The Logic of Free Choice
Axiomatizations of State-based Modal Logics

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written by

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Abstract

We examine modal logics employing state-based semantics. In this type of semantics, formulas are interpreted with respect to sets of possible worlds.

The logics studied extend classical modal logic with a special non-emptiness atom NE and with the inquisitive disjunction. We make use of two distinct state-based notions of modality which are equivalent when applied to classical formulas but which come apart in our non-classical setting.

We obtain sound and complete natural deduction systems for three state-based modal logics, and show that each of the logics is expressively complete for the set of state properties invariant under state k-bisimulation for some finite k.

One of the logics studied extends Aloni's [1, 3] bilateral state-based modal logic (**BSML**) with the inquisitive disjunction. This logic is bilateral: in addition to the positive support relation between states and formulas, a negative anti-support relation is used. The logic can be used to account for free choice (FC) inferences as Aloni does using **BSML**. The non-emptiness atom NE allows for the representation of a "pragmatic enrichment" of formulas by the principle "avoid stating a contradiction". Narrow-scope FC inferences are derived as entailments involving pragmatically enriched formulas. The bilateralism is associated with a negation which tracks the anti-support clauses; this is used to model the interactions between natural language negation and FC inferences. Wide-scope FC inferences and epistemic contradictions are captured in states possessing specific properties; we define these properties using inference rules.

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Chapter 1

Introduction

In natural language, a sentence such as "You may A or B" often appears to license an inference of "You may A and you may B":

> You may go to the beach or go to the cinema.
> $\rightsquigarrow$ You may go the beach and you may go to the cinema.

A standard formalization of this inference in deontic logic would be:

$$\Diamond(b \vee c) \to (\Diamond b \wedge \Diamond c)$$

This is not derivable classically. One straightforward way of accounting for these inferences would be to adopt some axiom that entails the above:

$$\Diamond(b \vee c) \to \Diamond b$$

This is problematic, however: $\Diamond b \to \Diamond(b \vee c)$ is a validity in classical modal logic, so the above would allow one to derive $\Diamond b \to (\Diamond b \wedge \Diamond c)$ and hence $\Diamond b \to \Diamond c$ for any b and c. Following von Wright [32], this apparent conflict between our linguistic intuitions and the precepts of logic has been called the *paradox of free choice permission* or simply the *paradox of free choice*; we will accordingly call inferences on the model of the above *Free Choice inferences* and the inference licensing phenomenon as a whole *Free Choice* (FC).[1]

Aloni, in her [1] and [3], proposes to employ a *bilateral state-based modal logic* (**BSML**) to account for FC and related linguistic phenomena. In state-based semantics, formulas are interpreted with respect to sets of possible worlds (these sets can be thought of as *information states*, hence

[1] This presentation of FC follows [1], which in turn follows [21].

"state-based") rather than the individual worlds used in classical Kripke semantics—in place of the classical

$M, w \vDash \phi \quad \phi$ is true at world $w \in W$ in model $M = (W, R, V)$

the following is a fundamental semantic notion:

$M, s \vDash \phi \quad \phi$ is supported by state $s \subseteq W$ in model $M = (W, R, V)$

Aloni's logic is also bilateral: assertability and rejectability are treated on a par, and each is associated with a primitive semantic notion. So in addition to support, representing assertability, we have:

$M, s \dashv \phi \quad \phi$ is anti-supported by state $s \subseteq W$ in model $M = (W, R, V)$

which represents rejectability of ϕ in s. The bilateralism is associated with a negation $\neg$ ($\neg\phi$ is assertable just in case ϕ is rejectable) and is used to account for how FC interacts with negation.

This thesis presents a sound and complete natural deduction system for a conservative extension of **BSML**—*bilateral state-based modal logic with global disjunction* (**BSML**$^{\mathbb{W}}$).[2] We also axiomatize two related systems: *state-based modal logic with global disjunction* (**SML**$^{\mathbb{W}}$), a unilateral variant in which the bilateral negation $\neg$ is replaced with a negation $\neg$ that only applies to the classical fragment of the logic; and *state-based globally modal logic with global disjunction* (**SGML**$^{\mathbb{W}}$), which similarly uses $\neg$ in place of $\neg$, but also makes use of modalities ($\Diamond$ and $\Box$, the *global modalities*) which are distinct from those employed by the other two logics ($\Diamond$ and $\Box$, the *flat modalities*). Our axiomatizations are based on pre-existing systems for logics which make use of $\neg$ and the global modalities. Considering **SML**$^{\mathbb{W}}$ and **SGML**$^{\mathbb{W}}$ helps us bridge the gap between the logics in the literature and **BSML**$^{\mathbb{W}}$; their axiomatizations may be thought of as intermediate steps on the path towards axiomatizing **BSML**$^{\mathbb{W}}$.

Let us briefly discuss these existing systems, as well as the origins of the model-theoretic ideas in **BSML**. Refer to Table 1.1 below for a list of the atoms and connectives in the logics we axiomatize and in the logics whose systems our axiomatizations are based on.

[2] The global disjunction $\mathbb{W}$ is also commonly known as the intuitionistic disjunction, and as the inquisitive disjunction. We discuss the rationale for axiomatizing the extension with $\mathbb{W}$ rather than the original **BSML** below.

Aloni's account of FC relies on combining the following ingredients: the *tensor disjunction* $\vee$, a generalization of the classical disjunction for state-based semantics; the special *non-emptiness atom* NE which is supported by a state just in case the state is non-empty; the flat modality $\Diamond$; and the bilateralism with its associated negation $\neg$. The negation functions somewhat independently from the other components and, as noted above, is used to model the special case of FC licensing phenomena interacting with (natural language) negation. The crux of Aloni's explanation rests on the interaction between $\vee$, NE and $\Diamond$; this is the feature of **BSML** that is most crucial and most novel.

Both $\vee$ and NE originate in dependence/team logic. The semantics standardly used for these logics is called *team semantics*. Team semantics for first-order logic was introduced by Väänänen in [27] on the basis of Hodges' [18] semantics for independence friendly logic [17]. In the first-order setting, team semantics involves interpreting formulas with respect to sets of assignments (*teams*) as opposed to the single assignments used in classical semantics. Transposing this idea to the propositional/modal context gives us interpretation with respect to sets of valuations or worlds as explained above—that is, team semantics for propositional or modal logic is essentially state-based semantics, and the teams used for interpretation are states. Propositional/modal team semantics was introduced by Väänänen in his work on modal dependence logic in [28].

In the dependence/team logic context, the tensor disjunction was already present in Hodges' [18]. It has also been independently proposed in assertability logic—see [12]. NE was introduced by Yang in [36] and Väänänen in [29]. In [38], Yang and Väänänen axiomatize propositional logics featuring both $\vee$ and NE. One of these is *strong propositional team logic* ($\mathbf{PT}^+$); our $\mathbf{SML}^{\mathbb{W}}$ and $\mathbf{SGML}^{\mathbb{W}}$ are modal versions of $\mathbf{PT}^+$, and the $\mathbf{PT}^+$ axiomatization forms the basis for all our systems.

Aloni developed the modality $\Diamond$ for her work in formal semantics; essentially the same notion is used in possibility semantics[3] [20] and it has been employed by Ciardelli for his inquisitive Kripke modal logics [5].[4] This modality is distinct from that used in modal dependence/team logics,

[3]In the context of possibility semantics and the Kripke semantics for intuitionistic logic (which we mention below in connection with $\mathbb{W}$), formulas are interpreted with respect to points in a partially ordered set. Since the power set of a set is a type of partially ordered set and state-based semantics is interpreted with respect to the elements of a power set, state-based semantics is a particular case of poset semantics.

[4]Ciardelli considers two distinct types of modal logics which are inquisitive in some sense—*inquisitive Kripke modal logics* such as **InqBK**, which are interpreted in Kripke models and use the flat modalities; and *inquisitive modal logics* such as **InqBM** which are interpreted in a different type of structure and use a different type of modality. We

Väänänen's [28] dependence logic modality $\diamonddot$ (the corresponding necessity-type modalities, $\Box$ and $\boxdot$, are likewise distinct). For reasons that will become clear when we define their semantics, we call $\Diamond$ and $\Box$ the flat modalities, and $\diamonddot$ and $\boxdot$ the global modalities.

The bilateralism and $\urcorner$ in **BSML** were inspired by truthmaker semantics [8]. There have also been other attempts to account for FC by making use of bilateralism—see [30]. Interestingly, the dual negation (and associated bilateralism) employed in the original formulations of dependence logic in [18] and [27] is essentially the same notion as $\urcorner$. In the dependence logic context this bilateralism was motivated by considerations in game-theoretic semantics (Hodges came to his notion of negation by adapting Hintikka's [15, 16] game-theoretic negation to his setting).

Yang and Väänänen's system for $\mathbf{PT}^+$ provides us with most of the rules we need for the interaction between $\vee$, NE, and the other connectives we utilize, but this system is not modal. Most components of $\mathbf{BSML}^{\mathbb{W}}$ have been employed in modal logics which have been axiomatized; NE, however, is a relatively recent innovation and we only have $\mathbf{PT}^+$ to draw from. Our main challenge, then, consists in accounting for how NE interacts with the modalities, and how it interacts with the other connectives in modal contexts.

Modal logics which have commonalities with $\mathbf{BSML}^{\mathbb{W}}$ (but which do not make use of NE) include Ciardelli's inquisitive Kripke modal logic **InqBK** [5] and Yang's *modal dependence logic with intuitionistic disjunction* ($\mathbf{MD}^{\mathbb{W}}$) [35]. **InqBK** uses $\Diamond$ but not $\vee$; Ciardelli does axiomatize an extension of non-modal inquisitive logic with $\vee$, but he does not consider such an extension of the modal logic. $\mathbf{MD}^{\mathbb{W}}$ makes use of $\vee$ and the global modalities. Our $\mathbf{SGML}^{\mathbb{W}}$ is essentially[5] $\mathbf{MD}^{\mathbb{W}}$ supplemented with NE. We get the modal rules for $\mathbf{SGML}^{\mathbb{W}}$ by building on Yang's $\mathbf{MD}^{\mathbb{W}}$-rules; some modifications then give us the modal rules for $\mathbf{SML}^{\mathbb{W}}$ and $\mathbf{BSML}^{\mathbb{W}}$.

On the modifications required for $\mathbf{BSML}^{\mathbb{W}}$: the anti-support clauses in **BSML** are defined in a way that ensures that De Morgan's laws and double negation elimination remain sound for $\urcorner$, and we define the anti-support clause for $\mathbb{W}$ in $\mathbf{BSML}^{\mathbb{W}}$ in accordance with this philosophy. These laws are then essentially all that is required to account for the behaviour of $\urcorner$ (we also use $\urcorner$-analogues of some $\neg$-rules from $\mathbf{PT}^+$).

Our systems, then, are based on those for $\mathbf{PT}^+$ and $\mathbf{MD}^{\mathbb{W}}$; before mov-

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only discuss the first type here.

[5] $\mathbf{MD}^{\mathbb{W}}$, being a dependence logic, also makes use of dependence atoms. Our $\mathbf{SGML}^{\mathbb{W}}$ does not feature these, but they are uniformly definable in it.

ing on, however, we should also mention *modal team logic* (**MTL**), first introduced in [26]. **MTL** is modal as per the name, and has been axiomatized by Lück in [25]. $⩔$ and NE are uniformly definable in **MTL**. **MTL** is, moreover, by an analogue of the van Benthem characterization theorem for the state-based setting, expressively complete for the set of all first-order definable state properties invariant under state bisimulation [23], where state bisimulation is a natural adaptation of the classical notion to the state-based setting. **MTL** makes use of the global modalities, however, and it is not clear whether it can uniformly define $\Diamond$. It is therefore similarly unclear how much insight Lück's rules can provide about the interaction between $\Diamond$, NE, and the other connectives. We note furthermore that **MTL** attains its great expressive power by employing the Boolean negation $\sim$:

$$M, s \models \sim \phi \text{ if and only if not } M, s \models \phi$$

which is not present in **BSML**; Lück's axiomatization also relies on this connective. In view of the intended applications of **BSML**, an axiomatization mainly in terms of the simpler atoms and connectives which Aloni's account makes essential use of would be preferable to an axiomatization featuring $\sim$.

Logic	Atoms	Connectives
Strong propositional team logic (**PT**$^+$)	p, NE	$\wedge, \vee, ⩔, \neg$
Modal dependence logic with $⩔$ (**MD**$^{⩔}$)	p, $= (\alpha_1, \ldots, \alpha_n, \beta)$	$\wedge, \vee, ⩔, \neg, ◈, ⊡$
State-based modal logic with $⩔$ (**SML**$^{⩔}$)	p, NE	$\wedge, \vee, ⩔, \neg, \Diamond, \Box$
State-based globally modal logic with $⩔$ (**SGML**$^{⩔}$)	p, NE	$\wedge, \vee, ⩔, \neg, ◈, ⊡$
Bilateral state-based modal logic (**BSML**)	p, NE	$\wedge, \vee, \neg, \Diamond, \Box$
BSML with $⩔$ (**BSML**$^{⩔}$)	p, NE	$\wedge, \vee, ⩔, \neg, \Diamond, \Box$

Table 1.1: Atoms and connectives in logics considered

We conclude our discussion of different systems. Table 1.1 lists the atoms and connectives of the logics we axiomatize, **BSML**, and the two logics on whose axiomatizations our systems are based: $\mathbf{PT}^+$ and $\mathbf{MD}^{\mathbb{W}}$.[6,7]

In addition to providing axiomatizations, we prove characterization theorems for our logics. We show, adapting a similar result in [14], that each of our logics characterizes the set of all state properties invariant under state k-bisimulation for some $k \in \mathbb{N}$. It is proved in [23] that a state property is invariant under state k-bisimulation for some $k \in \mathbb{N}$ if and only if it is first-order definable and invariant under state bisimulation; therefore, the analogue of the van Benthem theorem for **MTL** also holds for each of our logics, and each has the same expressive power as **MTL**.

We note in passing a simple but interesting consequence of this result. As we will see, the NE- and $\mathbb{W}$-free fragment of each of our logics is essentially classical modal logic; similarly, the $\sim$-free fragment of **MTL** is classical modal logic.[8] So given our result, classical modal logic supplemented with $\mathbb{W}$ and NE is equal in expressive power to classical modal logic supplemented with $\sim$. This is the modal analogue of a fact Yang and Väänänen point out in [38]: $\mathbf{PT}^+$ (classical propositional logic with NE and $\mathbb{W}$) is equal in expressive power to classical propositional logic with $\sim$.

Let us briefly discuss the global disjunction $\mathbb{W}$ in connection with expressive power. (As mentioned above, $\mathbb{W}$ is also called the intuitionistic disjunction due to its use in intuitionistic logic, as well as the inquisitive disjunction—in inquisitive semantics [5, 6] it is used to model the meanings of questions.) Regular **BSML** is *closed under unions*: if a (non-empty) collection of states supports a formula in the language of **BSML**, the state formed by the union of the collection will also support the formula. This also means that **BSML** cannot define properties which are not union closed. $\mathbb{W}$ can be used to define such properties; therefore, the logics we axiomatize (which all make use of $\mathbb{W}$) are strictly more expressive than **BSML**. As noted, we moreover prove they are expressively complete. This is one advantage of $\mathbf{BSML}^{\mathbb{W}}$ over **BSML**: some potentially useful properties and connectives may only be definable in the former. The primary reason we axiomatize this extension rather than Aloni's original logic, however, is that

[6] The semantics for most symbols in the table will be defined in Chapter 2. For the semantics of the dependence atoms of $\mathbf{MD}^{\mathbb{W}}$, see [35].

[7] Note that the table omits the $\bot$ atom of $\mathbf{PT}^+$ and $\mathbf{MD}^{\mathbb{W}}$ for readability. This is definable in terms of the other atoms and connectives listed.

We have also made a slight modification. The original, published version of $\mathbf{PT}^+$ in [38] does not feature the negation $\neg$ which in this thesis applies to all classical formulas—only proposition symbols may be negated in the original version of the system. The modified version in [34] which we also make use of does include $\neg$.

[8] The claim above holds for **MTL** as presented in [23]. It does not hold if the syntax is as in [25].

proving completeness is more straightforward with $\varovee$ in the syntax—we discuss this below. It may also be possible to use $\varovee$ to model the meanings of questions in $\mathbf{BSML}^{\varovee}$ as in inquisitive semantics.

With regard to expressive power, NE plays a role similar to that played by $\varovee$, but in the opposite direction. NE allows us to construct formulas (and hence properties) which are not *downward closed*. A formula is downward closed if whenever it is supported by a state, it must also be supported by all substates (subsets) of that state. The NE-free fragments of our logics are downward closed, meaning that these fragments are incapable of defining properties which are not downward closed.

