Algebraic models of type theory

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In the seminal work by Awodey and Warren it was shown that the intensional identity types of Martin-Löf dependent type theory can be modelled categorically using weak factorisation systems. In this interpretation the dependent types are modelled by *fibrations*, i.e. the *right* maps of a weak factorisation system. This work inspired a lot of further research into such categorical models of identity types. Recently it was adapted by Gambino and Larrea to the setting of *algebraic* weak factorisation systems who added interpretations of the dependent sum and product types of said type theory. In their work the dependent types are interpreted using the algebras of the pointed endofunctor of the system, and in the present work we show that the same approach also works when we instead use the algebras for the monad of the system.

Contents

Chapter 1

Introduction

In a series of papers published in the 70s and 80s Martin-Löf developed a version of dependent type theory with the aim of developing a foundation for constructive mathematics. It includes several types such as the dependent sum, product, and intensional identity types, and is now often referred to as Martin-Löf dependent type theory.

Arguably the most novel of these types are the intensional identity types, of which the inhabitants are witnesses of equality between terms. Considerations on how to interpret these identity types led to the development of what is called *homotopy type theory*, in which a type is viewed as a topological space, terms of a type as points in that space, and the equalities between terms as paths between them in the topological space.

Finding models of type theory that are true to this view has been an ongoing effort. In the work by Awodey and Warren in [10] it was shown that a homotopy theoretic model of the identity types can be obtained using weak factorisation systems (WFS). In this interpretation the dependent types are modelled by *fibrations*, i.e. the *right* maps of the weak factorisation system. There were however some coherence issues with this interpretation regarding substitution. This is because substitution is modelled using pullbacks, and while substitution commutes strictly with operations like type formation, pullbacks generally only commute up to natural isomorphism. This was later remedied in a continuation of this work by van den Berg en Garner in [8] by using a more structured variant of weak factorisation systems called *cloven* weak factorisation systems, in combination with a splitting construction first introduced by Hoffman in [14].

More recently Gambino and Larrea further adapted this approach to *algebraic* weak factorisation systems (AWFS) in [3], interpreting the dependent types using the algebras for the pointed endofunctor of the AWFS or in other words as the right maps of the underlying WFS of the AWFS, and adding interpretations of the dependent sum and product types.

In the present work we build on those results by Gambino and Larrea in [3], making use of algebraic weak factorisation systems to model dependent type theory. The key difference is that we will now use the algebras of the monad of the AWFS rather than for the underlying pointed endofunctor in order to interpret the dependent types. Since these are a subset of the algebras for the pointed endofunctors many of the same underlying ideas used in [3] still apply, and as such the structure of this work will closely mimic that in [3, Sections 1, 2, and 3].

Specifically the approach in those sections is as follows. For any AWFS there is a *comprehension* category induced by the algebras for the pointed endofunctor of the AWFS. It is shown that if this category is equipped with choices of structure for the sum, product, and identity types of Martin-Löf type theory in a suitably functorial way, then applying a splitting construction to this category yields a proper interpretation of the theory. Then, conditions are identified for AWFS which guarantee that such choices can be made. For the interpretation of sum types no additional assumptions need to be made as we can simply use the fact that the algebras are closed under composition. For the product types we need that the AWFS satisfies a *Frobenius condition*, which ensures that algebras are also closed under pushforward. For the identity types we need the notion of a *stable functorial choice of path objects* introduced by van den Berg and Garner in [8]. After proving that these conditions are indeed sufficient, an example of such an AWFS was exhibited on the category of groupoids.

In the present work we reproduce these steps just listed but instead using the comprehension category induced by the algebras for the monad of the AWFS. The interpretation of the sum and identity types will be almost exactly as in [3], most of the work is in formulating a suitable Frobenius condition. We do so in general terms and this will be the main contribution of this work. We then conclude by showing that the aforementioned AWFS on groupoids satisfies the additional properties that we have formulated.

The contents of this work are structured as follows. First we consider the specific fragment of Martin-Löf type theory that we wish to model in Chapter 2. In Chapter 3 we look at the notion of comprehension categories which are commonly used for interpreting type theory, it is one of the several equivalent ways of doing so which are pervasive in the literature. In the last section of this chapter, Section 3.3, we look at how the π-*clans* defined by Joyal in [7] give rise to such a category and along with it an interpretation of the sum and product types. Since weak factorisation systems are a strengthening of these clans we can this method as the basis for the rest of our work, exactly as was done in [1]. The several different notions of factorisation systems are then reviewed in Chapter 4, culminating in the definition of algebraic weak factorisation systems. In order to interpret product types we need to place an additional demand on the algebraic weak factorisation systems which is called a *Frobenius condition*. In Chapter 5 we look at this condition in the general setting of classes, categories, and double categories of maps and prove that at each level there is an equivalent phrasing of this condition in terms of pushforward rather than pullback functors. We then put everything together in Chapter 6 to phrase sufficient conditions for algebraic weak factorisation systems so they can used to obtain a model for the type theory outline in Chapter 2. We then exhibit such an factorisation system in Chapter 7, and finish with some concluding remarks in Chapter 8.

Chapter 2

Martin-Löf dependent type theory

The purpose of this chapter is to describe the variant of type theory we aim to model in the coming chapters, i.e. the fragment of Martin-Löf dependent type theory with sum, product, and intensional identity types. We will describe what a type theory is in general, what distinguishes the dependent variant, and then list the axioms we are interested in. The content of this chapter summarises some of the material from [12] and [13, Chapter 10] which is relevant for the rest of the present work.

2.1 Type theory

A type theory is a logical framework within which there are three primary notions; *types* which are akin to sets; *terms* which must have a certain type and may be thought of as the elements of the types; and contexts which are lists of (unique) variables declared to be of some type. Then there are generally four kinds of judgements, made with respect to a context Γ, as listed in the table below.

Judgement	Notation
A is a type	$\Gamma \vdash A$ type
Types A and B are equal	$\Gamma \vdash A = B$ type
The term t is of type A	$\Gamma \vdash t : A$
Terms t and s are equal	$\Gamma \vdash t = s : A$

As an example we might have a type $\mathbb{N}$ representing the natural numbers with some axioms $\Gamma \vdash \mathbb{N}$ type, $\Gamma \vdash 0 : \mathbb{N}$, and whenever $\Gamma \vdash n : \mathbb{N}$ also $\Gamma \vdash s(n) : \mathbb{N}$, along with further rules describing induction and recursion.

Most type theories then have a few well known *type constructors*, i.e. ways to create new types from old. For instance when $\Gamma \vdash A$ type and $\Gamma \vdash B$ type we have a type $\Gamma \vdash A \to B$ type the terms of which correspond to functions from A to B. Likewise we might have a type $\Gamma \vdash A \times B$ type corresponding to the cartesian product of A and B.

Specification of new types follows a general format, and the axioms of the type in question are often named after this format. For instance the three axioms listed above for the natural numbers are respectively called the formation and introduction rules. The general scheme is as follows (as listed in [12, Appendix A.2.4]):

- A *formation* rule which describes the contexts in which the type can be formed.
- The *introduction* rules stating how new elements of this type can be formed, or in other words what the canonical inhabitants of the type are.
- The *elimination* rules that state how elements of the type can be used, or how one can obtain a mapping out of the type.
- The *computation* rules that states how the introduction and elimination rules interact.
- Optionally a *uniqueness* principle.

Other than rules pertaining to specific kinds of types there are a number of general *structural* rules of a type theory. We will just list one such rule as an example, see for instance [12, Appendix A.2.2] for a more complete overview.

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, \Delta \vdash b : B}{\Gamma, x{:}A, \Delta \vdash b : B} \qquad \text{(Weakening)}$$

2.2 Dependent types

A dependent type theory is one in which the types are additionally allowed to depend on the variables in the context. For instance given some variable $x : A$ we might have a type $B(x)$ that makes reference to x, written as a judgement $x : A \vdash B(x)$ type. A commonly used example is the dependent type $x : \mathbb{N} \vdash \mathrm{Nat}(x)$ type of natural numbers up to x, i.e. for any particular natural number $n : \mathbb{N}$ we get a type $\mathrm{Nat}(x)$ of which the terms are natural numbers less than or equal to n.

In set theory such dependent types correspond to indexed families of sets $(B)_{x \in A}$. Equivalently such a family is given by a function $f : B \to A$ where B_x is given by $f^{-1}(x) := \{b \in B \mid f(b) = x\}$. This provides the intuition for interpreting a dependent type in category theory as a morphism $B \to A$.

2.3 Sum, product, and identity types

We are interested in modelling the dependent sum, product, and identity types which we will now describe. First we look at the sum types Σ, which behaves like the cartesian product of sets but where the second component of the pairs in it may depend on the first. For any type $B(x)$ depending on some $x : A$ we have a type $\Sigma_{x:A} B(x)$. We may construct (or, introduce) terms of this type by pairing up terms $a : A$ and $b : B(a)$ with a pairing operation explicitly denoted by p, i.e. $\mathrm{p}(a, b) : \Sigma_{x:A}B(x)$. We then have an induction principle that says that in order to define a map out of $\Sigma_{x:A}B(x)$ it suffices to define it on the canonical pairs, along with a computation principle that says that the resulting maps acts on the canonical pairs as we defined it. This results in the following rules.

$$\frac{\Gamma, x{:}A \vdash B(x) \text{ type}}{\Gamma \vdash \Sigma_{x:A}B(x) \text{ type}} \tag{Σ-Form.}$$

$$\Gamma, x{:}A, y{:}B(x) \vdash \mathrm{p}(x, y) : \Sigma_{x:A}B(x) \tag{Σ-Intro.}$$

$$\frac{\Gamma, z{:}\Sigma_{x:A}B(x) \vdash T(z) \text{ type} \qquad \Gamma, x{:}A, y{:}B(x) \vdash t : T\,(\mathrm{p}(x, y))}{\Gamma, z{:}\Sigma_{x:A}B(x) \vdash \mathrm{ind}(T, t, z) : T(z)} \tag{Σ-Elim.}$$

$$\frac{\Gamma, z{:}\Sigma_{x:A}B(x) \vdash T(z) \text{ type} \qquad \Gamma, x{:}A, y{:}B(x) \vdash t : T\,(\mathrm{p}(x, y))}{\Gamma, x{:}A, y{:}B(x) \vdash \mathrm{ind}(T, t, \mathrm{p}(x, y)) = t : T(\mathrm{p}(x, y))} \tag{Σ-Comp.}$$

Next are the product, or function, types Π which behave like sets of functions. The difference is that the domain of the function depends on its input. Set theoretically this corresponds to the situation where we have a set A indexing a family of sets $(B)_{x\in A}$, a function f for which $f(x) \in B_x$ is then an element of $\Pi_{x\in A}B_x$. So when we have a type $B(x)$ that depends on $x : A$ we have a type $\Pi_{x:A}B(x)$ of dependent functions. To construct an element of this type we use lambda abstraction; if for every $x : A$ we can construct a term $t : B(x)$ then we get a function $\lambda_{x:A}.t : \Pi_{x:A}B(x)$. Functions can be applied to elements in their domain so for this we have an application operation app, i.e. when $f : \Pi_{x:A}B(x)$ and $x : A$ then $\mathrm{app}(f, x) : B(x)$. Lastly there is a uniqueness principle stating the so called η-conversion rule that $\lambda x.\mathrm{app}(f, x) = f$. As follows the rules which formally express this.

$$\frac{\Gamma, x{:}A \vdash B(x) \text{ type}}{\Gamma \vdash \Pi_{x:A}B(x) \text{ type}} \tag{Π-Form.}$$

$$\frac{\Gamma, x{:}A \vdash t : B(x)}{\Gamma \vdash \lambda_{x:A}.t : \Pi_{x:A}B(x)} \tag{Π-Intro.}$$

$$\Gamma, x{:}A, f{:}\Pi_{x:A}B(x) \vdash \mathrm{app}(f, x) : B(x) \tag{Π-Elim.}$$

$$\frac{\Gamma, x{:}A \vdash t : B(x)}{\Gamma, x{:}A \vdash \mathrm{app}(\lambda_{x:A}.t, x) = t : B(x)} \tag{Π-Comp.}$$

$$\Gamma, f{:}\Pi_{x:A}B(x) \vdash \lambda_{x:A}.\mathrm{app}(f, x) = f : \Pi_{x:A}B(x) \tag{Π-Uniq.}$$

Lastly we look at the rules for Id-types. Unlike the previous two these have no clear counterpart in set theory. They were introduced to complete the *propositions-as-types* paradigm which is closely related to the well known Brouwer-Heyting-Kolmogorov interpretation. According to this paradigm types correspond to propositions and their inhabitants to proofs of those propositions. For example the type $A \times B$ may be understood as the proposition "A and B", and a term of this type as a pair of proofs of A and B respectively. The identity types then correspond to the notion of equality.

The idea is that for any type A and any two of its inhabitants $x, y : A$ we have a type $\mathrm{Id}_A(x, y)$, sometimes written as $x =_A y$ or just $x = y$. An inhabitant $p : \mathrm{Id}_A(x, y)$ can be thought of as a proof that x and y are equal. Since equality is reflexive we have for any $x : A$ an inhabitant $\mathrm{r}(x) : \mathrm{Id}_A(x, x)$. The induction principle says that in order to define a map out of $\mathrm{Id}_A(x, y)$ it suffices to define it on these canonical inhabitants $\mathrm{r}(x)$, and the computation rule states how this resulting map acts on those canonical inhabitants.

$$\Gamma, x{:}A, y{:}A \vdash \mathrm{Id}_A(x, y) \text{ type} \tag{Id-Form.}$$

$$\Gamma, x{:}A \vdash \mathrm{r}(x) : \mathrm{Id}_A(x, x) \tag{Id-Intro.}$$

$$\frac{\Gamma, x{:}A, y{:}A, p{:}\,\mathrm{Id}_A(x, y) \vdash T(x, y, p) \text{ type} \qquad \Gamma, x{:}A \vdash t : T(x, x, \mathrm{r}(x))}{\Gamma, x{:}A, y{:}A, p{:}\,\mathrm{Id}_A(x, y) \vdash \mathrm{j}_A(T, t, x, y, p) : T(x, y, p)} \tag{Id-Elim.}$$

$$\frac{\Gamma, x{:}A, y{:}A, p{:}\,\mathrm{Id}_A(x, y) \vdash T(x, y, p) \text{ type} \qquad \Gamma, x{:}A \vdash t : T(x, x, \mathrm{r}(x))}{\Gamma, x{:}A \vdash \mathrm{j}_A(T, t, x, x, \mathrm{r}(x)) = t : T(x, x, \mathrm{r}(x))} \tag{Id-Comp.}$$

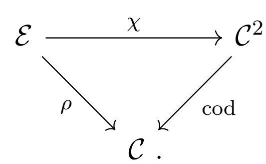
Types like products, disjoint unions, functions, etc., were introduced to type theory as mathematical notions and then later seen to correspond to logical notions. For the identity types it was the other way around, they were first introduced as a logical notion and then seen to correspond to a mathematical notion, namely that of *path spaces* in topology. This view led to the development of what is known as homotopy type theory, in which types are viewed as topological spaces with their inhabitants representing points in the space and the inhabitants of their identity types as paths in the space between those two points. Finding models of type theory that are true to this view is what led to the work on using weak factorisation systems to interpret type theory and as such the present work.

Chapter 3

Comprehension categories

There are several ways of interpreting dependent types in category theory which can be shown to be equivalent to one another. Of these various options we will work with comprehension categories, a notion which is closely related to that of Grothendieck fibrations. We will assume familiarity with Grothendieck fibrations, although a definition may be found in Appendix A.1. For an extensive treatment on the relation between type theory, fibrations, and comprehension categories the reader is referred to [13].

Definition 1. Consider a category $\mathcal{C}$ with a functor $\chi : \mathcal{E} \to \mathcal{C}^2$ and let ρ denote the composition $\mathrm{cod} \cdot \chi$. The pair $(\mathcal{C}, \chi)$ is called a *comprehension category* if ρ is a fibration and χ sends cartesian arrows to pullback squares in $\mathcal{C}$:



When the category $\mathcal{C}$ in question is clear we may omit reference to it and just speak of the comprehension category χ. We will denote $\mathrm{dom} \cdot \chi$ by τ, and sometimes simply write f instead of τf for arrows in $\mathcal{E}$. We say $(\mathcal{C}, \chi)$ is *cloven*, *normal*, or *split* whenever ρ is.

Remark 2. If $\mathcal{C}$ has pullbacks then cod is a fibration and $\chi : \rho \to \mathrm{cod}$ a fibered functor, but this need not be the case.

The elements Γ of $\mathcal{C}$ are interpreted as contexts and the objects A in its fibre $\rho(\Gamma)$ as dependent types derivable in context Γ. The functor τ models context extension of Γ with a fresh variable of type A so we denote $\tau(A)$ by $\Gamma.A$. The comprehension functor maps dependent types to the projection $\chi_A : \Gamma.A \to \Gamma$ that drops this variable.

To soundly model substitution we will need $\rho : \mathcal{E} \to \mathcal{C}$ to be a split fibration. Not every fibration can be made split, but every fibration is equivalent to some split fibration, which is obtained by applying the right adjoint of the inclusion $\mathbf{SpFib}(\mathcal{C}) \to \mathbf{Fib}(\mathcal{C})$ to ρ, see for instance [5]. This splitting can be extended to operate on comprehension categories so that for any comprehension category there is a split one that is equivalent to it, see [1, Chapter 2]. This result is an adaptation of the of the result by Hoffman in [14] for locally cartesian closed categories.

Aside from modelling the structural rules of type theory we want to model additional logical structure, namely the dependent sum, product, and identity types. In [1] Larrea distinguishes for each of these type formers three types of definitions.

1. A *choice* of the logical structure specifies the structure needed for interpreting the formation, introduction, elimination, and computation rules of the type, but leaves out conditions relating to substitution. This definition applies to any comprehension category.

2. A *strictly stable choice* expands on the first and is only interpretable in split comprehension categories because it requires the choices to cohere strictly with the split cleavage, i.e. with substitution. A split comprehension category satisfying this condition therefore properly models the type in question.

3. A *pseudo-stable choice* applies to any comprehension category and expands on the first point by adding conditions which ensure that the split comprehension category obtained by applying the right adjoint splitting satisfies the second point.

The pseudo-stability conditions for sum and product types are due to Hofmann in [14], and those for intensional identity types to Warren in [19]. It was shown by Gambino and Larrea in [3] that AWFS satisfying suitable conditions give rise to a comprehension category with such pseudo-stable choices.

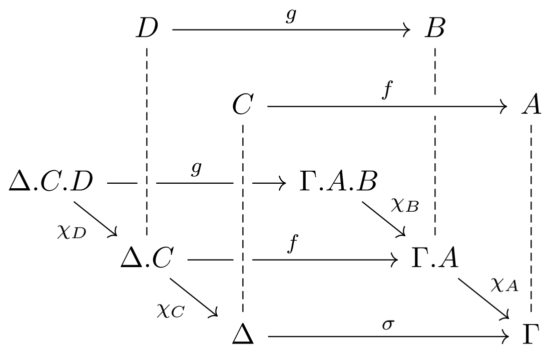
These definitions of pseudo-stable choices underlie the approach taken in this work: we find pseudo-stable choices for each of the kinds of logical structure that we want so that we have a proper interpretation after applying the right adjoint splitting. For this reason we only give the definition of pseudo-stability and leave out point 2 above, more details and proofs of these statements can be found in [1, Chapter 1 and 2]. Before proceeding we first consider a useful tool for describing the pseudo-stability conditions.

