Quantum quirks, classical contexts: Towards a Bohrification of effect algebras

MSc Thesis *(Afstudeerscriptie)*

written by

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under the supervision of **Chris Heunen** and **Nick Bezhanishvili**, and submitted to the Examinations Board in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:
28 September 2020

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Abstract

Two results of the following general form are proved: a functor from a category of algebras to the category of posets is essentially injective on objects above a certain size. The first result is for Boolean algebras and the functor taking each Boolean algebra to its poset of finite subalgebras. This strengthens and provides a novel proof for a result by Sachs, Filippov and Grätzer, Koh & Makkai. The second result is for finite MV-algebras and the functor taking each such algebra to its poset of partitions of unity.

The second result uses the dual equivalence of finite MV-algebras with finite multisets, as well as the correspondence between partition posets of finite multisets and setoid quotients. Thus the equivalence is constructed via the powerset functor for multisets, and setoid quotients are introduced. The equivalence is a special case of a more general duality proved by Cignoli, Dubuc and Mundici.

The primary interest of this work lies in algebras describing quantum observables, hence both results are viewed as statements about effect algebras. Since MV-algebras contain 'unsharp' (i.e. self-orthogonal) elements, the second result shows that sharpness is not a necessary condition for essential injectivity of the partitions of unity functor. Physically, this means that there are systems with unsharp effects (namely, those represented by finite MV-algebras) which can be faithfully reconstructed from all the possible measurements.

Acknowledgements

I would like to thank Chris Heunen for agreeing to supervise this work remotely, way before it was clear all supervision will take place remotely this spring and summer anyway. I would like to thank Nick Bezhanishvili for agreeing to be my local supervisor despite me not being sure what the project will be about. A very special thanks goes to John van de Wetering for suggesting the research area as well as providing useful references and comments at different stages of this work. I would like to thank Sam Staton for giving references to recent work in effect algebras. I am grateful to Yoàv and Harold for reading and giving feedback on parts of the final draft.

I acknowledge the financial support of the Amsterdam Science Talent Scholarship for the duration of the masters degree.

While this thesis is not a work on category theory as such, categorical language, concept and ideas permeate it throughout. I would, therefore, like to use the opportunity to thank everyone who has contributed to me learning this deep, exciting and difficult subject. Three people need to be mentioned here: Chris Heunen, whose spring 2017 course *Categories and quantum informatics* and a research project in the following summer at the University of Edinburgh were my first introduction to categories; Tom Leinster, whom I would like to thank for not turning down an eager undergraduate wanting to learn about sheaves and for invaluable advice on mathematical writing; Benno van den Berg, from whom I learned about categorical models for homotopy type theory. In this context, it would be intellectually dishonest to not mention the nLab[1], if for nothing else then for a quick and easy source of references. I would like to express my esteem for everyone who has contributed to this project.

Importantly, I would like to mention two academic communities at the University of Amsterdam: the ILLC and the Master of Logic community as well as the MSc Mathematics community, enriched by interactions between students from both degrees, as well as by the students from other universities. Everyone who is/was involved in organising seminars, talks, social events, or was just there to share their knowledge and ideas with others, deserves a round of applause; and for each n, if they deserve n rounds of applause, then they deserve $n + 1$ rounds too. I sincerely hope that the interactions between the two communities will continue.

Lastly, I personally thank Matteo, Mireia, Loe, Yoàv and Ann for support, friendship and a continued interest towards what I do.

[1]https://ncatlab.org

Contents

Chapter 1

Introduction and motivation

The guiding principle of this work is that "shape determines structure". More precisely, in many scenarios, a given structure has a class of substructures such that the poset of these substructures is sufficient to uniquely determine the full structure. Thus the mere 'shape' of how the substructures fit together (poset) dictates what the entire structure should be. The class of substructures is often 'nicer' than the full structure we started with; the substructures may either be computationally more manageable (as in the case of finite subalgebras of a Boolean algebra), or there could even be a difference in the epistemic status, as is the case for algebras arising from quantum observables, whose subalgebras and partitions represent the classically accessible information. While for the reconstruction itself it is irrelevant whether the substructures are nice or not, as both are just an element of a poset, the difference lies in accessibility. Thus, as an arbitrary Boolean algebra can be approximated by finite ones, likewise, an inaccessible part of reality can be approximated by accessible ones.

The area of order theory and computer science that is inter alia concerned with informational approximations of the above kind is known as *domain theory*. See Abramsky and Jung [1] for an introduction to the subject, and Section 4 of Heunen [32] for a brief description in the context of this work.

The main classes of structures we focus on are Boolean algebras, MV-algebras and effect algebras; the former two being special cases of the last one. Effect algebras are partial algebras which originate from theoretical physics: the idea is that not all observables are simultaneously measurable, so that the effect algebra operation is only defined for those elements which are 'orthogonal'.

Operational quantum mechanics takes the set of experimental outcomes, i.e. the physical events which may actually occur, as its primitive. Such events are called *effects*. Since effects represent possible measurement outcomes, we must be able to assign a probability to each effect. For this reason the general effects are sometimes called *unsharp*, to contrast them with *sharp* effects whose probability of occurring is either 0 or 1.

Connection to probabilities dictates that in the Hilbert space formulation of quantum mechanics, the set of effects is given by positive[1] operators A such that $0 \leq A \leq 1$, where 0 and 1 are the zero and identity operators, and the partial order is the pointwise order of the operators. For the details on this, see e.g. Introduction of Busch, Grabowski and Lahti [11]. If H is a Hilbert space, then the set of effects $\mathcal{E}(H)$ is closed under the orthocomplement $A' = 1 - A$, but it is not closed under addition of operators: we may add two operators A and B if and only if $A + B \leq 1$. This gives rise to the motivating (and eponymic) example of an *effect algebra*.

Motivated by a similar result for orthoalgebras due to Harding, Heunen, Lindenhovius and Navara [31] (Theorem 1.1), one might be tempted to think that there is a suitable class of 'classical' subalgebras of an effect algebra which determine it. We begin our discussion by showing that this is not the case, after which we motivate why it could be physically plausible to expect such a result for partitions of unity rather than subalgebras.

1.1 Mathematical motivation: Encoding structure in posets

Logically, this section should come after Sections 2.1 and 2.2, where effect algebras, orthoalgebras and MV-algebras are defined. However, the theorem and the two examples presented here are precisely the mathematical driving force for the rest of this research, so that they should be presented at the very beginning. The presentation is therefore made as light on the technical details as possible.

A subalgebra of an effect algebra is *Boolean* if it is a Boolean algebra whose structure is compatible with the effect algebra structure; namely, two elements are disjoint (their meet is 0) in the Boolean algebra if and only if they are orthogonal in the effect algebra, and in this case their (Boolean) join is equal to their (effect) sum. Let us denote the poset of Boolean subalgebras of an effect algebra E by $\mathrm{BSub}(E)$. One of the main results of Harding, Heunen, Lindenhovius and Navara [31] is the following:

[1] Positive semidefinite, to be precise.

Theorem 1.1. *If A is a proper orthoalgebra, then* $\mathrm{BSub}(A)$ *has enough directions and* $\mathrm{Dir}(\mathrm{BSub}(A))$ *is an orthoalgebra isomorphic to A.*

Here the condition of being proper says that A does not have maximal Boolean subalgebras which have at most four elements, and Dir denotes the set of what Harding et al. call directions. An important consequence of this is that an orthoalgebra A is determined by its poset of Boolean subalgebras. The result is in fact stronger than that: it provides a direct way to reconstruct A from $\mathrm{BSub}(A)$, namely by taking the set of directions of the latter with the appropriate orthoalgebra structure.

As Harding et al. point out, this result can be seen as classical measurement contexts (Boolean subalgebras) containing enough information about a quantum system with 'sharp' effects (since an important class of examples of orthoalgebras is given by the set of projections on a Hilbert space).

The rest of this section is devoted to showing that no such reconstruction result can hold for all effect algebras, even when we relax the Boolean condition and consider all subalgebras. Given an effect algebra, Example 1.2 constructs a non-isomorphic effect algebra without altering the Boolean subalgebra poset, thus showing that no result of the form of Theorem 1.1 holds for the class of all effect algebras. Example 1.3 strengthens this observation: we give a countable collection of non-isomorphic effect algebras which all have the same subalgebra poset (up to an isomorphism), not just of Boolean subalgebras but of all subalgebras. A reader unfamiliar with effect algebras (or unwilling to dwell on the technical details) is encouraged to skip the rest of this section and return back to the examples after reading Sections 2.1 and 2.2.

The examples make use of two important classes of effect algebras: Boolean algebras and finite chains with $n \geq 2$ elements denoted by L_n. In the former, two elements are orthogonal if they are disjoint (their meet is 0), in which case the effect algebra sum is given by the join. The latter may be seen as finite subalgebras of the unit interval, and two elements $a, b \in L_n$ are orthogonal if $a + b \leq 1$, in which case their effect algebra sum is just $a + b$. The orthosupplement of a is given by $a' = 1 - a$. See Examples 2.7, 2.18 and 2.19 for details.

Example 1.2. We begin by noting that the only Boolean subalgebra of $L_3 = \{0, \frac{1}{2}, 1\}$ is the two-element algebra $\{0, 1\}$.

Now let E be an arbitrary effect algebra. Consider the coproduct effect algebra $E \sqcup L_3$, which may be constructed by taking the disjoint union of the underlying sets and identifying 1's as well as the 0's. The orthogonality relation is then given by $x \perp y$ iff x and y are contained in one of the summands and are orthogonal in it. When the coproduct is constructed in this way, we claim that $B \subseteq E \sqcup L_3$ is a Boolean subalgebra iff $B \subseteq E$ and B is a Boolean subalgebra of E. The 'if' direction is clear, as any subalgebra of E is a subalgebra of $E \sqcup L_3$. The 'only if' direction is also straightforward: for if a Boolean subalgebra $B \subseteq E \sqcup L_3$ contains $\frac{1}{2}$, then $\frac{1}{2} \perp \frac{1}{2}$ implies $\frac{1}{2} = \frac{1}{2} \wedge \frac{1}{2} = 0$, which is a contradiction.

Thus we have observed that $\mathrm{BSub}(E \sqcup L_3) = \mathrm{BSub}(E)$ for any effect algebra E. So in particular, we may take E to be a Boolean algebra with more than four elements, so that the condition of $E \sqcup L_3$ being proper is satisfied. Then there is no procedure that uniquely recovers $E \sqcup L_3$ from $\mathrm{BSub}(E \sqcup L_3)$, as the same procedure would also recover E, which is not isomorphic to $E \sqcup L_3$.

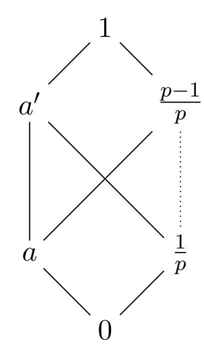
The following example is inspired by Example 2.4 in Riečanová [49], which was translated from D-posets to effect algebras by Dvurečenskij [22, Example 2.3].

Example 1.3. Let $p \geq 3$ be a prime number. Define an effect algebra E_p as follows. Let $\{a, a'\} \sqcup L_{p+1}$ be the underlying set. For $q, r \in L_{p+1}$, define $q \perp r$ iff q and r are orthogonal in L_{p+1}, and then $q \oplus r$ is the sum in L_{p+1}. Then, let

$$a \perp a, \quad a \perp \frac{1}{p}, \quad a \perp a' \quad \text{and}$$

$$a \oplus a = \frac{p-1}{p}, \quad a \oplus \frac{1}{p} = a', \quad a \oplus a' = 1.$$

It is straightforward to verify that this indeed defines an effect algebra. Below we depict the Hasse diagram of E_p.



Observe that the only subalgebras of E_p are $\{0, 1\}$, L_{p+1} and E_p, as L_{p+1} has no non-trivial subalgebras (since p is prime), and any subalgebra containing $\{a, a'\}$ must also contain $(a \oplus a)' = \frac{1}{p}$, whence it must be E_p. Thus the poset of subalgebras of E_p is the three element chain, independently of p. Thus we have a countable collection of effect algebras (one for each prime) with the same subalgebra poset. To add insult to injury, this is also the subalgebra poset of the chain L_5.

Note that Example 1.3 shows that the reconstruction is not possible already for MV-algebras: the chains L_p all have isomorphic subalgebra posets.

1.2 Physical motivation: Bohr's doctrine of classical concepts

Ever since the Kochen-Specker theorem [40], it has been clear that Boolean algebras do not provide adequate semantics for the logic of quantum observables: the heart of the proof is that there is a partial Boolean algebra of propositions describing a quantum system which is not embeddable into a (total) Boolean algebra. This has spawned a search for the 'correct' (partial) algebraic structure that would capture the observational and propositional nature of a quantum system. A promising such class of structures is that of effect algebras, having their origins in the operational approach to quantum mechanics.

On the other hand, there has been a significant amount of research to reconcile the fundamental non-commeasurability of quantum observables with the fact that the means by which we obtain information from a system are always classical. In more detail, any single measurement in an experimental setup can only access a subsystem of the full quantum system that consists of simultaneously measurable quantities. An individual measurement is therefore 'classical' in this sense. A classic example is the momentum-position pair, of which only one can be measured with arbitrary precision. This gives rise to the question: are such 'classical snapshots' sufficient to know the full quantum system? Similar considerations led Bohr to formulate his principle which came to be known as the *doctrine of classical concepts*: roughly, all and only physically relevant information about a quantum system is accessible via classical measurements. See Chapter I of Scheibe [52] for a detailed discussion of Bohr's views (specifically Sections I.2(e) and I.2(g) for the classical concepts); and Camilleri and Schlosshauer [13] for a more recent perspective.

The research paradigm that undertook the effort of making Bohr's doctrine precise is known as *topos quantum theory* [21], [33]. The approach taken by Heunen, Landsman and Spitters [33] starts from the formulation of quantum mechanics in terms of C^*-algebras, although the precise algebraic details are not important for our purposes. Suffice it to say that the hom-set $\mathbf{Hilb}(H, H)$ of continuous linear operators on a Hilbert space H has the structure of a C^*-algebra, and in fact by the Gelfand-Naimark theorem [27], any C^*-algebra embeds into one of this form.

The algebraic counterpart of non-commeasurability is non-commutativity, and conversely, commutativity can be regarded as classicality. Indeed, the C^*-algebra of a Hilbert space is in general non-commutative, while its commutative subalgebras can be regarded as the classically commeasurable sets of observables, or as 'classical contexts' which may be held fixed while the observables in that context are measured. The idea of Heunen et al. is to consider the poset of commutative subalgebras $\mathcal{C}(A)$ of a C^*-algebra A. The question is then twofold:

(1) Does the poset $\mathcal{C}(A)$ contain enough information about A so that it could be thought to fully capture all the physically relevant information?

(2) Is there a way to reason about A classically using $\mathcal{C}(A)$?

The former question is answered in the positive in e.g. Hamhalter [30, Theorem 3.4], so long as one accepts that the physically relevant information is the Jordan structure of A. See also Döring and Harding [20] for a similar result for von Neumann algebras, the PhD thesis of Lindenhovius [45] for an extensive discussion of the structure of $\mathcal{C}(A)$, as well as the list at the very beginning of the introduction of Heunen and Lindenhovius [34] for a collection of properties of A that can be recovered from $\mathcal{C}(A)$.

The latter question is answered by Heunen et al. by showing that the 'tautological' presheaf in the presheaf topos $[\mathcal{C}(A), \mathbf{Set}]$ defined by $C \mapsto C$ on objects and $(C \subseteq D) \mapsto (C \hookrightarrow D)$ on morphisms has the structure of a *commutative* C^*-algebra internally in the presheaf topos. The combination of these two facts allows one to reason about a quantum system as if it were classical. Heunen et al. call this process *Bohrification*. See the 2017 expository article of Heunen [32] for an overview of how different parts of this research program fit together.

It is the first of the above questions that serves as the high-level motivation for this work. While Harding et al. [31] were able to reconstruct an algebra representing a quantum system with sharp effects (projections) only, general systems also contain unsharp effects. Thus it is meaningful to ask the same question for effect algebras. As we have seen in the previous section, for effect algebras it is not even clear what this 'poset of classical substructures' should be, as even the poset of *all* subalgebras fails to determine some effect algebras, to say nothing of a subclass of subalgebras.

We suggest that instead of subalgebras, we should consider the poset of partitions of unity ordered by refinement (Definition 6.2). Physically this is motivated by their close link with *positive operator valued measures* (POVMs). Thus the main result of this work (Theorem 6.24) is a reconstruction result in a very special case. In the light of the examples in the previous section, the reconstruction result is somewhat surprising. Based on the examples one might think that some additional condition on effect algebras or MV-algebras are needed to make them physical. And indeed there are effect algebras and MV-algebras which have nothing to do with physical systems. Thus the reconstruction result (especially if Conjecture 7.1 turns out to hold) suggests that the notion of a (generalised) measurement may play a role in the theory of effect algebras and MV-algebras beyond its physical significance.

1.3 How to read this thesis

The remaining chapters are structured as follows.

Chapter 2 defines effect algebras, orthoalgebras and MV-algebras, as well as finite multisets and setoids. The reader already familiar with effect algebras/orthoalgebras or MV-algebras may skip Section 2.1 or Section 2.2, respectively, and refer back if necessary. Similarly, the reader familiar with multisets may skip Section 2.3, with the possible exception of construction of the multisets-to-setoids functor (right after Definition 2.24).

The entirety of Chapter 3 is dedicated to proving that the categories of finite multisets and finite MV-algebras are dually equivalent via the powerset functor (Theorem 3.8). This is a special case of a more general duality proved by Cignoli, Dubuc and Mundici [19]. If a reader is familiar with this duality, or is willing to take Theorem 3.8 on trust, then this chapter may be skipped.

Chapter 4 gives and unifies two ways of viewing effect algebras as presheaves. This is based on the work by Staton and Uijlen [55]. We use this to give an isomorphism criterion for effect algebras (Corollary 4.10).

Chapter 5 uses the isomorphism criterion for effect algebras from Chapter 4 to show that a Boolean algebra with more than four elements is uniquely determined by the poset of its finite subalgebras (Theorem 5.24). This is a strengthening of a known result with a new proof.

In Chapter 6, we first define partitions of unity, partition poset of a finite multiset and the setoid quotient of finite setoids. We then show that the partition posets of finite multisets coincide with the setoid quotients (Lemma 6.13). Finally, we use the duality of finite multisets with finite MV-algebras (Chapter 3) to show that any finite MV-algebra with more than four elements is uniquely determined by its poset of partitions of unity.

In the concluding Chapter 7, we suggest a conjecture that any effect algebra is determined by its poset of partitions of unity (Conjecture 7.1). We point out that the reconstruction results of Chapters 5 and 6 are both special cases of this very general conjecture.

Appendix A defines functors that are essentially injective/isomorphic on objects, as these notions are not entirely standard. Appendix B contains a short note on how to extend the setoid quotient (Definition 6.10) to a functor. Appendix C introduces finite product theories and establishes a Yoneda-like isomorphism criterion in this general context (Proposition C.29). This gives an alternative way to prove the reconstruction result for Boolean algebras (Theorem 5.24) without resorting to effect algebras.

1.4 Contributions

Two main results of this work are essential injectivity of the finite subalgebra functor on Boolean algebras with more than four elements (Theorem 5.24) and essential injectivity of the partitions of unity functor on finite MV-algebras with more than four elements (Theorem 6.24). The former is a strengthening of a result proved by Sachs [51], Filippov [24] and Grätzer, Koh and Makkai [29], while the latter, to the knowledge of the author, is completely new. The proof of the former result differs from the proof strategy of any of the quoted authors, and, the author believes, is more conceptual than the previous proofs.

As a technical tool for showing essential injectivity of the partitions of unity functor, setoid quotients are introduced (Definition 6.10) and are shown to coincide with the partition posets of finite multisets (Lemma 6.13).

The dual equivalence of finite multisets with finite MV-algebras (Theorem 3.8) is shown by constructing the powerset functor, which, albeit naturally isomorphic to (a restriction of) the construction of [19], is different in spirit.

Chapter 2

Structures at hand

This chapter introduces the structures that are the object of study of the remaining chapters. Section 2.1 focusses on two closely related partial algebraic structures: effect algebras and orthoalgebras. Section 2.2 defines MV-algebras and shows that these form a subcategory of effect algebras. Both sections outline the relation of these structures to Boolean algebras and posets. The topic of Section 2.3 are the combinatorial structures used as a tool in Chapter 6: finite multisets and finite setoids.

We summarise the relations between all the structures we will encounter in the diagram of Figure 2.1. The rest of the thesis can be seen as defining the objects of this diagram as well as proving results about them. The reader is encouraged to return to the diagram whenever a new structure is introduced, or a theorem is proved, to see how it fits the general picture.

There is a slight dishonesty in the caption of Figure 2.1: the setoid quotient SQuot is not a functor, as it is only defined on objects and there is no natural way to define it on morphisms. The possibilities of extending SQuot to a functor are the object of discussion of Appendix B.

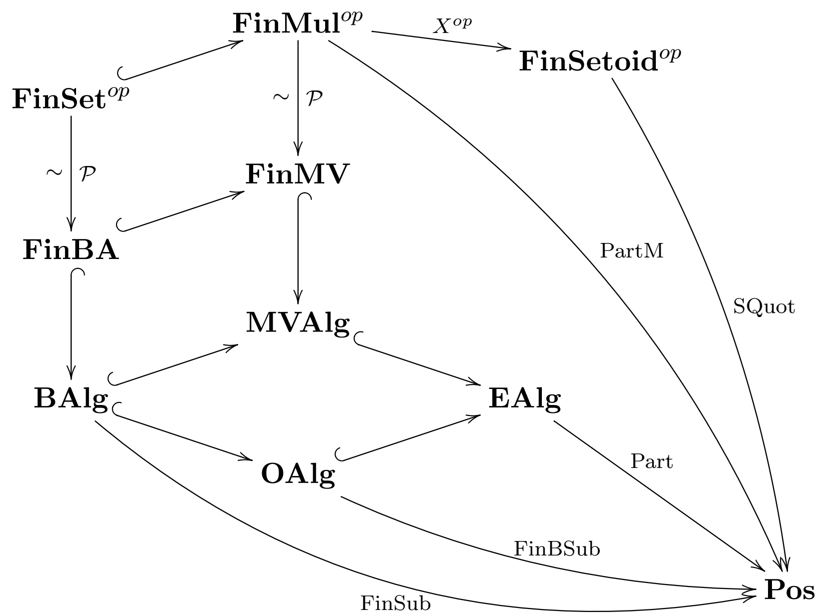


Figure 2.1: A road map of categories and functors.

Notation	Means	Where defined
FinSet	category of finite sets	-
FinMul	category of finite multisets	Definition 2.21
FinSetoid	category of finite setoids	Definition 2.24
BAlg	category of Boolean algebras	-
FinBA	category of finite Boolean algebras	-
MVAlg	category of MV-algebras	Section 2.2
FinMV	category of finite MV-algebras	-
EAlg	category of effect algebras	Section 2.1
OAlg	category of orthoalgebras	Section 2.1
Pos	category of posets	-
$\mathcal{P}$	powerset functor	Section 3.2
X	multisets-to-setoids functor	Section 2.3
FinSub	finite subalgebra poset	Theorem 5.24
FinBSub	finite Boolean subalgebra poset	Section 6.1
Part	partitions of unity functor	Section 6.1
PartM	multiset partitions functor	Definition 6.7
SQuot	setoid quotient	Definition 6.10

In Figure 2.1, $\mathcal{P}$: $\mathbf{FinSet}^{op} \to \mathbf{FinBA}$ stands for the usual powerset functor exhibiting the dual equivalence of finite sets and finite Boolean algebras. It can be seen as (naturally isomorphic to) the domain-codomain restriction of the powerset functor for multisets $\mathcal{P} : \mathbf{FinMul}^{op} \to \mathbf{FinMV}$, so that there is no confusion to denote them by the same symbol. We write $\hookrightarrow$ for the subcategory inclusion up to an equivalence, while $\sim$ stands for an equivalence of categories. The notation used in Figure 2.1 is explained in the table below it, where we also refer to the relevant part of the current text where the object in question is discussed.

2.1 Effect algebras and orthoalgebras

Effect algebras (and equivalent structures) were introduced by three different schools starting from the late 1980s in an attempt to capture the (partial) algebraic structure of the Hilbert space effects $\mathcal{E}(H)$. Interestingly, as Bennett and Foulis [5] point out, Boole in his *An investigation of the laws of thought*, considers $+$, the operation that is to become the join of a Boolean algebra, a *partial* operation that only applies to disjoint elements [7, Class II, p. 23]. Bennett and Foulis proceed to give a characterisation of Boolean algebras where the 'join' is indeed restricted to the orthogonal elements [5, Theorem 1.1]; this is in effect our observation in Example 2.7 that each Boolean algebra can be seen as an effect algebra.

In 1989, Giuntini and Greuling [28] summarised the properties of Hilbert space effects in the notion of a *weak orthoalgebra*. In 1994, Chovanec and Kôpka [41] introduced *D-posets* as posets with a partial difference operation. Later, both of these structures were shown equivalent to *effect algebras* as introduced by Bennett and Foulis [5] in 1994. For much more details on effect algebras and related structures, we refer the reader to Dvurečenskij and Pulmannová [23].

One aspect of effect algebras making them interesting is that they generalise both structures that are typically associated as providing semantics for a (quantum) logic, and structures that are closely associated with the study of probabilities. Examples of the former include Boolean algebras, orthomodular posets, orthomodular lattices, orthoalgebras and MV-algebras. Examples of the latter are the unit interval, σ-algebras, Hilbert space effects and effects in a C^*-algebra. For the discussion of some logical aspects of effect algebras, see e.g. Foulis and Pulmannová [26], Rad, Sharafi and Smets [48] and Chajda, Halaš and Länger [14]. For connections to categorical logic, see Jacobs [36]. The probabilistic aspects are developed in e.g. Westerbaan,

Westerbaan and van de Wetering [56] and Staton and Uijlen [55].

A special case of an effect algebra is an *orthoalgebra*, where every non-zero element is *sharp*, that is, not orthogonal to itself (cf. axiom (O4)). As mentioned in the introduction, sharp elements correspond to effects whose probability of occurring is either 0 or 1.

Definition 2.1 (Effect algebra)**.** An *effect algebra* is a partial algebra

$$(E, 0, 1, ', \perp, \oplus)$$

with a set E, constants 0 and 1 in E, a total unary operation $' : E \to E$, a binary relation $\perp \subseteq E \times E$, and a partial binary operation $\oplus : \perp \to E$, such that axioms (E1)-(E4) hold for all $a, b, c \in E$:

(E1) if $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$,

(E2) if $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$ and $a \perp (b \oplus c)$ as well as

$$(a \oplus b) \oplus c = a \oplus (b \oplus c),$$

(E3) $a \perp a'$ and $a \oplus a' = 1$, and if $a \perp b$ such that $a \oplus b = 1$, then $b = a'$,

(E4) if $a \perp 1$, then $a = 0$.

The unary operation $'$ is called *orthosupplementation* and we refer to a' as the *orthosupplement* of a. The domain of definition $\perp$ of $\oplus$ is called the *orthogonality relation* on E, and we say that $a, b \in E$ are *orthogonal* if and only if $a \perp b$. The operation $\oplus$ is referred to as the *orthogonal sum* or simply the *sum*.

Axioms (E1) and (E2) are referred to as (partial) *commutativity* and (partial) *associativity*. We refer to the second part of (E3) as *uniqueness* (of orthosupplements).

We will sometimes simply write $a \oplus b$ and leave the fact that $a \perp b$ implicit.

Definition 2.2. Let E and F be effect algebras. A function $f : E \to F$ is a *morphism of effect algebras* if $f(1) = 1$ and for all $a, b \in E$

$$a \perp b \text{ implies } f(a) \perp f(b) \text{ and } f(a \oplus b) = f(a) \oplus f(b).$$

Note that an effect algebra morphism $f : E \to F$ preserves orthosupplements: since $a \perp a'$, we have $f(a) \perp f(a')$ and

$$1 = f(a \oplus a') = f(a) \oplus f(a'),$$

whence by uniqueness of orthosupplements $f(a') = f(a)'$.

We denote the category of effect algebras by **EAlg**.

Definition 2.3 (Orthoalgebra)**.** We obtain a definition of an *orthoalgebra* from the definition of an effect algebra 2.1 by replacing (E4) with

(O4) if $a \perp a$, then $a = 0$.

