The nerve criterion and polyhedral completeness of intermediate logics

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Abstract

We investigate a recently-devised polyhedral semantics for intermediate logics, in which formulas are interpreted in n-dimensional polyhedra. An intermediate logic is *polyhedrally complete* if it is complete with respect to some class of polyhedra. We provide a necessary and sufficient condition for the polyhedral-completeness of a logic. This condition, which we call the Nerve Criterion, is expressed in terms of the so-called 'nerve' of a poset, a construction which we employ from polyhedral geometry.

The criterion allows for the investigation of the polyhedral completeness phenomenon using purely combinatorial methods. Utilising it, we show that there are continuum many intermediate logics that are not polyhedrally-complete. We also provide a countably infinite class of logics axiomatised by the Jankov-Fine formulas of 'starlike trees', which includes Scott's Logic, all of which are polyhedrally-complete.[1]

Contents

[1]This paper is based on [Ada19].

1 Introduction

The genesis of many connections between logic and geometry led to the discovery of topological semantics for intuitionistic and modal logic, as pioneered by Marshall Stone [Sto38], Tang Tsao-Chen [Tsa38], Alfred Tarski [Tar39] and John C. C. McKinsey [McK41]. This semantics is now well-known. In short, one starts with a topological space X, and interprets intuitionistic formulas inside the Heyting algebra of open sets of X, and modal formulas inside the modal algebra of subsets of X with $\Box$ interpreted as the topological interior operator. A celebrated result due to Tarski [Tar39] states that this provides a complete semantics for intuitionistic propositional logic (**IPC**) on the one hand, and the modal logic **S4** on the other. Moreover, one can even obtain completeness with respect to certain individual spaces. Specifically, McKinsey and Tarski showed [MT44] that for any separable metric space X without isolated points, if $\mathbf{IPC} \nvdash \phi$, then ϕ has a countermodel based on X, and similarly with **S4** in place of **IPC**. Later, this result was refined still further by Helena Rasiowa and Roman Sikorski, who showed that one can do without the assumption of separability [RS63].

This result traces out an elegant interplay between topology and logic; however, it simultaneously establishes limits on the power of this kind of interpretation. Indeed, examples of separable metric spaces without isolated points are the n-dimensional Euclidean space $\mathbb{R}^n$ and the Cantor space 2^ω. What McKinsey and Tarski's result shows is that — topologically speaking — the logics of these spaces are the same, namely **IPC** or **S4**. The upshot is that topological semantics does not allow logic to capture much of the geometric content of a space.

A natural idea is that, if we want to remedy the situation and allow for the capture of more information about a space, then we need an algebra finer than the Heyting algebra of open sets, or the modal algebra of arbitrary subsets with the interior operator. This idea was developed by Marco Aiello, Johan van Benthem, Guram Bezhanishvili and Mai Gehrke. They consider the modal logic of *chequered* subsets of $\mathbb{R}^n$: finite unions of sets of the form $\prod_{i=1}^n C_i$, where each $C_i \subseteq \mathbb{R}$ is convex ([ABB03] and [BBG03]; see also [BB07]).

This line of work was further developed in [Bez+18b], [Gab+17] and [Gab+18], which take this algebra-refinement idea one step further. To be able to capture some of the geometric content of a space, it is natural to restrict attention to topological spaces and subsets which are *polyhedra* (of arbitrary dimension). Moreover, the set $\mathrm{Sub_o}(P)$ of open subpolyhedra of P is a Heyting algebra under $\subseteq$ (and a similar result holds in the modal case). This allows for an interpretation of intuitionistic and modal formulas in $\mathrm{Sub_o}(P)$. The main result of [Bez+18b] is that more is true. A polyhedral analogue of Tarski's theorem holds: these polyhedral semantics are complete for **IPC** and **S4.Grz**. Furthermore, this approach delivers that logic can capture the dimension of the polyhedron in which it is interpreted, via the bounded depth formulas bd_n

[CZ97, Sec. 2.4]. In particular, the polyhedron P is n dimensional iff P validates bd_{n+1} and does not validate bd_{n+2} for $n \in \omega$ [Bez+18b].

In this paper we make further advances in the study of polyhedral semantics. We introduce and study polyhedral completeness for intermediate logics. We say that an intermediate logic L is polyhedrally complete if there is a class $\mathscr{C}$ of polyhedra such that L is the logic of $\mathscr{C}$. It follows from [Bez+18b] that **IPC** and the logic $\mathbf{BD}_n$ of bounded depth n, for each n are polyhedrally-complete. In this paper we construct infinitely many polyhedrally-complete logics and also show that there are continuum many polyhedrally incomplete ones.

To this end, for each poset F we define the *nerve*, $\mathcal{N}(F)$ of F as the collection of finite non-empty chains in F ordered by inclusion. The nerve will be our key concept relating logic with polyhedral geometry. The nerve construction is closely related to the operation of *barycentric subdivision* on a triangulation of a polyhedron. As was already noted in [Bez+18b], given a polyhedron P, its triangulation corresponds to a validity-preserving map from P onto F. In the algebraic terminology it corresponds to an embedding of the Heyting algebra of upsets of F into the Heyting algebra of open subpolyhedra of P. If a finite frame F is given by some triangulation Σ of a polyhedron P, then $\mathcal{N}(F)$ corresponds to a barycentric subdivision of Σ. Exploiting this relation we present a proof of the *Nerve Criterion for polyhedral completeness*: a logic L is complete with respect to some class of polyhedra if and only if it is the logic of a class of finite frames closed under taking nerves. Viewing this result in terms of Kripke frames, we can say that "the logic of a polyhedron is the logic of the iterated nerves of any one of its triangulations". The criterion yields many negative results, showing in particular that there are continuum-many non-polyhedrally-complete logics with the finite model property.

Using the Nerve criterion we will also expand the known domain of polyhedrally-complete logics. We consider logics defined using *starlike trees* as *forbidden configurations* — i.e. logics defined by the *Jankov-Fine formulas* of a collection of trees with a special property: trees which only branch at the root. Exploiting the Nerve Criterion, and a result by Zakharyaschev [Zak93] that all these logics have the finite model property, we prove that every such logic is polyhedrally-complete. This yields a countably infinite class of polyhedrally-complete logics of each finite height and of infinite height. This class includes Scott's logic **SL**. As forbidden configurations, starlike trees have a natural geometric meaning, expressing connectedness properties of polyhedral spaces. One might wonder if a generalisation is possible to arbitrary trees, or even to a wider class of frames. As to the latter, some negative results are known; see [Ada19, Corollary 4.12]. For the former, the situation is rather obscure, and it is not clear whether it is possible to account for the additional complexity introduced by allowing branching at higher points of the tree; see the discussion on 'general trees' in [Ada19, p. 61].

In a related paper [Ada+20] we look at the problem of polyhedral completeness from a different angle. We can start with some natural class of polyhedra and try to determine (axiomatize) its logic. This logic will by definition be polyhedrally complete, however, the question of axiomatization is highly non-trivial. In [Ada+20] we give an axiomatization of the logic of (n-dimensional) convex polyhedra via Jankov-Fine formulas of special star-like trees. In [Gab+19] a full characterization of 'flat' 2-dimensional polyhedral logics is announced in the setting of modal logic. In this paper we do not discuss the modal case. We note, however, that all the results proved in this paper transfer to the extensions of the modal logic **S4.Grz**.

In this paper we combine geometric methods with techniques from the logical combinatorics of finite frames, as well as combinatorial geometry, in order to deepen the exciting new link recently established between logic and polyhedra. This area is still in its infancy, and there are many interesting open problems and directions for future research. The natural ultimate goal would be a full classification of all polyhedrally-complete logics. But other directions present themselves, such as questions of decidability, or the intriguing prospect of using logical methods to prove classical theorems in geometry. We briefly explore these ideas and others in the conclusion.

The paper is organised as follows. In Section 2, we give the required background on intermediate logics and polyhedral geometry, fixing our notation. Section 3 presents the polyhedral semantics first defined in [Bez+18b], and in Section 3.3 we further elaborate on this link between logic and geometry at the level of morphisms. In Section 4, we present and prove the Nerve Criterion for polyhedral-completeness (Theorem 4.1), using techniques from rational polyhedral geometry. Making use of this criterion, Section 5 establishes that all stable logics (as defined in [BB09]) are polyhedrally-incomplete, of which there are continuum-many. Then in Section 6, we define the class of 'starlike' logics, and prove that each one is polyhedrally-complete. The techniques in the these two sections are entirely combinatorial. Finally, we conclude in Section 7 with some interesting directions for future research.

2 Preliminaries

The present paper deals with intermediate logics. In this section we remind the reader of the relational and algebraic semantics for such logics, and survey the definitions and results which will play their part in the forthcoming. As a main reference we use [CZ97]. On the other side of the link is polyhedral geometry, with which we assume rather less familiarity, and thus present in more detail.

2.1 Posets as Kripke frames

A *Kripke frame* for intuitionistic logic is simply a poset $(F, \leqslant)$. The validity relation $\vDash$ between frames and formulas is defined in the usual way, see, e.g., [CZ97, Ch. 2]. Given a class of frames $\mathbf{C}$, its *logic* is:

$$\mathrm{Logic}(\mathbf{C}) := \{\phi \text{ a formula } \mid \forall F \in \mathbf{C} \colon F \vDash \phi\}$$

Conversely, given a logic $\mathscr{L}$, define:

$$\mathrm{Frames}(\mathscr{L}) := \{F \text{ a Kripke frame} \mid F \vDash \mathscr{L}\}$$
$$\mathrm{Frames}_{\mathrm{fin}}(\mathscr{L}) := \{F \text{ a finite Kripke frame} \mid F \vDash \mathscr{L}\}$$

A logic $\mathscr{L}$ has the *finite model property* (fmp) if it is the logic of a class of finite frames. Equivalently, if $\mathscr{L} = \mathrm{Logic}(\mathrm{Frames}_{\mathrm{fin}}(\mathscr{L}))$.

Let us carve out some additional vocabulary and notation. Fix a poset F. For any $x \in F$, its *upset*, *downset*, *strict upset* and *strict downset* are defined, respectively, as follows.

$$\uparrow(x) := \{y \in F \mid y \geqslant x\}$$
$$\downarrow(x) := \{y \in F \mid y \leqslant x\}$$
$$\Uparrow(x) := \{y \in F \mid y > x\}$$
$$\Downarrow(x) := \{y \in F \mid y < x\}$$

For any set $S \subseteq F$, its *upset* and *downset* are defined, respectively, as follows.

$$\uparrow U := \bigcup_{x \in U} \uparrow(x)$$
$$\downarrow U := \bigcup_{x \in U} \downarrow(x)$$

A subframe $U \subseteq F$ is *upwards-closed* or a *generated subframe* if $U = \uparrow U$. It is *downwards-closed* if $\downarrow U = U$. The *Alexandrov topology* on F is the set $\mathrm{Up}\, F$ of its upwards-closed subsets. This constitutes a topology on F. In the sequel, we will freely switch between thinking of F as a poset and as a topological space. Note that the closed sets in this topology correspond to downwards-closed sets.

A *chain* in F is $X \subseteq F$ which as a subposet is linearly-ordered. The *length* of the chain X is $|X|$. A chain $X \subseteq F$ is *maximal* if there is no chain $Y \subseteq F$ such that $X \subset Y$ (i.e. such that X is a proper subset of Y). The *height* of F is the element of $\mathbb{N} \cup \{\infty\}$ defined by:

$$\mathsf{height}(F) := \sup\{|X| - 1 \mid X \subseteq F \text{ is a chain}\}$$

For notational uniformity, say that this value is also the *depth* of F, $\mathsf{depth}(F)$. For any $x \in F$, define its *height* and *depth* as follows.

$$\mathsf{height}(x) := \mathsf{height}(\downarrow(x))$$
$$\mathsf{depth}(x) := \mathsf{depth}(\uparrow(x))$$

The *height* of a logic $\mathscr{L}$ is the element of $\mathbb{N} \cup \{\infty\}$ given by:

$$\mathsf{height}(\mathscr{L}) := \sup\{\mathsf{height}(F) \mid F \in \text{Frames}(\mathscr{L})\}$$

A frame F has *uniform height n* if every top element has height n.

A *top element* of F is $t \in F$ such that $\mathsf{depth}(t) = 0$. The set of top elements in F is denoted by $\mathsf{Top}(F)$; let $\mathsf{Trunk}(F) := F \setminus \mathsf{Top}(F)$. For any $x, y \in F$, say that x is an *immediate predecessor* of y and that y is an *immediate successor* of x if $x < y$ and there is no $z \in F$ such that $x < z < y$. Write $\text{Succ}(x)$ for the collection of immediate successors of x.

The poset F is *rooted* if it has a minimum element, which is called the *root*, and is usually denoted by $\bot$. Define:

$$\text{Frames}_{\bot}(\mathscr{L}) := \{F \in \text{Frames}(\mathscr{L}) \mid F \text{ is rooted}\}$$
$$\text{Frames}_{\bot,\mathsf{fin}}(\mathscr{L}) := \{F \in \text{Frames}_{\mathsf{fin}}(\mathscr{L}) \mid F \text{ is rooted}\}$$

A *path* in F is a sequence $p = x_0 \cdots x_k$ of elements of F such that for each i we have $x_i < x_{i+1}$ or $x_i > x_{i+1}$. Write $p \colon x_0 \rightsquigarrow x_k$. The path p is *closed* if $x_0 = x_k$. The poset F is *path-connected* if between any two points there is a path.

Lemma 2.1. *For F a frame, it is path-connected if and only if it is connected as a topological space.*

Proof. See [BG11, Lemma 3.4]. □

A *connected component* of F is a subframe $U \subseteq F$ which is connected as a topological subspace and is such that there is no connected V with $U \subset V$.

Lemma 2.2. *Let F be a frame.*

(1) The connected components partition F.

(2) Connected components are downwards-closed and upwards-closed.

Proof. These are standard results in topology. See e.g. [Mun00, §25, p. 159]. □

An *antichain* in F is a subset $Z \subseteq F$ in which no two elements are comparable. The *width* $\mathsf{width}(F)$ of F is the cardinality of the largest antichain in F.

A function $f : F \to G$ is a *p-morphism* if for every $x \in F$ we have:

$$f(\uparrow(x)) = \uparrow(f(x))$$

Equivalently, f should satisfy the following conditions.

$$\forall x, y \in F \colon (x \leqslant y \Rightarrow f(x) \leqslant f(y)) \tag{Forth}$$
$$\forall x \in F \colon \forall z \in G \colon (f(x) \leqslant z \Rightarrow \exists y \colon (x \leqslant y \wedge f(y) = z)) \tag{Back}$$

An *up-reduction* from F to G is a surjective p-morphism f from an upwards-closed set $U \subseteq F$ to G. Write $f : F \circ\!\!\rightarrow G$.

Proposition 2.3. *If there is an up-reduction $F \circ\!\!\rightarrow G$ then* $\text{Logic}(F) \subseteq \text{Logic}(G)$. *In other words, if $G \nvDash \phi$ then $F \nvDash \phi$.*

Proof. See [CZ97, Corollary 2.8, p. 30 and Corollary 2.17, p. 32]. $\square$

Corollary 2.4. *If* $\mathbf{C}$ *is any collection of frames and* $\mathscr{L} = \mathrm{Logic}(\mathbf{C})$*, then:*

$$\mathscr{L} = \mathrm{Logic}(\mathrm{Frames}_{\perp}(\mathscr{L}))$$

Proof. First, $\mathscr{L} \subseteq \mathrm{Logic}(\mathrm{Frames}_{\perp}(\mathscr{L}))$. Conversely, suppose $\mathscr{L} \nvdash \phi$. Then there exists $F \in \mathbf{C}$ such that $F \nvDash \phi$, hence there is $x \in F$ such that $x \nvDash \phi$ (for some valuation on F), meaning that $\uparrow(x) \nvDash \phi$. Now, $\uparrow(x)$ is upwards-closed in F, hence $\mathsf{id}_{\uparrow(x)}$ is an up-reduction $F \multimap\!\!\rightarrow \uparrow(x)$. Then by Proposition 2.3, we get that $\uparrow(x) \vDash \mathscr{L}$, so that $\uparrow(x) \in \mathrm{Frames}_{\perp}(\mathscr{L})$. $\square$

A finite poset T is a *tree* if it has a root $\perp$, and every other $x \in T \setminus \{\perp\}$ has exactly one immediate predecessor. A *branch* in T is a maximal chain. Given any finite, rooted poset F, its *tree unravelling* $\mathscr{T}(F)$ is the set of its strict chains which contain the root. Define the function $\mathsf{last}\colon \mathscr{T}(F) \to F$ by:

$$X \mapsto \max(X)$$

Proposition 2.5. $\mathscr{T}(F)$ *is a tree and* last *is a p-morphism.*

Proof. See [CZ97, Theorem 2.19, p. 32]. $\square$

2.2 P-congruences

An alternative way of viewing a p-morphism $f : F \to G$ is as a kind of congruence relation on F (see [CZ97, p. 262]). This way of thinking will enable a convenient method of constructing p-morphisms.

A *p-congruence* on a frame F is an equivalence relation $\sim$ such that whenever $x \leqslant y$ we have $[x] \subseteq \downarrow[y]$. The *quotient frame* $F/{\sim}$ has as elements the equivalence classes of $\sim$, and its relation is given by:

$$[x] \leqslant [y] \quad \Longleftrightarrow \quad [x] \subseteq \downarrow[y]$$

The *quotient map* is $q\colon F \to F/{\sim}$, given by $x \mapsto [x]$.

Proposition 2.6. *The quotient map is a p-morphism.*

Proof. See [CZ97, Theorem 8.68(i), p. 263]. $\square$

Theorem 2.7 (First Isomorphism Theorem)**.** *Let* $f : F \to G$ *be a surjective p-morphism. Then relation* $\sim$ *on* F *defined by:*

$$x \sim y \quad \Longleftrightarrow \quad f(x) = f(y)$$

is a p-congruence, and moreover $F/{\sim} \cong G$ *via the map* $[x] \mapsto f(x)$*.*

Proof. See [CZ97, Theorem 8.68(ii), p. 263]. $\square$

Proposition 2.8. *Let* F *be a frame and* $\mathscr{W}$ *be a set of pair-wise disjoint subsets of* $\mathsf{Top}(F)$*. The relation* $\sim_{\mathscr{W}}$*, defined as follows, is a p-congruence.*

$$x \sim_{\mathscr{W}} y \quad \Longleftrightarrow \quad x = y \text{ or } \exists W \in \mathscr{W} : x, y \in W$$

Proof. This is immediate from the definition. $\square$

Definition 2.9. Define $F/\mathscr{W} := F/\sim_{\mathscr{W}}$. Relabel the element $[x] \in F/\mathscr{W}$ as x whenever $x \in F \setminus \bigcup \mathscr{W}$. Let $q_{\mathscr{W}}$ be the quotient map on $\sim_{\mathscr{W}}$.

2.3 Heyting algebras and co-Heyting algebras

A *Heyting algebra* is a tuple $(A, \wedge, \vee, \rightarrow, 0, 1)$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded lattice and $\rightarrow$, called the *Heyting implication*, satisfies:

$$c \leqslant a \rightarrow b \quad \Longleftrightarrow \quad c \wedge a \leqslant b$$

The validity relation $\vDash$ between Heyting algebras and formulas is defined in the usual way; the Logic notation is extended appropriately. The logic of a Heyting algebra is exactly the logic of its finitely generated subalgebras. Say that A is *locally-finite* if for every $S \subseteq A$ finite, the algebra $\langle S \rangle$ generated by S is finite. Topological spaces provide important examples of Heyting algebras: for every topological space X, its collection of open sets $\mathcal{O}(X)$ forms a Heyting algebra.

