{T} encodes F ($t_{\mathrm{T}}F$), the Equivalence Theorem yields that $\mathrm{T} \models F(t_{\mathrm{T}})$, which is just what it means for F to be a structural property of t_{T}. OT postulates that elements of structures are bound to their theories and, moreover, that they only encode those properties yielded by their hosting theories:[207] E-Incompleteness is an immediate consequence of OTS.

Furthermore, a version of Linnebo [2007]'s weak downwards identity dependence thesis also holds in OTS:[208]

> **(ODO)**: Any individuation of a mathematical object t in a structure T involves entities which also suffice for individuating the other objects in T.[209]

This can be seen as follows. Given the definition of identity for abstract objects, abstract objects are essentially characterized by the properties they encode. Let T be a mathematical theory, and let t_{T} be an element of T. By the Equivalence Theorem, t_{T} only encodes T-structural properties; that is, any individuation of t_{T} will be crucially grounded in facts of the form $\mathrm{T} \models F(t_{\mathrm{T}})$, involving T. However, the identity of any element of T is grounded similarly in T and its theorems and, as such, T suffices for the individuation of all the other elements in T. But then any individuation of t_{T} makes use of entities (T) which suffice to individuate any element of T, proving ODO. *A fortiori*, this also shows that elements weakly depend on their structures:

> **(ODS)**: Any individuation of a mathematical object t in a structure T involves entities which also suffice for individuating T.

We conclude the presentation of OTS by explicitly answering our three programmatic questions. Mathematical structures, i.e. theories on the background of OT, are abstract objects encoding all and only those propositional properties

respectively; the thesis that all their essential properties are structural may then be called *E-Incompleteness*.

[207] We notice resemblance to universalism in §2.2 in this respect.

[208] Also discussed at the end of MacBride's dilemma in §3.2.2. Nodelman and Zalta [2014, §4.2] also discusses this, holding that mathematical objects in a structure depend on the other objects in the same structure because all are grounded in the same kind of facts of the form $\mathrm{T} \models p$; we essentially prove that Linnebo's very weak dependence claim strictly holds in OTS.

[209] Linnebo [2007, p. 78].

corresponding to their truths (**Question 1**). 'Positions' or 'elements' of structures are themselves abstract objects encoding all and only those properties recognized by their hosting structures as holding of them; the identity of positions is governed by the structural-properties they encode (**Question 2**). Finally, reference to structures and their elements is – with some exceptions to be discussed below – singular reference (**Question 3**). We can discuss OTS against our canonical measuring stick.

3.4.2.2 Against the Canon. The *problem of Cross-identities* is decided in the negative by OTS: mathematical objects are bound to the theories/structures providing the properties they encode, there being no overlap between structures. Since the conception of mathematical objects is essentially bound to the theory/structure they are elements of, then objects from distinct structures encode different properties and hence are *distinct* abstract objects.[210] Notice that this is a theorem of OT and thus non-negotiable for OTS. The mathematical practice of 'identifying' (what on a non-eliminativist interpretation are) mathematical objects from different structures is explained in terms of embeddings.

We need to take a closer look into OTS' notion of elementhood before discussing the *Automorphism problem*. First, entities which are not absolutely discernible are *not* elements of structures: given the definition of elementhood – elements are those entities corresponding to singular terms of the theory which are absolutely-discernible in T from any other entity quantified over by the theory – distinct entities which are not discernible by properties in a theory are not elements of that theory. OTS tackles the Automorphism problem by denying that there are structures which contain elements which are not absolutely discernible. However, this raises the spectrum of denying solid mathematical facts: the group of additive integers, for instance, is undeniably a mathematical object – after all, denying the existence of such structures is one way to formulate the absurdities older non-eliminativist conceptions such as SGS were lead into. The crucial difference is subtle: OTS doesn't deny the existence of structures such as the additive group of integers or the complex field, neither ends up identifying symmetric elements; it rather holds that such entities are not *bona fide* elements of such structures, ordinary claims to the

[210] Nodelman and Zalta [2014, p. 59-60]. Zalta and Nodelman add:

> *To think otherwise is to suppose that abstract objects and relations are somehow out there, independent of our theories of them, waiting to be discovered.* (Nodelman and Zalta [2014, p. 59-60]) *Such relations are not 'out there waiting to be discovered', but are the way that our various theories of them describe them to be.* (Nodelman and Zalta [2014, p. 62])

According to OTS, mathematical objects are theory- or structure-laden in a constitutive sense: there is nothing to them outside the structure. In this sense, structures are indeed prior to their objects. This is a strong version of structuralism, if anything is. The authors point out that Frege's Caesar problem is also thereby avoided (Nodelman and Zalta [2014, footnote 18]).

contrary being regarded as spurious. But this cannot be the end of the story: for all properties $F_{\mathbb{C}}$, both $i_{\mathbb{C}}$ and $-i_{\mathbb{C}}$ (of complex analysis) would have $\mathbb{C} \models F(i_{\mathbb{C}})$ if and only if $\mathbb{C} \models F(-i_{\mathbb{C}})$ and, therefore, by the Equivalence Theorem, both $i_{\mathbb{C}}$ and $-i_{\mathbb{C}}$ would encode the same properties; hence, given the definition of identity for abstract objects, $i_{\mathbb{C}} = -i_{\mathbb{C}}$, which by Equivalence again would lead to $\mathbb{C} \models i_{\mathbb{C}} =_{\mathbb{C}} -i_{\mathbb{C}}$, which is absurd since $\mathbb{C} \models i \neq_{\mathbb{C}} -i$. However, that is only if $i_{\mathbb{C}}$ and $-i_{\mathbb{C}}$ are objects at all: Zalta and Nodelman hold that strong-indiscernibles are *not* elements, chiefly because they are *not objects* (abstract or otherwise) at all.[211]

This claim requires a semantic account of such terms, alongside imposing semantic adjustments to the construal of ordinary mathematical discourse. Zalta and Nodelman's suggestion is formulated as follows:[212]

> **(Normalizing Procedure)** Let T be a mathematical theory with singular terms t_1 and t_2 such that T $\vdash$ $t_1 \neq t_2$ while T $\vdash \forall F(F(t_1) \equiv F(t_2))$. Let $\varphi(...t_1...)$ be a formula of $\mathcal{L}^{\mathrm{T}}$ such that T $\vdash \varphi(...t_1...)$. Then (1) replace $\varphi(...t_1...)$ with $\exists x(\Phi \wedge \varphi(...x...))$, where Φ is the formula 'introducing' t_1 into the language (do the same for t_2, if necessary).[213] Only afterwards (2) apply the Importation rule.