3.1 Dependent tuples

In [1] a method is described of constructing a category of *dependent tuples* of a given comprehension category.

Definition 3. Given a comprehension category $\chi : \mathcal{E} \to \mathcal{C}^2$ we can form for each positive number n a *category of dependent tuples* $\mathrm{DT}_n(\chi)$. Objects are given by n-tuples $(A)_i$ of objects in $\mathcal{E}$ with $\rho(A_{i+1}) = \tau(A_i)$, and arrows $(B)_i \to (A)_i$ by n-tuples of arrows $(f)_i$ with

$f_i : B_i \to A_i$ and $\rho(f_{i+1}) = \tau(f_i)$. Composition is defined component-wise. As an example we visualise a morphism $(f, g) : (C, D) \to (A, B)$ in $\mathrm{DT}_2(\chi)$ over $\sigma : \Delta \to \Gamma$:



The projection functors $\rho_{n+1} : \mathrm{DT}_{n+1}(\chi) \to \mathrm{DT}_n(\chi)$ which drop the last component are all fibrations, so composing these with ρ gives a fibration $\hat{\rho} : \mathrm{DT}_n(\chi) \to \mathcal{C}$ where the $\hat{\rho}$-cartesian arrows are those tuples $(f)_i$ consisting of ρ-cartesian f_i.

The category of dependent pairs $\mathrm{DT}_2(\chi)$ is of particular interest for modelling the sum and product types.

3.2 Modelling additional logical structure

The definitions of pseudostability that we list in this section are there in their entirety for the sake of completeness, for our purposes we will primarily be interested in the aspects that are needed for modelling the type formation rules. Of secondary importance are the aspects related to modelling the introduction, elimination, and computation rules. This is because the choices we will make for these data in our comprehension category in Chapter 4 will be as those in [1, Section 2.7] for π-clans, thus inheriting all the other properties stated by these pseudo-stability definitions from the proofs given there. Of least importance are the coherence/naturality conditions which are there to ensure that the comprehension category obtained from the right adjoint splitting has the necessary properties to properly interpret the type theory. Since we will not be looking into this method the reader may safely ignore these, the interested reader is referred to [1, Chapter 2] for more details.

All these definitions are given with respect to some comprehension category $\chi : \mathcal{E} \to \mathcal{C}^2$. For convenience we use the abbreviations $\Gamma := \rho(A)$ and $\Delta := \rho(B)$ if $A, B \in \mathcal{E}$, and $\sigma := \rho(f)$ when $f : B \to A$.

3.2.1 Σ-types

Definition 4. A *pseudo-stable choice of* Σ*-types* $(\Sigma, \mathrm{p}, \mathrm{ind})$ on χ consists of the following.

1. A fibered functor $\Sigma : \hat{\rho} \to \rho$ modelling the type formation rule $(A, B) \mapsto \Sigma_A B$:

$$\begin{array}{ccc} \mathrm{DT}_2(\chi)_{\mathrm{c}} & \xrightarrow{\Sigma} & \mathcal{E}_{\mathrm{c}} \\ & {\scriptstyle\hat{\rho}}\searrow \quad \swarrow{\scriptstyle\rho} & \\ & \mathcal{C}\ . & \end{array}$$

   Here we let $\mathcal{E}_{\mathrm{c}}$ denote the wide subcategory of $\mathcal{E}$ spanned by its cartesian arrows, and likewise for $\mathrm{DT}_2(\chi)_{\mathrm{c}}$.

2. For each $(A, B) \in \mathrm{DT}_2(\chi)$ a pairing morphism as in the lower left commuting diagram:

$$\begin{array}{ccc} \Gamma.A.B & \xrightarrow{\mathrm{p}_{A,B}} & \Gamma.\Sigma_A B \\ & {\scriptstyle\chi_A\chi_B}\searrow \quad \swarrow{\scriptstyle\chi_{\Sigma_A B}} & \\ & \Gamma & \end{array} \qquad \begin{array}{ccc} \Delta.C.D & \xrightarrow{\mathrm{p}_{C,D}} & \Delta.\Sigma_C D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle\Sigma_f g} \\ \Gamma.A.B & \xrightarrow[\mathrm{p}_{A,B}]{} & \Gamma.\Sigma_A B\ . \end{array}$$

   Together these should constitute a natural transformation p in the sense that for each $(f, g) : (C, D) \to (A, B)$ the upper right naturality square commutes.

3. For each $(A, B) \in \mathrm{DT}_2(\chi)$ an operation that sends a dependent type $T \in \rho(\Gamma.\Sigma_A B)$ and $t : \Gamma.A.B \to \Gamma.\Sigma_A B.T$ satisfying $\chi_T t = \mathrm{p}_{A,B}$ to a section $\mathrm{ind}_{A,B}(T, t)$ of T making both triangles commute, as depicted on the left below:

$$\begin{array}{ccc} \Gamma.A.B & \xrightarrow{t} & \Gamma.\Sigma_A B.T \\ {\scriptstyle\mathrm{p}_{A,B}}\downarrow & \nearrow{\scriptstyle\mathrm{ind}_{A,B}(T,t)} & \downarrow{\scriptstyle\chi_T} \\ \Gamma.\Sigma_A B & \xrightarrow[1]{} & \Gamma.\Sigma_A B \end{array} \qquad \begin{array}{ccc} \Delta.\Sigma_C D & \xrightarrow{\mathrm{ind}_{C,D}(T',t')} & \Delta.\Sigma_C D.T' \\ {\scriptstyle\Sigma_f g}\downarrow & & \downarrow{\scriptstyle h} \\ \Gamma.\Sigma_A B & \xrightarrow[\mathrm{ind}_{A,B}(T,t)]{} & \Gamma.\Sigma_A B.T\ . \end{array}$$

   Again we also demand these morphisms satisfy a coherence condition. Consider $(f, g) : (C, D) \to (A, B)$ and let $h : T' \to T$ be cartesian over $\Sigma_f g$, then the universal property of the pullback underlying h gives us a section $t' : \Delta.C.D \to \Delta.\Sigma_C D.T'$ over the pairing morphism. Now the induced square as on the right above should commute.

3.2.2 Π-types

Next we consider Π-types, for which the definition is similar to that of the Σ-types.

Definition 5. A *pseudo-stable choice of* Π-*types* $(\Pi, \lambda, \mathrm{app})$ on χ consists of the following.

1. A fibered functor $\Pi : \hat{\rho} \to \rho$ modelling the type formation rule $(A, B) \mapsto \Pi_A B$:

$$\begin{array}{ccc} \mathrm{DT}_2(\chi)_\mathrm{c} & \xrightarrow{\Pi} & \mathcal{E}_\mathrm{c} \\ & {\scriptstyle \hat{\rho}} \searrow \quad \swarrow {\scriptstyle \rho} & \\ & \mathcal{C} \; . & \end{array}$$

2. For each $(A, B) \in \mathrm{DT}_2(\chi)$ an operation $\lambda_{A,B}$ that sends a section $t : \Gamma.A \to \Gamma.A.B$ to a section $\lambda_{A,B} t : \Gamma \to \Gamma.\Pi_A B$ as on the lower left in:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\lambda_{A,B} t} & \Gamma.\Pi_A B \\ & {\scriptstyle 1} \searrow \quad \swarrow {\scriptstyle \chi_{\Pi_A B}} & \\ & \Gamma & \end{array} \qquad \qquad \begin{array}{ccc} \Delta & \xrightarrow{\lambda_{C,D} t'} & \Delta.\Pi_C D \\ {\scriptstyle \sigma} \downarrow & & \downarrow {\scriptstyle \Pi_f g} \\ \Gamma & \xrightarrow[\lambda_{A,B} t]{} & \Gamma.\Pi_A B \; . \end{array}$$

   These morphisms should together satisfy a coherence condition. Consider $(f, g) : (C, D) \to (A, B)$ with g cartesian, then any section $t : \Gamma.A \to \Gamma.A.B$ induces a section $t' : \Delta.C \to \Delta.C.D$ by the universal property of the pullback square underlying g. Now the resulting square on the right above should commute.

3. An arrow $\mathrm{app}_{A,B} : \Gamma.A.\Pi_A B \to \Gamma.A.B$ satisfying $\chi_B \cdot \mathrm{app}_{A,B} = \chi_{\Pi_A B}$ where $\Gamma.A.\Pi_A B$ is obtained by choosing *any* lifting $\chi_{A,\Pi_A B} : \Pi_A B \to \Pi_A B$ of $\Pi_A B$ along χ_A. Note that we are abusing notation by writing $\Pi_A B$ for the domain of this arrow. For any section $t : \Gamma.A \to \Gamma.A.B$ the pullback underlying $\chi_{A,\Pi_A B}$ induces a section that we abusively denote by $\lambda_{A,B} t : \Gamma.A \to \Gamma.A.\Pi_A B$. We require that $\mathrm{app}_{A,B} \cdot \lambda_{A,B} t = t$, which expresses the computation rule of Π-types:

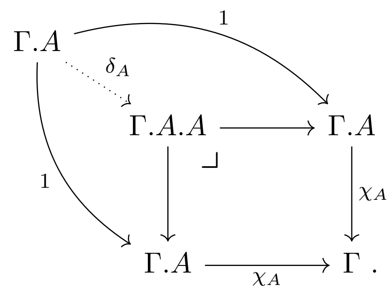
$$\begin{array}{ccc} \Gamma.A & \overset{\lambda_{A,B} t}{\dashrightarrow} & \Gamma.A.\Pi_A B \\ & {\scriptstyle t} \searrow \quad \swarrow {\scriptstyle \mathrm{app}_{A,B}} & \\ & \Gamma.A.B & \end{array} \qquad \qquad \begin{array}{ccc} \Delta.C.\Pi_C D & \xrightarrow{\mathrm{app}_{C,D}} & \Delta.C.D \\ {\scriptstyle \Pi_f g} \downarrow & & \downarrow {\scriptstyle g} \\ \Gamma.A.\Pi_A B & \xrightarrow[\mathrm{app}_{A,B}]{} & \Gamma.A.B \; . \end{array}$$

   Together these application morphisms should satisfy a coherence condition in the following sense. If $(f, g) : (C, D) \to (A, B)$ then the universal property of the pullback underlying $\chi_{A,\Pi_A B}$ induces an arrow which we abusively denote by $\Pi_f g : \Delta.C.\Pi_C D \to \Gamma.A.\Pi_A B$. Now we want that the resulting square on the right above commutes.

Remark 6. Since having pseudo-stable choices of sum and product types means having two (fibered) functors $\Sigma, \Pi : \mathrm{DT}_2(\chi)_\mathrm{c} \to \mathcal{E}_\mathrm{c}$ we might expect to find a functor in the other direction forming left and right (fibered) adjoints with them but there does not seem to exist such a functor.

3.2.3 Id-types

Lastly we consider the Id-types, for which the definition is more cumbersome than that of the Σ and Π types. To describe these we will assume a choice of cartesian lifts of χ_A along A for each $A \in \mathcal{E}$, as this enables us to model the context morphism $(\Gamma, x : A) \to (\Gamma, x : A, y : A)$ that puts $y := x$. Again we abuse notation and denote the domain of this cartesian lift by A, so that we can refer to the context $\Gamma.A.A$. The idea is that we now have for each $A \in \mathcal{E}$ a *diagonal* morphism δ_A induced by the pullback square underlying the cartesian lift of χ_A:

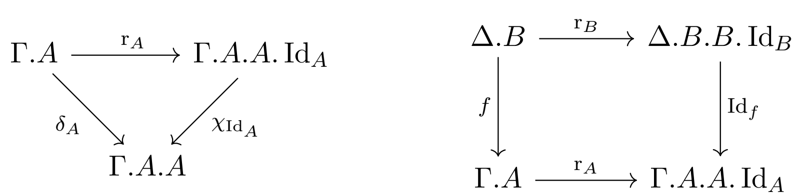


This δ_A models the aforementioned context morphism. It is so named because in **Set** such a map is given by $x \mapsto (x, x)$.

Definition 7. A *pseudo-stable choice of* Id*-types* $(\mathrm{Id}, \mathrm{r}, \mathrm{j})$ on χ consists of the following.

1. An endofunctor Id on $\mathcal{E}_c$ that models the type formation rule $A \mapsto \mathrm{Id}_A$. This assignment should be such that Id_A is over $\Gamma.A.A$ and Id_f over $f \times_{\rho(f)} f$.

2. For each $A \in \mathcal{E}$ a section r_A of Id_A over the diagonal morphism δ_A, depicted on the left below, modelling the introduction rule:



   These should satisfy a naturality condition in that for any cartesian $f : B \to A$ the diagram on the right above should commute.

3. An operation j_A which assigns to each pair (T, t) consisting of an object T in the fibre over $\Gamma.A.A.\,\mathrm{Id}_A$ and a section t of T over r_A a section $\mathrm{j}_A(T, t)$ of T as in the lower left diagram. Both these triangles should commute, of which the upper left states the

computation rule.

$$
\begin{array}{ccc}
\Gamma.A & \xrightarrow{\;t\;} & \Gamma.A.A.\operatorname{Id}_A.T \\
{\scriptstyle \mathrm{r}_A}\big\downarrow & \overset{\mathrm{j}_A(T,t)}{\nearrow} & \big\downarrow{\scriptstyle \chi_T} \\
\Gamma.A.A.\operatorname{Id}_A & \xrightarrow[\;1\;]{} & \Gamma.A.A.\operatorname{Id}_A
\end{array}
\qquad
\begin{array}{ccc}
\Delta.B.B.\operatorname{Id}_B & \xrightarrow{\mathrm{j}_B(T',t')} & \Delta.B.B.\operatorname{Id}_B.T' \\
{\scriptstyle \operatorname{Id}_f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
\Gamma.A.A.\operatorname{Id}_A & \xrightarrow[\mathrm{j}_A(T,t)]{} & \Gamma.A.A.\operatorname{Id}_A.T\ .
\end{array}
$$

These diagonal fillers should satisfy a coherence condition in the following sense. If in addition to the situation above we have cartesian $f : B \to A$ and $g : T' \to T$ such that g is over Id_f then the pullback square underlying g and the naturality condition of r induce a section t' of T'. The induced square as on the upper right above should commute.

Remark 8. Unlike the dependent sum and function types it seems difficult to phrase the formation rule of the Id-types as the existence of a fibered functor.

3.3 The comprehension category of π-clans

As the conclusion to this chapter we will examine an example of a comprehension category equipped with pseudo-stable choices of dependent sum and function types, given by the π-clans[1] as defined in [7]. The material in section is an abridged version of the contents of [1, Section 2.7] with some details added from [7]. We repeat it here not just to serve as an example but more importantly because it will form the basis of our work in Chapter 6. The strategy there will be the same as the one employed by Larrea in [1, Chapter 4].

Definition 9. A map $f : A \to B$ in a category $\mathcal{C}$ is *carrable* if the postcomposition functor $f_! : \mathcal{C}/A \to \mathcal{C}/B$ has a right adjoint f^*. If g is a map with codomain B then f^*g is called the *base change* of g along f. A set of morphisms $\mathcal{R} \subseteq \mathcal{C}_1$ is *closed under base change* if every map in $\mathcal{R}$ is carrable and the base change of an $\mathcal{R}$ map along any other map is again in $\mathcal{R}$.

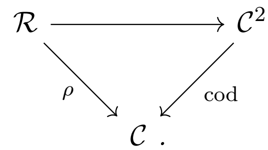
Here the functor f^* is just the usual pullback functor, so f being carrable means being able to pull back along f.

Definition 10. A *clan* is a category $\mathcal{C}$ with a set of morphisms $\mathcal{R} \subseteq \mathcal{C}_1$ which contains the isomorphisms in $\mathcal{C}$, and is closed under base change and composition. A morphism in $\mathcal{R}$ is called a *fibration.*

Given an object A in a clan $\mathcal{C}$ we let $\mathcal{R}(A)$ denote the full subcategory of $\mathcal{C}/A$ whose objects are fibrations with codomain A. This is called the *local clan* at A.

[1] It seems that in an earlier version of [7] these clans were instead called tribes, so what we define here as a clan is what is called a tribe in [1].

Proposition 11. *Let $\mathcal{C}$ be a clan and consider $\mathcal{R}$ as a full subcategory of $\mathcal{C}^2$. The inclusion of $\mathcal{R} \to \mathcal{C}^2$ is a comprehension category:*

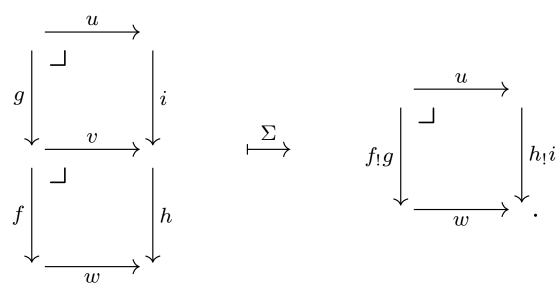


The cartesian morphisms of ρ are the pullback squares between fibrations, and its fibers are the local clans.

Any morphism $f : A \to B$ induces a functor $f^* : \mathcal{R}(B) \to \mathcal{R}(A)$ since $\mathcal{R}$ is closed under base change, and likewise if f is also a fibration then $f_! : \mathcal{R}(A) \to \mathcal{R}(B)$.

Proposition 12. *The comprehension category of Proposition 11 admits a pseudo-stable choice of Σ-types.*

Proof. A dependent tuple in this comprehension category is given by two composable fibrations f, g so we can define $\Sigma : (f, g) \mapsto f_! g$. Given two composable pairs of fibrations and pullback squares between them we have that the outer square is a pullback square between their compositions and we take this as the definition of Σ on arrows:



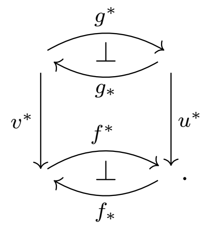
Writing $\Gamma := \rho(f)$ we have that $\Gamma.\Sigma_f g = \Gamma.f.g$, so for the pairing morphism we can just put $\mathrm{p}_{f,g} := 1_{\Gamma.f.g}$. Given a fibration h with codomain $\Gamma.\Sigma_f g$ and some morphism $t : \Gamma.f.g \to \Gamma.\Sigma_f g.h$ with $ht = 1_{\Gamma.f.g}$ we define $\mathrm{ind}_{f,g}(h, t) := t$. The coherence conditions are now easily verified. □

The choice of Π-types takes considerably more work. First we need an additional assumption on our clan.

Definition 13. A clan $\mathcal{C}$ is a π-*clan* if for every fibration $f : A \to B$ the pullback functor f^* has a right adjoint $f_* : \mathcal{R}(A) \to \mathcal{R}(B)$ called the *pushforward* functor.

These right adjoints will be used for constructing the functor Π. In order to define its action on morphisms we also need the notion of a Beck-Chevalley condition.

Definition 14. Consider a square of functors that commutes up to a natural isomorphism $f^*v^* \cong u^*g^*$, and right adjoints $f^* \dashv f_*$ and $g^* \dashv g_*$:



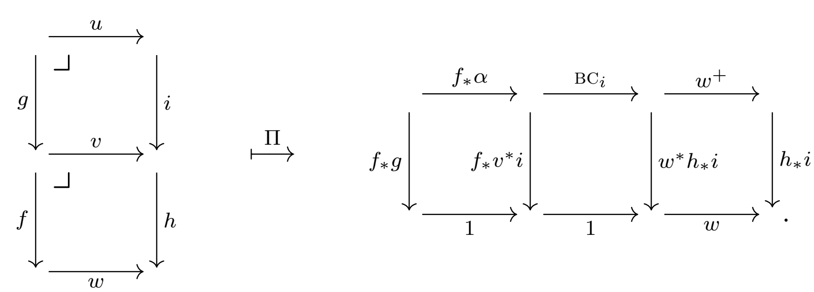
This square of functors is said to satisfy the *Beck-Chevalley condition* if the mate $v^*g_* \to f_*u^*$ of $f^*v^* \to u^*g^*$ is also an isomorphism.