Co-Heyting algebras are the duals of Heyting algebras. Specifically, a *co-Heyting algebra* is a tuple $(C, \wedge, \vee, \leftarrow, 0, 1)$ such that $(C, \wedge, \vee, 0, 1)$ is a bounded lattice, and $\leftarrow$, called the *co-Heyting implication*, satisfies:

$$a \leftarrow b \leqslant c \quad \Longleftrightarrow \quad a \leqslant b \vee c$$

For more information on co-Heyting algebras, the reader is referred to [MT46, §1] and [Rau74], where they are called 'Brouwerian algebras'.

A Heyting algebra A may be regarded as a category. Then its dual category A^{op} is a co-Heyting algebra. In the case of the Heyting algebra $\mathcal{O}(X)$ of open sets in a topological space, such a duality has a concrete realisation: the co-Heyting algebra $\mathcal{O}(X)^{\mathrm{op}}$ is the algebra $\mathscr{C}(X)$ of closed subsets of X.

2.4 Topological semantics

Given a topological space X, the collection of open sets $\mathcal{O}(X)$ of X forms a Heyting algebra. We take $\varnothing$, X, $\cap$ and $\cup$ for 0, 1, $\wedge$ and $\vee$, respectively, and define the Heyting implication $\rightarrow$ by:

$$U \rightarrow V := \mathrm{Int}(U^{\mathsf{C}} \cup V)$$

where Int denotes the topological interior operator, and $-^{\mathsf{C}}$ is complement operator.

Proposition 2.10. *With these assignments, $\mathcal{O}(X)$ is a Heyting algebra.*

Proof. See [CZ97, Proposition 8.31, p. 247]. □

This means that we can interpret formulas inside topological spaces. Write $X \vDash \phi$ for $\mathcal{O}(X) \vDash \phi$, and extend the other Heyting algebra notation to X. The completeness result mentioned in the introduction can now be written down explicitly.

Theorem 2.11 (McKinsey-Tarski Theorem). *Let X be any separable metrisable space without isolated points. Then* $\mathbf{IPC} = \mathrm{Logic}(X)$.

Proof. The original proof is in [MT44]. Helena Rasiowa and Roman Sikorski proved this result without the separability requirement [RS63]. For a newer, more topological proof, see [Bez+18a]. For some modern proofs of specific cases, see [BB07, §2.5, pp. 241–250]. □

The topological space X also comes with a co-Heyting algebra, namely its collection of closed sets $\mathscr{C}(X)$. Co-Heyting implication on $\mathscr{C}(X)$ is defined:

$$C \leftarrow D := \mathrm{Cl}(C \setminus D)$$

where Cl denotes the topological closure operator. Now, the present topological setting provides concrete realisation of the schema of dualities between Heyting and co-Heyting algebras. Indeed, the complement operator $-^{\mathsf{C}}$ gives an isomorphism $\mathcal{O}(X)^{\mathrm{op}} \cong \mathscr{C}(X)$.

2.5 Finite Esakia Duality

The Alexandrov topology allows us to associate to each poset F the Heyting algebra $\operatorname{Up} F$ consisting of its upwards-closed sets. The process forms part of a contravariant equivalence of categories, known as the Esakia Duality. The finite fragment of this duality relates finite posets with finite Heyting algebras.

The *spectrum* of a Heyting algebra A is defined:

$$\operatorname{Spec}(A) := \{X \subseteq A \mid X \text{ is a prime filter of } A \text{ as a distributive lattice}\}$$

This constitutes a poset under subset inclusion.

Theorem 2.12. *The maps* Up *and* Spec *are the object-level components of a duality between the category of finite Kripke frames with p-morphisms and the category of finite Heyting algebras with homomorphisms.*

Proof. See [DT66]. The original proof of the general Esakia duality can be found in [Esa74; Esa19]. Detailed proofs are also given in [CJ14] and [Mor05, §5]. In the finite case, we have isomorphisms $A \cong \operatorname{Up}\operatorname{Spec} A$ and $F \cong \operatorname{Spec}\operatorname{Up} F$ for any finite Heyting algebra A and finite poset F. The former is part of Brikhoff's Representation Theorem [Bir37]. Both isomorphisms may be found in [DP90, pp. 171-172]. □

Importantly, this duality is logic-preserving.

Proposition 2.13. *Let F be a frame and A be a finite Heyting algebra. Then:*

$$\operatorname{Logic}(F) = \operatorname{Logic}(\operatorname{Up} F)$$
$$\operatorname{Logic}(A) = \operatorname{Logic}(\operatorname{Spec} A)$$

Proof. For the first equality, see [CZ97, Corollary 8.5, p. 238], noting that our Kripke frames are special cases of what are there called 'intuitionistic general frames'. The second equality follows from the first using the finite Esakia duality. □

2.6 Jankov-Fine formulas as forbidden configurations

To every finite rooted frame Q, we associate a formula $\chi(Q)$, the *Jankov-Fine* formula of Q (also called its *Jankov-De Jongh formula*). The precise definition of $\chi(Q)$ is somewhat involved, but the exact details of this syntactical form are not relevant for our considerations. What matters to us is its notable semantic property.

Theorem 2.14. *For any frame F, we have that $F \vDash \chi(Q)$ if and only if F does not up-reduce to Q.*

Proof. See [CZ97, §9.4, p. 310], for a treatment in which Jankov-Fine formulas are considered as specific instances of more general 'canonical formulas'. An alternative proof can be found in [Bez06, §3.3, p. 56], which gives a complete definition of $\chi(Q)$. See also [BB09] for an algebraic version of this result. □

Jankov-Fine formulas formalise the intuition of 'forbidden configurations'. The formula $\chi(Q)$ 'forbids' the configuration Q from its frames.

The following consequence of Theorem 2.14 will come in handy later on.

Corollary 2.15. *Let $\mathscr{L} = \operatorname{Logic}(\mathbf{C})$ where $\mathbf{C}$ is a class of frames. Then:*

$$\operatorname{Frames}_{\perp,\mathrm{fin}}(\mathscr{L}) = \{F \text{ finite rooted frame} \mid \exists G \in \mathbf{C} \colon G \circ\!\!\rightarrow F\}$$

Proof. First, if F is a finite rooted frame such that there is $G \in \mathbf{C}$ and an up-reduction $G \circ\!\!\rightarrow F$, then by Proposition 2.3 we have that $F \in \operatorname{Frames}_{\perp,\mathrm{fin}}(\mathscr{L})$. Conversely take F finite and rooted, and assume that there is no $G \in \mathbf{C}$ with $G \circ\!\!\rightarrow F$. Then by Theorem 2.14, $G \vDash \chi(F)$ for every $G \in \mathbf{C}$; whence $\mathscr{L} \vdash \chi(F)$. By Theorem 2.14, $F \nvDash \chi(F)$ implying $F \nvDash \mathscr{L}$. This yields $F \notin \operatorname{Frames}_{\perp,\mathrm{fin}}(\mathscr{L})$. □

When the poset F has a root, the condition in Theorem 2.14 can be strengthened slightly. Let F and Q be finite posets, and let Q have root $\bot$. An up-reduction $f : F \to Q$ is *pointed* with *apex* $x \in F$ if we have $\mathrm{dom}(f) = \uparrow(x)$ and $f^{-1}\{\bot\} = \{x\}$.

Lemma 2.16. *If there is an up-reduction $F \circ\!\!\rightarrow Q$ then there is a pointed up-reduction $F \circ\!\!\rightarrow Q$.*

Proof. Take $f : F \circ\!\!\rightarrow Q$, and choose $x \in f^{-1}\{\bot\}$ maximal. Then $f|_{\uparrow(x)}$ is still a p-morphism, and is moreover a pointed up-reduction $F \circ\!\!\rightarrow Q$. $\square$

Corollary 2.17. *Let F, Q be finite posets, with Q rooted. Then $F \vDash \chi(Q)$ if and only if there is no pointed up-reduction $F \circ\!\!\rightarrow Q$.*

2.7 Some standard logics

The logic **IPC** is the standard intuitionistic propositional calculus. An *intermediate logic* is any consistent logic extending **IPC**. Classical logic, **CPC**, is the largest intermediate logic.

Proposition 2.18. **IPC** *is the logic of the class of all finite frames, i.e. it has the fmp.*

Proof. See [CZ97, Theorem 2.57, p. 49]. $\square$

For every $n \in \mathbb{N}$, let $\mathbf{BD}_n$ be the logic of all finite frames of height at most n. This has the following axiomatisation in terms of Jankov-Fine formulas. Let Ch_k be the chain (linear order) on $k+1$ elements.

Proposition 2.19. $\mathbf{BD}_n$ *is the logic axiomatised by* $\mathbf{IPC} + \chi(\mathrm{Ch}_k)$.

Proof. See [CZ97, Table 9.7, p. 317, and §9]. $\square$

Scott's Logic, **SL**, is usually axiomatised by the Scott sentence:

$$\mathbf{SL} = \mathbf{IPC} + \mathbf{IPC} + ((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$$

This logic can also be axiomatised using a forbidden configuration, as follows.

Proposition 2.20. $\mathbf{SL} = \mathbf{IPC} + \chi($$)$.

Proof. See [CZ97, Table 9.7, p. 317, and §9]. $\square$

2.8 Polytopes, polyhedra and simplices

Every polyhedron considered here lives in some Euclidean space $\mathbb{R}^n$. Take $x_0, \ldots, x_d \in \mathbb{R}^n$. An *affine combination* of $x_0, \ldots, x_d$ is a point $r_0 x_0 + \cdots + r_d x_d$, specified by some $r_0, \ldots, r_d \in \mathbb{R}$ such that $r_0 + \cdots + r_d = 1$. A *convex combination* is an affine combination in which additionally each $r_i \geqslant 0$. Given a set $S \subseteq \mathbb{R}^n$, its *convex hull* $\mathrm{Conv}\, S$ is the collection of convex combinations of its elements. A subspace $S \subseteq \mathbb{R}^n$ is *convex* if $\mathrm{Conv}\, S = S$. A *polytope* is the convex hull of a finite set. A *polyhedron* in $\mathbb{R}^n$ is a set which can be expressed as the finite union of polytopes. Note that every polyhedron is closed and bounded, hence compact.

A set of points $x_0, \ldots, x_d$ is *affinely independent* if whenever:

$$r_0 x_0 + \cdots + r_d x_d = \mathbf{0} \quad \text{and} \quad r_0 + \cdots + r_d = 0$$

we must have that $r_0, \ldots, r_d = 0$. This is equivalent to saying that the vectors:

$$x_1 - x_0, \ldots, x_d - x_0$$

are linearly independent. Simplices are the most basic polyhedra of each dimension. A *d-simplex* is the convex hull σ of $d+1$ affinely independent points $x_0, \ldots, x_d$, which we call its *vertices*. Write $\sigma = x_0 \cdots x_d$; its *dimension* is $\mathrm{Dim}\,\sigma := d$.

Proposition 2.21. *Every simplex determines its vertex set: two simplices coincide if and only if they share the same vertex set.*

Proof. See [Mau80, Proposition 2.3.3, p. 32]. □

A *face* of σ is the convex hull τ of some non-empty subset of $\{x_0, \dots, x_d\}$ (note that τ is then a simplex too). Write $\tau \preccurlyeq \sigma$, and $\tau \prec \sigma$ if $\tau \neq \sigma$.

Since $x_0, \dots, x_d$ are affinely independent, every point $x \in \sigma$ can be expressed uniquely as a convex combination $x = r_0x_0 + \cdots + r_dx_d$ with $r_0, \dots, r_d \geqslant 0$ and $r_0 + \cdots + r_d = 1$. Call the tuple $(r_0, \dots, r_d)$ the *barycentric coordinates* of x in σ. The *barycentre* $\widehat{\sigma}$ of σ is the special point whose barycentric coordinates are $(\frac{1}{d+1}, \dots, \frac{1}{d+1})$. The *relative interior* of σ is defined:

$$\operatorname{Relint} \sigma := \{r_0x_0 + \cdots + r_dx_d \in \sigma \mid r_0, \dots, r_d > 0\}$$

The relative interior of σ is 'σ without its boundary' in the following sense. The *affine subspace spanned by* σ is the set of all affine combinations of $x_0, \dots, x_d$. Then the relative interior of σ coincides with the topological interior of σ inside this affine subspace. Note that $\operatorname{Cl} \operatorname{Relint} \sigma = \sigma$, the closure being taken in the ambient space $\mathbb{R}^n$.

For any $X, Y \subseteq \mathbb{R}^n$, a function $X \to Y$ is an *affine map* if it is of the form $x \mapsto Mx + b$, where M is a linear transformation and $b \in \mathbb{R}^n$. Now let P, Q be polyhedra. A homeomorphism $f : P \to Q$ is *piecewise-linear* if there is a triangulation Σ of P such that for each $\sigma \in \Sigma$ the restriction $f|_\sigma$ is affine. Call such maps *PL homeomorphisms* for short.

Proposition 2.22. *The inverse of a PL homeomorphism is a PL homeomorphism.*

Proof. See [RS72, p. 6]. □

2.9 Triangulations

A *simplicial complex* in $\mathbb{R}^n$ is a finite set Σ of simplices satisfying the following conditions.

(a) Σ is $\prec$-downwards-closed: whenever $\sigma \in \Sigma$ and $\tau \prec \sigma$ we have $\tau \in \Sigma$.

(b) If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is either empty or a common face of σ and τ.

The *support* of Σ is the set $|\Sigma| := \bigcup \Sigma$. Note that by definition this set is automatically a polyhedron. We say that Σ is a *triangulation* of the polyhedron $|\Sigma|$. Notice that Σ is a poset under $\prec$, called the *face poset*. A *subcomplex* of Σ is subset which is itself a simplicial complex. Note that a subcomplex, as a poset, is precisely a downwards-closed set. Given $\sigma \in \Sigma$, its *open star* is defined:

$$\mathrm{o}(\sigma) := \bigcup \{\operatorname{Relint}(\tau) \mid \tau \in \Sigma \text{ and } \sigma \subseteq \tau\}$$

Proposition 2.23. *The relative interiors of the simplices in a simplicial complex Σ partition $|\Sigma|$. That is, for every $x \in |\Sigma|$, there is exactly one $\sigma \in \Sigma$ such that $x \in \operatorname{Relint} \sigma$.*

Proof. See [Mau80, Proposition 2.3.6, p. 33]. □

In light of Proposition 2.23, for any $x \in |\Sigma|$ let us write σ^x for the unique $\sigma \in \Sigma$ such that $x \in \operatorname{Relint} \sigma$.

Proposition 2.24. *Let Σ be a simplicial complex, take $\tau \in \Sigma$ and $x \in \operatorname{Relint} \tau$. Then no proper face $\sigma \prec \tau$ contains x. This means that σ^x is the inclusion-smallest simplex containing x.*

Proof. See [Bez+18b, Lemma 3.1]. □

The next result is a basic fact of polyhedral geometry, and is of fundamental importance in its connection with logic. For Σ a triangulation and S a subspace of the ambient Euclidean space $\mathbb{R}^n$, define:

$$\Sigma_S := \{\sigma \in \Sigma \mid \sigma \subseteq S\}$$

This, being a downwards-closed subset of Σ, is a subcomplex of Σ.

Lemma 2.25 (Triangulation Lemma)**.** *Any polyhedron admits a triangulation which simultaneously triangulates each of any fixed finite set of subpolyhedra. That is, for a collection of polyhedra $P, Q_1, \dots, Q_m$ such that each $Q_i \subseteq P$, there is a triangulation Σ of P such that Σ_{Q_i} triangulates Q_i for each i.*

Proof. See [RS72, Theorem 2.11 and Addendum 2.12, p. 16]. □

Remark 2.26. The term 'polyhedron' is ancient, and over the years it has acquired a variety of meanings. A remark on the present terminology is in order. In one very traditional usage (though still present in some fields today), 'polyhedron' is reserved for *convex* sets. Another possible restriction, in line with historical terminology, is that 'polyhedron' applies only to *three-dimensional solids*. As is standard in the field of piecewise-linear topology however, the usage in the present paper is not subject to these restrictions (c.f. classic textbooks [Sta67; RS72]).

Note however that the standard usage of 'polyhedron' is in fact more general than the present one. In PL topology, a 'polyhedron' is the union of a *locally-finite* simplicial complex. The latter is defined as a (possibly infinite) set Σ of simplices satisfying (a) and (b) in our definition of 'simplicial complex' above, subject to the condition that every point $x \in \bigcup \Sigma$ has an open neighbourhood which intersects only finitely-many simplices. Now, it is a standard fact that 'compact polyhedra' (in the more general sense) coincide with what we are referring to here as 'polyhedra' (see [RS72, Theorem 2.2, p. 12]). Hence we are effectively using the term 'polyhedron' as a shorthand for 'compact polyhedron'; such usage is common in the literature (see, e.g. [Mau80]).

2.10 Barycentric subdivision

Triangulations allow us in some ways to approximate the structure of a polyhedron. The finer the triangulation, the better the approximation. Barycentric subdivisions afford us a systematic way of generating finer and finer triangulations, starting from a base.

Let Σ, Δ be simplicial complexes. Δ is a *subdivision* or *refinement* of Σ, notation $\Delta \lhd \Sigma$, if $|\Sigma| = |\Delta|$ and every simplex of Δ is contained in a simplex of Σ.

Lemma 2.27. *If $\Delta \lhd \Sigma$ then for every $\sigma \in \Sigma$ we have:*

$$\sigma = \bigcup \{\tau \in \Delta \mid \tau \subseteq \sigma\}$$

Proof. Let $S := \{\tau \in \Delta \mid \tau \subseteq \sigma\}$. Clearly $\bigcup S \subseteq \sigma$. Conversely, for $x \in \sigma$, let $\tau^x \in \Delta$ be such that $x \in \operatorname{Relint} \tau^x$. Since Δ refines Σ, there is some $\rho \in \Sigma$ such that $\tau^x \subseteq \rho$; assume that ρ is inclusion-minimal with this property. It follows from [Spa66, §3, Lemma 3, p. 121] that $\operatorname{Relint} \tau^x \subseteq \operatorname{Relint} \rho$, meaning that $x \in \sigma \cap \operatorname{Relint} \rho$. By condition (b) on Σ, we have that $\sigma \cap \rho$ is face of ρ. But then by Proposition 2.24, $\rho \preccurlyeq \sigma$, since otherwise $\sigma \cap \rho$ would be a proper face of ρ containing $x \in \operatorname{Relint} \rho$. Therefore $\tau^x \subseteq \rho \subseteq \sigma$ so that $x \in \bigcup S$. □

The *barycentric subdivision* $\operatorname{Sd} \Sigma$ of Σ is particularly important kind of subdivision. The idea is that we put a new vertex at the barycentre of each simplex in Σ, then build up the rest of the simplicial complex around this. Spelling this in detail is somewhat involved, and the technical details will not be needed in this paper. Hopefully the examples in Figure 1 should provide the intuition behind the construction, but for a full definition we refer the reader to [Mun84, §15, p. 83].

3 The algebra of open subpolyhedra

With the preliminaries in place, we are in a position to establish a link between intuitionistic logic and polyhedra. For this, we will be following [Bez+18b].