The Normalizing Procedure eliminates apparent reference to symmetric entities through paraphrase of ordinary discourse. This way there are no genuine singular terms seemingly denoting strongly indiscernible entities in T, nor in the T-truths imported in OT: every structure is essentially rigid. What are then such terms as i and $-i$ in ordinary mathematical discourse? Following Shapiro [2008], Zalta and Nodelman suggest that such terms are implicitly bounded free variables. Such terms do not strictly speaking refer and, in ultimate analysis, they are not part of the theories mentioning them on their surface grammar. The solution provided to the Automorphism problem implicitly furnishes an answer to the *Individuation objection*: OT rules out distinct albeit indiscernible objects from its ontology. As such, the corresponding theories/structures are not absurdly eliminated themselves.

[211] The following summarizes it:

> *The point is that, ontologically speaking, there is no need to worry about what constitutes the numerical diversity of i and –i. 'i' and '–i' do not denote distinct abstract objects—they are arbitrary names used by mathematicians as labels on a structural symmetry of* $\mathbb{C}$*.* (Nodelman and Zalta [2014, p. 71])

See also Murphy [forthcoming] for a discussion of OTS and its treatment of apparent commitment to indiscernibles.

[212] Nodelman and Zalta [2014, p. 70-71].

[213] For instance, i in complex analysis is introduced through $i^2 + 1 = 0$. The suggestion is then as follows: for complex analysis replace any theorem $\varphi(...i...)$ with $\exists x(x^2 + 1 = 0 \wedge \varphi(...x...))$; here Φ is $x^2 + 1 = 0$. For a discussion of the group of integers, dense linear orders without endpoints and so on, see Nodelman and Zalta [2014, p. 66-69]. In short, their resolution in such cases is that some theories do not have any elements, but rather be constituted entirely by relations.

The *Permutation problem* doesn't arise in OTS: structures are constituted by their theorems/truths, while their elements are their definable entities, i.e. abstract objects serving as referents of singular terms and only encoding theory bound structural properties. Since elements are constituted by their relations which are themselves constituted by the structures/theories, no permutation of the domain could be meaningfully constructed. The *Circularity problem* is addressed[214] and replied: both the identities of structures and those of their elements are grounded in "invariances" in the use of singular terms, and that of relational and singular terms denoting their elements. Ultimately, these are grounded in further facts which do not themselves depend on their identities. Therefore any sort of ontological, circularity is avoided. The problem of *Structural Properties* is also solved: positions in structures do have non-structural properties exemplification-wise and, as such, no contradiction arises when identifying mathematical objects with positions in structures.[215] Concerning *MacBride's dilemma*, it can be easily seen that OTS may strive on both horns: it takes no damage from the Individuation objection while holding that mathematical objects are individuated by structural properties; meanwhile, it validates versions of both the Incompleteness thesis and Dependence thesis and hence it certainly avoids collapse into Platonism.

Concerning the *problem of Singular Reference*, OTS can secure singular reference for all mathematical terms. However, OTS has to recast some apparent singular terms purporting to refer to distinct indiscernible entities as bound variables. Reference to mathematical structures and their elements as opposed to systems is explained in terms of linguistic use: structures are certain objects wholly determined by the properties they encode, given by their axioms and their logical consequences; since systems would have distinct properties, then they are not the intended subject matter of mathematical theories. Finally, the crucial *Semantic objections* are only partly replied: OTS provides a natural account of singular reference to structures and their positions, providing for a distinctly non-eliminativist account validating the metasemantic motivation. However, concerning the second Semantic objection, OTS provides no further motivation for endorsing structures besides its presumed semantic virtues.

3.4.2.3 Summing up. We conclude Object Theoretic Structuralism taking stock in Figure 4.

Further objections have been leveraged against the underlying Object Theory,[216]

[214] Nodelman and Zalta [2014, p. 64].

[215] That is, since, plausibly, in predicating non-structural properties of mathematical objects the *exemplification* sense of predication is employed, rather than the *encoding* sense thereof. No incoherency is forthcoming either: 'only encoding structural properties' is not itself a property *encoded* by positions in structures, but one *exemplified* by them.

[216] Assadian [2016, p. 72ff] presents two objections, the first concerning OTS and its

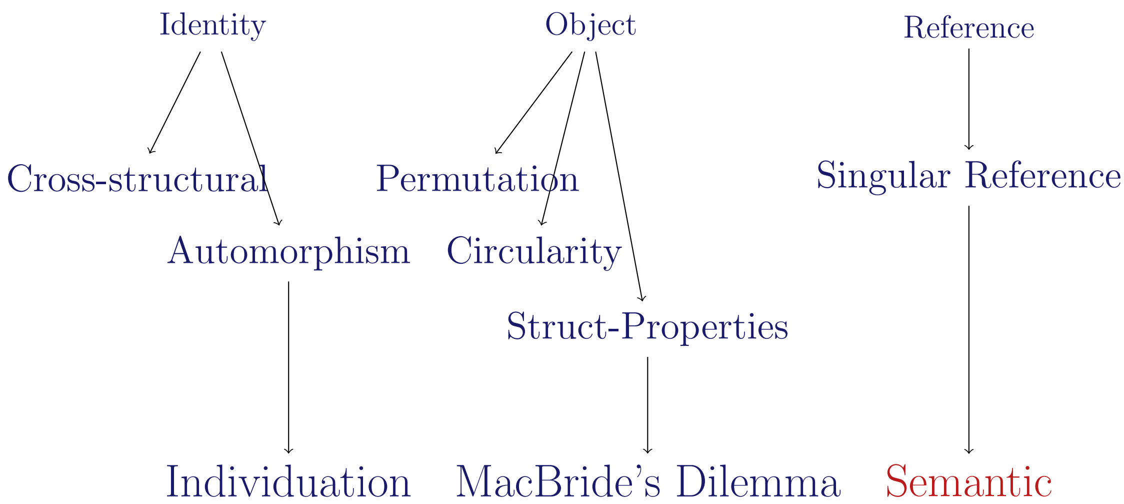


Figure 4: OTS against the canon

both concerning its performance withing a structuralist account of mathematics, as well as a general linguistic account of predication. However, this is not the place to review object Theory in its entirety.

3.4.3 Unlabeled Graph-theoretic Structuralism

In this section we present Hannes Leitgeb's account of structures as unlabeled graphs.[217] We first introduce Unlabeled Graph-theoretic Structuralism (UGS), followed by its assessment against the canonical problems and objections presented above; we conclude with a performance map as in the case of OTS.