It is shown by Hella et al. in [14] that classical modal logic (using the global modalities) extended with $\varovee$ (i.e. the NE-free fragment of $\mathbf{SGML}^{\varovee}$) is expressively complete for the set of all downward-closed state properties invariant under state k-bisimulation for some $k \in \mathbb{N}$. It may be possible to establish an analogue of this result for the $\varovee$-free fragment of one or all of our logics—i.e. to show, for instance, that **BSML** is expressively complete for the set of all union-closed state properties invariant under state k-bisimulation for some $k \in \mathbb{N}$. We leave this for future work.

Moving on from expressive power, we briefly describe our strategy for proving completeness before concluding.

Each of our natural deduction systems is based on that for $\mathbf{PT}^{+}$. Yang and Väänänen, in [38], prove the completeness of this logic by a method involving disjunctive normal forms. We adapt this strategy to the modal setting. We first show that for every model M, each state s, and each $k \in \mathbb{N}$, there is a formula that precisely characterizes the pair (M, s) up to k-bisimulation. We then prove that every formula is provably equivalent to some formula in a normal form defined in terms of these characteristic formulas and $\varovee$. (These normal-form formulas are also what we use to prove the characterization theorems.) Completeness then follows from the semantic properties of the characteristic formulas and the rules for $\varovee$. This is why we axiomatize the extension $\mathbf{BSML}^{\varovee}$ rather than **BSML**. Yang and Väänänen have also devised a method for adapting this strategy for logics which do not make use of $\varovee$. They use this to axiomatize the $\varovee$-free fragment of $\mathbf{PT}^{+}$. We hypothesize that excluding the rules involving $\varovee$ from our $\mathbf{BSML}^{\varovee}$ axiomatization and adding rules similar to those added by Yang and Väänänen for the $\varovee$-free fragment of $\mathbf{PT}^{+}$ would produce an axiomatization of **BSML**, and that adapting the proof in this thesis using the aforementioned method would then suffice for proving the completeness of the resulting system. This is also left for future work.

The thesis is structured as follows:

Chapter 2 presents the preliminaries. We define the syntax and semantics for the logics to be axiomatized. We discuss some state-semantic properties of formulas and use these properties to determine entailment relations between the flat modalities and the global ones. This allows us to show that each of our logics is a conservative extension of classical modal logic. We demonstrate how **BSML** and $\mathbf{BSML}^{\mathbb{W}}$ can account for FC, and why $\mathbf{SML}^{\mathbb{W}}$ and $\mathbf{SGML}^{\mathbb{W}}$ fail to do so. Finally, we list some results from classical modal logic we make use of in later chapters. These concern the standard notion of bisimulation and the characteristic formulas of classical modal logic—Hintikka formulas.

In Chapter 3 we prove the characterization theorems. We first show how to adapt the standard notion of bisimulation to the state-based setting and prove a state bisimulation invariance theorem for all of our logics. We then define characteristic formulas for states, and a disjunctive normal form for formulas. These enable us to prove the characterization theorems. In the second part of the chapter, we introduce a variant of FC—*wide-scope Free Choice*—and another linguistic phenomenon involving modalities—*epistemic contradictions*—and examine how Aloni proposes to account for these using **BSML**. Aloni's predictions only hold in states possessing certain properties; we make use of the characteristic formulas in showing how these properties can be defined using inference rules. We also point out here that our logics are not closed under uniform substitution.

Chapter 4 presents the natural deduction systems and soundness proofs.

In Chapter 5, we first prove weak completeness for each of the three logics as outlined above. For strong completeness, we make use of Lück's axiomatization of **MTL** [25]: this axiomatization is strongly complete, which implies that **MTL** is compact. As we noted above, **MTL** has the same expressive power as our logics. These facts together imply that our logics are also compact; strong completeness then follows from compactness and weak completeness.

Chapter 2

Preliminaries

In Section 2.1, we define the syntax and semantics for the logics to be axiomatized. We additionally define the syntax and a state-based semantics for classical modal logic—the other logics are all conservative extensions of classical modal logic, and we make extensive use of results that apply only to this fragment.

In Section 2.2, we define a few key state-semantic properties of formulas and use these to examine the semantics in more detail. We show how these properties tally with the syntax of the formulas. These results then allow us to establish some facts about the relationships between the logics required for the sequel: all of the logics extend classical modal logic, as noted above, and the state-based semantics for classical modal logic are in a sense reducible to the classical world-based semantics.

In Section 2.3, we show how $\mathbf{BSML}^{\mathbb{W}}$ can account for FC, and demonstrate why both the flat modality $\Diamond$ and the bilateral negation $\neg$ are necessary to procure the full range of Aloni's predictions.

In Section 2.4, we list the results we require from classical modal logic: we define Hintikka formulas and bisimulation, recall the relationship between the two, and show that there are only a finite number of Hintikka formulas of a given modal depth.

2.1 Syntax and Semantics

We assume throughout that Φ is a finite set of proposition symbols. Mention of Φ will for the most part be suppressed for brevity. Note, however, that some results will depend on the precise contents of Φ or the fact that Φ is finite; we make reference to Φ explicit when discussing these results to highlight this dependence.

Definition 2.1.1. (Syntax of ML, SML$^{\Vee}$, SGML$^{\Vee}$, BSML$^{\Vee}$) In all of the following, $p \in \Phi$.

- The set of formulas of *classical modal logic* **ML**(Φ) is generated as follows:

$$\alpha ::= p \mid \neg\alpha \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha) \mid \Diamond \alpha$$

- The set of formulas of *state-based modal logic with global disjunction* **SML**$^{\Vee}$(Φ) is generated as follows:

$$\phi ::= p \mid \neg\alpha \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \Vee \phi) \mid \Diamond \phi \mid \Box \phi \mid \text{NE}$$

  where $\alpha \in$ **ML**(Φ).

- The set of formulas of *state-based globally modal logic with global disjunction* **SGML**$^{\Vee}$(Φ) is generated as follows:

$$\phi ::= p \mid \neg\alpha \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \Vee \phi) \mid \diamondcdot \phi \mid \boxdot \phi \mid \text{NE}$$

  where $\alpha \in$ **ML**$^{\diamondcdot}$(Φ), with **ML**$^{\diamondcdot}$(Φ) generated as follows:

$$\alpha ::= p \mid \neg\alpha \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha) \mid \diamondcdot \alpha$$

- The set of formulas of *bilateral state-based modal logic with global disjunction* **BSML**$^{\Vee}$(Φ) is generated as follows:

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \Vee \phi) \mid \Diamond \phi \mid \text{NE}$$

We make use of the abbreviation **L**(Φ) to refer to the set of formulas which are in **SML**$^{\Vee}$(Φ), in **SGML**$^{\Vee}$(Φ) or in **BSML**$^{\Vee}$(Φ), i.e. **L**(Φ) = $\bigcup\{$**SML**$^{\Vee}$(Φ), **SGML**$^{\Vee}$(Φ), **BSML**$^{\Vee}$(Φ)$\}$. As mentioned above, we will frequently omit Φ and write simply **ML**, **ML**$^{\diamondcdot}$, **SML**$^{\Vee}$, **SGML**$^{\Vee}$, **BSML**$^{\Vee}$ and **L**.

ML$^{\diamondcdot}$, the negatable fragment of **SGML**$^{\Vee}$, consists of the formulas of **ML** with $\diamondcdot$ in place $\Diamond$, and so the only difference between **SGML**$^{\Vee}$ and **SML**$^{\Vee}$ is the modality. We will show in the next section that $\diamondcdot$ and $\Diamond$ are equivalent over classical formulas, and we could therefore have defined **ML** using $\diamondcdot$ instead of $\Diamond$.

Adhering to a convention already put into practice above, we will use ϕ, ψ, χ, γ, ν, η and ζ to refer to arbitrary formulas in the entirety of **L**, whereas α, β and δ are used exclusively to refer to arbitrary classical formulas (formulas in **ML**, **ML**$^{\diamondcdot}$, or the classical fragment of **BSML**$^{\Vee}$).

Models are standard Kripke models:

Definition 2.1.2. (Models and states) A *model M over* Φ is a triple $M = (W, R, V)$ where M is a set of *possible worlds*, $R \subseteq W \times W$ is an *accessibility relation* and $V : \Phi \to \wp(W)$ is a *valuation*. Subsets of W are called *states* on M.

When referring to an arbitrary model M, we will assume its components are named W, R and V without explicitly stating this (similarly, an arbitrary model M' is assumed to be (W', R', V'), and so on)—for instance:

Definition 2.1.3. Let M be a model. For any state s on M, let

$$R[s] := \{v \in W \mid \exists w \in s : wRv\} \text{ and}$$
$$R^{-1}[s] := \{w \in W \mid \exists v \in s : wRv\}$$

For any $w \in W$, let $R[w] := R[\{w\}]$ and $R^{-1}[w] := R^{-1}[\{w\}]$.

To define the semantics of the global modality $\Diamond\!\!\!\Diamond$, we make use of the notion of successor states:

Definition 2.1.4. (Successor states) Let M be a model. For any states s and t on M, t is a *successor state* of s, written sRt, if $t \subseteq R[s]$ and $s \subseteq R^{-1}[t]$.

Note that equivalently sRt if and only if $t \subseteq R[s]$ and for each $w \in s$: $t \cap R[w] \neq \varnothing$; i.e. every world in t has a predecessor in s, and every world in s has a successor in t. Note also that if sRt, then $s = \varnothing$ if and only if $t = \varnothing$.

Definition 2.1.5. (Semantics of $\mathbf{ML}, \mathbf{SML}^{\mathbb{W}}, \mathbf{SGML}^{\mathbb{W}}$, $\mathbf{BSML}^{\mathbb{W}}$) For a model M over Φ, a state s on M, and $\phi \in \mathbf{L}$, the notion of ϕ being *supported* by s in M, written $M, s \vDash \phi$ (or $s \vDash \phi$ when M is clear from the context), is defined recursively as follows:

$M, s \vDash p$	iff	$\forall w \in s : w \in V(p)$
$M, s \vDash \neg\alpha$	iff	$\forall w \in s : M, \{w\} \nvDash \alpha \quad (\alpha \in \mathbf{ML} \cup \mathbf{ML}^{\Diamond\!\!\!\Diamond})$
$M, s \vDash {\sim}\phi$	iff	$M, s \dashv \phi$
$M, s \vDash \phi \wedge \psi$	iff	$M, s \vDash \phi$ and $M, s \vDash \psi$
$M, s \vDash \phi \vee \psi$	iff	$\exists t, t' : t \cup t' = s$ and $M, t \vDash \phi$ and $M, t' \vDash \psi$
$M, s \vDash \phi \mathbb{W} \psi$	iff	$M, s \vDash \phi$ or $M, s \vDash \psi$
$M, s \vDash \Diamond\phi$	iff	$\forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing$ and $M, t \vDash \phi$
$M, s \vDash \Box\phi$	iff	$\forall w \in s : M, R[w] \vDash \phi$
$M, s \vDash \Diamond\!\!\!\Diamond\phi$	iff	$\exists t : sRt$ and $M, t \vDash \phi$
$M, s \vDash \boxdot\phi$	iff	$M, R[s] \vDash \phi$
$M, s \vDash \text{NE}$	iff	$s \neq \varnothing$

where $p \in \Phi$.

For $\phi \in \mathbf{BSML}^{\mathbb{W}}$, the notion of ϕ being *anti-supported* by s in M, written $M, s \dashv \phi$ (or $s \dashv \phi$), is defined recursively as follows:

$$
\begin{array}{lll}
M, s \dashv p & \text{iff} & \forall w \in s : w \notin V(p) \\
M, s \dashv \neg\phi & \text{iff} & M, s \vDash \phi \\
M, s \dashv \phi \wedge \psi & \text{iff} & \exists t, t' : t \cup t' = s \text{ and } M, t \dashv \phi \text{ and } M, t' \dashv \psi \\
M, s \dashv \phi \vee \psi & \text{iff} & M, s \dashv \phi \text{ and } M, s \dashv \psi \\
M, s \dashv \phi \mathbin{\mathbb{W}} \psi & \text{iff} & M, s \dashv \phi \text{ and } M, s \dashv \psi \\
M, s \dashv \Diamond\phi & \text{iff} & \forall w \in s : M, R[w] \dashv \phi \\
M, s \dashv \text{NE} & \text{iff} & s = \varnothing
\end{array}
$$

where $p \in \Phi$.

We write $M, s \nvDash \phi$ (or $s \nvDash \phi$) if $M, s \vDash \phi$ is not the case, and $M, s \not\dashv \phi$ (or $s \not\dashv \phi$) if $M, s \dashv \phi$ is not the case.

We define the following abbreviations:

<table>
<tr><td>ML</td><td rowspan="3">$\bot := p \wedge \neg p$</td><td></td><td></td><td rowspan="3">$\top\!\!\!\top := p \vee \neg p$</td><td>$\Box\phi := \neg \Diamond \neg\phi$</td></tr>
<tr><td>$\mathbf{SML}^{\mathbb{W}}$</td><td rowspan="3">$\top := \text{NE}$</td><td rowspan="3">$\bot\!\!\!\bot := \bot \wedge \text{NE}$</td><td rowspan="2"></td></tr>
<tr><td>$\mathbf{SGML}^{\mathbb{W}}$</td></tr>
<tr><td>$\mathbf{BSML}^{\mathbb{W}}$</td><td>$\bot := p \wedge ⫬p$</td><td>$\top\!\!\!\top := p \vee ⫬p$</td><td>$\Box\phi := ⫬ \Diamond ⫬\phi$</td></tr>
</table>

for some fixed $p \in \Phi$. Here $\bot$ is the *weak contradiction*, supported only by the empty state; $\bot\!\!\!\bot$, the *strong contradiction*, on the other hand, is supported by no state whatsoever. Analogously, $\top$, the *weak tautology*, is supported by all non-empty states; $\top\!\!\!\top$, the *strong tautology*, by all states.

We define the empty disjunctions for each of our logics in terms of these abbreviations:

$$\bigvee \varnothing := \bot \qquad\qquad \bigvee\!\!\!\!\bigvee \varnothing := \bot\!\!\!\bot$$

Definition 2.1.6. (Semantic entailment, equivalence and validity) For any set of formulas $\Gamma \cup \{\phi, \psi\} \in \mathbf{L}$, we say that:

- ψ is a *semantic consequence* of Γ, or Γ *semantically entails* ψ, written $\Gamma \vDash \psi$, if for all models M and all states s on M : if $M, s \vDash \gamma$ for all $\gamma \in \Gamma$, then $M, s \vDash \psi$. If $\{\phi\} \vDash \psi$, we also write $\phi \vDash \psi$.

- ϕ and ψ are *semantically equivalent*, written $\phi \equiv \psi$, if $\phi \vDash \psi$ and $\psi \vDash \phi$.

- ϕ is *semantically valid*, written $\vDash \phi$, if the empty set of formulas entails ϕ, i.e. $\varnothing \vDash \phi$.

We write $\phi \not\models \psi$ if $\phi \models \psi$ is not the case; $\phi \not\equiv \psi$ if $\phi \equiv \psi$ is not the case; $\Gamma \not\models \phi$ if $\Gamma \models \phi$ is not the case, and $\not\models \phi$ if $\models \phi$ is not the case.

Note that in the above definitions, the symbol $\models$ is being used ambiguously both with regard to which particular support relation it ultimately pertains to (i.e. to $\models_{\mathbf{ML}}$, $\models_{\mathbf{SML}^{\mathbb{W}}}$, $\models_{\mathbf{SGML}^{\mathbb{W}}}$, or $\models_{\mathbf{BSML}^{\mathbb{W}}}$), and with regard to whether its referent is the support relation or one of the auxiliary semantic notions. The syntax has been chosen so that if ϕ is a formula in two of the logics—$\phi \in L_1 \cap L_2$—then for any model M and any state s on M, $M, s \models_{L_1} \phi$ if and only if $M, s \models_{L_2} \phi$.[9] We may therefore also think of the semantics as being defined for the entirety of $\mathbf{L}$; we will occasionally use $\models$ in this manner (for instance, we may write $\phi \models \psi$ with ϕ and ψ formulas of different logics).

2.2 State-semantic Properties

In order to examine the semantics and the relationships between the logics in an effective manner, we make use of some commonly known state-semantic properties of formulas. These can be found in, for instance, [38]. For the most part the results in this section concerning these properties are adaptations of commonly-known results to the current setting; Proposition 2.2.10, which concerns the relationship between the modalities, is a new result.