Any square $(u, v) : f \to g$ in $\mathcal{C}$ between fibrations f and g induces such a situation as in this definition, and it is a well known fact that if the underlying square (u, v) is a pullback then the Beck-Chevalley condition is satisfied.

Lemma 15. *If $(u, v) : f \to g$ is a pullback square between fibrations f, g then the mate $v^*g_* \to f_*u^*$ of the canonical isomorphism $f^*v^* \to u^*g^*$ is an isomorphism.*

Proposition 16. *If C is a π-clan with pullbacks then its associated comprehension category of Proposition 11 admits a pseudo-stable choice of Π-types.*

Proof. Given a pair of composable fibrations f, g we now use the pushforward functor along f to define $\Pi : (f, g) \mapsto f_*g$. To define its action on morphisms we consider two pairs of composable fibrations f, g and h, i along with two pullback squares $(u, v) : g \to i$ and $(v, w) : f \to h$. Now Lemma 15 tells us there is a natural transformation $\textsc{bc} : f_*v^* \to w^*h_*$. Using this we can define Π on arrows as the composition of the squares on the right below:

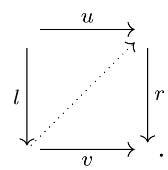


Here α is the cone morphism induced by v^*i. Some calculations then show that this composition is a pullback square and that this assignment is functorial, the reader is referred to [1, Lemma 2.7.8] for the details.

To define λ we consider a section $t : \Gamma.f \to \Gamma.f.g$. We want an arrow $\lambda t : 1 \to f_*g$, so if we assume for convenience that our choice of pullbacks preserves identities we can simply take the transpose $\lambda_{f,g}t := \bar{t} : 1 \to g$. Similarly we define $\mathrm{app}_{f,g} := \varepsilon_g : \Gamma.f.\Pi_f g \to \Gamma.f.g$. Now checking that these data satisfy the coherence conditions requires some lengthy calculations and again the reader is referred to [1, Lemma 2.7.8] for the details. $\square$

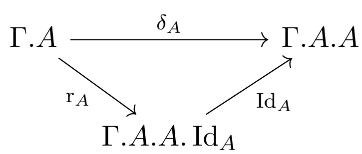
Unlike for the sum and product types a π-clan does not readily admit a pseudo-stable choice of Id-types. It is for this reason that we will use strengthened versions of clans in the coming chapters so we conclude this chapter by looking at the difficulties with interpreting Id-types in more detail.

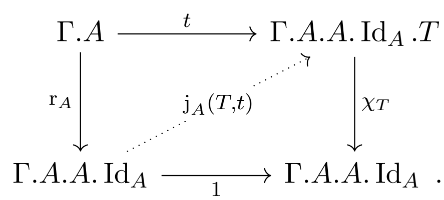
Since a type $\Gamma \vdash A$ type is modelled by a fibration $f : \Gamma.A \to \Gamma$ we need a fibration with codomain $\Gamma.A.A := \Gamma.A \times_\Gamma \Gamma.A$ to interpret the type Id_A. The idea of Awodey and Warren in [10] is to use a *weak factorisation system* $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ to obtain such a fibration. We will define this notion in the next chapter so for now we will just mention that $\mathcal{L}$ and $\mathcal{R}$ are classes of maps where $\mathcal{R}$ is a clan, every morphism f in $\mathcal{C}$ has a factorisation $f = r \cdot l$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$, and every morphism in $\mathcal{R}$ has a *right lifting property* for every map in $\mathcal{L}$. This lifting property states that if $l \in \mathcal{L}$, $r \in \mathcal{R}$, and $(u, v) : l \to r$ then there is a diagonal filler as drawn below:



This definition is very similar to that of a *tribe* in [7].

Now the factorisation property allows us to obtain the desired fibration because the diagonal morphism $\delta_A : \Gamma.A \to \Gamma.A.A$ factorises as $r \cdot l$, and so we can take r to interpret Id_A and l to interpret the reflexivity term r_A, as depicted on the left diagram below:





Lastly we can interpret the elimination terms $\mathrm{j}_A(T, t)$ by applying the lifting property of right maps to the square $(t, 1)$ as depicted on the right above.

So we see that in a weak factorisation system or a tribe we can interpret the structure of the types, but the problem is that in general these will not satisfy the additional coherence properties required by the pseudo-stability definition. To obtain these properties we will instead use *algebraic* weak factorisation systems, which we look at in the next chapter.

Chapter 4

Algebraic weak factorisation systems

In order to obtain a comprehension category with pseudo-stable choices of dependent sum, product, and identity types we will work with what are called *algebraic weak factorisation systems* (AWFS), a more structured variant of *weak factorization systems* (WFS) which in turn is a more structured variant of the clans we saw in Section 3.3. Any AWFS induces a comprehension category which is similar to those of clans and it is the overarching goal of this work to formulate conditions on an AWFS that ensure this comprehension category can be equipped with pseudo-stable choices of the Σ-, Π-, and Id-types.

This chapter will not contain any new results but rather provides an exposition of the three types of factorisation systems that are relevant to the coming chapters, each expanding on the last, namely: weak factorisation systems, functorial weak factorisation systems, and algebraic weak factorisation systems. Like with clans each of these factorisation systems has a distinguished class of maps $\mathcal{R}$ of some category $\mathcal{C}$ which are commonly called *fibrations* or *right* maps, but for the functorial WFS and AWFS these respectively have *categorical* and *double categorical* structure.

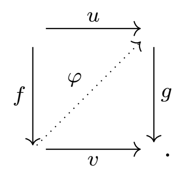
Further distinguishing these three kinds of factorisation systems from clans is that for each the class (or category, or double category) of right maps has a *right lifting property* against another class of maps in $\mathcal{C}$. This notion of lifting will play a central role in the next chapter and so we will consider for each of the three kinds of factorisation systems their associated notion of lifting at respectively the functional, categorical, or double categorical level.

This chapter is made up of three sections, each of which is devoted to explaining one of these factorisation systems and its associated notion of lifting. Many of these definitions and results can be found there [2] and [11] and the reader is referred there for a far more detailed exposition.

4.1 Weak factorisation systems

Central to the notion of factorisation systems is the idea of lifting properties of maps.

Definition 17. Given morphisms f, g in a category $\mathcal{C}$ we write $f \pitchfork g$ and say f has the *left lifting property* for g or equivalently that g has the *right lifting property* for f if every commuting square $(u, v) : f \to g$ has a diagonal filler φ making both triangles commute:



In such cases we say that φ is *lift* of (u, v) or a *solution* to the *lifting problem* (u, v).

Now the lifting relation $\pitchfork$, like any relation, induces two operations on the powerset $P(\mathcal{C}_1)$ given by $(-)^{\pitchfork} : J \subseteq \mathcal{C}_1 \mapsto \{g \in \mathcal{C}_1 \mid f \pitchfork g \text{ for all } f \in J\}$ and ${}^{\pitchfork}(-)$ defined dually. Moreover, letting $\mathcal{P}(\mathcal{C}_1)$ denote the category corresponding to the partial order $(P(\mathcal{C}_1), \subseteq)$ we have that these induce an adjunction:

$$\mathcal{P}(\mathcal{C}_1) \underset{(-)^{\pitchfork}}{\overset{{}^{\pitchfork}(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{P}(\mathcal{C}_1)^{\mathrm{op}} . \tag{4.1.1}$$

An adjunction between posets is known as a *Galois connection* and the one described above is sometimes referred to as the Galois connection between orthogonal classes of maps. It is an instance of a general result regarding relations, see e.g. [18, Proposition 7].

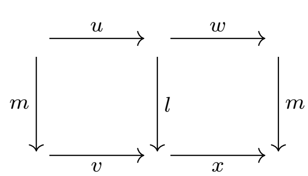
With these notions in place we can define what a weak factorisation system is, of which an AWFS is an algebraisized version.

Definition 18. A *weak factorisation system* $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ consists of two classes of maps, one class $\mathcal{L}$ of *left* maps and one $\mathcal{R}$ of *right* maps, which satisfy the following conditions.

1. Every morphism f in $\mathcal{C}$ factors as $f = r \cdot l$ with $r \in \mathcal{R}$ and $l \in \mathcal{L}$.

2. We have $l \pitchfork r$ for any $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

3. Both $\mathcal{L}$ and $\mathcal{R}$ are retract closed. This means for instance that if $l \in \mathcal{L}$ and we have

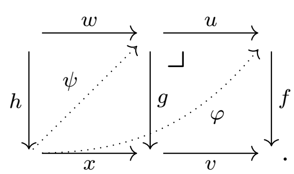


such that the top and bottom arrows compose to identities, then $m \in \mathcal{L}$.

It is well known that conditions 2 and 3 above are equivalent to the statement that $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\pitchfork}$. This provides a very useful way of proving membership for either class and it can be used to show that $\mathcal{R}$ is closed under composition and pullbacks, and contains all the isomorphisms. We given an example of this which we will use later.

Lemma 19. *If $J, K \subseteq \mathcal{C}_1$ and $K = J^{\pitchfork}$ then K is closed under pullbacks.*

Proof. Let $(u, v) : g \to f$ be a pullback square with $f \in K = J^{\pitchfork}$. To show $g \in J^{\pitchfork}$ we consider $h \in J$ and $(w, x) : h \to g$:

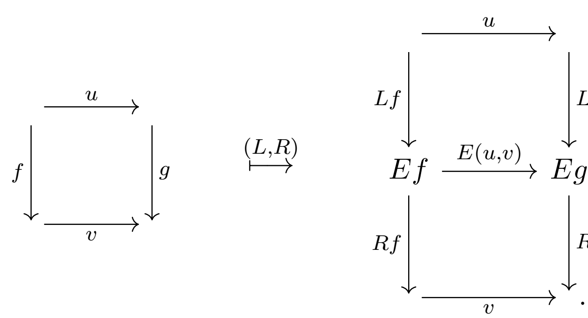


The rectangle (uw, vx) has a solution φ so that the pullback square induces an arrow ψ which can easily be seen to solve (w, x). $\square$

4.2 Functorial weak factorisation systems

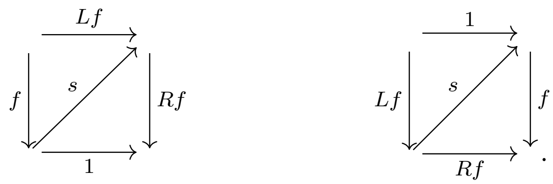
Next we expand on this definition using the notion of functorial factorisation.

Definition 20. Briefly put a *functorial factorisation* on a category $\mathcal{C}$ is a section $\mathcal{C}^2 \to \mathcal{C}^3$ of the composition functor. Such a functor is equivalently defined as a pair of endofunctors L and R on $\mathcal{C}^2$ satisfying $\operatorname{cod} L = \operatorname{dom} R$, $\operatorname{dom} L = \operatorname{dom}$, $\operatorname{cod} R = \operatorname{cod}$, and $f = Rf \cdot Lf$ for all morphisms f. This pair also induces a functor $E := \operatorname{cod} L = \operatorname{dom} R : \mathcal{C}^2 \to C$. These components are best understood by looking at how they factor squares of $\mathcal{C}$:



Definition 21. A *functorial weak factorisation system* on a category $\mathcal{C}$ is a WFS $(\mathcal{L}, \mathcal{R})$ with a functorial factorisation satisfying $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ for any morphism f in $\mathcal{C}$.

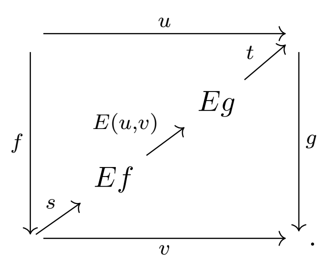
The functors L and R of such a factorisation provide a pointing and copointing for each other, meaning that there are natural transformations $\eta : 1 \to R$ and $\varepsilon : L \to 1$, of which the components at some f are given by $(Lf, 1)$ and $(1, Rf)$ respectively. Now we can interpret the meaning of a (R, η)-algebra by taking the definition of an algebra for a monad but dropping the multiplication condition, see Definition 67. Spelling we this out we see that a morphism f is given (R, η)-algebra structure by a solution s to $(Lf, 1)$ and dually (L, ε)-coalgebra structure by a solution s to $(1, Rf)$:



Let us call these algebras and coalgebras R-maps and L-maps. Now as shown in [11, Lemma 2.8] we have the following.

Lemma 22. *For a functorial* WFS *the set of L-maps coincides with $\mathcal{L}$ and the set of R-maps with $\mathcal{R}$.*

Proof. First we note that if $f \in \mathcal{L}$, then because $Rf \in \mathcal{R}$ by assumption there is a lift s of $(Lf, 1)$. We argue similarly that every right map is an R-map. Now for the other direction we consider an L-map (f, s), an R-map (g, t) and a square $(u, v) : f \to g$. An easy calculation shows that $t \cdot E(u, v) \cdot s$ is a solution to (u, v):

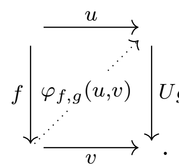


The argument for R-maps is the same. □

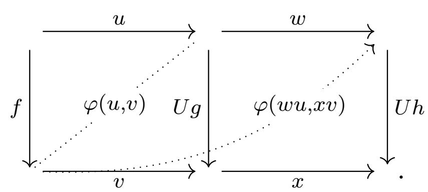
This result shows in what sense a functorial WFS is more structured than a regular one; the factorisation of morphisms in f as a left map followed by a right map, as well as the lifts between left and right maps, are given explicitly rather than merely postulated to exist. Moreover as mentioned in the introduction to this chapter we now have categorical structure on the classes of left and right maps due to the notions of algebra and coalgebra morphisms of Definition 67. Given two R-maps (f, s), (g, t) a morphism between them is a morphism between the underlying arrows $(u, v) : f \to g$ that also commutes with the algebra structure, meaning that $u \cdot s = t \cdot E(u, v)$. The definition of morphisms between L-maps is similar. This yields categories which we denote by L-**Map** and R-**Map**.

Now it is natural to ask whether we have $L\text{-}\mathbf{Map} \cong {}^{\pitchfork}R\text{-}\mathbf{Map}$ and $R\text{-}\mathbf{Map} \cong L\text{-}\mathbf{Map}^{\pitchfork}$ for a categorical version of the adjunction 4.1.1. We will see that this is not the case and this is one reason why one might want to further strengthen the notion of functorial WFS. Before we proceed we consider the categorical version of 4.1.1.

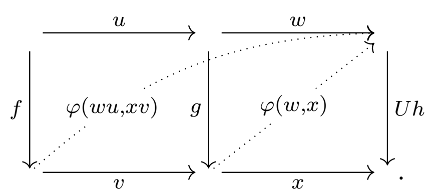
Definition 23. Let $\mathcal{C}$ be a category and $U : \mathcal{J} \to \mathcal{C}^2$ a functor. We define a category ${}^{\pitchfork}\mathcal{J}$ with as objects pairs (f, φ_{f-}) where f is a morphism in $\mathcal{C}$ and φ_{f-} a *left $\mathcal{J}$ lifting operation*. Such an operation assigns for each $g \in \mathcal{J}$ and $(u, v) : f \to Ug$ a diagonal filler $\varphi_{f,g}(u, v)$:



This assignment should be natural in the sense that when $\alpha : g \to h$ in $\mathcal{J}$ is over a square $(w, x) : Ug \to Uh$ in C^2 and $(u, v) : f \to g$ then $w \cdot \varphi_{f,g}(u, v) = \varphi_{f,h}(wu, xv)$:



The arrows between (f, φ_{f-}) and (g, φ_{g-}) in ${}^{\pitchfork}\mathcal{J}$ are given by arrows between the underlying morphisms $(u, v) : f \to g$ which cohere with the lifting operations in the sense that if $h \in \mathcal{J}$ and $(w, x) : g \to Uh$ then $\varphi_{g,h}(w, x) \cdot v = \varphi_{f,h}(wu, xv)$:



There is a dual notion of a *right $\mathcal{J}$ lifting operation* and a category $\mathcal{J}^{\pitchfork}$.

This construction comes with a functor ${}^{\pitchfork}U : {}^{\pitchfork}\mathcal{J} \to \mathcal{C}^2$ that forgets the lifting operation, and if $F : \mathcal{J} \to \mathcal{K}$ over $\mathcal{C}$ then we have ${}^{\pitchfork}F : {}^{\pitchfork}\mathcal{K} \to {}^{\pitchfork}\mathcal{J}$ given by $(f, \varphi_{f-}) \mapsto (f, \varphi_{f(F-)})$. This means we get a functor ${}^{\pitchfork}(-) : \mathbf{Cat}/\mathcal{C}^2 \to (\mathbf{Cat}/\mathcal{C}^2)^{\mathrm{op}}$ and similarly one $(-)^{\pitchfork}$ in the other direction, together constituting an adjunction [2, Proposition 15]:

$$\mathbf{Cat}/\mathcal{C}^2 \underset{(-)^{\pitchfork}}{\overset{{}^{\pitchfork}(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathbf{Cat}/\mathcal{C}^2)^{\mathrm{op}} \ . \tag{4.2.1}$$

These functors are sometimes called the *orthogonality* functors.

Now it is not hard to verify the lifts of L-maps against R-maps are natural in maps between R-maps, so that we get a functor $L\text{-}\mathbf{Map} \to {}^{\pitchfork}R\text{-}\mathbf{Map}$. There is a functor in the other direction as well but as mentioned these are in general not inverse to one another, and the same goes for $R\text{-}\mathbf{Map}$ and $L\text{-}\mathbf{Map}^{\pitchfork}$.

4.3 Algebraic weak factorisation systems

We have now developed the necessary vocabulary to define AWFS. This notion was introduced by Grandis and Tholen in [16] under the name *natural weak factorisation system* for the purpose of defining a variant of functorial WFS of which the classes $\mathcal{L}$ and $\mathcal{R}$ are closed under all colimits and limits (taken in $\mathcal{C}^2$) respectively. The problem with functorial WFS in this regard, as they identified, is that the lifts between the left and right maps are not chosen *naturally*. The solution is to extend the pointed and copointed endofunctors (R, η) and (L, ε) to a monad (R, η, μ) and comonad (L, ε, δ) respectively, which is achieved by the following definition.

Definition 24. An *algebraic weak factorisation system* (L, R) on a category $\mathcal{C}$ consists of the following data.

1. A functorial factorisation (L, R) on $\mathcal{C}$.
2. Natural transformations $\mu : RR \to R$ and $\delta : L \to LL$ which extend (R, η) to a monad $\mathsf{R} := (R, \eta, \mu)$ and (L, ε) to a comonad $\mathsf{L} := (L, \varepsilon, \delta)$.
3. The pair (L, R) satisfies a distributive law.

The last point was added by Garner in his work on AWFS in [17]. Since this condition will not directly play a role in our work we may safely ignore it.