3.4.3.1 Theory. *Unlabeled Graph Theory* (UGT)[218] is a second-order axiomatic theory of unlabeled graphs, formulated in terms of three primitive notions: $Graph(G)$, $Vertex(v, G)$ and $Connected(v, w, G)$, meant to express 'G is an unlabeled graph', 'v is a vertex in G' and 'v and w are connected by an edge in G', respectively.[219] The first order domain of UGT contains graphs and vertices therein; variables x, y range over first order entities in general, while v, w are

'mathematical' virtues, involving mixed predication in mathematical contexts; another objection involves faulty attribution of abstract referents in cases of failed reference to concrete entities.

[217]This theory is introduced in writing in Leitgeb [forthcoming,a] and Leitgeb [forthcoming,b]: the former introduces the theory of unlabeled graphs as *sui generis* structures (UGT), while the latter discusses many of the objections leveraged against previous versions of non-eliminativism.

[218]Introduced in Leitgeb [forthcoming,a, §4].

[219]Labeled and unlabeled graphs are usually defined set theoretically. Leitgeb [forthcoming,a, §3] contains an informative discussion of labeled and unlabeled graphs in the mathematical literature. UGT provides an alternative to the mathematical orthodoxy, treating unlabeled graphs as *sui*

reserved for vertices and G, G' for graphs. UGT's second order domain contains extensional properties (sets/classes), relations and functions on its first order domain, for which X, R and f are dedicated variables, respectively. Three simplifying assumptions are made: graphs are undirected, graphs contain no loops and there is no overlap of graphs; all these will be reflected in UGT axioms which can be later dropped or modified for further extensions.

UGT's *logical system* is governed by four logical principles underlying full SOL[220] including full second order comprehension schemas for relations and functions,[221] the principle of identity of indiscernibles (PII) governing the identity of graphs and vertices,[222] extensionality governing the identity of properties, relations and functions[223] and, finally, the choice axiom.[224] Given comprehension and extensionality, the class $V(G)$ of all vertices in a graph G exists and is unique and can be defined as follows:

(Vertex class) $\forall G \forall X(V(G) = X \leftrightarrow \forall x(X(x) \leftrightarrow Vertex(x, G)))$

The *non-logical axioms* of UGT are divided into *general* and *existential axioms*. The first general axiom of UGT reflects the simplifying assumptions that graphs are undirected (iii) and do not reflexive edges (ii):

(G1) $\forall G \forall v, w(Connected(v, w, G) \rightarrow$

(i) $Vertex(v, G) \wedge Vertex(w, G) \wedge$

(ii) $v \neq w \wedge$

(iii) $Connected(w, v, G))$

The second axiom states that vertices are not themselves graphs and graphs are disjoint:

(G2) $\forall G \forall v(Vertex(v, G) \rightarrow$

(i) $\neg Graph(v) \wedge$

(ii) $\neg \exists G'(G \neq G' \wedge Vertex(v, G')))$

The third axiom provides a distinctively structuralist identity criterion for graphs in term of isomorphisms:

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generis mathematical objects. Unless explicitly mentioned otherwise, 'graph' stands for 'unlabeled graph' in what follows.

[220] $\forall G ...$ and $\exists G ...$ abbreviate $\forall x(Graph(x) \rightarrow ...)$ and $\exists x(Graph(x) \wedge ...)$, respectively.

[221] **(L1)** $\exists R^n \forall x_1, ..., x_n(R^n(x_1, ..., x_n) \leftrightarrow \varphi[x_1, ..., x_n])$, with R^n not free in φ; Functional(φ) $\rightarrow \exists f \forall v, w(f(v) = w \leftrightarrow \varphi[v, w])$, f not free in φ.

[222] **(L2)** $\forall x, y(x = y \leftrightarrow \forall X(X(x) \leftrightarrow X(Y)))$.

[223] **(L3)** $\forall R^n, S^n(R^n = S^n \leftrightarrow \forall x_1, ..., x_n(R^n(x_1, ..., x_n) \leftrightarrow S^n(x_1, ..., x_n)))$, $\forall f, g(f = g \leftrightarrow \forall x(f(x) = g(x)))$.

[224] **(L4)** $\forall R^{n+1}(\forall x_1, ..., x_n \exists y R^{n+1}(x_1, ..., x_n, y) \rightarrow \exists f^n \forall x_1, ..., x_n R^{n+1}(x_1, ..., x_n, f(x_1, ..., x_n)))$.

(G3) $\forall G, G'(G = G' \leftrightarrow G \cong G')$[225]

The first existential axiom asserts the existence of the trivial-graph (G_0), i.e. a graph with only one vertex and no edges:

(E1) $\exists G \exists! v Vertex(v, G)$

The second and third existential axioms of UGT correspond to ordinary mathematical operations on graphs, using "old" graphs to build up new ones by adding (new and isolated) vertices and edges (between preexisting vertices), respectively:

(E2) $\forall G \exists G' \exists v'(Vertex(v', G') \land Isolated(v', G') \land$
$\exists f(Isomorphism(f, G, G' - \{v'\})))$[226]

(E3) $\forall G \forall v, w((Vertex(v, G) \land Vertex(w, G) \land$
$v \neq w \land \neg Connected(v, w, G)) \rightarrow$

(i) $\exists G' \exists v' \exists w'(Connected(v', w', G') \land$

(ii) $\exists f(Isomorphism(f, G, G' - \{v', w'\}) \land v' = f(v) \land w' = f(w))))$

We can already prove the existence and uniqueness of each finite unlabeled graph, i.e. unlabeled graphs with finitely many vertices[227], starting with the trivial-graph (E1) and going bottom-up adding vertices and/or edges (by E2 and E3), while relying on the structuralist identity axiom for graphs (G3) for uniqueness. For instance, the existence and uniqueness of the dumbbell graph G_1 (the graph with two vertices and no edges) is granted by the following consequence the previous axioms:

(G_1) $\exists! G_1 \exists v_1, v_2(v_1 \neq v_2 \land Vertex(v_1, G_1) \land Vertex(v_2, G_1) \land$
$\neg Connected(v_1, v_2) \land \forall w(Vertex(w, G_1) \rightarrow (w = v_1 \lor w = v_2))$

$G_1$is the result of taking the graph G_0 (by E1 and G3) and adding an isolated vertex (by E2) concluding that it is unique (by G3). Similar results grant the existence and uniqueness of all finite graphs. Moreover, facts concerning the number of automorphisms of a certain graph as well as facts concerning the cardinality of

[225]Where isomorphism ($\cong$) between graphs is defined as the existence of a bijective and structure-preserving map, the latter two being defined in terms of the primitive vocabulary of UGT (Leitgeb [forthcoming,a, p. 19]).