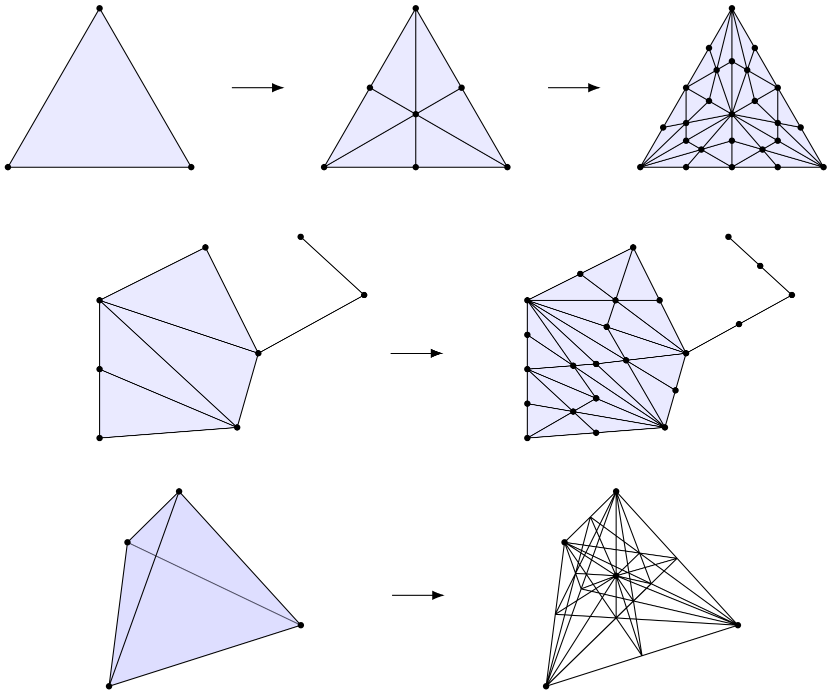


Figure 1: Examples of barycentric subdivision (the right-most tetrahedron is drawn without filled-in faces to aid clarity)

3.1 Polyhedral semantics

Given a polyhedron P, let Sub$P$ denote the collection of its subpolyhedra.

Theorem 3.1. Sub$P$ *is a co-Heyting algebra, and a subalgebra of* $\mathscr{C}(X)$.

Proof. See [Bez+18b, Corollary 3.8]. The proof makes fundamental use of the Triangulation Lemma. □

Any subpolyhedron of P is by definition compact, and hence closed. Therefore it is not surprising, once the algebraic nature of Sub$P$ is established, that it turns out to be a *co-Heyting* algebra. In topology and logic, on the other hand, it is more conventional to work with open sets and *Heyting* algebras. Thus, it is natural at this point to switch to the Heyting algebra dual to Sub$P$, which has the following concrete realisation.

Given a polyhedron P, an *open subpolyhedron* of P is the complement of a (compact) subpolyhedron of P. Denote by $\mathrm{Sub_o}P$ the collection of open subpolyhedra in P. It is evidently the dual of Sub$P$, and Theorem 3.1 yields the following.

Theorem 3.2. $\mathrm{Sub_o}P$ *is a Heyting algebra, and a subalgebra of* $\mathscr{O}(X)$.

Once we have a Heyting algebra, we can start interpreting logics. For any formula ϕ, say that $P \vDash \phi$ if and only if $\mathrm{Sub_o}P \vDash \phi$ as a Heyting algebra. Theorem 3.2 then tells us that this interpretation is sound.

Call an intermediate logic *polyhedrally-complete* if it is the logic of some class of polyhedra. The remainder of the paper will be devoted to exploring what it means for a logic to be polyhedrally-complete.

In [Bez+18b], it is shown that **IPC** is polyhedrally-complete, being the logic of all polyhedra, while $\mathbf{BD}_n$ is the logic of all polyhedra of dimension at most n. It is also noted that all polyhedrally-complete logics must have the finite model property. This will also follow from Theorem 3.7 below, since triangulations are always finite.

3.2 Triangulation subalgebras

Triangulations of polyhedra have an important algebraic correspondent. Let Σ be a triangulation of P. Then $\Sigma \subseteq \operatorname{Sub} P$. Let $\mathrm{P_c}(\Sigma)$ be the sublattice of $\operatorname{Sub_o}(P)$ generated by Σ.

Lemma 3.3. $\mathrm{P_c}(\Sigma)$ *is a co-Heyting subalgebra of* $\operatorname{Sub} P$.

Proof. See [Bez+18b, Lemma 3.6]. □

Call any algebra of the form $\mathrm{P_c}(\Sigma)$ a *triangulation subalgebra*. The following lemma allows us to interrogate the ostensibly intractable structure $\operatorname{Sub} P$ by examining its triangulation algebras, all of which are finite.

Lemma 3.4. *Every finitely-generated subalgebra of* $\operatorname{Sub} P$ *is contained in some triangulation algebra.*

Proof. See [Bez+18b, Lemma 3.2]. Essentially, this is the content of the Triangulation Lemma 2.25. □

Turning now to the dual, every triangulation Σ of a polyhedron P gives rise to a sub-Heyting algebra $\mathrm{P_o}(\Sigma)$, which we also call a *triangulation subalgebra*, generated by the complements of the simplices in Σ. Lemma 3.4 gives us the following fact about $\operatorname{Sub_o} P$.

Corollary 3.5. $\operatorname{Sub_o} P$ *is a locally-finite Heyting algebra.*

Proof. This follows from the dual of Lemma 3.4 and the fact that triangulation subalgebras are finite. □

The algebra $\mathrm{P_o}(\Sigma)$ is somewhat hard to visualise, but in fact it is exactly to dual (in the sense of the finite Esakia Duality) of Σ, regarded as a Kripke frame.

Lemma 3.6. *The map:*

$$\begin{aligned} \gamma^{\uparrow}\colon \operatorname{Up}\Sigma &\to \mathrm{P_o}(\Sigma) \\ U &\mapsto \bigcup_{\sigma \in U} \operatorname{Relint}(\sigma) \end{aligned}$$

is an isomorphism of Heyting algebras.

Proof. See [Bez+18b, Lemma 4.3]. □

Now, $\operatorname{Logic}(P)$ is the logic of its finitely-generated subalgebras, which by the dual of Lemma 3.4, is the logic of its triangulation algebras. Combining this with our duality result Lemma 3.6, we obtain the following characterisation.

Theorem 3.7. *The logic of a polyhedron is the logic of its triangulations.*

The following additional facts about triangulation algebras will be useful later on.

Lemma 3.8. *(1) Triangulation algebras determine their corresponding triangulations. That is, for any two triangulations Σ and Δ, if* $\mathrm{P_o}(\Sigma) = \mathrm{P_o}(\Delta)$ *then* $\Sigma = \Delta$.

(2) If Σ and Δ are triangulations which are isomorphic as posets then $\mathrm{P_o}(\Sigma) \cong \mathrm{P_o}(\Delta)$.

(3) If Δ refines Σ, then $\mathrm{P_o}(\Sigma)$ *is a subalgebra of* $\mathrm{P_o}(\Delta)$.

Proof. (1) It follows from conditions (a) and (b) on simplicial complexes that $\mathrm{P_c}(\Sigma)$ consists exactly of the unions of elements of Σ, and similarly for Δ. Assume $\mathrm{P_o}(\Sigma) = \mathrm{P_o}(\Delta)$, so that $\mathrm{P_c}(\Sigma) = \mathrm{P_c}(\Delta)$, and take $\sigma \in \Sigma$. Then $\sigma \in \mathrm{P_c}(\Delta)$, so $\sigma = \bigcup S$ for some $S \subseteq \Delta$, and similarly each $\tau \in S$ is $\tau = \bigcup T_\tau$ for some $T_\tau \subseteq \Sigma$. Hence:

$$\sigma = \bigcup \bigcup_{\tau \in S} T_\tau$$

But then by condition (b) on Σ, every $\rho \in \bigcup_{\tau \in S} T_\tau$ must either be equal to σ or be a proper face of σ. Since $\operatorname{Relint}\sigma$ contains no proper face of σ, we must have $\sigma \in T_\tau$ for some $\tau \in S$. But then $\sigma \subseteq \tau \subseteq \sigma$, and so $\sigma \in \Delta$. Applying this argument also in the other direction, we get that $\Sigma = \Delta$.

(2) This follows from Lemma 3.6.

(3) By Lemma 2.27, every $\sigma \in \Sigma$ is the union of simplices in Δ. Whence $\Sigma \subseteq \mathrm{P_c}(\Delta)$. Therefore, by definition $\mathrm{P_c}(\Sigma) \subseteq \mathrm{P_c}(\Delta)$. By symmetry $\mathrm{P_c}(\Delta) \subseteq \mathrm{P_c}(\Sigma)$. □

3.3 PL homeomorphisms and polyhedral maps

Let us now consider the relationship between logic and polyhedral geometry on the level of morphisms.

A map $f : P \to Q$ is a *PL embedding* if $f(P)$ is a polyhedron and $f : P \to f(P)$ is a PL homeomorphism.

Let P be a polyhedron and F be a poset. A function $f : P \to F$ is a *polyhedral map* if the preimage of any open set in F is an open subpolyhedron of P. Note that such a function is continuous.

Proposition 3.9. *Let $f : P \to F$ be a function from a polyhedron P to a finite poset F, and write $f^* := f^{-1}[-] \colon \mathscr{P}(F) \to \mathscr{P}(P)$ for the inverse image function.*

(1) The function f is polyhedral if and only if f^* descends to a lattice homomorphism $f^* \colon \mathrm{Up}\,F \to \mathrm{Sub_o}P$.

(2) The function f is polyhedral and open if and only if f^* descends to a homomorphism of Heyting algebras $f^* \colon \mathrm{Up}\,F \to \mathrm{Sub_o}P$.

Proof. Clearly f^* is a homomorphism of Boolean algebras, so (1) follows from the definitions. As for (2), let us first assume that f is polyhedral and open, and take $U, V \in \mathrm{Up}\,F$ with the aim of showing that $f^*(U \to V) = f^*(U) \to f^*(V)$. The left-to-right inclusion follows from the fact that f^* is a lattice homomorphism. For the right-to-left, writing X^{C} for the complement of X, we have the following chain of inclusions.

$$
\begin{aligned}
f[f^*(U) \to f^*(V)] &= f\left[\mathrm{Int}\big(f^{-1}[U]^{\mathsf{C}} \cup f^{-1}[V]\big)\right] \\
&\subseteq \mathrm{Int}\big(f\left[f^{-1}[U]^{\mathsf{C}} \cup f^{-1}[V]\right]\big) && (f \text{ is open}) \\
&= \mathrm{Int}\big(f\left[f^{-1}[U^{\mathsf{C}} \cup V]\right]\big) \\
&\subseteq \mathrm{Int}(U^{\mathsf{C}} \cup V) \\
&= U \to V
\end{aligned}
$$

Applying $f^* = f^{-1}$ to both sides, we get that $f^*(U) \to f^*(V) \subseteq f^*(U \to V)$.

For the converse implication, assume that f^* is a Heyting algebra homomorphism. By (1), f is polyhedral, so take $W \subseteq F$ with the aim of showing that $f^{-1}[\mathrm{Int}\,W] = \mathrm{Int}(f^{-1}[W])$. First let $A := \mathrm{Int}((\uparrow W)^{\mathsf{C}} \cup W) \cup \mathrm{Int}(W^{\mathsf{C}})$ and $B := \mathrm{Int}\,W$. A routine calculation verifies that $A^{\mathsf{C}} \cup B = W$, and moreover that $A, B \in \mathrm{Up}\,F$. Then:

$$
\begin{aligned}
f^{-1}[\mathrm{Int}\,W] &= f^*[A \to B] \\
&= f^*[A] \to f^*[B] && (f^* \text{ is a homomorphism}) \\
&= \mathrm{Int}(f^*[A]^{\mathsf{C}} \cup f^*[B]) \\
&= \mathrm{Int}(f^*[A^{\mathsf{C}} \cup B]) \\
&= \mathrm{Int}(f^{-1}[W])
\end{aligned}
$$

□

Proposition 3.10. *Any PL homeomorphism $f : P \to Q$ between polyhedra, along with its inverse $g : Q \to P$, induce mutually inverse isomorphisms of Heyting algebras $f^* \colon \mathrm{Sub_o}Q \to \mathrm{Sub_o}P$ and $g^* \colon \mathrm{Sub_o}P \to \mathrm{Sub_o}Q$.*

Proof. The inverse image of a subpolyhedron under a PL homeomorphism is again a subpolyhedron [RS72, Corollary 2.5, p. 13], meaning the inverse image of an open subpolyhedron is an open subpolyhedron. Furthermore, homeomorphisms are open maps. Hence $f^* \colon \mathscr{P}(Q) \to \mathscr{P}(P)$ and $g^* \colon \mathscr{P}(P) \to \mathscr{P}(Q)$ descend to functions as in the statement. These are mutually inverse isomorphisms of lattices by definition.

The fact that they also preserve Heyting implication follows just as in the proof of Proposition 3.9. □

Corollary 3.11. *If P and Q are PL homeomorphic then* $\mathrm{Logic}(P) = \mathrm{Logic}(Q)$.

Let Σ be a simplicial complex and F be a poset. Given any function $f : \Sigma \to F$, define the map $\widehat{f} : |\Sigma| \to F$ by:

$$\widehat{f}(x) := f(\sigma^x)$$

Proposition 3.12. *When $f : \Sigma \to F$ is a p-morphism, $\widehat{f} : |\Sigma| \to F$ is an open polyhedral map.*

Proof. For any $U \in \mathrm{Up}\, F$, we have that:

$$\widehat{f}^{-1}[U] = \bigcup\{\mathrm{Relint}\,\sigma \mid \sigma \in \Sigma \text{ and } \sigma \in f^{-1}[U]\} = \gamma^{\uparrow}(f^{-1}[U])$$

Since f is monotonic, $f^{-1}[U]$ is upwards-closed in Σ, whence as above $\widehat{f}^{-1}[U]$ is an open sub-polyhedron of $|\Sigma|$. Now take an open set $W \subseteq |\Sigma|$, with the aim of showing that $\widehat{f}[W]$ is open. Define:

$$\Sigma \# W := \{\sigma \in \Sigma \mid \mathrm{Relint}(\sigma) \cap W \neq \varnothing\}$$

Then:

$$\widehat{f}[W] = \{f(\sigma^x) \mid x \in W\} = f[\Sigma \# W]$$

If $\sigma \in \Sigma \# W$ and $\sigma \preccurlyeq \tau$, then as $\sigma \subseteq \tau = \mathrm{Cl}\,\mathrm{Relint}\,\tau$ and W is open, we have $\tau \in \Sigma \# W$; i.e. $\Sigma \# W$ is upwards-closed. But now, f is open and so $\widehat{f}[W]$ is also upwards-closed. □

4 The Nerve Criterion

Given a poset F, its *nerve*, $\mathscr{N}(F)$, is the collection of finite non-empty chains in F ordered by inclusion.

The following theorem is one of the main contributions of the paper:

Theorem 4.1 (The Nerve Criterion)**.** *A logic is polyhedrally-complete if and only if it is the logic of a class of finite frames closed under the nerve construction $\mathscr{N}$.*

The utility of the Nerve Criterion is that it transforms logic-geometric questions into questions about finite posets, to which finite combinatorial methods are applicable.

The proof of the Nerve Criterion is given in Section 4.5, and for it we will need to import several results from polyhedral geometry. The heart of the argument is the classical link between nerves and barycentric subdivision.

Let Σ be a simplicial complex. The *kth derived subdivision of* Σ, denoted by $\Sigma^{(k)}$, is the result of applying the barycentric subdivision operation k-times on Σ. I.e. $\Sigma^{(k)} = \mathrm{Sd}^k\,\Sigma$. Now let A be a triangulation subalgebra of $\mathrm{Sub}_{\mathrm{o}}P$ for some polyhedron P. By Lemma 3.8 (1), there is a unique triangulation Σ of P such that $A = \mathrm{P}_{\mathrm{o}}(\Sigma)$. For any $k \in \mathbb{N}$, let $A^{(k)} := \mathrm{P}_{\mathrm{o}}(\Sigma^{(k)})$.

Theorem 4.2. *Let P be a polyhedron and let A be any triangulation subalgebra of* $\mathrm{Sub}_{\mathrm{o}}P$*. For any finitely-generated subalgebra B of* $\mathrm{Sub}_{\mathrm{o}}P$*, there is $k \in \mathbb{N}$ such that B is isomorphic to a subalgebra of $A^{(k)}$.*

Sections 4.1–4.4 will be devoted to proving this theorem.

4.1 Rational polyhedra and unimodular triangulations

The intuition behind Theorem 4.2 is that any triangulation can be approximated from any other by taking iterated barycentric subdivisions. The difficulty one might face with spelling out such an intuition is dealing with the 'continuum nature' of $\mathbb{R}^n$. It might be imagined that, if we start with a triangulation Σ on irrational vertices and try to approximate it using the iterated barycentric subdivisions of a triangulation on rational vertices, the approximations would never quite capture all of Σ. The approach taken here is effectively to show that it suffices to restrict attention to the rational case. In order to make this idea precise, we need some definitions. For these, we will mainly be following [Mun11].

A polytope in $\mathbb{R}^n$ is *rational* if it may be written as the convex hull of finitely many points in $\mathbb{Q}^n \subseteq \mathbb{R}^n$. A polyhedron in $\mathbb{R}^n$ is *rational* if it may be written as a union of a finite collection of rational polytopes. A simplicial complex Σ is *rational* if it consists of rational simplices. Note that when this is the case, $|\Sigma|$ is a rational polyhedron.

For any $x \in \mathbb{Q}^n \subseteq \mathbb{R}^n$, there is a unique way to write out x in coordinates as $x = (\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n})$ such that for each i, we have $p_i, q_i \in \mathbb{Z}$ coprime. The *denominator* of x is defined:

$$\mathrm{Den}(x) := \mathrm{lcm}\{q_1, \dots, q_n\}$$

Note that $\mathrm{Den}(x) = 1$ if and only if x has integer coordinates. Letting $q = \mathrm{Den}(x)$, the *homogeneous correspondent* of x is defined to be the integer vector:

$$\widetilde{x} := \left(\frac{qp_1}{q_1}, \dots, \frac{qp_n}{q_n}, q\right)$$

A rational d-simplex $\sigma = x_0 \cdots x_d$ is *unimodular* if there is an $(n+1)\times(n+1)$ matrix with integer entries whose first d columns are $\widetilde{x_0}, \dots, \widetilde{x_d}$, and whose determinant is ± 1. This is equivalent to requiring that the set $\{\widetilde{x_0}, \dots, \widetilde{x_d}\}$ can be completed to a $\mathbb{Z}$*-module basis* of $\mathbb{Z}^{d+1}$. A simplicial complex is *unimodular* if each one of its simplices is unimodular.

4.2 Farey subdivisions

In order to obtain the main result concerning barycentric subdivisions, we go via another kind of subdivision which is more amenable to the rational case.

Proposition 4.3. *For any $x, y \in \mathbb{Q}^n$, there is a unique $m \in \mathbb{Q}^n$ such that $\widetilde{m} = \widetilde{x} + \widetilde{y}$, and this lies in the relative interior of the* 1*-simplex* Conv $\{x, y\}$.

Proof. Let $H_{n+1} \subseteq \mathbb{R}^{n+1}$ be the hyperplane specified by:

$$H_{n+1} := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 1\}$$

Identify $\mathbb{Q}^n$ with the set of rational points of H_{n+1}. Under this identification, $\widetilde{m} = \widetilde{x} + \widetilde{y}$ lies in the affine cone:

$$\{a\widetilde{x} + b\widetilde{y} \mid a, b > 0\}$$

A routine computation then proves the geometrically evident fact that m is the point of intersection of the line spanned in R^{n+1} by the vector $\widetilde{m}$, with the hyperplane H_{n+1}; from which the result follows. $\square$

For $x, y \in \mathbb{Q}^n$, let this $m \in \mathbb{Q}^n$ be their *Farey mediant*. The Farey mediant behaves in a similar way to the barycentre of x and y.