Note that unlike in the definition of functorial WFS we do not require that the functorial factorisation is part of a WFS. A functorial factorisation almost constitutes a WFS (as we saw in Lemma 22) except that in general we do not have $Lf \in L\text{-}\mathbf{Map}$ and $Rf \in R\text{-}\mathbf{Map}$. However as we will see the multiplication and comultiplication give us these properties, which makes it redundant to require that this functorial factorisation is part of a WFS.

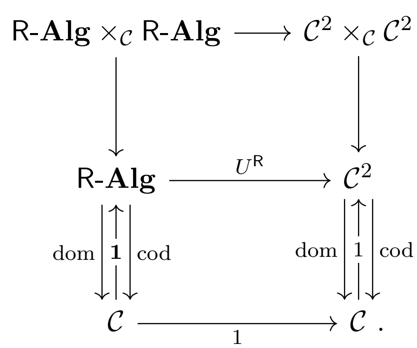
Let us consider what this definition constitutes and in what sense it can be considered a factorisation system. It follows from the unit condition of the monad R that for a morphism f in $\mathcal{C}$ the second component of μ_f is the identity. Therefore μ is really just a natural transformation $\mu : ER \to E$, and similarly $\delta : E \to EL$. Now as with a functorial WFS we get the following categories.

Definition 25. An R-*map* is an algebra for (R, η) and an R-*algebra* is an algebra for (R, η, μ), i.e. an R-map (f, s) that in addition satisfies $s \cdot E(s, 1) = s \cdot \mu_f$. Morphisms between R-algebras are the same as the morphisms between R-maps, so that we have categories $\mathsf{R}\text{-}\mathbf{Map}$ and $\mathsf{R}\text{-}\mathbf{Alg}$ along with a fully faithful inclusion $\mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{R}\text{-}\mathbf{Map}$. Dually we have the notion of L-*maps*, L-*coalgebras*, and an inclusion of categories $\mathsf{L}\text{-}\mathbf{Coalg} \to \mathsf{L}\text{-}\mathbf{Map}$.

The monad axiom stating associativity of multiplication expresses that for any morphism f its R image has an R-algebra structure (Rf, μ_f) and similarly the axiom for the comultiplication of L states that each Lf has L-coalgebra structure. Furthermore it is easy to verify that the sets of L-maps and R-maps are retract closed, and as we have seen in Lemma 22 that $f \pitchfork g$ for every L-map (f, s) and R-map (g, t). This means that the L-maps and R-maps are the left and right classes of a (functorial) WFS, which is often called the *underlying* WFS of the AWFS. We can now also verify that $\mathsf{L}\text{-}\mathbf{Map} \cong {}^{\pitchfork}\mathsf{R}\text{-}\mathbf{Alg}$ and $\mathsf{R}\text{-}\mathbf{Map} \cong \mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$. In this sense there is somewhat of a mismatch; for WFS we have $\mathcal{R} = \mathcal{L}^{\pitchfork}$ and $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$ so we might have expected to find an isomorphisms between $\mathsf{L}\text{-}\mathbf{Map}$ and ${}^{\pitchfork}\mathsf{R}\text{-}\mathbf{Map}$, and likewise for $\mathsf{R}\text{-}\mathbf{Map}$ and $\mathsf{L}\text{-}\mathbf{Map}^{\pitchfork}$. While we do have functors between these categories they do not form an isomorphism in general.

Now since Lf and Rf have L-coalgebra and R-algebra structure respectively, and since we have lifts of coalgebras against algebras exactly as in Lemma 22, we might wonder whether $\mathsf{L}\text{-}\mathbf{Coalg}$ and $\mathsf{R}\text{-}\mathbf{Alg}$ also form the classes of a WFS. This is not the case because these classes are not retract closed (in fact their retract closures are the classes of L-maps and R-maps respectively) so in this sense the notion of AWFS is weaker than that of a WFS. What we do get is an analog of $\mathcal{R} = \mathcal{L}^{\pitchfork}$ and $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$, but then on a double categorical level. To see how this works we need to consider the double categorical structure of the algebras and coalgebras.

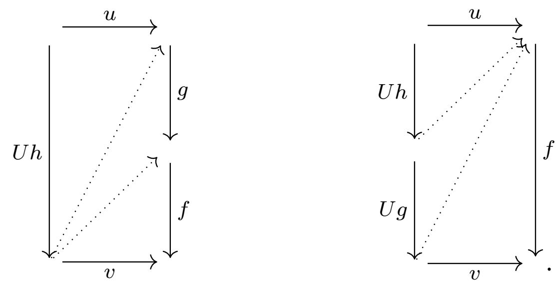
Given two R-algebras $\boldsymbol{f} : A \to B$ and $\boldsymbol{g} : B \to C$ there is a canonical R-algebra structure $\boldsymbol{g} \cdot \boldsymbol{f}$ on the composition $g \cdot f$. Additionally there is for each object $A \in C$ a unique R-algebra structure $\mathbf{1}_A$ on 1_A. These operations satisfy all the further requirements needed to make $\mathsf{R}\text{-}\mathbf{Alg}$ the arrow category of a *double category* $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$, i.e. an internal category in $\mathbf{Cat}$. The reader unfamiliar with double categories may find some more information on them in Appendix A.3. We now also have a forgetful double functor $U^{\mathsf{R}} : \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$:



Dually we have a double category $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}$ of L-coalgebras. These statements are not easily verified but they are proven in [2, Lemma 1 and Section 2.8]. In the same work a double categorical version of the adjunction 4.2.1 is constructed which we will now consider.

First we note if $U : \mathcal{J} \to \mathcal{C}^2$ we can construct a double category $\mathcal{J}^{\pitchfork\pitchfork}$ with object category $\mathcal{C}$ and arrow category $\mathcal{J}^{\pitchfork}$. Its identity functor associates to $A \in \mathcal{C}$ a right $\mathcal{J}$ lifting operation

φ_{-1_A} defined by $\varphi_{f,1_A}(u,v) = v$. Given (f, φ_{-f}), (g, φ_{-g}) for composable f and g we can define a lifting operation φ_{-fg} on their composition by $\varphi_{h,fg} = \varphi_{h,g}(u, \varphi_{h,f}(gu, v))$, as on the left of:



Now if we have instead a double functor $U : \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ then we can define a double category $\mathbb{J}^{\pitchfork}$ as a double subcategory of $\mathcal{J}_1^{\pitchfork}$ where the arrow category $\mathcal{J}_1^{\pitchfork}$ is defined as the subcategory of $\mathcal{J}_1^{\pitchfork}$ whose objects satisfy the additional property that they respect composition in $\mathbb{J}$ in the following sense. If $g, h \in \mathcal{J}_1$ are vertically composable maps and $(u, v) : U(gh) \to f$ then we should have $\varphi_{gh,f}(u,v) = \varphi_{g,f}(\varphi_{h,f}(u, vg), v)$.

Of course we have dual constructions for the left lifting operations, together yielding a double categorical version of the adjunction 4.2.1:

$$\mathbf{Dbl}/\mathbb{S}\mathbf{q}(\mathcal{C}) \underset{(-)^{\pitchfork}}{\overset{^{\pitchfork}(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathbf{Dbl}/\mathbb{S}\mathbf{q}(\mathcal{C}))^{\mathrm{op}} \ . \tag{4.3.1}$$

It is shown in [2, Proposition 20] that there are double isomorphisms $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg} \cong {}^{\pitchfork}\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ and $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \cong \mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork}$.

Chapter 5

The Frobenius condition

We saw in Section 3.3 that in order to interpret product types using clans, we needed that the pushforward of a fibration along a fibration is again a fibration. In Chapter 6 we will see that the forgetful functor $U^{\mathsf{R}} : \mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^2$ is a comprehension category that can be used to interpret type theory in a similar way as with clans. The fibrations are then given by R-algebras, so in order to interpret product types we will need that the pushforward of an R-algebra along an R-algebra is again an R-algebra. In practice this condition can be difficult to verify but luckily it admits an equivalent phrasing in terms of the pullback functors which is easier to verify. This phrasing says that the pullback of an L-coalgebra along an R-algebra is again an L-coalgebra and is often called a *Frobenius property* or *Frobenius condition*. The phrasing and proof of this equivalence for WFS is well known but the analog for AWFS which we give here is new.

The work in this chapter draws inspiration from the results and methods of Gambino and Sattler in [4] where it is shown that this statement holds when phrased for the L- and R-maps of an AWFS.

We begin in Section 5.1 by reviewing the statement for WFS as it commonly appears in the literature. We then propose a slightly different perspective which we argue simplifies the statement and proof and allows for easier generalisations to categories and double categories of maps. Then in Section 5.2 we precisely state and prove this equivalence. Next in Section 5.3 we show that the components of this proof have analogs in the cases of categories and double categories of maps, and then use these for double categories to give a proof for that case in Section 5.4. Lastly we look at what is needed for lifting the *Beck-Chevalley* morphism in Section 5.5, since this is needed for interpreting the product types as we saw in Section 3.3.

5.1 The Frobenius condition for weak factorisation systems

The Frobenius condition for WFS states that the the pullback of a left map along a right map is again a left map. In other words, when we have a pullback square $(i, f) : h \to g$ then if $f \in \mathcal{R}$ and $g \in \mathcal{L}$, then $h \in \mathcal{L}$:

$$\begin{array}{ccc} \cdot & \xrightarrow{i} & \cdot \\ {\scriptstyle h}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ \cdot & \xrightarrow[f]{} & \cdot \end{array} \quad . \tag{5.1.1}$$

A choice of pullbacks induces for every morphisms a pullback functor, and if these in addition have right adjoint pushforward functors then the Frobenius condition admits an equivalent rephrasing in terms of these pushforward functors.

Lemma 26. *For a* WFS $(\mathcal{L}, \mathcal{R})$ *the following are equivalent.*

1. *The pullback of an* $\mathcal{L}$ *map along an* $\mathcal{R}$ *map is an* $\mathcal{L}$ *map.*
2. *The pushforward of an* $\mathcal{R}$ *map along an* $\mathcal{R}$ *map is an* $\mathcal{R}$ *map.*

The square 5.1.1 suggests that in the context of a choice of pullbacks the Frobenius condition states that for any cospan formed by $f \in \mathcal{R}$ and $g \in \mathcal{L}$ we should have $f^*g \in \mathcal{L}$, in other words: the *object* components of the pullback functors induced by $\mathcal{R}$ maps preserve $\mathcal{L}$ maps. This perspective is also present in the definition of the *functorial Frobenius condition* used in [8], [4, Definition 6.1], and [3, Definition 2.8], which is stated as the existence of a lift $\tilde{P}$ of the pullback functor P that maps a cospan (f, g) to f^*g:

$$\begin{array}{ccc} \mathsf{R}\text{-}\mathbf{Map} \times_{\mathcal{C}} \mathsf{L}\text{-}\mathbf{Map} & \overset{\tilde{P}}{\dashrightarrow} & \mathsf{L}\text{-}\mathbf{Map} \\ \downarrow & & \downarrow \\ \mathcal{C}^2 \times_{\mathcal{C}} \mathcal{C}^2 & \xrightarrow{P} & \mathcal{C}^2 \end{array} \quad .$$

In the present work we will shift the focus from the object component of the pullback functors to the *arrow* component, in the sense that we phrase the Frobenius condition as saying that if $f : A \to B$ is an $\mathcal{R}$ map, $u : C \to B$ an arbitrary map, and $g : D \to C$ an $\mathcal{L}$ map, then f^*g is an $\mathcal{L}$ map. The reason for making this shift is that it seems to clarify the statement and proof of Lemma 26 and considerably eases the task of finding an analogous condition for AWFS.

Before stating and proving the rephrasing formally we consider how this shift affects Lemma 26. Suppose we phrase the Frobenius condition not on the level of a WFS but on the level of a fixed arbitrary morphism f in the underlying category $\mathcal{C}$, i.e. we say f satisfies the object (resp. arrow) Frobenius property if the object (resp. arrow) component of f^* preserves $\mathcal{L}$ maps. Similarly we say f satisfies the object (resp. arrow) pushforward property if f_* preserves $\mathcal{R}$ maps. Now we can ask ourselves whether the equivalence of Lemma 26 still holds on the level of a fixed map f for both the arrow and object variants. In other words, is it the case that a map f satisfies the object (resp. arrow) Frobenius property if and only if it satisfies the object (resp. arrow) pushforward property? It seems that this is only the case for the arrow variant of the conditions, even though the proof of this fact is largely the same as the proof for Lemma 26. Note also that this equivalence holds for an *arbitrary* f, rather than for an $\mathcal{R}$ map.

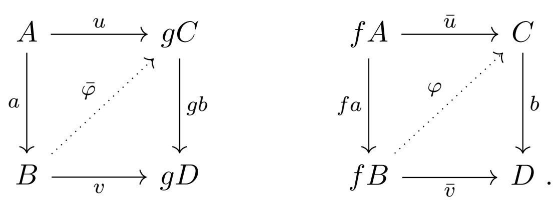
5.2 The Frobenius equivalence for classes of maps

We will now carefully formalize and prove our rephrasing so that we may then generalize the result to AWFS.

For $f \dashv g : \mathcal{D} \to \mathcal{C}$ there is a relation between $\pitchfork$ on $\mathcal{C}$ and $\mathcal{D}$ which is expressed by the following two lemmas[1].

Lemma 27. *For morphisms a in $\mathcal{C}$ and b in $\mathcal{D}$ we have $fa \pitchfork b$ if and only if $a \pitchfork gb$.*

Proof. For left to right we note that $(u, v) : a \to gb$ induces a square $(\bar{u}, \bar{v}) : fa \to b$ by transposing, so by assumption there is a lift $\varphi : fB \to C$. Transposing this back gives us a solution $\bar{\varphi} : B \to gC$ to (u, v):



The other direction is dual. ∎

From this we obtain a kind of change of base lemma for $\pitchfork$, which is the analog of [2, Proposition 21].

Lemma 28. *Let $J \subseteq \mathcal{C}_1$ and $K \subseteq \mathcal{D}_1$, then $f(J) \subseteq {}^{\pitchfork}K$ if and only if $g(K) \subseteq J^{\pitchfork}$.*

Proof. From the previous lemma we get ${}^{\pitchfork}g(K) = f^{-1}({}^{\pitchfork}K)$ and so

$$\begin{aligned} f(J) \subseteq {}^{\pitchfork}K &\text{ iff } J \subseteq f^{-1}({}^{\pitchfork}K) \\ &\text{ iff } J \subseteq {}^{\pitchfork}g(K) \\ &\text{ iff } g(K) \subseteq J^{\pitchfork}. \end{aligned}$$

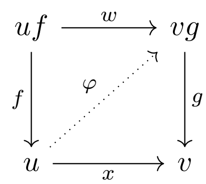
∎

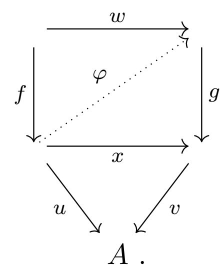
[1]These also appear on Joyal's CatLab site at https://ncatlab.org/joyalscatlab/published/Weak+factorisation+systems as Lemma 4.1 and Proposition 4.2, the latter of which is phrased for WFS.

Next we need a result regarding commutativity of our slicing operation defined earlier and the lifting operations. Let $J \subseteq \mathcal{C}_1$ and $A \in \mathcal{C}_0$, we define $J/A := \{(f, u) \in (\mathcal{C}/A)_1 \mid f \in J\}$ i.e. the set of arrows in $\mathcal{C}/A$ whose underlying morphism is a member of J.

Lemma 29. *Let $J \subseteq \mathcal{C}_1$ and $A \in \mathcal{C}_0$, then $(J^{\pitchfork})/A = (J/A)^{\pitchfork}$ and $({}^{\pitchfork}J)/A \subseteq {}^{\pitchfork}(J/A)$.*

Proof. Let us first consider when for some $(f, u), (g, v) \in (\mathcal{C}/A)_1$ it holds that $(f, u) \pitchfork (g, v)$. A commuting square between (f, u) and (g, v) in $\mathcal{C}/A$ consists of a pair of maps $w : uf \to vg$ and $x : u \to v$ such that $gw = xf$ and looks as on the left of:



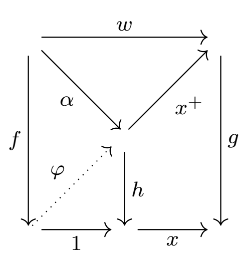


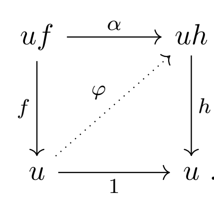
This situation paints a picture in $\mathcal{C}$ as on the right above. Composition in $\mathcal{C}/A$ is just composition in $\mathcal{C}$ so a solution φ to the square on the left is a solution to $(w, x) : f \to g$ and vice versa (but note that $(f, u) \pitchfork (g, v)$ does not imply $f \pitchfork g$). With that in mind it is easy to verify our claims. Consider for instance $(\boldsymbol{g}, v) \in (J^{\pitchfork})/A$, then if we have a square (w, x) as on the left above for some (f, u) with $f \in J$ then we simply take the lift of $(w, x) : f \to g$ as our solution to $(w, x) : (f, u) \to (g, v)$. Conversely if $(g, v) \in (J/A)^{\pitchfork}$ and $(w, x) : f \to g$ for some $f \in J$ then lifting $(w, x) : (f, vx) \to (g, v)$ provides us with our solution. Lastly if $(\boldsymbol{f}, u) \in ({}^{\pitchfork}J)/A$ we can again just solve $(w, x) : f \to g$ to obtain a solution. $\square$

One inclusion is missing from the previous lemma. For this we need that the ambient category $\mathcal{C}$ has pullbacks of J maps and that the pullback of a J map is again in J. It is well known that sets in the image of $(-)^{\pitchfork}$ satisfy this property, as we also saw in Lemma 19.

Lemma 30. *Let $\mathcal{C}$ be a category with pullbacks, $J \subseteq \mathcal{C}_1$ be closed under these pullbacks, and $A \in \mathcal{C}_0$; then ${}^{\pitchfork}(J/A) \subseteq ({}^{\pitchfork}J)/A$.*

Proof. Let $(f, u) \in {}^{\pitchfork}(J/A)$ and $(w, x) : f \to g$ for some $g \in J$. By assumption there is a pullback square $(x^+, x) : h \to g$ with h underlying some $\boldsymbol{h} \in J$. This means $(\boldsymbol{h}, u) \in J/A$, and so we can obtain a solution φ to $(\alpha, 1) : (f, u) \to (h, u)$ as depicted on the right below:





Now $x^+\varphi$ is a solution to (w, x) because $x^+\varphi f = x^+\alpha = w$ and $gx^+\varphi = xh\varphi = x$. □

In summary we have the following corollary.

Corollary 31. *If $\mathcal{C}$ has pullbacks and $J, K \subseteq \mathcal{C}_1$ are classes of maps with $J = {}^{\pitchfork}K$ and $K = J^{\pitchfork}$ then $(J/A)^{\pitchfork} = (J^{\pitchfork})/A$ and ${}^{\pitchfork}(K/A) = ({}^{\pitchfork}K)/A$ for any $A \in \mathcal{C}$.*

We are now ready to give the proof of our rephrasing of Lemma 26. Let $\mathcal{C}$ be a category with a choice of pullbacks and $J, K \subseteq \mathcal{C}_1$ such that $J = {}^{\pitchfork}K$ and $K = J^{\pitchfork}$.

Definition 32. A map $f : A \to B$ in $\mathcal{C}$ satisfies the *Frobenius property* with respect to (J, K) if $f^*(J/B) \subseteq (J/A)$, i.e. if the arrow component of f^* preserves J maps. Likewise we say it satisfies the *pushforward property* if $f_*(K/A) \subseteq (K/B)$.