Definition 2.2.1. Let $\phi \in \mathbf{L}$.

- ϕ has the *downward closure property* (or ϕ is *downward closed*) if for any model M, if $M, s \models \phi$ and $t \subseteq s$, then $M, t \models \phi$.
- ϕ has the *union closure property* (or ϕ is *union closed*) if for any model M and any non-empty set of states S on M, if $M, s \models \phi$ for all $s \in S$, then $M, \bigcup S \models \phi$.
- ϕ has the *empty state property* if for any model M we have $M, \varnothing \models \phi$.
- ϕ has the *flatness property* (or ϕ is *flat*) if for any model M we have $M, s \models \phi$ if and only if $M, \{w\} \models \phi$ for all $w \in W$.

Note the following relationship between the properties:

[9] The symbols that are given distinct definitions in different logics, and hence the symbols for which differences in support may arise, are $\Box$, $\bot$ and $\mathbb{T}$. It is easy to see that for any logic L, $M, s \models_L \bot$ if and only if $s = \varnothing$; $M, s \models_L \mathbb{T}$ is always the case; and $M, s \models \Box\phi$ if and only if for all $w \in s$, $M, R[w] \models \phi$.

Proposition 2.2.2. Let $\phi \in \mathbf{L}$. Then ϕ has the flatness property if and only if ϕ has the downward closure, union closure and empty state properties.

Proof.
$\Rightarrow$: Assume that ϕ has the flatness property.

- ϕ is downward closed: Let M be some model, and s be a state on M. Assume $M, s \vDash \phi$ and let $t \subseteq s$. By flatness, $M, \{w\} \vDash \phi$ for all $w \in s$ and therefore $M, \{w\} \vDash \phi$ for all $w \in t$, so that by flatness $M, t \vDash \phi$.

- ϕ is union closed: Assume that for some model M and some non-empty set of states S on M, we have $M, s \vDash \phi$ for all $s \in S$. By flatness, we have $M, \{w\} \vDash \phi$ for all $w \in \bigcup S$, so that by flatness, $M, \bigcup S \vDash \phi$.

- ϕ has the empty state property: Let M be some model. It is vacuously the case that for all $w \in \varnothing$, $M, \{w\} \vDash \phi$, so that by flatness, $M, \varnothing \vDash \phi$.

$\Leftarrow$: Assume that ϕ has the downward closure, union closure and empty state properties. Let M be some model, and s be a state on M.

Assume $M, s \vDash \phi$. If $s \neq \varnothing$, we have by downward closure that $M, \{w\} \vDash \phi$ for all $w \in s$; if $s = \varnothing$, this is vacuously true. Either way, then, $M, \{w\} \vDash \phi$ for all $w \in s$.

Conversely, assume $M, \{w\} \vDash \phi$ for all $w \in s$. If $s = \varnothing$, then $M, s \vDash \phi$ by the empty state property. If $s \neq \varnothing$, then $M, s \vDash \phi$ by union closure. Either way, then, $M, s \vDash \phi$.

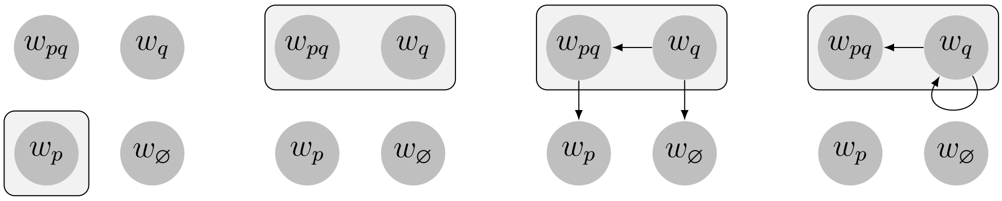
So we have $M, s \vDash \phi$ if and only if $M, \{w\} \vDash \phi$ for all $w \in s$. □

We now take a closer look at the semantics for each of the logics.

ML

Note that most of our state-semantical clauses for classical modal logic (those for p, $\neg$ and $\Diamond$) express conditions pertaining to what obtains individually at each world in the state (i.e. the conditions are of the form: "for each world in the state, X is the case"); see Figure 2.1 for some examples.[10]

[10] In all figures throughout the thesis, the circled area indicates the state s, and the name of each world shows which proposition symbols are supported by the singleton set containing that world (e.g. $\{w_{pq}\} \vDash p$ and $\{w_{pq}\} \vDash q$). The R-relation is indicated using arrows.



(a) $s \vDash p \wedge \neg q$ (b) $s \nvDash p$ and $s \nvDash \neg p$ (c) $s \vDash \Diamond p$ and $s \nvDash \Box p$ (d) $s \nvDash \Diamond p$ and $s \vDash \Box q$

Figure 2.1: Examples of **ML** semantics

The sole exceptions are the conjunction and the tensor disjunction, but in the context of semantics for **ML** only, these clauses could in fact be equivalently expressed as:

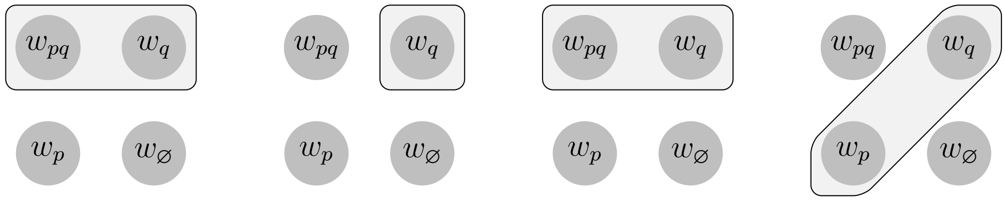
$$
\begin{array}{lll}
M, s \vDash \alpha \wedge \beta & \text{iff} & \forall w \in s : M, \{w\} \vDash \alpha \text{ and } M, \{w\} \vDash \beta \\
M, s \vDash \alpha \vee \beta & \text{iff} & \forall w \in s : M, \{w\} \vDash \alpha \text{ or } M, \{w\} \vDash \beta
\end{array}
$$

So in this classical setting, all support conditions are equivalent to conditions of the form: for all worlds in the state, something obtains. This implies that all formulas in **ML** are flat and hence that they also have the downward closure, union closure and empty set properties; we prove this in Corollary 2.2.9. It is for this reason, as well as the fact that the support conditions at a singleton state coincide with the classical truth conditions for the world in that state, that the state-based semantics for classical formulas is reducible to the classical semantics—support in states reduces to truth in worlds. We will formalize this observation later (Proposition 2.2.16).

$\mathbf{SML}^{\mathbb{W}}$

State-based modal logic with global disjunction ($\mathbf{SML}^{\mathbb{W}}$) is **ML** extended with NE and $\mathbb{W}$. When the clause for NE is added, the clauses for the conjunction and the tensor disjunction may no longer be rephrased as described above, and their "non-flat", genuinely state-based behaviour becomes apparent.

The tensor disjunction is supported by a state if the state can be split into two (possibly non-disjoint) substates, each of which supports one of the disjuncts. In Figure 2.2(a), since $\{w_{pq}\} \vDash p \wedge \text{NE}$ and $\{w_q\} \vDash q \wedge \text{NE}$, we have $s \vDash (p \wedge \text{NE}) \vee (q \wedge \text{NE})$, but it is not the case that for all $w \in s$, $\{w\} \vDash (p \wedge \text{NE}) \vee (q \wedge \text{NE})$ since this fails for w_q—this is an example of a formula that is not downward closed. It also clearly does not have the empty state property. (Note that all of this also applies to the conjunction



(a) $s \vDash (p \wedge \text{NE}) \vee (q \wedge \text{NE})$ (b) $s \nvDash (p \wedge \text{NE}) \vee (q \wedge \text{NE})$ (c) $s \vDash p \mathbin{⩛} q$ (d) $s \nvDash p \mathbin{⩛} q$

Figure 2.2: Examples of $\mathbf{SML}^{⩛}$ semantics

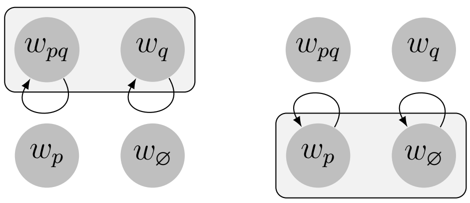
of this formula with itself, so that both disjunction and conjunction are now genuinely state-based.)

Figures 2.2(c) and 2.2(d) illustrate the global disjunction. Note that in 2.2(d), we do have $\{w\} \vDash p \mathbin{⩛} q$ for each $w \in s$—this is an example of a formula that is not union closed.

For $\mathbf{SML}^{⩛}$, the semantics for $\Box$ is given explicitly, whereas for **ML**, $\Box$ is the $\neg$-dual of $\Diamond$ (i.e. $\Box\phi := \neg \Diamond \neg\phi$). Given that $\neg$ may at present only precede classical formulas, one would have to extend the semantics for it in order to procure duality again. The most natural generalization would not function in the way intended given the presence of NE—see the discussion concerning the $\mathbf{SGML}^{⩛}$-modalities below.

$\mathbf{SGML}^{⩛}$

State-based globally modal logic with global disjunction ($\mathbf{SGML}^{⩛}$) is $\mathbf{SML}^{⩛}$ with the global modalities ⟐ and ⧈ in place of the flat modalities $\Diamond$ and $\Box$. The following demonstrates the differences between the two sets of modalities:



(a) ⟐$((p \wedge \text{NE}) \vee (q \wedge \text{NE})) \nvDash \Diamond((p \wedge \text{NE}) \vee (q \wedge \text{NE}))$ (b) $\Diamond(p \mathbin{⩛} \neg p) \nvDash$ ⟐$(p \mathbin{⩛} \neg p)$

Figure 2.3: Non-equivalence of the modalities

In Figure 2.3(a), we have:

- $s \models \diamonddot((p \wedge \textsc{ne}) \vee (q \wedge \textsc{ne}))$: Clearly sRs and $M, s \models (p \wedge \textsc{ne}) \vee (q \wedge \textsc{ne})$.
- $s \not\models \Diamond((p \wedge \textsc{ne}) \vee (q \wedge \textsc{ne}))$: The only non-empty subset of $R[w_q]$ is $\{w_q\}$, and since $\{w_q\} \not\models (p \wedge \textsc{ne}) \vee (q \wedge \textsc{ne})$, there are no non-empty subsets t of $R[w_q]$ such that $t \models (p \wedge \textsc{ne}) \vee (q \wedge \textsc{ne})$. Therefore $s \not\models \Diamond((p \wedge \textsc{ne}) \vee (q \wedge \textsc{ne}))$.

Note that we also have $s \models \boxdot((p \wedge \textsc{ne}) \vee (q \wedge \textsc{ne}))$ but $s \not\models \Box((p \wedge \textsc{ne}) \vee (q \wedge \textsc{ne}))$.

In Figure 2.3(b), we have:

- $s \models \Diamond(p \mathbin{\mathbb{V}} \neg p)$: Note that $\{w_p\} \subseteq R[w_p]$ is non-empty and that since $\{w_p\} \models p$ we have $\{w_p\} \models p \mathbin{\mathbb{V}} \neg p$. Similarly $\{w_\varnothing\} \subseteq R[w_\varnothing]$ is non-empty and since $\{w_\varnothing\} \models \neg p$, we have $\{w_\varnothing\} \models p \mathbin{\mathbb{V}} \neg p$. So for each $u \in s$ there is a non-empty $t \subseteq R[u]$ such that $t \models p \mathbin{\mathbb{V}} \neg p$; therefore $s \models \Diamond(p \mathbin{\mathbb{V}} \neg p)$.
- $s \not\models \diamonddot(p \mathbin{\mathbb{V}} \neg p)$: Let t be such that sRt. Then clearly $t = s = \{w_p, w_\varnothing\}$. Since $\{w_p\} \models p$ and $\{w_\varnothing\} \not\models p$, we have $t \not\models (p \mathbin{\mathbb{V}} \neg p)$. Therefore $s \not\models \diamonddot(p \mathbin{\mathbb{V}} \neg p)$.

Again, we also have $s \models \Box(p \mathbin{\mathbb{V}} \neg p)$ but $s \not\models \boxdot(p \mathbin{\mathbb{V}} \neg p)$.

We can now also see the rationale for our names of the modalities: $\Diamond$ is flat in that for any s, we have $s \models \Diamond\phi$ if and only if for all $w \in s : \{w\} \models \Diamond\phi$, and similarly for $\Box$. The global modalities are global in that the above does not hold and for $\diamonddot\phi$ or $\boxdot\phi$ to be supported in a state s, the state as a whole must bear a relationship to some other state (a successor state or $R[s]$) which in turn supports ϕ. (The clause for the global disjunction $\mathbin{\mathbb{V}}$ similarly looks at the state as whole.)

As with $\mathbf{SML}^{\mathbb{V}}$, the modalities in $\mathbf{SGML}^{\mathbb{V}}$ are defined separately. In her [35], Yang discusses some logics which contain the global modalities, lack NE, and are capable of expressing the intuitionistic negation

$$M, s \models \neg_\varnothing \phi \qquad \text{iff} \qquad \forall t \subseteq s : \text{if } M, t \models \phi, \text{ then } t = \varnothing$$

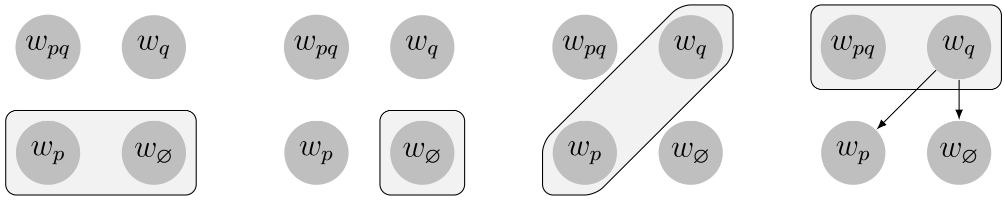
In these systems, $\boxdot\phi$ is equivalent to $\neg_\varnothing \diamonddot \neg_\varnothing \phi$, and the intuitionistic notion generalizes the classical $\neg$-notion in the sense that in these systems, $s \models \neg\alpha$ if and only if $s \models \neg_\varnothing \alpha$ for all classical α. So $\boxdot$ is the $\neg_\varnothing$-dual of $\diamonddot$, with $\neg_\varnothing$ a natural generalization of $\neg$. But in the presence of NE, $\neg_\varnothing$ would not work as intended: for instance, we would have $\varnothing \models \neg_\varnothing \phi$ for all ϕ, but $\varnothing \not\models \boxdot\textsc{ne}$, so that $\boxdot\textsc{ne} \not\equiv \neg_\varnothing \diamonddot \neg_\varnothing \textsc{ne}$. Variations of the notion such as

$$M, s \models \neg_{\perp\!\!\!\perp} \phi \qquad \text{iff} \qquad \forall t \subseteq s : M, t \not\models \phi$$

are equally problematic (note that $\neg_{\perp\!\!\!\perp}$ does not generalize $\neg$). Figure 2.3(a) also shows why notions along these lines will not do for duality. We have $s \vDash \boxdot((p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE}))$, but since $\{w_q\} \nvDash (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$ and $\varnothing \nvDash (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$, we have $\{w_q\} \vDash \neg_i((p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE}))$ for $i \in \{\varnothing, \perp\!\!\!\perp\}$. Then since $\{w_q\}R\{w_q\}$, we have $\{w_q\} \vDash \diamond\!\!\!\!\Diamond \neg_i((p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE}))$, so $s \nvDash \neg_i \diamond\!\!\!\!\Diamond \neg_i((p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE}))$. Similar considerations apply for the modalities in $\mathbf{SML}^{\vee\!\!\!\vee}$.

$\mathbf{BSML}^{\vee\!\!\!\vee}$

Bilateral state-based modal logic with global disjunction is $\mathbf{SML}^{\vee\!\!\!\vee}$ with the bilateral negation $\rightharpoondown$ replacing $\neg$. Below are some examples of the semantics for $\rightharpoondown$:



(a) $s \vDash \rightharpoondown q$

(b) $s \vDash \rightharpoondown(p \vee q)$ and $s \vDash \rightharpoondown(p \vee\!\!\!\vee q)$

(c) $s \vDash \rightharpoondown(p \wedge q)$ and $s \vDash \rightharpoondown((p \vee \rightharpoondown \mathrm{NE}) \wedge (q \vee \rightharpoondown \mathrm{NE}))$

(d) $s \vDash \rightharpoondown \Diamond q$

Figure 2.4: Examples of semantics for $\rightharpoondown$

We will show below that for all classical formulas α we have $\neg\alpha \equiv \rightharpoondown\alpha$; here we note some other interesting properties of the negation.