Using the notion of Farey mediant, one can define the notion of a Farey subdivision. Just as in the case of barycentric subdivision, the precise formulation is somewhat involved, while the technical details are not so important for the present paper. Thus, as before, we will present the idea, coupled with some diagrams, in order to give the essential intuition. For a complete definition, we refer the reader to [Mun11, §5.1, p. 55].

Let Σ_1, Σ_2 be rational simplicial complexes in $\mathbb{R}^n$. Then Σ_2 is an *elementary Farey subdivision* of Σ_1 if it is obtained from Σ_1 by subdividing exactly one of its 1-simplices Conv $\{x, y\}$ through the introduction of the Farey mediant m of x and y as the single new vertex of Σ_2. If Σ_2 can be obtained from Σ_1 through finitely many successive elementary Farey subdivisions, then we say Σ_2 is a *Farey subdivision* of Σ_1. See Figure 2 for examples of this operation.

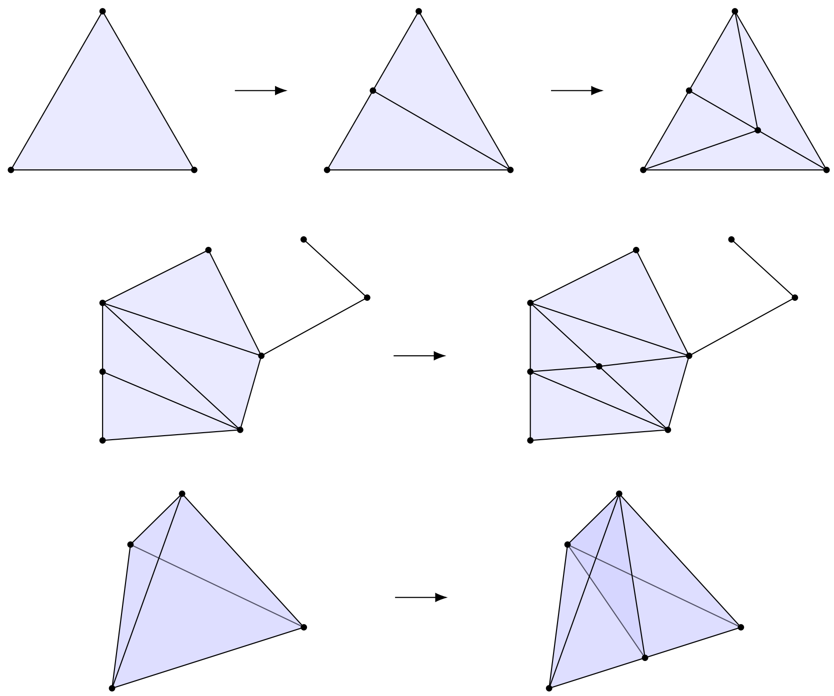


Figure 2: Examples of elementary Farey subdivisions

To relate Farey subdivisions with barycentric subdivisions, note that one may define an *elementary barycentric subdivision* analogously to the Farey case, by taking a single 1-simplex and adding a new vertex at its barycentre. The following technical lemma will be useful below; its proof uses the details of the full definition of elementary Farey and barycentric subdivision.

Lemma 4.4. *Let Σ, Δ be simplicial complexes with Σ rational, assume that $\gamma\colon \Sigma \to \Delta$ is an isomorphism of Σ and Δ as posets, and take a 1-simplex $\sigma \in \Sigma$. Then the elementary Farey subdivision of Σ along σ and the elementary barycentric subdivision of Δ along $\gamma(\sigma)$ are isomorphic as posets.*

Proof. Indeed, at the level of posets, elementary Farey subdivision and elementary barycentric subdivision are the same operation: we take a 1-simplex and add a new vertex somewhere in its interior, then construct the rest of the complex around this. For more details see [Ale30, §III]. □

The following is a fundamental fact of rational polyhedral geometry, and captures the idea of 'rational approximation'.

Lemma 4.5 (The De Concini-Procesi Lemma)**.** *Let P be a rational polyhedron, and let Σ be a unimodular triangulation of P. There exists a sequence $(\Sigma_i)_{i\in\mathbb{N}}$ of unimodular triangulations of P with $\Sigma_0 = \Sigma$ such that:*

(a) For each $i \in \mathbb{N}$, Σ_{i+1} is an elementary Farey subdivision of Σ_i, and

(b) For any rational polyhedron $Q \subseteq P$, there is $i \in \mathbb{N}$ such that Σ_i triangulates Q.

Proof. See [Mun11, Theorem 5.3, p. 57]. □

4.3 From $\mathbb{R}$ to $\mathbb{Q}$

We will now see how to relate general polyhedra to rational polyhedra, and general simplicial complexes to unimodular simplicial complexes.

Lemma 4.6. *Let P be a polyhedron, and let Σ be a triangulation of P. There exist an integer $n \in \mathbb{N}$, a rational polyhedron $Q \subseteq \mathbb{R}^n$, and a unimodular triangulation Δ of Q such that P and Q are PL-homeomorphic via a map that induces an isomorphism of Σ and Δ as posets.*

Proof. This is a standard argument. Fix a bijection β from the vertices of Σ to the standard basis of $\mathbb{R}^n$, where n is the number of vertices in Σ. Take a simplex $\sigma = x_0 \cdots x_d$ in Σ. Note that the points $\beta(x_0), \ldots, \beta(x_d)$ are affinely independent; let $\alpha(\sigma)$ be the d-simplex spanned by their convex hull: $\alpha(\sigma) := \mathrm{Conv}\{\beta(x_0), \ldots, \beta(x_d)\}$. Since the vertices of $\alpha(\sigma)$ are standard basis elements, $\alpha(\sigma)$ is a unimodular simplex by definition. Let $f_\sigma \colon \sigma \to \alpha(\sigma)$ be the linear map determined by $f_\sigma(x_i) = \beta(x_i)$ for each i, and let $g_\sigma \colon \alpha(\sigma) \to \sigma$ be its inverse, determined by $g_\sigma(\beta(x_i)) = x_i$.

Now, let $Q := \bigcup_{\sigma \in \Sigma} \alpha(\sigma)$. For any simplices $\sigma \preccurlyeq \tau$, the map f_σ agrees with f_τ on σ. Hence we may glue these maps together to form a map $f : P \to Q$, i.e. $f(x) = f_\sigma(x)$, where σ is any simplex of Σ containing x. Similarly, we may glue together the maps g_σ for $\sigma \in \Sigma$ to form an inverse to f. By definition f is a PL homeomorphism. Finally, note that $\Delta := \{\alpha(\sigma) \mid \sigma \in \Sigma\}$ is a triangulation of Q, and that f induces the poset isomorphism $\sigma \mapsto \alpha(\sigma)$ between Σ and Δ. □

Lemma 4.7. *Let Σ be a unimodular triangulation of the rational polyhedron P, and suppose Σ' is a Farey subdivision of Σ. There is a triangulation Δ of P which is isomorphic as a poset to Σ', and $k \in \mathbb{N}$ such that $\Sigma^{(k)}$ refines Δ.*

Proof. The proof works by replacing each elementary Farey subdivision by an elementary barycentric subdivision. We induct on the number $m \in \mathbb{N}^{>0}$ of elementary Farey subdivisions needed to obtain Σ' from Σ. If $m = 1$, let $\mathrm{Conv}\{x, y\}$ be the 1-simplex of Σ being subdivided through its Farey mediant. Then the first barycentric subdivision $\Sigma^{(1)}$ of Σ refines the elementary barycentric subdivision Σ^* of Σ along $\mathrm{Conv}\{x, y\}$. By Lemma 4.4, Σ^* and Σ' are isomorphic.

For the induction step, suppose $m > 1$, and write $(\Sigma_i)_{i=0}^m$ for the finite sequence of triangulations connecting $\Sigma = \Sigma_0$ to $\Sigma' = \Sigma_m$ through elementary Farey subdivisions. By the induction hypothesis, there is $k \in \mathbb{N}$ such that $\Sigma^{(k)}$ refines a triangulation Δ isomorphic to Σ_{m-1}; let us fix one such isomorphism γ. Let $\mathrm{Conv}\{x, y\}$ be the 1-simplex of Σ_{m-1} that must be subdivided through its Farey mediant in order to obtain Σ_m. Let further σ be the simplex of Δ that corresponds to $\mathrm{Conv}\{x, y\}$ through the isomorphism γ. Since the 1-simplices are exactly the height-1 elements of Δ, we get that σ is a 1-simplex. Then $\Sigma^{(k+1)}$ refines Δ^*, the latter denoting the elementary barycentric subdivision of Δ along σ. But Δ is isomorphic to Σ_{m-1}, and therefore by Lemma 4.4, Δ^* is isomorphic to Σ_m. □

Lemma 4.8 (Beynon's Lemma)**.** *Let P be a rational polyhedron, and let Σ be a triangulation of P. There exists a rational triangulation of P which is isomorphic as a poset to Σ.*

Proof. This is the main result of [Bey77]. □

4.4 Putting it all together

It is time to combine all our ingredients and prove the main theorem of the chapter.

Proof of Theorem 4.2. Let Σ be the triangulation of P such that $A = \mathrm{P_o}(\Sigma)$. Using Lemma 4.6, Lemma 3.8 (2) and Proposition 3.10 we may assume without loss of generality that P is rational and Σ is unimodular. By Lemma 3.4, there is a triangulation Δ of P such that B is isomorphic to a subalgebra of $\mathrm{P_o}(\Delta)$. By Beynon's Lemma 4.8 and Lemma 3.8 (2), we may assume that Δ is rational (and hence each member of B is, too). By the De Concini-Procesi Lemma 4.5, there is a Farey subdivision Σ' of Σ that refines Δ. Therefore by Lemma 3.8 (3), B is isomorphic to a subalgebra of $\mathrm{P_o}(\Sigma')$. By Lemma 4.7, there is $k \in \mathbb{N}$ such that $\Sigma^{(k)}$ refines Σ' up to isomorphism. Hence by Lemma 3.8 (3) again, $A^{(k)}$ contains a subalgebra isomorphic to $\mathrm{P_o}(\Sigma')$, and therefore also a subalgebra isomorphic to B. □

4.5 Bringing nerves back onto the stage

Let us now see how to attain the Nerve Criterion from Theorem 4.2. The reason that the nerve construction is relevant here is the following.

Proposition 4.9. *Let Σ be a simplicial complex. The barycentric subdivision of Σ is isomorphic as a poset to the nerve of Σ:*

$$\mathrm{Sd}\,\Sigma \cong \mathscr{N}(\Sigma)$$

Proof. Let us give an intuitive proof as to why this is the case. For more detail, we refer the reader to [Mau80, Proposition 2.5.10, p. 51] and [RW12, §3].

In our informal definition, the construction of the barycentric subdivision of a simplicial complex Σ involved putting a new vertex at the barycentre of each simplex of Σ, and constructing the rest of $\mathrm{Sd}\,\Sigma$ around this. Let us consider in a little more detail what this involves. For each simplex $\sigma \in \Sigma$, we have a new 0-simplex, which we will label $\{\sigma\}$. The first step in 'building up the rest of $\mathrm{Sd}\,\Sigma$' would be to add in some 1-simplices. A little reflection and diagram staring (consider again Figure 1) indicates that we should put a 1-simplex between $\{\sigma\}$ and $\{\tau\}$ exactly when $\sigma \prec \tau$ or $\tau \prec \sigma$, i.e. when $\{\sigma,\tau\}$ is a chain in Σ. Let us label such a new 1-simplex $\{\sigma,\tau\}$. The next stage would be to add in some 2-simplices. Some further reflection and diagram staring should indicate that we should add a 2-simplex connecting σ, τ and ρ exactly when $\{\sigma,\tau,\rho\}$ is a chain in Σ. Label such a 2-simplex by $\{\sigma,\tau,\rho\}$. Continuing in this fashion, we eventually arrive at an isomorphism $\mathrm{Sd}\,\Sigma \cong \mathscr{N}(\Sigma)$. □

Corollary 4.10. *For P a polyhedron and Σ a triangulation of P we have:*

$$\mathrm{Logic}(P) = \mathrm{Logic}(\mathscr{N}^k(\Sigma) \mid k \in \mathbb{N})$$

Proof. Indeed:

$$\begin{aligned}
\mathrm{Logic}(P) &= \mathrm{Logic}(\mathrm{Sub_o}P) \\
&= \mathrm{Logic}(A \mid A \text{ finitely-generated subalgebra of } \mathrm{Sub_o}P) && \text{(Lemma 3.4)} \\
&= \mathrm{Logic}(\mathrm{P_o}(\Sigma^{(k)}) \mid k \in \mathbb{N}) && \text{(Theorem 4.2)} \\
&= \mathrm{Logic}(\Sigma^{(k)} \mid k \in \mathbb{N}) && \text{(as above)} \\
&= \mathrm{Logic}(\mathscr{N}^k(\Sigma) \mid k \in \mathbb{N}) && \text{(Proposition 4.9)}
\end{aligned}$$

□

For the converse direction of the Nerve Criterion, we will need the following construction, described in [Bez+18b]. Let F be a finite poset. Using the nerve, we define its *geometric realisation* via a simplicial complex. Enumerate $F = \{x_1,\dots,x_m\}$, and let $e_1,\dots,e_m$ be the standard basis vectors of $\mathbb{R}^m$. The *simplicial complex induced by F* is defined:

$$\nabla F := \{\mathrm{Conv}\{e_{i_1},\dots,e_{i_k}\} \mid \{x_{i_1},\dots,x_{i_k}\} \in \mathscr{N}(F)\}$$

Now, the map $\max\colon \mathscr{N}(F) \to F$, which sends a chain to is maximum element, is a p-morphism. Since $\nabla F \cong \mathscr{N}(F)$ as posets, this induces an open polyhedral map $|\nabla F| \to F$, meaning that $\mathrm{Logic}(|\nabla F|) \subseteq \mathrm{Logic}(F)$.

Proof of Theorem 4.1, the Nerve Criterion. Assume that $\mathscr{L}$ is the logic of a class $\mathbf{C}$ of polyhedra. For each $P \in \mathbf{C}$ fix a triangulation Σ_P, and let:

$$\mathbf{C}^* := \{\mathscr{N}^k(\Sigma_P) \mid P \in \mathbf{C} \text{ and } k \in \mathbb{N}\}$$

Then:

$$\begin{aligned}
\mathrm{Logic}(\mathbf{C}^*) &= \bigcap_{P\in\mathbf{C}} \mathrm{Logic}(\mathscr{N}^k(\Sigma_P) \mid k \in \mathbb{N}) \\
&= \bigcap_{p\in\mathbf{C}} \mathrm{Logic}(P) && \text{(Corollary 4.10)} \\
&= \mathrm{Logic}(\mathbf{C}) = \mathscr{L}
\end{aligned}$$

Conversely, assume that $\mathscr{L} = \text{Logic}(\mathbf{D})$, where $\mathbf{D}$ is a class of finite frames closed under $\mathscr{N}$. Let:

$$\mathbf{D}_* := \{|\nabla(F)| : F \in \mathbf{D}\}$$

We will show that $\mathscr{L} = \text{Logic}(\mathbf{D}_*)$. First suppose that $\mathscr{L} \nvdash \phi$, so that $F \not\models \phi$ for some $F \in \mathbf{D}$. Then we have that $|\nabla(F)| \not\models \phi$, so that $\text{Logic}(\mathbf{D}_*) \nvdash \phi$. Conversely, suppose that $\text{Logic}(\mathbf{D}_*) \nvdash \phi$, so that $|\nabla(F)| \not\models \phi$ for some $F \in \mathbf{D}$. By definition $\nabla(F)$ is a triangulation of $|\nabla(F)|$, hence by Corollary 4.10 there is $k \in \mathbb{N}$ such that $\nabla(F)^{(k)} \not\models \phi$. But $\nabla(F) \cong \mathscr{N}(F)$ by definition, and so by Proposition 4.9 we get $\mathscr{N}^{k+1}(F) \cong \nabla(F)^{(k)}$. Thus, as $\mathbf{D}$ is closed under $\mathscr{N}$, we get that $\mathscr{L} \nvdash \phi$. ☐

5 Polyhedrally incomplete logics

In this section, we use the Nerve Criterion to provide a negative result concerning polyhedral completeness, showing that every stable logic is polyhedrally-incomplete, of which there are contiuum many.

A logic $\mathscr{L}$ is *stable* if $\text{Frames}_\perp(\mathscr{L})$ is closed under monotone images (see [BB17], where stable logics are first defined).

Proposition 5.1. *The following well-known logics[2] are all stable.*

(i) The logic of weak excluded middle, $\mathbf{KC} = \mathbf{IPC} + (\neg p \vee \neg\neg p)$.

(ii) Gödel-Dummett logic, $\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p)$.

(iii) $\mathbf{LC}_n = \mathbf{LC} + \mathbf{BD}_n$.

(iv) The logic of bounded width n, $\mathbf{BW}_n = \mathbf{IPC} + \bigvee_{i=0}^{n}(p_i \to \bigvee_{j\neq i} p_j)$.

(v) The logic of bounded top width n, defined:

$$\mathbf{BTW}_n := \bigwedge_{0\leqslant i<j\leqslant n} \neg(\neg p_i \wedge \neg p_j) \to \bigvee_{i=0}^{n}(\neg p_i \to \bigvee_{j\neq i} \neg p_j)$$

(vi) The logic of bounded cardinality n, defined:

$$\mathbf{BC}_n := p_0 \vee (p_0 \to p_1) \vee ((p_0 \wedge p_1) \to p_2) \vee \cdots \vee ((p_0 \wedge \cdots \wedge p_{n-1}) \to p_n)$$

Proof. See [BB17, Theorem 7.3]. ☐

In fact:

Theorem 5.2. *There are continuum-many stable logics.*

Proof. See [BB17, Theorem 6.13]. ☐

Theorem 5.3. *Every stable logic has the finite model property.*

Proof. See [BB17, Theorem 6.8]. ☐

Hence, stable logics are good candidates for polyhedrally-complete logics. However:

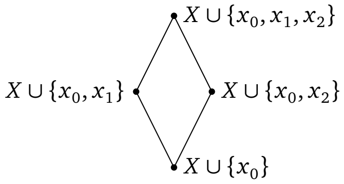
Theorem 5.4. *If $\mathscr{L}$ is a stable logic other than* $\mathbf{IPC}$*, and* $\text{Frames}(\mathscr{L})$ *contains a frame of height at least* 2*, then $\mathscr{L}$ is not polyhedrally-complete.*

Proof. Let $\mathscr{L}$ be a polyhedrally-complete stable logic of height at least 2. We show that $\mathscr{L} = \mathbf{IPC}$.

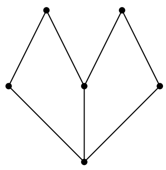
By the Nerve Criterion 4.1, there is a class $\mathbf{C}$ of finite frames closed under $\mathscr{N}$ such that $\mathscr{L} = \text{Logic}(\mathbf{C})$. Since $\text{Frames}(\mathscr{L})$ contains a frame of height at least 2, we must have $\mathscr{L} \nvdash \mathbf{BD}_1$. Since $\mathscr{L} = \text{Logic}(\mathbf{C})$, there is therefore $F \in \mathbf{C}$ such that $\text{height}(F) \geqslant 2$. This means there are $x_0, x_1, x_2 \in F$ with $x_0 < x_1 < x_2$. Without loss of generality, we may assume that x_2 is a top element and that x_1 is an immediate predecessor of x_2

[2]For more information on these logics see [CZ97, Table 4.1, p. 112].