Proposition 33 (Frobenius equivalence)**.** *The Frobenius and pushforward properties are equivalent.*

Proof. Let $f : A \to B$, then we have:

$$\begin{aligned} f^*(J/B) \subseteq (J/A) &\text{ iff } f^*(J/B) \subseteq {}^{\pitchfork}(K/A) && \text{(by } J = {}^{\pitchfork}K \text{ and Corollary 31)} \\ &\text{ iff } f_*(K/A) \subseteq (J/B)^{\pitchfork} && \text{(by Lemma 28)} \\ &\text{ iff } f_*(K/A) \subseteq (K/B). && \text{(by } J^{\pitchfork} = K \text{ and Corollary 31)} \end{aligned}$$
□

5.3 Analogs for categories and double categories of maps

The Frobenius equivalence of Proposition 33 is stated at the level of *functions* $J \to \mathcal{C}_1$ and we will now phrase and prove analogs at the level of *functors* $\mathcal{J} \to \mathcal{C}^2$ and *double functors* $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$. Generalizing set theoretic constructions like this is a common practice in category theory and the recipe is simple: rephrase everything in terms of existence of arrows and commutativity of diagrams.

For instance our construction of J/A can be seen as the pullback of the inclusion $J \to \mathcal{C}_1$ along the arrow component of the domain functor $\mathrm{dom} : \mathcal{C}/A \to \mathcal{C}$ which sends arrows with codomain A to their domain, and acts as the inclusion on arrows between them. This evidently has analogs at the categorical and double categorical levels as illustrated below:

$$\begin{array}{ccc} J/A & \longrightarrow & J \\ \downarrow & \lrcorner & \downarrow \\ (\mathcal{C}/A)_1 & \xrightarrow{\mathrm{dom}_1} & \mathcal{C}_1 \end{array} \qquad \begin{array}{ccc} \mathcal{J}/A & \longrightarrow & \mathcal{J} \\ \downarrow & \lrcorner & \downarrow \\ (\mathcal{C}/A)^2 & \xrightarrow{\mathrm{dom}^2} & \mathcal{C}^2 \end{array} \qquad \begin{array}{ccc} \mathbb{J}/A & \longrightarrow & \mathbb{J} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{S}\mathbf{q}(\mathcal{C}/A) & \xrightarrow{\mathbb{S}\mathbf{q}(\mathrm{dom})} & \mathbb{S}\mathbf{q}(\mathcal{C}) \end{array} .$$

The definition given by the the middle square coincides with the definition of slicing used in [4, Sections 5 and 6] although it is not explicitly formulated as a pullback there.

Likewise we can consider $f^*(J/B) \subseteq (J/A)$ to say that $f^*{\restriction}(J/B) : J/B \to J/A$ which in turn is to say there *exists* a function $\boldsymbol{f}^* : J/B \to J/A$ which fits into a diagram over $f^* : (C/A)_1 \to (C/B)_1$ as in the bottom left square below. Again we easily find analogs as shown in the middle and right squares:

$$\begin{array}{ccc} J/B & \overset{\boldsymbol{f}^*}{\dashrightarrow} & J/A \\ \downarrow & & \downarrow \\ (\mathcal{C}/B)_1 & \xrightarrow{f_1^*} & (\mathcal{C}/A)_1 \end{array} \qquad \begin{array}{ccc} \mathcal{J}/B & \overset{\boldsymbol{f}^*}{\dashrightarrow} & \mathcal{J}/A \\ \downarrow & & \downarrow \\ (\mathcal{C}/B)^2 & \xrightarrow{f^{*2}} & (\mathcal{C}/A)^2 \end{array} \qquad \begin{array}{ccc} \mathbb{J}/B & \overset{\boldsymbol{f}^*}{\dashrightarrow} & \mathbb{J}/A \\ \downarrow & & \downarrow \\ \mathbb{S}\mathbf{q}(\mathcal{C}/B) & \xrightarrow{\mathbb{S}\mathbf{q}(f^*)} & \mathbb{S}\mathbf{q}(\mathcal{C}/A) \end{array} .$$

Likewise we can phrase our change of base lemma 28 as saying that given an adjunction $f \dashv g : \mathcal{D} \to \mathcal{C}$ and inclusions $J \to \mathcal{C}_1$ and $K \to \mathcal{D}_1$ there is a bijection between inclusions $J \to {}^{\pitchfork}K$ over $f_1 : \mathcal{C}_1 \to \mathcal{D}_1$ and $K \to J^{\pitchfork}$ over $g_1 : \mathcal{D}_1 \to \mathcal{C}_1$:

$$\begin{array}{ccc} J & \longrightarrow & {}^{\pitchfork}K \\ \downarrow & & \downarrow \\ \mathcal{C}_1 & \xrightarrow{f_1} & \mathcal{D}_1 \end{array} \qquad \begin{array}{ccc} K & \longrightarrow & J^{\pitchfork} \\ \downarrow & & \downarrow \\ \mathcal{D}_1 & \xrightarrow{g_1} & \mathcal{C}_1 \end{array} .$$

We have already encountered the analogs of the Galois connection of the lifting operations, and in [2, Proposition 21] Bourke and Garner extend the adjunction 4.3.1 to account for change of base. In particular it states that given an adjunction $f \dashv g$ and double functors $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$, $\mathbb{K} \to \mathbb{S}\mathbf{q}(\mathcal{D})$, there is a bijection between double functors $\mathbb{J} \to {}^{\Pitchfork}\mathbb{K}$ over $\mathbb{S}\mathbf{q}(f)$ and $\mathbb{K} \to \mathbb{J}^{\Pitchfork}$ over $\mathbb{S}\mathbf{q}(g)$ as in the diagrams below:

$$\begin{array}{ccc} \mathbb{J} & \longrightarrow & {}^{\Pitchfork}\mathbb{K} \\ \downarrow & & \downarrow \\ \mathbb{S}\mathbf{q}(\mathcal{C}) & \xrightarrow{\mathbb{S}\mathbf{q}(f)} & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{array} \qquad \begin{array}{ccc} \mathbb{K} & \longrightarrow & \mathbb{J}^{\Pitchfork} \\ \downarrow & & \downarrow \\ \mathbb{S}\mathbf{q}(\mathcal{D}) & \xrightarrow{\mathbb{S}\mathbf{q}(g)} & \mathbb{S}\mathbf{q}(\mathcal{C}) \end{array} .$$

This also implies such a result for the categorical level, which is stated explicitly in [4, Proposition 5.7].

Now what remains is phrasing analogs of the property that a class of maps $J \to \mathcal{C}_1$ is closed under pullbacks, and that the classes in the image of $(-)^{\pitchfork}$ satisfy this property. In [4, Proposition 5.4] a clever way of phrasing that a functor $U : \mathcal{J} \to \mathcal{C}^2$ is closed under pullbacks is given: we require that U is a comprehension category, i.e. that $\operatorname{cod} U$ is a Grothendieck fibration. It is then shown that this implies there is an isomorphism ${}^{\pitchfork}(\mathcal{J}/A) \cong ({}^{\pitchfork}\mathcal{J})/A$ for any $A \in \mathcal{C}$. In the next section we look at this result in more detail and give an analog for double categories of maps $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$.

5.3.1 Commutativity of slicing and the orthogonality functors

We consider a category of maps $U : \mathcal{J} \to \mathcal{C}^2$. Much like in the case of Lemma 29 we get 3 of the 4 required functors without further pullback related requirements on $\mathcal{J}$ or $\mathcal{C}$.

Lemma 34. *For $\mathcal{J} \to \mathcal{C}^2$ and $A \in \mathcal{C}$ there is an isomorphism $(\mathcal{J}/A)^{\pitchfork} \cong (\mathcal{J}^{\pitchfork})/A$ and a functor $({}^{\pitchfork}\mathcal{J})/A \to {}^{\pitchfork}(\mathcal{J}/A)$.*

Proof. In each case we use the corresponding construction in Lemma 29 to define a lifting operation for which the required properties are then easily proven. □

For the last one, as shown in [4, Proposition 5.4], we need that $\operatorname{cod} U$ is a Grothendieck fibration.

Lemma 35. *If $\operatorname{cod} U$ is a Grothendieck fibration then $({}^{\pitchfork}\mathcal{J})/A \to {}^{\pitchfork}(\mathcal{J}/A)$ has an inverse.*

Proof. The proof is as that of Lemma 30; we use the construction of the lift given there to define a lifting operation, only now we take a cartesian lift rather than just a pullback. Naturality of the resulting lifting operation is then proven using the universal property of that lift. □

Lastly we need that the functors in the essential image of $(-)^{\pitchfork}$ satisfy this additional property, which is shown by the following lemma. We note that $\mathcal{C}$ need not necessarily have all pullbacks, but at least for the morphisms in $\mathcal{J}^{\pitchfork}$ under consideration.

Lemma 36. *Let $U : \mathcal{J} \to \mathcal{C}^2$, then $\operatorname{cod} U^{\pitchfork}$ is a Grothendieck fibration.*

Proof. Let $\boldsymbol{f} \in \mathcal{J}^{\pitchfork}$ be over f in $\mathcal{C}^2$. Given a cospan (v, f) we take the pullback of v^*f:

$$\begin{array}{ccc} \cdot & \xrightarrow{v^+} & \cdot \\ {\scriptstyle v^*f}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \cdot & \xrightarrow[v]{} & \cdot \end{array}.$$

Now we can use the lift constructed in Lemma 19 to define a right $\mathcal{J}$ lifting operation for v^*g, which is unique with respect to the property that it makes (v^+, v) a morphism in $\mathcal{J}^{\pitchfork}$. It is then easily verified that this lifting operation satisfies the naturality condition, and that (v^+, v) is cartesian. □

In summary we have the following corollary.

Corollary 37. *Let $\mathcal{J} \to \mathcal{C}^2$ and $\mathcal{K} \to \mathcal{C}^2$ satisfy $\mathcal{J} \cong {}^{\pitchfork}\mathcal{K}$ and $\mathcal{K} \cong \mathcal{J}^{\pitchfork}$, then for any $A \in \mathcal{C}$ there are isomorphisms $(\mathcal{J}/A)^{\pitchfork} \cong (\mathcal{J}^{\pitchfork})/A$ and ${}^{\pitchfork}(\mathcal{K}/A) \cong ({}^{\pitchfork}\mathcal{K})/A$.*

The case for a double category $U : \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ is now very similar. With no additional assumptions on $\mathcal{C}$ we obtain a double isomorphism $(\mathbb{J}/A)^{\pitchfork} \cong (\mathbb{J}^{\pitchfork})/A$ and a double functor $({}^{\pitchfork}\mathbb{J})/A \to {}^{\pitchfork}(\mathbb{J}/A)$.

Lemma 38. *For $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ there is a double isomorphism $(\mathbb{J}/A)^{\pitchfork} \cong (\mathbb{J}^{\pitchfork})/A$ and a double functor $({}^{\pitchfork}\mathbb{J})/A \to {}^{\pitchfork}(\mathbb{J}/A)$.*

Proof. The strategy is to take the lifting operations of Lemma 34 and show they satisfy all the additional requirements by making use of the additional assumptions. This requires some straightforward verifications which we omit. □

In order to obtain the missing functor we should again demand that $\mathcal{C}$ has (enough) pullbacks and that $\operatorname{cod} U_1$ is a Grothendieck fibration, only now we also need that it respects vertical composition in $\mathbb{J}$ in the following sense. Let $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{J}_1$ be vertically composable morphisms of $\mathbb{J}$, and consider a cospan (v, fg). Now we could either first take the cartesian lift with respect to $\boldsymbol{f}$, and then lift the result along $\boldsymbol{g}$, or we could lift directly along the vertical composition $\boldsymbol{fg}$; we want these lifts to coincide.

Lemma 39. *If $\operatorname{cod} U_1$ is a Grothendieck fibration which respects vertical composition in $\mathbb{J}$ then $({}^{\pitchfork}\mathbb{J})/A \to {}^{\pitchfork}(\mathbb{J}/A)$ has an inverse.*

Proof. The argument is entirely analogous to that of Lemma 35 and Lemma 30. Again the verification requires some lengthy but straightforward calculations which we omit. □

Lemma 40. *Let $U : \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$, then $\operatorname{cod} \cdot (U^{\pitchfork})_1$ is a Grothendieck fibration which respects vertical composition of $\mathbb{J}^{\pitchfork}$.*

Proof. The argument is largely the same as in Lemma 36; we consider $\boldsymbol{f} \in \mathcal{J}_1^{\pitchfork}$ over f and a cospan (v, f), then there is a unique lifting operation on v^*f defined as before, unique w.r.t. the property that square (v, v^+) underlies a morphism in $\mathcal{J}_1^{\pitchfork}$. To check the condition regarding composition we consider vertically composable $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{J}_1^{\pitchfork}$ and a cospan (v, fg):

$$
\begin{array}{ccc}
\cdot & \xrightarrow{v^{++}} & \cdot \\
{\scriptstyle v^*g}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\
\cdot & \xrightarrow{v^{+}} & \cdot \\
{\scriptstyle v^*f}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\
\cdot & \xrightarrow[v]{} & \cdot
\end{array}
.
$$

Since both of these are squares of $\mathbb{J}^{\pitchfork}$ we can vertically compose them, which means (v^{++}, v) is a square of $\mathbb{J}^{\pitchfork}$ from $v^*f \cdot v^*g \to fg$. Therefore since $v^*(fg)$ is unique with respect to this property it follows that it is equal to the vertical composition of v^*f and v^*g. □

In summary we have the following corollary.

Corollary 41. *Let $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ and $\mathbb{K} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ satisfy $\mathbb{J} \cong {}^{\pitchfork}\mathbb{K}$ and $\mathbb{K} \cong \mathbb{J}^{\pitchfork}$, then for any $A \in \mathcal{C}$ there are isomorphisms $(\mathbb{J}/A)^{\pitchfork} \cong (\mathbb{J}^{\pitchfork})/A \cong \mathbb{K}/A$ and ${}^{\pitchfork}(\mathbb{K}/A) \cong ({}^{\pitchfork}\mathbb{K})/A \cong \mathbb{J}/A$.*

5.4 The Frobenius equivalence for double categories of maps

We now have the necessary results to establish categorical and double categorical versions of Proposition 33. We only state the one for double categories explicitly.

Definition 42. A morphism $f : A \to B$ satisfies the *Frobenius property* with respect to $(\mathbb{J}, \mathbb{K})$ if there exists a lift $\boldsymbol{f}^* : \mathbb{J}/B \to \mathbb{J}/A$ over $\mathbb{S}\mathbf{q}(f^*)$, and similarly the *pushforward property* if there is a lift $\boldsymbol{f}_* : \mathbb{K}/A \to \mathbb{K}/B$ over $\mathbb{S}\mathbf{q}(f_*)$.

The analog of Lemma 28 is given by [2, Proposition 21] which is phrased by first extending the adjunction 4.3.1 using the categories $\mathbf{Dbl}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}})$ and $\mathbf{Dbl}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}})$. An object in $\mathbf{Dbl}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}})$ is a category $\mathcal{C}$ together with a double functor $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$, and an arrow between two objects $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ and $\mathbb{K} \to \mathbb{S}\mathbf{q}(\mathcal{D})$ is an adjunction $f \dashv g : \mathcal{D} \to \mathcal{C}$ with a lift $\boldsymbol{f} : \mathbb{J} \to \mathbb{K}$ over $\mathbb{S}\mathbf{q}(f)$. The category $\mathbf{Dbl}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}})$ is defined dually.

Proposition 43. *Let $\mathbb{J}$ and $\mathbb{K}$ be double categories over $\mathbb{S}\mathbf{q}(\mathcal{C})$ for some $\mathcal{C}$ with pullbacks satisfying $\mathbb{J} \cong {}^{\pitchfork}\mathbb{K}$ and $\mathbb{K} \cong \mathbb{J}^{\pitchfork}$, then the Frobenius and pushforward properties w.r.t. $(\mathbb{J}, \mathbb{K})$ are equivalent.*

Proof. Consider a map $f : A \to B$. Using Corollary 41 and [2, Proposition 21] we have

$$\begin{aligned}
\mathbf{Dbl}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}})(\mathbb{J}/B, \mathbb{J}/A) &\cong \mathbf{Dbl}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}})(\mathbb{J}/B, (\mathbb{K}/A)^{\pitchfork}) \\
&\cong \mathbf{Dbl}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}})(\mathbb{K}/A, {}^{\pitchfork}(\mathbb{J}/B)) \\
&\cong \mathbf{Dbl}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}})(\mathbb{K}/A, \mathbb{K}/B),
\end{aligned}$$

where we consider morphisms in $\mathbf{Dbl}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}})$ and $\mathbf{Dbl}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}})$ with respect to the adjunction $f^* \dashv f_*$. □

This proposition demonstrates the benefit of having rephrased the Frobenius property for classes of maps as in Definition 32. It made it clear what the analogous statement for categories and double categories of maps should be and how we should prove it. While the original statement of Lemma 26 may just as easily be proven directly this is certainly not the case for Proposition 43.

5.5 Lifting the Beck-Chevalley isomorphism

In the preceding sections we have considered lifts of functors to categories of maps, i.e. when $f : \mathcal{C} \to \mathcal{D}$ is a functor and there are categories of maps $\mathcal{J} \to \mathcal{C}^2$ and $\mathcal{K} \to \mathcal{D}^2$ then there may be a lift $\boldsymbol{f}$ of f^2. Given two such lifts we can also consider lifts of natural transformations. By this we mean the following. Consider functors $f, g : \mathcal{C} \to \mathcal{D}$, a natural transformation $\alpha : f \to g$, categories of maps $\mathcal{J} \to \mathcal{C}$ and $\mathcal{K} \to \mathcal{D}$, and lifts $\boldsymbol{f}, \boldsymbol{g}$ of f, g, then there may be a natural transformation $\boldsymbol{\alpha} : \boldsymbol{f} \to \boldsymbol{g}$ of which the components are over those of α:

$$\begin{array}{ccc} \mathcal{J} & \overset{\boldsymbol{f}}{\underset{\boldsymbol{g}}{\rightrightarrows}} \; \Downarrow \boldsymbol{\alpha} & \mathcal{K} \\ \downarrow & & \downarrow \\ \mathcal{C}^2 & \overset{f^2}{\underset{g^2}{\rightrightarrows}} \; \Downarrow \alpha^2 & \mathcal{D}^2 \end{array} .$$

We conclude this chapter by looking at how we can lift the natural transformation BC defined in Section 3.3 in this way, so that we can mimic the approach used there for interpreting Π-types in the next chapter using AWFS. A method for doing so was outlined in [4, Section 6], and we will see that the exact same core ideas apply to our situation. However since in *loc. cit.* the focus is on categories rather than double categories we do have to make some small adaptations to account for that.

Roughly speaking the strategy is to additionally assume that a class of maps satisfies the Frobenius property *functorially*, which requires that maps between them are preserved by the Frobenius construction. This can then be used to lift a mate of BC which in turn implies BC lifts as well. To begin we explicitly note a consequence of [2, Proposition 21].