As Aloni [3] notes, there is a failure of replacement of equivalent formulas under negation:[11,12]

Fact 2.2.3. For $\mathbf{BSML}^{\vee\!\!\!\vee}$:

[11] Failure of replacement also holds for the dual negation of dependence logic as pointed out in, for instance, [22].

[12] Aloni defines her weak contradiction as $\bot_A := \rightharpoondown\mathrm{NE}$ and her strong contradiction as $\perp\!\!\!\perp_A := \mathrm{NE} \wedge \rightharpoondown\mathrm{NE}$. She can then express Fact 2.2.3 by saying, first, that negating the strong contradiction yields the weak tautology: $\rightharpoondown \perp\!\!\!\perp_A \equiv \top$; but negating the weak tautology gives us the weak contradiction rather than the strong contradiction again: $\perp\!\!\!\perp_A \equiv \rightharpoondown\rightharpoondown \perp\!\!\!\perp_A \not\equiv \rightharpoondown\top \equiv \bot_A$. And similarly, negating the strong tautology yields the weak contradiction: $\rightharpoondown \top\!\!\!\top \equiv \bot_A$; but negating the weak contradiction gives the weak tautology rather than the strong tautology: $\top\!\!\!\top \equiv \rightharpoondown\rightharpoondown \top\!\!\!\top \not\equiv \rightharpoondown\bot_A \equiv \top$.

Note that $\bot_A$ and $\perp\!\!\!\perp_A$ cannot be defined in $\mathbf{SML}^{\vee\!\!\!\vee}$ or $\mathbf{SGML}^{\vee\!\!\!\vee}$. We have chosen to define $\bot$ and $\perp\!\!\!\perp$ in a uniform manner in all logics to simplify the presentation; this means our contradictions are different from Aloni's.

- $\neg(\text{NE} \land \neg\text{NE}) \equiv \text{NE}$ but $\neg\neg(\text{NE} \land \neg\text{NE}) \not\equiv \neg\text{NE}$:

$$\begin{array}{lllllllll} \neg(\text{NE} \land \neg\text{NE}) & \equiv & \neg\text{NE} \lor \neg\neg\text{NE} & \equiv & \neg\text{NE} \lor \text{NE} & \equiv & \text{NE} & \equiv & \top \\ \neg\neg(\text{NE} \land \neg\text{NE}) & \equiv & (\text{NE} \land \neg\text{NE}) & \equiv & \bot\!\!\!\bot & \not\equiv & \neg\text{NE} & \equiv & \bot \end{array}$$

- $\neg \bot\!\!\!\bot \equiv \neg\text{NE}$ but $\neg\neg \bot\!\!\!\bot \not\equiv \neg\neg\text{NE}$:

$$\begin{array}{lllllllll} \neg \bot\!\!\!\bot & \equiv & \bot & \equiv & \neg\text{NE} & & & & \\ \neg\neg \bot\!\!\!\bot & \equiv & \bot\!\!\!\bot & \not\equiv & \neg\neg\text{NE} & \equiv & \text{NE} & \equiv & \top \end{array}$$

It is easy to see from the semantic clauses that other types of replacement of equivalents may be carried out safely:

Fact 2.2.4. For any $\phi, \psi, \chi \in \mathbf{BSML}^{\mathbb{W}}$ such that $\phi \equiv \psi$, we have $\phi \land \chi \equiv \psi \land \chi$; $\phi \lor \chi \equiv \psi \lor \chi$; $\phi \mathbin{\mathbb{W}} \chi \equiv \psi \mathbin{\mathbb{W}} \chi$; $\Diamond\phi \equiv \Diamond\psi$; and $\Box\phi \equiv \Box\psi$.

The following will be crucial for the axiomatization:

Fact 2.2.5. (Double negation elimination and De Morgan's laws for BSML$^{\mathbb{W}}$) For any $\phi, \psi \in \mathbf{BSML}^{\mathbb{W}}$:

- $\neg\neg\phi \equiv \phi$
- $\neg(\phi \lor \psi) \equiv \neg(\phi \mathbin{\mathbb{W}} \psi) \equiv \neg\phi \land \neg\psi$
- $\neg(\phi \land \psi) \equiv \neg\phi \lor \neg\psi$

Given that the above holds, $\mathbf{BSML}^{\mathbb{W}}$ formulas can be arranged into negation normal form, which will simplify our proofs by induction on the syntax.

Fact 2.2.6. (Negation normal form for BSML$^{\mathbb{W}}$) For any $\phi \in \mathbf{BSML}^{\mathbb{W}}$, there is a formula $\psi \in \mathbf{BSML}^{\mathbb{W}}$ such that $\phi \equiv \psi$ and in ψ, all occurrences of $\neg$ either precede atomic formulas ($p \in \Phi$ or NE) or form part of an occurrence of the sequence $\neg \Diamond \neg$ (i.e. a part of $\Box$).

Proof. By induction on the complexity of ϕ.

- $\phi = p$ or $\phi = \text{NE}$. ϕ is already in negation normal form.
- $\phi = \psi \land \chi$, $\phi = \psi \lor \chi$, $\phi = \psi \mathbin{\mathbb{W}} \chi$, or $\phi = \Diamond\psi$. These cases follow immediately from the induction hypothesis applied to ψ and χ and Fact 2.2.4.
- $\phi = \neg\psi$. We consider different cases:

- $\phi = \neg p$ or $\phi = \neg \text{NE}$. ϕ is already in negation normal form.
- $\phi = \neg\neg\chi$. We have $\neg\neg\chi \equiv \chi$, and then the conclusion follows by the induction hypothesis applied to χ.
- $\phi = \neg(\chi \wedge \eta)$, $\phi = \neg(\chi \vee \eta)$, or $\phi = \neg(\chi \mathbin{\mathbb{W}} \eta)$. The conclusion follows by De Morgan's Laws for $\mathbf{BSML}^{\mathbb{W}}$, the induction hypothesis applied to $\neg\chi$ and $\neg\eta$, and Fact 2.2.4.
- $\phi = \neg \Diamond \chi$. Note that $\neg \Diamond \chi \equiv \Box\neg\chi$:

$$\begin{array}{rlcl} & M, s \models \neg \Diamond \chi & \iff & M, R[s] \dashv \chi \\ \iff & M, R[s] \models \neg\chi & \iff & M, R[s] \dashv \neg\neg\chi \\ \iff & M, s \models \neg \Diamond \neg\neg\chi & \iff & M, s \models \Box\neg\chi \end{array}$$

  By the induction hypothesis, we have $\neg\chi \equiv \eta$ for some η in negation normal form. The conclusion then follows from Fact 2.2.4. □

We note in passing here that the same holds for the other logics; that this is the case follows from the negation normal form for classical modal logic, the fact that $\mathbf{SML}^{\mathbb{W}}$ and $\mathbf{SGML}^{\mathbb{W}}$ extend $\mathbf{ML}$ (Proposition 2.2.13), and the correspondence between state semantics and classical semantics for $\mathbf{ML}$ (Proposition 2.2.16).

Fact 2.2.7. (Negation normal form for $\mathbf{ML}, \mathbf{SML}^{\mathbb{W}}, \mathbf{SGML}^{\mathbb{W}}$) For any $\phi \in \mathbf{ML}, \mathbf{SML}^{\mathbb{W}}$ or $\mathbf{SGML}^{\mathbb{W}}$, there is a formula ψ in the same logic such that $\phi \equiv \psi$ and in ψ, all occurrences of $\neg$ either precede proposition symbols ($p \in \Phi$) or form part of an occurrence of a sequence $\neg \Diamond \neg\alpha$ or a sequence $\neg \mathbin{\Diamond\!\!\!\!\cdot\,} \neg\alpha$ (i.e. a part of $\Box\alpha$ or $\boxdot\alpha$), where $\alpha \in \mathbf{ML} \cup \mathbf{ML}^{\Diamond\!\!\!\!\cdot\,}$.

(Note that while in general $\Box$ is not defined as $\neg \Diamond \neg$ in $\mathbf{SML}^{\mathbb{W}}$, and $\boxdot$ not as $\neg \mathbin{\Diamond\!\!\!\!\cdot\,} \neg$ in $\mathbf{SGML}^{\mathbb{W}}$, these sequences do play the part of the boxes in the classical fragments of the logics ($\mathbf{ML}$ and $\mathbf{ML}^{\Diamond\!\!\!\!\cdot\,}$). So a formula in one of these logics is in negation normal form when all occurrences of $\neg$ either precede proposition symbols or form part of a box-sequence in the classical fragment of the logic.)

The following will not be proved until Chapter 3, but we include it here to help illuminate the nature of the bilateralism in $\mathbf{BSML}^{\mathbb{W}}$:

Proposition 3.3.9. For any formula $\phi \in \mathbf{BSML}^{\mathbb{W}}$, any model M and any state s on M, if $M, s \models \phi$, then for any state t on M, if $M, t \dashv \phi$, then $s \cap t = \varnothing$ (and in particular, if $M, s \models \phi$ and $M, s \dashv \phi$, then $s = \varnothing$).

Extending classical modal logic

We now link the properties with the syntax and show that all our logics extend **ML**.

Proposition 2.2.8. For any formula $\phi \in \mathbf{L}$:

- If ϕ does not contain NE, ϕ has the downward closure property and the empty state property.
- If ϕ does not contain $\mathbb{W}$, ϕ has the union closure property.

Proof. By induction on the complexity of ϕ. This is easy to see and a commonly known result for most of **L**, so we show only a few cases and remark that since by Fact 2.2.6 any $\phi \in \mathbf{BSML}^{\mathbb{W}}$ may be assumed to be in negation normal form, the only cases involving $\neg$ or bilateralism we need to consider are the negated atomic ones (and similarly for $\neg$ by Fact 2.2.7).

- $\phi = p$. For all models M and states s on M, $M, s \vDash p$ if and only if $w \in V(p)$ for all $w \in s$ if and only if $M, \{w\} \vDash p$ for all $w \in s$. So ϕ is flat, and therefore by Proposition 2.2.2 it has the downward closure, union closure and empty state properties.
- $\phi = \neg p$ or $\phi = \neg p$. This case is analogous to that for $\phi = p$.
- $\phi =$ NE. If for some model M and non-empty collection of states S on M we have $M, s \vDash$ NE for all $s \in S$, then for each $s \in S$, $s \neq \varnothing$, and therefore $\bigcup S \neq \varnothing$ so that $M, \bigcup S \vDash$ NE.
- $\phi = \neg$NE. If for some model M and non-empty collection of states S on M we have $M, s \vDash \neg$NE for all $s \in S$, then for each $s \in S$, $s = \varnothing$, and therefore $\bigcup S = \varnothing$ so that $M, \bigcup S \vDash \neg$NE.
- $\phi = \Diamond\psi$.
  - Downward closure: If $\Diamond\psi$ does not contain NE, then by the induction hypothesis, ψ is downward closed. Assume that $M, s \vDash \Diamond\psi$ and let $t \subseteq s$. By $M, s \vDash \Diamond\psi$ there is some s' such that sRs' and $M, s' \vDash \psi$; fix such an s'. Then note:
    - $R[t] \subseteq R[t]$.
    - $t \subseteq R^{-1}[R[t]]$: If $t = \varnothing$, this is trivially the case. Otherwise let $w \in t$. Since sRs', we have $s \subseteq R^{-1}[s']$, so that since $w \in t \subseteq s$, there is some $v \in s'$ such that wRv. Since $w \in t$, we have $v \in R[t]$, and therefore $w \in R^{-1}[R[t]]$. Since w was arbitrary, $t \subseteq R^{-1}[R[t]]$.

  * $M, R[t] \vDash \psi$: Since sRs' we have $R[s] \subseteq s'$ so that $R[t] \subseteq s'$. Then by downward closure, $M, R[t] \vDash \psi$.

  So $tR(R[t])$ and $M, R[t] \vDash \psi$; therefore $M, t \vDash \Diamond\psi$.

- Empty state property: If $\Diamond\psi$ does not contain NE, then by the induction hypothesis, ψ has the empty state property. Let M be a model. By the empty state property, $M, \varnothing \vDash \psi$. Clearly $\varnothing R \varnothing$, so $M, \varnothing \vDash \Diamond\psi$.
- Union closure: If $\Diamond\psi$ does not contain $\mathbb{W}$, then by the induction hypothesis, ψ is union closed. Assume that for some model M and non-empty collection of states S on M we have $M, s \vDash \Diamond\psi$ for all $s \in S$. Then for each $s \in S$, there is some s' such that sRs' and $M, s' \vDash \psi$; fix such an s' for each $s \in S$. Let $u := \bigcup_{s \in S} s'$. Then:

  * $u \subseteq R[\bigcup S]$: If $u = \varnothing$, this is trivially the case. Otherwise let $w \in u$. Then for some $s \in S$ we have $w \in s'$, so $w \in R[s]$ and therefore $w \in R[\bigcup S]$. w was arbitrary, so $u \subseteq R[\bigcup S]$.
  * $\bigcup S \subseteq R^{-1}[u]$: If $\bigcup S = \varnothing$, this is trivially the case. Otherwise let $w \in \bigcup S$. Then for some $s \in S$ we have $w \in s$, so $w \in R^{-1}[s']$. Clearly $R^{-1}[s'] \subseteq R^{-1}[u]$, so $w \in R^{-1}[u]$. Since w was arbitrary, $\bigcup S \subseteq R^{-1}[u]$.
  * $M, u \vDash \psi$: We have it that $S' = \{s' \mid s \in S\}$ is a non-empty collection of states such that for all $s' \in S' : M, s' \vDash \psi$. Therefore, by union closure and noting that $u = \bigcup S'$, we have $M, u \vDash \psi$.

  So $(\bigcup S)Ru$ and $M, u \vDash \psi$; therefore $M, \bigcup S \vDash \Diamond\psi$. □

By Propositions 2.2.2 and 2.2.8, we have now shown that all classical formulas are flat:

Corollary 2.2.9. For any $\alpha \in \mathbf{ML} \cup \mathbf{ML}^{\Diamond}$, α has the union closure, downward closure, empty state and flatness properties.

For the modalities we have:

Proposition 2.2.10. Let $\phi \in \mathbf{L}$. Then:

1. If ϕ is downward closed, then:

   a) $\Diamond\phi \vDash \Diamond\phi$ and

   b) $\boxdot\phi \vDash \Box\phi$.

2. If ϕ is union closed and has the empty state property, then:

   a) $\Diamond\phi \models \diamonddot\phi$ and

   b) $\Box\phi \models \boxdot\phi$.

So if ϕ has all three properties, then $\Diamond\phi \equiv \diamonddot\phi$ and $\Box\phi \equiv \boxdot\phi$.

Proof. 1. a) Let $M, s \models \diamonddot\phi$ (note that if $\phi = \bot\!\!\bot$, ϕ is downward closed but $M, s \models \diamonddot \bot\!\!\bot$ can never be the case, so $\diamonddot\phi \models \Diamond\phi$ holds trivially). Then there is a t such that sRt and $M, t \models \phi$.

Case 1: $s \neq \varnothing$. Fix some $w \in s$. Since $s \subseteq R^{-1}[t]$, there is some $v \in t$ such that wRv. By downward closure, $M, \{v\} \models \phi$; note that clearly $\{v\}$ is non-empty. Since w was arbitrary, we therefore have $M, s \models \Diamond\phi$.

Case 2: $s = \varnothing$. Then trivially $M, s \models \Diamond\phi$.

In either case, then, $M, s \models \Diamond\phi$.

b) Let $M, s \models \boxdot\phi$ (as above, the case in which $\phi = \bot\!\!\bot$ holds trivially). Then $M, R[s] \models \phi$.

Case 1: $s \neq \varnothing$. Fix some $w \in s$. By downward closure, $M, R[w] \models \phi$. Since w was arbitrary, we have $M, s \models \Box\phi$.

Case 2: $s = \varnothing$. Then trivially $M, s \models \Box\phi$.

In either case, then, $M, s \models \Box\phi$.

2. a) Let $M, s \models \Diamond\phi$.

Case 1: $s \neq \varnothing$. Since $M, s \models \Diamond\phi$, for each $w \in s$ there is a non-empty $t_w \subseteq R[w]$ such that $M, t_w \models \phi$; fix such a t_w for each $w \in s$. Let $t := \bigcup_{w \in s} t_w$. Then:

- $t \subseteq R[s]$.
- $s \subseteq R^{-1}[t]$: Since $s \neq \varnothing$, we can fix some $w \in s$. Then there is a non-empty $t_w \subseteq R[w]$ such that $t_w \subseteq t$, and so there is some $v \in t$ such that wRv. w was arbitrary, so $s \subseteq R^{-1}[t]$.
- $M, t \models \phi$: Since s is non-empty, $\{t_w \mid w \in s\}$ is non-empty, so that by union closure and noting that $t = \bigcup\{t_w \mid w \in s\}$, we have $M, t \models \phi$.