It is easy to see that every orthoalgebra is an effect algebra. Indeed, we only need to check the axiom (E4): if $a \perp 1$, then by (E3) $a \perp (a \oplus a')$ and by (E2) $a \perp a$, whence (O4) gives $a = 0$.

A morphism of orthoalgebras is then just a morphism of effect algebras. We write **OAlg** for the resulting category.

In the following proposition, we record some immediate consequences of the axioms for an effect algebra.

Proposition 2.4. *Let E be an effect algebra. The following hold for all $a, b, c \in E$:*

(i) $a'' = a$,

(ii) $0' = 1$,

(iii) $a \perp 0$, *and* $a \oplus 0 = a$,

(iv) $a \oplus b = 0$ *implies* $a = b = 0$,

(v) $a \oplus b = a \oplus c$ *implies* $b = c$.

Statement (iv) is referred to as *positivity*, and statement (v) as the *cancellation property.*

Proof. (i) By (E3) and (E1), $a' \oplus a = 1$, so that by the uniqueness part of (E3) $a = a''$.

(ii) By (E3), $1 \perp 1'$, so that by (E4) $1' = 0$, whence $0' = 1$ by (i).

(iii) By (E3) and (ii), $a \oplus a' = 1 = 0'$, so that $0 \perp (a \oplus a')$, whence by (E2) $a \perp 0$ and

$$(a \oplus 0) \oplus a' = 0 \oplus (a \oplus a') = 0 + 0' = 1.$$

Uniqueness part of (E3) then gives $a \oplus 0 = a'' = a$.

(iv) Observe that the uniqueness part of (E3) yields that

$$(a \oplus b)' \oplus a = b',$$

so that $a \oplus b = 0$ implies (using (ii)) that $a \perp 1$, whence $a = 0$. By (iii) this gives $b' = 1$, so that $b = 0$.

(v) As before, observe that

$$(a \oplus b)' \oplus a = b' \quad \text{and} \quad (a \oplus c)' \oplus a = c',$$

so that $a \oplus b = a \oplus c$ gives $b' = c'$, whence $b = c$.

□

Given an effect algebra E, define a relation $\leq$ on E by $a \leq b$ iff there is a $c \in E$ such that $a \perp c$ and $a \oplus c = b$.

Proposition 2.5. *For every effect algebra, $(E, \leq)$ is a bounded poset with the least and greatest elements 0 and 1.*

Proof. Reflexivity follows from the fact that $a \oplus 0 = a$ ((iii) of Proposition 2.4).

For transitivity, suppose $a \leq b$ and $b \leq d$, so that there are c and e with $a \oplus c = b$ and $b \oplus e = d$. By associativity, this gives

$$d = (a \oplus c) \oplus e = a \oplus (c \oplus e),$$

showing $a \leq d$.

For antisymmetry, suppose $a \leq b$ and $b \leq a$ witnessed by $a \oplus c = b$ and $b \oplus d = a$. Using associativity and (iii) of Proposition 2.4, this gives

$$a \oplus 0 = a = (a \oplus c) \oplus d = a \oplus (c \oplus d),$$

so that by the cancellation property ((v) of Proposition 2.4) we get $c \oplus d = 0$. Now (iv) of Proposition 2.4 gives that $c = d = 0$, so that $a = b$.

Since $0 \oplus a = a$ for all a, we have $0 \leq a$ for all a, showing that 0 is the bottom element. Since $a \oplus a' = 1$ for all a, we have $a \leq 1$ for all a, showing that 1 is the top element. □

Thus we may use all the standard terminology about partial orders with relation to effect algebras. In particular, observe that an effect algebra map is order-preserving, as it preserves orthogonality. This partial order interacts with the effect algebra operations as follows.

Proposition 2.6. *The following properties hold for any elements a and b of an effect algebra*

(i) $a \leq b$ *iff* $b' \leq a'$,

(ii) $a \perp b$ *iff* $a \leq b'$.

Proof. (i) If $a \leq b$, let c be such that $a \oplus c = b$. Then $b' = (a \oplus c)'$, so that $a' = c \oplus (a \oplus c)' = c \oplus b'$, showing $b' \leq a'$. The converse follows by the fact that $'$ is involutive ((i) or Proposition 2.4).

(ii) If $a \perp b$, then $b' = a \oplus (a \oplus b)'$, so that $a \leq b'$. If $a \leq b'$, let c be such that $a \oplus c = b'$. Since $b' \perp b$, axiom (E2) yields $a \perp b$.

□

Example 2.7. Every Boolean algebra can be seen as an orthoalgebra (and hence an effect algebra). Indeed, if B is a Boolean algebra, define $a \perp b$ iff $a \wedge b = 0$, in which case $a \oplus b := a \vee b$. The orthosupplement is just the Boolean complement, the constants 0 and 1 are the same as in B.

The axioms (E1)-(E3) and (O4) then follow easily from the properties of the Boolean operations. The partial order of an effect algebra obtained from a Boolean algebra coincides with the usual order. Namely, $a \wedge b = a$ iff $a \vee b = b$ iff these is a $c \in B$ with $a \wedge c = 0$ and $a \vee c = b$ (for the 'only if' direction, take $c = b \wedge a'$). Thus there is no ambiguity to write $a \leq b$ for elements of a Boolean algebra.

In fact, the category of Boolean algebras **BAlg** is a full subcategory of **EAlg** (and hence of **OAlg**) when Boolean algebras are viewed as effect algebras in the above way. Indeed, if $f : B \to C$ is a morphism of Boolean algebras, then it is a morphism of effect algebras since it preserves the Boolean operations, thus orthogonality and the effect algebra operations. Conversely, if $f : B \to C$ is a morphism of effect algebras, then $f(1) = 1$ and $f(a') = f(a)'$. Since f preserves the order, for all $a, b \in B$ we have

$$f(a) \vee f(b) \leq f(a \vee b).$$

We wish to show the reverse inequality. Observe that

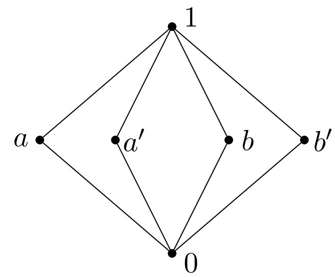
$$a \vee b = a \vee (b \wedge a') \quad \text{and} \quad a \perp (b \wedge a'),$$

so that $f(a) \perp f(b \wedge a')$ and $f(a \vee b) = f(a) \vee f(b \wedge a')$. Using once more the fact that f is order-preserving, we have $f(b \wedge a') \leq f(b)$, whence we obtain

$$f(a \vee b) = f(a) \vee f(b \wedge a') \leq f(a) \vee f(b),$$

showing that f preserves $\vee$. By a de Morgan law it follows that f preserves $\wedge$, and is thus a morphism of Boolean algebras.

Example 2.8. As an example of an orthoalgebra which is not a Boolean algebra, consider the 'glueing' of two four-element Boolean algebras $A = \{0, 1, a, a'\}$ and $B = \{0, 1, b, b'\}$ such that the constants 0 and 1 coincide and $x \perp y$ iff $x, y \in X$ for either $X = A$ or $X = B$ such that and $x \perp y$ in X, in which case the sum is defined as the sum in X. This is in fact the coproduct of A and B in **EAlg**, which in the effect algebra literature is known as the *horizontal sum*. The Hasse diagram of the resulting orthoalgebra is depicted below.



Examples of effect algebras that are not orthoalgebras are discussed in the next section.

2.2 MV-algebras

MV-algebras were introduced by Chang [16] as algebraic models of Łukasievicz many-valued logics (MV stands for 'many-valued'). Subsequently, Chang [15] showed that the $\aleph_0$-valued Łukasievicz logic is complete with respect to the class of all MV-algebras. In fact Chang's result is stronger than this: a formula is provable iff it is valid in all MV-algebras iff it is valid in linearly ordered MV-algebras (cf. Proposition 2.15) iff it is valid in the unit interval (see Example 2.18).

We refer the reader to Cignoli, d'Ottaviano and Mundici [17] for more details and for a systematic introduction to MV-algebras. See also Mundici's tutorial [47] containing a nice game-theoretic motivation.

Definition 2.9. An *MV-algebra* is an algebra $(M, 0, 1,', +)$ with constants 0 and 1, a unary operation $' : M \to M$ and a binary operation $+ : M \times M \to M$ such that the following axioms hold:

(MV1) $a + (b + c) = (a + b) + c$, *(associativity)*

(MV2) $a + b = b + a$, *(commutativity)*

(MV3) $a + 0 = a$,

(MV4) $a + 1 = 1$,

(MV5) $a'' = a$,

(MV6) $0' = 1$,

(MV7) $a + a' = 1$,

(MV8) $(a' + b)' + b = (a + b')' + a$.

We will refer to the operation $+$ as the sum.

If a is an element of an MV-algebra and $n \in \mathbb{N}$, we use the abbreviation $n \cdot a$ or na for the n-fold sum of a with itself (with the convention that $0 \cdot n = 0$).

Since MV-algebras are models of an algebraic theory (e.g. in the sense of Definition C.10), the MV-algebra morphisms are just the usual maps of algebras, namely the signature-preserving functions. We denote the category of MV-algebras by **MVAlg**.

Given an MV-algebra M, we define two more binary operations

$$\begin{aligned} \wedge, \vee : M \times M &\to M \\ a \wedge b &:= \left(\left(a + b'\right)' + b' \right)' \\ a \vee b &:= \left(a' + b\right)' + b. \end{aligned}$$

Note that commutativity of $+$ together with (MV8) yield that both $\wedge$ and $\vee$ are commutative. We define relations $\leq$ and $\perp$ on M by $a \leq b$ iff $a = a \wedge b$ and $a \perp b$ iff $a \leq b'$. Define an operation $\oplus : \perp \to M$ simply by $a \oplus b := a + b$. We then have the following.

Lemma 2.10. *In any MV-algebra, $a \perp b$ iff $a' + b' = 1$. Equivalently, $a \leq b$ iff $a' + b = 1$.*

Proof. We prove the first statement. If $a \perp b$, then $a = a \wedge b' = ((a + b)' + b)'$, so that

$$a' + b' = (a + b)' + b + b' = (a + b)' + 1 = 1.$$

If $a' + b' = 1$, then

$$a \wedge b' = \left((a + b)' + b\right)' = \left(\left(a' + b'\right)' + a' \right)' = \left(0 + a'\right)' = a,$$

so that $a \perp b$. □

In particular, Lemma 2.10 implies that $\perp$ is a symmetric relation. The following properties of how the sum interacts with $\leq$ are now immediate.

Proposition 2.11. *For any elements a, b and c of an MV-algebra, we have*

(i) $a \leq a + b$,

(ii) if $a \leq b$, *then* $a + c \leq b + c$.

Proof. (i) We compute: $(a+b) \wedge a = \big((a+b+a')' + a'\big)' = (0+a')' = a$.

(ii) Suppose $a \leq b$, so that $a' + b = 1$. Then

$$(a+c)' + (b+c) = \big(a' + c'\big)' + a' + b = \big(a' + c'\big)' + 1 = 1,$$

so that $a + c \leq b + c$.

□

Lemma 2.12. *Let a be an element in an MV-algebra. Then $b = a'$ if and only if*

- $a + b = 1$ *and*
- $a' + b' = 1$.

Proof. The 'only if' direction is immediate. If the two equalities hold, then (MV8) gives

$$b = (a+b)' + b = \big(a' + b'\big)' + a' = a'.$$

□

Proposition 2.13. *For any MV-algebra M, the tuple $(M, 0, 1, ', \perp, \oplus)$ is an effect algebra. Moreover, the partial order of this effect algebra is precisely the relation $\leq$.*

Proof. We first show that the axioms (E1)-(E4) are satisfied.

(E1) If $a \perp b$, then $b \perp a$ by symmetry of $\perp$ and $a \oplus b = b \oplus a$ by commutativity of $+$.

(E2) Suppose $a \perp b$ and $(a+b) \perp c$, so that by Lemma 2.10 $a' + b' = 1$ and $(a+b)' + c' = 1$. Then

$$b' + c' = \big(a' + b'\big)' + b' + c' = (a+b)' + a + c' = 1 + a = 1,$$

showing $b \perp c$, and

$$\begin{aligned} a' + (b+c)' &= \big(a'+b'\big)' + a' + (b+c)' \\ &= (a+b)' + b + (b+c)' \\ &= (a+b)' + c' + \big(b'+c'\big)' \\ &= 1 + \big(b'+c'\big)' \\ &= 1, \end{aligned}$$

showing $a \perp (b \oplus c)$. Then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ is just associativity of $+$.

(E3) Since $1 = a + a' = a'' + a'$, we have $a \perp a'$ by Lemma 2.10. Uniqueness is simply Lemma 2.12.

(E4) If $a \perp 1$, then by Lemma 2.10, $a' + 0 = a' = 1$, whence $a = 0$.

Now let $a, b \in M$. Suppose there is a $c \in M$ such that $a + c = b$ (and $a \perp c$). Then

$$b \wedge a = \Big(\big(a + c + a'\big)' + a'\Big)' = a,$$

showing $a \leq b$. Conversely, if $a \leq b$ so that by Lemma 2.10 $a' + b = 1$, then let $c := (b' + a)'$. We have

$$a' + c' = a' + b' + a = 1 \quad \text{and} \quad a + c = a + \big(b' + a\big)' = b + \big(b + a'\big)' = b,$$

and the former is equivalent to $a \perp c$ once more by Lemma 2.10. Thus $\leq$ is precisely the partial order of the effect algebra. □

We therefore call elements a and b of an MV-algebra *orthogonal* when they are orthogonal in the corresponding effect algebra, that is, if $a \perp b$ iff $a \leq b'$ iff $a' + b' = 1$.

The following Corollary is now immediate from (iv) and (v) of Proposition 2.4.

Corollary 2.14. *The following hold for all elements a, b, c of an MV-algebra:*

(i) if $a \leq b'$ and $a + b = 0$, then $a = b = 0$,

(ii) if $b, c \leq a'$ and $a + b = a + c$, then $b = c$.

Proposition 2.15. *For any MV-algebra M, the tuple $(M, 0, 1, \leq, \wedge, \vee)$ is a bounded lattice.*

Proof. The fact that $(M, 0, 1, \leq)$ is a bounded poset follows from Propositions 2.13 and 2.5.

Now by Lemma 2.10,

$$c \leq a \wedge b \quad \text{iff} \quad c' + \Big(\big(a + b'\big)' + b'\Big)' = 1,$$

and similarly $c \leq a, b$ iff $c' + a = c' + b = 1$. Thus suppose $c \leq a \wedge b$. Then

$$\begin{aligned} c' + a &= a + c' + \bigg(c' + \Big(\big(a + b'\big)' + b'\Big)'\bigg)' \\ &= a + \big(a + b'\big)' + b' + \Big(c + \big(a + b'\big)' + b'\Big)' \\ &= b + \big(a' + b\big)' + b' + \Big(c + \big(a + b'\big)' + b'\Big)' \\ &= 1, \end{aligned}$$

and similarly $c' + b = 1$, showing $c \leq a, b$. Conversely, suppose $c \leq a, b$. Then

$$\begin{aligned} c' + \Big(\big(a + b'\big)' + b'\Big)' &= c' + \big(c' + b\big)' + \Big(\big(a + b'\big)' + b'\Big)' \\ &= b' + \big(c + b'\big)' + \Big(\big(a + b'\big)' + b'\Big)' \\ &= \big(c + b'\big)' + a + b' + \big(a + b' + b\big)' \\ &= \big(c' + b\big)' + a + c' \\ &= 1, \end{aligned}$$

showing $c \leq a \wedge b$. Thus we have shown that $c \leq a, b$ iff $c \leq a \wedge b$, so that $\wedge$ satisfies the universal property of the meet.

For $\vee$, observe that we have $a \vee b = (a' \wedge b')'$, so that $a \vee b \leq c$ iff $c' \leq a' \wedge b'$ iff $c' \leq a', b'$ iff $a, b \leq c$. □

Remark 2.16. In fact, any lattice obtained from an MV-algebra in the above way is distributive. For a proof, see e.g. Dvurečenskij and Pulmannová [23, Proposition 2.2.4].

Thus any MV-algebra can be seen as an effect algebra whose partial order is a distributive lattice. Since the meet and join are defined in terms of the MV-algebra operations, it follows that any MV-algebra morphism is also a lattice morphism. Since orthogonality is defined in terms of the order and $\oplus$ is the restriction of $+$, every MV-algebra morphism is also an effect algebra morphism. This exhibits **MVAlg** as a subcategory of **EAlg**.

Example 2.17. Every Boolean algebra is an MV-algebra by taking $'$ to be the Boolean complement and $+$ to be the join. It is then immediate that the axioms (MV1)-(MV7) are satisfied, while (MV8) becomes

$$(a \wedge b') \vee b = (a' \wedge b) \vee a,$$

so that by distributivity (and by the fact that $'$ is the complement) both sides reduce to $a \vee b$. Since preservation of joins, complements and 1 is sufficient for a map to be a Boolean algebra morphism, the category **BAlg** is a full subcategory of **MVAlg**.

Example 2.18. An important example of an MV-algebra is the real unit interval $[0, 1]$ with 'truncated addition' $a \oplus b := \min(a+b, 1)$ (we write $\oplus$ for the MV-algebra operation to distinguish it from addition of real numbers) and $a' := 1 - a$. Again, it is easy to see that the axioms (MV1)-(MV7) hold and (MV8) follows by case analysis.

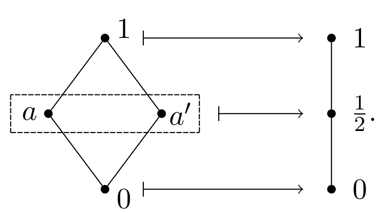
The unit interval gives an example of an effect algebra which is not an orthoalgebra: we have $a \perp b$ iff $a \leq 1 - b$ iff $a + b \leq 1$, so that for example $\frac{1}{3} \perp \frac{1}{3}$ while $\frac{1}{3} \neq 0$.

Example 2.19. Another important class of examples of MV-algebras are the subalgebras of $[0, 1]$ generated by $\frac{1}{n-1}$ for each natural number $n \geq 2$. For each natural number $n \geq 2$, we denote such a subalgebra by L_n. Note that L_n has n elements, and its underlying set is given by

$$L_n = \left\{ \frac{m}{n-1} : m \in \mathbb{N} \text{ and } m \leq n-1 \right\}.$$

Thus L_n is just a chain of n elements. It turns out that these are precisely the finite subalgebras of $[0, 1]$, see Section 3.5 of Cignoli, d'Ottaviano and Mundici [17]. Moreover, we have $L_n \subseteq L_m$ if and only if $n-1$ divides $m-1$.

Remark 2.20. Note that while **BAlg** is a full subcategory of both **MVAlg** and **EAlg**, the MV-algebras **MVAlg** are *not* a full subcategory of **EAlg**. Consider, for instance, a map f from the four-element Boolean algebra $\{0, 1, a, a'\}$ to MV-algebra L_3 defined below:



This is a morphism of effect algebras as $\frac{1}{2} \perp \frac{1}{2}$ and $\frac{1}{2} \oplus \frac{1}{2} = 1$, but not a morphism of MV-algebras, as we have $a = a \vee a$ but

$$f(a) = \frac{1}{2} \neq f(a) \oplus f(a) = 1.$$

2.3 Finite multisets and setoids

What matters about a set is how many elements it has. More formally, the only structural (i.e. isomorphism invariant) property of a set is its cardinality. The idea of a multiset comes from adding one more invariant to this picture: what matters about a multiset is how many elements it has *and* the type of each element. Thus a good picture of a multiset is simply a set of sets; we discuss this by showing that multisets may equivalently be viewed as setoids (Proposition 2.25), which in turn correspond to partitions of sets.

While we are mostly interested in the order-theoretic properties of multisets and setoids, they most frequently appear in combinatorics and computer science. For multisets in combinatorics and computer science, see e.g. Hickman [35], Chapter 5 in Knuth [39], Section 1.2 in Stanley [54] and Section I.2 in Flajolet and Sedgewick [25]. For setoids in type theory, see e.g. Barthe, Capretta and Pons [3] and Kinoshita and Power [38].

Our definition of the category of multisets largely follows the finite case of Cignoli, Dubuc and Mundici [19].

Definition 2.21 (Finite multisets)**.** The category of *finite multisets* **FinMul** has as its objects pairs (A, η), where A is a finite set and

$$\eta : A \to \mathbb{N} \setminus \{0\}$$

is a function.

A morphism from (A, η) to (B, μ) is given by a pair of functions (f, ϕ)

$$f : A \to B$$
$$\phi : A \to \mathbb{N} \setminus \{0\}$$

such that $(\mu f) \cdot \phi = \eta$, where multiplication is defined pointwise.

The identity morphism on (A, η) is just $(\mathrm{id}_A, 1)$, where $1 : A \to \mathbb{N} \setminus \{0\}$ is the constant function with value 1. Given morphisms

$$(A, \eta) \xrightarrow{(f,\phi)} (B, \mu) \xrightarrow{(g,\psi)} (C, \nu),$$

their composition is defined by

$$(g, \psi) \circ (f, \phi) \coloneqq (gf, (\psi f) \cdot \phi).$$

Given a multiset (A, η), we refer to A as the *types* of the multiset. For each type $a \in A$, we refer to $\eta(a)$ as the *multiplicity* of the type a, and call η the *multiplicity function.* The *cardinality* of (A, η) is given by $\sum_{a \in A} \eta(a)$. We will often represent a multiset by

$$\left\{a_1^{n_1}, \dots, a_k^{n_k}\right\},$$

where each a_i is a type and n_i the multiplicity of that type. Sometimes, when the cardinality is small, we will simply repeat the types to represent multiplicity, so that $\{a, a, b, b, b\}$ is the same multiset as $\left\{a^2, b^3\right\}$. We will also use the isomorphism invariant representation: a finite list of positive integers:

$$(n_1, \dots, n_k),$$

whose length k corresponds to the number of types and each n_i is the multiplicity of ith type. We adopt the convention that the list is non-decreasing. With this convention, such lists are in bijection with isomorphism classes of finite multisets. Thus the above multiset $\{a, a, b, b, b\}$ is given (up to isomorphism) by the list $(2, 3)$.

With this terminology and notation, a morphism of finite multisets is a function on types such that the multiplicity of each type in the domain is divisible by the multiplicity of its image. For example, there is a morphism of multisets

$$\left\{a^3, b^5, c^6\right\} \to \left\{d, e, f^3, g^7\right\}$$

sending $a, c \mapsto f$ and $b \mapsto d$, but no morphisms from $\left\{a^3\right\}$ to $\left\{a^2\right\}$.

Remark 2.22. There is no good (or intrinsic) reason to define morphisms of multisets the way we have defined them. In fact any multiset of cardinality n corresponds uniquely (up to isomorphism) to an equivalence relation on a finite set with n elements, and this captures the essence of the multiset: the only thing that matters is the number of types and the multiplicity of each type. Thus it would be perhaps more natural to define multisets as setoids (see Definition 2.24 and Proposition 2.25). The reason for choosing the more restricted version of morphisms is that the category **FinMul** is dually equivalent to the category of finite MV-algebras (see Theorem 3.8).

Definition 2.23 (Submultisets)**.** Let (A, η) be a finite multiset. A *submultiset* is a function $\sigma : A \to \mathbb{N}$ such that $\sigma \leq \eta$ in the pointwise order.

If it is clear that we are talking about multisets, we will simply say subset rather than submultiset. Note that, unlike in the definition of a multiset, we allow subsets to contain types with zero multiplicity. This is mostly a choice we made out of convenience, so that all subsets have the same domain. Equivalently, we could have defined a submultiset as a multiset whose set part is a subset of A and whose multiplicity function is below η. Given a subset σ of (A, η), we write $\operatorname{supp}(\sigma)$ for its *support*, that is,

$$\operatorname{supp}(\sigma) \coloneqq \{a \in A : \sigma(a) \neq 0\}.$$

We will denote the set of all subsets of (A, η) by $\mathcal{P}(A, \eta)$ and call it the *powerset* of (A, η). The powerset is partially ordered by the pointwise order of the subsets. Observe that if (A, η) is a set (that is, η is constantly 1), then $\mathcal{P}(A, \eta)$ is (isomorphic to) the poset of subsets of A in the usual sense. We can define complements, unions and intersection of submultisets in much the same way as for sets. While for a set this results in a Boolean algebra, for a multiset we obtain an MV-algebra. This is done in detail in Lemmas 3.6 and 3.7.

We now move on to discuss setoids and how multisets can be viewed as setoids.

Definition 2.24 (Finite setoids)**.** The category of *finite setoids* **FinSetoid** has as its objects finite sets equipped with an equivalence relation $(S, \sim)$, and as its morphisms functions that preserve the equivalence relation.

There is a functor $X : \mathbf{FinMul} \to \mathbf{FinSetoid}$ given by

$$(A, \mu) \mapsto (X_A, \sim),$$

on objects, where

$$X_A \coloneqq \coprod_{a \in A} \coprod_{i=1}^{\eta(a)} \{(i, a)\}$$

and the equivalence relation is defined by $(i, a) \sim (j, b)$ iff $a = b$. There is no canonical choice on how to define X on morphisms; we choose to map $(A, \eta) \xrightarrow{(f,\phi)} (B, \mu)$ to

$$\begin{aligned} Xf : (X_A, \sim) &\to (X_B, \sim) \\ (i, a) &\mapsto (i \mod \mu(f(a)), f(a)). \end{aligned}$$

Proposition 2.25. *The functor $X : \mathbf{FinMul} \to \mathbf{FinSetoid}$ is faithful and essentially isomorphic on objects.*

Proof. For faithfulness, suppose

$$(f, \phi), (g, \psi) : (A, \eta) \to (B, \mu)$$

are morphisms of multisets such that $Xf = Xg$. It follows that for each $a \in A$, we have $f(a) = g(a)$, so that $f = g$. Hence $\mu f \cdot \phi = \eta = \mu f \cdot \psi$, and since $\mu > 0$ we get $\phi = \psi$.

Now define a function $Y : \mathcal{O}b(\mathbf{FinSetoid}) \to \mathcal{O}b(\mathbf{FinMul})$ by

$$(S, \sim) \mapsto (S/\sim , \eta) \, ,$$

where $\eta([a]) \coloneqq |[a]|$ for each equivalence class in $S/\sim$. It is straightforward to see that $(A, \eta) \simeq YX(A, \eta)$ and $(S, \sim) \simeq XY(S, \sim)$ for all finite multisets (A, η) and finite setoids $(S, \sim)$. □

Chapter 3

Finite multisets and finite MV-algebras

The aim of this chapter is to establish a connection between finite multisets and finite MV-algebras. Namely, we show that every powerset of a finite multiset is an MV-algebra (Lemma 3.6), and that the powerset construction extends to a contravariant functor. We then show that this powerset functor is an equivalence, thus establishing a dual equivalence of the categories of finite multisets and finite MV-algebras (Theorem 3.8). This extends the well-known fact that the usual powerset functor gives a dual equivalence of categories of finite sets and finite Boolean algebras.

The duality of this chapter is a special case of the duality between locally finite MV-algebras and certain completion of multisets proved by Cignoli, Dubuc and Mundici [19]. This extends the usual Stone duality. For the detailed discussion of the finite case of the above duality, see Section 3 of [18]. We remark that the powerset functor that we construct in this chapter is not exactly the restriction of the functor exhibiting the duality in [18], albeit it is naturally isomorphic to it.

3.1 Atoms in MV-algebras

This section collects results about atoms and sums of atoms in an MV-algebra that will be needed to prove the duality of the next section.

Given an MV-algebra M, we denote its set of atoms by $\mathrm{At}(M)$. First, observe that a is an atom if and only if a' is a coatom, as we have $b \leq a$ iff $a' \leq b'$.