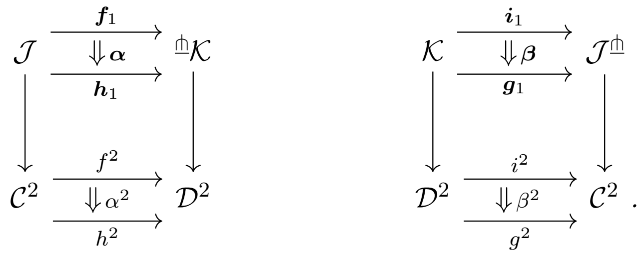
Lemma 44. *For an adjunction $f \dashv g : \mathcal{D} \to \mathcal{C}$ and $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$, $\mathbb{K} \to \mathbb{S}\mathbf{q}(\mathcal{D})$, there is a bijection between lifts $\boldsymbol{f}$ and $\boldsymbol{g}$ as in the diagrams:*

$$\begin{array}{ccc} \mathbb{J} & \xrightarrow{\boldsymbol{f}} & {}^{\underline{\pitchfork}}\mathbb{K} \\ \downarrow & & \downarrow \\ \mathbb{S}\mathbf{q}(\mathcal{C}) & \xrightarrow{\mathbb{S}\mathbf{q}(f)} & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{array} \qquad \begin{array}{ccc} \mathbb{K} & \xrightarrow{\boldsymbol{g}} & \mathbb{J}^{\underline{\pitchfork}} \\ \downarrow & & \downarrow \\ \mathbb{S}\mathbf{q}(\mathcal{D}) & \xrightarrow{\mathbb{S}\mathbf{q}(g)} & \mathbb{S}\mathbf{q}(\mathcal{C}) \end{array} .$$

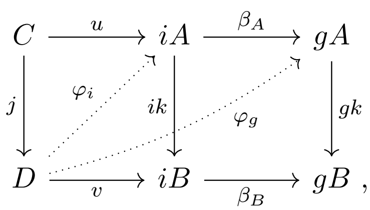
A complete proof can be found in [2, Proposition 21] but the argument is essentially as in Lemma 27. For instance if we have such a lift $\boldsymbol{f}$ then $\boldsymbol{g}$ can be defined by transposing the lifts provided by $\boldsymbol{f}$.

If we have two adjunctions between $\mathcal{C}$ and $\mathcal{D}$ with lifts that correspond under the bijection of the previous lemma then there is also a bijection between lifts of natural transformations between them, as stated by the next lemma. This is the same statement as [3, Proposition 5.8] but then phrased for $\underline{\pitchfork}$ instead of $\pitchfork$ or in other words for classes of maps with lifting operations that respect the vertical composition operation of their respective double category. As we will see the proof is virtually the same.

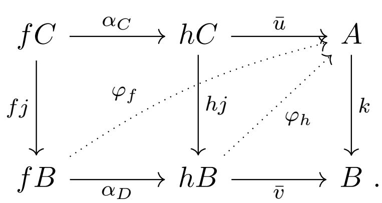
Lemma 45. *Consider* $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ *and* $\mathbb{K} \to \mathbb{S}\mathbf{q}(\mathcal{D})$ *and functors* $f, h : \mathcal{C} \to \mathcal{D}$ *and* $g, i : \mathcal{D} \to \mathcal{C}$ *such that* $f \dashv g$ *and* $h \dashv i$*. Suppose that* f *and* g *have lifts which correspond to each other according to the bijection in Lemma 44, and likewise for* h *and* i*. Then if* $\alpha : f \to h$ *and* $\beta : i \to g$ *are mates there is a lift* $\boldsymbol{\alpha} : \boldsymbol{f}_1 \to \boldsymbol{h}_1$ *of* $\alpha^2 : f^2 \to h^2$ *if and only if there is a lift* $\boldsymbol{\beta} : \boldsymbol{i}_1 \to \boldsymbol{g}_1$*, as in the following diagrams where* $\mathcal{J}$ *and* $\mathcal{K}$ *denote the arrow categories of* $\mathbb{J}$ *and* $\mathbb{K}$*:*



Proof. We show left to right, the other direction is dual. Consider $\boldsymbol{k} \in \mathcal{K}$ over $k : A \to B$. We want to show (β_C, β_D) underlies a morphism in $\mathcal{J}^{\pitchfork}$, so we consider $\boldsymbol{j} \in \mathcal{J}$ over $j : C \to D$ and a square $(u, v) : j \to ik$. Now we should show that $\beta_A \cdot \varphi_i = \varphi_g$ as in



where φ_i and φ_g denote the evident lifts. Transposing yields the following picture in $\mathcal{D}$:



Our assumption that α lifts means $\varphi_h \cdot \alpha_D = \varphi_f$. Since we assumed f and g correspond to each other according to the bijection in Lemma 44 we have $\varphi_f = \bar{\varphi}_g$, and similarly $\varphi_h = \bar{\varphi}_i$. Using these equations, the fact that α and β are mates, and the usual properties of adjunctions, we can now compute that $\beta_A \cdot \varphi_i = \varphi_g$ as desired. □

Now let us reconsider our goal. Given a pullback square $(u, v) : f \to g$ where the pullback functors along f and g have right adjoints, the canonical isomorphism $f^* v^* \to u^* g^*$ has a mate $\text{BC} : v^* g_* \to f_* u^*$ which by the Beck-Chevalley condition is an isomorphism. What we want is to lift BC when f and g satisfy the Frobenius property with respect to some double categories $(\mathbb{J}, \mathbb{K})$ as in Definition 42, in order to be able to reproduce the argument in 16 for AWFS in the next chapter. The strategy is to first instantiate Lemma 45 to conclude that we can lift BC by lifting one of its mates. Then in order to lift this mate we need to functorialize the Frobenius property.

Consider $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ and $\mathbb{K} \to \mathbb{S}\mathbf{q}(\mathcal{D})$ satisfying $\mathbb{J} \cong {}^{\pitchfork}\mathbb{K}$ and $\mathbb{K} \cong \mathbb{J}^{\pitchfork}$. Note that for any morphism $u : A \to B$ in $\mathcal{C}$ there is a lift $\boldsymbol{u}_! : \mathbb{J}/A \to \mathbb{J}/B$ of $u_!$ and as such, by applying the same reasoning as in Proposition 43, a lift $\boldsymbol{u}^* : \mathbb{K}/B \to \mathbb{K}/A$ of u^*. Now for a pullback square

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D \end{array}$$

where f and g satisfy the Frobenius property w.r.t. $(\mathbb{J}, \mathbb{K})$ we get two adjunctions $u_! f^* \dashv f_* u^*$ and $g^* v_! \dashv v^* g_*$ between $\mathcal{C}/B$ and $\mathcal{C}/C$ according to which the canonical transformation $\alpha : u_! f^* \to g^* v_!$ and $\text{BC} : v^* g_* \to f_* u^*$ form mates.

Since f satisfies the Frobenius property and $u_!$ lifts regardless we get a double functor $\boldsymbol{u}_! \boldsymbol{f}^* : \mathbb{J}/B \to \mathbb{J}/C$, and similarly a lift $\boldsymbol{f}_* \boldsymbol{u}^*$ in the other direction. Moreover these correspond to each other via the bijection in Lemma 44 by construction. Analogously we have corresponding lifts $\boldsymbol{g}^* \boldsymbol{v}_!$ and $\boldsymbol{v}^* \boldsymbol{g}_*$. Hence we may now instantiate Lemma 45 to this situation. We let $\mathcal{J}$ and $\mathcal{K}$ denote the arrow categories of $\mathbb{J}$ and $\mathbb{K}$ respectively and abusively write $\boldsymbol{f}$ instead of $\boldsymbol{f}_1$ for the arrow component of a double functor $\boldsymbol{f}$. Now we have that α lifts if and only if BC lifts as below:

$$\begin{array}{ccc} \mathcal{J}/B & \overset{\boldsymbol{u}_!\boldsymbol{f}^*}{\underset{\boldsymbol{g}^*\boldsymbol{v}_!}{\rightrightarrows}}\ \Downarrow \boldsymbol{\alpha} & \mathcal{J}/C \\ \downarrow & & \downarrow \\ (\mathcal{C}/B)^2 & \overset{(u_! f^*)^2}{\underset{(g^* v_!)^2}{\rightrightarrows}}\ \Downarrow \alpha^2 & (\mathcal{C}/C)^2 \end{array} \qquad \begin{array}{ccc} \mathcal{K}/C & \overset{\boldsymbol{v}^*\boldsymbol{g}_*}{\underset{\boldsymbol{f}_*\boldsymbol{u}^*}{\rightrightarrows}}\ \Downarrow \textbf{BC} & \mathcal{K}/B \\ \downarrow & & \downarrow \\ (\mathcal{C}/C)^2 & \overset{(v^* g_*)^2}{\underset{(f_* u^*)^2}{\rightrightarrows}}\ \Downarrow \text{BC}^2 & (\mathcal{C}/B)^2 \end{array}$$

Now the last step is to see how we can lift α. For this we first functorialize the Frobenius property.

Definition 46. Let $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ and $\mathbb{K} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ satisfy $\mathbb{J} \cong {}^{\pitchfork}\mathbb{K}$ and $\mathbb{K} \cong \mathbb{J}^{\pitchfork}$, we say a category $\mathcal{I} \to \mathcal{C}^2$ satisfies the *functorial Frobenius property* w.r.t. $(\mathbb{J}, \mathbb{K})$ if for every $\boldsymbol{f} \in \mathcal{I}$ the underlying morphism f satisfies the Frobenius property in a way that is functorial in morphisms of $\mathcal{I}$. Consider the situation drawn below for $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{I}$. The dotted arrows are induced by the universal property of the pullback. The functoriality of the condition states that if the morphism $h \to i$ underlies a morphism of $\mathcal{J}$ maps, and $f \to g$ one of $\mathcal{I}$ maps, then the induced square $f^*h \to g^*i$ also underlies a morphism of $\mathcal{J}$ maps:

$$\begin{array}{ccccccccccc}
\cdot & \xrightarrow{f^*h} & \cdot & \longrightarrow & \cdot & \xrightarrow{f} & \cdot & \longleftarrow & \cdot & \xleftarrow{h} & \cdot \\
\vdots & & \vdots & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdot & \xrightarrow[g^*i]{} & \cdot & \longrightarrow & \cdot & \xrightarrow[g]{} & \cdot & \longleftarrow & \cdot & \xleftarrow[i]{} & \cdot
\end{array}.$$

Proposition 47. *If $\mathcal{I} \to \mathcal{C}^2$ satisfies the functorial Frobenius property with respect to $(\mathbb{J}, \mathbb{K})$ and $(u, v) : f \to g$ is a pullback square underlying a morphism $\boldsymbol{f} \to \boldsymbol{g}$ in $\mathcal{I}$, then $\text{BC} : v^*g_* \to f_*u^*$ lifts to $\text{BC} : \boldsymbol{v}^*\boldsymbol{g}_* \to \boldsymbol{f}_*\boldsymbol{u}^*$.*

Proof. By the preceding discussion it suffices to show that the canonical transformation $u_!f^* \to g^*v_!$ lifts. Spelling out the definition of this transformation, we see that the statement that it lifts in this way is just a particular instance of the functoriality property in Definition 46. □

This concludes our work on the Beck-Chevalley condition. We finish this chapter with a short digression on a variant of the Frobenius condition that is sometimes used in the literature, and is called the *strong Frobenius condition* in [1, Definition B.6.2].

5.6 The strong Frobenius condition

A strengthening of the Frobenius condition used in [1, Definition B.6.2] and similarly in [9, Proposition 5.5] is that in addition to the pullback of an L-map along an R-map yielding an L-map the pullback square in question should be a morphism of L-maps:

$$\begin{array}{ccc}
\cdot & \xrightarrow{\varepsilon_g} & \cdot \\
{\scriptstyle f^*g}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\
\cdot & \xrightarrow[f]{} & \cdot
\end{array}. \qquad (5.6.1)$$

Since the top arrow of this square is the component at g of the counit $\varepsilon : f_!f^* \to 1$ of the adjunction $f_! \dashv f^*$ we can understand this condition as saying that this transformation lifts to $\boldsymbol{f}_!\boldsymbol{f}^* \to 1$ in the sense described in the previous section:

$$\begin{array}{ccc}
\cdot & \xrightarrow{\varepsilon_{vg}} & \cdot \\
{\scriptstyle f^*g}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\
\cdot & \xrightarrow{\varepsilon_v} & \cdot \\
{\scriptstyle f^*v}\downarrow & \lrcorner & \downarrow{\scriptstyle v} \\
\cdot & \xrightarrow[f]{} & \cdot
\end{array}.$$

To see why this is, consider when v is the identity. Writing subscript 0 and 1 to differentiate explicitly between the object and arrow components of f^* we have that $f_0^* g$ gets L-map structure from the isomorphism $(1, f_0^* 1) : f_1^* g \cong f_0^* g$. This makes $(1, f_0^* 1)$ and its inverse morphisms of L-maps. Therefore by assumption $(\varepsilon_g, \varepsilon_1)$ is also a morphism of L-maps, and hence so is their composition $(\varepsilon_g, \varepsilon_1) \cdot (1, (f_0^* 1)^{-1}) = (\varepsilon_g, f)$:

$$\begin{array}{ccccc} \cdot & \xrightarrow{1} & \cdot & \xrightarrow{\varepsilon_g} & \cdot \\ {\scriptstyle f_0^* g}\downarrow & & \downarrow{\scriptstyle f_1^* g} & & \downarrow{\scriptstyle g} \\ \cdot & \xrightarrow[(f_0^* 1)^{-1}]{} & \cdot & \xrightarrow[\varepsilon_1]{} & \cdot \end{array} .$$

This means that we can indeed interpret the strong Frobenius condition of [1, Definition B.6.2], which states that the square in 5.6.1 is a morphism of L-maps, as a special case of the demand that the counit of $f_! \dashv f^*$ lifts in this way.

The statement of [1, Proposition 4.2.3] it that the strong Frobenius condition implies the adjunction $f^* \dashv f_*$ for some R-**Map** $\boldsymbol{f} : A \to B$ lifts to $\mathsf{R}\text{-}\mathbf{Map}(A) \to \mathsf{R}\text{-}\mathbf{Map}(B)$, where $\mathsf{R}\text{-}\mathbf{Map}(A)$ denotes the fiber of $\mathrm{cod} \cdot U^{\mathsf{R}}$ over A. The argument is that we can use the variant of Lemma 45 for regular categories of maps to lift both the unit and counit of $f^* \dashv f_*$ by lifting the counit and unit of $f_! \dashv f^*$, since they are mates. It is then argued that we can use the strong Frobenius condition to lift the unit and counit of $f_! \dashv f^*$. In light of the observation that the strong Frobenius condition states that the counit of $f_! \dashv f^*$ lifts we would expect that for this result the strong Frobenius condition [1, Definition B.6.2] would need to be further strengthened to demand that the unit of $f_! \dashv f^*$ also lifts. A close inspection of the proof of [1, Proposition 4.2.3] reveals a small oversight that suggests that this is indeed the case; in the calculation that serves to show that the unit lifts the equality $\eta_g \varepsilon_{fg} = 1$ is used for some g with codomain A whereas it is only $\varepsilon_{fg} \eta_g = 1$ that holds in general.

Chapter 6

Interpreting type theory in awfs

In this chapter we mimic the constructions of the pseudo-stable choices made for π-clans in Section 3.3 for the R-algebras of an AWFS. The choices of the dependent sum and product types will be exactly as for clans, except that now because the forgetful functor $U^{\mathsf{R}} : \mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^2$ is not full we need to make sure that the morphisms used in those constructions are morphisms of R-algebras. Having done this we make use of the additional structure that an AWFS offers compared to a clan in order to also construct a pseudo-stable choice of Id-types. The strategy for doing this will exactly mimic the approach taken in [1, Section 4.3], but then phrased for R-algebras instead of R-maps.

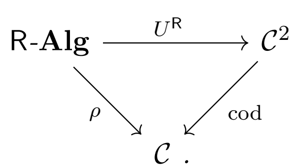
6.1 The comprehension category induced by an awfs

The category of R-algebras for an AWFS is similar to a clan in that it contains all isomorphisms and that the pullback of an R-algebra along any map is again an R-algebra. It also similarly induces a comprehension category. The following definitions and propositions are as in [2, Section 3.4].

Definition 48. A functor $p : \mathcal{J} \to \mathcal{C}^2$ is a *discrete pullback fibration* if for every $\boldsymbol{f} \in \mathcal{J}$ and every pullback square $(u, v) : g \to f$ there is a unique arrow $\varphi : \boldsymbol{g} \to \boldsymbol{f}$ in $\mathcal{J}$ over (u, v) which in addition is cartesian.

Lemma 49. *The forgetful functor $U^{\mathsf{R}} : \mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^2$ is a discrete pullback fibration.*

Corollary 50. *Given an* AWFS (L, R) *on a category $\mathcal{C}$ with pullbacks we have a comprehension category:*



Its cartesian arrows are given by pullback squares underlying R*-algebra morphisms.*

Let us denote the fiber of an object $A \in \mathcal{C}$ by $\mathsf{R}\text{-}\mathbf{Alg}(A)$, i.e. the subcategory of $\mathcal{C}/A$ whose objects are R-algebras and whose morphisms are R-algebra morphisms over the identity on A.

6.2 Σ-types

The composition operation that gives R-algebras their double categorical structure provides us with a way of interpreting dependent sums just as in the example of clans in Section 3.3.

Proposition 51. *There exists a pseudo-stable choice of* Σ*-types for the comprehension category induced by* (L, R)*.*

Proof. Consider an R-algebra $\boldsymbol{f} : A \to B$, then vertical composition with $\boldsymbol{f}$ gives a functor $\boldsymbol{f}_! : \mathsf{R}\text{-}\mathbf{Alg}(A) \to \mathsf{R}\text{-}\mathbf{Alg}(B)$. We use this to define the action of Σ on objects: $(\boldsymbol{f}, \boldsymbol{g}) \mapsto \boldsymbol{f}_!\boldsymbol{g}$. For its action on morphisms we proceed as in the case of clans:

$$\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ {\scriptstyle \boldsymbol{g}}\downarrow & \lrcorner & \downarrow{\scriptstyle \boldsymbol{i}} \\ \cdot & \xrightarrow{v} & \cdot \\ {\scriptstyle \boldsymbol{f}}\downarrow & \lrcorner & \downarrow{\scriptstyle \boldsymbol{h}} \\ \cdot & \xrightarrow{w} & \cdot \end{array} \quad \stackrel{\Sigma}{\longmapsto} \quad \begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ {\scriptstyle \boldsymbol{f}_!\boldsymbol{g}}\downarrow & \lrcorner & \downarrow{\scriptstyle \boldsymbol{h}_!\boldsymbol{i}} \\ \cdot & \xrightarrow{w} & \cdot \end{array} .$$

The square on the right is an R-algebra morphism because it is the vertical composition of two squares in $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$. The rest of the choices are made as for clans in Proposition 12. □

6.3 Π-types

Like with clans, the interpretation of Π-types is not so straightforward and requires us to place more demands on the AWFS underlying the comprehension category in order to obtain pushforward functors along the R-algebras. This is achieved by the following definition of an AWFS satisfying the Frobenius condition, which by Proposition 43 compares to regular AWFS like π-clans compare to regular clans. However rather than directly assuming the existence of right adjoint pushforward functors as done in the definition of π-clans we use a Frobenius condition. This is because the Frobenius property is easier to verify in practice. Since $\mathsf{R}\text{-}\mathbf{Alg}$ is not a full subcategory of $\mathcal{C}^2$ we are additionally tasked with showing that the Beck-Chevalley isomorphism lifts to a morphism of R-algebras. We saw in Section 5.5 that this follows from the functorial Frobenius condition of Definition 46.