So sRt and $M, t \models \phi$; therefore $M, s \models \diamonddot\phi$.

Case 2: $s = \varnothing$. Then clearly $sR\varnothing$. Since ϕ has the empty state property, $M, \varnothing \models \phi$, and so $M, s \models \diamonddot\phi$.

In either case, then, $M, s \models \diamonddot\phi$.

b) Let $M, s \models \Box\phi$.

Case 1: $s \neq \varnothing$. We have it that for all $w \in s : M, R[w] \models \phi$. Since s is non-empty, $\{R[w] \mid w \in s\}$ is non-empty, so by union closure and noting that $R[s] = \bigcup\{R[w] \mid w \in s\}$, we have $M, R[s] \models \phi$. Therefore $M, s \models \boxdot\phi$.

Case 2: $s = \varnothing$. Then $R[s] = \varnothing$. Since ϕ has the empty state property, $M, \varnothing \models \phi$, and so $M, s \models \boxdot\phi$.

In either case, then, $M, s \models \boxdot\phi$. □

It now follows that **SGML**$^{\mathbb{W}}$ extends **ML**. First, by Propositions 2.2.8 and 2.2.10:

Corollary 2.2.11. For any $\phi \in \mathbf{L}$:

1. If ϕ does not contain NE, then $\Diamonddot\phi \models \Diamond\phi$ and $\boxdot\phi \models \Box\phi$.

2. If ϕ does not contain $\mathbb{W}$, then $\Diamond\phi \models \Diamonddot\phi$ and $\Box\phi \models \boxdot\phi$.

Therefore, if ϕ does not contain NE or $\mathbb{W}$, and in particular if $\phi \in \mathbf{ML} \cup \mathbf{ML}^{\Diamonddot}$, then $\Diamond\phi \equiv \Diamonddot\phi$ and $\Box\phi \equiv \boxdot\phi$.

Definition 2.2.12. Define a map $* : \mathbf{ML} \to \mathbf{ML}^{\Diamonddot}$ by $\Diamond \mapsto \Diamonddot$ (i.e. $*(\alpha)$ is α with each $\Diamond$ replaced by a $\Diamonddot$).

We will also write α^* for $*(\alpha)$, and we write A^* for $\{\alpha^* \mid \alpha \in \mathrm{A}\}$ (where $\mathrm{A} \subseteq \mathbf{ML}$). Note again that while $\boxdot$ is not in general the $\neg$-dual of $\Diamonddot$ in **SGML**$^{\mathbb{W}}$, this is the case in the **ML**$^{\Diamonddot}$-fragment. Therefore $*$ is a one-to-one map between **ML** and **ML**$^{\Diamonddot}$, and given Corollary 2.2.11:

Proposition 2.2.13. (SGML$^{\mathbb{W}}$ is a conservative extension of ML) For any $\alpha \in \mathbf{ML} : \alpha \equiv \alpha^*$.

Similarly, $\intprod$ and $\neg$ are equivalent when applied to classical formulas, and therefore **BSML**$^{\mathbb{W}}$ also extends **ML**:

Definition 2.2.14. Define a map $** : \mathbf{ML} \to \mathbf{BSML}^{\mathbb{W}}$ by $\neg \mapsto \intprod$ (i.e. $**(\alpha)$ is α with each $\neg$ replaced by a $\intprod$).

We again write α^{**} for $**(\alpha)$ and A^{**} for $\{\alpha^{**} \mid \alpha \in \mathrm{A}\}$. This is a one-to-one map between **ML** and the NE- and $\mathbb{W}$-free fragment of **BSML**$^{\mathbb{W}}$ (call this fragment **ML****), and:

Proposition 2.2.15. (BSML$^{\mathbb{W}}$ is a conservative extension of ML) For any $\alpha \in \mathbf{ML}$, we have $\alpha \equiv \alpha^{**}$.

Proof. By induction on the complexity of α (note that we may assume that α is in negation normal form):

- $\alpha = p$. $p^{**} = p$. The **ML** and **BSML**$^{\mathbb{W}}$ semantic clauses for p are identical so $\alpha \equiv \alpha^{**}$.

- $\alpha = \neg p$. $(\neg p)^{**} = \lrcorner p$. We have $M, s \vDash \neg p$ if and only if for all $w \in s : w \notin V(p)$ if and only if $M, s \dashv p$ if and only $M, s \vDash \lrcorner p$, so $\alpha \equiv \alpha^{**}$.

- $\alpha = \beta \wedge \delta$, $\alpha = \beta \vee \delta$ or $\alpha = \Diamond\beta$. Clearly $(\beta \wedge \delta)^{**} = \beta^{**} \wedge \delta^{**}$, $(\beta \vee \delta)^{**} = \beta^{**} \vee \delta^{**}$ and $(\Diamond\beta)^{**} = \Diamond\beta^{**}$. The result then follows from the induction hypothesis and the fact that for each of $\wedge$, $\vee$ and $\Diamond$, the **ML** and **BSML**$^{\mathbb{W}}$ semantic clauses are identical.

- $\alpha = \Box\beta$. We have:

$$
\begin{aligned}
& M, s \vDash \Box\beta \\
\Longleftrightarrow\ & M, s \vDash \neg \Diamond \neg\beta \\
\Longleftrightarrow\ & \forall w \in s : M, \{w\} \nvDash \Diamond\neg\beta \\
\Longleftrightarrow\ & \forall w \in s : \nexists\, t \subseteq R[w] : t \neq \varnothing \text{ and } M, t \vDash \neg\beta \\
\Longleftrightarrow\ & \forall w \in s : \nexists\, t \subseteq R[w] : t \neq \varnothing \text{ and } \forall v \in t : M, \{v\} \nvDash \beta \\
\Longleftrightarrow\ & \forall w \in s : \forall t \subseteq R[w] : t = \varnothing \text{ or } \exists v \in t : M, \{v\} \vDash \beta \\
\Longleftrightarrow\ & \forall w \in s : \forall v \in R[w] : M, \{v\} \vDash \beta \\
\Longleftrightarrow\ & \forall w \in s : M, R[w] \vDash \beta && \text{Corollary 2.2.9} \\
\Longleftrightarrow\ & \forall w \in s : M, R[w] \vDash \beta^{**} && \text{hypothesis} \\
\Longleftrightarrow\ & \forall w \in s : M, R[w] \dashv \lrcorner\beta^{**} \\
\Longleftrightarrow\ & M, s \dashv \Diamond\lrcorner\beta^{**} \\
\Longleftrightarrow\ & M, s \vDash \lrcorner \Diamond \lrcorner\beta^{**} \\
\Longleftrightarrow\ & M, s \vDash (\neg \Diamond \neg\beta)^{**} \\
\Longleftrightarrow\ & M, s \vDash \alpha^{**}
\end{aligned}
$$

□

Given that **SML**$^{\mathbb{W}}$ clearly extends **ML** in this manner, we have now shown that all of the logics do so. We will call all members of $\mathbf{CML}(\Phi) := \mathbf{ML}(\Phi) \cup \mathbf{ML}^{\circledast}(\Phi) \cup \mathbf{ML}(\Phi)^{**}$ classical formulas. Note that Corollary 2.2.9 applies also to $\mathbf{ML}^{**}$—all formulas in this set have the union closure, downward closure, empty state and flatness properties.

In order for us to make use of the classical results to be introduced in Section 2.4, it remains to link the state-based semantics for classical modal logic with the classical semantics. Given Corollary 2.2.9, we get the following simply by noting that the state-based clauses for a singleton state match exactly with the classical clauses for a world:

Proposition 2.2.16. For any model M and any $\alpha \in \mathbf{CML}$, we have:

$$M, s \vDash \alpha \qquad \Longleftrightarrow \qquad M, w \vDash \alpha \text{ for all } w \in s$$

(where $\vDash$ on the left is the state-based support relation, and $\vDash$ on the right is the truth relation from classical modal logic.)
In particular, for any $w \in W : M, \{w\} \vDash \alpha \iff M, w \vDash \alpha$.

We then also have it that for classical formulas, entailment and equivalence as classically defined coincide with our state-semantic definitions:

Fact 2.2.17. For any $\mathrm{B} \cup \{\alpha, \beta\} \subseteq \mathbf{CML}$:

$$\mathrm{B} \vDash \alpha \iff \mathrm{B} \vDash_C \alpha, \text{ and therefore } \alpha \equiv \beta \iff \alpha \equiv_C \beta$$

Where

$$\begin{aligned} \mathrm{B} \vDash_C \alpha \quad &:\Longleftrightarrow \quad \forall (M, w) : (\forall \beta \in \mathrm{B} : M, w \vDash \beta) \Rightarrow M, w \vDash \alpha \\ \alpha \equiv_C \beta \quad &:\Longleftrightarrow \quad \alpha \vDash_C \beta \text{ and } \beta \vDash_C \alpha \end{aligned}$$

Proof. $\Rightarrow$: Assume $\mathrm{B} \vDash \alpha$. Fix some (M, w) and assume $M, w \vDash \beta$ for each $\beta \in \mathrm{B}$. By Proposition 2.2.16, $M, \{w\} \vDash \beta$ for each $\beta \in \mathrm{B}$. Then by $\mathrm{B} \vDash \alpha$ we have $M, \{w\} \vDash \alpha$, so that by Proposition 2.2.16, $M, w \vDash \alpha$.

$\Leftarrow$: Assume $\mathrm{B} \vDash_C \alpha$. Fix some (M, s) and assume $M, s \vDash \beta$ for each $\beta \in \mathrm{B}$. By Proposition 2.2.16, $M, w \vDash \beta$ for each $\beta \in \mathrm{B}$. By $\mathrm{B} \vDash_C \alpha$ we have $M, w \vDash \alpha$ for each $w \in s$, so that by Proposition 2.2.16, $M, s \vDash \alpha$. □

We may therefore speak simply of entailment and equivalence and may always omit the C-subscripts.

2.3 Accounting for FC

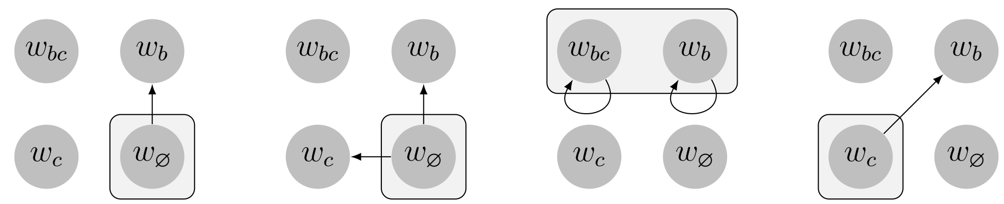
Aloni [3] hypothesizes that in certain situations, the effects of pragmatic principles on semantics can be modelled by a systematic "intrusion" of these principles into the process of meaning composition, and that FC inferences are the result of such an intrusion. She proposes that the intruding principle in the case of FC is "avoid stating a contradiction", (derivable, for instance, from the Gricean maxim of Quality [11]), and that this could be formalized in **BSML** (or $\mathbf{BSML}^{\mathbb{W}}$) as NE (i.e. as $\neg\bot_A$; see footnote 11).

In situations in which an intrusion of “avoid stating a contradiction” is triggered, the usual formalizations of natural language expressions become “pragmatically enriched” by the intrusion. For a formula ϕ in the NE-free fragment of $\mathbf{BSML}^{\mathbb{W}}$, the formula $\phi^+ \in \mathbf{BSML}^{\mathbb{W}}$ pragmatically enriched by an intrusion of the principle is defined recursively as follows:

$$
\begin{array}{lll}
p^+ & := & p \wedge \text{NE} \\
(\neg\phi)^+ & := & \neg\phi^+ \wedge \text{NE} \\
(\phi \wedge \psi)^+ & := & (\phi^+ \wedge \text{NE}) \wedge (\psi^+ \wedge \text{NE}) \\
(\phi \vee \psi)^+ & := & (\phi^+ \wedge \text{NE}) \vee (\psi^+ \wedge \text{NE}) \\
(\phi \mathbin{\mathbb{W}} \psi)^+ & := & (\phi^+ \wedge \text{NE}) \mathbin{\mathbb{W}} (\psi^+ \wedge \text{NE}) \\
(\Diamond\phi)^+ & := & \Diamond\, \phi^+ \wedge \text{NE}
\end{array}
$$

Aloni then claims that FC inferences are justified in the sense that the following holds for all $\phi, \psi \in \mathbf{BSML}^{\mathbb{W}} : (\Diamond(\phi \vee \psi))^+ \vDash \Diamond\phi \wedge \Diamond\psi$. For let $M, s \vDash (\Diamond(\phi \vee \psi))^+$, i.e. $M, s \vDash \Diamond((\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE})) \wedge \text{NE}$. Let $w \in s$. Then since $M, s \vDash \Diamond((\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE}))$, there is some non-empty $t \subseteq R[w]$ such that $M, t \vDash (\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE})$. Therefore there are some t_1, t_2 such that $t = t_1 \cup t_2$; $M, t_1 \vDash \phi \wedge \text{NE}$; and $M, t_2 \vDash \psi \wedge \text{NE}$. Then $t_1 \neq \varnothing$ and $M, t_1 \vDash \phi$; and $t_2 \neq \varnothing$ and $M, t_2 \vDash \psi$; and note also that $t_1 \subseteq R[w]$ and $t_2 \subseteq R[w]$. Since w was arbitrary, this is the case for all $w \in s$, and therefore $M, s \vDash \Diamond\phi \wedge \Diamond\psi$.[13]

Let us examine an example.



(a) $s \vDash \Diamond(b \vee c)$ $s \nvDash (\Diamond(b \vee c))^+$

(b) $s \vDash (\Diamond(b \vee c))^+$

(c) $s \vDash (\Diamond\!\!\!\!\diamond(b \vee c))^+$ $s \nvDash \Diamond\!\!\!\!\diamond b \wedge \Diamond\!\!\!\!\diamond c$

(d) $\forall(t \neq \varnothing) \subseteq s :$ $t \nvDash \Diamond(b \vee c)^+$ and $s \nvDash \neg\Diamond\, b \wedge \neg\Diamond\, c$

Figure 2.5: FC example and failure of FC for $\Diamond\!\!\!\!\diamond$ and $\neg$

I tell you “You may go to beach or go to the cinema” (with going to the beach represented by b and going to the cinema represented by c). If the

[13]Note that FC inferences are not predicted to be licensed for the global disjunction in the current system. Accounts of FC (specifically narrow-scope FC, which we are presently discussing—see Section 3.3 for wide-scope FC) using the global disjunction and a notion of modality distinct from ours can be found in [2] and [5].

situation is as in Figure 2.5(a), then while $s \models \Diamond(b \vee c)$ does hold and so in the classical logician's sense you may go to the beach or go to the cinema, there is no permissible world in which you may go to the cinema. Recall that the tensor disjunction is supported in a state if the state can be split into two substates, with each substate supporting one of the disjuncts. The disjunction $b \vee c$ is only permissible in s in the sense that the permissible scenario $\{w_b\}$ of going to the beach may be thought to consist of $\{w_b\}$ and the impossible scenario $\varnothing$ which vacuously supports c. On the other hand in Figure 2.5(b) we have $s \models (\Diamond(b \vee c))^+$ and hence $s \models \Diamond b \wedge \Diamond c$. Both options are realized in permissible states.

Figure 2.5(c) demonstrates why $\Diamond\!\!\!\!\cdot\,$ fails to model FC in the manner $\Diamond$ does: we have that $s \models (\Diamond\!\!\!\!\cdot\,(b \vee c))^+$ (assuming that $(\Diamond\!\!\!\!\cdot\,(b \vee c))^+$ is defined in the expected way), but since the only successor state of s is s itself and since $s \not\models c$, we have $s \not\models \Diamond\!\!\!\!\cdot\, c$, and therefore $s \not\models \Diamond\!\!\!\!\cdot\, b \wedge \Diamond\!\!\!\!\cdot\, c$.

In order to understand why the bilateral negation is required, we consider the following:

You may not go to the beach or go to the cinema.
$\rightsquigarrow$ You may not go the beach and you may not go to the cinema.

$$\neg \Diamond (b \vee c) \to (\neg \Diamond b \wedge \neg \Diamond c)$$

As with our original example of FC, this inference usually appears to be licensed in natural language.