Lemma 3.1. *If a and b are distinct atoms of an MV-algebra, then $a' + b = a'$.*

Proof. Since both a' and b' are coatoms, by (i) of Proposition 2.11 we have that either $a'+b = a'$ or $a'+b = 1$, and either $b'+a = b'$ or $b'+a = 1$. Observe that we cannot have both $a' + b = 1$ and $b' + a = 1$, as then Lemma 2.12 would imply $a = b$. Thus if $a' + b = 1$, then we must have $b' + a = b'$, in which case we compute:

$$b = b + \big(a' + b\big)' = a + \big(b' + a\big)' = a + b,$$

which is a contradiction, since a and b are distinct atoms. Thus we must indeed have $a' + b = a'$ (and $b' + a = b'$). □

Proposition 3.2. *If a and b are distinct atoms of an MV-algebra, then for all $n, m \in \mathbb{N}$ we have*

$$(n \cdot a)' + m \cdot b = (n \cdot a)'.$$

Proof. If $m = 0$, then this is a triviality, and if $n = 0$, then this reduces to (MV4). Hence we may assume that $m, n \geq 1$. We now claim that

$$b + (n \cdot a)' = (n \cdot a)'. \tag{3.3}$$

Since the $(n \cdot a)'$ is always below $b+(n \cdot a)'$, it suffices to show the left-to-right inequality. For clarity, let us write $\alpha := n \cdot a$. Observe that by Lemma 3.1, we have that $b' + \alpha = b'$, and we wish to show

$$b + \alpha' \leq \alpha'.$$

We thus compute:

$$\begin{aligned}
\alpha' \wedge \big(b + \alpha'\big) &= \Big(\big(\alpha' + (b + \alpha')'\big)' + \big(b + \alpha'\big)'\Big)' \\
&= \Big(\big(b' + (b' + \alpha)'\big)' + \big(b + \alpha'\big)'\Big)' \\
&= \Big(\big(b' + b\big)' + \big(b + \alpha'\big)'\Big)' \\
&= \Big(\big(b + \alpha'\big)'\Big)' \\
&= b + \alpha',
\end{aligned}$$

whence the desired inequality (and hence equality) follows. To conclude, we just observe that

$$(n \cdot a)' + m \cdot b = (n \cdot a)'$$

follows by repeatedly applying equation (3.3). □

We say that a finite multiset $\mathcal{A} = (A, \eta)$ is a *multiset of atoms* of an MV-algebra if the set of types A is a subset of atoms. Given a multiset of atoms $\mathcal{A} = (A, \eta)$, define its *sum* as $\sum \mathcal{A} \coloneqq \sum_{a \in A} \eta(a) \cdot a$. This is well-defined by associativity and commutativity of the MV-algebra sum.

Corollary 3.4. *Let a be an atom of an MV-algebra, and let $\mathcal{A}$ be a multiset of atoms such that a is not a type of $\mathcal{A}$. Then $n \cdot a$ and $\sum \mathcal{A}$ are orthogonal for any $n \in \mathbb{N}$.*

Proof. The terms in the sum $\sum \mathcal{A}$ are of the form $m \cdot b$ for some $m \in \mathbb{N}$ and an atom b not equal to a, so that Proposition 3.2 yields

$$(n \cdot a)' + \sum \mathcal{A} = (n \cdot a)',$$

whence $\sum \mathcal{A} \leq (n \cdot a)'$. □

Corollary 3.5. *Let $(a_1, \ldots, a_k)$ be a finite sequence of distinct atoms in an MV-algebra, and let $(n_1, \ldots, n_k)$ be a sequence of integers. Then*

$$\sum_{i=1}^{k} n_i \cdot a_i = \bigvee_{i=1}^{k} n_i \cdot a_i.$$

Proof. We argue by induction on k. If $k = 1$, there is nothing to show. Now suppose the statement is true for some k, and consider sequences as above of length $k + 1$. Then, writing c for $\bigvee_{i=1}^{k} n_i \cdot a_i = \sum_{i=1}^{k} n_i \cdot a_i$ we get

$$\begin{aligned}
\bigvee_{i=1}^{k+1} n_i \cdot a_i &= (n_{k+1} \cdot a_{k+1}) \vee c \\
&= \big((n_{k+1} \cdot a_{k+1})' + c\big)' + c \\
&= n_{k+1} \cdot a_{k+1} + c \\
&= \sum_{i=1}^{k+1} n_i \cdot a_i,
\end{aligned}$$

where the third equality uses Corollary3.4 as well as the induction hypothesis. We conclude that the statement holds for all k. □

3.2 Duality of FinMV and FinMul

Recall from Section 2.3 that we denote the powerset of a multiset (A, η) by $\mathcal{P}(A, \eta)$. We begin the task of showing the titular duality by noticing that each powerset is indeed an MV-algebra.

Lemma 3.6. *For every multiset (A, η), the powerset $\mathcal{P}(A, \eta)$ has the structure of an MV-algebra. Namely, we define the constant* 0 *as* $0 : A \to \mathbb{N}$ *(the constant function with value* 0*), the constant* 1 *as* η*, and for subsets* σ *and* τ *we define* σ' *and* $\sigma + \tau$ *by*

$$\sigma'(a) \coloneqq \eta(a) - \sigma(a)$$
$$(\sigma + \tau)(a) \coloneqq \sigma(a) \oplus \tau(a) \coloneqq \min\left(\sigma(a) + \tau(a), \eta(a)\right)$$

for each $a \in A$.

Note that we denote by $\oplus$ the truncated addition of natural numbers, and by $+$ both addition of submultisets and of natural numbers.

Proof. In this proof only, we use square brackets in addition to the usual parentheses for legibility.

The axioms (MV2)-(MV7) are immediate from the definitions. For (MV1), let σ, τ and ν be subsets and let $a \in A$. We need to show that

$$\sigma(a) \oplus [\tau(a) \oplus \nu(a)] = [\sigma(a) \oplus \tau(a)] \oplus \nu(a).$$

First, if $\tau(a) + \nu(a) \geq \eta(a)$, then the left-hand side is equal to $\eta(a)$, and

$$[\sigma(a) \oplus \tau(a)] \oplus \nu(a) \geq \tau(a) \oplus \nu(a) = \eta(a),$$

so that right-hand side is also equal to $\eta(a)$. Thus suppose $\tau(a)+\nu(a) \leq \eta(a)$, so that $\tau(a) \oplus \nu(a) = \tau(a) + \nu(a)$. If $\sigma(a) + \tau(a) + \nu(a) \geq \eta(a)$, then the left-hand side is again equal to $\eta(a)$, and so is the right-hand side, as either already $\sigma(a) + \tau(a) \geq \eta(a)$ or the right-hand side reduces to the same case as the left-hand side. If $\sigma(a) + \tau(a) + \nu(a) \leq \eta(a)$, then in particular $\tau(a) + \nu(a) \leq \eta(a)$ and $\sigma(a) + \tau(a) \leq \eta(a)$ and the statement reduces to associativity of addition of positive integers.

For (MV8), let σ and τ be subsets and let $a \in A$. Without loss of generality, suppose that $\sigma(a) \leq \tau(a)$. We need to show that

$$[\eta(a) - [\eta(a) - \sigma(a)] \oplus \tau(a)] \oplus \tau(a) = [\eta(a) - [\eta(a) - \tau(a)] \oplus \sigma(a)] \oplus \sigma(a).$$

Since $\sigma(a) \leq \tau(a)$, we have that

$$\eta(a) - \sigma(a) + \tau(a) \geq \eta(a),$$
$$\eta(a) - \tau(a) + \sigma(a) \leq \eta(a).$$

Thus the left-hand side reduces to

$$[\eta(a) - \eta(a)] \oplus \tau(a) = \tau(a),$$

and the right-hand side to

$$[\tau(a) - \sigma(a)] \oplus \sigma(a) = \tau(a),$$

hence we conclude. $\square$

The assignment of powerset to each multiset extends to a functor

$$\mathcal{P} : \mathbf{FinMul}^{op} \to \mathbf{FinMV}$$

as follows. Let $(f, \phi) : (A, \eta) \to (B, \mu)$ be a morphism of finite multisets. We then define a map (we omit ϕ from the map for clarity)

$$\begin{aligned}\mathcal{P}f : \mathcal{P}(B, \mu) &\to \mathcal{P}(A, \eta)\\ \sigma &\mapsto \sigma f \cdot \phi,\end{aligned}$$

where multiplication is defined pointwise. First observe that the map is well-defined:

$$\sigma f \cdot \phi \leq \mu f \cdot \phi = \eta,$$

where the inequality holds since σ is a subset of (B, μ) and the equality since (f, ϕ) is a morphism in **FinMul**. Thus $\mathcal{P}f(\sigma)$ is indeed a subset of (A, η). We now need to check that $\mathcal{P}f$ is a morphism of MV-algebras. It is clear that $\mathcal{P}f$ sends zero to zero and μ to η. To see that $\mathcal{P}f$ preserves $+$ we need to show that for all $\sigma, \tau \in \mathcal{P}(B, \mu)$ we have

$$(\sigma + \tau)f \cdot \phi = \sigma f \cdot \phi + \tau f \cdot \phi,$$

which means for all $a \in A$

$$(\sigma(f(a)) \oplus \tau(f(a))) \cdot \phi(a) = (\sigma(f(a)) \cdot \phi(a)) \oplus (\tau(f(a)) \cdot \phi(a)).$$

But this is immediate, for if $\sigma(f(a)) + \tau(f(a)) \leq \mu(f(a))$, then

$$\sigma(f(a)) \cdot \phi(a) + \tau(f(a)) \cdot \phi(a) \leq \eta(a),$$

so that the equality holds by distributivity of multiplication. Similarly, if $\sigma(f(a)) + \tau(f(a)) \geq \mu(f(a))$, then the left-hand side is equal to $\eta(a)$, and since $\sigma(f(a)) \cdot \phi(a) + \tau(f(a)) \cdot \phi(a) \geq \eta(a)$ so is the right-hand side. Preservation of $'$ is even simpler; we simply compute

$$\begin{aligned}\mathcal{P}f(\sigma') &= \mathcal{P}f(\mu - \sigma)\\ &= (\mu - \sigma)f \cdot \phi\\ &= \mu f \cdot \phi - \sigma f \cdot \phi\\ &= \eta - \sigma f \cdot \phi\\ &= (\sigma f \cdot \phi)'\\ &= (\mathcal{P}f(\sigma))',\end{aligned}$$

where subtraction is defined pointwise, and it is well-defined at each step since subsets are below the multiplicity functions.

Lemma 3.7. *The joins and meets in the powerset MV-algebra $\mathcal{P}(A, \eta)$ are given by*

$$(\sigma \wedge \tau)(a) = \min(\sigma(a), \tau(a)),$$
$$(\sigma \vee \tau)(a) = \max(\sigma(a), \tau(a)).$$

Proof. First, observe that the MV-algebra order of $\mathcal{P}(A, \eta)$ coincides with the pointwise order. Indeed, by Lemma 2.10, σ is below τ in the MV-algebra order iff $\sigma' + \tau = \eta$ iff $(\eta - \sigma) + \tau = \eta$ iff for all $a \in A$ we have

$$\min(\eta(a) - \sigma(a) + \tau(a), \eta(a)) = \eta(a)$$

iff $\eta(a) - \sigma(a) + \tau(a) \geq \eta(a)$ for all $a \in A$ iff $\sigma(a) \leq \tau(a)$ for all $a \in A$ iff $\sigma \leq \tau$.

It is then easy to see that the pointwise minimum of σ and τ verifies the universal property of the meet in the pointwise order: $\nu \leq \sigma, \tau$ iff $\nu(a) \leq \min(\sigma(a), \tau(a))$ for all $a \in A$. Similarly for the join. □

Theorem 3.8. *The powerset functor $\mathcal{P} : \mathbf{FinMul}^{op} \to \mathbf{FinMV}$ is an equivalence of categories.*

Proof. Let us fix finite multisets (A, η) and (B, μ).

Faithful: Suppose $(f, \phi), (g, \psi) : (A, \eta) \to (B, \mu)$ are morphisms such that $\mathcal{P}f = \mathcal{P}g$. Explicitly, this means that for all $\sigma \in \mathcal{P}(B, \mu)$ we have

$$\sigma f \cdot \phi = \sigma g \cdot \psi.$$

For each $a \in A$, let $\chi_{f(a)} : B \to \mathbb{N}$ denote the characteristic function of $f(a)$, that is, $\chi_{f(a)}(b) = 1$ iff $b = f(a)$ and is zero otherwise. Then $\chi_{f(a)}$ is a subset of (B, μ), so that

$$\chi_{f(a)} f \cdot \phi = \chi_{f(a)} g \cdot \psi.$$

Evaluating both sides at a yields

$$\phi(a) = \chi_{f(a)}(g(a)) \cdot \psi(a).$$

Since $\phi(a) \neq 0$, we must have $\chi_{f(a)}(g(a)) \neq 0$, whence $\chi_{f(a)}(g(a)) = 1$, so that $f(a) = g(a)$ and $\phi(a) = \psi(a)$. Thus we conclude that $(f, \phi) = (g, \psi)$, as required.

Full: Let $F : \mathcal{P}(B,\mu) \to \mathcal{P}(A,\eta)$ be a morphism of MV-algebras. For each $b \in B$, let $\chi_b : B \to \mathbb{N}$ be the characteristic function of b. Observe that we may decompose μ as

$$\sum_{b\in B} \mu(b) \cdot \chi_b = \mu,$$

where the notation $\mu(b) \cdot \chi_b$ stands for the $\mu(b)$-fold sum of χ_b with itself. Since F is an MV-algebra morphism, we get that

$$\sum_{b\in B} \mu(b) \cdot F\chi_b = \eta. \tag{3.9}$$

Hence given $a \in A$, since $\eta(a) \neq 0$, the above equality implies that there is a $b \in B$ such that $F\chi_b(a) \neq 0$. Such b is moreover unique: suppose that we have $c \in B$ such that also $F\chi_c(a) \neq 0$. From the fact that F preserves joins we have

$$F\chi_b \wedge F\chi_c = F(\chi_b \wedge \chi_c),$$

and from Lemma 3.7 we have that the left-hand side is non-zero (that is, not the zero function), as $\min(F\chi_b(a), F\chi_c(a)) \neq 0$. Thus, since F preserves the zero function, $\chi_b \wedge \chi_c \neq 0$, which again by Lemma 3.7 implies that $b = c$.

We therefore define

$$\begin{aligned} f : A &\to B \\ a &\mapsto b \text{ s.t. } F\chi_b(a) \neq 0, \end{aligned}$$

which is well-defined by existence and uniqueness of such b. We moreover define

$$\begin{aligned} \phi : A &\to \mathbb{N} \setminus \{0\} \\ a &\mapsto F\chi_{f(a)}(a). \end{aligned}$$

Now given $a \in A$, we have

$$\mu(f(a)) \cdot \phi(a) = \mu(f(a)) \cdot F\chi_{f(a)}(a) = \eta(a)$$

by equation (3.9) and by choice of $f(a)$. Thus $(f, \phi) : (A, \eta) \to (B, \mu)$ is a morphism of multisets. Observe that on the characteristic functions $\mathcal{P}f$ acts as

$$\mathcal{P}f(\chi_b)(a) = \chi_b(f(a)) \cdot \phi(a) = F\chi_b(a),$$

so that $\mathcal{P}f(\chi_b) = F\chi_b$ for all $b \in B$. To conclude, we compute for $\sigma \in \mathcal{P}(B, \mu)$:

$$\begin{aligned}\mathcal{P}f(\sigma) &= \mathcal{P}f\left(\sum_{b\in B} \sigma(b) \cdot \chi_b\right) \\ &= \sum_{b\in B} \sigma(b) \cdot \mathcal{P}f(\chi_b) \\ &= \sum_{b\in B} \sigma(b) \cdot F\chi_b \\ &= F\left(\sum_{b\in B} \sigma(b) \cdot \chi_b\right) \\ &= F\sigma.\end{aligned}$$

Essentially surjective on objects: Let M be a finite MV-algebra. Observe that, since M is finite, for each atom $a \in \mathrm{At}(M)$, there is the least $n \in \mathbb{N} \setminus \{0\}$ such that $n \cdot a = (n+1) \cdot a$. Thus let $\mu : \mathrm{At}(M) \to \mathbb{N} \setminus \{0\}$ be the map sending each atom to such n. We can thus define a map

$$\begin{aligned}\iota : \mathcal{P}(\mathrm{At}(M), \mu) &\to M \\ \sigma &\mapsto \sum_{a\in \mathrm{At}(M)} \sigma(a) \cdot a.\end{aligned}$$

We claim that ι is an isomorphism of MV-algebras. We first show that it is a bijection. Thus let $m \in M$, and let

$$\mathcal{A}_m \coloneqq \{a \in \mathrm{At}(M) : a \leq m\}$$

the set of atoms that lie below m. Since M is finite, $\mathcal{A}_m$ is non-empty. For each $a \in \mathcal{A}_m$, either there is the greatest n such that $n \cdot a \leq m$, or $\mu(a) \leq m$. In the former case we define $\sigma(a) \coloneqq n$ and in the latter $\sigma(a) \coloneqq \mu(a)$. This defines a subset $\sigma : \mathrm{At}(M) \to \mathbb{N}$ once we let $\sigma(a) \coloneqq 0$ if $a \notin \mathcal{A}_m$. Now

$$\iota(\sigma) = \sum_{a\in\mathcal{A}_m} \sigma(a) \cdot a \leq m,$$

where the inequality follows by Corollary 3.5, noting that m is an upper bound for $\mathcal{A}_m$. Thus there is a $c \in M$ such that

$$\sum_{a\in\mathcal{A}_m} \sigma(a) \cdot a + c = m.$$

But now all the atoms that are below c are also below m, and thus appear in the sum on the left-hand side. Since the coefficients $\sigma(a)$ were so chosen that either $(\sigma(a)+1)\cdot a \nleq m$ or

$$(\sigma(a)+1)\cdot a = (\mu(a)+1)\cdot a = \mu(a)\cdot a = \sigma(a)\cdot a,$$

we conclude that c is absorbed into the sum on the left-hand side, whence

$$m = \sum_{a\in\mathcal{A}_m} \sigma(a)\cdot a = \iota(\sigma).$$

Thus ι is surjective.

For injectivity, suppose $\sigma, \tau \in \mathcal{P}(\mathrm{At}(M), \mu)$ are such that $\iota(\sigma) = \iota(\tau)$. Explicitly, this means

$$\sum_{a\in\mathrm{At}(M)} \sigma(a)\cdot a = \sum_{a\in\mathrm{At}(M)} \tau(a)\cdot a.$$

Let $b \in \mathrm{At}(M)$ be some atom, and rewrite the above equality as

$$\sigma(b)\cdot b + \sum_{a\in\mathrm{At}(M)\setminus\{b\}} \sigma(a)\cdot a = \tau(b)\cdot b + \sum_{a\in\mathrm{At}(M)\setminus\{b\}} \tau(a)\cdot a.$$

Now let us add $\sum_{a\in\mathrm{At}(M)\setminus\{b\}} \mu(a)\cdot a$ on both sides. By the choice of μ, each coefficient $\sigma(a)$ and $\tau(a)$ is absorbed into $\mu(a)$, hence we obtain:

$$\sigma(b)\cdot b + \sum_{a\in\mathrm{At}(M)\setminus\{b\}} \mu(a)\cdot a = \tau(b)\cdot b + \sum_{a\in\mathrm{At}(M)\setminus\{b\}} \mu(a)\cdot a.$$

By Corollary 3.4, both $\sigma(b)\cdot b$ and $\tau(b)\cdot b$ are orthogonal to the sum, whence by the cancellation property ((ii) of Corollary 2.14) we get

$$\sigma(b)\cdot b = \tau(b)\cdot b.$$

Since $\sigma, \tau \leq \mu$, we conclude that $\sigma(b) = \tau(b)$. This yields that $\sigma = \tau$, showing injectivity.

It remains to show that ι is an MV-algebra morphism. It is clear that $\iota(0) = 0$. Next, we have

$$\iota(\mu) = \sum_{a\in\mathrm{At}(M)} \mu(a)\cdot a \geq m$$

for any $m \in M$ by surjectivity of ι, so we conclude that $\iota(\mu) = 1$. Now let $\sigma, \tau \in \mathcal{P}(\mathrm{At}(M), \mu)$. We compute

$$\begin{aligned}\iota(\sigma + \tau) &= \sum_{a \in \mathrm{At}(M)} (\sigma(a) \oplus \tau(a)) \cdot a \\ &= \sum_{a \in \mathrm{At}(M)} (\sigma(a) + \tau(a)) \cdot a \\ &= \sum_{a \in \mathrm{At}(M)} \sigma(a) \cdot a + \sum_{a \in \mathrm{At}(M)} \tau(a) \cdot a \\ &= \iota(\sigma) + \iota(\tau),\end{aligned}$$

where the second equality follows by considering two cases for each $a \in \mathrm{At}(M)$: 1) $\sigma(a) + \tau(a) \leq \mu(a)$ when $\sigma(a) \oplus \tau(a) = \sigma(a) + \tau(a)$, and 2) $\sigma(a)+\tau(a) \geq \mu(a)$, when $\sigma(a) \oplus \tau(a) = \mu(a)$ but also $(\sigma(a)+\tau(a)) \cdot a = \mu(a) \cdot a$ by the choice of $\mu(a)$. Thus ι preserves $+$.

Finally, let $\sigma \in \mathcal{P}(\mathrm{At}(M), \mu)$, and let $\tau \in \mathcal{P}(\mathrm{At}(M), \mu)$ be the unique subset such that $\iota(\tau) = \iota(\sigma)'$ (using the fact that ι is a bijection), so that we have $\iota(\tau + \sigma) = \iota(\mu)$. Thus, by injectivity, we have $\tau + \sigma = \mu$, so that for each $a \in \mathrm{At}(M)$ we get $\tau(a) + \sigma(a) \geq \mu(a)$. Next,

$$\tau(a) \cdot a \leq \sum_{b \in \mathrm{At}(M)} \tau(b) \cdot b = \iota(\tau) = \iota(\sigma)' = \left(\sum_{b \in \mathrm{At}(M)} \sigma(b) \cdot b \right)' \leq (\sigma(a) \cdot a)',$$

for each $a \in \mathrm{At}(M)$, showing orthogonality of $\tau(a) \cdot a$ and $\sigma(a) \cdot a$. On the other hand, we have

$$\sigma(a) \cdot a + (\sigma(a) \cdot a)' = \sum_{b \in \mathrm{At}(M)} \mu(b) \cdot b,$$

so that $(\mu(a) - \sigma(a)) \cdot a \leq (\sigma(a) \cdot a)'$, showing orthogonality of $\sigma'(a) \cdot a = (\mu(a) - \sigma(a)) \cdot a$ and $\sigma(a) \cdot a$. We then write

$$\tau(a) \cdot a + \sigma(a) \cdot a = \mu(a) \cdot a = \sigma'(a) \cdot a + \sigma(a) \cdot a,$$

which the cancellation property ((ii) of Corollary 2.14) yields $\tau(a) \cdot a = \sigma'(a) \cdot a$. Since both $\tau \leq \mu$ and $\sigma' \leq \mu$, we conclude that $\tau(a) = \sigma'(a)$ for all $a \in \mathrm{At}(M)$, whence $\tau = \sigma'$. Thus for any subset σ we have $\iota(\sigma') = \iota(\sigma)'$, showing that ι preserves $'$.

Thus we have shown that ι is an MV-algebra morphism and a bijection. Since MV-algebras are models of an algebraic presentation in the sense of

Definition C.10, the forgetful functor reflects isomorphisms (this is an easy consequence of Proposition C.3, for a proof see e.g. Proposition 3.3.3 in Borceux [9]). Thus we conclude that ι is an isomorphism of MV-algebras. $\square$

Chapter 4

Categorical embeddings of effect algebras

In this short chapter, we consider two ways to view effect algebras as presheaves. The first one is a very general categorical construction known as the *nerve functor*. The second one, the *test functor*, is more specific to effect algebras (or more generally to pointed partial commutative monoids), and was defined by Staton and Uijlen [55] in their work on nonlocality and contextuality.

We show that the nerve functor and the test functor are naturally isomorphic. As a corollary to this and a result of Staton and Uijlen showing that the test functor is full and faithful, we obtain the fact that the category of *finite* Boolean algebras is dense in the category of effect algebras. This demonstrates that effect algebras are a very natural extension of Boolean algebras, and is furthermore used to obtain the main result of Chapter 5.

4.1 Dense subcategories

The nerve functor has its origins in topological categories and classifying spaces [53]. We, however, consider the purely categorical version and outline its connection to dense subcategories. We warn the reader that our usage of the term 'nerve' is not entirely standard, as the subcategory in the definition of the nerve is typically required to be the category of simplices or some other category of 'shapes'. The terminology and constructions introduced in this section vaguely follow those of Adámek and Rosický [2], especially Section 1.B.

Definition 4.1 (Canonical colimit)**.** Let $\mathcal{A}$ be a small, full subcategory of a category $\mathcal{C}$. For an object $C \in \mathcal{C}$, the *canonical diagram* of C with respect to $\mathcal{A}$ is the forgetful functor

$$D : \mathcal{A}/C \longrightarrow \mathcal{C},$$

where $\mathcal{A}/C$ denotes the comma category which is the full subcategory of the coslice category $\mathcal{C}/C$ on those elements whose object part lies in $\mathcal{A}$.

We say that C is a *canonical colimit of $\mathcal{A}$-objects* provided that the canonical diagram has a colimit with colimit object C and coprojections

$$D\left(A \xrightarrow{f} C\right) \xrightarrow{f} C,$$

where $f : A \to C$ ranges through the objects of $\mathcal{A}/C$.

Definition 4.2 (Dense subcategory)**.** A small, full subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is *dense* if every object of $\mathcal{C}$ is a canonical colimit of $\mathcal{A}$-objects.

Definition 4.3 (Nerve functor)**.** Let $\mathcal{A}$ be a small, full subcategory of a category $\mathcal{C}$. The *nerve functor*

$$N_{\mathcal{A}} : \mathcal{C} \to [\mathcal{A}^{op}, \mathbf{Set}]$$

is defined by restriction of the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{op}, \mathbf{Set}]$ as

$$C \mapsto y_C|_{\mathcal{A}^{op}}$$

on objects, and as

$$\left(C \xrightarrow{f} C'\right) \mapsto y_f|_{\mathcal{A}^{op}},$$

where $y_f|_{\mathcal{A}^{op}}$ is the restriction of y_f to $y_C|_{\mathcal{A}^{op}} \to y'_C|_{\mathcal{A}^{op}}$. This is well-defined since $\mathcal{A}$ is a full subcategory.

We record the following simple observation connecting dense subcategories to nerve functors.

Proposition 4.4. *Let $\mathcal{A}$ be a small, full subcategory of a category $\mathcal{C}$. Then $\mathcal{A}$ is dense if and only if the nerve functor $N_{\mathcal{A}}$ is full and faithful.*

Proof. Let C and D be objects of $\mathcal{C}$. The maps $C \to D$ are in bijection with the natural transformations $y_C|_{\mathcal{A}^{op}} \to y_D|_{\mathcal{A}^{op}}$ via the assignment $f \mapsto f \circ -$ if and only if the cocone over the canonical diagram of C with vertex C and legs

$$D\left(A \xrightarrow{g} C\right) \xrightarrow{g} C$$

is the initial object in the category of cocones over the canonical diagram of C. But the latter condition says precisely that C is the canonical colimit of $\mathcal{A}$-objects. □

Corollary 4.5 (Yoneda lemma for dense subcategories)**.** *Let $\mathcal{A}$ be a small, full and dense subcategory of a category $\mathcal{C}$. Then for any objects $B, C \in \mathcal{C}$ we have $B \simeq C$ if and only if*

$$\mathcal{C}(A, B) \simeq \mathcal{C}(A, C)$$

naturally in $A \in \mathcal{A}$.

Proof. This is saying that $B \simeq C$ iff $N_{\mathcal{A}}(B) \simeq N_{\mathcal{A}}(C)$, which follows since $N_{\mathcal{A}}$ is full and faithful by Proposition 4.4, and since full and faithful functors are essentially injective on objects (Proposition A.2). □

4.2 Tests and denseness

Here we largely repeat the definitions of Staton and Uijlen [55] in order to be able to state Theorem 4.7.

The following definition should be compared to that of a partition of unity (Definition 6.2). To be completely precise, the definition below should be inductive. However, given the similarity to the definition of a partition of unity, we omit this; and we trust that the reader can follow the presentation without this technical precision, filling in the details if necessary.

Definition 4.6 (n-test)**.** Let E be an effect algebra and let $n \in \mathbb{N}$. An *n-test* is a list of elements of E of length n

$$(e_1, \ldots, e_n)$$

such that their sum $\bigoplus_{i=1}^{n} e_i$ exists and is equal to 1.