[226]Here (and, *ceteris paribus*, in E3 below) $Isomorphism(f, G, G' - \{v'\})$ is an abbreviation for a complex formula in the language of UGT stating that is an isomorphism between G and the subgraph of G' resulting after removing the vertex v' form G'. This complication is due to the requirement that graphs are disjoint (G1) and thus that strictly speaking the new graph G' doesn't result by adding a vertex (or edge) to G; UGT handles such cases by essentially establishing the existence of an embedding $h : G' \rightarrow G$.

[227]Leitgeb [forthcoming,a, p. 20], **Theorem 1**.

graphs are also derivable, using the second order resources, from the above above. Leitgeb's **Metatheorem 1** establishes the consistency of the above axioms relative to ZFC.[228] Leitgeb showcases the power of UGT up to this point by showing that various graph-theoretic notions can be defined[229] and relations between graphs and other mathematical entities can be studied.[230]

Taking subgraphs of given graphs is another ordinary operation of graphs yielding new ones from old, which suggests capturing the idea in a subgraph axiom:

(E4) $\forall G \forall X((\forall v(X(v) \rightarrow Vertex(v, G))) \rightarrow$
$\exists G' \exists f(Isomorphism(f, G|_X, G')))$[231]

Other natural operations on graphs such as the union of two graphs and product of two graphs[232] can be formulated in UGT and added as axioms supplementing the above list. Last but not least, infinite graphs are brought into existence by an infinite graph axiom postulating the existence of an infinite (countable) graph:

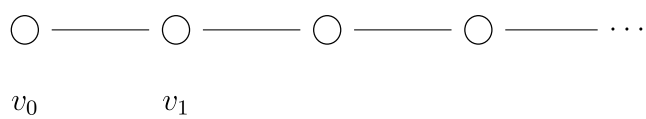
(E∞) $\exists G \exists v_0, v_1(Vertex(v_0, G) \wedge Vertex(v_1, G) \wedge Connected(v_0, v_1, G) \wedge$

(i) $\forall w(Connected(w, w_0, G) \rightarrow w = w_1) \wedge$

(ii) $\exists f(Isomorphism(f, G, G - \{v_0\}) \wedge f(v_0) = v_1))$[233]

Graphs postulated by E∞ look as follows:

G_∞:



[228] Leitgeb [forthcoming,a, p. 22].

[229] Leitgeb defines subgraphs and proper subgraphs (Leitgeb [forthcoming,a, p. 24], **D4** and **D5**, respectively).

[230] This is possible by adding the axioms governing the mathematical objects of interest to UGT, extending the first order domain of with the newly added mathematical entities. Adding the axioms PA^2 to UGT, for instance, one can define walks in a graph (Leitgeb [forthcoming,a, p. 23], **D3**) and go on to label graphs by natural numbers, define connectedness of graphs and so on.

[231] $Isomorphism(f, G|_X, G')$ is an abbreviation of a lengthy UGT formula (Leitgeb [forthcoming,a, p. 24]) to the effect that there is an isomorphism between the restriction of G to those of its nodes in a class X and G'. The notion discussed above, $Isomorphism(f, G, G' - \{v\})$, is a particular case of the one here. Like before, the intricacies are due to disjointness of graphs (G2).

[232] Which Leitgeb [forthcoming,a, p. 25] recommends adding as axioms **E5** and **E6**. Further axioms mirroring natural operations such as composition of graphs, taking n-cubes of graphs etc. would make for further axioms **E7**, **E8** etc.

[233] The infinite graph G_∞ is 'built up' employing Dedekind infinity: there is a bijection between the graph and a proper subgraph of it.

By the identity criteria for graphs, G_∞ is unique. Once G_∞ has been added to the first order domain, many more infinite graphs are brought into the picture applying the existential axioms introduced above. This open-endedly concludes UGT and Leitgeb's constructive endeavour with an eye on further extensions.

The main argument favouring UGT over set theoretic reconstructions of unlabeled graphs is that the former, unlike the latter, resembles the mathematician's practice when engaging with graphs (Leitgeb [forthcoming,a, p. 35]) and, as such, if UGT is otherwise satisfactory, then it should have the upper hand as an account of graphs. Moreover, the success of UGT provides the non-eliminativist with a powerful example, shedding some hope against objections aiming to undermine the very coherence of non-eliminativism.

UGT provides an account of a certain kind of mathematical objects, namely unlabeled graphs, as sui generis objects whose identity is governed by isomorphism (G3). Leitgeb's suggestion is then the following: UGT may be employed as an account of *sui generis mathematical structures in general*; we label this view *Unlabeled Graph-theoretic Structuralism* (UGS). Structures are not set theoretic entities endowed with a domain of positions and relations on them and, in this sense, structures are not systems: unlike structures, systems are entities in the second order domain of UGT, largely conceived of set theoretically. However, structures are positional entities: given a structure S (we switch to structure notation from this point on), $V(\mathrm{S})$ (defined above) is a set containing all and only the vertices of S, which are in turn mathematical objects which serve of referents of mathematical terms; this answers our **Question 1**. While the identity of structures is governed by the genuinely structuralist principle identifying isomorphic structures, the identity of positions is governed by PII (L2); we will see how UGS avoids the Automorphism problem when assessing it against the canon. In this sense, mathematical objects are *not* structural objects: they possess many non-structural properties such as 'being the RGB number of my favourite color' and so on.[234] As such, the Incompleteness claim[235] fails for positions in structures. However, what sets apart positions from Platonic individuals is a version of Linnebo's Dependence claim. We detail John Wigglesworth [2018b]'s account of the matter in the next paragraph to complete our answer to **Question 2**. Before that, answering **Question 3**, we mention that UGS largely provides a singular reference account of mathematical terms, with a caveat to be mentioned when discussing the problem of Singular Reference in the next section.

John Wigglesworth [2018b]'s furnishes an account of dependence in UGS[236] We

[234] Even though structures are unique up to isomorphism, structures may be isomorphic to systems.

[235] See the end of §3.2.2.4, discussing Linnebo [2007]'. Probably a meaningful version of the E-Incompleteness thesis can be formulated in terms of the essential properties of vertices, similar to the one discussed in the previous sections concerning OTS.

[236] Wigglesworth's wider aim is to provide an account of dependence in mathematical

remind the reader that Linnebo [2007] identifies two dependence claims endorsed by early non-eliminativists and which can distinguish their objects as positions in structures from Platonic abstracta. Linnebo construes these claims as weak identity dependence claims. Recasting them in the idiom of grounding, Wigglesworth construes (metaphysical) dependence as a relation of partial ground, whereas grounding is understood in Finean terms as a metaphysically explanatory relation. In this sense, a collection of facts Γ *fully grounds* a fact F if and onlu if the obtaining of the facts in Γ completely explain the obtaining of F. A fact G *partially grounds* another fact F is and only if G is part of a collection of facts Γ which fully grounds F. Wigglesworth holds that the grounding relation involved in ODO is partial, whereas that involved in ODS is the relation of full ground. Roughly speaking, given a structure S and a position x in it, Linnebo's dependence claims become the following:

(ODO) The identity of x partly grounds the identity of any other position in S;

(ODS) The identity of S fully grounds the identity of every position therein.