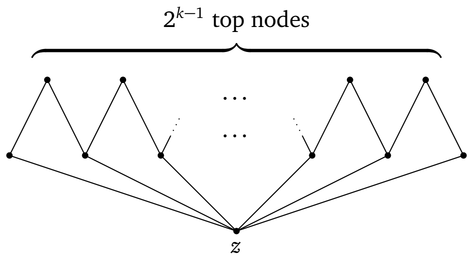
and x_0 an immediate predecessor of x_1. Now, by assumption $\mathcal{N}^k(F) \in \mathbf{C}$ for every $k \in \mathbb{N}$. Let us examine the structure of these frames a little. Note that $\{x_0, x_1, x_2\}$ is a chain. Let X be a maximal chain in $\Downarrow(x_0)$. We have the following relations occurring in $\mathcal{N}(F)$.



Moreover, by assumptions on x_0, x_1, x_2 and X, we have that $X \cup \{x_0, x_1, x_2\}$ is a top element of $\mathcal{N}(F)$, with $X \cup \{x_0, x_1\}$ and $X \cup \{x_0, x_2\}$ immediate predecessors, and $X \cup \{x_0\}$ an immediate predecessor of those. So, we may apply this argument once more, to obtain the following structure sitting at the top of $\mathcal{N}^2(F)$.



Iterating, we see that at the top of $\mathcal{N}^k(F)$ we have the following structure.



Let z be the base element of this structure, as indicated. Now, take $k \in \mathbb{N}$ and let $\{t_1, \dots, t_m\}$ be the top nodes of $\mathcal{N}^k(F)$ produced by this construction, where $m = 2^{k-1}$. By Proposition 2.3, $\uparrow(z) \in \mathrm{Frames}_\perp(\mathcal{L})$.

Let now G be an arbitrary poset with up to m elements $\{y_1, \dots, y_m\}$ (possibly with duplicates) plus a root $\perp$. Define $f : \uparrow(z) \to G$ as follows.

$$x \mapsto \begin{cases} y_i & \text{if } x = t_i, \\ \perp & \text{otherwise.} \end{cases}$$

Then f is monotonic. Since $\mathcal{L}$ is stable, this means that $G \in \mathrm{Frames}_\perp(\mathcal{L})$. Thus (since, by Proposition 2.18 and Corollary 2.4, **IPC** is the logic of finite rooted frames) we get that $\mathcal{L} = \mathbf{IPC}$. $\lightning$ $\square$

6 Polyhedrally complete logics: starlike completeness

In this section, we use the Nerve Criterion to establish a class of logics which are polyhedrally-complete. These logics are axiomatised using the forbidden configuration method of Jankov-Fine formulas. The proofs in this section largely involve combinatorial manipulations of posets.

6.1 Starlike trees

A tree T is a *starlike tree* if every $x \in T \setminus \{\bot\}$ has at most one immediate successor. The terminology 'starlike' comes from graph theory [WS79]. If we were to place the root of a starlike tree at the centre of a diagram and arrange its branches radially outward, it would look like a star.

It will be useful to carve out some notation with which we can conveniently point to each starlike tree (up to isomorphism). Note that a starlike tree is determined by the multiset of its branch heights. The following notation is inspired by that used in the theory of multisets.

Let $n_1, \ldots, n_k, m_1, \ldots, m_k \in \mathbb{N}^{>0}$, with $n_1, \ldots, n_k$ distinct. Then let us define $T = \langle n_1^{m_1} \cdots n_k^{m_k} \rangle$ as the starlike tree with the property that if we remove the root $\bot$ we are left with exactly, for each i, m_i chains of length n_i. Let $\langle \epsilon \rangle = \bullet$, the singleton poset. Call $\alpha = n_1^{m_1} \cdots n_k^{m_k}$ (or ϵ) the *signature* of T. We will always assume that $n_1 > n_2 > \cdots > n_k$.

In other words, $T = \langle n_1^{m_1} \cdots n_k^{m_k} \rangle$ is composed of, for each i, m_i branches of length $n_i + 1$. See Figure 3 for some examples of starlike trees together with their signatures. We will sometimes write 1^0 for ϵ.

Let $\alpha = n_1^{m_1} \cdots n_k^{m_k}$ be a signature. The *length* of α is defined as $|\alpha| := m_1 + \cdots + m_k$. Let $|\epsilon| := 0$. For $j \leqslant |\alpha|$, the *jth height*, $\alpha(j)$, is n_i, where:

$$m_1 + \cdots + m_{i-1} \leqslant j < m_1 + \cdots + m_i$$

Let α and β be signatures. Say that $\alpha \leqslant \beta$ if $|\alpha| \leqslant |\beta|$ and for every $j \leqslant |\alpha|$ we have $\alpha(j) \leqslant \beta(j)$. Visually, this means that if we represent $\alpha = n_1^{m_1} \cdots n_k^{m_k}$ on a grid as a block n_1-tall and m_1-wide, followed by a block n_2-tall and m_2-wide, and so on, and similarly for β, that β covers α. Considering the examples in Figure 3, we have the following relations:

$$1^3 < 3 \cdot 1^2 < 3^2 \cdot 2 \cdot 1, \quad 2 < 3 \cdot 1^2$$

Remark 6.1. When $\alpha = n_1^{m_1} \cdots n_k^{m_k}$ and β are signatures, we have $\alpha \leqslant \beta$ if and only if $|\alpha| \leqslant |\beta|$ and for every $i \leqslant k$, we have:

$$\beta(m_1 + \cdots + m_i) \geqslant n_i$$

Proposition 6.2. *If $\alpha \leqslant \beta$ then there is a p-morphism $\langle \beta \rangle \to \langle \alpha \rangle$.*

Proof. Let us first fix labellings on $\langle \alpha \rangle$ and $\langle \beta \rangle$. Label the root of $\langle \alpha \rangle$ with $\bot$. We may arrange the branches of $\langle \alpha \rangle$ in a sequence so that the jth branch has height $\alpha(j)$. Let us label the non-root elements of the jth branch in ascending order as $a(j,1), \ldots, a(j, \alpha(j))$, and similarly for $\langle \beta \rangle$, with $b(j,i)$ for $j \leqslant |\beta|$ and $i \leqslant \beta(j)$.

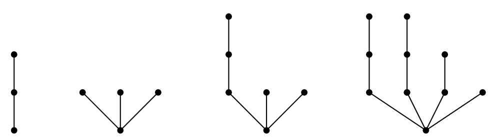
Now, define $f : \langle \beta \rangle \to \langle \alpha \rangle$ as follows. Note, for $j \leqslant |\alpha|$, we have $\alpha(j) \leqslant \beta(j)$. For $i \leqslant \beta(j)$ let:

$$f(b(j,i)) := a(j, \min(i, \alpha(j)))$$

For $j > |\alpha|$ and $i \leqslant \beta(j)$, let:

$$f(b(j,i)) := a(1, \alpha(1))$$

A routine calculation shows that f is a p-morphism. □



$\langle 2 \rangle$ $\langle 1^3 \rangle$ $\langle 3 \cdot 1^2 \rangle$ $\langle 3^2 \cdot 2 \cdot 1 \rangle$

Figure 3: Some examples of starlike trees

Note that the starlike tree $\langle k \rangle$ is the chain on $k+1$ elements, Ch_k. We will use this former notation for chains from now on. For $k \in \mathbb{N}^{>0}$, the *k-fork* is the starlike tree $\langle 1^k \rangle$.

6.2 Starlike logics

We are now in a position to define the principle class of logics that will be investigated in this section. Let $\mathscr{S} := \{\alpha \text{ signature} \mid \alpha \neq 1^2\}$. Take $\Lambda \subseteq \mathscr{S}$ (possibly infinite). The *starlike logic* $\mathbf{SFL}(\Lambda)$ based on Λ is the logic axiomatised by **IPC** plus $\chi(\langle \alpha \rangle)$ for each $\alpha \in \Lambda$. Write $\mathbf{SFL}(\alpha_1, \dots, \alpha_k)$ for $\mathbf{SFL}(\{\alpha_1, \dots, \alpha_k\})$.

Proposition 6.3. $\mathbf{SL} = \mathbf{SFL}(2 \cdot 1)$. *So Scott's logic is a starlike logic.*

Proof. See [CZ97, §9 and Table 9.7, p. 317]. □

Let us examine what $\mathbf{SFL}(\Lambda)$ 'means' in terms of its class of frames. The formula $\chi(\langle \alpha \rangle)$ turns out to express a kind of connectedness property. Let us first see some new terminology.

Let F be a finite poset. Define $\mathfrak{C}(F)$ to be the set of connected components of F. The *connectedness type* $\mathrm{c}(F)$ of F is the signature $n_1^{m_1} \cdots n_k^{m_k}$ such that $\mathfrak{C}(F)$ contains for each i exactly m_i sets of height $n_i - 1$, and nothing else. Let $\mathrm{c}(\varnothing) := \epsilon$.

Remark 6.4. Note that when F is connected, $\mathrm{c}(F) = n+1$, where $n = \mathsf{height}(F)$.

Let $\alpha > \epsilon$ be a signature. An *α-partition* of F is a partition:

$$F = C_1 \sqcup \cdots \sqcup C_{|\alpha|}$$

into open sets such that C_j has height at least $\alpha(j) - 1$. For notational uniformity, say that F has an ϵ-partition if $F = \varnothing$.

Remark 6.5. So an α-partition is an open partition in which the number and heights of the connected components are specified by α.

Lemma 6.6. *A finite poset F has an α-partition if and only if $\alpha \leqslant \mathrm{c}(F)$.*

Proof. Let $\beta := \mathrm{c}(F)$, and write $\alpha = n_1^{m_1} \cdots n_k^{m_k}$. We may assume $\beta > \epsilon$. Then we can partition F into its connected components:

$$F = \hat{C}_1 \sqcup \cdots \sqcup \hat{C}_{|\beta|}$$

such that $\hat{C}_j$ has height $\beta(j) - 1$. Take $\alpha \leqslant \beta$. We construct an α-partition $(C_j \mid j \leqslant |\alpha|)$ in blocks. First, since $\alpha \leqslant \beta$, we have that $\beta(m_1) \geqslant n_1$. This means that each of $\hat{C}_1, \dots, \hat{C}_{m_1}$ has height at least n_1. Let $C_1, \dots, C_{m_1}$ be these components $\hat{C}_1, \dots, \hat{C}_{m_1}$. Next, we have that $\beta(m_1 + m_2) \geqslant n_2$, meaning that each of $\hat{C}_{m_1+1}, \dots, \hat{C}_{m_1+m_2}$ has height at least n_2. Let $C_{m_1+1}, \dots, C_{m_1+m_2}$ be these components. Continue constructing $(C_j \mid j \leqslant |\alpha|)$ in this fashion. Note that we don't run out, since $|\alpha| \leqslant |\beta|$. Finally, take the remaining $|\beta| - |\alpha|$ components and add them to C_1.

Conversely, assume that $(C_j \mid j \leqslant |\alpha|)$ is an α-partition of F. First note that since this is an open partition, we must have that $|\alpha| \leqslant |\mathfrak{C}(F)| = |\beta|$. Now consider C_1. Let:

$$\Gamma := \{l \leqslant |\beta| : \hat{C}_l \subseteq C_1\}$$

Since C_1 is open and closed, for each $\hat{C}_l$, either $\hat{C}_l \subseteq C_1$ or $\hat{C}_l \cap C_1 = \varnothing$. Hence:

$$C_1 = \bigcup_{l \in \Gamma} \hat{C}_l$$

Because each $\hat{C}_l$ is upwards- and downwards-closed, this means that:

$$\mathsf{height}(C_1) = \max\left\{\mathsf{height}(\hat{C}_l) \;\middle|\; l \in \Gamma\right\}$$

Therefore, as $\beta(1)$ is maximal in $\{\beta(j) \mid j \leqslant |\beta|\}$, we get that $\alpha(1) \leqslant \beta(1)$.

Applying this argument inductively on $F \setminus C_1$, we get that $\alpha \leqslant \beta = \mathrm{c}(F)$. □

Corollary 6.7. *When F is connected, F has an α-partition if and only if $\alpha = k$, where $k \leqslant \mathsf{height}(F) + 1$.*

Let F be a poset and α be a signature. F is *α-connected* if there is no $x \in F$ such that there is an α-partition of $\Uparrow(x)$.

Remark 6.8. By Lemma 6.6, this is equivalent to requiring that $\alpha \nleqslant \mathrm{c}(\Uparrow(x))$ for each $x \in F$.

We can now express the meaning of $\chi(\langle\alpha\rangle)$ on frames.

Proposition 6.9. *For F a finite poset and α any signature, $F \vDash \chi(\langle\alpha\rangle)$ if and only if F is α-connected.*

Proof. First label the elements of $\langle\alpha\rangle$ as in the proof of Proposition 6.2. Assume that $F \nvDash \chi(\langle\alpha\rangle)$. Then by Corollary 2.17 there is a pointed up-reduction $f : F \to \langle\alpha\rangle$ with apex x. This means that $f^{-1}[\langle\alpha\rangle \setminus \{\bot\}] = \Uparrow(x)$. For each $j \leqslant |\alpha|$, let:

$$C_j := f^{-1}\{a(j,1), \dots, a(j,\alpha(j))\}$$

Since $\{a(j,1), \dots, a(j,\alpha(j))\}$ is upwards-closed, so is C_j. Note that the C_j's are disjoint. Hence $(C_j \mid j \leqslant k)$ is an open partition of $\Uparrow(x)$. Now, pick $x_1 \in f^{-1}\{a(j,1)\}$. Since f is a p-morphism, there is $x_2 \in f^{-1}\{a(j,2)\}$ with $x_1 < x_2$. Continuing in this fashion, we find a chain of length $\alpha(j)$ in C_j, whence $\mathsf{height}(C_j) \geqslant \alpha(j) - 1$. But then $(C_j \mid j \leqslant k)$ is an α-partition of $\Uparrow(x)$, meaning that F is not α-connected.

Conversely, assume that F is not α-connected, so that there is $x \in F$ and an α-partition $(C_j \mid j \leqslant k)$ of $\Uparrow(x)$. For each C_j, we have, by definition, that $\mathsf{height}(C_j) \geqslant \alpha(j) - 1$. Hence by Proposition 2.19 there is a p-morphism $f_j : C_j \to \langle\alpha(j) - 1\rangle$. Define $f : \uparrow(x) \to \langle\alpha\rangle$ as follows.

$$y \mapsto \begin{cases} \bot & \text{if } y = x, \\ f_j(y) & \text{if } y \in C_j \end{cases}$$

Then f is a p-morphism, so an up-reduction $F \circ\!\!\to \langle\alpha\rangle$. ☐

Remark 6.10. In particular it follows that $\mathbf{BD}_n = \mathbf{IPC} + \chi(\langle n+1\rangle)$. This is just Proposition 2.19 of course.

The last matter to resolve before moving on to consider the completeness of starlike logics is their number. For this we make use of Higman's Lemma. A *quasi-well-order* is a preorder which is well-founded and has no infinite antichain. Given a preorder I, let $I^{<\omega}$ be the set of finite sequences of elements of I ordered by $(x_1, \dots, x_n) \leqslant (y_1, \dots, y_m)$ if and only if there is $f : \{1, \dots n\} \to \{1, \dots, m\}$ injective such that for each $k \leqslant n$ we have $x_k \leqslant y_{f(k)}$.

Lemma 6.11 (Higma's Lemma, [**higman52**])**.** *If I is a quasi-well-order then so is $I^{<\omega}$.*

Proposition 6.12. *There are exactly countably-many starlike logics.*

Proof. It suffices to show that there is no infinite antichain of starlike trees with respect to p-morphic reduction. In light of Proposition 6.2, it therefore suffices to show that there is no infinite antichain of signatures with respect to the ordering defined on them. Now, we can recast signatures as (monotonic decreasing) finite sequences of integers. Indeed, the signature α is determined by the sequence $(\alpha(1), \dots, \alpha(|\alpha|))$. In this way, the set of signatures is seen to be a suborder of $\omega^{<\omega}$. Now, $(\omega, \leqslant)$ is clearly a quasi-well-order, and hence by Higman's Lemma **??**, so is $\omega^{<\omega}$. Thus there is no infinite antichain of signatures, as required. ☐

6.3 Starlike completeness

The main theorem to be proved in this section is the following.

Theorem 6.13. *Every starlike logic is polyhedrally-complete.*

As an immediate consequence, we obtain:

Corollary 6.14. *Scott's Logic is polyhedrally-complete.*

Remark 6.15. The starlike logic $\mathbf{SFL}(2 \cdot 1, 1^3)$ is particularly important geometrically. In [Ada+20], it is shown that this is the logic of all convex polyhedra.

In order to prove Theorem 6.11, we introduce the following new validity concept on frames. Let F be a poset and ϕ be a formula. F *nerve-validates* ϕ, notation $F \vDash_{\mathscr{N}} \phi$, if for every $k \in \mathbb{N}$ we have $\mathscr{N}^k(F) \vDash \phi$.

Remark 6.16. Since, as already remarked in Section 4.5, we always have the p-morphism $\max\colon \mathscr{N}(G) \to G$, for every G, by Proposition 2.3 this is equivalent to requiring that $\mathscr{N}^k(F) \vDash \phi$ for infinitely-many $k \in \mathbb{N}$.

Lemma 6.17. *A logic $\mathscr{L}$ is polyhedrally-complete if and only if it has the finite model property and every rooted finite frame of $\mathscr{L}$ is the up-reduction of a poset which nerve-validates $\mathscr{L}$.*

Proof. Assume that $\mathscr{L}$ is polyhedrally-complete. Then by the Nerve Criterion 4.1 it is the logic of a class $\mathbf{C}$ of finite frames which is closed under $\mathscr{N}$, and so has the fmp. Then by Corollary 2.15, every finite rooted frame F of $\mathscr{L}$ is the up-reduction of some $F' \in \mathbf{C}$. Since $\mathbf{C} \subseteq \mathrm{Frames}(\mathscr{L})$ and is closed under $\mathscr{N}$, such an F' nerve-validates $\mathscr{L}$.

Conversely, let $\mathbf{C}$ be the class of all finite rooted frames which nerve-validate $\mathscr{L}$. Note that $\mathbf{C}$ is closed under $\mathscr{N}$. Further, clearly $\mathscr{L} \subseteq \mathrm{Logic}(\mathbf{C})$. To see the reverse inclusion, suppose that $\mathscr{L} \nvdash \phi$. Since $\mathscr{L}$ has the fmp, there is $F \in \mathrm{Frames}_{\perp,\mathrm{fin}}(\mathscr{L})$ such that $F \nvDash \phi$. By assumption, F is the up-reduction of $F' \in \mathbf{C}$. Then by Proposition 2.3, $F' \nvDash \phi$, meaning that $\mathrm{Logic}(\mathbf{C}) \nvdash \phi$. $\square$

Lemma 6.18. *Every starlike logic has the finite model property.*

Proof. In [Zak93, Corollary 0.11], Zakharyaschev shows that every logic axiomatised by the Jankov-Fine formulas of trees has the finite model property. $\square$

With Lemma 6.16, we can now use Lemma 6.15 to produce a proof of Theorem 6.11. Given a rooted finite frame F of $\mathbf{SFL}(\Lambda)$, we proceed as follows.

(1) We examine what it means for a frame to nerve-validate $\chi(\langle\alpha\rangle)$.

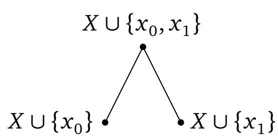
(2) We see that it can be assumed that F is *graded* (a structural property of posets defined below).