Definition 52. An AWFS (L, R) satisfies the *functorial Frobenius condition* if $\mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^2$ satisfies the functorial Frobenius property with respect to $(\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}, \mathsf{R}\text{-}\mathbb{A}\mathbf{lg})$.

Remark 53. The Frobenius property of $f : A \to B$ for some WFS $(\mathcal{C}, \mathcal{L}, \mathcal{R})$ now states that $f^*(\mathcal{L}/B) \subseteq (\mathcal{L}/A)$. Since $\mathcal{L}/A$ and $\mathcal{R}/A$ are the classes of maps of the *slice* WFS $(\mathcal{C}/A, \mathcal{L}/A, \mathcal{R}/A)$, this property for f might then be understood as saying that f^* constitutes a morphism of WFS between the slice WFS on $\mathcal{C}/B$ and $\mathcal{C}/A$ in some suitable sense. Considering instead an AWFS we have that the double category $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}/A$ is formed as:

$$\begin{array}{ccc} \mathsf{R}\text{-}\mathbb{A}\mathbf{lg}/A & \longrightarrow & \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \\ \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle U^{\mathsf{R}}} \\ \mathbb{S}\mathbf{q}(\mathcal{C}/A) & \xrightarrow{\mathbb{S}\mathbf{q}(\mathrm{dom})} & \mathbb{S}\mathbf{q}(\mathcal{C}) \end{array} .$$

This situation is discussed more generally in [2, Section 4.5], and it is shown there that the pullback square above yields a *slice* AWFS $(\mathsf{L}/A, \mathsf{R}/A)$ on $\mathcal{C}/A$ with $(\mathsf{R}/A)\text{-}\mathbb{A}\mathbf{lg} \cong \mathsf{R}\text{-}\mathbb{A}\mathbf{lg}/A$. In light of this interpretation we might expect the Frobenius property of a map $f : A \to B$ to state that f^* underlies a morphism of slice AWFS $(\mathcal{C}/B, \mathsf{L}/B, \mathsf{R}/B) \to (\mathcal{C}/A, \mathsf{L}/A, \mathsf{R}/A)$, but this is not immediately apparent as the definition of morphism between AWFS (see for instance [2, Section 2.9]) is somewhat involved.

Since R-algebras are stable under pullback we know that any morphism $f : A \to B$ gives rise to a functor $\boldsymbol{f}^* : \mathsf{R}\text{-}\mathbf{Alg}(B) \to \mathsf{R}\text{-}\mathbf{Alg}(A)$. If f underlies an R-algebra $\boldsymbol{f}$ then we can also define a pushforward functor $\boldsymbol{f}_* : \mathsf{R}\text{-}\mathbf{Alg}(A) \to \mathsf{R}\text{-}\mathbf{Alg}(B)$ as follows. Let $\boldsymbol{g}$ be an R-algebra with codomain A, then $(\boldsymbol{g}, 1) \in \mathsf{R}\text{-}\mathbf{Alg}/A$ so since $\boldsymbol{f}$ satisfies the Frobenius property there is an R-algebra structure on $f_{*1}g$ and hence by the isomorphism $(1, f_*1) : f_{*1}g \cong f_{*0}g$ we obtain one on $f_{*0}g$; this determines the action of $\boldsymbol{f}_*$ on objects. Similarly we can show that if $u : \boldsymbol{g} \to \boldsymbol{h}$ in $\mathsf{R}\text{-}\mathbf{Alg}(A)$ then $f_*u : \boldsymbol{f}_*\boldsymbol{g} \to \boldsymbol{f}_*\boldsymbol{h}$ in $\mathsf{R}\text{-}\mathbf{Alg}(B)$.

Proposition 54. *If (L, R) satisfies the functorial Frobenius condition then there exists a pseudo-stable choice of Π-types for the comprehension category it induces.*

Proof. We define Π on objects by $(\boldsymbol{f}, \boldsymbol{g}) \mapsto \boldsymbol{f}_*\boldsymbol{g}$. For its action on morphisms we should show that the composition of the squares on the right below is a morphism of R-algebras:

$$\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ {\scriptstyle \boldsymbol{g}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \boldsymbol{i}} \\ \cdot & \xrightarrow{v} & \cdot \\ {\scriptstyle \boldsymbol{f}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \boldsymbol{h}} \\ \cdot & \xrightarrow[w]{} & \cdot \end{array} \quad \overset{\Pi}{\longmapsto} \quad \begin{array}{ccccccc} \cdot & \xrightarrow{f_*\alpha} & \cdot & \xrightarrow{\mathrm{BC}_i} & \cdot & \xrightarrow{w^+} & \cdot \\ {\scriptstyle \boldsymbol{f}_*\boldsymbol{g}}\big\downarrow & & \big\downarrow{\scriptstyle \boldsymbol{f}_*\boldsymbol{v}^*\boldsymbol{i}} & & \big\downarrow{\scriptstyle \boldsymbol{w}^*\boldsymbol{h}_*\boldsymbol{i}} & & \big\downarrow{\scriptstyle \boldsymbol{h}_*\boldsymbol{i}} \\ \cdot & \xrightarrow[1]{} & \cdot & \xrightarrow[1]{} & \cdot & \xrightarrow[w]{} & \cdot \end{array} .$$

It is easily checked that the square $(\alpha, 1) : g \to v^*i$ is an algebra morphism and therefore so is $(f_*\alpha, 1)$; the square in the middle is because (L, R) satisfies the functorial Frobenius condition and we saw in Section 5.5 that this suffices to lift the BC transformation; and the square on the right is because it is a pullback square. This means the outer rectangle is an algebra morphism too. The rest of the proof proceeds as the choice of Π-types for π-clans in Proposition 16. □

6.4 Id-types

The method we will use for interpreting Id-types originates from [10] and was further developed in [8]. The idea is that dependent types are interpreted as the right maps of a WFS $(\mathcal{L}, \mathcal{R})$. Then given an $\mathcal{R}$ map $f : A \to \Gamma$ we factorise the diagonal δ_f as

$$A \xrightarrow{rf} I(f) \xrightarrow{\rho f} A \times_\Gamma A \ ,$$

where $rf \in \mathcal{L}$ and $\rho f \in \mathcal{R}$. This means ρf can interpret the identity type Id_A, i.e. the formation rule; and rf the reflexivity term, i.e. the introduction rule; and lastly the lifts of left maps against right maps the elimination terms $\mathrm{j}_A(T, t)$:

$$\begin{array}{ccc} \Gamma.A & \xrightarrow{t} & \Gamma.A.A.\mathrm{Id}_A.T \\ {\scriptstyle \mathrm{r}_A}\downarrow & \overset{\mathrm{j}_A(T,t)}{\nearrow} & \downarrow{\scriptstyle \chi_T} \\ \Gamma.A.A.\mathrm{Id}_A & \xrightarrow[1]{} & \Gamma.A.A.\mathrm{Id}_A \end{array} \ .$$

In order to realise the coherence conditions of the elimination terms we need these lifts to be structured instead of merely postulated to exist, which is why more structured versions of WFS were introduced in the context of modelling identity types. In [8] the notion of *cloven* WFS was used, for which the left and right classes of maps have categorical structure denoted by $\mathcal{L}$**-Map** and $\mathcal{R}$**-Map**, and the factorisation above should be given by a functor $(r, \rho) : \mathcal{R}\textbf{-Map} \to \mathcal{L}\textbf{-Map} \times_\mathcal{C} \mathcal{R}\textbf{-Map}$ for which the right leg ρ preserves pullback squares. This same approach is used in [3, Lemma 2.9] but then based on the underlying WFS of an AWFS so the signature of the functor is accordingly changed to $\mathsf{R}\textbf{-Map} \to \mathsf{L}\textbf{-Map} \times_\mathcal{C} \mathsf{R}\textbf{-Map}$. In the present work we will also use the same approach so since we are working with AWFS and our left and right classes are given by L-**Coalg** and R-**Alg** we instead use a functor with the signature $\mathsf{R}\textbf{-Alg} \to \mathsf{L}\textbf{-Coalg} \times_\mathcal{C} \mathsf{R}\textbf{-Alg}$.

Definition 55. A *functorial factorisation of the diagonal* on a category $\mathcal{C}$ is a functor $P = (r, \rho) : \mathcal{C}^2 \to \mathcal{C}^2 \times_\mathcal{C} \mathcal{C}^2$ such that for each morphism $f : A \to B$ we have $\rho f \cdot rf = \delta_f$ where δ_f is the diagonal morphism $A \to A \times_B A$. The action of P should be specified as follows:

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D \end{array} \quad \stackrel{P}{\longmapsto} \quad \begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle rf}\downarrow & & \downarrow{\scriptstyle rg} \\ If & \xrightarrow{I(u,v)} & Ig \\ {\scriptstyle \rho f}\downarrow & & \downarrow{\scriptstyle \rho g} \\ A \times_B A & \xrightarrow{u \times_v u} & C \times_D C \end{array} .$$

Definition 56. A functorial factorisation of the diagonal $P = (r, \rho)$ is called *stable* when its right leg ρ preserves pullback squares, i.e. when $(u, v) : f \to g$ is a pullback square then so is $(I(u, v), u \times_v u) : \rho f \to \rho g$.

Definition 57. An AWFS (L, R) on a category $\mathcal{C}$ has a *stable functorial choice of path objects* (sfpo) if it has a stable functorial factorisation of the diagonal P which lifts to a functor $P : \mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{L}\text{-}\mathbf{Coalg} \times_{\mathcal{C}} \mathsf{R}\text{-}\mathbf{Alg}$:

$$\begin{array}{ccc} \mathsf{R}\text{-}\mathbf{Alg} & \xrightarrow{P} & \mathsf{L}\text{-}\mathbf{Coalg} \times_{\mathcal{C}} \mathsf{R}\text{-}\mathbf{Alg} \\ \downarrow & & \downarrow \\ \mathcal{C}^2 & \xrightarrow{P} & \mathcal{C}^2 \times_{\mathcal{C}} \mathcal{C}^2 \end{array} .$$

Remark 58. One might wonder what causes the need for an sfpo when the functorial factorisation accompanying the AWFS already provides us with such a functorial assignment $f \mapsto (L\delta_f, R\delta_f)$. The problem with this, as pointed out in [8, Remark 3.3.4], is that this choice will rarely be stable.

Proposition 59. *The comprehension category induced by an* AWFS *with an spfo* (I, r, ρ) *has a pseudo-stable choice of* Id*-types.*

Proof. The functor Id is given by $f \mapsto \rho_f$, so that functoriality follows from functoriality of ρ, and preservation of cartesian morphisms from the stability condition. The reflexivity morphism is given by rf and the j terms as the lifts given by the AWFS. The proofs of pseudo-stability are now exactly as in [3, Lemma 2.9]. □

6.5 Modelling type theory

Putting the results of the previous sections together we obtain the following.

Theorem 60. *The right adjoint splitting of the comprehension category induced by an* AWFS (L, R) *that satisfies the functorial Frobenius condition and is equipped with an sfpo yields a model of Martin-Löf type theory with sum, product, and identity types.*

This is the version of [3, Theorem 2.12] phrased for the algebras of the monad of an AWFS rather than for the algebras for the pointed endofunctor, and accomplishes the main objective of this work. In the next chapter we will look at an example of such an AWFS on the category **Grpd** of groupoids.

Chapter 7

The groupoid model

In [3, Section 3] Gambino and Larrea revisit the Hofmann-Streicher model [15] of type theory on the category **Grpd** of Groupoids. They do so by equipping **Grpd** with an AWFS satisfying the appropriate conditions for interpreting type theory, i.e. one satisfying the functorial Frobenius condition and equipped with a stable functorial choice of path object for the R-maps of the AWFS. In this chapter we describe this construction and show that it additionally satisfies these conditions phrased for the R-algebras, thereby obtaining an example of an AWFS mentioned in Theorem 60.

We start with the definition of a comma category.

Definition 61. Given functors $f : \mathcal{C} \to \mathcal{E}$, $g : \mathcal{D} \to \mathcal{E}$ we can define their *comma category* $f \downarrow g$ with objects given by triples (α, c, d) where $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $\alpha : fc \to gd$ in $\mathcal{E}$, and arrows by $(\beta, \gamma) : (\alpha, c, d) \to (\alpha', c', d')$ where $\beta : c \to c'$, $\gamma : d \to d'$ are morphisms satisfying $\alpha' \cdot f\beta = g\gamma \cdot \alpha$. Each such category comes equipped with two projection functors $\pi_{\mathcal{C}} : f \downarrow g \to \mathcal{C}$ and $\pi_{\mathcal{D}} : f \downarrow g \to \mathcal{D}$.

For our purposes we will consider the case where the first component is an identity, i.e. we consider just one functor $f : \mathcal{C} \to \mathcal{D}$ and denote $1_{\mathcal{D}} \downarrow f$ as just $\downarrow\! f$. The objects of $\downarrow\! f$ can now be thought of as the situations in $\mathcal{D}$ which should have a cartesian lift in $\mathcal{C}$ for f to be a fibration:

$$\begin{array}{ccc} & & c \\ & & \vdots \\ d & \xrightarrow{\alpha} & fc\ . \end{array} \tag{7.0.1}$$

The assignment $f \mapsto \downarrow\! f$ is functorial and can be expanded to a functorial factorization (C_t, F) on **Cat**. The left leg sends f to the functor $\mathsf{C}_t f : c \mapsto (1_{fc}, fc, c)$ and the right leg to the projection functor $Ff := \pi_{\mathcal{D}}$, so that f factorizes as

$$\mathcal{C} \xrightarrow{\mathsf{C}_t f} \downarrow\! f \xrightarrow{Ff} \mathcal{D}.$$

Now f is an F-map if it has has a lift for the square as depicted on the lower left below:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{1} & \mathcal{C} \\ {\scriptstyle \mathsf{C}_t f}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle f} \\ \downarrow f & \xrightarrow[Ff]{} & \mathcal{D} \end{array}, \qquad \begin{array}{ccc} & & c \\ & & \vdots \\ d & \xrightarrow{\alpha} & fc \end{array} \quad \overset{s}{\longmapsto} \quad \begin{array}{ccc} s(\alpha, d, c) & \xrightarrow{(\alpha,1)} & c \\ \vdots & & \vdots \\ d & \xrightarrow{\alpha} & fc \end{array}.$$

So there is a functor s which given a situation as in 7.0.1 finds an object in the preimage of d and a lift of the arrow α because $(\alpha, 1) : (\alpha, d, c) \to (1_{fc}, fc, c)$. A calculation shows $s(\alpha, 1)$ is f-cartesian over α, so that this assignment provides a normal cleavage for f. The morphisms between F-maps are given by cleavage preserving maps.

To complete the definition of the monad we need a multiplication $\mu_f : \downarrow Ff \to \downarrow f$. The objects of the category $\downarrow Ff$ look as follows:

$$\begin{array}{ccccc} & & & & c \\ & & & & \vdots \\ e & \xrightarrow{\beta} & d & \xrightarrow{\alpha} & fc \end{array}.$$

The functor μ_f simply composes α and β, and the statement that f is an F-algebra translates to the statement that f is a split fibration.

Next let us consider the C_t-maps, i.e. maps f equipped with a lift s of the lower left square:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathsf{C}_t f} & \downarrow f \\ {\scriptstyle f}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle Ff} \\ \mathcal{D} & \xrightarrow[1]{} & \mathcal{D} \end{array}, \qquad d \quad \overset{s}{\longmapsto} \quad \begin{array}{ccc} & & \tilde{f}d \\ & & \vdots \\ d & \xrightarrow{n_d} & f\tilde{f}d \end{array}.$$

Such an s associates to each $d \in \mathcal{D}$ an object $\tilde{f}d \in \mathcal{C}$ along with an arrow $n_d : d \to f\tilde{f}d$. This notation is suggestive: s constitutes a pair $(\tilde{f}, n)$ with $\tilde{f} : \mathcal{D} \to \mathcal{C}$ a retraction of f and $n : 1 \to f\tilde{f}$ a natural transformation satisfying $nf = 1$. The morphisms in C_t-**Map** between $(f, \tilde{f}, n)$ and $(g, \tilde{g}, m)$ are given by $(u, v) : f \to g$ satisfying $u\tilde{f} = \tilde{g}v$ and $vn = mv$.

To see what the comultiplication $\delta_f : \downarrow f \to \downarrow \mathsf{C}_t f$ looks like we first note that spelling out the definition of $\downarrow \mathsf{C}_t f$ reveals that $\downarrow \mathsf{C}_t f \cong \downarrow (f \cdot \mathrm{dom})$, so its objects look like:

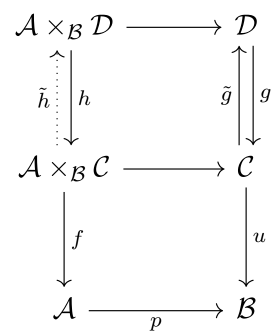
$$\begin{array}{ccccc} & & c & \xrightarrow{\beta} & c' \\ & & \vdots & & \\ d & \xrightarrow{\alpha} & fc & & \end{array}.$$

This means we can just put $\delta_f := \downarrow(\mathrm{id}, 1)$ where $\mathrm{id} : \mathcal{C} \to \mathcal{C}^2$, i.e. a tuple (α, d, c) is send to $(\alpha, d, 1_c)$. Triples of the form $(n_d, d, \tilde{f}d)$ can be mapped into $\downarrow \mathsf{C}_t f$ in a second canonical way given by the endomorphism $\tilde{f}n_d$ on $\tilde{f}d$. The property of being a C_t-coalgebra states these choices coincide, i.e. that $\tilde{f}n = 1$. Together with $nf = 1$ this tells us $\tilde{f} \dashv f$ with unit n and counit 1. Such a pair $(f, \tilde{f})$ is sometimes called a *lali*, short for left adjoint left inverse, and are also discussed in [2, Section 4.2].

Now we show that this AWFS restricted to the category **Grpd** satisfies the condition of Definition 52. The argument is mostly the same as in [1, Proposition 4.5.4] but expanded to work with lalis and split fibrations.

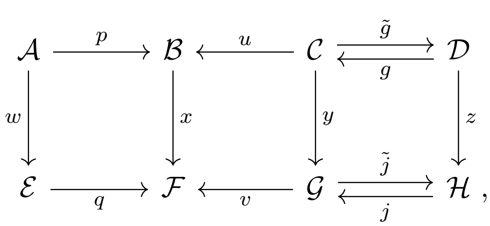
Lemma 62. *The* AWFS (C_t, F) *on* **Grpd** *satisfies the functorial Frobenius condition.*

Proof. We consider a split fibration $p : \mathcal{A} \to \mathcal{B}$, a functor $u : \mathcal{C} \to \mathcal{B}$, and a lali $(g, \tilde{g}) : \mathcal{C} \to \mathcal{D}$ with unit n:



First we construct a functor $f : \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \to \mathcal{A}$. Let $(a, c) \in \mathcal{A} \times_{\mathcal{B}} \mathcal{C}$, then $un_c : uc = pa \to ug\tilde{g}c$ has a cocartesian lift along a of which we take the codomain as the definition of $f(a, c)$. This induces a functor $\tilde{h} : \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \to \mathcal{A} \times_{\mathcal{B}} \mathcal{D}$ given by $(a, c) \mapsto (f(a, c), \tilde{g}c)$, and it is straightforward to verify that $(\tilde{h}, h)$ is a lali. In fact to construct this $\tilde{h}$ it suffices for p to be a *normal* fibration rather than a split fibration. We need the additional assumption that it is split precisely to show that this construction preserves composition of lalis. This is again a straightforward but somewhat tedious verification which we omit.