Let us denote the collection of n-tests of E by $T(E)(n)$. This defines a functor from finite sets (which we identify with $\mathbb{N}$) to **Set**

$$\begin{aligned} T(E) : \mathbb{N} &\to \mathbf{Set} \\ n &\mapsto T(E)(n) \\ \left(n \xrightarrow{f} m\right) &\mapsto (T(E)(n) \to T(E)(m)) \\ (e_1, \ldots, e_n) \quad &\mapsto \left(\bigoplus_{i \in f^{-1}(j)} e_i \right)_{j=1,\ldots,m} \end{aligned}$$

for each effect algebra E. This, in turn, extends to the *test functor*

$$\begin{aligned} T : \mathbf{EAlg} &\to [\mathbb{N}, \mathbf{Set}] \\ E &\mapsto T(E) \\ (\alpha : E \to F) &\mapsto T(\alpha), \end{aligned}$$

where $T(\alpha) : T(E) \to T(F)$ is the natural transformation with components

$$\begin{aligned} T(\alpha)_n : T(E)(n) &\to T(F)(n) \\ (e_1, \ldots, e_n) &\mapsto (\alpha(e_1), \ldots, \alpha(e_n)). \end{aligned}$$

Staton and Uijlen showed the following.

Theorem 4.7 (Corollary 10 in [55])**.** *The test functor $T : \mathbf{EAlg} \to [\mathbb{N}, \mathbf{Set}]$ is full and faithful.*

Since the powerset functor $\mathcal{P} : \mathbb{N}^{op} \to \mathbf{FinBA}$ is an equivalence of categories, we have an equivalence of categories

$$- \circ \mathcal{P}^{op} : [\mathbf{FinBA}^{op}, \mathbf{Set}] \to [\mathbb{N}, \mathbf{Set}].$$

The following isomorphism is already implicit in Staton's and Uijlen's work; here we state it explicitly and give a detailed proof.

Proposition 4.8. *The test functor $T : \mathbf{EAlg} \to [\mathbb{N}, \mathbf{Set}]$ is naturally isomorphic to the nerve functor composed with the above equivalence:*

$$N_{\mathbf{FinBA}}(-) \circ \mathcal{P}^{op} : \mathbf{EAlg} \to [\mathbb{N}, \mathbf{Set}].$$

Proof. We wish to define a natural transformation

$$\mu : N_{\mathbf{FinBA}}(-) \circ \mathcal{P}^{op} \to T.$$

Each component

$$\mu_E : N_{\mathbf{FinBA}}(E) \circ \mathcal{P}^{op} \to T(E)$$

is itself a natural transformation, whose components we define by

$$\begin{aligned} (\mu_E)_n : \mathbf{EAlg}(\mathcal{P}n, E) &\to T(E)(n) \\ f &\mapsto (f(\{i\}))_{i \in n}\,. \end{aligned}$$

This is clearly an isomorphism: the inverse is given by mapping an n-test $(e_1, \ldots, e_n)$ to the effect algebra morphism uniquely determined by mapping each atom $\{i\}$ to e_i. It remains to show naturality of μ_E and μ.

For a function $g : n \to m$, we need to show that the diagram

$$\begin{array}{ccc} \mathbf{EAlg}(\mathcal{P}n, E) & \xrightarrow{(\mu_E)_n} & T(E)(n) \\ \big\downarrow{\scriptstyle -\circ g^{-1}} & & \big\downarrow{\scriptstyle T(E)(g)} \\ \mathbf{EAlg}(\mathcal{P}m, E) & \xrightarrow{(\mu_E)_m} & T(E)(m) \end{array}$$

commutes. Here g^{-1} stands for the preimage. Via the down-right path, a morphism of effect algebras $f : \mathcal{P}n \to E$ is mapped to the m-test $\left(fg^{-1}(i)\right)_{i \in m}$, and via right-down to $\left(\bigoplus_{k \in g^{-1}(i)} f(\{k\})\right)_{i \in m}$. Using the fact that f is a morphism of effect algebras, we have for each $i \in m$

$$\bigoplus_{k \in g^{-1}(i)} f(\{k\}) = f\left(\bigcup_{k \in g^{-1}(i)} \{k\}\right) = fg^{-1}(i),$$

showing commutativity.

For an effect algebra morphism $h : E \to F$, we need to show that the following diagram of natural transformations commutes:

$$\begin{array}{ccc} N_{\mathbf{FinBA}}(E) \circ \mathcal{P}^{op} & \xrightarrow{\mu_E} & T(E) \\ \big\downarrow{\scriptstyle h\circ -} & & \big\downarrow{\scriptstyle Th} \\ N_{\mathbf{FinBA}}(F) \circ \mathcal{P}^{op} & \xrightarrow{\mu_F} & T(F) \end{array}$$

where $h \circ -$ is the natural transformation whose every component is $h \circ -$. The proof of commutativity is a matter of computation using the fact that natural transformations are composed pointwise. □

Corollary 4.9. *The category* **FinBA** *is a dense subcategory of* **EAlg**.

Proof. By Proposition 4.8 and Theorem 4.7, the nerve $N_{\mathbf{FinBA}} : \mathbf{EAlg} \to [\mathbf{FinBA}, \mathbf{Set}]$ is full and faithful, so that this follows by Proposition 4.4. □

Corollary 4.10. *For any effect algebras E and F, we have $E \simeq F$ if and only if*

$$\mathbf{EAlg}(B, E) \simeq \mathbf{EAlg}(B, F)$$

naturally in finite Boolean algebras B.

Proof. Just apply Corollary 4.5 to the situation of Corollary 4.9. □

Chapter 5

Subalgebras of Boolean algebras

The aim of this chapter is to prove that any Boolean algebra with more than two elements is determined by the poset of its finite subalgebras (Theorem 5.24). This is a strengthening of an analogous result using all subalgebras, which was first proved by Sachs [51, Theorem 4] in 1961, then more systematically via colimits by Grätzer, Koh and Makkai [29, Corollary 2] in 1972. It also follows from more general considerations of Filippov [24, p. 54] from 1965, and from a stronger result for orthoalgebras by Harding et al. (Theorem 1.1 in the introduction) from 2019. What allows us to obtain the proof using only the finite subalgebras is Corollary 4.10, or alternatively, Proposition C.37.

Throughout this chapter, let B be a Boolean algebra and $\mathrm{FinSub}(B)$ the poset of its finite subalgebras ordered by inclusion. We will denote the least and the greatest elements of B by 0 and 1, the complement of $a \in B$ by a', join and meet in B by $\vee$ and $\wedge$.

5.1 Subalgebra lattice and order isomorphisms

A non-empty intersection of finite subalgebras of a Boolean algebra B is again a finite subalgebra, so that $\mathrm{FinSub}(B)$ has non-empty meets given by intersections. The join of a finite collection of finite subalgebras is given by the least subalgebra containing each subalgebra in the collection. Since finitely generated Boolean algebras are finite, the join is itself a finite subalgebra. This makes $\mathrm{FinSub}(B)$ a lattice, and in fact a sublattice of the lattice of *all* subalgebras of B. We will denote the meet and the join in $\mathrm{FinSub}(B)$

by $\cap$ and $\vee$. We view $\mathrm{FinSub}(B)$ as living in the category of posets **Pos**. The isomorphisms in **Pos** will be referred to as *order isomorphisms*.

Proposition 5.1. *Let $\iota : P \to Q$ be an order isomorphism. Then for any element $s \in P$, the restriction*

$$\iota|_{\downarrow s} : \downarrow s \to \iota(\downarrow s)$$

is also an order isomorphism.

Proof. We also have the restriction

$$\iota^{-1}|_{\downarrow \iota(s)} : \downarrow \iota(s) \to \iota^{-1}(\downarrow \iota(s)).$$

In fact it suffices to show that (1) $\downarrow \iota(s) = \iota(\downarrow s)$ and (2) $\iota^{-1}(\downarrow \iota(s)) = \downarrow s$, as then we have an order-preserving map with an order-preserving inverse.

For (1), we have $a \in \downarrow \iota(s)$ iff $a \leq \iota(s)$ iff $\iota^{-1}(a) \leq s$ iff $\iota^{-1}(a) \in \downarrow s$ iff $a \in \iota(\downarrow s)$.

For (2), we compute $b \in \iota^{-1}(\downarrow \iota(s))$ iff $\iota(b) \in \downarrow \iota(s)$ iff $\iota(b) \leq \iota(s)$ iff $b \leq s$ iff $b \in \downarrow s$. □

Proposition 5.2. *An order isomorphism between two lattices is always a lattice isomorphism.*

Proof. Let $\iota : L \to M$ be an order isomorphism between lattices L and M. Then $m \leq \iota(x \wedge y)$ iff $\iota^{-1}(m) \leq x \wedge y$ iff $\iota^{-1}(m) \leq x$ and $\iota^{-1}(m) \leq y$ iff $m \leq \iota(x)$ and $m \leq \iota(y)$ iff $m \leq \iota(x) \wedge \iota(y)$, whence $\iota(x \wedge y) = \iota(x) \wedge \iota(y)$. The argument for the join is dual. □

The least element of $\mathrm{FinSub}(B)$ is the two-element Boolean subalgebra $\{0, 1\}$. The atoms of $\mathrm{FinSub}(B)$ are the four-element subalgebras $\{0, 1, a, a'\}$, where $a \notin \{0, 1\}$. Following Harding et al. [31], we make the following conventions.

Definition 5.3 (Basic element)**.** We say that an element of a poset with a least element is *basic* if it is either an atom or the least element.

For a poset P with a least element, we denote the subposet of P consisting of basic elements by $\mathrm{Bas}(P)$. A straightforward calculation shows that an order isomorphism sends basic elements to basic elements.

Definition 5.4 (Basic elements of $\mathrm{FinSub}(B)$)**.** Let B be a Boolean algebra, and let $a \in B$. We denote the corresponding basic element of $\mathrm{FinSub}(B)$ by

$$x_a := \{0, 1, a, a'\}.$$

Note that indeed an element of FinSub(B) is basic if and only if it is of the form x_a for some $a \in B$.

The main objective of this section is to show that any order isomorphism between FinSub(B) and FinSub(C) induces an order isomorphism between the sets of basic elements of the Boolean algebras B and C, as long as B and C have more than four elements. To this end, we wish to find an order-theoretic characterisation of those basic elements $x_a \in \text{FinSub}(B)$ where either a or a' is basic in B.

Definition 5.5 (Ideal subalgebra)**.** Let B be a Boolean algebra, and let $I \subseteq B$ be an ideal. Denote $I' \coloneqq \{a' : a \in I\}$. It is immediate that $I \cup I'$ is a subalgebra of B. A subalgebra of this form is called an *ideal subalgebra* of B.

When we write $I \cup I' \in \text{FinSub}(B)$, we mean that I is a finite ideal, and in such case we refer to $I \cup I'$ as a *finite ideal subalgebra.*

Lemma 5.6. *Let $I \cup I' \in \text{FinSub}(B)$ be an ideal subalgebra, and let $Y \in \text{FinSub}(B)$ be an arbitrary finite subalgebra. Then their join is given by*

$$\left(I \cup I'\right) \vee Y = \left\{a' \wedge (b \vee y) : a, b \in I, y \in Y\right\}.$$

Proof. The right-to-left inclusion is clear, as the join is a subalgebra. For the left-to-right inclusion, note that $I \cup I'$ and Y are both contained in the set on the right-hand side (RHS for short). Thus it suffices to show that the RHS is a subalgebra.

Let $z = a' \wedge (b \vee y)$ for some $a, b \in I$ and $y \in Y$. Then

$$z' = a \vee (b' \wedge y') = (a \vee b') \wedge (a \vee y') = (a' \wedge b)' \wedge (a \vee y').$$

Since I is an ideal, $a' \wedge b \in I$, and since Y is a subalgebra, $y' \in Y$. Hence z' is of the required form and is in the RHS.

Now let $z = a' \wedge (b \vee y)$ and $w = \alpha' \wedge (\beta \vee \gamma)$ be in the RHS, that is, $a, b, \alpha, \beta \in I$ and $y, \gamma \in Y$. Then

$$\begin{aligned} z \wedge w &= (a' \wedge \alpha') \wedge (b \vee y) \wedge (\beta \vee \gamma) \\ &= (a \vee \alpha)' \wedge ((b \wedge (\beta \vee \gamma)) \vee (y \wedge \beta) \vee (y \wedge \gamma)). \end{aligned}$$

Now $a \vee \alpha$, $b \wedge (\beta \vee \gamma)$, $y \wedge \beta$ and $(b \wedge (\beta \vee \gamma)) \vee (y \wedge \beta)$ are all in I, as I is an ideal, while $y \wedge \gamma \in Y$, as Y is a subalgebra. Thus $z \wedge w$ is in the RHS.

Thus the RHS is closed under complement and meet, from which by de Morgan laws it follows that it is closed under join; whence it is a subalgebra, as required. $\square$

Definition 5.7 ((Dual) modular element)**.** An element x of a lattice L is

- *modular* if for all $z, y \in L$ with $x \leq z$ we have

$$(x \vee y) \wedge z = x \vee (y \wedge z),$$

- *dual modular* if it is modular and for all $w, y \in L$ with $w \leq y$ we have

$$(w \vee x) \wedge y = w \vee (x \wedge y).$$

The following is Lemma 2 in Sachs [51], where it is proved for any ideal subalgebra in the lattice of *all* subalgebras.

Lemma 5.8. *Any finite ideal subalgebra of a Boolean algebra B is a modular element of* $\mathrm{FinSub}(B)$*.*

Proof. Let $I \cup I'$ be an ideal subalgebra, and let $Z, Y \in \mathrm{FinSub}(B)$ be such that $I \cup I' \subseteq Z$. We need to show that

$$\big((I \cup I') \vee Y\big) \cap Z = \big(I \cup I'\big) \vee (Y \cap Z),$$

that is, by Lemma 5.6, we need to show that

$$\big\{a' \wedge (b \vee y) : a, b \in I, y \in Y\big\} \cap Z = \big\{a' \wedge (b \vee w) : a, b \in I, w \in Y \cap Z\big\}.$$

The right-to-left inclusion is immediate, as $I \cup I' \subseteq Z$ and Z is a subalgebra. For the left-to-right inclusion, suppose that $a, b \in I$ and $y \in Y$ are such that $a' \wedge (b \vee y) \in Z$. Then $b' \in I' \subseteq Z$, so that

$$b' \wedge (a' \wedge (b \vee y)) = a' \wedge b' \wedge y = (a \vee b)' \wedge y \in Z.$$

Next, $a \vee b \in I$, as I is an ideal, consequently $(a \vee b) \wedge y \in I \subseteq Z$. Writing $z \coloneqq a \vee b$, we have that both $z \wedge y \in Z$ and $z' \wedge y \in Z$, so that

$$y = (z \wedge y) \vee (z' \wedge y) \in Z,$$

whence $y \in Y \cap Z$ and $a' \wedge (b \vee y)$ is in the set on the right-hand side. $\square$

The following proposition is included out of general curiosity and for the sake of completeness. As it will play no role in the remaining results, it may be skipped by a hasty reader.

Proposition 5.9. *Any finite ideal subalgebra of a Boolean algebra B is dual modular in* $\mathrm{FinSub}(B)$*.*

Proof. Let $I \cup I'$ be a finite ideal subalgebra. Then it is modular by Lemma 5.8. Now let $W, Y \in \mathrm{FinSub}(B)$ be such that $W \subseteq Y$. We need to show that

$$\big(W \vee \big(I \cup I'\big)\big) \cap Y = W \vee \big(\big(I \cup I'\big) \cap Y\big)\,,$$

that is, by Lemma 5.6, we need to show that

$$\big\{a' \wedge (b \vee w) : a, b \in I, w \in W\big\} \cap Y = W \vee \big(\big(I \cup I'\big) \cap Y\big)\,.$$

For the right-to-left inclusion, it suffices to show that W and $(I \cup I') \cap Y$ are contained in the subalgebra on the left-hand side, which is immediate. For the left-to-right inclusion, suppose that $a, b \in I$ and $w \in W$ are such that $a' \wedge (b \vee w) \in Y$. Since $w \in W \subseteq Y$ and Y is a subalgebra, we have $w' \in Y$. Thus we get

$$\begin{aligned} w' \wedge (a' \wedge (b \vee w)) &= a' \wedge b \wedge w' \in Y, \\ w' \vee (a' \wedge (b \vee w)) &= a' \vee w' \in Y. \end{aligned}$$

Moreover, since $b \in I$ and I is an ideal, we obtain $a' \wedge b \wedge w' \in I$, whence $a' \wedge b \wedge w' \in (I \cup I') \cap Y$. Similarly, $a' \vee w' \in (I \cup I') \cap Y$. We now rewrite the expression we started with as

$$\begin{aligned} a' \wedge (b \vee w) &= a' \wedge ((b \wedge w') \vee w) \\ &= (a' \wedge b \wedge w') \vee (a' \wedge w) \\ &= (a' \wedge b \wedge w') \vee ((a' \vee w') \wedge w), \end{aligned}$$

which is a Boolean combination of elements in $(I \cup I') \cap Y$ and W, so that it is contained in $W \vee ((I \cup I') \cap Y)$, as required. □

The following is Theorem 1 from Sachs [51], with the modification that there it is presented for dual modular elements (cf. Corollary 5.14) in the lattice of all subalgebras.

Theorem 5.10. *A finite subalgebra X of a Boolean algebra B is modular in* $\mathrm{FinSub}(B)$ *if an only if it is an ideal subalgebra.*

Proof. The 'if' direction is Lemma 5.8. For the 'only if' direction, let $X \in \mathrm{FinSub}(B)$ be modular. Define a subset of B by

$$I \coloneqq \{p \in B : \downarrow p \subseteq X\}\,.$$

First observe that I is an ideal: it is non-empty as $0 \in I$, downwards closed as $p \in I$ and $y \in B$ with $y \leq p$ imply $\downarrow y \subseteq \downarrow p \subseteq X$, so that $y \in I$. For upwards directedness, let $p, y \in I$. Then $a \in \downarrow (p \vee y)$ iff $a \leq p \vee y$ iff

$$a = a \wedge (p \vee y) = (a \wedge p) \vee (a \wedge y).$$

By assumption $a \wedge p \in \downarrow p \subseteq X$ and $a \wedge y \in \downarrow y \subseteq X$, whence $a \in X$ as X is a subalgebra. Thus $\downarrow (p \vee y) \subseteq X$, so that $p \vee y \in I$, as required.

Note that $I \subseteq X$, and since X is a subalgebra, $I \cup I' \subseteq X$. We now wish to show that the reverse containment holds. Hence let $a \in X$, and suppose towards a contradiction that $a \notin I \cup I'$. Since $a \notin I$, we have $\downarrow a \nsubseteq X$, and since $a \notin I'$, we get $a' \notin I$ so that $\downarrow a' \nsubseteq X$. We may thus choose some $y \leq a$ and $z \leq a'$ such that $y, z \notin X$.

Using the notation of Definition 5.4, we claim that

$$X \vee x_y \subseteq X \vee x_{y \vee z} \tag{5.11}$$

In fact it is sufficient to see that $y \in X \vee x_{y \vee z}$, which follows by noting that

$$y = a \wedge y = a \wedge (y \vee z),$$

using that $y \leq a$ and $a \wedge z = 0$. Similarly, we have

$$z = a' \wedge (y \vee z) \in X \vee x_{y \vee z}.$$

On the other hand, we claim that

$$X \vee x_y \subseteq \{b \in B : b \wedge a' \in X\} \tag{5.12}$$

Clearly X is contained in the set on the right-hand side. Since $y \wedge a' = 0 \in X$ and $y' \wedge a' = a' \in X$, so is x_y. Thus it suffices to show that the set on the right-hand side is a subalgebra. Hence suppose $b \wedge a' \in X$. Then $b' \vee a \in X$, so that $b' \wedge a' = (b' \vee a) \wedge a' \in X$. Next, if $b \wedge a', c \wedge a' \in X$, then immediately $(b \wedge c) \wedge a' \in X$. Thus it is indeed a subalgebra, proving the claimed inclusion (5.12).

Since $z = a' \wedge (y \vee z) \notin X$, we conclude that $y \vee z \notin X \vee x_y$ using (5.12). Thus we have that

$$x_{y \vee z} \cap (X \vee x_y) = \{0, 1\}. \tag{5.13}$$

Finally, we apply the assumption that X is modular to $X \subseteq X \vee x_y$ and $x_{y \vee z}$:

$$(X \vee x_{y \vee z}) \cap (X \vee x_y) = X \vee (x_{y \vee z} \cap (X \vee x_y)),$$

whence $X \vee x_y = X$, where we used equations (5.11) and (5.13). But this is the sought-after contradiction, as this implies $y \in X$.

Thus we conclude that $X \subseteq I \cup I'$, yielding $X = I \cup I'$, which exhibits X as an ideal subalgebra. $\square$

Corollary 5.14. *In the lattice of finite subalgebras of a Boolean algebra, modular and dual modular elements coincide.*

Proof. Immediate consequence of Theorem 5.10 and Proposition 5.9. □

Corollary 5.15. *Let B be a Boolean algebra. A basic element $x_a \in \mathrm{FinSub}(B)$ is modular iff either a or a' is basic in B.*

Proof. First suppose (without loss of generality) that $a \in B$ is basic. Then $x_a = \downarrow a \cup \uparrow a'$ is an ideal subalgebra and hence modular by Theorem 5.10.

Conversely, if x_a is modular, then, by Theorem 5.10, $\{0, 1, a, a'\} = I \cup I'$ for some ideal $I \subseteq B$. Thus either $a \in I$ or $a' \in I$. The former yields that either $a = 1$, so that a' is basic, or $a \neq 1$, in which case a is basic. Similarly for the latter. □

Note that we have now achieved one of our goals and characterised those elements of $\mathrm{FinSub}(B)$ which are equal to x_a for some basic $a \in B$ in purely order-theoretic therms. Namely, these are precisely those elements which are both basic and modular.

The following result excludes the possibility of recovering Boolean algebras that are too 'small' from their posets of subalgebras.

Proposition 5.16. *A Boolean algebra B has an element a such that both a and a' are basic if and only if B has at most four elements.*

Proof. The 'if' direction follows by inspecting the Boolean algebras with 1, 2 and 4 elements. For the 'only if' direction, let $a \in B$ be such that both a and a' are basic, and let $x \in B$. Let us write x as

$$x = (x \wedge a) \vee (x \wedge a').$$

Since a and a' are basic, we have $x \wedge a \in \{0, a\}$ and $x \wedge a' \in \{0, a'\}$, whence $x \in \{0, 1, a, a'\}$. We therefore conclude that $B \subseteq \{0, 1, a, a'\}$, as required. □

We are now ready to state and prove the main result of this section.

Theorem 5.17. *Let B and C be Boolean algebras with more than four elements. Then any order isomorphism between* $\mathrm{FinSub}(B)$ *and* $\mathrm{FinSub}(C)$ *induces an order isomorphism between* $\mathrm{Bas}(B)$ *and* $\mathrm{Bas}(C)$.

Proof. Let $\iota : \mathrm{FinSub}(B) \to \mathrm{FinSub}(C)$ be an order isomorphism. For any $b \in \mathrm{Bas}(B)$, the element $x_b \in \mathrm{FinSub}(B)$ is basic and modular by Corollary 5.15. Since an order isomorphism is also a lattice isomorphism (Proposition 5.2), ι preserves both basic and modular elements, so that $\iota(x_b)$ is basic and modular. Thus there is a unique pair c, c' in C such that

$\iota(x_b) = x_c$. Once more by Corollary 5.15 one of c and c' is basic in C. By Proposition 5.16, exactly one of them is basic. Let us denote the one that is basic by $\bar{\iota}(b)$. Thus we have defined a mapping

$$\begin{aligned} \bar{\iota} : \mathrm{Bas}(B) &\to \mathrm{Bas}(C) \\ b &\mapsto c \in \mathrm{Bas}(C) \text{ s.t. } x_c = \iota(x_b). \end{aligned}$$

We have that $\bar{\iota}(0) = 0$, and since the atoms are incomparable with each other, the map is order-preserving. Starting with ι^{-1}, we get the map in the other direction:

$$\begin{aligned} \overline{\iota^{-1}} : \mathrm{Bas}(C) &\to \mathrm{Bas}(B) \\ c &\mapsto b \in \mathrm{Bas}(B) \text{ s.t. } x_b = \iota^{-1}(x_c) \end{aligned}$$

(such b is guaranteed to exist and is unique). We then have

$$\bar{\iota}\left(\overline{\iota^{-1}}(c)\right) = \bar{\iota}(b) = d \in \mathrm{Bas}(C)$$

with $x_d = \iota(x_b) = x_c$, where we used that ι is an isomorphism. Thus $d = c$ and $\bar{\iota}\left(\overline{\iota^{-1}}(c)\right) = c$. The other composition follows similarly. Thus $\bar{\iota}$ is indeed an order isomorphism. □

Note that the restriction on the size of the Boolean algebras is necessary here, as for algebras with at most four elements the subalgebra x_a may contain more than one basic element, so that the induced mapping is not uniquely defined.

Corollary 5.18. *Let B and C be finite Boolean algebras with more than four elements. Then an order isomorphism $\iota : \mathrm{FinSub}(B) \to \mathrm{FinSub}(C)$ induces a Boolean algebra isomorphism*

$$\begin{aligned} \gamma : B &\to C \\ b &\mapsto \bigvee \{\bar{\iota}(a) : a \in \mathrm{Bas}(B) \text{ and } a \leq b\}, \end{aligned}$$

where $\bar{\iota} : \mathrm{Bas}(B) \to \mathrm{Bas}(C)$ is the induced isomorphism of Theorem 5.17.

Proof. This is straightforward from the facts that $\bar{\iota}$ restricts to an isomorphism of atoms and that any finite Boolean algebra is isomorphic to the powerset of its atoms. □

Lemma 5.19. *Let B and C be finite Boolean algebras. Suppose that $\alpha, \beta : \mathrm{FinSub}(B) \to \mathrm{FinSub}(C)$ are lattice morphisms which agree on the subalgebras x_b where b is basic in B. Then $\alpha = \beta$.*

Proof. Since for every subalgebra $X \in \mathrm{FinSub}(B)$ we have $X = \bigvee_{a \in X} x_a$ and everything is finite, it is sufficient to show that α and β agree on the subalgebras x_a for each $a \in B$.

Given $a \in B$, let us denote by $\mathcal{B}_a$ the set of all basic elements below a:

$$\mathcal{B}_a \coloneqq \{b \in \mathrm{Bas}(B) : b \leq a\}.$$

We first show that for each $a \in B$,

$$\bigvee_{b \in \mathcal{B}_a} x_b = \left\{ \bigvee S : S \subseteq \mathcal{B}_a \right\} \cup \left\{ \bigwedge T' : T \subseteq \mathcal{B}_a \right\}, \tag{5.20}$$

where $T' \coloneqq \{t' : t \in T\}$. The right-to-left inclusion is clear. For the left-to-right inclusion, observe that each x_b for $b \in \mathcal{B}_a$ is contained in the set on the right-hand side (RHS for short). Thus it suffices to show that the RHS is a subalgebra. It is manifestly closed under complements. Now suppose x and y are in the RHS. If $x = \bigvee S$ for some $S \subseteq \mathcal{B}_a$, then $x \wedge y = \bigvee_{s \in S}(s \wedge y)$ is also in the RHS, as each $s \wedge y$ is either 0 or s, since each s is basic. Thus it remains to consider the case when both x and y are meets of complements of elements in $\mathcal{B}_a$, but then certainly so is $x \wedge y$. Thus the RHS is indeed a subalgebra, and we obtain the left-to-right inclusion.