(3) Using this additional structure, we construct a frame F' and the p-morphism $F' \to F$, with the property that $F' \vDash_{\mathscr{N}} \mathbf{SFL}(\Lambda)$

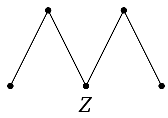
The reader will have noticed that the difork $\langle 1^2 \rangle$ is omitted from the definition of a starlike logic, and consequently from the Main Theorem 6.11. In fact, polyhedral semantics is quite fond of this tree: when we take it as a forbidden configuration, the resulting landscape of polyhedrally-complete logics is as sparse as possible, as is shown below.

Proposition 6.19. *Let $\mathscr{L}$ be a polyhedrally-complete logic containing* $\mathbf{SFL}(1^2)$. *Then* $\mathscr{L} = \mathbf{CPC}$*, the maximum logic.*

Proof. Suppose for a contradiction that $\mathscr{L}$ is a polyhedrally-complete logic containing $\mathbf{SFL}(1^2)$ other than $\mathbf{CPC}$. By the Nerve Criterion 4.1, $\mathscr{L} = \mathrm{Logic}(\mathbf{C})$ where $\mathbf{C}$ is a class of finite posets closed under $\mathscr{N}$. Since $\mathscr{L} \neq \mathbf{CPC}$, there must be $F \in \mathbf{C}$ with $\mathsf{height}(F) \geqslant 1$. This means that F has a chain $x_0 < x_1$. As in the proof of Theorem 5.4, we may assume that x_1 is a top element of F and that x_0 is an immediate predecessor of x_1. Take X a maximal chain in $\Downarrow(x_0)$. Then, as in that proof, we obtain the following structure lying at the top of $\mathscr{N}(F)$.



Applying the nerve once more, we obtain the following structure at the top of $\mathcal{N}^2(F)$.



Since **C** is closed under $\mathcal{N}$, we get that $\mathcal{N}^2(F) \in \mathrm{Frames}(\mathcal{L})$. But $\Uparrow(Z)$ maps p-morphically onto $\langle 1^2 \rangle$, contradicting that $\mathcal{L} \vdash \chi(\langle 1^2 \rangle)$. ↯ $\square$

6.4 Nerve-validation

While validating $\chi(\langle \alpha \rangle)$ corresponds to α-connectedness (as shown in Proposition 6.9), *nerve*-validating $\chi(\langle \alpha \rangle)$ corresponds to α-*nerve*-connectedness. Let F be a poset and $x < y$ in F. The *diamond* and *strict diamond* of x and y are defined, respectively:

$$\updownarrow(x, y) := \uparrow(x) \cap \downarrow(y)$$
$$\Updownarrow(x, y) := \updownarrow(x, y) \setminus \{x, y\}$$

A poset F is *α-diamond-connected* if there are no $x < y$ in F such that there is an α-partition of $\Updownarrow(x, y)$. The poset F is *α-nerve-connected* if it is α-connected and α-diamond-connected.

With a slight conceptual change, α-connectedness and α-diamond-connectedness can be harmonised as follows. For any poset F, we take a new element ∞, and let $\check{F} := F \cup \{\infty\}$, where ∞ lies above every element of F. Then F is α-nerve-connected if and only if there are no $x < y$ in $\check{F}$ for which there is an α-partition of $\Updownarrow(x, y)$.

Theorem 6.20. *Let F be a finite poset and take $\alpha \in \mathcal{S}$. Then $F \vDash_{\mathcal{N}} \chi(\langle \alpha \rangle)$ if and only if F is α-nerve-connected.*

Proof. Assume that F is not α-nerve-connected with the aim of showing $F \nvDash_{\mathcal{N}} \chi(\langle \alpha \rangle)$. Choose $x < y$ in $\check{F}$ such that $\Updownarrow(x, y)$ has an α-partition. That is, there is an open partition $(C_j \mid j \leqslant |\alpha|)$ of $\Updownarrow(x, y)$ such that $\mathsf{height}(C_j) = \alpha(j)$. Choose a chain $X \subseteq F$ which is maximal with respect to (i) $x, y \in X$ (ignoring the case $y = \infty$), and (ii) $X \cap \Updownarrow(x, y) = \varnothing$. I will show that $\Uparrow(X)^{\mathcal{N}(F)}$ has an α-partition. Note that by maximality of X, elements $Y \in \Uparrow(X)^{\mathcal{N}(F)}$ are determined by their intersection $Y \cap \Updownarrow(x, y)$. For $j \leqslant |\alpha|$, let:

$$\widehat{C}_j := \{Y \in \Uparrow(X)^{\mathcal{N}(F)} \mid Y \cap C_j \neq \varnothing\}$$

Take $j, l \leqslant |\alpha|$ distinct. Since both C_j and C_l are upwards- and downwards-closed in $\Updownarrow(x, y)$, there is no chain $Y \in \Uparrow(X)^{\mathcal{N}(F)}$ such that $Y \cap C_j \neq \varnothing$ and $Y \cap C_l \neq \varnothing$. This means that:

(1) $\widehat{C}_j$ and $\widehat{C}_l$ are disjoint.

(2) For any $Y \in \Uparrow(X)^{\mathcal{N}(F)}$ we have $Y \in \widehat{C}_j$ if and only if $Y \cap \Updownarrow(x, y) \subseteq C_j$. Hence each $\widehat{C}_j$ is upwards- and downwards-closed in $\Uparrow(X)^{\mathcal{N}(F)}$.

Furthermore, since $(C_j \mid j \leqslant |\alpha|)$ covers $\Updownarrow(x, y)$, we get that $(\widehat{C}_j \mid j \leqslant |\alpha|)$ covers $\Uparrow(X)^{\mathcal{N}(F)}$. Finally, any maximal chain in $\widehat{C}_j$ is a sequence of chains $Y_0 \subset \cdots \subset Y_l$ such that $|Y_{i+1} \setminus Y_i| = 1$; this then corresponds to a maximal chain in C_j. Therefore:

$$\mathsf{height}(\widehat{C}_j) = \mathsf{height}(C_j)$$

Ergo $(\widehat{C}_j \mid j \leqslant |\alpha|)$ is an α-partition of $\Uparrow(X)^{\mathcal{N}(F)}$, meaning that $\mathcal{N}(F)$ is not α-connected. Then, by Proposition 6.9, $\mathcal{N}(F) \nvDash \chi(\langle \alpha \rangle)$, hence by definition $F \nvDash_{\mathcal{N}} \chi(\langle \alpha \rangle)$.

For the converse direction, we will show that if F is α-nerve-connected, then so is $\mathcal{N}(F)$, which will give the result by induction (note that α-nerve-connectedness is

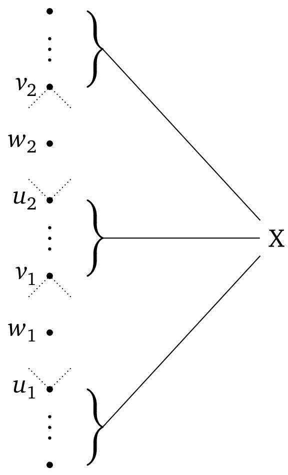


Figure 4: The set-up when X has more than one gap

stronger than α-connectedness, and hence by Proposition 6.9 if $\mathscr{N}^k(F)$ is α-nerve-connected then $\mathscr{N}^k(F) \vDash \chi(\langle\alpha\rangle))$. So assume that F is α-nerve-connected. We will first prove α-connectedness. Take $X \in \mathscr{N}(F)$ with the aim of showing that $\Uparrow(X)^{\mathscr{N}(F)}$ has no α-partition.

Firstly, assume that X has more than one 'gap'; that is, there are distinct $w_1, w_2 \in F \setminus X$ such that $X \cup \{w_1\}$ and $X \cup \{w_2\}$ are still chains, but such that there exists $z \in X$ with $w_1 < z < w_2$. Take $Y, Z \in \Uparrow(X)^{\mathscr{N}(F)}$. We will use the two gaps to juggle elements between the two sets so as to provide a path $Y \rightsquigarrow Z$ which never touches X (i.e. lies in $\Uparrow(X)^{\mathscr{N}(F)}$). For $i \in \{1,2\}$, let $u_i \in X \cap \Downarrow(w_i)$ be greatest and $v_i \in X \cap \Uparrow(w_i)$ be least. See Figure 4 for a representation of the situation. Now, without loss of generality, we may assume that $Y \cap \Updownarrow(u_1, v_1) \neq \varnothing$ (we may add w_1 to Y, noting that $w_1 \in \Updownarrow(u_1, v_1)$). Similarly, we may assume that $Y \cap \Updownarrow(u_2, v_2) \neq \varnothing$, and likewise for Z. We then have the following path in $\Uparrow(X)^{\mathscr{N}(F)}$ (note that some of the sets along the path may be equal, but in all cases the path is still there):

$$\begin{tikzcd}[column sep=tiny]
Y \arrow[dr, dash] & & (Y \setminus \Updownarrow(u_1,v_1)) \cup \{w_1\} \arrow[dr, dash] \arrow[dl, dash] & & (Z \setminus \Updownarrow(u_1,v_1)) \cup \{w_1\} \arrow[dr, dash] \arrow[dl, dash] & & Z \arrow[dl, dash] \\
& Y \setminus \Updownarrow(u_1,v_1) & & X \cup \{w_1\} & & Z \setminus \Updownarrow(u_1,v_1) &
\end{tikzcd}$$

Here, the gap $\Updownarrow(u_2, v_2)$ is used ensure that $Y \setminus \Updownarrow(u_1, v_1)$ and $Z \setminus \Updownarrow(u_1, v_1)$ are not equal to X, and the fact that we have $v_1 \leqslant z \leqslant u_2$ ensures that all these sets are indeed in $\mathscr{N}(F)$. Hence, $\Uparrow(X)^{\mathscr{N}(F)}$ is path-connected so connected. Therefore, by Corollary 6.7, it suffices to show that $\mathsf{height}(\Uparrow(X)^{\mathscr{N}(F)}) < \mathsf{height}(F)$. But this is immediate from the definition of $\mathscr{N}$.

Hence we may assume that X has exactly one gap (when X has no gaps, $\Uparrow(X)^{\mathscr{N}(F)} = \varnothing$). This means that there are $x, y \in X$ with $x < y$ such that $X \cap \Updownarrow(x, y) = \varnothing$ and X is maximal outside of $\Updownarrow(x, y)$. As before then, elements $Y \in \Uparrow(X)^{\mathscr{N}(F)}$ are determined by their intersection $Y \cap \Updownarrow(x, y)$. Suppose that $\Uparrow(X)^{\mathscr{N}(F)}$ has an α-partition $(\widehat{C}_j \mid j \leqslant |\alpha|)$. For each $j \leqslant |\alpha|$, let:

$$C_j := \bigcup \widehat{C}_j \cap \Updownarrow(x, y)$$

Note that $\bigcup_{j \leqslant |\alpha|} C_j = \Updownarrow(x, y)$. For each $j \leqslant |\alpha|$, since $\widehat{C}_j$ is downwards-closed, we have that, for $z \in \Updownarrow(x, y)$:

$$z \in C_j \quad \Leftrightarrow \quad \exists Y \in \widehat{C}_j : z \in Y \quad \Leftrightarrow \quad X \cup \{z\} \in \widehat{C}_j$$

This means in particular that the C_j's are pairwise disjoint. Further, if $z \in C_j$ and $w \in \Updownarrow(x, y)$ with $w < z$, then $X \cup \{w, z\}$ is a chain, and so as $\widehat{C}_j$ is upwards-closed, we have $X \cup \{w, z\} \in \widehat{C}_j$, meaning that $w \in C_j$; similarly when $w > z$. Whence each C_j is upwards- and downwards-closed. Finally, as above, maximal chains in $\widehat{C}_j$ correspond to maximal chains in C_j of the same length, whence:

$$\mathsf{height}(\widehat{C}_j) = \mathsf{height}(C_j)$$

But then $(C_j \mid j \leqslant |\alpha|)$ is an α-partition of $\Updownarrow(x, y)$, contradicting the fact that F is α-nerve-connected. $\lightning$

This shows that $\mathscr{N}(F)$ is α-connected. What about α-diamond-connectedness? In fact we can show this without using any assumptions on F. Take $X, Y \in \mathscr{N}(F)$ with $X \subset Y$ We will show that $\Updownarrow(X, Y)^{\mathscr{N}(F)}$ has no α-partition. We may assume that $|Y \setminus X| \geqslant 2$, otherwise $\Updownarrow(X, Y)^{\mathscr{N}(F)} = \varnothing$. Note that this means in particular that $\alpha > 1$, since F is α-connected. If $|Y \setminus X| = 2$, then $\Updownarrow(X, Y)^{\mathscr{N}(F)}$ is the antichain on two elements, which, since $\alpha \neq 1^2$ by assumption, has no α-partition. So assume that $|Y \setminus X| \geqslant 3$; we will show that in fact $\Updownarrow(X, Y)^{\mathscr{N}(F)}$ is connected. Take distinct $Z, W \in \Updownarrow(X, Y)^{\mathscr{N}(F)}$. Choose $z \in Z \setminus X$ and $w \in W \setminus X$. Since $|Y \setminus X| \geqslant 3$, we have that $X \cup \{z, w\} \in \Updownarrow(X, Y)^{\mathscr{N}(F)}$. Hence the following is a path in $\Updownarrow(X, Y)^{\mathscr{N}(F)}$:

$$\begin{tikzcd}[column sep=small]
Z \arrow[dr, dash] & & X \cup \{z, w\} \arrow[dl, dash] \arrow[dr, dash] & & W \arrow[dl, dash] \\
& X \cup \{z\} & & X \cup \{w\} &
\end{tikzcd}$$

Therefore, $\Updownarrow(X, Y)^{\mathscr{N}(F)}$ is connected. Finally, note that:

$$\mathsf{height}(\Updownarrow(X, Y)^{\mathscr{N}(F)}) \leqslant \mathsf{height}(\mathscr{N}(F)) = \mathsf{height}(F) \qquad \square$$

Remark 6.21. Note that the proof shows an interesting property of the formulas $\chi(\langle\alpha\rangle)$: we have $F \vDash_{\mathscr{N}} \chi(\langle\alpha\rangle)$ if and only if $\mathscr{N}(F) \vDash \chi(\langle\alpha\rangle)$. This is not true in general. For example, formulas expressing bounded width can take many iterations of the nerve construction to become falsified.

6.5 Graded posets

The next step is to show that we can put $F \in \mathrm{Frames}_{\perp,\mathrm{fin}}(\mathbf{SFL}(\Lambda))$ into a special form. The following definition comes from combinatorics (see e.g. [Sta97, p. 99]).

Definition 6.22 (Graded poset)**.** A *rank function* on a poset F is a map $\rho \colon F \to \mathbb{N}$ such that:

(i) whenever x is minimal in F, we have $\rho(x) = 0$,

(ii) whenever y is the immediate successor of x, we have $\rho(y) = \rho(x) + 1$.

If F is non-empty and has a rank function, then it is *graded*.

The notion of gradedness has a strong visual connection. When a poset is graded, we can draw it out in well-defined layers such that any element's immediate successors lie entirely in the next layer up.

Proposition 6.23. *Let F be a finite poset.*

(1) F is graded if and only if for every $x \in F$, all maximal chains in $\downarrow(x)$ have the same length.

(2) When F is graded, $\rho(x) = \mathsf{height}(x)$ for every $x \in F$, and $\mathsf{height}(F) = \max \rho[F]$.

(3) Rank functions, when they exist, are unique.

Proof. (1) See [Sta97, p. 99]. Assume that F is graded, and take X a maximal chain in $\downarrow(x)$ for some $x \in F$. Let $k = \rho(x)$ We will show that $|X| = k+1$. Since X is a chain, the ranks of each of its elements are distinct. Since X is maximal, $x \in X$. Suppose for a contradiction that there is $j < k$ such that there is no $x \in X$ of rank j. We may assume that j is minimal with this property. We can't have $j = 0$, since otherwise X wouldn't contain any minimal element, so wouldn't be a maximal chain. Hence, there is $y \in X$ with $\rho(y) = j-1$. Let z be next in X after y. Then y has an immediate successor w such that $w \leqslant z$. By definition, $\rho(w) = j$, so $w \notin X$. But $X \cup \{w\}$ is a chain, contradicting the maximality of X. $\lightning$ Therefore, $|X| = k+1$.

Conversely, define $\rho : F \to \mathbb{N}$ by:

$$x \mapsto \mathsf{height}(x)$$

Let us check that ρ is a rank function. (i) Clearly, when x is minimal, $\rho(x) = 0$. (ii) Suppose for a contradiction that there are $x, y \in F$, with y an immediate successor of x, such that $\rho(y) \neq \rho(x)+1$. First, by definition, $\rho(y) > \rho(x)$, so we must have $\rho(y) > \rho(x)+1$. Choose maximal chains $X \subseteq \downarrow(x)$, $Y \subseteq \downarrow(y)$. Note that by assumption:

$$|Y| > |X| + 1$$

But now, since y is an immediate successor of x, both $X \cup \{y\}$ and Y are maximal chains in $\downarrow(y)$ of different heights. $\lightning$

(2) This follows from the proof of (1).

(3) This follows from (2). □

Corollary 6.24. *(1) Every tree is graded.*

(2) For any finite poset F, its nerve $\mathscr{N}(F)$ is graded, with rank function given by $\rho(X) = |X| - 1$.

Proof. For (2), note that for any $X \in \mathscr{N}(F)$ we have $\mathsf{height}(X) = |X| - 1$. □

6.6 Gradification in the presence of Scott's tree

The task now is, given a finite rooted frame F of $\mathbf{SFL}(\Lambda)$, to find a finite graded rooted frame F' of $\mathbf{SFL}(\Lambda)$ and a p-morphism $f : F' \to F$. We will do this using two different methods, depending on whether or not we have Scott's tree $\langle 2 \cdot 1 \rangle$ present. Let us first consider the case $2 \cdot 1 \in \Lambda$. The following lemmas show us that this case is not too complicated.

Lemma 6.25. *Take $\Lambda \subseteq \mathscr{S}$ such that $2 \cdot 1 \in \Lambda$ but $n \notin \Lambda$ for any $n \in \mathbb{N}$.*

(1) If there is no $k \in \mathbb{N}^{>0}$ such that $1^k \in \Lambda$, then $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(2 \cdot 1)$.

(2) Otherwise, let $k \in \mathbb{N}^{>0}$ be minimal such that $1^k \in \Lambda$. Then $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(2 \cdot 1, 1^k)$.

Proof. (1) Take $\alpha \in \Lambda$. Then by assumption $\alpha(1) \geqslant 2$, hence, as $\alpha \neq n$, we have $2 \cdot 1 \leqslant \alpha$. Then by Proposition 6.2 there is a p-morphism $\langle \alpha \rangle \to \langle 2 \cdot 1 \rangle$. Hence by the semantic meaning of Jankov-Fine formulas, Theorem 2.14, we have that any frame validating $\chi(\langle 2 \cdot 1 \rangle)$ will also validate $\chi(\langle \alpha \rangle)$. This means that $\mathbf{SFL}(\Lambda) \subseteq \mathbf{SFL}(2 \cdot 1)$. The converse direction is immediate.