To make these precise, Wigglesworth provides an account of the *identities* of unlabeled graphs and their vertices. We render this talk in terms of essence: what Wigglesworth calls the *identity of a structure S* ($Id(\mathrm{S})$), we understand as the *essence* of S, and similarly for positions. Given a structure S and a position n of S, Wigglesworth defines:

$$Id(\mathrm{S}) := \{Graph(\mathrm{S}')|\mathrm{S}' \cong \mathrm{S}\} = \{\mathrm{S}\}$$
$$Id(n) := \{(n_1, n_2)|Connected(n_1, n_2, \mathrm{S}) \wedge (n = n_1 \vee n = n_2)\} = E_n^{\mathrm{S}}$$ [237]

structuralism, showcasing the peculiarities of mathematical objects as opposed to ordinary entities. Drawing on Linnebo [2007] discussion and account of dependence in terms of Abstraction principles, Wigglesworth tries to provide an alternative which is not biased in favour of either *in re* or *ante rem* versions of no-eliminativism. Finally, proofs of the Linnebo's ODO and ODS weak identity dependence claims are provided.

[237] Supporting these definitions, we can quote Leitgeb saying:

> *While unlabeled graphs are what they are in virtue of their structure, which is reflected by their identity criterion (G3), vertices in unlabeled graphs are what they are in virtue of the structure (the unlabeled graph) they belong to, and the identity and difference relations for the nodes of a graph belong to that very structure. There is quite simply more to the structure of the unlabeled graph G1 than what is tracked by the mere existence of an automorphism that sends a to b: just taken by itself, the existence of such an automorphism would be compatible both with a = b and with a a ≠ b , and there is no reason whatsoever to preclude either of these cases "ex cathedra", let alone on alleged "structuralist" grounds.* (Leitgeb [forthcoming,b, p. 5])

E is reminiscent of the set theoretic treatment of graphs as ordered pairs of a non-empty collection

In plain English, the identity of unlabeled graphs is its isomorphism class, which by G3 is the singleton of the graph; the identity of a vertex is the collection of edges involving it. while that of vertices is given as the collection of edges involving them. The Dependence claims are then rendered as follows:

(ODO) For all vertices n_1, n_2 of a graph S, the fact that $Id(n_1) = E_{n_1}$ partially grounds the fact that $Id(n_2) = E_{n_2}$;

(ODS) For all mathematical objects n of a graph S, the fact that S has identity $Id(\mathrm{S})$ fully grounds the fact that $Id(n) = E_n$.

Let's first consider ODO. Let $\Gamma_{n'} = \{Id(n) = E_n | n \in V(G) \wedge n \neq n'\}$[238] i.e. the collection of (all but one) facts to the effect that the identity of some vertex n of G is E_n. Then ODO can be reformulated as follows (since n_1 was universally quantified):

(ODO) For all vertices n_2 of a graph G, the obtaining of all the facts in Γ_{n_2} fully grounds the fact that $Id(n_2) = E_{n_2}$.[239]

Metaphysicians commonly agree that grounding claims entail corresponding necessity claims: if Δ fully grounds F, then necessarily if all the facts in Δ obtain, then F obtains. In the case of ODO we have:

(ODO$_\Box$) For all vertices n_2 of S: $\Box(\Gamma_{n_2}$ obtain $\rightarrow Id(n_2) = E_{n_2})$[240]

Discussing the domain of $\Box$, Wigglesworth suggests that $\Box$ ranges over *possible unlabeled graphs*.[241] Interpreting the necessity involved as such arguably gives us both of the following:

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of vertices and another possibly empty of edges. G_1 above, for instance, would be set theoretically characterized as $G_1 = \langle D, E\rangle$. However, this characterization can be recovered to some extent in UGT. In this sense, D would be $V(G_1)$ (which is uniquely characterized as shown above) and E would be given by Comprehension (L1) as the relation corresponding to the formula $Connected(x, y, G_1)$. We will sometimes employ the set theoretic notation, without thereby committing to sets.

[238] The necessity claim corresponding to ODO would be trivialized - and, with it, ODO itself - if the fact that $Id(n_2) = E_{n_2}$ would be itself in Γ_{n_2}. See below.

[239] This is a slightly modified version of Wigglesworth formulation. The original version was mentioning n_1 explicitly; however, that would be redundant since the identity fact corresponding to n_1 is itself part of Γ.

[240] There is already a slight problem here. The ODO$_\Box$ would be trivialized if the fact that $Id(n_2) = E_{n_2}$ would be itself in Γ_{n_2}; therefore, following the argument concluding that the necessity claim entails ODO, ODO itself would thus be trivialized. This suggest that Γ_{n_2} must not contain the identity fact corresponding to n_2. However, if we do not include the identity fact corresponding to n_2 in Γ_{n_2}, then it is unclear what we should do in the case of G_0, the graph with only one vertex and no edges: on the face of it, this graph would provide a counterexample to ODO if adding of nodes is not mentioned, as Wigglesworth omits to do. One solution would be to consider graphs which differ from G_0 in that they have more vertices, which would allow a similar proof to the one in Wigglesworth [2018b, p. 230]. However, G_0 would have to be mentioned as a special case.

[241] Metaphysical necessity would trivialize the matter in this case since supposedly if mathematical entities exist, they exist necessarily and have their identity as such.