To conclude, we claim that for each $a \in B$,

$$x_a = \left(\bigvee_{b \in \mathcal{B}_a} x_b \right) \cap \left(\bigvee_{d \in \mathcal{B}_{a'}} x_d \right).$$

Showing this is indeed sufficient, as α and β are assumed to preserve finite meets and joins. First, since $a = \bigvee_{b \in \mathcal{B}_a} b$ and $a' = \bigvee_{d \in \mathcal{B}_{a'}} d$, the subalgebra x_a is contained in the right-hand side. Now suppose y is contained in the subalgebra on the right-hand side. Using equation (5.20), there are four options:

(1) $y = \bigvee S = \bigvee T$,

(2) $y = \bigvee S = \bigwedge T'$,

(3) $y = \bigwedge S' = \bigvee T$,

(4) $y = \bigwedge S' = \bigwedge T'$,

for some $S \subseteq \mathcal{B}_a$ and $T \subseteq \mathcal{B}_{a'}$. In the first case, $y \leq a$ and $y \leq a'$, whence $y = 0$. In the second case, $y \leq a$ and $a \leq y$, so that $y = a$. In the third case, $a' \leq y$ and $y \leq a'$, so $y = a'$. In the fourth case, $a' \leq y$ and $a \leq y$, so that $y = 1$. Thus we conclude that $y \in \{0, a, a', 1\} = x_a$, as required. □

Theorem 5.21. *Let B and C be Boolean algebras with more than four elements, and suppose that $\iota : \mathrm{FinSub}(B) \to \mathrm{FinSub}(C)$ is an order isomorphism. Let $X, Y \in \mathrm{FinSub}(B)$ be finite subalgebras with more than four elements such that $X \subseteq Y$. Denote by $\gamma_X : X \to \iota X$ the Boolean algebra isomorphism induced by $\iota|_{\mathrm{FinSub}(X)} : \mathrm{FinSub}(X) \to \mathrm{FinSub}(\iota X)$ as in Corollary 5.18, and similarly for $\gamma_Y : Y \to \iota Y$. Then*

$$\gamma_X = \gamma_Y|_X.$$

Proof. First note that $\mathrm{FinSub}(X) = \downarrow X$, where the down-set is taken in $\mathrm{FinSub}(B)$, so that $\iota|_{\mathrm{FinSub}(X)} : \mathrm{FinSub}(X) \to \mathrm{FinSub}(\iota X)$ is indeed an order isomorphism by Proposition 5.1, whence it follows that the statement of the proposition makes sense.

By definition of γ_Y, we have that $\gamma_Y[x_b] = \iota(x_b)$, where $\gamma_Y[-]$ denotes the image, for any basic element $b \in Y$. By Lemma 5.19, we have $\gamma_Y[-] = \iota|_{\mathrm{FinSub}(Y)}$, and similarly, $\gamma_X[-] = \iota|_{\mathrm{FinSub}(X)}$. Thus we obtain

$$\gamma_X[-] = \left(\iota|_{\mathrm{FinSub}(Y)}\right)|_{\mathrm{FinSub}(X)} = (\gamma_Y[-])\,|_{\mathrm{FinSub}(X)}.$$

We will next show that γ_Y maps basic elements of X to basic elements of ιX. Thus suppose that $a \in \mathrm{Bas}(X)$. The above equality tells us that $\gamma_Y(a) \in \iota X$. Now let $z \in \iota X$ be such that $z \leq \gamma_Y(a)$, so that $\gamma_Y^{-1}(z) \leq a$. Again by the above equality, $\gamma_Y^{-1}(z) \in X$. Since a is basic, either $\gamma_Y^{-1}(z) = 0$ or $\gamma_Y^{-1}(z) = a$, whence either $z = 0$ or $z = \gamma_Y(a)$, showing that $\gamma_Y(a)$ is indeed basic.

Now given $a \in \mathrm{Bas}(X)$, we have that

$$\gamma_Y[x_a] = \{0, 1, \gamma_Y(a), \gamma_Y(a)'\} = \{0, 1, \gamma_X(a), \gamma_X(a)'\} = \gamma_X[x_a].$$

Since X has more than four elements, a' is not basic by Proposition 5.16, consequently $\gamma_X(a)'$ is not basic. Since γ_Y is an isomorphism and $\gamma_Y(a)$ is basic, we conclude that $\gamma_Y(a) = \gamma_X(a)$.

Since γ_Y agrees with γ_X on the basic elements of X, we conclude that γ_Y agrees with γ_X on all of X, which is what we had to show. □

Corollary 5.22. *Let B and C be Boolean algebras with more than four elements. Then any order isomorphism $\iota : \mathrm{FinSub}(B) \to \mathrm{FinSub}(C)$ induces a Boolean algebra isomorphism*

$$\gamma_X : X \to \iota X$$

on each finite subalgebra X of B. This family is moreover natural in X in the sense that whenever X and Y are finite subalgebras such that $X \subseteq Y$, the diagram below commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad \gamma_X \quad} & \iota X \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{\quad \gamma_Y \quad} & \iota Y \end{array}$$

where the vertical arrows are the inclusions.

Proof. If X has more than four elements, this is just Theorem 5.21. To define the isomorphism on a four-element subalgebra X, simply take any finite subalgebra Y properly containing X (which exists by assumption) and define $\gamma_X \coloneqq \gamma_Y|_X$. By Theorem 5.21, this is independent of the choice of Y. Similarly for the two-element subalgebra, although in that case there is of course a unique isomorphism. □

5.2 Boolean algebras are determined by their subalgebra poset

In the previous section we saw that an order isomorphism between two finite subalgebra posets induces a family of maps between the finite subalgebras that looks something like a natural transformation (Corollary 5.22). The task of the following lemma is to make this 'something like' precise.

Lemma 5.23. *Let B and C be Boolean algebras with more than four elements. Then any order isomorphism between* $\mathrm{FinSub}(B)$ *and* $\mathrm{FinSub}(C)$ *induces a natural isomorphism*

$$\alpha : \mathbf{BAlg}(-, B) \to \mathbf{BAlg}(-, C),$$

where the hom-functors are restricted to the finite Boolean algebras.

Proof. Let $\iota : \mathrm{FinSub}(B) \to \mathrm{FinSub}(C)$ be an order isomorphism. Given an $F \in \mathbf{FinBA}$ and a Boolean algebra morphism $s : F \to B$, let us write $\tilde{s} : F \to \mathrm{im}\ s$ for the Boolean algebra morphism agreeing with s whose codomain is restricted to the image of s. Let us also write $\gamma_s \coloneqq \gamma_{\mathrm{im}\ s}$, where $\gamma_{\mathrm{im}\ s} : \mathrm{im}\ s \to \iota(\mathrm{im}\ s)$ is the isomorphism from Corollary 5.22 induced by $\iota|_{\mathrm{im}\ s}$. Finally, we write $j_s : \iota(\mathrm{im}\ s) \hookrightarrow C$ for the inclusion. Note that j_s is a Boolean algebra morphism since $\iota(\mathrm{im}\ s)$ is a subalgebra.

For each finite Boolean algebra F, we define a map

$$\alpha_F : \mathbf{BAlg}(F, B) \to \mathbf{BAlg}(F, C)$$
$$s \mapsto j_s \circ \gamma_s \circ \tilde{s}.$$

To see that this is natural, let $g : G \to F$ be a Boolean algebra morphism. We need to show that $\alpha_F(s)g = \alpha_G(sg)$, for each $s : F \to B$. This amounts to

$$j_s \circ \gamma_s \circ \tilde{s} \circ g = j_{sg} \circ \gamma_{sg} \circ \widetilde{sg}.$$

First note that $\widetilde{sg} = \tilde{s}g$. Next, im $(sg) \subseteq$ im s, so that γ_{sg} and γ_s agree on the image of sg, and $\iota(\text{im }(sg)) \subseteq \iota(\text{im }s)$, so that j_{sg} and j_s agree on $\iota(\text{im }(sg))$. The desired equality follows.

It remains to see that each α_F is an isomorphism. For injectivity, let $s, t : F \to B$ be morphisms such that $\alpha_F(s) = \alpha_F(t)$, so that we have

$$j_s \circ \gamma_s \circ \tilde{s} = j_t \circ \gamma_t \circ \tilde{t}.$$

Since $\gamma_s \circ \tilde{s}$ is surjective and both j_s and j_t are inclusions, we get that for any $a \in \iota(\text{im }s)$ there is a $k \in F$ such that

$$a = (\gamma_s \circ \tilde{s})(k) = (\gamma_t \circ \tilde{t})(k) \in \iota(\text{im }t),$$

so that $\iota(\text{im }s) \subseteq \iota(\text{im }t)$. Reversing the argument we obtain the other inclusion, so that $\iota(\text{im }s) = \iota(\text{im }t)$, whence $j_s = j_t$ and im $s =$ im t. The latter implies that $\gamma_s = \gamma_t$, so that we have shown

$$j_s \circ \gamma_s \circ \tilde{s} = j_s \circ \gamma_s \circ \tilde{t}.$$

By injectivity of j_s and γ_s, we get $\tilde{s} = \tilde{t}$, whence $s = t$, as required.

For surjectivity, let $u : F \to C$ be a morphism. As before, denote by $\tilde{u} : F \to \text{im } u$ the restriction of u to its image. Write

$$\beta \coloneqq \gamma_{\iota^{-1}(\text{im } u)} : \iota^{-1}(\text{im } u) \to \text{im } u$$

and $e : \iota^{-1}(\text{im } u) \hookrightarrow B$ for the inclusion. Now let $r \coloneqq e\beta^{-1}\tilde{u}$. We thus have im $r = \iota^{-1}(\text{im } u)$, so that $\gamma_r = \beta$. For an $a \in F$ we compute

$$\alpha_F(r)(a) = (j_r \circ \gamma_r)(r(a)) = \gamma_r(\gamma_r^{-1}(u(a))) = u(a),$$

whence $\alpha_F(r) = u$, showing surjectivity. □

The discussion of this chapter is summarised in the following.

Theorem 5.24. *Let B and C be Boolean algebras with more than four elements. Then $B \simeq C$ if and only if* $\mathrm{FinSub}(B) \simeq \mathrm{FinSub}(C)$. *In other words, the functor*

$$\begin{aligned} \mathrm{FinSub} : \mathbf{BAlg} &\to \mathbf{Pos} \\ D &\mapsto \mathrm{FinSub}(D) \\ (D \xrightarrow{f} H) &\mapsto f[-], \end{aligned}$$

where $f[-]$ denotes the image, is essentially injective on Boolean algebras with more than four elements.

Proof. The 'only if' direction is immediate, as functors preserve isomorphisms. The 'if' direction is Lemma 5.23 together with Corollary 4.10, using the fact that **BAlg** is a full subcategory of **EAlg** (see the discussion after Example 2.7). □

It is possible to prove Theorem 5.24 using Proposition C.37 instead of Corollary 4.10. In such case one notes that a version of Lemma 5.23 with the hom-functors are restricted to the free and finitely generated Boolean algebras may be proved by just replacing every occurrence of 'finite' with 'free and finitely generated'. Thus it is not necessary to pass via effect algebras.

Chapter 6

Partition posets

The culmination of this chapter, and the main contribution of the thesis, is the reconstruction theorem for finite MV-algebras from the poset of partitions of unity (Theorem 6.24). It turns out it is easier to work with multisets, and even easier with setoids. To this end, we define partitions of unity of an arbitrary effect algebra and partitions of finite multisets in Section 6.1. We also discuss the usual partition lattice of a finite set as a special case of a poset of multiset partitions. In Section 6.2, we introduce setoid quotients as quotient posets of finite partition lattices, and show that these are exactly the partition posets of finite multisets. We then prove some preliminary lemmas about setoid quotients. Section 6.3 proves the main result using the constructions and lemmas of the two preceding sections.

There is a close connection between partitions of unity and measurements in operational quantum mechanics. The most general notion of a measurement in quantum mechanics is given by a *positive operator valued measure* (POVM for short) [11], [50]. In addition to theoretical generality, POVMs are very close to experimental setups; see examples in the Introduction of Busch, Grabowski and Lahti [11, p. I.1.2] as well as Brandt [10] for another example. In the finite case, POVMs are collections of positive operators which sum up to the identity. For a Hilbert space effect algebra, each partition of unity gives rise to a unique image of a discrete POVM, and conversely, each image of a discrete POVM uniquely determines a partition of unity (see Remark 6.3). Generalising this, Jenča [37] gives a necessary and sufficient condition under which a subset of an effect algebra is contained in the image of a Boolean algebra.

6.1 Partitions of unity

We begin by making the idea of a multiset whose types come from an effect algebra precise.

Let E be an effect algebra. We say that $\mathcal{A} = (A, \eta)$ is a *multiset of elements* of E if $\mathcal{A}$ is a finite multiset such that $A \subseteq E$. If $\mathcal{A} = (A, \eta)$ and $\mathcal{B} = (B, \mu)$ are multisets of elements of E, we define their *union* as the multiset of elements $\mathcal{A} \cup \mathcal{B} \coloneqq (A \cup B, \nu)$, where $\nu : A \cup B \to \mathbb{N} \setminus \{0\}$ is defined by

$$a \mapsto \begin{cases} \eta(a) \text{ if } a \in A \setminus B, \\ \mu(a) \text{ if } a \in B \setminus A, \\ \eta(a) + \mu(a) \text{ if } a \in A \cap B. \end{cases}$$

Note that this definition does not coincide with the union (join) of submultisets of some ambient multiset.

Definition 6.1 (Summable multiset)**.** Let E be an effect algebra. We define the collection of *summable* multisets of elements $\mathfrak{S}_E$ as well as the sum function

$$\bigoplus : \mathfrak{S}_E \to E$$

by recursion as follows:

- $\varnothing \in \mathfrak{S}_E$ and $\bigoplus \varnothing \coloneqq 0$,
- if $\mathcal{A} \in \mathfrak{S}_E$, then for every $a \in E$, we have $\{a\} \cup \mathcal{A} \in \mathfrak{S}_E$ iff $a \perp \bigoplus \mathcal{A}$, in which case we define

$$\bigoplus (\{a\} \cup \mathcal{A}) \coloneqq a \oplus \bigoplus \mathcal{A}.$$

Note that every singleton set is summable, and its sum is equal to the unique element of the set. A simple induction shows that summability is well-defined, that is, independent of the order in which a summable multiset is obtained in the above recursion. Thus for a summable multiset of elements $\mathcal{A} = (A, \eta)$ we have

$$\bigoplus \mathcal{A} = \bigoplus_{a \in A} \eta(a) \cdot a.$$

Definition 6.2 (Partition of unity)**.** Let E be an effect algebra. A multiset of elements $\mathcal{P}$ is a *partition of unity* if it is summable, 0 is not a type of $\mathcal{P}$ and $\bigoplus \mathcal{P} = 1$.

Remark 6.3. Every effect algebra morphism $f : B \to E$ whose domain is a finite Boolean algebra B gives rise to a partition of unity $\mathcal{B} \coloneqq (f(\mathrm{At}(B)), \alpha)$, where $\alpha : f(\mathrm{At}(B)) \to \mathbb{N} \setminus \{0\}$ is given by

$$\alpha(e) \coloneqq \left| f^{-1}(e) \cap \mathrm{At}(B) \right| .$$

In fact, morphisms from finite Boolean algebras to effect algebras coming from a Hilbert space correspond to discrete POVMs. Indeed, a POVM can be defined as a countable join preserving effect algebra morphism

$$\Sigma \to \mathcal{E}(H)$$

from a σ-algebra Σ to a Hilbert space effect algebra $\mathcal{E}(H)$ [50, Definition 1]. Since the finite σ-algebras are exactly the Boolean algebras, in the finite (i.e. discrete) case this becomes just an effect algebra morphism $B \to \mathcal{E}(H)$. Thus, for a Hilbert space effect algebra, there is a one-to-one correspondence between partitions of unity and images of discrete POVMs.

Next, we define a partial order on the set of partitions of unity of E. In what follows, X is the multisets-to-setoids functor from Section 2.3.

Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of unity. We say that $\mathcal{P} \leq \mathcal{Q}$ if there exists a surjection $f : X(\mathcal{P}) \to X(\mathcal{Q})$ such that for each $(j, a) \in X(\mathcal{Q})$ we have

$$\bigoplus_{(i,b) \in f^{-1}(j,a)} b = a.$$

If we want to make the choice of a surjection explicit, we write $f : \mathcal{P} \leq \mathcal{Q}$ for $\mathcal{P} \leq \mathcal{Q}$ witnessed by a surjection f.

Proposition 6.4. *The relation $\leq$ defines a partial order on the set of partitions of unity of E.*

Proof. Reflexivity is clear: just take the surjection in the definition to be the identity map. For transitivity, let $f : \mathcal{P} \leq \mathcal{Q}$ and $g : \mathcal{Q} \leq \mathcal{R}$. Then $gf : X(\mathcal{P}) \to X(\mathcal{R})$ is a surjection, and given $(j, a) \in X(\mathcal{R})$ we have

$$\begin{aligned} \bigoplus_{(i,b) \in (gf)^{-1}(j,a)} b &= \bigoplus_{(i,b) \in f^{-1}(g^{-1}(j,a))} b \\ &= \bigoplus_{(k,c) \in g^{-1}(j,a)} \bigoplus_{(i,b) \in f^{-1}(k,c)} b \\ &= \bigoplus_{(k,c) \in g^{-1}(j,a)} c \\ &= a. \end{aligned}$$

For antisymmetry, let $\mathcal{P} = (P, \eta)$ and $\mathcal{Q} = (Q, \mu)$ be partitions, and suppose $f : \mathcal{P} \leq \mathcal{Q}$ and $g : \mathcal{Q} \leq \mathcal{P}$. Since gf is a surjection and $X(\mathcal{P})$ is finite, gf is in fact a bijection, and similarly so is fg. Thus f and g must in fact be injective, hence both are bijections. Now let $(i, b) \in X(\mathcal{P})$. We have

$$b = \bigoplus_{(j,a) \in g^{-1}(i,b)} a,$$

whence $g^{-1}(i, b) = (j, b)$ for some j since g is a bijection. Thus $P = Q$, and for each $b \in P$, the map g restricts to a bijection between $\coprod_{i=1}^{\eta(b)} \{(i, b)\}$ and $\coprod_{j=1}^{\mu(b)} \{(j, b)\}$, whence $\eta(b) = \mu(b)$, showing $\mathcal{P} = \mathcal{Q}$. $\square$

We denote the poset of partitions of unity of E by $\mathrm{Part}(E)$, and say that $\mathrm{Part}(E)$ is *ordered by refinement*: the relation $\mathcal{P} \leq \mathcal{Q}$ holds precisely when the elements of $\mathcal{Q}$ can be decomposed into summands in order to obtain $\mathcal{P}$. This extends to the *partitions of unity functor*

$$\mathrm{Part} : \mathbf{EAlg} \to \mathbf{Pos}$$

by sending a morphism of effect algebras $f : E \to F$ to the 'multi-image' function $\mathrm{Part}_f : \mathrm{Part}(E) \to \mathrm{Part}(F)$ defined by

$$\mathrm{Part}_f(P, \eta) := (fP \setminus \{0\}, \eta_f)\,,$$

where $\eta_f : fP \setminus \{0\} \to \mathbb{N} \setminus \{0\}$ is defined by

$$\eta_f(a) := \sum_{b \in f^{-1}(a) \cap P} \eta(b).$$

Since effect algebra morphisms preserve the effect algebra operation and 1, this is indeed a map of partitions of unity.

We need to check that Part_f is order preserving. Thus suppose $g : (P, \eta) \leq (Q, \mu)$. Given $c \in fQ \setminus \{0\}$, let us number the elements in the restricted preimage $f^{-1}(c) \cap Q$ as

$$f^{-1}(c) \cap Q = \{a_1, \ldots, a_k\},$$

so that we have $\sum_{i=1}^{k} \mu(a_i) = \mu_f(c)$. This induces a partition of the set with $\mu_f(c)$ elements into k parts; precisely, define a map

$$\begin{aligned} \phi_c : \{1, \ldots, \mu_f(c)\} &\to \{a_1, \ldots, a_k\} \\ \{1, \ldots, \mu(a_1)\} &\mapsto a_1 \\ \{\mu(a_i) + 1, \ldots, \mu(a_{i+1})\} &\mapsto a_{i+1}. \end{aligned}$$

Now we may define a surjection $\hat{g} : X(\mathrm{Part}_f(P, \eta)) \to X(\mathrm{Part}_f(Q, \mu))$ by letting for each $(j, c) \in X(\mathrm{Part}_f(Q, \mu))$

$$\hat{g}^{-1}(j, c) \coloneqq \left\{(i, f(b)) : (i, b) \in g^{-1}(j \mod \mu(\phi_c(j)),\ \phi_c(j))\right\}.$$

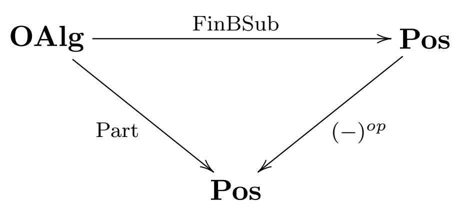
By construction (and by surjectivity of g), the sets $\hat{g}^{-1}(j, c)$ are a disjoint cover of $X(\mathrm{Part}_f(P, \eta))$, so that this indeed defines a surjection $\hat{g}$. Moreover, we have

$$\bigoplus_{(i,a)\in\hat{g}^{-1}(j,c)} a = \bigoplus_{(i,b)\in g^{-1}(j \mod \mu(\phi_c(j)),\ \phi_c(j))} f(b) = f(\phi_c(j)) = c,$$

so that we indeed have $\hat{g} : \mathrm{Part}_f(P, \eta) \leq \mathrm{Part}_f(Q, \mu)$.

The following proposition shows that in the case of orthoalgebras, the poset of partitions of unity is particularly nice: namely, it is obtained by turning the poset of finite Boolean subalgebras upside down.

Proposition 6.5. *Let A be an orthoalgebra. Then* $\mathrm{Part}(A) \simeq \mathrm{FinBSub}(A)^{op}$*. In fact the diagram*



commutes up to a natural isomorphism.

Proof. Since in any orthoalgebra $a \perp a$ implies $a = 0$, any partition of unity is in fact a set; in what follows we therefore make no difference between a set and its corresponding multiset or setoid. Thus for each orthoalgebra A we may define a map

$$\rho_A : \mathrm{Part}(A) \to \mathrm{FinBSub}(A)^{op}$$
$$P \mapsto \left\{\bigoplus S : S \subseteq P\right\}.$$

Since P is a partition of unity, $\rho_A(P)$ contains 1 and is closed under $'$. Since P is a set (rather than a multiset), $\rho_A(P)$ is closed under $\oplus$. It is moreover isomorphic to the powerset algebra of P, hence a Boolean subalgebra of A.

We claim that $P \leq Q$ iff $\rho_A(Q) \subseteq \rho_A(P)$. First suppose that $f : P \leq Q$. Then, given $S \subseteq Q$, we have $f^{-1}S \subseteq P$ and $\bigoplus S = \bigoplus f^{-1}S$, showing

$\rho_A(Q) \subseteq \rho_A(P)$. Conversely, suppose $\rho_A(Q) \subseteq \rho_A(P)$. This means that for every $q \in Q$, there is a subset $f^{-1}(q) \subseteq P$ such that $\bigoplus f^{-1}(q) = q$. Since distinct elements in Q are orthogonal, the subsets $f^{-1}(q)$ are disjoint. Moreover, since $\bigoplus Q = 1$, every element of P appears in some set $f^{-1}(q)$. Thus this defines a surjection $f : P \to Q$ witnessing $P \leq Q$.

Next, given a finite Boolean subalgebra B of A, the set of atoms $\mathrm{At}(B)$ is a partition of unity such that $\rho_A(\mathrm{At}(B)) = B$. Thus we have shown that ρ_A is an order isomorphism for each orthoalgebra A. It remains to see that $\rho : \mathrm{Part} \to \mathrm{FinBSub}(-)^{op}$ with components so defined is a natural transformation. Thus let $f : A \to B$ be an orthoalgebra morphism. We need to show that the diagram

$$\begin{array}{ccc}
\mathrm{Part}(A) & \xrightarrow{\quad \rho_A \quad} & \mathrm{FinBSub}(A)^{op} \\
\Big\downarrow{\scriptstyle \mathrm{Part}_f} & & \Big\downarrow{\scriptstyle f[-]^{op}} \\
\mathrm{Part}(B) & \xrightarrow{\quad \rho_B \quad} & \mathrm{FinBSub}(B)^{op}
\end{array}$$

commutes. That is, we need to show that for all partitions of unity $P \in \mathrm{Part}(A)$ we have

$$f\left\{\bigoplus S : S \subseteq P\right\} = \left\{\bigoplus T : T \subseteq fP \setminus \{0\}\right\}.$$

But this is immediate, as f is an orthoalgebra morphism and hence preserves $\oplus$, and since any subset of P is summable. Thus we are done. □

Before defining partitions of a finite multiset, we briefly discuss the poset of partitions of a finite set. This is a well-known structure in combinatorics, and is usually known as the *finite partition lattice*. This lattice is important both as a special case of the partition poset of a finite multiset, and as the starting point of constructing a setoid quotient, which is defined in the next section as a certain quotient of a finite partition lattice.

Definition 6.6 (Partition of a set)**.** Let X be a finite set. A *partition* P of X is a collection of subsets of X (i.e. $P \subseteq \mathcal{P}X$) such that

- $\varnothing \notin P$,
- if $A, B \in P$, then $A \cap B = \varnothing$,
- $\bigcup P = X$.

Given a finite set X, and partitions P and Q, we define $P \leq Q$ iff

$$\forall A \in P \; \exists B \in Q \text{ with } A \subseteq B.$$

It is clear that $\leq$ is a partial order on the set of partitions, whose least element is the partition containing all the singleton sets of $\mathcal{P}X$, and the greatest element is $\{\{X\}\}$.

Since the sets in a partition are disjoint, the set $B \in Q$ as above with the property that $A \subseteq B$ is unique. Since the sets in a partition cover X, we also have that if $P \leq Q$, then every set in Q is the union of a unique collection of sets from P. This observation shows that we have $P \leq Q$ if and only if there is a surjection $f : P \to Q$ such that for all $B \in Q$ we have

$$\bigcup_{A \in f^{-1}(B)} A = B.$$

We will write $f : P \leq Q$ to indicate that $P \leq Q$ witnessed by a surjection $f : P \to Q$ (compare this to the definition of the partial order of partitions of unity immediately after Definition 6.2).

We now wish to extend the notion of a partition poset to finite multisets. For this, we use the duality between finite multisets and finite MV-algebras and the partitions of unity functor.

Definition 6.7 (Multiset partition functor)**.** Define the *multiset partition functor*

$$\mathrm{PartM} : \mathbf{FinMul}^{op} \to \mathbf{Pos}$$

as $\mathrm{PartM} := \mathrm{Part} \circ \mathcal{P}$, where $\mathcal{P}$ is the powerset functor and Part is the restriction of the partitions of unity functor to finite MV-algebras.

Given a multiset $\mathcal{A}$, we refer to $\mathrm{PartM}(\mathcal{A})$ as the *partition poset* of $\mathcal{A}$, and to the elements of $\mathrm{PartM}(\mathcal{A})$ as *partitions* of $\mathcal{A}$. See Example 6.14 for an example of a multiset partition poset. We remark that defining partitions in this way results in the most 'liberal' definition of a multiset partition. Namely, such a partition is a 'multiset of multisets', that is, repetition is allowed within a multiset in a partition as well as within the partition itself, so that the partition can contain multiple copies of the same multiset. For a discussion of different choices for a partition of a multiset as well as their asymptotic properties, see Bender [4].

Remark 6.8. Definitions 6.6 and 6.7 are consistent: if we view a finite set as a multiset, then its powermultiset is isomorphic to its powerset, so that the partitions of unity poset of the powerset algebra is isomorphic to the poset of partitions in the sense of Definition 6.6. Thus when talking about partitions of a finite set X, we will write $\mathrm{PartM}(X)$ for its partition poset and will treat the elements of $\mathrm{PartM}(X)$ as in Definition 6.6.

Proposition 6.9. *For any finite set X, the poset of partitions* $\mathrm{PartM}(X)$ *is a lattice, which is dually isomorphic to* $\mathrm{FinSub}(\mathcal{P}X)$.

Proof. Since every Boolean algebra is an orthoalgebra, the isomorphism is a special case of Proposition 6.5. Then the fact that the subalgebra poset is a lattice yields that so is the partition poset. □

6.2 Setoid quotients

Setoid quotients and their order-theoretic properties are the main ingredient in the proof of the reconstruction theorem of the next section. We begin by lifting the equivalence relation of a setoid to the finite partition lattice.

Let $(X, \sim)$ be a finite setoid. We extend the equivalence relation $\sim$ to the powerset of X: let $S, T \subseteq X$ be subsets, we say $S \sim T$ iff there is a bijection $f : S \to T$ such that for all $s \in S$ we have $s \sim f(s)$. Similarly, for partitions $P, Q \in \mathrm{PartM}(X)$, we let $P \sim Q$ iff there is a bijection $g : P \to Q$ such that for all $S \in P$ we have $S \sim g(S)$.