Aloni's account predicts this by noting that $(\neg\Diamond(\phi\vee\psi))^+ \models \neg\Diamond\phi\wedge\neg\Diamond\psi$. For let $M, s \models (\neg\Diamond(\phi\vee\psi))^+$, i.e. $M, s \models \neg(\Diamond((\phi\wedge\text{NE})\vee(\psi\wedge\text{NE}))\wedge\text{NE})\wedge\text{NE}$. It is easy to see that then $M, s \models \neg\Diamond((\phi\wedge\text{NE})\vee(\psi\wedge\text{NE}))$, so $M, s \models\!\!\mid \Diamond((\phi\wedge\text{NE})\vee(\psi\wedge\text{NE}))$, so that for all $w \in s$ we have $M, R[w] \models\!\!\mid (\phi\wedge\text{NE})\vee(\psi\wedge\text{NE})$. Let $w \in s$. Then by the above, $M, R[w] \models\!\!\mid \phi\wedge\text{NE}$ and $M, R[w] \models\!\!\mid \psi\wedge\text{NE}$, so that $M, R[w] \models\!\!\mid \phi$ and $M, R[w] \models\!\!\mid \psi$. Since w was arbitrary, we then have both $M, s \models \neg\Diamond\phi$ and $M, s \models \neg\Diamond\psi$, so $M, s \models \neg\Diamond\phi\wedge\neg\Diamond\psi$.

For $\mathbf{SML}^{\mathbb{W}}$, which lacks the bilateral negation, the pragmatically enriched formula $(\neg\Diamond(\phi\vee\psi))^+$ cannot be defined since $\neg$ may only precede classical formulas. Figure 2.5(d) additionally demonstrates that no obvious generalization of $\neg$ can account for cases like this in the manner $\mathbf{BSML}^{\mathbb{W}}$ does. We have (assuming that $\Diamond(b \vee c)^+$ is defined in the expected way) that:

- For each non-empty $t \subseteq s$ we have $t \not\models \Diamond(b\vee c)^+$: Clearly $\{w_b\} \not\models (b\vee c)^+$ so that since $R[w_c] = \{w_b\}$, there are no non-empty subsets t of $R[w_c]$ such that $t \models (b \vee c)^+$. Therefore $\{w_c\} \not\models \Diamond(b \vee c)^+$. Since $s = \{w_c\}$, we have it that for each non-empty $t \subseteq s$, $t \not\models \Diamond(b \vee c)^+$.

- $s \not\models \neg \Diamond b \land \neg \Diamond c$: Since $\{w_b\} \models b$ so that $\{w_c\} \models \Diamond b$, we have $s \not\models \neg \Diamond b$, and therefore $s \not\models \neg \Diamond b \land \neg \Diamond c$.

2.4 Bisimulation and Hintikka Formulas

The bisimulation relation between pointed models captures what is essential for classical modal equivalence[14], and Hintikka formulas are a syntactic characterization of the same. In the next chapter we use these notions to define state-based analogues of them; here we simply list the required definitions and theorems. These are all either standard classical results found, for instance, in [10], or straightforward consequences of such.

Note that we formulate many definitions and results in this section in terms of the language of **ML**, but in light of the equivalence of the formulas in these fragments we could equally well have done so in terms of the language of $\mathbf{ML}^{\diamondsuit}$ or that of $\mathbf{ML}^{**}$.

Definition 2.4.1. (Pointed models) A *pointed model over* Φ is a pair (M, w) where M is a model over Φ and $w \in W$.

Definition 2.4.2. (Modal depth) The *modal depth* $md(\phi)$ of a formula of $\phi \in \mathbf{L}$ is defined recursively as follows:

- $md(p) = md(\text{NE}) = 0$ for $p \in \Phi$
- $md(\neg\alpha) = md(\alpha)$
- $md(\neg\psi) = md(\psi)$
- $md(\psi \land \chi) = md(\psi \lor \chi) = md(\psi \veebar \chi) = max\{md(\psi), md(\chi)\}$
- $md(\Diamond\psi) = md(\Box\psi) = md(\diamondsuit\psi) = md(\boxdot\psi) = md(\psi) + 1$

Definition 2.4.3. (Classical modal equivalence) Let (M, w) and (M', w') be pointed models.

- Let $k \in \mathbb{N}$. (M, w) and (M', w') are *k-equivalent*, written $M, w \equiv_k M', w'$, if for all $\alpha \in \mathbf{ML}$ with $md(\alpha) \leq k$: $M, w \models \alpha \iff M', w' \models \alpha$.
- (M, w) and (M', w') are **ML**-equivalent, written $M, w \equiv^{\mathbf{ML}} M', w'$, if $M, w \equiv_k M', w'$ for all $k \in \mathbb{N}$.

[14]The relation we make use of is also called *finite bisimulation* in the literature to distinguish it from a stronger relation which can capture equivalence in infinitary modal logic [10]. The distinction does not matter for our purposes, so we speak simply of bisimulation.

We write $M, w \not\equiv_k M', w'$ if $M, w \equiv_k M', w'$ is not the case, and $M, w \not\equiv^{\mathbf{ML}} M', w'$ if $M, w \equiv^{\mathbf{ML}} M', w'$ is not the case.

Note that $\vDash$ in the above (and below) is the truth relation from classical modal logic. Note also that $M, w \equiv^{\mathbf{ML}} M', w'$ iff for all $\alpha \in \mathbf{ML} : M, w \vDash \alpha \iff M', w' \vDash \alpha$.

Definition 2.4.4. (Bisimilarity) Let $k \in \mathbb{N}$, and let (M, w) and (M', w') be pointed models over Φ. The *k-bisimilarity* relation—we write $M, w \leftrightarroweq_k M', w'$ and say (M, w) and (M', w') are *k-bisimilar* if the relation holds—is defined recursively as follows:

- $M, w \leftrightarroweq_0 M, w$ iff for all $p \in \Phi$ we have $M, w \vDash p \iff M', w' \vDash p$.
- $M, w \leftrightarroweq_{k+1} M', w'$ iff $M, w \leftrightarroweq_0 M', w'$ and
  - for all $v \in R[w]$ there is a $v' \in R'[w']$ such that $M, v \leftrightarroweq_k M', v'$
  - for all $v' \in R'[w']$ there is a $v \in R[w]$ such that $M, v \leftrightarroweq_k M', v'$

(M, w) and (M', w') are *bisimilar*, written $M, w \leftrightarroweq M', w'$, if $M, w \leftrightarroweq_k M', w'$ for all $k \in \mathbb{N}$.

We write $M, w \not\leftrightarroweq_k M', w'$ if $M, w \leftrightarroweq_k M', w'$ is not the case, and $M, w \not\leftrightarroweq M', w'$ if $M, w \leftrightarroweq M', w'$ is not the case.

When (M, w) and (M', w') are bisimilar, we also say that w and w' are bisimilar if the models are clear from the context.

The definition of Hintikka formulas makes use of the fact that Φ is finite:

Definition 2.4.5. (Hintikka formulas) Let $k \in \mathbb{N}$ and let (M, w) be a pointed model over Φ. We define the *k-th Hintikka formula* of (M, w), denoted by $\chi^k_{M,w}$, recursively as follows:[15]

$$\begin{aligned} \chi^0_{M,w} &:= \bigwedge\{p \mid p \in \Phi, w \in V(p)\} \wedge \bigwedge\{\neg p \mid p \in \Phi, w \notin V(p)\} \\ \chi^{k+1}_{M,w} &:= \chi^k_{M,w} \wedge \bigwedge_{v \in R[w]} \Diamond \chi^k_{M,v} \wedge \Box \bigvee_{v \in R[w]} \chi^k_{M,v} \end{aligned}$$

We will also call $\chi^k_{M,w}$ a *Hintikka formula of degree k*.

We will write χ^k_w in place of $\chi^k_{M,w}$ when M is clear from the context.

[15] A note on notation: when a large connective such as $\bigvee$ is followed by a formula inside a pair of parentheses, as in $\bigvee(\phi)$, the parentheses delimit the scope of the connective. If there are no parentheses, the scope of the connective is limited to what immediately follows the connective. So in $\bigvee_{i \in I} \phi_i \wedge \psi$, only the ϕ_i is inside the scope of the disjunction.

Intuitively, the first conjunct χ^0_w of a Hintikka formula provides a complete description of what is the case at w; the conjunct $\bigwedge_{v\in R[w]} \Diamond\chi^k_v$ of a formula of degree $k+1$ lists all things of modal depth k which are possible in w; and the conjunct $\Box \bigvee_{v\in R[w]} \chi^k_v$ states that nothing else of modal depth k is possible.

We list some straightforward consequences of the definition:

Fact 2.4.6. Let $k \in \mathbb{N}$ and let (M, w) be a pointed model. Then $md(\chi^k_w) \leq k$.

Securing the following fact requires Φ to be finite; we quantify over sets of proposition symbols here to make the more general claim explicit:

Fact 2.4.7. For any finite set of proposition symbols Φ and any $k \in \mathbb{N}$, there are only finitely many non-equivalent k-th Hintikka formulas of pointed models over Φ.

Proof. By induction on $k \in \mathbb{N}$:

- $k = 0$. It is easy to see from the definition that there are $2^{|\Phi|}$ non-equivalent 0-th Hintikka formulas of pointed models over Φ.

- $k+1$. By the induction hypothesis there are only finitely many non-equivalent k-th Hintikka formulas of pointed models over Φ, say n-many.

  Since any $\alpha, \beta \in \mathbf{ML}$ are equivalent iff $\Diamond\alpha$ and $\Diamond\beta$ are equivalent, there are also n-many non-equivalent formulas of the form $\Diamond\chi^k_{M,v}$, where M is a model over Φ. If $\phi, \psi \in \mathbf{ML}$ are equivalent, then their conjunction is equivalent to both of them; therefore there are at most 2^n-many non-equivalent formulas of the form $\bigwedge_{(M,v)\in\mathcal{P}} \Diamond\chi^k_{M,v}$, where $\mathcal{P}$ is a set of pointed models over Φ (for each of the n equivalence types, the conjunction either includes formula(s) of that type or it does not).

  Similarly there are at most 2^n-many non-equivalent formulas of the form $\Box \bigvee_{(M,v)\in\mathcal{P}} \chi^k_{M,v}$, where $\mathcal{P}$ is a set of pointed models over Φ.

  Clearly, then, there are less than $n \cdot 2^{2n}$ formulas of the form

  $$\chi^k_{M,w} \wedge \bigwedge_{v\in R[w]} \Diamond \chi^k_{M,v} \wedge \Box \bigvee_{v\in R[w]} \chi^k_{M,v}$$

  where M is a pointed model over Φ; and hence only finitely many $k+1$-th Hintikka formulas of pointed models over Φ. □

The key result concerning Hintikka formulas and bisimulation is the following—k-bisimilarity and Hintikka formulas of degree k encapsulate k-equivalence, and bisimilarity encapsulates modal equivalence (securing the entirety of this proposition also requires Φ to be finite):

Proposition 2.4.8. Let $k \in \mathbb{N}$ and let (M, w) and (M, w') be pointed models over Φ. Then:

$$\begin{array}{lllll} M, w \equiv_k M', w' & \iff & M, w \leftrightarroweq_k M', w' & \iff & M', w' \models \chi^k_w \\ M, w \equiv^{\mathbf{ML}} M', w' & \iff & M, w \leftrightarroweq M', w' & \iff & \forall n \in \mathbb{N} : M', w' \models \chi^n_w \end{array}$$

There are certain useful methods of using known models to construct new ones in such a way that the new models have worlds that are bisimilar to worlds in the old ones and hence make true the same formulas. We make use of the following method (see, for instance [4]):

Definition 2.4.9. (Disjoint unions) A collection $\mathcal{M}_I = \{M_i = (W_i, R_i, V_i) \mid i \in I\}$ of models over Φ is *disjoint* if $\bigcap_{i \in I} W_i = \varnothing$.

The *disjoint union* of a disjoint collection $\mathcal{M}_I$ of models over Φ is the model $\biguplus \mathcal{M}_I = (W, R, V)$, where $W = \bigcup_{i \in I} W_i$, $R = \bigcup_{i \in I} R_i$ and $V(p) = \bigcup_{i \in I} V_i(p)$ for each $p \in \Phi$.

The disjoint union of a collection $\mathcal{M}_I$ of models that is not disjoint is formed by indexing the domains of the models in $\mathcal{M}_I$ to make the collection disjoint (and modifying the accessibility relations and valuations accordingly), and then proceeding as above.

Note that for simplicity we refer to the worlds in the domain of a disjoint union $\biguplus \mathcal{M}_I = (W, R, V)$ of a non-disjoint collection $\mathcal{M}_I$ as though they were the original unindexed worlds—that is, we speak as though W were simply $\bigcup_{i \in I} W_i$ (and similarly for R and V).

We then have the following:[16]

Proposition 2.4.10. For any collection $\mathcal{M}_I = \{M_i = (W_i, R_i, V_i) \mid i \in I\}$, for every $i \in I$ and every $w \in W_i$:

- for all $n \in \mathbb{N} : \chi^n_{M_i, w} = \chi^n_{\biguplus \mathcal{M}_I, w}$; and so
- $M_i, w \leftrightarroweq \biguplus \mathcal{M}_I, w$ and
- $M_i, w \equiv^{\mathbf{ML}} \biguplus \mathcal{M}_I, w$.

[16] Note that [4] includes the claims about bisimulation and equivalence. We have added the slightly stronger claim concerning identity of Hintikka formulas; that it holds is easy to see by a simple induction.

Chapter 3

Characterization Theorems

In Section 3.1 we adapt bisimulation to our state-based setting and prove a state bisimulation invariance theorem for all of our logics. The notion of state bisimilarity we examine was introduced in [14] and [24]. (See also [7] for a discussion of notions of state-based bisimulation in the context of inquisitive logic.)

In Section 3.2 we define state-based analogues of Hintikka formulas, and prove that these formulas precisely characterize states up to state k-bisimulation for finite k. We use these formulas to define the normal forms required for the completeness proof, and use the normal forms to prove that each of our logics characterizes the set of state properties closed[17] under state k-bisimulation for some $k \in \mathbb{N}$.

The first two sections closely follow [14], in which it is proved that $\mathbf{ML}^{\otimes}$ with $\mathbb{W}$ characterizes the set of state properties which are downward closed as well as being closed under state k-bisimulation for some $k \in \mathbb{N}$. We also draw on [13] for the structure of the characteristic formulas.

In Section 3.3, we introduce wide-scope Free Choice inferences and epistemic contradictions and explain how Aloni proposes to account for these using **BSML**. Her explanation relies on certain state properties; we make use of the characteristic formulas defined in Section 3.2 to show how these properties can be defined in terms of inference rules in $\mathbf{BSML}^{\mathbb{W}}$. We also point out in this section that our logics are not closed under uniform substitution.

[17]Note that following Hella et al. [14] we will speak in terms of *closure* of properties under bisimulation rather than the invariance of properties, as in the introduction. These are the same notion.

3.1 Bisimulation Invariance

The goal of this section is to prove that if two pointed state models are state bisimilar, then they support the same **L**-formulas—that is, they are **L**-equivalent.

Definition 3.1.1. (Pointed state models and state properties) A *pointed state model (over* Φ*)* is a pair (M, s) where M is a model over Φ and s is a state on M. We write $\mathcal{M}(\Phi)$ (or simply $\mathcal{M}$ when Φ is clear from the context) for the class of all pointed state models over Φ and we call subsets of $\mathcal{M}$ *state properties*.

Definition 3.1.2. (Equivalence of pointed state models) Let $(M, s), (M', s') \in \mathcal{M}$, and let L be some logic.

- Let $k \in \mathbb{N}$. (M, s) and (M', s') are *k-equivalent in L*, written $M, s \equiv^L_k M', s'$, if for all $\phi \in L$ with $md(\phi) \leq k$: $M, s \vDash \phi \iff M', s' \vDash \phi$.
- (M, s) and (M', s') are *L-equivalent*, written $M, s \equiv^L M', s'$, if $M, s \equiv^L_k M', s'$ for all $k \in \mathbb{N}$.

We write $M, s \not\equiv^L_k M', s'$ if $M, s \equiv^L_k M', s'$ is not the case, and $M, s \not\equiv^L M', s'$ if $M, s \equiv^L M', s'$ is not the case.

We omit the L from the notation when it is clear from the context. Note that $M, s \equiv^L M', s'$ iff for all $\phi \in L : M, s \vDash \phi \iff M', s' \vDash \phi$.

We will treat the entirety of **L** as a single logic in this section: our bisimulation invariance theorem will pertain to k-equivalence in **L**, which clearly entails k-equivalence in each of our logics. Using the results in the next section we will be able to show that two pointed state models are k-equivalent in one of our logics if and only if they are k-equivalent in all (Corollary 3.2.10).