ODO $\Rightarrow$ ODO$_\Box$

ODO$_\Box$ $\Rightarrow$ ODO

The latter is commonly regarded as false when $\Box$ is rendered as metaphysical necessity. Counterexamples involve cases in which metaphysically necessary existents figure in the consequent of the embedded conditional. Consider Socrates and the number 2: if the number 2 exists necessarily, then necessarily if Socrates exists, then the number 2 exists. But of course we wouldn't want to conclude that the existence of Socrates (fully) grounds existence facts concerning numbers. However, given the interpretation at hand having $\Box$ range over the domain of possible graphs, such consequences do not follow, since we allow for failure of metaphysical necessities – the number 2, for instance, may fail to have the identity that it actually has, although, if it exists, then it is presumably metaphysically necessary that it has its actual identity.[242] Provided that this argument is successful (which we won't debate here) a proof of ODO$_\Box$ would arguably be a proof of ODO. Wigglesworth proves the counterpositive of ODO$_\Box$:

(ODO$_\Box$) For all vertices n_2 of S: $\Box(Id(n_2) \neq E_{n_2} \rightarrow \Gamma$ don't obtain)

Wigglesworth's proof proceeds as follows:

Proof. Consider a graph G with n_2 in $V(G)$ and let G' be like G except that $Id(n_2) = E^{G'}_{n_2} \neq E_{n_2}$.[243] This means that for some x, y vertices of both G and G' with $n_2 = x$ or $n_2 = y$, either (i) $E^G \ni (x, y) \notin E^{G'}$ or (ii) $E^{G'} \ni (x, y) \notin E^G$.[244] Without loss of generality, suppose $n_2 = x$. Notice that then $y \neq n_2$ (since we explicitly ruled out loops in G1 above),[245] so let y be some vertex $n_3 \neq n_2$. Then $Id(n_3) = E^{G'}_{n_3} \neq E_{n_3}$, since (n_2, n_3) belongs to E_{n_3} but not to $E^{G'}_{n_3}$ in the (i) case, or vice versa in the (ii) case. So $Id(n_3) \neq E_{n_3}$. But notice that $Id(n_3) = E_{n_3}$ is a fact in Γ (since $n_3 \neq n_2$). Hence some fact in Γ fails to obtain, so Γ don't obtain, completing the proof. $\Box$

Wigglesworth proceeds similarly to show ODS$_\Box$ and, following the previous argument, ODS itself.

(ODS$_\Box$) For all vertices of G and n in $V(G)$, for all graphs G':

$G' \in Id(G) \rightarrow E_n = E^{G'}_n$

[242] A related worry is discussed: since Socrates doesn't exist in any unlabeled graph, than his non-existence would seemingly be grounded in any fact whatsoever. However, Wigglesworth comments that this only shows that one should not evaluate existence grounding statements concerning Socrates - or any medium sized physical object - with respect to possible graphs.

[243] We assume that *actually* $Id(n_2) = E^G_{n_2}$ since, *actually*, n_2 is a vertex of G; we write E_{n_2} instead of $E^G_{n_2}$, but $E^{G'}_{n_2}$ when $G' \neq G$.

[244] Where $E^G = \bigcup_{n \text{ in } V(G)} E^G_n$.

[245] Maybe except for the identity relation Wigglesworth [2018b, p. 231].

Proof. Let G' be arbitrary and suppose that $G' \in Id(G) = \{G\}$. Therefore $G' = G$ and hence $E_n^{G'} = E_n$, as required.[246] $\square$

3.4.3.2 Against the Canon. In general, Leitgeb claims that "UGT does not suffer from any problem of identity."[247] The *problem of Cross-structural identities* is settled in the negative by G2: vertices belong to one and only one graph. It is suggested that ordinary mathematical statements seemingly identifying positions from distinct structures rely on semantic indeterminacy and should be understood in terms of existing embeddings connecting the positions concerned. However, unlike the case of FAS and OTS, since G2 is not used in any of the results concerning graph existence and uniqueness, if some such identification is found irresistible, then G2 may be dropped and UGT could proceed unhindered. Further on, UGS' solution to the *Automorphism problem* relies on a conception of structures in which identity is an intrinsic relation.[248] For instance, UGT yields that the dumbbell graph (G_1) is unique and can be defined as follows:

$$G_1 := \iota G(\exists v_1, v_2(v_1 \neq v_2 \wedge Vertex(v_1, G) \wedge Vertex(v_2, G) \wedge \neg Connected(v_1, v_2) \wedge \forall w(Vertex(w, G) \to (w = v_1 \vee w = v_2)))$$ [249]

G_1 is the unique entity in the first order domain of UGT satisfying the embedded condition, which itself yields that G_1 contains two and only two distinct positions: Leitgeb holds that vertices depend on graphs for their identity, while graphs come along with their own identity relation in the sense just specified.[250] It is also clear that PII doesn't cause troubles in this setting: as mentioned above, the general Incompleteness claim fails for positions in UGS. In this sense, positions in structures contain non-structural properties, including haecceities (by second order comprehension), which in turn suffice for avoiding the absurd collapse of symmetric vertices.[251] The *Individuation objection* is also thereby replied by

[246] Wigglesworth proves again the counterpositive (Wigglesworth [2018b, p. 231-232]). However, the direct proof presented here is seemingly a better choice. In general, Wigglesworth account suffers from several problems, of which we only mention one here. Presumably, for x, y in some structure G, we assume that we have $Id(x) = Id(y) \to x = y$; this is what a true identity property or essence should grant in the least. However, consider symmetric vertices, say those of G_1 above and label them a and b. Notice that $Id(a) = E_a = \varnothing = E_b = Id(b)$. But then it follows that $a = b$, which is false since $a \neq b$ in virtue of the nature of G_1 (as explained above). And similarly for G_2 etc. This is a version of the Automorphism Problem and it shows that E_n cannot be an identity property or essence for n.

[247] Leitgeb [forthcoming,b, p. 3].

[248] As first suggested in Leitgeb and Ladyman [2008].

[249] Adding the ideology of definite descriptions to UGT.

[250] Leitgeb [forthcoming,b, p. 4].

[251] Leitgeb [forthcoming,b, p. 10] discusses a potential confusion that might seemingly lead to the Automorphism problem in UGT. The crucial morale avoiding this pitfall is that sets, second order entities, are *not* graphs, first order entities. For instance, $G_0 \neq V(G_0)$, even if the *vertices* of G_0

essentially rejecting the early structuralist slogans holding that positions in structures only contain structural albeit non-identity involving properties.

Regarding the *Permutation problem*, Leitgeb holds that S^{π} is crucially not a structure itself, but a system: structures are unique up to isomorphism and, therefore, any permutation on their domain results in systems thereof which are, unlike structures, second order entities of UGT.[252] Embarrassment of riches is thereby avoided since classes are not equally suitable reductions or referents of mathematical structures and mathematical terms (more on this below when we discuss the problem of Singular Reference). Concerning the *Circularity problem*, Leitgeb holds that structures and their positions are ontologically individuated as a whole, positions and structures depending on each other. Similarly, structures and their positions are grasped altogether at once, being utterly impossible to separate them in epistemic order.[253] The *problem of Structural Properties* is replied by essentially embracing its conclusion: mathematical objects, positions in structures, do have non-structural properties, against the early structuralist intuitions, but without thereby giving up non-eliminativism. Finally, concerning *MacBride's dilemma*, UGS rejects the structuralist slogans (for reasons which are independent of the Automorphism or Structural properties problems) and so it is a candidate for the *old news* horn concluding to collapse into Platonism. However, UGS satisfies a version of the Dependence claim and, as such, positions are unlike Platonic objects in that they are crucially dependent on their hosting structures. Concerning the status of positions in structures as objects, Leitgeb[254] argues that vertices exist in UGT according to ontological criteria that made a career in philosophical debates – e.g. "to be is to be the value of a bound variable" – at least since Quine on: positions, i.e. vertices, just like structures, i.e. graphs, figure in the UGT's first order domain of quantification.