Note that we slightly abuse the notation by writing $\sim$ for all the three equivalence relations; there should, however, be no ambiguity whether we mean a relation on elements, sets or sets of sets. We refer to the equivalence relation so obtained as a *setoid equivalence* on $\mathrm{PartM}(X)$. If we want to make the choice of bijection explicit, we will write $g : P \sim Q$ for $P \sim Q$ witnessed by g. Further, we adopt the convention that if $g : P \sim Q$, then for each $S \in P$, the bijection witnessing $S \sim g(S)$ is denoted by g_S.

Definition 6.10 (Setoid quotient)**.** Let $(X, \sim)$ be a finite setoid. Its *setoid quotient* is the quotient poset $\mathrm{PartM}(X)/\sim$. That is, the elements of the poset are the equivalence classes of partitions $[P]$ under the setoid equivalence, and $[P] \leq [Q]$ iff $P' \leq Q'$ for some partitions P' and Q' with $P' \sim P$ and $Q' \sim Q$.

We discuss the possibilities of extending the setoid quotient to a functor in Appendix B.

Proposition 6.11. *Let $\mathrm{PartM}(X)/\sim$ be the setoid quotient of some finite setoid $(X, \sim)$. Then for any partitions P and Q, the following are equivalent*

(1) $[P] \leq [Q]$,

(2) there is a partition Q' such that $Q' \sim Q$ and $P \leq Q'$,

(3) there is a partition P' such that $P' \sim P$ and $P' \leq Q$.

Proof. Clearly, both (2) and (3) imply (1). Thus suppose (1), so that there are $\hat{P}$ and $\hat{Q}$ with $f : \hat{P} \sim P$ and $g : \hat{Q} \sim Q$ such that $h : \hat{P} \leq \hat{Q}$. We define

$$Q' \coloneqq \left\{ \cup f \left[h^{-1} S \right] : S \in \hat{Q} \right\}.$$

Now $S \sim \cup f \left[h^{-1} S \right]$ for each $S \in \hat{Q}$. Moreover, this defines a bijection $\hat{Q} \to Q'$, so that $Q' \sim \hat{Q} \sim Q$. By construction, $P \leq Q'$, showing (2). Similarly, we define

$$P' \coloneqq \left\{ g_{h(T)}[T] : T \in \hat{P} \right\},$$

which satisfies the properties of (3). □

Remark 6.12. There is a perhaps more natural way to construct the setoid quotient. For this, given a finite set X, observe that $\mathrm{PartM}(X)$ is isomorphic to a sublattice of the subgroup lattice of the automorphism group $\mathrm{Aut}(X)$ via

$$\begin{aligned} \Gamma : \mathrm{PartM}(X) &\to \mathrm{SubG}(X) \\ P &\mapsto \Gamma_P \coloneqq \{ g \in \mathrm{Aut}(X) : \forall S \in P, g[S] = S \}, \end{aligned}$$

where $\mathrm{SubG}(X)$ denotes the lattice of subgroups of $\mathrm{Aut}(X)$ ordered by inclusion. It is straightforward to check that each Γ_P is a subgroup, and that for all partitions P and Q, we have $P \leq Q$ iff $\Gamma_P \leq \Gamma_Q$ (Birkhoff [6, p. 97] gives this as an Exercise 8a). Now given a finite setoid $(X, \sim)$, the collection of $\sim$-equivalence classes E defines a partition of X. The corresponding subgroup Γ_E acts on $\Gamma(\mathrm{PartM}(X))$ by conjugation; in other words, viewing Γ_E as a category with one object $\bullet$, we have a functor

$$\begin{aligned} \mathcal{E} : \Gamma_E &\to \mathbf{Pos} \\ \bullet &\mapsto \Gamma(\mathrm{PartM}(X)) \\ g &\mapsto \left(\Gamma_P \mapsto g \Gamma_P g^{-1} \right). \end{aligned}$$

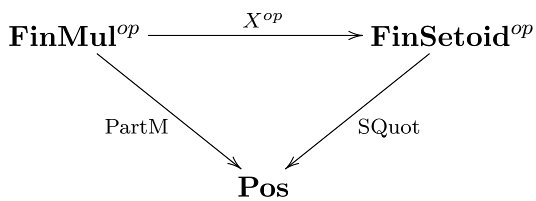
The setoid quotient $\mathrm{PartM}(X)/\!\sim$ is then nothing but the colimit of $\mathcal{E}$.

The following lemma establishes the connection between setoid quotients and multiset partition posets. Since the technicalities of the proof get a little messy, it may be useful to have a look at Example 6.14 while reading the proof.

Lemma 6.13. *The partition poset of any finite multiset is isomorphic to a finite setoid quotient: namely, for every finite multiset $\mathcal{A}$ we have*

$$\mathrm{PartM}(\mathcal{A}) \simeq \mathrm{SQuot}(X(\mathcal{A})).$$

In other words, the diagram



commutes on objects up to an isomorphism.

Note that strictly speaking we have not defined $\mathrm{SQuot} : \mathbf{FinSetoid}^{op} \to \mathbf{Pos}$, however, it is defined on every object of $\mathbf{FinSetoid}^{op}$, since every finite setoid has a setoid quotient.

Proof. Let $\mathcal{A} = (A, \eta)$ be a multiset with cardinality n, and let $(X, \sim) = X(A, \eta)$ be the corresponding setoid. We wish to define an order isomorphism

$$C : \mathrm{PartM}(X)\big/\sim \xrightarrow{\sim} \mathrm{PartM}(\mathcal{A}).$$

Given an equivalence class $[P] \in \mathrm{PartM}(X)\big/\sim$, each set $S \in P$ corresponds to a submultiset $\sigma_S : A \to \mathbb{N}$ by letting

$$\sigma_S(a) := |\{i : (i, a) \in S\}|.$$

We thus define a partition $C([P]) = (C_P, \rho_P)$ of $\mathcal{A}$ by

$$C_P := \{\sigma \in \mathcal{P}\mathcal{A} : \{S \in P : \sigma_S = \sigma\} \neq \varnothing\}$$

$$\begin{aligned} \rho_P : C_P &\to \mathbb{N} \setminus \{0\} \\ \sigma &\mapsto |\{S \in P : \sigma_S = \sigma\}|. \end{aligned}$$

This is well-defined: if $P \sim Q$, then for each submultiset σ, the sets $S \in P$ such that $\sigma_S = \sigma$ are in bijection with sets $T \in Q$ such that $\sigma_T = \sigma$, whence $C([P]) = C([Q])$.

Now suppose $[P] \leq [Q]$, that is, there are partitions P' and Q' with $P' \sim P$ and $Q' \sim Q$ such that $f : P' \leq Q'$. We claim that $C([P]) \leq C([Q])$. By the fact that C is well-defined, it is equivalent to show that $C([P']) \leq$

$C([Q'])$. Observe that there is a bijection from P' to $X(C([P']))$ given by mapping $S \in P'$ to (i, σ_S) (the choice of $C([P'])(\sigma)$ guarantees that there are as many indices i as there are sets S which map to the same submultiset). Thus we obtain a surjection

$$X(C([P'])) \simeq P' \xrightarrow{f} Q' \simeq X(C([Q'])).$$

Given $(j, \tau) \in X(C([Q']))$, we have that (i, σ) is in the preimage of this surjection if and only if S corresponding to (i, σ) is in the preimage of T corresponding to (j, τ). Since the union of all such (disjoint) S is T, we conclude that the sum of all such σ is τ. Thus we indeed have $C([P]) \leq C([Q])$, showing that C is order preserving.

Conversely, suppose P and Q are partitions of X such that $f : C([P]) \leq C([Q])$. Let us fix some bijections $g : P \to X(C([P]))$ and $h : Q \to X(C([Q]))$ which map S to (i, σ_S). Now define

$$Q' \coloneqq \{\cup g^{-1} f^{-1} h(T) : T \in Q\}.$$

Observe that for $T \in Q$:

$$\pi_2 h(T) = \sum_{(i,\sigma) \in f^{-1}h(T)} \sigma = \sum_{S \in g^{-1}f^{-1}h(T)} \pi_2 g(S),$$

where π_2 stands for projection on the second component. Thus, for each $a \in A$, there is a one-to-one correspondence between elements of T whose second component is a and such elements in $\cup g^{-1} f^{-1} h(T)$, in other words, $T \sim \cup g^{-1} f^{-1} h(T)$. We therefore have $Q' \sim Q$. Moreover, given $S \in P$ we have $h^{-1} f g(S) \in Q$ and

$$S \subseteq \cup g^{-1} f^{-1} h h^{-1} f g(S)$$

since f is a surjection. Thus $P \leq Q'$, whence $[P] \leq [Q]$.

Since C both preserves and reflects the order, it is injective. It remains to show surjectivity. Given a partition (D, δ) of $\mathcal{A}$, let us enumerate the subsets in D as $\sigma_1, \ldots, \sigma_k$ such that each subset appears with multiplicity dictated by δ. For each $a \in A$, let

$$\mathfrak{S}_a \coloneqq \{\sigma_i : a \in \operatorname{supp}(\sigma_i)\}.$$

Let us enumerate the sets in $\mathfrak{S}_a$ as $\sigma_1^a, \ldots, \sigma_{z_a}^a$ (this enumeration need not have anything to do with the enumeration of D), so that we have

$$\sum_{i=1}^{z_a} \sigma_i^a(a) = \eta(a)$$

for all $a \in A$. This induces a partition of the set with $\eta(a)$ elements into z_a parts; precisely, define a function

$$\begin{aligned} \phi_a : \{1, \ldots, \eta(a)\} &\to \mathfrak{S}_a \\ 1, \ldots, \sigma_1^a(a) &\mapsto \sigma_1^a \\ \sigma_i^a(a) + 1, \ldots, \sigma_{i+1}^a(a) &\mapsto \sigma_{i+1}^a. \end{aligned}$$

We may, of course, view the codomain of ϕ_a as all of $\{\sigma_1, \ldots, \sigma_k\}$. We use this to define a subset D'_j of X for every $j = 1, \ldots, k$:

$$D'_j := \bigcup_{a \in A} \bigcup_{i \in \phi_a^{-1}(\sigma_j)} \{(i, a)\}.$$

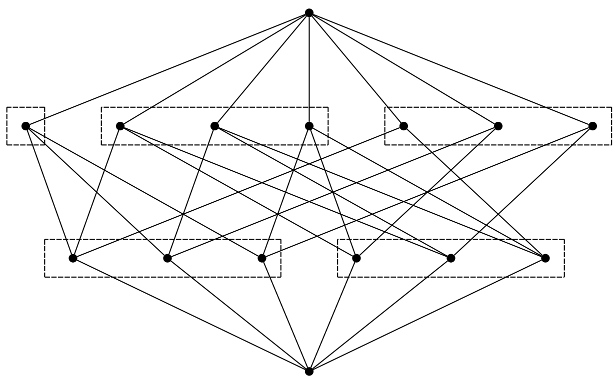
Finally, we obtain a partition of X by taking

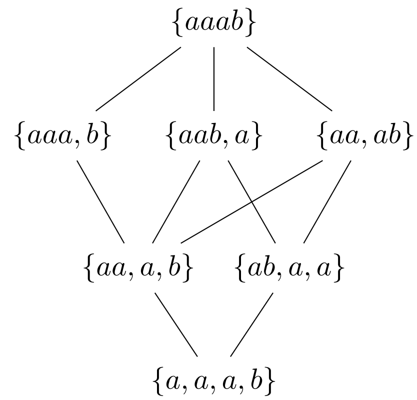
$$D' := \left\{ \left\{ D'_j \right\} : j = 1, \ldots, k \right\}.$$

Thus D' contains 'the same' subsets as D, except that we replace the occurrence of each type a with (i, a), taking care to use each index $1, \ldots, \eta(a)$ exactly once. It is thus an easy consequence of this construction that $C([D']) = (D, \delta)$, showing surjectivity. □

Since the functor X is essentially isomorphic on objects (Proposition 2.25), the above lemma in fact shows that not only is every multiset partition poset isomorphic to a setoid quotient, but also conversely, each setoid quotient comes from a multiset partition poset (up to an isomorphism). Thus setoid quotients and multiset partition posets really are the same thing.

Example 6.14. Consider the finite multiset $\{a, a, a, b\}$, so that the corresponding setoid is $\{(1, a); (2, a); (3, a); (1, b)\}$. In the figure below, we depict on the right the multiset partition poset (we omit the set brackets between subsets), that is, the poset of partitions of unity of the powerset. On the left, we depict the finite partition lattice of the four element set of the setoid. The dashed boxes indicate the equivalence classes under the setoid equivalence. Note that the quotient poset under this equivalence gives a poset isomorphic to the one on the right, as dictated by Lemma 6.13.





We now move on to discussing the order-theoretic properties of setoid quotients. To this end, it will be useful to view them as graded posets.

Definition 6.15 (Graded poset)**.** A poset P is *graded* if there is a function $\rho : P \to \mathbb{N}$ (called the *rank function*) such that

- ρ is strictly order-preserving: if $x, y \in P$ with $x < y$, then $\rho(x) < \rho(y)$, and

- if $x, y \in P$ are such that y covers x, then $\rho(y) = \rho(x) + 1$.

The value of the rank function $\rho(x) \in \mathbb{N}$ at an element $x \in P$ is referred to as the *rank* of x. In a graded poset, there are no infinite descending chains, as this would contradict the fact that $\mathbb{N}$ has the least element. If (P, ρ) is a graded poset and P has the least element 0, we adopt the convention that $\rho(0) = 0$. With this convention, we have the following uniqueness result.

Proposition 6.16. *Let P be a poset with the least element 0. If both $\rho : P \to \mathbb{N}$ and $\phi : P \to \mathbb{N}$ are rank functions such that $\rho(0) = \phi(0) = 0$, then $\rho = \phi$.*

Proof. Let $a \in P$. Since there are no infinitely descending chains, there is a finite chain

$$a = a_n > a_{n-1} > \cdots > a_0 = 0$$

such that a_i covers a_{i-1} for each $i \geq 1$. Since the rank function exists, any such chain has the same length. By definition of the rank function, we must have $\rho(a) = n = \phi(a)$. $\square$

Thus for a graded poset with the least element we may speak of *the* rank of an element without explicitly mentioning the rank function.

Observe that for any finite set X with n elements, the finite partition lattice $\mathrm{PartM}(X)$ is a graded poset, and that the rank of a partition P is just $n - |P|$. A partition P covers Q if and only if P is obtained from Q by taking the union of exactly two sets. Since in the setoid quotient $\mathrm{SQuot}(X, \sim)$ partitions are identified only if they have the same cardinality, partitions in any given equivalence class have the same cardinality. It follows that $\mathrm{SQuot}(X, \sim)$ is a graded poset with the rank of an equivalence class $[P]$ given by $n - |P|$.

Given a finite set X and a pair of distinct elements $x, y \in X$, let us denote by

$$a_{xy} \coloneqq \{\{x, y\}\} \cup \{\{z\} : z \in X \setminus \{x, y\}\}$$

the partition containing the two element set $\{x, y\}$ and no other non-singleton sets. Observe that these are exactly the atoms of $\mathrm{PartM}(X)$ (upon identifying a_{xy} with a_{yx}). Since any setoid quotient only identifies elements of the same rank, it follows that the atoms of $\mathrm{SQuot}(X, \sim)$ are the equivalence classes of the form

$$[a_{xy}].$$

Lemma 6.17. *Let $(X, \sim)$ be a finite setoid with more than two elements. There is only one $\sim$-equivalence class (namely, X) if and only if the setoid quotient* $\mathrm{SQuot}(X, \sim)$ *has the second least element.*

Proof. The setoid quotient has the second least element if and only if it has exactly one atom. In other words, for all elements $x, y, z, w \in X$ such that $x \neq y$ and $z \neq w$ we have

$$a_{xy} \sim a_{zw}.$$

Then the 'only if' direction is clear: if all element are in the same equivalence class, then any two two-element sets are equivalent, hence all atoms are equivalent.

Thus suppose $\mathrm{SQuot}(X, \sim)$ has exactly one atom. Let $x, y, z \in X$ such that x, y and z are all distinct (such a triple exists since X has at least three elements). Then we have $a_{xy} \sim a_{xz} \sim a_{yz}$, whence $\{x, y\} \sim \{x, z\} \sim \{y, z\}$, wherefrom it follows that $x \sim y \sim z$. Repeating this for the triple w, t, x, where $w, t \in X$ are any distinct elements with $w, t \neq x$, we conclude that all of X lies in a single equivalence class. $\square$

Corollary 6.18. *Let $\mathcal{A} = (A, \eta)$ be a finite multiset with cardinality larger than two. Then $|A| = 1$ if and only if the partition poset* $\mathrm{PartM}(\mathcal{A})$ *has the second least element.*

Proof. This is immediate by Lemmas 6.17 and 6.13: $\mathrm{PartM}(\mathcal{A})$ has the second least element iff $\mathrm{SQuot}(X(\mathcal{A}))$ has one iff $X(\mathcal{A})$ has exactly one equivalence class iff $\mathcal{A}$ has exactly one type. □

Lemma 6.19. *Let $(X, \sim)$ be a finite setoid and let E be some $\sim$-equivalence class. Let us denote the partition of X containing E and no other non-singleton sets (or only the singleton sets if E is a singleton) by*

$$P_E := \{E\} \cup \{\{x\} : x \in X \setminus E\}.$$

Then

$$\downarrow [P_E] \simeq \mathrm{SQuot}(E, \sim|_E),$$

where the downset on the left-hand side is taken in $\mathrm{SQuot}(X, \sim)$ *and* $\sim|_E$ *is the restriction of* $\sim$ *to* E*, in other words, the total equivalence relation on* E*.*

Proof. Observe that the equivalence class $[P_E]$ contains exactly one element: namely, P_E. Thus by Proposition 6.11, $[Q] \leq [P_E]$ if and only if $Q \leq P_E$. But this occurs precisely when

$$Q = Q_E \cup \{\{x\} : x \in X \setminus E\},$$

where Q_E is a partition of E. Thus we may define an order isomorphism

$$\begin{aligned} \downarrow [P_E] &\to \mathrm{SQuot}(E, \sim|_E) \\ [Q] &\mapsto [Q_E]. \end{aligned}$$

This is well-defined, as any $g : R \sim Q$ restricts to $R_E \sim Q_E$. Moreover, $[R] \leq [Q]$ iff there is R' with $R' \sim R$ and $R' \leq Q$ iff $R_E \sim Q_E$ iff $[R_E] \leq [Q_E]$. Since surjectivity is immediate, this is indeed an order isomorphism. □

6.3 Finite MV-algebras from partitions

We are finally ready to prove the advertised reconstruction result for finite MV-algebras. In fact we prove it for finite multisets, from which the desired result immediately follows.

Theorem 6.20. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite multisets with cardinality greater than two. Then $\mathcal{A} \simeq \mathcal{B}$ if and only if* $\mathrm{PartM}(\mathcal{A}) \simeq \mathrm{PartM}(\mathcal{B})$*. In other words, the functor*

$$\mathrm{PartM} : \mathbf{FinMul}^{op} \to \mathbf{Pos}$$

is essentially injective on multisets with cardinality greater than two.

Proof. Let $\mathcal{A} = (A, \eta)$ and $\mathcal{B} = (B, \mu)$ be finite multisets such that $\mathrm{PartM}(\mathcal{A}) \simeq \mathrm{PartM}(\mathcal{B})$. By Lemma 6.13, $\mathrm{SQuot}(X(\mathcal{A})) \simeq \mathrm{SQuot}(X(\mathcal{B}))$. Observe that the height of $\mathrm{PartM}(X(\mathcal{B}))$ is equal to the cardinality of $X(\mathcal{B})$ and hence to the cardinality of $\mathcal{B}$. Since the setoid quotient only identifies elements of the same rank, it follows that the height of $\mathrm{SQuot}(X(\mathcal{B}))$ is the cardinality of $\mathcal{B}$. Thus, as an order isomorphism preserves the height of a poset, we conclude that $\mathcal{A}$ and $\mathcal{B}$ have the same cardinality. Let us denote this cardinality by c.

Next, let us count the number of atoms in $\mathrm{SQuot}(X(\mathcal{B}))$. Recall that the atoms are of the form

$$[a_{xy}] = [\{\{x, y\}\} \cup \{\{z\} : z \in X \setminus \{x, y\}\}].$$

Writing $m := |B|$, we claim that the number of equivalence classes of this form is

$$\binom{m}{2} + |\{b \in B : \mu(b) \geq 2\}|\,.$$

Indeed, the first term in the above expression accounts for those equivalence classes where $x \nsim y$, while the second term for those which have $x \sim y$. Let us denote the cardinality of those types in B whose multiplicity is exactly 1 by

$$\beta := |\{b \in B : \mu(b) = 1\}|\,.$$

Then, evaluating the binomial coefficient above, we obtain that the number of atoms in $\mathrm{SQuot}(X(\mathcal{B}))$ is

$$\frac{1}{2}m(m-1) + m - \beta = \frac{1}{2}m(m+1) - \beta.$$

Similarly, writing $n := |A|$ and

$$\alpha := |\{a \in A : \eta(a) = 1\}|\,,$$

the number of atoms in $\mathrm{SQuot}(X(\mathcal{A}))$ is

$$\frac{1}{2}n(n+1) - \alpha.$$

Since an order isomorphism preserves atoms, we obtain an equality

$$n(n+1) + 2\beta = m(m+1) + 2\alpha.$$

Without loss of generality, let us assume that $m \geq n$, say $m = n + k$ for some $k \in \mathbb{N}$. Upon substituting this into the above equality, we obtain

$$2\beta = k^2 + k + 2nk + 2\alpha. \tag{6.21}$$

Since $\beta \leq m = n + k$, we get

$$2n + 2k \geq k^2 + k + 2nk + 2\alpha. \tag{6.22}$$

Observe that if $k > 1$, then $k^2 + k > 2k$ and $2nk > 2n$ (as $n \neq 0$). But this means that the right-hand side of the inequality (6.22) is strictly above $2n + 2k$, contradicting the same inequality. Thus in fact we must have either $k = 1$ or $k = 0$. Suppose for a contradiction that $k = 1$, so that (6.22) becomes

$$2n + 2 \geq 2 + 2n + 2\alpha,$$

which yields $\alpha = 0$. Then equation (6.21) gives $\beta = n + 1 = m$. Thus every type of $\mathcal{B}$ has multiplicity 1, so that $c = m$. On the other hand, every type of $\mathcal{A}$ has multiplicity at least 2, so that $c \geq 2n$. But we have $c = m = n + 1$, which in combination with the previous inequality gives $1 \geq n$, whence $c \leq 2$, which contradicts the assumption that $c > 2$. Thus we must have $k = 0$, which yields $n = m$ and $\alpha = \beta$.

Thus we have shown that $\mathcal{A}$ and $\mathcal{B}$ have the same number of types, and the same number of types whose multiplicity is exactly one.

Let us write the multisets in the isomorphism invariant notation as

$$\mathcal{A} = (n_1, \ldots, n_k) \quad \mathcal{B} = (m_1, \ldots, m_k),$$

where k is the common number of types. Recall that we make the convention that $n_1 \leq \cdots \leq n_k$, and similarly for $\mathcal{B}$. Now suppose $n_k \geq 4$, and let $a \in A$ be the type such that $\eta(a) = n_k$. Let us denote by $[a]$ the equivalence class of $X(\mathcal{A})$ induced by a, and the corresponding partition as in Lemma 6.19 by

$$P_a \coloneqq \{[a]\} \cup \{\{x\} : x \in [a]^c\}.$$

By the same lemma,

$$\downarrow [P_a] \simeq \mathrm{SQuot}([a], \sim),$$

where $\sim$ is the total equivalence relation on $[a]$. Note that the rank of P_a is $n_k - 1 \geq 3$, so this is also the rank of $[P_a]$. Since $\mathrm{SQuot}(X(\mathcal{A})) \simeq \mathrm{SQuot}(X(\mathcal{B}))$, there is an element of rank $n_k - 1$ in $\mathrm{SQuot}(X(\mathcal{B}))$ such that its downset is isomorphic to $\mathrm{SQuot}([a], \sim)$. Let us denote this element by $[P_a]'$.

We now claim that for any partition P in $[P_a]'$, all the elements in non-singleton sets of P are equivalent. We will first argue that there is at least one set in P with cardinality at least three. Thus suppose for a contradiction that all sets in P are either singletons or two-element sets. Let $\{x, y\}; \{z, w\} \in P$

be two-element sets. By Lemma 6.17, SQuot($[a]$, $\sim$) has the second least element, hence so does $\downarrow [P_a]'$. But this yields that $\{x, y\} \sim \{z, w\}$. Thus any two-element sets in P are equivalent, which implies that $\downarrow [P_a]'$ has exactly one coatom. Thus also SQuot($[a]$, $\sim$) has exactly one coatom, which is a contradiction, since by assumption $|[a]| \geq 4$, so that the coatoms

$$\{\{a\}, \{a\}^c\} \quad \text{and} \quad \{\{a, a\}, \{a, a\}^c\}$$

are distinct. Thus there is at least one set $S \in P$ with cardinality at least 3. An argument very similar to the proof of Lemma 6.17 shows that all elements in S are equivalent. Moreover, taking a two-element subset $\{x, y\} \subseteq S$ and any two-element subset $\{z, w\}$ of any other non-singleton set of P yields $\{x, y\} \sim \{z, w\}$, whence $x \sim y \sim z \sim w$. Thus we have indeed shown that all the elements in non-singleton sets of P are equivalent.

Note that the cardinality of the elements in non-singleton sets of P is bounded below by n_k. Indeed, there is a chain (up to setoid equivalence)

$$P_0 < P_1 < \cdots < P_{n_k - 1} = P$$

where P_0 is the bottom element, such that each P_{i+1} covers P_i, so that P_{i+1} is obtained from P_i by taking the union of two sets. Since this is done $n_k - 1$ times, the least number of elements that is used is n_k (it is exactly n_k when P has a unique non-singleton set). It follows that $\mathcal{B}$ has a type whose multiplicity is at least n_k. But it cannot have a type with a higher multiplicity, for reversing this argument would imply that $\mathcal{A}$ has a type with multiplicity strictly larger than n_k. Thus $\mathcal{B}$ has a type with multiplicity exactly n_k.

Repeating the above argument for the type $d \in A$ with $\eta(d) = n_{k-1}$ (assuming $n_{k-1} \geq 4$), we get that $\mathcal{B}$ has a type with multiplicity at least n_{k-1}. As before, let us denote by $[P_d]'$ the image of $[P_d]$ under the order isomorphism. Since a and d are distinct types with multiplicity greater than two, $[P_a]$ and $[P_d]$ are incomparable. Suppose towards a contradiction that there is a partition in $[P_d]'$ with an element whose multiplicity is strictly greater than n_{k-1}. Then there is a type $e \in A$ with multiplicity strictly greater than n_{k-1} and $[P_d] \leq [P_e]$. But the only type of $\mathcal{A}$ whose multiplicity is (possibly) greater than n_{k-1} is a, which then contradicts the fact that $[P_a]$ and $[P_d]$ are incomparable. Thus we conclude that $\mathcal{B}$ has a type whose multiplicity is exactly n_{k-1}. Progressing in this way, we eventually obtain that $\mathcal{A}$ and $\mathcal{B}$ have the same number of types of any multiplicity strictly greater than 3.

Now suppose $\mathcal{A}$ has s_2 types with multiplicity 2 and s_3 types with multiplicity 3. Similarly, let t_2 and t_3 be the number of types with multiplicity

2 and 3 in $\mathcal{B}$. The argument thus far shows that $\mathcal{A}$ and $\mathcal{B}$ have the same cardinality, the same number of types, and the same number of types of any other multiplicity than 2 or 3. Thus the combined number of types of multiplicity 2 and 3 must be the same, as well as the cardinality of the submultiset containing all and only the types with multiplicity 2 or 3. In equations, we must have:

$$s_2 + s_3 = t_2 + t_3$$
$$2s_2 + 3s_3 = 2t_2 + 3t_3.$$

Solving the above system of equations we obtain $s_2 = t_2$ and $s_3 = t_3$. We therefore finally conclude that $\mathcal{A} \simeq \mathcal{B}$. □

Remark 6.23. In light of Remark 6.12, Theorem 6.20 says that if $\mathcal{E} : \Gamma_E \to$ **Pos** and $\mathcal{E}' : \Gamma_{E'} \to$ **Pos** are two group actions obtained from setoids $(X, \sim)$ and $(X', \sim')$ such that their colimits are isomorphic, then in fact $\Gamma_E \simeq \Gamma_{E'}$, or equivalently, $(X, \sim) \simeq (X', \sim')$.