It remains be shown that this construction is functorial in morphisms between split fibrations, so we consider the following situation:



with $(w, x) : p \to q$ a morphism of split fibrations and $(y, z) : (\tilde{g}, g) \to (\tilde{j}, j)$ a morphism of lalis. This induces the following square between the constructed lalis:

$$\begin{array}{ccc} \mathcal{A} \times_{\mathcal{B}} \mathcal{D} & \xrightarrow{(wu'h)\times(zp^{++})} & \mathcal{E} \times_{\mathcal{F}} \mathcal{H} \\ {\scriptstyle \tilde{h}}\uparrow\downarrow{\scriptstyle h} & & {\scriptstyle \tilde{k}}\uparrow\downarrow{\scriptstyle k} \\ \mathcal{A} \times_{\mathcal{B}} \mathcal{C} & \xrightarrow[(wu')\times(yp^{+})]{} & \mathcal{E} \times_{\mathcal{F}} \mathcal{G} \ . \end{array}$$

Here u' denotes the first projection $\mathcal{A} \times_{\mathcal{B}} \mathcal{E} \to \mathcal{A}$, and likewise p^+ and p^{++} the second projections of $\mathcal{A} \times_{\mathcal{B}} \mathcal{E}$ and $\mathcal{A} \times_{\mathcal{B}} \mathcal{F}$ respectively. Now a short calculation using the fact that (w, x) is a morphism of split fibrations and that (y, z) is a morphism of lalis we can show that $(wu'h) \times (zp^{++}) \cdot \tilde{h} = \tilde{k} \cdot (wu') \times (yp^+)$, which suffices (as noted in [2, Section 4.2]) to show that $((wu'h) \times (zp^{++}), (wu') \times (yp^+))$ is a morphism of lalis. □

Next we consider the stable functorial choice of path objects of Definition 57. We construct a functorial factorization $P = (r, \rho) : \mathcal{C}^2 \to \mathcal{C}^2 \times_{\mathcal{C}} \mathcal{C}^2$ satisfying $\rho f \cdot rf = \delta_f$, and then show it can be lifted to a functor $\mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{R}\text{-}\mathbf{Alg} \times_{\mathcal{C}} \mathsf{L}\text{-}\mathbf{Coalg}$. The construction of P follows a general procedure called the *interval path-object factorisation* which is described in [1, Appendix C.3].

Lemma 63. *The* AWFS (C_t, F) *on* **Grpd** *has a stable functorial choice of path objects.*

Proof. For each functor $f : \mathcal{C} \to \mathcal{D}$ we may define the category $Pf \subseteq \mathcal{C}^2$ as the full subcategory of arrows α in $\mathcal{C}$ that satisfy $f\alpha = 1$. Doing so we get functors $rf := \mathrm{id} : \mathcal{C} \to Pf$ and $\rho f := \mathrm{dom} \times \mathrm{cod} : Pf \to \mathcal{C} \times_{\mathcal{D}} \mathcal{C}$ which together form a functorial factorization $\mathcal{C}^2 \to \mathcal{C}^2 \times_{\mathcal{C}} \mathcal{C}^2$ of the diagonal δ_f.

To see that the left leg r produces lalis we note that cod is left inverse to id, and that there is a natural transformation $n : 1 \to \mathrm{id} \cdot \mathrm{cod}$ with components $n_\alpha := (\alpha, 1)$. It follows easily from the definition of n that $\mathrm{cod}\, n = 1$ and $n\, \mathrm{id} = 1$ so that $(\mathrm{cod}, \mathrm{id})$ is indeed a lali.

Next we consider whether ρ produces split fibration. For this we need that $\mathcal{C}$ is a groupoid, and that f maps arrows in Pf to identities. Let $\alpha : c \to d$ in Pf and $(u, v) : (a, b) \to (c, d)$ in $\mathcal{C} \times_{\mathcal{D}} \mathcal{C}$:

$$\begin{array}{ccc} a & \xrightarrow{u} & c \\ {\scriptstyle v^{-1}\alpha u}\downarrow & & \downarrow{\scriptstyle \alpha} \\ b & \xrightarrow[v]{} & d \ . \end{array}$$

We obtain an arrow $v^{-1}\alpha u : a \to b$ which is mapped to the identity by f so that it is in Pf above (a, b), as well as a cartesian arrow $(u, v) : v^{-1}\alpha u \to \alpha$ above (u, v). This gives us a cleavage which, since it is just a kind of inclusion, is easily seen to be split.

That the functor P preserves pullbacks is most easily seen in the general construction so the reader is reffered to [1, Appendix C.3] for further details. □

Chapter 8

Conclusions

In [3] and [1] Gambino and Larrea identified conditions for AWFS that ensure that the comprehension category $\mathsf{R}\text{-}\mathbf{Map} \to \mathcal{C}^2$ of its R-maps can be equipped with pseudo-stable choices of the sum, product, and identity types of Martin-Löf dependent type theory. These conditions are that the AWFS satisfies a *Frobenius condition* and has a *stable functorial choice of path objects*.

In the preceding chapters we saw that this same approach can be used when we instead use the comprehension category $\mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^2$. In particular we found that for the interpretation of the sum and identity types the same ideas applied as used in [3] and elsewhere in the literature: the sum types are modelled by composition and the identity types by using a stable functorial choice of path objects, only now phrased for R-algebras rather than R-maps. Most of the work was in finding an analog of the Frobenius condition in Chapter 5. We formulated this condition at a general level for double categories of maps and proved it is equivalent to a *pushforward* condition. In the context of AWFS this condition says that the R-algebras are closed under pushforward, and from there we could use the same strategy as in [3] to model the Π-types. Having established this we then showed that the AWFS on groupoids described in [3] satisfies the stronger conditions that we phrased, thus serving as an example of a model of the fragment of Martin-Löf dependent type theory arising from our results.

A clear objective for future work is to find more examples of AWFS satisfying the conditions we identified in Chapter 6, i.e. those satisfying the functorial Frobenius condition and having a stable functorial choice of path objects. Ideally these examples would have L-coalgebras and R-algebras which are not retract closed, thus making full use of the results of the present work.

Perhaps a good way of finding such examples would be to use the *Beck theorem* for AWFS proven by Bourke and Garner in [2, Theorem 6]. This result characterises the essential image of $(-)\text{-}\mathbb{A}\mathbf{lg} : \mathbf{AWFS}_{\mathrm{lax}} \to \mathbf{Dbl}^2$ and the authors of [2] argue that this theorem is essential for a smooth handling of AWFS, as it allows one to specify an AWFS by constructing a double functor of the correct form which is easier in practice. As such the best way to find AWFS satisfying the functorial Frobenius condition might be to translate the condition along the equivalence given by the Beck theorem. If the observation we made in Remark 53 is correct then it might provide a way for doing this translation, as it expresses the Frobenius condition as the existence of certain arrows in the category of AWFS. Seeing which arrows these correspond to in $\mathbf{Dbl}^2$ might then help to find the desired characterisation.

Appendix A

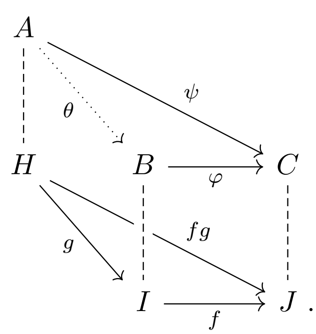
Category theory background

For the self-containedness of this work we list some oft-used notions and results.

A.1 Fibrations

These definitions as well as much more information on fibrations are found in [5].

Definition 64. Let $\rho : \mathcal{E} \to \mathcal{B}$ be a functor, and $\varphi : B \to C$ an arrow in $\mathcal{E}$. We say φ is ρ-*cartesian* over $\rho(\varphi) = f : I \to J$ if for every $g : H \to I$ and $\psi : A \to C$ with $\rho(\psi) = fg$, there exists $\theta : A \to B$ unique with respect to the property that $\rho(\theta) = g$ and $\varphi \cdot \theta = \psi$:



We use dashed lines to visualize the image of ρ. Reversing the direction of all the arrows we get the dual notion of *cocartesian* arrows in $\mathcal{E}$.

Definition 65. A *Grothendieck fibration*, often just called a *fibration*, is a functor $p : \mathcal{E} \to \mathcal{B}$ such that for every $C \in \mathcal{E}$ and $f : I \to \rho(C)$ there is a cartesian arrow φ over f, called the *cartesian lift* of f along C. If $p^{\text{op}} : \mathcal{E}^{\text{op}} \to \mathcal{B}^{\text{op}}$ is a fibration then p is called an *opfibration*, meaning it has cocartesian lifts, and if it is both a fibration and opfibration then it is said to be a *bifibration*.

A choice function associating each $f : I \to \rho(C)$ to a cartesian lift of f along C is called a *cleavage* for ρ. A fibration together with a cleavage is called a *cloven* fibration. If a cleavage preserves identities we say it is *normal*, and if in addition it preserves composition we say it is *split*. A fibration is *normal* or *split* if it has a normal or split cleavage.

The quintessential example is the codomain functor $\mathrm{cod} : \mathcal{C}^2 \to \mathcal{C}$ which is a fibration if and only if $\mathcal{C}$ has pullbacks. The cartesian arrows of cod are exactly the pullback squares in $\mathcal{C}$. Cleavages for cod correspond to choices of pullbacks, and a split cleavage means this choice is strictly associative.

Given two fibrations over the same base $\pi : \mathcal{D} \to \mathcal{B}$, $\rho : \mathcal{E} \to \mathcal{B}$ we say a functor $f : \mathcal{D} \to \mathcal{E}$ is a *fibered* functor $f : \pi \to \rho$ if $\rho \cdot f = \pi$ and f preserves cartesian arrows in the sense that if φ is π-cartesian then $f\varphi$ is ρ-cartesian. Fibrations over $\mathcal{B}$ together with fibered functors between them constitute a category $\mathbf{Fib}(\mathcal{B})$. Similarly we can define morphisms between split fibrations as fibered functors preserving the cleavage, and this yields a category $\mathbf{SpFib}(\mathcal{B})$ of split fibrations over $\mathcal{B}$. The functor $\mathbf{SpFib}(\mathcal{B}) \to \mathbf{Fib}(\mathcal{B})$ that forgets cleavages has both a left and right adjoint. In [1] this right adjoint is extended to operate on comprehension categories, thus providing a way of obtaining a split comprehension category from a non-split one, called its *right adjoint splitting*.

A.2 Monads

Definition 66. A *monad* T on a category $\mathcal{C}$ is a triple (T, η, μ) consistsing of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta : 1 \to T$, $\mu : TT \to T$ called the *unit* and *multiplication* of the monad respectively, for which all of the following diagrams commute:

$$\begin{array}{ccccc} T & \xrightarrow{T\eta} & TT & \xleftarrow{\eta T} & T \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1} & \\ & & T & & \end{array} \qquad \begin{array}{ccc} TTT & \xrightarrow{\mu T} & TT \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ TT & \xrightarrow[\mu]{} & T \end{array}.$$

Dualizing this definition we get the concept of a *comonad* (T, ε, δ) on $\mathcal{C}$ with counit $\varepsilon : T \to 1$ and comultiplication $\delta : T \to TT$.

Definition 67. Let T be a monad on a category $\mathcal{C}$, then a T-*algebra* is a pair (X, f) consisting of an object $X \in \mathcal{C}$ and an arrow $f : TX \to X$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle f} \\ & & X \end{array} \qquad \begin{array}{ccc} TTX & \xrightarrow{Tf} & TX \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle f} \\ TX & \xrightarrow[f]{} & X \end{array}.$$

By dropping the condition for the multiplication we get a notion of algebras for the pointed endofunctor (T, η), which we refer to as *T-maps*.

A *morphism* of T-algebras $(X, f), (Y, g)$ is an arrow $h : X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} TX & \xrightarrow{Th} & TY \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[h]{} & Y \end{array} .$$

The algebras and morphisms between them induce a category T-**Alg** named the *Eilenberg-Moore category* of T. We can use the same definition of morphisms between T-maps, and we call the induced category T-**Map**. Since the definition of morphisms between T-algebras or T-maps is the same, we have a full and faithful inclusion T-**Alg** $\to$ T-**Map**.

For a comonad T we have dual definitions of *T-coalgebras*, morphisms between them, and a resulting co-Eilenberg-Moore category T-**Coalg**. The category of coalgebras for the copointed endofunctor (T, ε) is (perhaps somewhat confusingly) also called T-**Map**.

A.3 Double categories

In any category $\mathcal{C}$ with pullbacks one has the notion of categories internal to $\mathcal{C}$. This definition is obtained by expressing the category axioms in terms of the existence of certain objects, arrows, and commuting squares in $\mathcal{C}$.

Definition 68. An *internal category* in $\mathcal{C}$ is made up of the following parts:

$$A_1 \times_{A_0} A_1 \xrightarrow{\quad \circ \quad} A_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{\mathrm{id}} \\ \xrightarrow[t]{} \end{array} A_0 .$$

The object A_0 represents the objects of the internal category and A_1 the morphisms, the operations of source-, target-, and identity arrow assignment are modelled by the morphisms s, t, and id respectively. The composition of arrows is given by $\circ$, where the pullback is taken with respect to s and t. All the usual axioms are now expressed by certain compositions of arrows being equal, and they should of course all be true in $\mathcal{C}$.

If we have two internal categories (A_0, A_1) and (B_0, B_1) then an *internal functor* consists of $f_0 : A_0 \to B_0$, which represents its action on objects, and $f_1 : B_0 \to B_1$, which represents its action on arrows. The obvious diagrams expressing things like preservation of composition should all commute. Together these notions give us the category of internal categories in $\mathcal{C}$.

Given two internal functors $f, g : (A_0, A_1) \to (B_0, B_1)$ we have a notion of *internal natural transformation* $\alpha : f \to g$. This is given by an arrow $\alpha : A_0 \to B_1$ in $\mathcal{C}$ making the related diagrams commute.

Now since **Cat** has pullbacks we can look at what constitutes an internal category in **Cat**, which are called double categories.

Definition 69. A *double category* $\mathbb{A}$ is an internal category in **Cat**. The objects in its object category $\mathcal{A}_0$ are called the *objects* of $\mathbb{A}$ and the arrows between them its *horizontal* arrows. The objects of its arrow category $\mathcal{A}_1$ are called its *vertical arrows*, and the arrows between them its *squares*. This terminology can be explained by visualizing an arrow $\alpha : f \to g$ in $\mathcal{A}_1$ as yielding the following 'square' in $\mathcal{A}_0$:

$$\begin{array}{ccc} sf & \xrightarrow{s\alpha} & sg \\ {\scriptstyle f}\big\downarrow & \overset{\alpha}{\Longrightarrow} & \big\downarrow{\scriptstyle g} \\ tf & \xrightarrow[t\alpha]{} & tg \end{array} .$$

To distinguish f and g from the morphisms in $\mathcal{A}_0$ we denote them with double arrowheads. Such squares can be horizontally composed using the composition of $\mathcal{A}_1$ and vertically using the composition functor $\mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \to \mathcal{A}_1$. Together with the notion of a *double functor* and *double natural transformation* this yields the 2-category **Dbl** of double categories.

A useful way to think about double categories is as ordinary categories that have been extended with a second set of morphisms between any two of its objects, one of them being the horizontal set and the other the vertical, along with a notion of composition for this second set. Viewed this way it is easy to see that any ordinary category $\mathcal{C}$ has a trivial double categorical structure in which this second set is equal to the first. In other words for any category $\mathcal{C}$ there is a double category $\mathbb{S}\mathbf{q}(\mathcal{C})$ with object category $\mathcal{C}$ and arrow category $\mathcal{C}^2$. This gives us the object part of a 2-functor $\mathbb{S}\mathbf{q} : \mathbf{Cat} \to \mathbf{Dbl}$. Lastly we mention that since **Cat** has limits so does **Dbl**. In particular we will use pullbacks of double categories.

Definition 70. Let $f : \mathbb{C} \to \mathbb{E}$ and $g : \mathbb{D} \to \mathbb{E}$, then their *double pullback* $\mathbb{C} \times_{\mathbb{E}} \mathbb{D}$ with object category $\mathcal{C}_0 \times_{\mathcal{E}_0} \mathcal{D}_0$ and arrow category $\mathcal{C}_1 \times_{\mathcal{E}_1} \mathcal{D}_1$. All of its operations are also defined componentwise, for instance its identity functor is given by $\mathrm{id}_{\mathbb{C}\times_{\mathbb{E}}\mathbb{D}} := \mathrm{id}_{\mathbb{C}} \times_{\mathrm{id}_{\mathbb{E}}} \mathrm{id}_{\mathbb{D}}$.

Bibliography

[1] Larrea, M. F. *Models of dependent type theory from algebraic weak factorisation systems.* PhD thesis. University of Leeds, 2019.

[2] Bourke, J., and Garner, R. *Algebraic weak factorisation systems I: Accessible awfs.* Journal of Pure and Applied Algebra, 220(1):108-147, Jan 2016.

[3] Gambino, N., and Larrea, M.F. *Models of Martin-Löf type theory from algebraic weak factorisation systems.* arXiv: 1906.01491[math.CT], 2019.

[4] Gambino, N., and Sattler, C. *The Frobenius condition, right properness, and uniform fibrations.* Journal of Pure and Applied Algebra, 221(12):3027-3068, Dec 2017.

[5] Streicher, T. *Fibred categories à la Jean Bénabou.* arXiv: 1801.02927[math.CT], 2018.

[6] nLab authors. *Beck-Chevalley condition.* http://ncatlab.org/nlab/show/Beck-Chevalley%20condition Revision 25, Jan 2021.

[7] Joyal, A. *Notes on clans and tribes.* arXiv: 1710.10238[math.CT], 2017.

[8] Berg, B. v. d., and Garner, R. *Topological and simplicial models of identity types.* ACM transactions on computational logic (TOCL), 13(1):3-44, 2012.

[9] Berg, B. v. d., and Faber, E. *Effective Kan fibrations in simplicial sets.* arXiv: 2009.12670[math.CT], 2020.

[10] Awodey, S., and Warren, M. *Homotopy theoretic models of identity types.* Mathematical Proceedings of the Cambridge Philosophical Society 146, 1, 45-55, 2009.

[11] Riehl, E. *Algebraic model structures.* New York Journal of Mathematics 17, 173-231, 2011.

[12] *Homotopy type theory: Univalent foundations of mathematics.* The Univalent Foundations Program, Institute for Advanced Study, https://homotopytypetheory.org, 2013.

[13] Jacobs, B. *Categorical logic and type theory.* Elsevier volume 141, 1999.

[14] Hofmann, M. *On the interpretation of type theory in locally cartesian closed categories.* International Workshop on Computer Science Logic, 427-441. Springer, 1994.

[15] Hofmann, M., and Streicher, T. *The groupoid interpretation of type theory.* Twenty-five years of constructive type theory (Venice, 1995), Oxford University Press, 83-111, 1998.

[16] Grandis, M., and Tholen, W. *Natural weak factorisation systems.* Archivum Mathematicum, 42(4), 2006.

[17] Garner, R. *Cofibrantly generated natural weak factorisation systems.* arXiv: math/0702290[math.CT], 2007.

[18] Erné, M., Koslowski, J., Austin, M., and Strecker, G. *A primer on Galois connections.* Annals of the New York Academy of Sciences, 704:103-125, 1993.

[19] Warren, M. *Homotopy Theoretic Aspects of Constructive Type Theory.* PhD thesis, Carnegie Mellon University, 2008.