Leitgeb considers the problem of Singular Reference in §3.1; however, given the way we understand and formulate it in this essay, Leitgeb's reply misses the point when arguing that it is dispelled once the Permutation problem is successfully replied. As we understand it, the Permutation problem, if applicable, undermines reference to structures *even if* it has already been established that mathematical reference is attracted to structures rather than systems. However, as we already explained, we understand the morale of Assadian [2018]'s (somewhat misleadingly labeled the) permutation plight as highlighting that even if the Permutation objection is replied and structures are thereby unique up to isomorphism, the non-eliminativist faces the quest of explaining why would mathematical vocabulary pick out or be attracted to structures at all, rather than pick out or be

are the *elements* of $V(G_0)$, being thereby isomorphic: the former is a first order entity, the latter is a class, which is a second order entity instantiating or exemplifying, in this case, G_0.

[252] Leitgeb [forthcoming,b, p. 11ff].

[253] Leitgeb [forthcoming,b, p. 13].

[254] Leitgeb [forthcoming,b, p. 13].

attracted to systems instead. Leitgeb fails to engage with *this* problem. Leitgeb highlights crucial differences between graphs and systems:[255]

1. Graphs are simple, while systems are composed;
2. Isomorphic graphs are identical (G3), while the identity of systems is governed by extensionality (L3);
3. Taking subgraphs requires the existence of a suitable structure preserving map, while any non-empty subset of a set serves as the domain of a system;
4. Graphs, unlike systems, have no set theoretic structure;
5. Graphs cannot be defined from systems, while systems may be defined in terms of graphs and their vertices.

All these aspects set apart structures from systems in UGT; however, UGS is called upon to explain why structures, rather than systems, attract the reference of mathematical terms. An answer to this problem unavoidably appeals to differences between the two types of objects: however, it asks for more than a list of differences, requiring an explanation concerning the way these differences contribute to increased eligibility or reference magnetism on the side of structures/graphs with respect to the mathematical vocabulary. Although highlighting the above differences, Leitgeb doesn't provide any such explanation. We can speculate that certain use facts provide an explanation of reference magnetism, as follows. Given methodological structuralism, mathematical practice only determine its subject matter up to isomorphism. This certainly doesn't exclude that mathematical structures have radically richer identity criteria, in spite of our ignorance or deficient means of expression (i.e. invoking epistemic or semantic insufficiencies, respectively, instead of a sort of ontological incompleteness). However, if there are suitable mathematical objects whose identity criterion is isomorphism, then such objects are more eligible referents for the mathematical vocabulary as employed in the mathematical practice. As such, given difference number 2 on Leitgeb's list, graphs, provided that they exist, rather than systems, attract mathematical reference. This is certainly a tentative account; however, it provides the prospect that UGS could answer the Singular Reference problem.

Concerning the other front of the Singular Reference problem, Leitgeb provides an arbitrary reference account for terms purportedly referring to symmetric positions of non-rigid structures, along the lines of Schiemer and Gratzl [2016]'s account presented in §2.1 above.[256] For instance, looking at G_1 once again,

[255] Leitgeb [forthcoming,b, p. 11-12].
[256] Leitgeb [forthcoming,b, p. 15-16].

Leitgeb's suggestion is to add the axioms of ϵ-calculus to UGT and then define as follows:

$$a := \epsilon v(Vertex(v, G_1))$$
$$b := \epsilon v(Vertex(v, G_1) \wedge v \neq a)$$

Informally, the idea is to have mathematical terms purporting to denote symmetric positions pick out one of these arbitrarily.[257] In spite of appeal to arbitrary reference in such cases, Leitgeb[258] argues that UGS, unlike eliminativism, can secure singular reference to structures themselves and positions in rigid structures; that is, she can live up to to non-eliminativist semantic standards covering most part of mathematical discourse. When she seemingly doesn't do so and appeals to arbitrary reference instead, however, she is only reflecting in her account peculiar features of the mathematical structures concerned, unlike the eliminativist who would construe all mathematical discourse in terms of such peculiarities.

However, Assadian [2018] complains against arbitrary reference on two grounds.[259] First, arbitrary reference is not a distinctively structuralist sort thereof[260] and, as such, it voids the metasemantic motivation for endorsing structures. However, as Leitgeb points out, this worry is not substantiated: arbitrary reference is rather a telling exception to the mostly singular sort of reference secured by UGS. Assadian's second complaint highlights that arbitrary reference brings about primitive semantic facts, which flies against philosophical orthodoxy; we suspend judgement on this issue and postpone its inspection to further work. We can now notice that this thereby also answers the first horn of the *Semantic objection*: the non-eliminativist is presumably able deliver singular reference for mathematical terms, at least for the most part. The second horn of the objection is not tackled in Leitgeb's essays; the question persists whether the semantic motivation suffices for endorsing structures and, if not – as the specter of the second Semantic objection threatens – whether there is any additional motivation for non-eliminativism besides the semantic desiderata.

3.4.3.3 Summing up. Figure 5 summarizes the performance of UGS against the objections leveraged against early versions of non-eliminativism:

[257]Leitgeb prefers this account over an account of such terms as free variables performing simulated reference (as we labeled it in §3.2.3.2), in light of two arguments. First, unlike the simulated reference account, epsilon terms are not variables and thus not bound to be a temporary stipulation within a process of existential elimination. Second, an explicit definition of terms referring to symmetric positions as epsilon terms allows us to deductively derive their distinctness from the the identity of the graph itself. This feature provides us with an understanding of the way symmetric positions are distinct *in virtue of* the identity of the very structure they figure in.

[258]Leitgeb [forthcoming,b, p. 18].

[259]Assadian [2018, p. 3212].

[260]See e.g. its employment by relativists, §2.1 above.