(2) Take $\alpha \in \Lambda$. If $\alpha(1) \geqslant 2$ then by Proposition 6.2 there is a p-morphism $\langle \alpha \rangle \to \langle 2 \cdot 1 \rangle$. If $\alpha(1) < 2$. Since $\alpha \neq \epsilon$, we have $\alpha(1) = 1$, meaning that $\alpha = 1^l$ for some $l \in \mathbb{N}^{>0}$. By assumption $k \leqslant l$. But then $1^k \leqslant \alpha$, giving that there is a p-morphism $\langle \alpha \rangle \to \langle 1^k \rangle$. It follows that for any $\alpha \in \Lambda$, $\langle \alpha \rangle$ up-reduces to either $\langle 2 \cdot 1 \rangle$ or $\langle 1^k \rangle$. By Theorem 2.14, any frame validating $\chi(\langle 2 \cdot 1 \rangle)$ and $\chi(\langle 1^k \rangle)$ will also validate $\chi(\langle \alpha \rangle)$. This implies that $\mathbf{SFL}(\Lambda) \subseteq \mathbf{SFL}(2 \cdot 1, 1^k)$. The converse direction is obvious. □

Corollary 6.26. *Take $\Lambda \subseteq \mathscr{S}$ such that $2 \cdot 1 \in \Lambda$ and there is $n \in \mathbb{N}$ with $n \in \Lambda$; assume that n is the minimal such natural number.*

(1) If there is no $k \in \mathbb{N}^{>0}$ such that $1^k \in \Lambda$, then $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(n, 2 \cdot 1)$.

(2) Otherwise, let $k \in \mathbb{N}^{>0}$ be minimal with $1^k \in \Lambda$. Then $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(n, 2 \cdot 1, 1^k)$.

Proof. This follows from Lemma 6.23 and the fact that when $n_1 < n_2$ every frame validating $\chi(\langle n_1 \rangle)$ also validates $\chi(\langle n_2 \rangle)$. □

Using this, the 'meaning' of $\mathbf{SFL}(\Lambda)$ can be expressed relatively simply. Note that this meaning is expressed in terms of the depth of elements $x \in F$. Up until this point we have mainly been concerned with the height of elements.

Lemma 6.27. *Take $\Lambda \subseteq \mathscr{S}$ such that $2 \cdot 1 \in \Lambda$, and let F be a finite poset. Let $n \in \mathbb{N}$ be minimal such that $n \in \Lambda$, or ∞ if no such signature is present. Similarly, let $k \in \mathbb{N}^{>0}$ be minimal with $1^k \in \Lambda$, or ∞. Then $F \vDash \mathbf{SFL}(\Lambda)$ if and only if the following three conditions are satisfied for every $x \in F$.*

(i) We have $\mathsf{height}(F) < n$.

(ii) Whenever $\mathsf{depth}(x) = 1$*, we have* $|\Uparrow(x)| < k$.

(iii) Whenever $\mathsf{depth}(x) > 1$*, the set $\Uparrow(x)$ is connected.*

Proof. By Corollary 6.24 and the fact that $F \vDash \chi(\langle n \rangle)$ if and only if $\mathsf{height}(F) \leqslant n-1$, it suffices to treat the case $n = \infty$. Now by Lemma 6.23, $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(2 \cdot 1, 1^k)$ when $k < \infty$, and $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(2 \cdot 1)$ otherwise.

Assume that $F \vDash \mathbf{SFL}(\Lambda)$. (ii) In the case $k < \infty$, take $x \in F$ with $\mathsf{depth}(x) = 1$. Note that $\Uparrow(x)$ is an antichain, so $(\{y\} \mid y \in \Uparrow(x))$ is an open partition of $\Uparrow(x)$. Since $x \vDash \chi(\langle 1^k \rangle)$, by Lemma 6.6 and Proposition 6.9 we must have $|\Uparrow(x)| < k$. (iii) Now take $x \in F$ with $\mathsf{depth}(x) > 1$, and suppose for a contradiction that $\Uparrow(x)$ is disconnected. Then we can partition $\Uparrow(x)$ into disjoint upwards-closed sets U, V. Since $\mathsf{depth}(x) > 1$, one of U and V (say U) must have height at least 1. But then (U, V) is a $(2 \cdot 1)$-partition of $\Uparrow(x)$, contradicting that $F \vDash \chi(\langle 2 \cdot 1 \rangle)$ by Proposition 6.9. $\lightning$

Conversely, assume that $F \nvDash \mathbf{SFL}(\Lambda)$ We will show that one of (ii) and (iii) is violated. If $F \nvDash \chi(\langle 2 \cdot 1 \rangle)$, then by Proposition 6.9 there is $x \in F$ and a $(2 \cdot 1)$-partition (U, V) of $\Uparrow(x)$. But then $\mathsf{height}(U) \geqslant 1$, meaning that $\mathsf{depth}(x) > 1$, and furthermore $\Uparrow(x)$ is disconnected, violating (iii). So let us assume that $k < \infty$, that $F \vDash \chi(\langle 2 \cdot 1 \rangle)$ but that $F \nvDash \chi(\langle 1^k \rangle)$. Again, we get $x \in F$ and a 1^k-partition $(C_1, \ldots, C_k)$ of $\Uparrow(x)$. We must have that $\mathsf{height}(C_1) = 0$, otherwise $(C_1, C_2 \cup \cdots \cup C_k)$ is a $(2 \cdot 1)$-partition of $\Uparrow(x)$. Similarly $\mathsf{height}(C_i) = 0$ for every $i \leqslant k$. This means that $\mathsf{depth}(x) = 1$, and that $|\Uparrow(x)| \geqslant k$, violating (ii). □

Theorem 6.28. *Let $\Lambda \subseteq \mathscr{S}$ be such that $2 \cdot 1 \in \Lambda$. Let F be a finite rooted poset such that $F \vDash \mathbf{SFL}(\Lambda)$. Then there is a finite graded rooted poset F' and a p-morphism $f : F' \to F$ such that $F' \vDash \mathbf{SFL}(\Lambda)$.*

This is the the 'gradification' theorem. Let us outline the construction before coming to the full proof.

- We first split F up into its tree unravelling $\mathscr{T}(F)$.
- We then lengthen branches so that the tree has a uniform height.
- Lastly, we join top nodes of this tree in order to recover any α-connectedness that we lost.

See Figure 5 for an example of this process.

Proof. Let $n := \mathsf{height}(F)$. We may assume $\epsilon \notin \Lambda$. If $2 \in \Lambda$, then by Remark 6.10, $n \leqslant 1$, so F is already graded. So assume that $2 \notin \Lambda$.

Start with the tree unravelling $T = \mathscr{T}(F)$ of F. Form a new tree T_0 by replacing each top node $t \in \mathsf{Top}(T)$ with a chain of new elements $t^*(0), \ldots, t^*(m_t)$, where

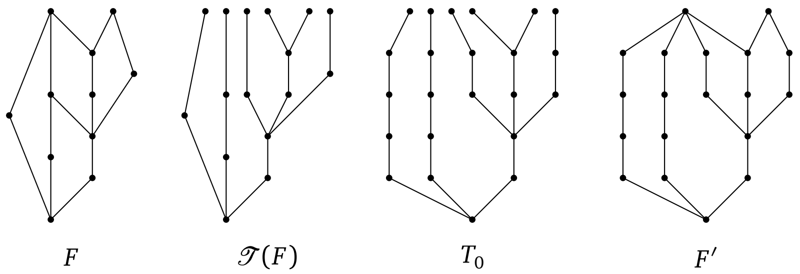


Figure 5: An example of gradification in the presence of Scott's tree

$m_t = n - \mathsf{height}(t)$. The relations between these new elements and the rest of T is as follows:

$$t^*(0) < \cdots < t^*(m_t),$$
$$x < t^*(0) \quad \Leftrightarrow \quad x < t \qquad \forall x \in T$$

Note that in T_0 all branches have the same length $n+1$. Define the p-morphism $g\colon T_0 \to T$ by:

$$x \mapsto \begin{cases} x & \text{if } x \in \mathsf{Trunk}(T), \\ \mathsf{last}(t) & \text{if } x = t^*(i) \text{ for some } t \in \mathsf{Top}(T) \text{ and } i \leqslant m_t \end{cases}$$

Form F' from T_0 by identifying, for top nodes $t, s \in \mathsf{Top}(T)$, the elements $t^*(m_t)$ and $s^*(m_s)$ whenever $\mathsf{last}(t) = \mathsf{last}(s)$. That is, let $F' := T_0/\mathscr{W}$, where:

$$\mathscr{W} := \{\{t^*(m_t) \mid \mathsf{last}(t) = u\} \mid u \in \mathsf{Top}(F)\}$$

Note that we have a p-morphism $f = \mathsf{last} \circ g \circ q_{\mathscr{W}}\colon F' \to F$. Furthermore, F is clearly finite and rooted. As to gradedness, take $x \in F'$ with the aim of showing that all maximal chains in $\downarrow(x)$ are of the same length, utilising Proposition 6.21. If $x \in \mathsf{Trunk}(F')$, then $\downarrow(x)^{F'}$ is a linear order. So assume that $x \in \mathsf{Top}(F')$. Then any maximal chain X in $\downarrow(x)$ corresponds to a branch of T_0, and therefore has length $n+1$.

Let us now use Lemma 6.25 to verify that our construction preserves α-connectedness for $\alpha \in \Lambda$ and complete the proof. Let $k \in \mathbb{N}^{>0}$ be minimal such that $1^k \in \Lambda$, or ∞ if no such signature is present. For $u \in \mathsf{Top}(F)$ let $\widehat{u}$ be the equivalence class of those elements $t^*(m_t)$ such that $\mathsf{last}(t) = u$. Note that by construction, for $x \in \mathsf{Trunk}(T)$ and $u \in \mathsf{Top}(F)$:

$$x < \widehat{u} \quad \Leftrightarrow \quad \mathsf{last}(x) < u \tag{$\star$}$$

We need to check the three conditions of Lemma 6.25.

(i) Note that $\mathsf{height}(F') = \mathsf{height}(F)$.

(ii) For any $x \in F'$ with $\mathsf{depth}(x) = 1$, either $x \in \mathsf{Trunk}(T)$ or $x = t^*(n_t - 1)$ for some top node $t \in T$. In the former case, the fact that $|\Uparrow(x)| \leqslant k$ follows from ($\star$) and the fact that $|\Uparrow(\mathsf{last}(x))^F| \leqslant k$. In the latter case we have $\Uparrow(x) = \left\{\widehat{\mathsf{last}(t)}\right\}$.

(iii) Similarly, for any $x \in F'$ with $\mathsf{depth}(x) > 1$, either $x \in \mathsf{Trunk}(T)$ or $x = t^*(r)$ for some top node $t \in T$ and $r < n_t - 1$. In the latter case, $\Uparrow(x)$ is a chain, so connected. For the former case, it suffices to show that any two top elements $\widehat{u}, \widehat{v} \in \Uparrow(x)$ are connected by a path in $\Uparrow(x)$. Note that $\mathsf{depth}(\mathsf{last}(x))^F > 1$. Now, since $F \vDash \chi(\langle 2 \cdot 1 \rangle)$, by Lemma 6.25 there is a path $u \rightsquigarrow v$ in $\Uparrow(\mathsf{last}(x))^F$. We may assume that this path is of form given in Figure 6 (a), where $w_0, \ldots, w_k$ are top nodes in F. Using ($\star$), this path then translates into a path $\widehat{u} \rightsquigarrow \widehat{v}$ as in Figure 6 (b), where $y_i \in \mathsf{last}^{-1}\{a_i\} \cap \Uparrow(x)$ for each i. $\square$

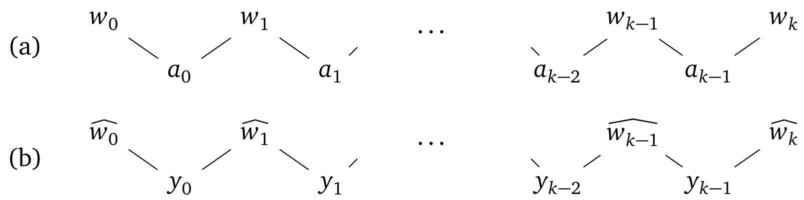


Figure 6: The form of the paths in $\Uparrow(\mathsf{last}(x))^F$ and $\Uparrow(x)^{F'}$

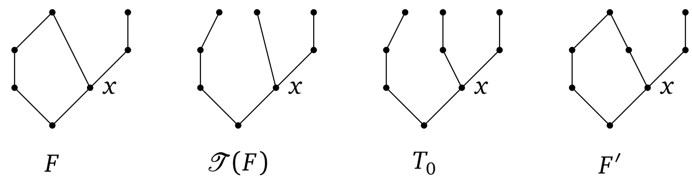


Figure 7: The technique in the proof of Theorem 6.26 does not work in general

6.7 Gradification without Scott's tree

Now that the situation $2 \cdot 1 \in \Lambda$ has been dealt with, let us turn to the case $2 \cdot 1 \notin \Lambda$. Unfortunately, the proof of Theorem 6.26 crucially relied on the fact that the original frame F was $(2 \cdot 1)$-connected. Consider for instance the frame F given in Figure 7, which at x is not $(2 \cdot 1)$-connected. If we apply the construction to F, we end up with a frame F' in which x sits below two connected components of height 1, that is, $\mathrm{c}(\Uparrow(x)^{F'}) = 2^2$.[3] Hence F' is not 2^2-connected, while F is. Taking $2 \cdot 1$ away from Λ is a double-edged sword however, since it allows for more complex constructions in F'.

The following reusable lemma will come in handy a couple of times.

Lemma 6.29. *Let $f : F' \to F$ be a surjective p-morphism between finite posets, and take $x \in F'$. Assume that for any $y, z \in \mathrm{Succ}(x)$ there is a path $y \rightsquigarrow z$ in $\Uparrow(x)$ whenever there is a path $f(y) \rightsquigarrow f(z)$ in $\Uparrow(f(x))$. Then:*

$$\mathfrak{C}(\Uparrow(x)) = \{f^{-1}[C] \mid C \in \mathfrak{C}(\Uparrow(f(x)))\}$$

In particular, if $\mathsf{height}(f^{-1}[C]) = \mathsf{height}(C)$ for any $C \in \mathfrak{C}(\Uparrow(f(x)))$ then:

$$\mathrm{c}(\Uparrow(x)) = \mathrm{c}(\Uparrow(f(x)))$$

Proof. Note that, since f is a p-morphism and F and F' are finite, $\{f^{-1}[C] \mid C \in \mathfrak{C}(\Uparrow(f(x)))\}$ is a partition of $\Uparrow(x)$ into upwards- and downwards-closed sets. So it suffices to show that $f^{-1}[C]$ is connected for every $C \in \mathfrak{C}(\Uparrow(f(x)))$. Take $y_0, z_0 \in f^{-1}[C]$. Since $f^{-1}[C]$ is downwards-closed in $\Uparrow(x)$, there are $y, z \in \mathrm{Succ}(x) \cap f^{-1}[C]$ such that $y \leqslant y_0$ and $z \leqslant z_0$. Then $f(y), f(z) \in C$, so by assumption there is a path $f(y) \rightsquigarrow f(z)$ in $\Uparrow(f(x))$. But then by assumption there is a path $y \rightsquigarrow z$ in $\Uparrow(x)$, which lies in $f^{-1}[C]$ since the latter is upwards- and downwards-closed. $\square$

Theorem 6.30. *Let $\Lambda \subseteq \mathscr{S}$ be such that $2 \cdot 1 \notin \Lambda$. Let F be a finite rooted poset such that $F \vDash$ **SFL**(Λ). Then there is a finite graded rooted poset F' and a p-morphism $f : F' \to F$ such that $F' \vDash$ **SFL**(Λ).*

The construction works in two steps as follows (see Figure 8 for an example).

- Again, we start by splitting F up into its tree unravelling $\mathscr{T}(F)$.

[3] Recall that $\mathfrak{C}(F)$ is the set of connected components of F and that $\mathrm{c}(F)$ of F is the signature $n_1^{m_1} \cdots n_k^{m_k}$ such that $\mathfrak{C}(F)$ contains for each i exactly m_i sets of height $n_i - 1$, and nothing else.

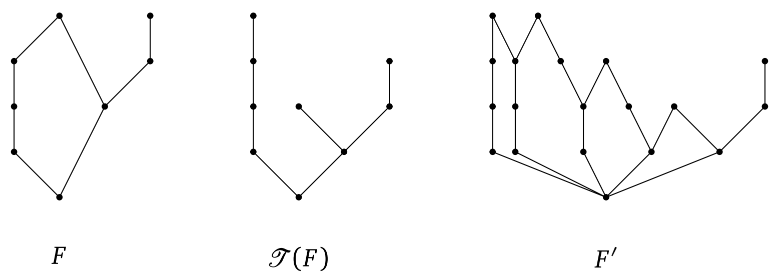


Figure 8: An example of gradification in the absence of Scott's tree.

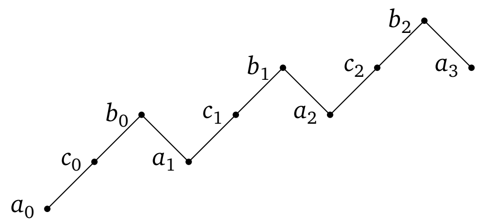


Figure 9: The relations between the zigzag points in case $l = 3$.

- Then, in order to connect the frame back up again while ensuring that it remains graded, we construct 'zigzag roller-coasters' connecting top nodes of different heights.

Proof of Theorem 6.28. As in the proof of Theorem 6.26, we may assume that $\epsilon, 1, 2 \notin \Lambda$.

Start with $T = \mathscr{T}(F)$. For every two distinct $p, q \in \mathsf{Top}(T)$ such that $\mathsf{last}(p) = \mathsf{last}(q) = t$, we will build a 'roller-coaster' structure $Z(p,q)$, which will furnish a bridge between p and q. Every such structure $Z(p,q)$ is independent, so that they can all be added to T at the same time. First note that by Corollary 6.22, T is graded; let $\rho\colon T \to \mathbb{N}$ be its rank function.

Now, take distinct $p, q \in \mathsf{Top}(T)$ such that $\mathsf{last}(p) = \mathsf{last}(q) = t$. Let $l := \rho(q) - \rho(p)$. By swapping p and q, we may assume that $l \geqslant 0$. We will join p and q with a zigzagging path, which consists of lower points $a_0, \dots, a_l$, upper points $b_0, \dots, b_{l-1}$ and intermediate points $c_0, \dots, c_{l-1}$. The relations between these points are as follows (see Figure 9).

$$a_i < c_i < b_i, \qquad a_{i+1} < b_i$$

Consider $p \wedge q$ (i.e. the intersection of p and q, regarded as strict chains containing the root), and let $k := \rho(p) - \rho(p \wedge q) - 1$. Note that $k \geqslant 0$ since p and q are incomparable. Moreover, $k \geqslant 1$. Indeed, suppose for a contradiction that $k = 0$, so that p is an immediate successor of $p \wedge q$. Then $\mathsf{last}(p)$ is an immediate successor of $\mathsf{last}(p \wedge q)$. But $\mathsf{last}(q) = \mathsf{last}(p)$, so we have, as strict chains:

$$p = (p \wedge q) \cup \{\mathsf{last}(p)\} = (p \wedge q) \cup \{\mathsf{last}(q)\} = q$$

contradicting that p and q are distinct. $\lightning$

To ensure that the new poset F' is still graded, we need to dangle some scaffolding down from the zigzag path to $p \wedge q$. Below each lower point a_i we will dangle a chain of $k + i - 1$ points $d(i,1), \dots, d(i, k+i-1)$. The relations are as follows:

$$d(i,1) < d(i,2) < \cdots < d(i, k+i-1) < a_i$$

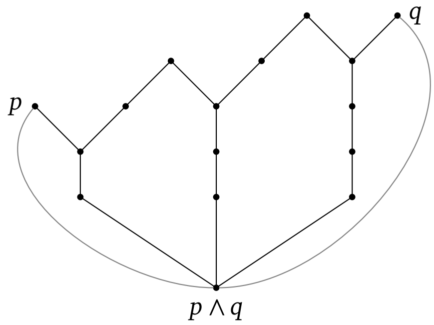


Figure 10: The zigzag path and the ladder structure in place.