Theorem 6.24. *The functor*

$$\text{Part} : \textbf{FinMV} \to \textbf{Pos}$$

is essentially injective on algebras with more than four elements.

Proof. This is immediate from Theorem 6.20 using that $\mathcal{P}$ is an equivalence of categories (Theorem 3.8). □

Chapter 7

Epilogue

We have seen that Boolean algebras with more than four elements are determined by the poset of their finite subalgebras, and that finite MV-algebras with more than four elements are determined by the poset of their partitions of unity. By Proposition 6.5, the poset of finite subalgebras of any Boolean algebra is dually isomorphic to the poset its partitions of unity. Using this, we may translate the first result to obtain that a Boolean algebra (with more than four elements) is determined by the poset of its partitions of unity. This provides some evidence for the claim that partitions of unity are the right kind of structure to consider if we wish to obtain an analogous result for arbitrary effect algebras.

Question (1) posed in the introduction (reformulated for effect algebras) has thus been answered in a special case, which nonetheless involves unsharp effects: these are elements of an effect algebra which are self-orthogonal. Reformulated in terms of physics, these correspond to effects with probability strictly between 0 and 1. This motivates us to extend the analysis of partition posets to larger classes of effect algebras, as a finite MV-algebra is always contained in the range of a single POVM after all.

7.1 Future work

Inspired by the special cases considered here, we formulate the following conjecture.

Conjecture 7.1. *Let E and F be effect algebras such that* $\mathrm{Part}(E)$ *and* $\mathrm{Part}(F)$ *do not have minimal elements of cardinality less than* 2*. Then*

$E \simeq F$ if and only if $\mathrm{Part}(E) \simeq \mathrm{Part}(F)$*. In other words, the functor*

$$\mathrm{Part} : \mathbf{EAlg} \to \mathbf{Pos}$$

is essentially injective on effect algebras which do not have minimal partitions of unity of cardinality less than 2*.*

Proof strategy: Again, it is the 'if' direction that is non-trivial. Thus suppose $\iota : \mathrm{Part}(E) \to \mathrm{Part}(F)$ is an order isomorphism. We wish to use Corollary 4.10. As was observed in Remark 6.3, for a finite Boolean algebra B, any effect algebra morphism $f : B \to E$ gives rise to a partition of unity $\mathcal{B} := (f(\mathrm{At}(B)), \alpha)$.

Thus we obtain a partition of unity $\iota(\mathcal{B})$ of F which has the same cardinality as $\mathcal{B}$. We now wish to define an effect algebra morphism $B \to F$ by mapping $\mathrm{At}(B)$ to the types of $\iota(\mathcal{B})$ assigning to each type as many atoms as is its multiplicity. However, it is unclear whether there is a canonical choice of such a morphism. To obtain this proof, we would need to identify an order-theoretic invariant corresponding to some 'nice' types of partitions (as modular elements of $\mathrm{FinSub}(B)$ correspond to ideal subalgebras of B in Section 5.1). The proof of Theorem 6.20 (in particular Lemma 6.19) suggests that this potential invariant might have something to do with partitions with only one type and a large enough cardinality. ⌟

The results presented here and the above proof strategy, specifically Corollary 4.10, give some reasons to suspect that the proposed conjecture might be true. In light of Remark 6.3, however, the physical consequence of this would be rather strong. Namely, it would imply that not even all, but just the discrete measurements alone would suffice to determine any physical system. Yet this is not entirely implausible, as we are allowed to refine the discrete measurements arbitrarily many times, thus approximating the continuous operator valued measures. A connection between refinements of measurements and coarse-graining is discussed in Section 2.3 of Busch and Quadt [12].

The next natural step would be to see whether any parts of the proof strategy of Theorem 6.24 extend to the infinite case. Since any MV-algebra is lattice-ordered (Proposition 2.15), it may be useful to look at related results for lattices. For instance, Filippov proved the following.

Theorem 7.2 (Theorem 5.4 in [24])**.** *Let L and L' be complemented lattices where the complements are unique. Then any isomorphism between the lattices of sublattices of L and L' is induced either by an isomorphism or a dual isomorphism between L and L'.*

This, of course, does not immediately apply to MV-algebras: while the orthosupplements are unique, they are not, in general, complements.

Possible other mathematical developments are listed below.

- One prospect for future research is to give a reconstruction procedure for a (finite) MV-algebra M from $\mathrm{Part}(M)$ similar to that of Harding et al. [31]. Currently the theorem just states that if $\mathrm{Part}(N) \simeq \mathrm{Part}(N)$, then $N \simeq M$, while the result of Harding et al. is stronger than this: given a proper orthoalgebra A and a poset P isomorphic to $\mathrm{BSub}(A)$, then the set of directions $\mathrm{Dir}(P)$ has the structure of an orthoalgebra which is isomorphic to A. A reconstruction of this kind should certainly be possible at least for the finite MV-algebras.

- Yet another mathematically interesting question, the answer to which is not even known for Boolean algebras, is what are those morphisms of Boolean algebras such that the functor FinSub is full and faithful when restricted to the wide subcategory of these morphisms. This is strictly stronger than the result proved here, as a full and faithful functor is always essentially injective on objects. Of course, the same question also makes sense for the partitions of unity functor.

- A completely different turn to the theory discussed here could be given by bringing in topology and Stone duality in particular. Namely, the dual version of Theorem 5.24 should be: if the Stone spaces X and X' with more than four clopen sets have isomorphic posets of finite images, then X and X' are homeomorphic. Making precise sense of what the 'posets of finite images' are could be an interesting (possibly easy) project. Perhaps expressing this result in topological terms could also shed some light on the previous question.

Appendix A

Essentially injective and essentially isomorphic functors

This brief section defines essentially injective/isomorphic functors, and is included to serve as a reference to a precise definition.

While essentially surjective functors abound in standard texts on category theory, the related notions of essentially injective and isomorphic functors are quite rare. The author suspects that the reason for this might be the fact that in pure category theory full and faithful functors are more natural and easier to work with (cf. Proposition A.2).

Definition A.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and let $\mathfrak{C}$ be an isomorphism-closed subclass of objects of $\mathcal{C}$. We say that F is

- *essentially surjective on objects* if for every object $D \in \mathcal{O}b(\mathcal{D})$, there is an object $C \in \mathcal{O}b(\mathcal{C})$ such that $F(C) \simeq D$,
- *essentially injective on* $\mathfrak{C}$*-objects* if for any objects $C, B \in \mathfrak{C}$, having $F(C) \simeq F(B)$ implies $C \simeq B$,
- *essentially isomorphic on objects* if there is a function $g : \mathcal{O}b(\mathcal{D}) \to \mathcal{O}b(\mathcal{C})$ such that for all objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$ we have $C \simeq gF(C)$ and $D \simeq Fg(D)$.

It is clear that a functor is essentially isomorphic on objects if and only if it is essentially surjective and essentially injective on all objects in $\mathcal{C}$ (assuming that we have the axiom of choice for classes).

The following proposition gives an important class of examples of essentially injective functors. The proof is a straightforward exercise in basic category theory.

Proposition A.2. *A full and faithful functor is essentially injective on all objects.*

The converse is, of course, not true, even for essentially isomorphic functors, as isomorphism on objects does not give us any handle on the morphisms.

Appendix B

Local isomorphisms of setoids

Here we briefly discuss the possibility of extending the setoid quotient (Definition 6.10) to a functor.

Recall that we defined a morphism of setoids (Definition 2.24) to be an equivalence relation preserving function. Since the setoid quotient is a quotient poset of the finite partition lattice, a natural candidate for a functor from setoids to posets would be the (contravariant) preimage functor, which takes a partition to the set of non-empty preimages of the elements in the partition. This is indeed how we will define SQuot on morphisms. The problem is that, without a restriction on morphisms, the induced mapping is not well-defined. To see this, consider the setoid $\{x_1, x_2, y\}$ with $x_1 \sim x_2$. Define an endomorphism by

$$\begin{aligned} x_1, x_2 &\mapsto y, \\ y &\mapsto x_1. \end{aligned}$$

Then we have equivalent partitions

$$\{x_1 y, x_2\} \sim \{x_2 y, x_1\}$$

under the setoid equivalence, while their preimages

$$\{x_1 x_2 y\} \nsim \{x_1 x_2, y\}$$

are not equivalent.

One option would be to restrict the setoid morphisms to those contained in the essential image of the multisets-to-setoids functor X. This would, however, result in essentially the same functor as PartM. Instead of this, we consider a more restricted, yet more intrinsic to setoids, class of maps.

Definition B.1 (Local isomorphism)**.** A morphism of setoids $f : (S, \sim) \to (T, \sim')$ is a *local isomorphism* if for every equivalence class $[a]$ we have $f([a]) = [f(a)]$, and the restriction

$$f|_{[a]} : [a] \to [f(a)]$$

is a bijection.

We denote the category of finite setoids and local isomorphisms by **LocSetoid**.

Remark B.2. The definition of local isomorphism corresponds to that of a local homeomorphism in topology. If for a setoid $(S, \sim)$ we take the $\sim$-equivalence classes as the basis, this defines a topology on X. Then the local isomorphisms between two setoids are precisely the local homeomorphisms in this topology. It is a well-known fact in sheaf theory that local homeomorphisms into some fixed space are equivalent to sheaves over that space (e.g. Mac Lane and Moerdijk [46, Corollary II.6.3]).

$$\begin{aligned}
\text{SQuot} : \mathbf{LocSetoid}^{op} &\to \mathbf{Pos} \\
(X, \sim) &\mapsto \text{PartM}(X)/\!\sim \\
\Big((X, \sim) \xrightarrow{f} (Y, \sim')\Big) &\mapsto \big(\ \text{PartM}(Y)/\!\sim' \ \to \text{PartM}(X)/\!\sim\ \big) \\
&\qquad [P] \quad \mapsto \big[\{f^{-1}S : S \in P \text{ and } f^{-1}S \neq \varnothing\}\big] \,.
\end{aligned}$$

Let us denote the image of the setoid map f under the functor by SQuot_f. Clearly, $\text{SQuot}_f([P])$ is a partition. We need to check that SQuot_f is well-defined. Thus suppose $g : P' \sim P$. Let us denote

$$f^{-1}P \coloneqq \{f^{-1}S : S \in P \text{ and } f^{-1}S \neq \varnothing\},$$

and similarly for P'. We wish to show that $f^{-1}P' \sim f^{-1}P'$.

Note that if $T \in f^{-1}P'$, then there is an $x \in X$ such that $f(x) \in T$, so that $g_T(f(x)) \in g(T)$. Since $f(x) \sim g_T(f(x))$ and $[f(x)] = f([x])$, there exists a $y \in [x]$ such that $f(y) = g_T(f(x))$, so that $f(y)$ is in $g(T)$. Hence $f^{-1}g(T) \neq \varnothing$ so that $f^{-1}g(T) \in f^{-1}P$.

Further, if $T, T' \in f^{-1}P'$ are such that $f^{-1}T = f^{-1}T'$, then $ff^{-1}T = ff^{-1}T' \subseteq T, T'$, whence $T = T'$, since distinct sets in P' are disjoint. Thus we may define

$$\begin{aligned} g' : f^{-1}P' &\to f^{-1}P \\ f^{-1}T &\mapsto f^{-1}g(T). \end{aligned}$$

This is a bijection: if $f^{-1}g(T) = f^{-1}g(T')$, then $T = T'$ as g is a bijection, and given $f^{-1}S \in f^{-1}P$ we have seen that $f^{-1}g^{-1}(S) \in f^{-1}P'$. Moreover, we have $f^{-1}T \sim f^{-1}g(T)$, since for each equivalence class $[x]$ in $\mathrm{PartM}(X)/\sim$, we have $[x] \cap f^{-1}T \simeq [x] \cap f^{-1}g(T)$, using the fact that f restricts to an isomorphism $[x] \to [f(x)]$. Thus $f^{-1}P' \sim f^{-1}P'$, as we wanted to show.

Appendix C

Finite product theories

Here we introduce a categorical approach to universal algebra by defining finite limit theories and by studying some of their properties. Our main aim is to characterise the representable models of any given finite product theory; these will turn out to correspond precisely to the models given by the 'free algebra functor' on finite sets, i.e. to free and finitely generated models. We mostly focus on the constructions and results needed in Chapter 5. As a result, the treatment here is far from complete, and the reader interested in learning the categorical approach to universal algebra in full detail is referred to Borceux [9, Ch. 3], which is also the main reference here.

Finite product theories are also known as *algebraic theories*, reflecting the fact that the models are algebras, and as *Lawvere theories* after William Lawvere, who introduced them in [43] in 1963 (see also Lawvere [42]).

C.1 Finite product theories and their models

Definition C.1 (Finite product theory)**.** A *finite product theory* $\mathcal{T}$ is a locally small category with a countable set of objects

$$\left\{T^0, T^1, \ldots, T^n, \ldots\right\},$$

where T^n is the n-th power of the fixed object T.

We say that $\mathcal{T}$ is *generated by* T and that T is a *generator* of $\mathcal{T}$.

Definition C.2 (Models)**.** Let $\mathcal{T}$ be a finite product theory. Denote by $\mathrm{Mod}_{\mathcal{T}}$ the full subcategory of the functor category $[\mathcal{T}, \mathbf{Set}]$ consisting of finite product preserving functors. The objects of $\mathrm{Mod}_{\mathcal{T}}$ are referred to as *models* of $\mathcal{T}$ and morphisms (natural transformations) as *homomorphisms* of $\mathcal{T}$-models.

Proposition C.3. *Let $\mathcal{T}$ be an algebraic theory, and let $\alpha : F \to G$ be a morphism in* $\mathrm{Mod}_{\mathcal{T}}$*. Then the diagram*

$$\begin{array}{ccc} F(T^n) & \xrightarrow{\alpha_{T^n}} & G(T^n) \\ \simeq \downarrow & & \downarrow \simeq \\ F(T)^n & \xrightarrow{(\alpha_T)^n} & G(T)^n \end{array}$$

commutes for any $n \in N$, where the vertical maps are canonical isomorphisms induced by the fact that F and G preserve products.

Proof. Composing both maps with the i:th projection $G(T)^n \to G(T)$, we obtain $\alpha_T \circ F\pi_i$ and $G\pi_i \circ \alpha_{T^n}$, where $\pi_i : T^n \to T$ is the i:th projection, which are equal by naturality of α. □

Proposition C.3 says that a morphism of models of a finite product theory is uniquely determined by its T-component.

Since representable functors $y^{T^n} \coloneqq (T^n, -)$ preserve limits, in particular they preserve finite products. Thus representables are always models of any finite product theory.

Definition C.4 (Representable model)**.** We call a model of a finite product theory *representable* if it is (isomorphic to) a representable functor.

Representable models play quite a special role in the category of models of a finite product theory and are in fact our main motivation for taking the categorical perspective on universal algebra. The simple, yet crucial, observation is the following.

Proposition C.5. *Let $\mathcal{T}$ be an algebraic theory, let $G \in \mathrm{Mod}_{\mathcal{T}}$, and let n be a finite set with n elements. Then there is a natural isomorphism*

$$\mathrm{Mod}_{\mathcal{T}}\left(y^{T^n}, G\right) \simeq \mathbf{Set}(n, G(T)),$$

where y^{T^n} denotes the representable functor with representing object T^n.

Proof. Simply observe

$$\mathrm{Mod}_{\mathcal{T}}\left(y^{T^n}, G\right) \simeq G(T^n) \simeq G(T)^n \simeq \mathbf{Set}(n, G(T)),$$

where we used the Yoneda lemma, that G preserves finite products and that **Set** is cartesian closed. □

Note that the above proposition states that there is an adjunction between $\mathrm{Mod}_{\mathcal{T}}$ and finite sets. The left adjoint of this gives the free model on finitely many generators. In Corollary C.22, we will extend this adjunction to **Set**.

C.2 Relation to set-based universal algebra

We now make precise the relation of finite product theories to theories of algebras as considered in universal algebra.

Definition C.6 (Language)**.** A *language of an algebraic theory* is a set $\mathcal{L}$ which is a disjoint union of a countable set `Var`, whose elements are called *variables*, and for each $n \in \mathbb{N}$, of a set $\mathcal{O}_n$ whose elements are called *function symbols of arity* n.

Definition C.7 (Terms)**.** Let $\mathcal{L}$ be a language of an algebraic theory. The set of *$\mathcal{L}$-terms* is inductively generated by:

- each variable is a term;
- if $f \in \mathcal{O}_n$ and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

Definition C.8 (Model in a language)**.** Let $\mathcal{L}$ be a language of an algebraic theory. By an *$\mathcal{L}$-model* we mean a tuple $(M, \{|\text{-}|_n\}_{n\in\mathbb{N}})$, where M is a set and each

$$|\text{-}|_n : \mathcal{O}_n \to \mathbf{Set}(M^n, M)$$

is a function, where $\mathbf{Set}(M^n, M)$ denotes the collection of maps $M^n \to M$. Since each function symbol has a unique arity, we will often denote $|\text{-}|_n$ simply by $|\text{-}|$ and an $\mathcal{L}$-model by $(M, |\text{-}|)$.

A *homomorphism* of $\mathcal{L}$-models $(M, |\text{-}|_M)$ and $(N, |\text{-}|_N)$ is a function $\eta : M \to N$ such that for any $f \in \mathcal{O}_n$ and any elements $x_1, \ldots, x_n$ in M

$$\eta(|f|_M(x_1, \ldots, x_n)) = |f|_N(\eta(x_1), \ldots, \eta(x_n)).$$

Note that $\mathcal{L}$-models and their homomorphisms form a category $\mathcal{L}$-Mod. Each such category comes equipped with an evident forgetful functor $U : \mathcal{L}\text{-Mod} \to \mathbf{Set}$ sending $(M, |\text{-}|)$ to M.

The following is the notion of an algebraic theory in the sense of set-based model theory or universal algebra. We use the word 'presentation' following Borceux [9, p. 3.2] to avoid confusion with finite product theories.

Definition C.9 (Algebraic presentation)**.** Let $\mathcal{L}$ be a language of an algebraic theory. An *axiom* in $\mathcal{L}$ is an expression of the form $t = s$, where s and t are $\mathcal{L}$-terms. An *algebraic presentation* is given by a pair $(\mathcal{L}, A)$, where $\mathcal{L}$ is a language of an algebraic theory, and A is a collection of axioms in $\mathcal{L}$.

Now given an $\mathcal{L}$-model $(M, |\text{-}|)$, any function $\nu : \mathtt{Var} \to M$ (interpretation of the variables) may be inductively extended to an interpretation -^{ν} of the terms:

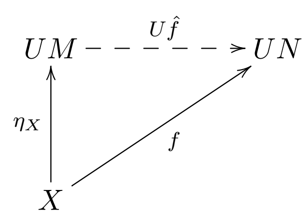
- for $x \in \mathtt{Var}$ let $x^{\nu} \coloneqq \nu(x)$;
- if $f \in \mathcal{O}_n$ and the terms $t_1, \ldots, t_n$ already have interpretations $t_1^{\nu}, \ldots, t_n^{\nu}$, we set
$$f(t_1, \ldots, t_n)^{\nu} \coloneqq |f|(t_1^{\nu}, \ldots, t_n^{\nu}).$$

A model of an algebraic presentation is then just a model that realizes the axioms in the following sense.

Definition C.10 (Model of a presentation)**.** Let $\mathcal{T} = (\mathcal{L}, A)$ be an algebraic presentation. A *model* of $\mathcal{T}$ is an $\mathcal{L}$-model $(M, |\text{ - }|)$ such that for any interpretation $\nu : \mathtt{Var} \to M$ and for any axiom $s = t$ in A, we have that $s^{\nu} = t^{\nu}$ when the interpretation is inductively extended to all $\mathcal{L}$-terms.

Given an algebraic presentation $\mathcal{T} = (\mathcal{L}, A)$, we denote the full subcategory of $\mathcal{L}$-models whose objects are $\mathcal{T}$-models by $\mathrm{Mod}^p_{\mathcal{T}}$. In what follows we shall refer to objects in $\mathrm{Mod}^p_{\mathcal{T}}$ just by the set M, and the collection of functions $|\text{ - }|_M$ is omitted but understood.

Definition C.11 (Free models)**.** Let $\mathcal{T}$ be an algebraic presentation and let X be a set. We say that a model $M \in \mathrm{Mod}^p_{\mathcal{T}}$ is a *free model on* X if there is a function $\eta_X : X \to UM$ such that for any other model $N \in \mathrm{Mod}^p_{\mathcal{T}}$ and any function $f : X \to UN$, there is a unique homomorphism of models $\hat{f} : M \to N$ such that the diagram



commutes.

Using the universal property in the above definition, it is easy to see that if a free model on X exists, then it is unique up to an isomorphism. In fact existence of free models for every set X is equivalent to the forgetful functor $U : \mathrm{Mod}^p_{\mathcal{T}} \to \mathbf{Set}$ having a left adjoint and η being the unit (see e.g. Theorem 2.3.6 in Leinster [44]). We will in fact show in the context of finite product theories that the forgetful functor does have a left adjoint (Corollary C.22) and hence there is a free model on every set.

Definition C.12 (Finitely generated models)**.** Let $\mathcal{T}$ be an algebraic presentation. A model $M \in \mathrm{Mod}^p_{\mathcal{T}}$ is *finitely generated* if it is a quotient of some free model on a finite set.

Here 'quotient' may be taken to mean either an epi or a regular epi, as these turn out to coincide in $\mathrm{Mod}^p_{\mathcal{T}}$ [9, Corollary 3.5.3].

Proposition C.13 (Proposition 3.2.9 in Borceux [9])**.** *Let $\mathcal{T}$ be an algebraic presentation, and let $\mathcal{F}$ be the full subcategory of $\mathrm{Mod}^p_{\mathcal{T}}$ containing those models which are both free and finitely generated. Then $\mathcal{F}^{op}$ has finite products and its skeleton $\mathcal{T}'$ is a finite product theory. Moreover, there is an equivalence of categories*

$$\mathrm{Mod}^p_{\mathcal{T}} \simeq \mathrm{Mod}_{\mathcal{T}'}.$$

Given a finite product theory $\mathcal{T}$ generated by T, we obtain a language $\mathcal{L}$ of an algebraic theory as follows. Fix some countable set $\texttt{Var}$, and for each $n \in \mathbb{N}$ we simply let $\mathcal{O}_n := \mathcal{T}(T^n, T)$. To extend this to an algebraic presentation, we define the following family of functions. Let us denote the set of all $\mathcal{L}$-terms whose variables are amongst $x_1, \ldots, x_m$ by $\texttt{Term}(x_1, \ldots, x_m)$. Given $t \in \texttt{Term}(x_1, \ldots, x_m)$, define a function

$$\phi_t : \texttt{Term}(x_1, \ldots, x_m) \to \mathcal{O}_m$$

by induction as follows:

- $\phi_t(x_i) = \pi_i$, where $\pi_i : T^m \to T$ is the i:th projection, or $\phi_t(x_1) = \mathrm{id}_T$ if $m = 1$;
- if $f \in \mathcal{O}_n$ and $s_1, \ldots, s_n$ are in $\texttt{Term}(x_1, \ldots, x_m)$ such that $\phi_t(s_j)$ has been defined for each $j = 1, \ldots, n$, we let

$$\phi_t(f(s_1, \ldots, s_n)) = f \circ (\phi_t(s_1), \ldots, \phi_t(s_n)).$$

We then define a set of axioms A by stipulating that for each tuple of variables $x_1, \ldots, x_m$ and for each $t \in \texttt{Term}(x_1, \ldots, x_m)$, the axiom

$$t = \phi_t(t)(x_1, \ldots, x_m)$$

is in A. Thus we obtain an algebraic presentation $\mathcal{T}' \coloneqq (\mathcal{L}, A)$.

Proposition C.14 (Proposition 3.3.4 in Borceux [9])**.** *Let a finite product theory $\mathcal{T}$ generated by T be given. Define a corresponding algebraic presentation $\mathcal{T}'$ as above. Then there is an equivalence of categories*

$$\mathrm{Mod}_{\mathcal{T}} \simeq \mathrm{Mod}^{p}_{\mathcal{T}'}.$$

Moreover, the forgetful functor $U : \mathrm{Mod}^{p}_{\mathcal{T}'} \to \mathbf{Set}$ corresponds to the functor $\mathrm{Mod}_{\mathcal{T}} \to \mathbf{Set}$ of evaluation at T.

The key observation in the proof of the above proposition is that the axioms in A are so chosen that models in $\mathrm{Mod}^{p}_{\mathcal{T}'}$ extend to a product preserving functor. In detail, given $M \in \mathrm{Mod}^{p}_{\mathcal{T}'}$, define $F : \mathcal{T} \to \mathbf{Set}$ by $F(T^n) \coloneqq M^n$ on objects, $Ff \coloneqq |f|$ on morphisms $f : T^m \to T$ and as the product map on morphisms $T^m \to T^n$. Now F preserves the identities, as $\pi_i(x_1, \dots, x_m) = x_i$ is an axiom so that $|\pi_i|(x_1, \dots, x_m) = x_i$ for all $x_1, \dots, x_m \in M$ whence their product is the identity. By definition of F on the product maps, it suffices to see that it preserves the composition of the maps that are of the form

$$T^n \xrightarrow{f} T^m \xrightarrow{g} T,$$

to see that F preserves composition of all maps. We thus have to show that

$$|g| \circ (|\pi_1 f|, \dots, |\pi_m f|) = |gf|.$$

To this end, let $\bar{x}$ be an arbitrary n-tuple of distinct variables, so that we have a term

$$t \coloneqq g((\pi_1 f)(\bar{x}), \dots, (\pi_m f)(\bar{x})).$$

Now $\phi_t(t) = gf$, that is,

$$g((\pi_1 f)(\bar{x}), \dots, (\pi_m f)(\bar{x})) = (gf)(\bar{x})$$

is an axiom, whence the desired equality follows.

To summarise, Propositions C.13 and C.14 show that finite product theories and presentations of algebraic theories carry the same information in the sense that from one we can produce the other in such a way that their categories of models are equivalent. From now on we solely focus on the finite product theories.

C.3 Limits and colimits in the categories of models

Let us fix some arbitrary finite product theory $\mathcal{T}$ generated by T.

If a functor $F : \mathcal{C} \to \mathcal{D}$ has a left adjoint, then it is straightforward to show that it preserves limits. The converse, however, requires the assumption that $\mathcal{C}$ is complete as well as some size restrictions. This well-known result in category theory is known as the *General adjoint functor theorem* (or *GAFT*). We state the version of Leinster [44, Theorem 6.3.10]; Borceux gives a different (but equivalent) condition [8, Theorem 3.3.3].

Theorem C.15 (General adjoint functor theorem)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If F has a left adjoint, then it preserves limits. Moreover, in case $\mathcal{C}$ is complete and locally small, and for each $D \in \mathcal{D}$ the comma category $(D \Rightarrow F)$ has a weakly initial set, F preserves limits* if and only if *it has a left adjoint.*

Definition C.16 (Replete subcategory)**.** A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is *replete* if $\mathcal{D}$ is closed under isomorphisms. Precisely, if $D \in \mathcal{D}$ and $C \in \mathcal{C}$ are isomorphic, then also $C \in \mathcal{D}$.

Note that $\mathrm{Mod}_{\mathcal{T}}$ is a replete subcategory of $[\mathcal{T}, \mathbf{Set}]$.

Definition C.17 (Reflective subcategory)**.** A replete subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is *reflective* if the inclusion functor $i : \mathcal{D} \to \mathcal{C}$ has a left adjoint. In such case the left adjoint is called the *reflection* of i.

A reflective subcategory inherits (co)completeness, as indicated by the following proposition.