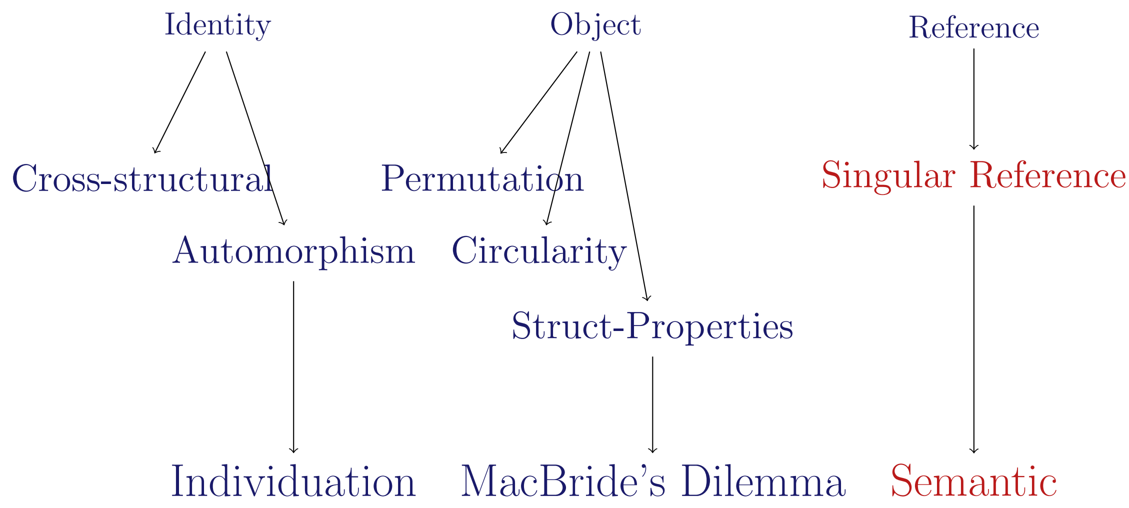

Figure 5: UGS against the canon

In spite of our tentative reply, we flag the problem of Singular Reference for further work. Moreover, in spite of the fact that we find Leitgeb's reply to the first horn of the Semantic objection satisfactory – holding that the non-eliminativist, unlike the eliminativist, can provide a singular reference account for most structures and mathematical theories, while when it cannot it is rather because of matching a peculiarity of the structure concerned – we flag the Semantic objection since nothing has been said concerning the second strand thereof to defend the non-eliminativist, and the present author sees nothing in sight.

4 Conclusion

We have now achieved our goal and reviewed in some detail three contemporary positionalist approaches to mathematical structuralism which,for the most part, succeed in overcoming many of the concerns plaguing early versions of non-eliminativism. We hereby conclude the present essay with a short summary and an eye on further work.

After introducing the topic and essential piece of terminology, we followed a quasi-historical route to modern mathematical structuralism: starting with Paul Benacerraf's seminal articles, we discussed eliminativism introducing useful ideology along the way, and we formulated the discontents which feed a line of reasoning being frequently invoked by non-eliminativists to motivate their view, labeled the 'metasemantic motivation' (inspired by Assadian [2018, p. 3205]). Further on, we dived into non-eliminativism and introduced an early articulation thereof, Stewart Shapiro's *Sui Generis* Structuralism, followed by an extensive discussion of many of the the problems and ensuing objections leveraged against it; we gathered together all these in a canon aiming to assess newly emerging articulations against it. However, we pointed out that most of items in our canon only plague positionalist versions of non-eliminativism and thereby took a detour through non-positionalism; after presenting a particular articulation of it, we concluded with several misgivings motivating the return to positionalism and review newly emerging versions thereof. In this order, we introduced in some detail Øystein Linnebo and Richard Pettigrew's Fregean Abstractionist Structuralism, Edward Zalta and Uri Nodelman's Object Theoretic Structuralism and Hannes Leitgeb's Graph Theoretic Structuralism. Assessing each of these views against our canonical concerns, we found that, for the most part, each of these is successfully replied: only Abstractionism is plagued by a problem outside our reference cluster, namely Permutation, while Object Theoretic Structuralism only faces the second strand of the Semantic objection calling for extra-semantic motivation for non-eliminativism. However, unlike the other two, Object Theoretic Structuralism received further criticism, some of it related to the underlying Object Theory itself; a full assessment is but material for further work.

Before saying a few words about loose ends and further work, we highlight the status of our humble thesis, and qualify it. As we said in §1, there is indeed gas left in the non-eliminativist structuralist tank. However, the Semantic objection, both the version mounted by Bahram Assadian in recent work, as well the version extending Richard Pettigrew's argument, threaten to undermine commitment to structures entirely; with the metasemantic motivation gone – either because non-eliminativists cannot live up to their promises and provide a singular reference account of mathematical terms (the first strand of the Semantic objection), or otherwise because eliminativists can do better than expected in providing a literal interpretation of ordinary mathematical discourse (its second strand) – ontological parsimony seemingly recommends eliminativism on independent grounds and, as such, non-eliminativism would seemingly stand defeated.

An inquiry into a notion of 'face value' or 'literal' construal of ordinary mathematical discourse (as suggested in §2.2) would certainly be consequential concerning the strength of the second strand of the Semantic objection, and, as such, probably for the fate of non-eliminativism in general. Moreover, interwoven with this, another crucial task for assessing the second strand of the Semantic objection is an enquiry into syntactic and semantic criteria which can be effectively employed to distinguish between singular terms and free variables in ordinary mathematical discourse (as suggested at the end of §3.2.3.2). Meanwhile, looking for alternative, non-semantic motivation for non-eliminativism is a task that is mostly independent of the former two. However, instead of looking for non-semantic grounds for believing in structures, we can also enrich the semantic motivation itself. For instance, linking the discussion of non-eliminativism with that concerning the proper construal and semantics of instantial terms could in principle provide further grounds for endorsing structures: if, for instance, an account of instantial terms identifying them with Finean arbitrary objects is successful then, arguably given the centrality of instantial terms in mathematical practice (as documented by e.g. Martino [2018]), this would shed some plausibility over Leon Horsten [2019]'s Generic Structuralism. The present essay is significantly incomplete by omitting the latter from among the positionalist views considered; besides largely completing the picture of contemporary non-eliminativist positionalist views, the introduction of arbitrary objects – as originally proposed in Fine and Tennant [1983], and Fine [1985] and employed in formulating a version of abstractionist structuralism in Fine [1998] related but unlike Generic Structuralism and Fregean Abstractionism – would have provided us with the means of completing the assessment of Assadian [2018]'s Singular Reference problem and, as such, it would have impacted on the first strand of the Semantic objection against non-eliminativism. We flag this task here for further work.

Last but by far not least, as mentioned in passing at the beginning of this essay, a joint assessment of scientific and mathematical structuralism would be a most interesting endeavour. Finally, all the items making up our paragraph long list of notable omissions are in sight for further work, most interestingly – from our enthusiastic point of view – category theoretic structuralism, homotopy type theoretic structuralism and Modal Set-Theoretic Structuralism.

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