Finally, let $\mathrm{Z}(p,q)$ denote the whole structure of the zigzag path plus the dangling scaffolding. Attach $\mathrm{Z}(p,q)$ to T by adding the following relations and closing under transitivity (see Figure 10).

$$a_0 < p, \qquad a_l < q, \qquad \forall i : p \wedge q < d(i,1)$$

Let F' be the result of adding $\mathrm{Z}(p,q)$ to T for every pair p,q, and define the function $f : F' \to F$ by:

$$f(x) := \begin{cases} \mathsf{last}(x) & \text{if } x \in T \\ \mathsf{last}(p) & \text{if } x \in Z(p,q) \text{ for some } p,q \end{cases}$$

First, let us see that f is a p-morphism. The (Forth) condition follows from the fact that last is monotonic, and that:

- if $x \leqslant y$ with $x \in T$ and $y \in Z(p,q)$, then by construction $x \leqslant p \wedge q$, meaning that $f(x) = \mathsf{last}(x) \leqslant \mathsf{last}(p \wedge q) \leqslant \mathsf{last}(p) = f(y)$, and
- if $x \leqslant y$ with $x \in Z(p,q)$ and $y \in T$, then by construction $y \in \{p,q\}$, so that $f(x) = \mathsf{last}(p) = f(y)$.

The (Back) condition follows from the fact that last is open, and that each $Z(p,q)$ maps to a top node.

Second, for any pair p,q, we can extend the rank function ρ to the new structure $\mathrm{Z}(p,q)$ as follows (as indicated by the heights of the nodes in Figure 10):

$$\begin{aligned} \rho(a_i) &= \rho(p) + i - 1 \\ \rho(b_i) &= \rho(p) + i + 1 \\ \rho(c_i) &= \rho(p) + i \\ \rho(d(i,j)) &= \rho(p \wedge q) + j \end{aligned}$$

To see that, thus extended, ρ is still a rank function, it suffices to check that the newly-ranked $Z(p,q)$ fits into T as a ranked structure. That is, we need to check the following equations.

$$\begin{aligned} \rho(p) &= \rho(a_0) + 1 \\ \rho(q) &= \rho(a_l) + 1 \\ \rho(d(i,1)) &= \rho(p \wedge q) + 1 \end{aligned}$$

But these follow by definition. In this way we see that F' is graded.

Finally, it remains to be shown that $F \vDash \mathbf{SFL}(\Lambda)$. So take $x \in F$. First, whenever $x \in Z(p,q)$ for some p,q, by construction $\Uparrow(x)$ is α-connected for every signature other than ϵ, 1^2, $2 \cdot 1$ and k where $k \geqslant \mathsf{height}(F) + 1$. Hence we may assume that $x \in T$.

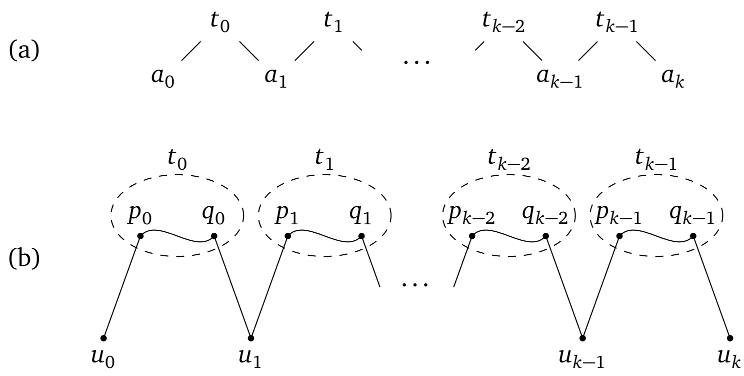


Figure 11: The form of the paths in $\Uparrow(\mathsf{last}(x))$ and $\Uparrow(x)$

Let us use Lemma 6.27. Take $y, z \in \mathrm{Succ}(x)$ such that there is a path $f(y) \rightsquigarrow f(z)$ in $\Uparrow(\mathsf{last}(x))$, with the aim of finding a path $y \rightsquigarrow z$ in $\Uparrow(x)$.

Assume that $y \in Z(p,q)$ for some p,q. Then since $y \in \mathrm{Succ}(x)$ and $x \in T$, by construction $x = p \wedge q$. All of $Z(p,q)$ is connected in $\Uparrow(x)$, hence there is a path $y \rightsquigarrow p$. Let $p' \in T$ be the immediate successor of x which lies below p (this exists since T is a tree). Then we have a path $y \rightsquigarrow p'$ in $\Uparrow(x)$. Therefore, we may assume that $y \in T$, and similarly that $z \in T$.

So, we have a path $\mathsf{last}(y) \rightsquigarrow \mathsf{last}(z)$. We now proceed in a similar fashion to the proof of Theorem 6.26. We may assume that the path $\mathsf{last}(y) \rightsquigarrow \mathsf{last}(z)$ has the form in Figure 11 (a), where $t_0, \ldots, t_k$ are top nodes in F. Let $u_0 := y$ and $u_k := z$. For each $i \in \{1, \ldots, k-1\}$, choose $u_i \in \mathsf{last}^{-1}\{a_i\}$. For $i \in \{0, \ldots, k-1\}$, take $p_i, q_i \in \mathsf{last}^{-1}\{t_i\}$ such that $u_i \leqslant p_i$ and $u_{i+1} \leqslant q_i$. For each such i, since $\mathsf{last}(p_i) = \mathsf{last}(q_i)$, there is a path $p_i \rightsquigarrow q_i$ which lies in $Z(p_i, q_i)$, and hence lies in $\Uparrow(x)$. Compose all these paths as in Figure 11 to form a path $y \rightsquigarrow z$ in $\Uparrow(x)$ as required.

It now remains to show that if $C \in \mathfrak{C}(\Uparrow(\mathsf{last}(x)))$, then $\mathsf{height}(f^{-1}[C]) = \mathsf{height}(C)$. First, since f is a p-morphism, $\mathsf{height}(f^{-1}[C]) \geqslant \mathsf{height}(C)$. Conversely, let $X \subseteq f^{-1}[C]$ be a maximal chain. Assume X intersects with some $Z(p,q)$. Then we can replace the part $X \cap (Z(p,q) \cup \{p,q\})$ with the unique maximal chain in $\Uparrow(p \wedge q)^T$ containing q (this exists since T is a tree). Then by construction this does not decrease the length of X nor does it move X outside of $f^{-1}[C]$ (since the latter is upwards- and downwards-closed). Therefore, we may assume that $X \subseteq T$, so X corresponds to a chain $\mathsf{last}[X]$ of the same length in C.

Therefore, by Lemma 6.27 we get that $\mathrm{c}(\Uparrow(x)) = \mathrm{c}(\Uparrow(\mathsf{last}(x)))$. Applying Lemma 6.6, we have that $\Uparrow(x)$ has an α-partition if and only if $\Uparrow(\mathsf{last}(x))$ has an α-partition. $\square$

6.8 Nervification

We now find ourselves, having suitably prepared F, in a position to make use of its additional graded structure. The general method of the final construction, in which we transform F into a frame which nerve-validates $\mathbf{SFL}(\Lambda)$, is the same as in Theorem 6.26 and Theorem 6.28. We begin with the tree unravelling $\mathscr{T}(F)$, perform some alterations, then rejoin top nodes. A key difference here is that we won't rejoin every top node to every other top node whose 'last' value is the same. Instead, we line up all the top nodes mapping to the same element and link each top node to at most two other top nodes: its neighbours. See Figure 12 for an example of the construction.

Definition 6.31. Let T be a finite tree. Then for each $x \in T$, we have that $\downarrow(x)$ is a chain. For $k \leqslant \mathsf{height}(x)$, let $x^{(k)}$ be the element of this chain which has height k. Let $x^{(-k)}$ be the element which has height $\mathsf{height}(x) - k$.

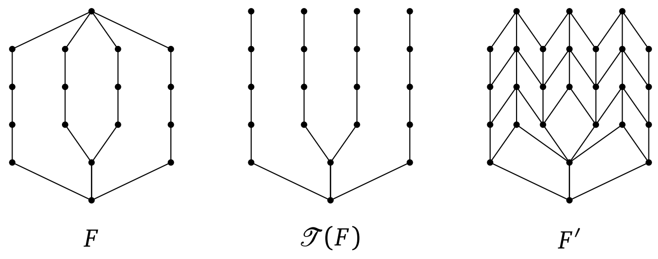


Figure 12: An example of nervification, using the graded structure of F

Definition 6.32. For $n \in \mathbb{N}$, let $\mathscr{S}_n := \mathscr{S} \setminus \{1^k \mid k < n\}$.

Theorem 6.33. *Take $\Lambda \subseteq \mathscr{S}$ and let F be a finite graded rooted poset of height n such that $F \vDash$ **SFL**(Λ). Then there is a poset F' and a p-morphism $f : F' \to F$ such that $F' \vDash$ **SFL**(Λ) and such that F' is α-diamond-connected for every $\alpha \in \mathscr{S}_n$.*

Proof of Theorem 6.31. We may assume that $\epsilon, 1 \notin \Lambda$. Further, if $2 \in \Lambda$, then $\mathsf{height}(F) = 1$, so F is already α-diamond-connected for every $\alpha \in \mathscr{S}_n$. Hence we may assume that $2 \notin \Lambda$.

Once more, start with $T = \mathscr{T}(F)$. Chop off the top nodes: let $T' := \mathsf{Trunk}(T)$. For each $t \in \mathsf{Top}(F)$, we will add a new structure $W(t)$, which lies only above elements of T'. Let $\rho : F \to \mathbb{N}$ be the rank function on F. Note that $\rho \circ \mathsf{last} : T \to \mathbb{N}$ is the rank function on T.

Take $t \in \mathsf{Top}(F)$. Enumerate $\mathsf{last}^{-1}\{t\} = \{p_1, \ldots, p_m\}$. For each $i \leqslant m-1$, define:

$$r_i := p_i \wedge p_{i+1}$$
$$l_i := \rho(\mathsf{last}(r_i))$$
$$k_i := \rho(t) - \rho(\mathsf{last}(r_i)) - 1$$

Note that $k_i \geqslant 1$ just as in the proof of Theorem 6.28. Since F is graded and T is a tree, we have that:

$$|\Updownarrow(r_i, p_i)^T| = |\Updownarrow(r_i, p_{i+1})^T| = k_i$$

In other words, $p_i^{(l_i)} = p_{i+1}^{(l_i)} = r_i$. We will construct a 'chevron' structure which joins $p_i^{(-1)}$ to $p_{i+1}^{(-1)}$. For each $i \leqslant m-1$, take new elements $a(i,1), \ldots, a(i,k_i)$, and add them to T' using the following relations.

$$a(i,1) < \cdots < a(i,k_i), \qquad \forall j \leqslant k_i : p_i^{(l+j)}, p_{i+1}^{(l+j)} < a(i,j)$$

Let $W(t)$ be this new structure (i.e. the chain $\{a(i,1) < \cdots < a(i,k_i)\}$ in place). See Figure 13 and Figure 14 for examples of this process of adding chevrons.

The process of adding $W(t)$ is independent for each $t \in \mathsf{Top}(F)$. Let F' be the result of adding every $W(t)$ to T'. Define $f : F' \to F$ by:

$$f(x) := \begin{cases} \mathsf{last}(x) & \text{if } x \in T' \\ t & \text{if } x \in W(t) \text{ for some } t \in \mathsf{Top}(F) \end{cases}$$

Since we have made sure that each $W(t)$ contains, for each $p_i \in \mathsf{last}^{-1}\{t\}$, a node above $p_i^{(-1)}$ which maps to t, and that all of the new structure maps to a top node, f is a p-morphism.

Let us see that $F' \vDash$ **SFL**(Λ). Take $x \in F'$. If $x \in W(t)$ for some t, then $\Uparrow(x)$ is either empty or a chain, hence $\Uparrow(x) \vDash$ **SFL**(Λ). So we assume that $x \in T'$. The verification is now very similar to that in Theorem 6.28, making use of Lemma 6.27. Take $y, z \in \mathrm{Succ}(x)$ such that there is a path $f(y) \rightsquigarrow f(z)$ in $\Uparrow(\mathsf{last}(x))$. As in the proof of Theorem 6.28, by construction of $W(t)$ we may assume that $y, z \in T'$. Just

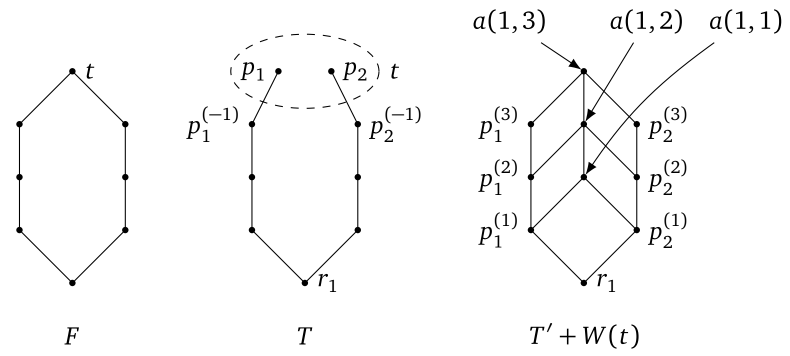


Figure 13: The chevron structure in a case with two branches.

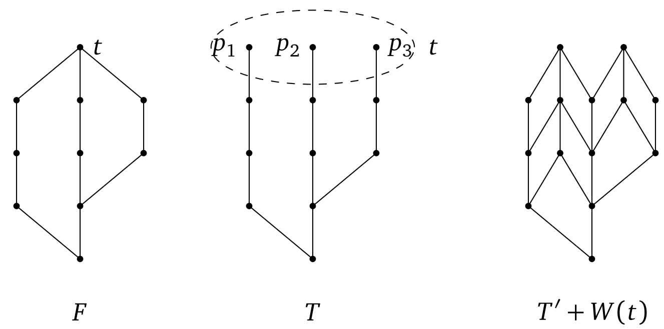


Figure 14: The chevron structure in a more complex case involving three branches.

as in that proof, we can construct a path $y \rightsquigarrow z$ from the path $f(y) \rightsquigarrow f(z)$, using the fact that whenever $t \in \Uparrow(\mathsf{last}(x)) \cap \mathsf{Top}(F)$, any $w, v \in f^{-1}\{t\}$ are connected by a path in $\Uparrow(x)^{F'}$ (this is how we constructed F'). It is straightforward then to check that if $C \in \mathfrak{C}(\Uparrow(\mathsf{last}(x)))$ we have $\mathsf{height}(f^{-1}[C]) = \mathsf{height}(C)$, giving that:

$$\mathsf{c}(\Uparrow(x)) = \mathsf{c}(\Uparrow(\mathsf{last}(x)))$$

To complete the proof, let us see that F' is α-diamond-connected for every $\alpha \in \mathscr{S}_n$. Take $x, y \in F'$ with $x < y$ and consider $\Updownarrow(x, y)$. There are several cases.

(a) Case $y \in T'$. We have that $\Updownarrow(x, y)^{F'} = \Updownarrow(x, y)^{T'}$, which is linearly-ordered since T' is a tree; hence it is connected and of height at most $n - 2$.

Hence $y = a(i, j)$ for $a(i, j) \in W(t)$ a new element. Let p_i, p_{i+1}, r_i, l_i be as above.

(b) Case $x \in W(t)$. Note that by construction $\Updownarrow(x, y)$ is linearly-ordered.

(c) Case $x = p_i^{(l+e)}$ for some e. If we have $\mathsf{height}(\Updownarrow(x, y)) = 1$, then $e = i - 1$ and $\Updownarrow(x, y)$ is the antichain on two elements, which is α-connected. Otherwise, by construction, $a(i, j-1) \in \Updownarrow(x, y)$ which is connected to everything.

(d) Case $x = p_{i+1}^{(l+e)}$ for some e. This is symmetric.

(e) Case $x = r_i$. Again, if $\mathsf{height}(\Updownarrow(x, y))) = 1$ then $j = 1$ and $\Updownarrow(x, y)$ is the antichain on two elements, otherwise $a(i, 1) \in \Updownarrow(x, y)$ which is connected to everything.

(f) Otherwise, $x < r_i$ (since T' is a tree). Then $r_i \in \Updownarrow(x, y)$ which is connected to everything. □

6.9 Putting it all together

After a fair bit of labour, we now have all the ingredients we need for our proof. Let us put them together.

Proof of Theorem 6.11. By Lemma 6.15 and Lemma 6.16, we need to show that every finite rooted frame of **SFL**(Λ) is the up-reduction of one which nerve-validates **SFL**(Λ); in fact this up-reduction is just a p-morphism. So take such a frame F. We may assume that F is graded: when we have $2 \cdot 1 \in \Lambda$, apply Theorem 6.26, otherwise apply Theorem 6.28. Then by Theorem 6.31, there is a frame F' and a p-morphism $f : F' \to F$ such that F' is α-nerve-connected for every $\alpha \in \Lambda$ (note that by Remark 6.10 we must have $\Lambda \subseteq \mathscr{S}_n$ where $n = \mathsf{height}(F)$). Then, by Theorem 6.18, F' nerve-validates **SFL**(Λ), which completes the proof. □

7 Conclusion

We hope to have demonstrated that the Heyting algebra $\mathrm{Sub}_{\mathrm{o}}P$ opens up a rich connection between logic and polyhedral geometry, which is given life by the sustained import of geometrical ideas. The link between triangulations and nerves utilised in [Bez+18a] for polyhedral completeness for **IPC** and **S4.Grz** has been developed further in this paper culminating in the Nerve Criterion. This is a product of the unison of logic with non-trivial arguments from rational polyhedral geometry.

The Nerve Criterion is exploited to chart out a class of polyhedrally-complete logics axiomatised by the Jankov-Fine formulas of *starlike trees*. The proof that a starlike logic is polyhedrally-complete utilises a number of combinatorial techniques on finite posets. Such logics have a clear geometric meaning and play an important part in polyhedral semantics. Indeed, the largest starlike logic $\mathbf{PL}_n$ of height n is shown in [Ada+20] to coincide with the logic of convex polyhedra of dimension n, while the logic of all convex polyhedra

$$\mathbf{PL} = \mathbf{SFL}(2 \cdot 1, 1^3) = \mathbf{IPC} + \chi(\langle 2 \cdot 1 \rangle) + \chi(\langle 1^3 \rangle)$$

The proofs of these results blend combinatorial and geometric ideas, and serve as a fitting culmination of the various strands of this new approach.

Polyhedral semantics for intermediate and modal logic is a very young area, and there are many open problems and directions for future research. We pick out just a few of these.

One ultimate goal would be a complete classification of polyhedrally complete logics. The results in this paper and in [Ada+20] take several steps towards such a classification, and chart out key features of the landscape. Identifying more polyhedrally complete logics would be the next immediate task in this direction.

The possibility of moving to a richer language is always available to us. One motivation for this is that with the present semantics, logic cannot capture any of the homology of the polyhedron in which it is interpreted. This is because formula satisfaction is always local in a polyhedron (this fact is not so pronounced in the present paper, where satisfaction at points of a polyhedron is eschewed in favour of the more abstract notion of triangulation). Homology seems a rather natural aspect for a logic to express; indeed, its axiomatic method is a well-developed line of research (see [Hat02, §2.3, p. 160]). Perhaps the addition of a universal modality will enable this expression.

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