Proposition C.18 (Propositions 3.5.3 and 3.5.4 [8])**.** *Let $\mathcal{C}$ be a (finitely) (co)complete category. Then any reflective subcategory of $\mathcal{C}$ is (finitely) (co)complete.*

Proposition C.19. *The category $\mathrm{Mod}_{\mathcal{T}}$ is complete, and the limits are computed pointwise. Moreover, the forgetful functor $U : \mathrm{Mod}_{\mathcal{T}} \to \mathbf{Set}$ preserves limits.*

Proof. Let $D : \mathbf{I} \to \mathrm{Mod}_{\mathcal{T}}$ be a diagram and let $i : \mathrm{Mod}_{\mathcal{T}} \to [\mathcal{T}, \mathbf{Set}]$ be the inclusion functor. Then the diagram $\hat{D} := i \circ D$ has a limit $\varprojlim_{\mathbf{I}} \hat{D}$ in $[\mathcal{T}, \mathbf{Set}]$, as functor categories are complete. Moreover, $\varprojlim_{\mathbf{I}} \hat{D}$ is computed pointwise. We now wish to show that $\varprojlim_{\mathbf{I}} \hat{D} \in \mathrm{Mod}_{\mathcal{T}}$, in other words, that

$\lim\limits_{\leftarrow \mathbf{I}} \hat{D}$ preserves finite products. Thus let $G : \mathbf{J} \to \mathcal{T}$ be a diagram, where $\mathbf{J}$ is a finite discrete category, such that $\lim\limits_{\leftarrow \mathbf{J}} G$ exists. We wish to show that

$$\lim_{\leftarrow \mathbf{I}} \hat{D} \left(\lim_{\leftarrow \mathbf{J}} G \right) = \lim_{\leftarrow \mathbf{J}} \left(\lim_{\leftarrow \mathbf{I}} \hat{D} \circ G \right).$$

Thus we compute

$$\begin{aligned} \lim_{\leftarrow \mathbf{I}} \hat{D} \left(\lim_{\leftarrow \mathbf{J}} G \right) &= \lim_{\leftarrow \mathbf{I}} \left(D(\text{-}) \left(\lim_{\leftarrow \mathbf{J}} G \right) \right) \\ &= \lim_{\leftarrow \mathbf{I}} \left(\lim_{\leftarrow \mathbf{J}} D(\text{-})(G(\text{-})) \right) \\ &= \lim_{\leftarrow \mathbf{J}} \left(\lim_{\leftarrow \mathbf{I}} D(\text{-})(G(\text{-})) \right) \\ &= \lim_{\leftarrow \mathbf{J}} \left(\lim_{\leftarrow \mathbf{I}} \hat{D} \circ G \right), \end{aligned}$$

where in the first and last equalities we used that limits in $[\mathcal{T}, \mathbf{Set}]$ are computed pointwise, in the second equality that $D(i)$ preserves finite products for each $i \in \mathbf{I}$, and in the third equality that limits commute with limits.

Next we want to show that $\lim\limits_{\leftarrow \mathbf{I}} \hat{D}$ verifies the universal property of the limit of D. Given a functor $F : \mathcal{T} \to \mathbf{Set}$, let us denote the constant diagram on F by $\Delta_F : \mathbf{I} \to [\mathcal{T}, \mathbf{Set}]$. Observe:

$$\begin{aligned} \mathrm{Mod}_{\mathcal{T}} \left(F, \lim_{\leftarrow \mathbf{I}} \hat{D} \right) &= [\mathcal{T}, \mathbf{Set}] \left(F, \lim_{\leftarrow \mathbf{I}} \hat{D} \right) \\ &= [\mathbf{I}, [\mathcal{T}, \mathbf{Set}]](\Delta_F, \hat{D}) \\ &= [\mathbf{I}, \mathrm{Mod}_{\mathcal{T}}] (\Delta_F, D), \end{aligned}$$

where the first and the last equalities hold since $\mathrm{Mod}_{\mathcal{T}}$ is a full subcategory of $[\mathcal{T}, \mathbf{Set}]$ and the middle equality is the definition of a limit. Thus we conclude that $\lim\limits_{\leftarrow \mathbf{I}} D$ exists and is equal to $\lim\limits_{\leftarrow \mathbf{I}} \hat{D}$. Since $\lim\limits_{\leftarrow \mathbf{I}} \hat{D}$ is computed pointwise, so is $\lim\limits_{\leftarrow \mathbf{I}} D$.

Since limits are computed pointwise and the forgetful functor is given by evaluation at T, it is immediate that it preserves limits. □

Theorem C.20. $\mathrm{Mod}_{\mathcal{T}}$ *is a reflective subcategory of* $[\mathcal{T}, \mathbf{Set}]$.

Proof. We wish to apply the General adjoint functor theorem C.15 to the inclusion functor $i : \mathrm{Mod}_{\mathcal{T}} \to [\mathcal{T}, \mathbf{Set}]$.

By Proposition C.19, $\mathrm{Mod}_{\mathcal{T}}$ is complete. The same proposition shows that i preserves limits. Moreover, Proposition C.3 shows that $\mathrm{Mod}_{\mathcal{T}}$ is locally small, as any natural transformation in $\mathrm{Mod}_{\mathcal{T}}$ is determined by its

T-component, and all possible such components form a set. Thus in order to conclude using GAFT, it remains to show that for each functor $P : \mathcal{T} \to \mathbf{Set}$, the comma category $(P \Rightarrow \mathrm{Mod}_{\mathcal{T}})$ has a weakly initial set, which we now do.

First observe that the comma category $(P \Rightarrow \mathrm{Mod}_{\mathcal{T}})$ is just the full subcategory of the coslice category $P/[\mathcal{T}, \mathbf{Set}]$ on those objects whose functor part lies in $\mathrm{Mod}_{\mathcal{T}}$. Given any such P, we define a collection of objects in this comma category as follows:

$$\mathfrak{P} \coloneqq \left\{ (H, \phi) : H(T) \subseteq \coprod_{n\in\mathbb{N}} \mathcal{T}(T^n, T) \times \mathcal{P}(T)^n \text{ and } \forall n \in \mathbb{N}, H(T^n) = H(T)^n \right\}.$$

We first argue that $\mathfrak{P}$ is a set. There is a set of possible choices for $H(T)$, namely, the powerset of the set in the above condition. Once $H(T)$ is fixed, this defines H on the objects of $\mathcal{T}$. On the morphism $f : T^n \to T^m$, there is a set of choices for the function $Hf : H(T^n) \to H(T^m)$. Thus the cardinality of possible definitions of H on morphisms is bounded by the cardinality of $\coprod_{n,m\in\mathbb{N}}(\mathcal{T}(T^n, T^m) \times \mathbf{Set}(H(T^n), H(T^m)))$. Combining this with the fact that there is a set of choices for $H(T)$, we conclude that there is a set of choices for H. Now suppose H is fixed. Then there is a set of possible choices for the natural transformation $\phi : P \to H$, whose cardinality is simply bounded by that of $\coprod_{n\in\mathbb{N}} \mathbf{Set}(P(T^n), H(T^n))$. Thus we conclude that $\mathfrak{P}$ is indeed a set.

To show that $\mathfrak{P}$ is weakly initial, let $\beta : P \to F$ be an object in the comma category. Since every finite product preserving functor is naturally isomorphic to one that preserves the products strictly, we may without loss of generality suppose that $F(T^n) = F(T)^n$. Write $X \coloneqq \beta_T(P(T))$, and let Y be the subset of $F(T)$ consisting of those elements that are obtained by applying functions in the image of F to a tuple of elements in X. Precisely, we let

$$Y \coloneqq \{y \in F(T) : \exists n \in \mathbb{N}(\exists g : T^n \to T, \exists x_1, \ldots, x_n \in X(y = Fg(x_1, \ldots, x_n)))\}.$$

Note that Y is closed under the application of functions in the F-image. Indeed, if $f : T^n \to T$, and $y_1, \ldots, y_n$ are in Y, so that $y_i = Fg_i\left(x_1^{(i)}, \ldots, x_{k_i}^{(i)}\right)$ for each $i = 1, \ldots, n$, then we let $k \coloneqq \sum k_i$ and define $\delta_i : T^k \to T^{k_i}$ to be the relevant projection (precisely $\pi_j\delta_i = \pi_{k_i \cdot j}$, where we slightly abuse the notation by not labelling the projections for distinct products). Now define $g : T^k \to T^n$ as the product map with components $\pi_i g = g_i \delta_i$. With this notation, we have

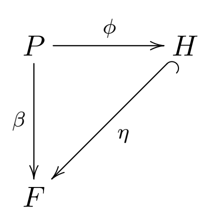
$$\begin{aligned} Ff(y_1, \dots, y_n) &= Ff\left(Fg_1\left(x_1^{(1)}, \dots, x_{k_1}^{(1)}\right), \dots, Fg_n\left(x_1^{(n)}, \dots, x_{k_n}^{(n)}\right)\right) \\ &= (Ff \circ (F(g_1\delta_1), \dots, F(g_n\delta_n)))\left(x_1^{(1)}, \dots, x_{k_n}^{(n)}\right) \\ &= F(fg)\left(x_1^{(1)}, \dots, x_{k_n}^{(n)}\right), \end{aligned}$$

which is in Y.

We can thus define $H \in \mathrm{Mod}_{\mathcal{T}}$ by $H(T^n) := Y^n$, and on morphisms $f : T^n \to T$ we let $Hf := (Ff)|_{Y^n}$, which uniquely extends to all morphisms by requiring that H preserves finite products. In order to define a natural transformation $\phi : P \to H$, note that

$$\pi_i^{FT} \beta_{T^n} = F\pi_i \circ \beta_{T^n} = \beta_T \circ P\pi_i$$

since F strictly preserves products and by naturality of β. Thus each component in the tuple lying in the image of β_{T^n} is in fact in the image of β_T, hence in X, so that the image of β_{T^n} is contained in $X^n \subseteq Y^n$. We thus simply put $\phi_{T^n}(a) := \beta_{T^n}(a)$ for each $a \in P(T^n)$. Since β is natural, and H on morphisms is the restriction of F, we get that ϕ is natural. We now have a commutative diagram,



where η is the inclusion natural transformation.

As currently defined, H need not be in $\mathfrak{P}$. We conclude the argument by showing that there is a functor in $\mathfrak{P}$ that is naturally isomorphic to H. Observe that the cardinality of Y is bounded by that of $\coprod_{n\in\mathbb{N}} \mathcal{T}(T^n, T) \times X^n$. Since the cardinality of X is bounded by $P(T)$, the cardinality of Y is bounded by that of $\coprod_{n\in\mathbb{N}} \mathcal{T}(T^n, T) \times \mathcal{P}(T)^n$. Thus we may choose a set $Y' \subseteq \coprod_{n\in\mathbb{N}} \mathcal{T}(T^n, T) \times \mathcal{P}(T)^n$ such that there is an isomorphism $h : Y \xrightarrow{\sim} Y'$. Now define $H' \in \mathrm{Mod}_{\mathcal{T}}$ by $H'(T^n) := (Y')^n$, and on morphisms $f : T^n \to T^m$ by $H'f := h^m Hf \circ h^{-n}$, where $h^{-n} := (h^n)^{-1} = (h^{-1})^n$. Now there is an evident family of isomorphisms $\alpha : H \to H'$ with components $\alpha_{T^n} := h^n$. It is an immediate consequence of the definitions that α is a natural isomorphism.

Now $H' \in \mathfrak{P}$, and since β factors through H, it also does through H'. Thus $\mathfrak{P}$ is indeed a weakly initial set for $\mathrm{Mod}_{\mathcal{T}}$. $\square$

Corollary C.21. $\mathrm{Mod}_{\mathcal{T}}$ *is cocomplete.*

Proof. Since $[\mathcal{T}, \mathbf{Set}]$ is cocomplete, this follows by Proposition C.18 and Theorem C.20. $\square$

Corollary C.22. *The forgetful functor* $U : \mathrm{Mod}_{\mathcal{T}} \to \mathbf{Set}$ *has a left adjoint.*

Proof. Let $G \in \mathrm{Mod}_{\mathcal{T}}$ and let $S \in \mathbf{Set}$. We compute

$$\begin{aligned}
\mathbf{Set}(S, G(T)) &\simeq \mathbf{Set}\left(\coprod_{x\in S}\{x\}, G(T)\right) \\
&\simeq \prod_{x\in S} \mathbf{Set}(1, G(T)) \\
&\simeq \prod_{x\in S} \mathrm{Mod}_{\mathcal{T}}\left(y^{T^1}, G\right) \\
&\simeq \mathrm{Mod}_{\mathcal{T}}\left(\coprod_{x\in S} y^{T^1}, G\right),
\end{aligned}$$

where in the third equality we used Proposition C.5 and in the last one the fact that $\mathrm{Mod}_{\mathcal{T}}$ is cocomplete. We conclude that the functor $F : \mathbf{Set} \to \mathrm{Mod}_{\mathcal{T}}$ given by $F(S) \coloneqq \coprod_{x\in S} y^{T^1}$ is the sought-after left adjoint. $\square$

We call the left adjoint F to the forgetful functor the *free model functor*.

C.4 Representable models and the Yoneda lemma

Recall that we defined free models (Definition C.11) and finitely generated models (Definition C.12) for an algebraic presentation. In light of the fact that any category of models of a finite product theory is equivalent to a category of models an algebraic presentation (Proposition C.14), the definitions translate to models of finite product theories almost word to word: we just need to replace every occurrence of 'algebraic presentation' with 'finite product theory', $\mathrm{Mod}^p_{\mathcal{T}}$ with $\mathrm{Mod}_{\mathcal{T}}$ and view the forgetful functor U as evaluation at T. The following proposition is an immediate consequence of existence of the free-forgetful adjunction $F \dashv U$.

Proposition C.23. *Let X be a set. Then the free model on X exists and is given by $F(X)$ and $\eta_X : X \to UF(X)$, where η is the unit of the adjunction $F \dashv U$.*

Proof. As remarked after Definition C.11, existence of free models for every set is equivalent to the forgetful functor having a left adjoint, which we showed in Corollary C.22. □

Proposition C.24. *The representable model y^{T^n} is (isomorphic to) the free model $F(n)$ for every $n \in \mathbb{N}$.*

Proof. This is essentially a rephrasing of Proposition C.5; for any model G and $n \in \mathbb{N}$ we have

$$\mathrm{Mod}_{\mathcal{T}}\left(y^{T^n}, G\right) \simeq \mathbf{Set}(n, G(T)) \simeq \mathbf{Set}(n, UG) \simeq \mathrm{Mod}_{\mathcal{T}}(F(n), G),$$

whence $y^{T^n} \simeq F(n)$. □

The following intuitively plausible fact has quite a non-trivial proof, which uses exactness properties of $\mathrm{Mod}_{\mathcal{T}}$ not covered here. The reader is referred to sections 3.5, 3.7 and 3.8 of Borceux [9] for the details.

Proposition C.25 (Lemma 3.8.4, [9])**.** *The free and finitely generated models of $\mathcal{T}$ are precisely (up to isomorphism) the free models $F(n)$ on finite sets n.*

Corollary C.26. *The representable models of $\mathcal{T}$ are precisely (up to isomorphism) the free and finitely generated ones.*

Proof. Immediate from propositions C.24 and C.25. □

Let us denote the full subcategory of $\mathrm{Mod}_{\mathcal{T}}$ on free and finitely generated models by $\mathrm{ffgMod}_{\mathcal{T}}$. We then have the following characterisation (cf. Proposition C.13).

Corollary C.27. *Every finite product theory $\mathcal{T}$ is equivalent to $\mathrm{ffgMod}_{\mathcal{T}}^{op}$.*

Proof. We know that the Yoneda embedding $y : \mathcal{T}^{op} \to [\mathcal{T}, \mathbf{Set}]$ is full and faithful. Corollary C.26 shows that it is also essentially surjective on objects when restricted to $\mathrm{ffgMod}_{\mathcal{T}}$. □

We are now ready to formulate 'the Yoneda lemma for algebras'. Particularly, we refer to the formulation of the Yoneda lemma stating that two functors $X, Y : \mathcal{C} \to \mathbf{Set}$, where $\mathcal{C}$ is a small category, are isomorphic if and only if the homsets $\mathcal{C}(y^A, X)$ and $\mathcal{C}(y^A, Y)$ are isomorphic naturally in $A \in \mathcal{C}$. For this, we establish a piece of (very standard) terminology.

Let $G \in \mathrm{Mod}_{\mathcal{T}}$ be a model of a finite product theory. We then have a functor

$$\begin{aligned} \mathrm{Mod}_{\mathcal{T}}(-, G) : \mathrm{ffgMod}^{op} &\to \mathbf{Set} \\ F(n) &\mapsto \mathrm{Mod}_{\mathcal{T}}(F(n), G) \\ (F(m) \xrightarrow{g} F(n)) &\mapsto - \circ g, \end{aligned}$$

Where we use the fact that the free and finitely generated models are precisely the F-image of finite sets. For models G and H, we say that the hom-sets $\mathrm{Mod}_{\mathcal{T}}(F(n), G)$ and $\mathrm{Mod}_{\mathcal{T}}(F(n), H)$ are *isomorphic naturally in* n if there is a natural isomorphism

$$\alpha : \mathrm{Mod}_{\mathcal{T}}(-, G) \to \mathrm{Mod}_{\mathcal{T}}(-, H).$$

Explicitly, this means there is an $\mathbb{N}$-indexed family of isomorphisms

$$\alpha_n : \mathrm{Mod}_{\mathcal{T}}(F(n), G) \to \mathrm{Mod}_{\mathcal{T}}(F(n), H)$$

in **Set** such that for any homomorphism of models $g : F(m) \to F(n)$ the diagram

$$\begin{array}{ccc} \mathrm{Mod}_{\mathcal{T}}(F(n), G) & \xrightarrow{\alpha_n} & \mathrm{Mod}_{\mathcal{T}}(F(n), H) \\ \downarrow{\scriptstyle -\circ g} & & \downarrow{\scriptstyle -\circ g} \\ \mathrm{Mod}_{\mathcal{T}}(F(m), G) & \xrightarrow{\alpha_m} & \mathrm{Mod}_{\mathcal{T}}(F(m), H) \end{array} \tag{C.28}$$

commutes.

Proposition C.29 (Yoneda lemma for algebras)**.** *Let G and H be models of a finite product theory $\mathcal{T}$. Then $G \simeq H$ if and only if*

$$\mathrm{Mod}_{\mathcal{T}}(F(n), G) \simeq \mathrm{Mod}_{\mathcal{T}}(F(n), H)$$

naturally in n.

Proof. If we view

$$\mathrm{Mod}_{\mathcal{T}}\left(y^{(-)}, G\right) : \mathcal{T} \to \mathbf{Set}$$

as the composition $\mathrm{Mod}_{\mathcal{T}}(-, G) \circ y$, where $y : \mathcal{T}^{op} \to [\mathcal{T}, \mathbf{Set}]$ is the Yoneda embedding, then the Yoneda lemma says that $G \simeq H$ if and only if there is a natural isomorphism

$$\mathrm{Mod}_{\mathcal{T}}\left(y^{(-)}, G\right) \xrightarrow{\sim} \mathrm{Mod}_{\mathcal{T}}\left(y^{(-)}, H\right).$$

By Proposition C.24, the above natural isomorphism exists if and only if there is a natural isomorphism

$$\mathrm{Mod}_{\mathcal{T}}(-, G) \xrightarrow{\sim} \mathrm{Mod}_{\mathcal{T}}(-, H),$$

which thus concludes the proof. □

Using the universal property of free algebras, we can rewrite the above theorem as Corollary C.32. To this end, let us denote the natural bijection

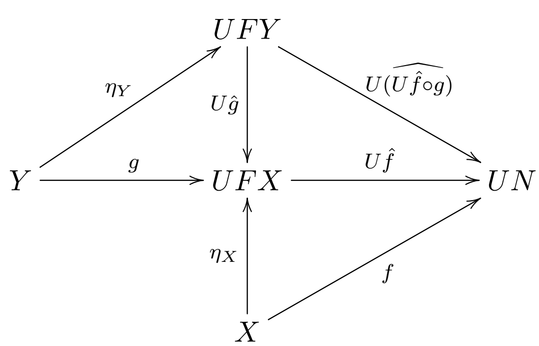
$$\mathbf{Set}(X, UN) \xrightarrow{\sim} \mathrm{Mod}_{\mathcal{T}}(FX, N)$$

by $\widehat{-}$. Equivalently, this is the bijection induced by the universal property of Definition C.11.

Lemma C.30. *Naturality of $\widehat{-}$, or equivalently, the universal property of Definition C.11, amounts to having for all functions $f : X \to UN$ and $g : Y \to UFX$ the identity*

$$\widehat{(U\hat{f} \circ g)} = \hat{f}\hat{g}. \tag{C.31}$$

Proof. We have that in the diagram



the bottom triangle and the top left triangle commute, so that by uniqueness in Definition C.11 also the top right triangle commutes. By faithfulness of U, we obtain the desired identity (C.31).

Conversely, assuming there is a collection of bijections $\widehat{-} : \mathbf{Set}(X, UN) \xrightarrow{\sim} \mathrm{Mod}_{\mathcal{T}}(FX, N)$ such that identity (C.31) holds, it is straightforward to show that the universal property of Definition C.11 holds. Since we will not use this fact we omit the details. □

Corollary C.32. *Let G and H be models of a finite product theory $\mathcal{T}$. Then $G \simeq H$ if and only if there is an $\mathbb{N}$-indexed family of isomorphisms*

$$\alpha_n : \mathbf{Set}(n, GT) \to \mathbf{Set}(n, HT)$$

in **Set** *such that for any function $f : m \to F(n)(T)$, the diagram below commutes:*

$$\begin{array}{ccc} \mathbf{Set}(n, GT) & \xrightarrow{\alpha_n} & \mathbf{Set}(n, HT) \\ \downarrow{\scriptstyle \widehat{-}\circ f} & & \downarrow{\scriptstyle \widehat{-}\circ g} \\ \mathbf{Set}(m, HT) & \xrightarrow{\alpha_m} & \mathbf{Set}(m, HT) \end{array} \tag{C.33}$$

where $\widehat{-} \circ f : \mathbf{Set}(n, GT) \to \mathbf{Set}(m, GT)$ is given by $s \mapsto U\hat{s} \circ f$.

Proof. Given a family of isomorphisms

$$\alpha_n : \mathbf{Set}(n, G) \to \mathbf{Set}(n, H)$$

as in the statement of the Corollary, define a family of maps

$$\hat{\alpha}_n : \mathrm{Mod}_{\mathcal{T}}(F(n), G) \to \mathrm{Mod}_{\mathcal{T}}(F(n), H)$$

by $\hat{\alpha}_n(\hat{s}) := \widehat{\alpha_n(s)}$. Since $\widehat{-} : \mathbf{Set}(n, G) \to \mathrm{Mod}_{\mathcal{T}}(F(n), D)$ is a bijection for any model D, this defines the map uniquely, and the resulting map is moreover a bijection.

Conversely, given a family of isomorphisms

$$\hat{\alpha}_n : \mathrm{Mod}_{\mathcal{T}}(F(n), G) \to \mathrm{Mod}_{\mathcal{T}}(F(n), H),$$

natural in n, define a family of maps

$$\alpha_n : \mathbf{Set}(n, G) \to \mathbf{Set}(n, H)$$

by $\widehat{\alpha_n(s)} := \hat{\alpha}_n(\hat{s})$, which again defines a unique isomorphism since $\widehat{-}$ is a bijection.

Now commutativity of the squares (C.28) and (C.33) is a matter of a diagram chase, using commutativity of the other square as well as repeatedly applying Lemma C.30. $\square$

Thus an algebra (seen as a model of a finite product theory) is completely determined by particularly 'nice' subalgebras, namely by the finitely generated ones, which patch together in a natural way. This result could be seen as fitting the general pattern of local data determining the global entity: starting from a set of finite elements, we generate all those elements that are accessible 'locally', i.e. by applying the operations in the language. The above corollary tells us that knowing all such local information results in knowing the full, or 'global', algebra.

C.5 Example: Boolean algebras

Let us write $\mathcal{B}$ for the finite product theory of Boolean algebras, G for its generating object and **ffgBA** for the full subcategory of **BAlg** consisting of free and finitely generated Boolean algebras. Thus we have an equivalence of categories $\mathrm{Mod}_{\mathcal{B}} \simeq \mathbf{BAlg}$, and an embedding (full and faithful functor) $\mathbf{BAlg} \hookrightarrow [\mathcal{B}, \mathbf{Set}]$.

Lemma C.34. *The free and finitely generated Boolean algebras are precisely the finite Boolean algebras with 2^{2^n} elements for some natural number n.*

Proof. By Proposition C.25, free and finitely generated models are precisely of the form $F(n)$ for some $n \in \mathbb{N}$. We claim that the left adjoint F to the forgetful functor on the finite sets is given by the double powerset functor $\mathcal{PP}$, and that the unit $\eta_n : n \to \mathcal{PP}(n)$ on the finite sets is given by

$$k \mapsto \mathcal{U}_k := \{U \subseteq n : k \in U\}.$$

We first observe that the atoms of $\mathcal{PP}(n)$ (i.e. singletons $\{U\}$ for some $U \subseteq n$) can be written as

$$\{U\} = \bigcap_{k \in U} \mathcal{U}_k \cap \bigcap_{j \in U^c} \mathcal{U}_j^c.$$

It is clear that $U \in \mathcal{U}_k$ and $U \in \mathcal{U}_j^c$ for each $k \in U$ and $j \in U^c$. Conversely, suppose $V \in \mathcal{U}_k$ and $V \in \mathcal{U}_j^c$ for each $k \in U$ and $j \in U^c$. But then $k \in U$ implies $k \in V$, so that $U \subseteq V$, and $j \in U^c$ implies $j \in V^c$, so that $V \subseteq U$. Thus $V = U$. Any element $X \in \mathcal{PP}(n)$ can therefore be written as

$$X = \bigcup_{U \in X} \left(\bigcap_{k \in U} \mathcal{U}_k \cap \bigcap_{j \in U^c} \mathcal{U}_j^c \right), \tag{C.35}$$

where each union and intersection is finite.

By uniqueness of free models, it suffices to show that $\mathcal{PP}(n)$ and η_n verify the universal property of Definition C.11. Thus let B be a Boolean algebra and let $f : n \to UB$ be a function. Commutativity of the diagram in Definition C.11 now amounts to $\hat{f}$ sending each $\mathcal{U}_k$ to $f(k)$. We claim that this condition uniquely defines a Boolean algebra morphism $\hat{f} : \mathcal{PP}(n) \to B$. Precisely, we define:

- $\hat{f}(\varnothing) := 0$;
- for each $k \in n$, let $\hat{f}(\mathcal{U}_k) := f(k)$ and $\hat{f}(\mathcal{U}_k^c) := f(k)'$;

- if $X, Y \in \mathcal{PP}(n)$ are such that $\hat{f}(X)$ and $\hat{f}(Y)$ are defined, then let $\hat{f}(X \cap Y) := \hat{f}(X) \wedge \hat{f}(Y)$ and $\hat{f}(X \cup Y) := \hat{f}(X) \vee \hat{f}(Y)$.

Note that the second clause is well-defined, as $\mathcal{U}_k = \mathcal{U}_j$ if and only if $k = j$. By equation (C.35), the three clauses define $\hat{f}$ on all of $\mathcal{PP}(n)$. By the third clause, $\hat{f}$ preserves meets and joins, and a straightforward inductive argument together with the de Morgan laws shows that $\hat{f}(X^c) = \hat{f}(X)'$. Thus $\hat{f}$ is indeed a Boolean algebra map, and by construction $U\hat{f} \circ \eta_n = f$. Now any other Boolean algebra morphism which sends each $\mathcal{U}_k$ to $f(k)$ satisfies the above clauses and is thus equal to $\hat{f}$.

We have thus shown that the free and finitely generated Boolean algebras are precisely the double powerset algebras $\mathcal{PP}(n)$ on finite sets n. We conclude by observing that $\mathcal{PP}(n)$ has 2^{2^n} elements. □

We have thus proved the following:

Corollary C.36. *The representable Boolean algebra (in the sense of Definition C.4) corresponding to y^{G^n} is the finite Boolean algebra with 2^{2^n} elements.*

This notion of representability is not to be confused with that of Stone representation theorem, stating that every Boolean algebra can be represented as the algebra of clopens of a Stone space induced by the ultrafilters of the Boolean algebra.

For each $n \in \mathbb{N}$, let us write $\mathfrak{F}_n$ for the Boolean algebra with 2^{2^n} elements.

Proposition C.37 (Yoneda lemma for Boolean algebras)**.** *Let B and C be Boolean algebras. Then $B \simeq C$ if and only if*

$$\mathbf{BAlg}(\mathfrak{F}_n, B) \simeq \mathbf{BAlg}(\mathfrak{F}_n, C)$$

naturally in n.

Proof. Just specialise $\mathcal{T} = \mathcal{B}$ in Proposition C.29 and use Corollary C.36. □

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