State bisimilarity is a natural generalization of world-based bisimilarity:

Definition 3.1.3. (State bisimilarity) Let $k \in \mathbb{N}$, and $(M, s), (M', s') \in \mathcal{M}$. (M, s) and (M', s') are *state k-bisimilar*, written $M, s \leftrightarroweq_k M', s'$, if

- for each $w \in s$ there is some $w' \in s'$ such that $M, w \leftrightarroweq_k M', w'$ and
- for each $w' \in s'$ there is some $w \in s$ such that $M, w \leftrightarroweq_k M', w'$.

(M, s) and (M', s') are *state bisimilar*, written $M, s \leftrightarroweq M', s'$, if $M, s \leftrightarroweq_k M', s'$ for all $k \in \mathbb{N}$.

We write $M, s \not\leftrightarroweq_k M', s'$ if $M, s \leftrightarroweq_k M', s'$ is not the case, and $M, s \not\leftrightarroweq M', s'$ if $M, s \leftrightarroweq M', s'$ is not the case.

When (M, s) and (M', s') are state bisimilar, we also say simply that they are bisimilar. If the models are clear from the context, we also say that s and s' are bisimilar.

We now list some properties of state bisimulation we will require. The following is a straightforward consequence of the definition:

Fact 3.1.4.

- For any $k \in \mathbb{N}$, $\leftrightarroweq_k$ is an equivalence relation on $\mathcal{M}$.
- $\leftrightarroweq$ is an equivalence relation on $\mathcal{M}$.

The following two results are from [14]. Proposition 3.1.5 follows from the fact that for the classical k-bisimilarity relation, $M, w \leftrightarroweq_k M', w'$ implies $M, w \leftrightarroweq_n M', w'$ for all $n < k$. For the proof of Proposition 3.1.6, see [14].

Proposition 3.1.5. Let $k \in \mathbb{N}$, and $(M, s), (M', s') \in \mathcal{M}$. If $M, s \leftrightarroweq_k M', s'$, then $M, s \leftrightarroweq_n M', s'$ for all $n < k$.

Proposition 3.1.6. Let $k \in \mathbb{N}$, and let $(M, s), (M', s') \in \mathcal{M}$ be such that $M, s \leftrightarroweq_{k+1} M', s'$. Then:

(i) for each t such that sRt there is a t' such that $s'R't'$ and $M, t \leftrightarroweq_k M', t'$;

(ii) for each t' such that $s'Rt'$, there is a t such that sRt and $M, t \leftrightarroweq_k M', t'$;

(iii) $M, R[s] \leftrightarroweq_k M', R'[s']$;

(iv) for all $s_1, s_2 \subseteq s$ such that $s = s_1 \cup s_2$ there are $s'_1, s'_2 \subseteq s'$ such that $s' = s'_1 \cup s'_2$; $M, s_1 \leftrightarroweq_{k+1} M', s'_1$; and $M, s_2 \leftrightarroweq_{k+1} M', s'_2$.

We also have:

Proposition 3.1.7. Let $k \in \mathbb{N}$, and $(M, s), (M', s') \in \mathcal{M}$. Then $M, s \leftrightarroweq_k M', s'$ iff

(i) for each $t \subseteq s$, there is a $t' \subseteq s'$ such that $M, t \leftrightarroweq_k M', t'$ and

(ii) for each $t' \subseteq s'$, there is a $t \subseteq s$ such that $M, t \leftrightarroweq_k M', t'$.

Proof. $\Rightarrow$: Assume $M, s \leftrightarroweq_k M', s'$ and let $t \subseteq s$. Let

$$t' = \{w' \in s' \mid \exists w \in s : M, w \leftrightarroweq_k M, w'\}$$

We show that $M, t \leftrightarroweq_k M', t'$. Clearly for every $w' \in t'$ there is some $w \in t$ such that $M, w \leftrightarroweq_k M', w'$. Conversely, let $w \in t \subseteq s$. Since $M, s \leftrightarroweq_k M', s'$, by the definition of k-bisimulation there is, for every $w \in s$, a $w' \in s'$ such that $M, w \leftrightarroweq_k M', w'$; therefore, fix a $w' \in s'$ such that $M, w \leftrightarroweq_k M', w'$. Then by the definition of t', we have $w' \in t'$. Since w was arbitrary, we have it that for every $w \in t$, there is a $w' \in t'$ such that $M, w \leftrightarroweq_k M', w'$. Therefore $M, t \leftrightarroweq_k M', t'$. Since t was arbitrary, there is for each $t \subseteq s$ some $t' \subseteq s'$ such that $M, t \leftrightarroweq_k M', t'$. The proof of (ii) is similar.

$\Leftarrow$: Assume that (i) and (ii) hold. Then in particular, there is a $t' \subseteq s'$ such that $M, s \leftrightarroweq_k M', t'$, and a $t \subseteq s$ such that $M, t \leftrightarroweq_k M', s'$; fix such a t' and t. Then for every $w \in s$, there is a $w' \in t' \subseteq s$ such that $M, w \leftrightarroweq_k M', w'$, and for every $w' \in s'$, there is a $w \in t \subseteq s$ such that $M, w \leftrightarroweq_k M', w'$; therefore $M, s \leftrightarroweq_k M', s'$. □

We can now prove our theorem—support for formulas in **L** is invariant under state bisimulation:

Theorem 3.1.8. (Bisimulation invariance) For any $(M, s), (M', s') \in \mathcal{M}$ and any $k \in \mathbb{N}$, if $M, s \leftrightarroweq_k M', s'$, then $M, s \equiv_k^{\mathbf{L}} M', s'$, and therefore $M, s \equiv_k^{L} M', s'$ for $L \in \{\mathbf{SML}^{\mathbb{W}}, \mathbf{SGML}^{\mathbb{W}}, \mathbf{BSML}^{\mathbb{W}}\}$.

Proof. Fix $(M, s), (M', s') \in \mathcal{M}$. We show that for any $\phi \in \mathbf{L}$, if $M, s \leftrightarroweq_k M', s'$ for $k = md(\phi)$, then $M, s \vDash \phi$ if and only if $M', s' \vDash \phi$; the conclusion then clearly follows. We show this by induction on the complexity of ϕ, noting that we may assume that ϕ is in the negation normal form for whichever logic it belongs to.

- $\phi = p$. We have it that $md(\phi) = 0$ so assume that $M, s \leftrightarroweq_0 M', s'$. Assume that $M, s \vDash p$. Then for all $w \in s$ we have $M, w \vDash p$. Since $M, s \leftrightarroweq_0 M', s'$, we have it that for each $w' \in s'$ there is a $w \in s$ such that $M, w \leftrightarroweq_0 M', w'$. Therefore, for all $w' \in s'$ we have $M', w' \vDash p$, and therefore $M', s' \vDash p$. The other direction is similar, so $M, s \vDash p$ if and only if $M', s' \vDash p$.

- $\phi = \neg p$ or $\phi = {\sim} p$. This case is analogous to that for $\phi = p$.

- $\phi = \textsc{ne}$. $md(\phi) = 0$ so assume $M, s \leftrightarroweq_0 M', s'$. Assume $M, s \vDash \textsc{ne}$. Then $s \neq \varnothing$; fix some $w \in s$. By $M, s \leftrightarroweq_0 M', s'$, there is some $w' \in s'$ such that $M, s \leftrightarroweq_0 M', s'$. Therefore $s' \neq \varnothing$ so that $M', s' \vDash \textsc{ne}$. The other direction is similar, so $M, s \vDash \textsc{ne}$ if and only if $M', s' \vDash \textsc{ne}$.

- $\phi = {\sim}\textsc{ne}$. $md(\phi) = 0$ so assume $M, s \leftrightarroweq_0 M', s'$. Assume $M, s \vDash {\sim}\textsc{ne}$. Then $s = \varnothing$. By $M, s \leftrightarroweq_0 M', s'$ we have $s' = \varnothing$ so that $M', s' \vDash {\sim}\textsc{ne}$. The other direction is similar, so $M, s \vDash {\sim}\textsc{ne}$ if and only if $M', s' \vDash {\sim}\textsc{ne}$.

- $\phi = \psi \wedge \chi$. Assume that $M, s \leftrightarroweq_k M', s'$ for $k = md(\psi \wedge \chi) = max\{md(\psi), md(\chi)\}$. Assume $M, s \vDash \psi \wedge \chi$. Then $M, s \vDash \psi$ and $M, s \vDash \chi$. By Proposition 3.1.5, $M, s \leftrightarroweq_m M', s'$ for $m = md(\psi)$ and $M, s \leftrightarroweq_n M', s'$ for $n = md(\chi)$. Therefore by the induction hypothesis, $M', s' \vDash \psi$ and $M', s' \vDash \chi$, so that $M', s' \vDash \psi \wedge \chi$. The other direction is similar, so $M, s \vDash \psi \wedge \chi$ if and only if $M', s' \vDash \psi \wedge \chi$.

- $\phi = \psi \vee \chi$. Assume that $M, s \leftrightarroweq_k M', s'$ for $k = md(\psi \vee \chi) = max\{md(\psi), md(\chi)\}$. Assume $M, s \vDash \psi \vee \chi$. Then for some $s_1, s_2 \subseteq s$ we have $M, s_1 \vDash \psi$, $M, s_2 \vDash \chi$ and $s = s_1 \cup s_2$.

  By Proposition 3.1.6 (iv), there are $s'_1, s'_2 \subseteq s'$ such that $s' = s'_1 \cup s'_2$, $M, s_1 \leftrightarroweq_k M', s'_1$, and $M, s_2 \leftrightarroweq_k M', s'_2$. Therefore by Proposition 3.1.5 we have $M, s_1 \leftrightarroweq_m M', s'_1$ for $m = md(\psi)$ and $M, s_2 \leftrightarroweq_n M', s'_2$ for $n = md(\chi)$. By the induction hypothesis, $M', s'_1 \vDash \psi$ and $M', s'_2 \vDash \chi$, so that $M', s' \vDash \psi \vee \chi$. The other direction is similar, so $M, s \vDash \psi \vee \chi$ if and only if $M', s' \vDash \psi \vee \chi$.

- $\phi = \psi \mathbin{w} \chi$. Analogous to the case for $\phi = \psi \wedge \chi$.

- $\phi = \Diamond\psi$. Assume that $M, s \leftrightarroweq_k M', s'$ for $k = md(\Diamond\psi) = md(\psi) + 1$. Assume $M, s \vDash \Diamond\psi$. Fix some $w' \in s'$. By $M, s \leftrightarroweq_k M', s'$, there is some $w \in s$ such that $M, w \leftrightarroweq_k M', w'$; fix such a w. Since $M, s \vDash \Diamond\psi$, there is a non-empty $t \subseteq R[w]$ such that $M, t \vDash \psi$; fix such a t. By $M, w \leftrightarroweq_k M', w'$ we have $M, \{w\} \leftrightarroweq_k M', \{w'\}$ so that by Proposition 3.1.6 (iii) applied to $M, \{w\}$ and $M', \{w'\}$, we get $M, R[w] \leftrightarroweq_{k-1} M', R'[w']$. Then by Proposition 3.1.7 there is some $t' \subseteq R'[w']$ such that $M, t \leftrightarroweq_{k-1} M', t'$. By the induction hypothesis, $M', t' \vDash \psi$. Since $t \neq \varnothing$ and $M, t \leftrightarroweq_{k-1} M', t'$, we have $t' \neq \varnothing$. Since w' was arbitrary we therefore have that $M', s' \vDash \Diamond\psi$. The other direction is similar, so $M, s \vDash \Diamond\psi$ if and only if $M', s' \vDash \Diamond\psi$.

- $\phi = \Box\psi$. Assume that $M, s \leftrightarroweq_k M', s'$ for $k = md(\Box\psi) = md(\psi) + 1$. Assume $M, s \vDash \Box\psi$. Fix some $w' \in s'$. Since $M, s \leftrightarroweq_k M', s'$, there is a $w \in s$ such that $M, w \leftrightarroweq_k M', w'$; fix such a w. Since $M, s \vDash \Box\psi$, we have $M, R[w] \vDash \psi$. By $M, w \leftrightarroweq_k M', w'$ we have $M, \{w\} \leftrightarroweq_k M', \{w'\}$ so that by Proposition 3.1.6 (iii) applied to $M, \{w\}$ and $M', \{w'\}$, we get $M, R[w] \leftrightarroweq_{k-1} M', R'[w']$. Then by the induction hypothesis, $M', R'[w'] \vDash \psi$. Since w' was arbitrary, $M', s' \vDash \Box\psi$. The other direction is similar, so $M, s \vDash \Box\psi$ if and only if $M', s' \vDash \Box\psi$.

- $\phi = \Diamond\!\!\!\Diamond\psi$. Assume that $M, s \leftrightarroweq_k M', s'$ for $k = md(\Diamond\!\!\!\Diamond\psi) = md(\psi) + 1$. Assume $M, s \vDash \Diamond\!\!\!\Diamond\psi$. Then there is some t such that sRt and $M, t \vDash \psi$;

fix such a t. Since $M, s \leftrightarrows_k M', s'$, by Proposition 3.1.6 (i), there is a t' such that $s'Rt'$ and $M, t \leftrightarrows_{k-1} M', t'$. By the induction hypothesis, $M', t' \models \psi$, so that $M', s' \models \Diamond\psi$. The other direction is similar, so $M, s \models \Diamond\psi$ if and only if $M', s' \models \Diamond\psi$.

- $\phi = \boxdot\psi$. Assume that $M, s \leftrightarrows_k M', s'$ for $k = md(\boxdot\psi) = md(\psi) + 1$. Assume $M, s \models \boxdot\psi$. Then $M, R[s] \models \psi$. Since $M, s \leftrightarrows_k M', s'$, by Proposition 3.1.6 (iii), $M, R[s] \leftrightarrows_{k-1} M', R'[s']$. By the induction hypothesis, $M', R'[s'] \models \psi$, so that $M', s' \models \boxdot\psi$. The other direction is similar, so $M, s \models \boxdot\psi$ if and only if $M', s' \models \boxdot\psi$. □

Corollary 3.1.9. For any $(M, s), (M', s') \in \mathcal{M}$, if $M, s \leftrightarrows M', s'$, then $M, s \equiv^{\mathbf{L}} M', s'$, and therefore $M, s \equiv^{L} M', s'$, where L is $\mathbf{SML}^{\mathbb{W}}$, $\mathbf{SGML}^{\mathbb{W}}$ or $\mathbf{BSML}^{\mathbb{W}}$.

We will prove the converse of Theorem 3.1.8 and that of Corollary 3.1.9 in Corollary 3.2.9.

3.2 Characterization Theorems

Let us first make precise the notion of characterization of sets of properties for our setting.

Definition 3.2.1. A formula $\phi \in \mathbf{L}$ *defines* a property $\mathcal{P} \subseteq \mathcal{M}$ if

$$\mathcal{P} = ||\phi|| := \{(M, s) \in \mathcal{M} \mid M, s \models \phi\}$$

A property $\mathcal{P} \subseteq \mathcal{M}$ is *definable* (by a formula) in a logic L if there is a formula ϕ in the language of L such that $\mathcal{P} = ||\phi||$.

Definition 3.2.2. (Characterization) Let $\mathbb{P}$ be a set of state properties. A logic L *characterizes* $\mathbb{P}$ if $\mathbb{P}$ is the set of state properties definable in L, i.e. if

$$\mathbb{P} = \{||\phi|| \subseteq \mathcal{M} \mid \phi \text{ is a formula in the language of } L\}.$$

If L characterizes a set $\mathbb{P}$ of state properties, we also say that L is *expressively complete* for $\mathbb{P}$.

Our goal is to show that each of our logics characterizes the set of all state properties that are closed under k-bisimulation for some $k \in \mathbb{N}$.

Definition 3.2.3. (Closure under k-bisimulation) Let $k \in \mathbb{N}$. A state property $\mathcal{P} \subseteq \mathcal{M}$ is *closed under k-bisimulation* if $(M, s) \in \mathcal{P}$ and $M, s \leftrightarrows_k M', s'$ together imply $(M', s') \in \mathcal{P}$.

We also define closure under full bisimulation; we make use of this in Section 3.3.

Definition 3.2.4. (Closure under bisimulation) A state property $\mathcal{P} \subseteq \mathcal{M}$ is *closed under bisimulation* if $(M, s) \in \mathcal{P}$ and $M, s \leftrightarroweq M', s'$ together imply $(M', s') \in \mathcal{P}$.

In order to show the characterization results, we first need to define characteristic formulas for states. These are the state-based counterparts to the standard Hintikka formulas.

Definition 3.2.5. (Characteristic formulas) For any $(M, s) \in \mathcal{M}(\Phi)$ and any $k \in \mathbb{N}$, define the *k-th characteristic formulas* $\Theta^k_{M,s} \in \mathbf{SML}^{\mathbb{W}}$, $(\Theta^k_{M,s})^* \in \mathbf{SGML}^{\mathbb{W}}$ and $(\Theta^k_{M,s})^{**} \in \mathbf{BSML}^{\mathbb{W}}$ of (M, s) as follows:

- If $s \neq \varnothing$, let

$$\begin{aligned} \Theta^k_{M,s} &:= \bigvee_{w \in s} (\chi^k_w \wedge \text{NE}) \\ (\Theta^k_{M,s})^* &:= \bigvee_{w \in s} ((\chi^k_w)^* \wedge \text{NE}) \\ (\Theta^k_{M,s})^{**} &:= \bigvee_{w \in s} ((\chi^k_w)^{**} \wedge \text{NE}) \end{aligned}$$

- If $s = \varnothing$, let $\Theta^k_{M,s} = (\Theta^k_{M,s})^* = (\Theta^k_{M,s})^{**} := \bot$ (i.e. $\Theta^k_{M,s} = (\Theta^k_{M,s})^* := p \wedge \neg p$ and $(\Theta^k_{M,s})^{**} := p \wedge {-}p$).

We also call $\Theta^k_{M,s}$, $(\Theta^k_{M,s})^*$ and $(\Theta^k_{M,s})^{**}$ *characteristic formulas of degree* k.

Recall that $*$ and $**$ are the maps we used to show that $\mathbf{SGML}^{\mathbb{W}}$ and $\mathbf{BSML}^{\mathbb{W}}$ are conservative extensions of **ML**. Clearly it follows from Propositions 2.2.13 and 2.2.15 that $\Theta^k_{M,s} \equiv (\Theta^k_{M,s})^* \equiv (\Theta^k_{M,s})^{**}$. We will sometimes use $\Theta^k_{M,s}$ to refer to any one of these three formulas. When results are framed this way, these results are such that they then follow for all three formulas due to the equivalence described above. When the precise syntax is important, we disambiguate. We also sometimes abuse the notation for Hintikka formulas in this fashion: χ^k_w may also refer to $(\chi^k_w)^*$ and $(\chi^k_w)^{**}$. We will write Θ^k_s when M is clear from the context. Note that this definition relies on the fact that Φ is finite (since Φ is finite, there are only finitely many non-equivalent Hintikka formulas of pointed models over Φ (Fact 2.4.7), and hence we may take the disjunction in Θ^k_s to be finite even when s is infinite).

It is easy to see from Facts 2.4.6 and 2.4.7 that we get state-based analogues:

Fact 3.2.6. Let $k \in \mathbb{N}$ and $(M, s) \in \mathcal{M}$. Then $md(\Theta^k_s) = md((\Theta^k_s)^*) = md((\Theta^k_s)^{**}) \leq k$.

Fact 3.2.7. For any finite set of proposition symbols Φ and any $k \in \mathbb{N}$, there are only finitely many non-equivalent k-th characteristic formulas of pointed state models over Φ.

Proof. By Fact 2.4.7, there are only finitely many non-equivalent k-th Hintikka formulas of pointed models over Φ, say n-many. If $\chi^k_w \equiv \chi^k_{w'}$, then $(\chi^k_w \wedge \text{NE}) \vee (\chi^k_{w'} \wedge \text{NE}) \equiv \chi^k_w \wedge \text{NE} \equiv \chi^k_{w'} \wedge \text{NE}$. Therefore there are at most 2^n-many non-equivalent formulas of the form $\bigvee_{(M,w)\in\mathcal{P}}(\chi^k_w \wedge \text{NE})$, where $\mathcal{P}$ is a set of pointed models over Φ. Clearly, then, there are only finitely many non-equivalent k-th characteristic formulas of pointed state models over Φ. □

Characteristic formulas of degree k precisely characterize pointed state models up to k-bisimulation:

Proposition 3.2.8. For any $(M, s) \in \mathcal{M}$ and any $k \in \mathbb{N}$, we have it that $M', s' \vDash \Theta^k_s$ iff $M', s' \vDash (\Theta^k_s)^*$ iff $M', s' \vDash (\Theta^k_s)^{**}$ iff $M, s \leftrightarrows_k M', s'$.

Proof. We prove $M', s' \vDash \Theta^k_s$ iff $M, s \leftrightarrows_k M', s'$. The conclusion then follows since $\Theta^k_{M,s} \equiv (\Theta^k_{M,s})^* \equiv (\Theta^k_{M,s})^{**}$.

Case 1 : $s = \varnothing$. Then $M, s \leftrightarrows_k M', s' \iff s' = \varnothing \iff M', s' \vDash \bot$.

Case 2: $s \neq \varnothing$.

- $\Leftarrow$: Fix some M', s' such that $M, s \leftrightarrows_k M', s'$. Let $w \in s$. Fix some $w' \in s'$ such that $M, w \leftrightarrows_k M', w'$. By Proposition 2.4.8, $M', w' \vDash \chi^k_w$. Then by Proposition 2.2.16 we have $M', \{w'\} \vDash \chi^k_w$ and therefore $M', \{w'\} \vDash \chi^k_w \wedge \text{NE}$. Since w was arbitrary, we have it that for each $w \in s$ (and hence for each $\chi^k_w \in \{\chi^k_v \mid v \in s\}$), there is a $\{w'\} \subseteq s'$ such that $M', \{w'\} \vDash \chi^k_w \wedge \text{NE}$; we can similarly show that for each $\{w'\} \subseteq s'$ there is a $w \in s$ (and hence a $\chi^k_w \in \{\chi^k_v \mid v \in s)$ such that $M', \{w'\} \vDash \chi^k_w \wedge \text{NE}$. Together these imply $M', s' \vDash \Theta^k_s$.

- $\Rightarrow$: Fix some M', s' such that $M', s' \vDash \Theta^k_s$. Then there are subsets $s'_w \subseteq s'$ (where $w \in s$) such that $M', s'_w \vDash \chi^k_w \wedge \text{NE}$ and $s' = \bigcup_{w\in s} s'_w$.

  Fix some $v' \in s'$. Then there is some $w \in s$ such that $v' \in s'_w$. Since $M', s'_w \vDash \chi^k_w \wedge \text{NE}$, we have $M', s'_w \vDash \chi^k_w$ so that by Proposition 2.2.16, $M', v' \vDash \chi^k_w$. Then by Proposition 2.4.8, $M, w \leftrightarrows_k M', v'$. Since v' was arbitrary, we have it that for any $v' \in s'$, there is a $w \in s$ such that $M, w \leftrightarrows_k M', v'$.

Let $w \in s$. Then there is some $s'_w \subseteq s'$ such that $M', s'_w \models \chi^k_w \wedge \textsc{ne}$. Then $M', s'_w \models \textsc{ne}$, so there is some $v' \in s'_w$. By Proposition 2.2.16, $M', v' \models \chi^k_w$, so that by Proposition 2.4.8, $M, w \leftrightarrows_k M', v'$. So for any $w \in s$, there is a $v' \in s'$ such that $M, w \leftrightarrows_k M', v'$.

Combining these two points, we get $M, s \leftrightarrows_k M', s'$. □

We now get a full state-based counterpart to Proposition 2.4.8 (this depends on Φ being finite):

Corollary 3.2.9. Let $k \in \mathbb{N}$, let $(M, s), (M', s') \in \mathcal{M}(\Phi)$, and let $L \in \{\mathbf{SML}^{\mathbb{W}}, \mathbf{SGML}^{\mathbb{W}}, \mathbf{BSML}^{\mathbb{W}}\}$. Then:

$$M, s \equiv^L_k M', s' \iff M, s \leftrightarrows_k M', s' \iff M', s' \models \Theta^k_s$$
$$M, s \equiv^L M', s' \iff M, s \leftrightarrows M', s'$$

Proof. By Proposition 3.2.8, $M, s \leftrightarrows_k M', s' \iff M', s' \models \Theta^k_s$. By Theorem 3.1.8, $M, s \equiv^L_k M', s' \implies M, s \leftrightarrows_k M', s'$.

If $M, s \equiv^L_k M', s'$, then since $M, s \models \Theta^k_s$ (by Fact 3.1.4 and Proposition 3.2.8) and since $md(\Theta^k_s) \leq k$ (by Fact 3.2.6), we have $M', s' \models \Theta^k_s$. Then by $M, s \leftrightarrows_k M', s' \iff M', s' \models \Theta^k_s$ we have $M, s \leftrightarrows_k M', s'$. So $M, s \equiv^L_k M', s' \implies M, s \leftrightarrows_k M', s'$.

Then also $M, s \equiv^L M', s' \iff \forall k \in \mathbb{N} : M, s \equiv^L_k M', s' \iff \forall k \in \mathbb{N} : M, s \leftrightarrows_k M', s' \iff M, s \leftrightarrows M', s'$. □

Corollary 3.2.10. Let $k \in \mathbb{N}$ and let $(M, s), (M', s') \in \mathcal{M}(\Phi)$. Then:

$$M, s \equiv^{\mathbf{SML}^{\mathbb{W}}}_k M', s' \Leftrightarrow M, s \equiv^{\mathbf{SGML}^{\mathbb{W}}}_k M', s' \Leftrightarrow M, s \equiv^{\mathbf{BSML}^{\mathbb{W}}}_k M', s'$$
$$M, s \equiv^{\mathbf{SML}^{\mathbb{W}}} M', s' \Leftrightarrow M, s \equiv^{\mathbf{SGML}^{\mathbb{W}}} M', s' \Leftrightarrow M, s \equiv^{\mathbf{BSML}^{\mathbb{W}}} M', s'$$

Given the above, we may now simply write $M, s \equiv_k M', s'$ and $M, s \equiv M', s'$.

We define the normal-form formulas used in the completeness proof and in the characterization theorems using the characteric formulas and the global disjunction. These formulas characterize sets of pointed state models (state properties) in essentially the same way that characteristic formulas characterize single pointed state models.

Definition 3.2.11. (Normal form) We say that a formula in the language of $\mathbf{SML}^{\mathbb{W}}$, of $\mathbf{SGML}^{\mathbb{W}}$, or of $\mathbf{BSML}^{\mathbb{W}}$ is in *normal form* when it is in the form

$$\mathbb{W}_{(M,s) \in \mathcal{F}} \Theta^k_s$$

where $\mathcal{F}$ is a finite subset of $\mathcal{M}$. We call formulas of this form *normal-form formulas of degree k*.

We now use formulas of this type to prove the characterization theorems (these again depend on Φ being finite):

Proposition 3.2.12. For any $\mathcal{P} \subseteq \mathcal{M}(\Phi)$, if $\mathcal{P}$ is closed under k-bisimulation for some $k \in \mathbb{N}$, then $\mathcal{P}$ is definable $\mathbf{SML}^{\mathbb{W}}(\Phi)$, in $\mathbf{SGML}^{\mathbb{W}}(\Phi)$ and in $\mathbf{BSML}^{\mathbb{W}}(\Phi)$.

Proof. Fix $\mathcal{P}$. By Fact 3.2.7, there are only finitely many non-equivalent characteristic formulas of degree k. Therefore we can find a finite subset $\mathcal{F}$ of $\mathcal{P}$ such that for each $(M, s) \in \mathcal{P}$, there is some $(M', s') \in \mathcal{F}$ such that $\Theta^k_{M,s} \equiv \Theta^k_{M',s'}$. Let $\phi := \mathbb{W}_{(M,s)\in\mathcal{F}} \Theta^k_{M,s} \in L$.

- Let $(M, s) \in \mathcal{P}$. Then $M, s \vDash \Theta^k_s$ by Proposition 3.2.8. Fix some $(M', s') \in \mathcal{F}$ such that $\Theta^k_{M,s} \equiv \Theta^k_{M',s'}$. Then $M, s \vDash \Theta^k_{M',s'}$ so that $M, s \vDash \phi$. Therefore $\mathcal{P} \subseteq ||\phi||$.

- Let (M, s) be such that $M, s \vDash \phi$. Then there is some $(M', s') \in \mathcal{F} \subseteq \mathcal{P}$ such that $M, s \vDash \Theta^k_{s'}$. By Proposition 3.2.8, $M, s \leftrightarroweq_k M', s'$, so that $(M, s) \in \mathcal{P}$ by closure under k-bisimulation. Therefore $||\phi|| \subseteq \mathcal{P}$. □

Theorem 3.2.13. (Characterization theorems) $\mathbf{SML}^{\mathbb{W}}(\Phi)$, $\mathbf{SGML}^{\mathbb{W}}(\Phi)$ and $\mathbf{BSML}^{\mathbb{W}}(\Phi)$ each characterizes the set

$$\mathbb{B} := \{\mathcal{P} \subseteq \mathcal{M}(\Phi) \mid \mathcal{P} \text{ is closed under } k\text{-bisimulation for some } k \in \mathbb{N}\}.$$

Proof. We want to prove $\mathbb{B} = \{||\phi|| \mid \phi \text{ is a formula in the language of } L\}$, for each of our logics L. The right-to-left inclusions follow from Theorem 3.1.8. The left-to-right inclusions follow from Proposition 3.2.12. □

We note in passing that as a consequence of the van Benthem characterization theorem (see, for instance, [4]), classical modal logic characterizes the set of pointed model properties which are closed under k-bisimulation for some $k \in \mathbb{N}$ (where characterization, pointed model properties, and closure under k-bisimulation are defined analogously to our state-based notions.) Due to this, we get the following for state-based classical modal logic (essentially this fact is pointed out in, for instance [35]):

Definition 3.2.14. A state property $\mathcal{P} \subseteq \mathcal{M}$ is *flat* if $(M, s) \in \mathcal{P}$ if and only if for all $w \in s : (M, \{w\}) \in \mathcal{P}$.

Proposition 3.2.15. $\mathbf{ML}(\Phi)$, $\mathbf{ML}^{\oslash}(\Phi)$ and $\mathbf{ML}(\Phi)^{**}$ each characterizes the set

$$\mathbb{F} := \{\mathcal{P} \subseteq \mathcal{M}(\Phi) \mid \mathcal{P} \text{ is flat \& closed under } k\text{-bisimulation for some } k \in \mathbb{N}\}.$$

Proof. We show the result for **ML**; the other results then follow by Propositions 2.2.13 and 2.2.15. We want to prove $\mathbb{F} = \{||\alpha|| \mid \alpha \in \mathbf{ML}\}$. The right-to-left inclusion follows from Theorem 3.1.8 and Corollary 2.2.9.

Let $\mathcal{P} \in \mathbb{F}$, and fix $k \in \mathbb{N}$ such that $\mathcal{P}$ is closed under k-bisimulation. Let $\mathcal{P}' := \{(M, w) \mid \exists(M, s) \in \mathcal{P} : w \in s\}$. Let (M', w') be such that $M, w \leftrightarroweq_k M', w'$. By flatness we have $(M, \{w\}) \in \mathcal{P}$, and by $M, w \leftrightarroweq_k M', w'$ we have $M, \{w\} \leftrightarroweq_k M', \{w'\}$; therefore $(M', \{w'\}) \in \mathcal{P}$. But then $(M', w') \in \mathcal{P}'$. So $\mathcal{P}'$ is closed under k-bisimulation. Then by the consequence of the van Benthem characterization theorem noted above, there is a formula $\phi \in \mathbf{ML}$ such that $(M, w) \in \mathcal{P}'$ if and only if $M, w \vDash \alpha$.

Then $(M, s) \in \mathcal{P}$ iff $\forall w \in s : (M, w) \in \mathcal{P}'$ iff $\forall w \in s : M, w \vDash \alpha$ iff (by Proposition 2.2.16) $M, s \vDash \alpha$. So $\mathcal{P} = ||\alpha||$. Therefore $\mathbb{F} \subseteq \{||\alpha|| \mid \alpha \in \mathbf{ML}\}$. □

3.3 Wide-scope FC and Epistemic Contradictions

We are now in a position to discuss two further linguistic phenomena which Aloni [1] proposes to account for using **BSML**.[18] The first is *wide-scope* Free Choice (first conceptualized as a variant of FC by Zimmermann in [39])[19]:

> You may go to the beach or you may go to the cinema.
>
> ⇝ You may go the beach and you may go to the cinema.
>
> $(\Diamond b \vee \Diamond c) \to (\Diamond b \wedge \Diamond c)$

As with our original example (an instance of *narrow-scope* FC), the inference above appears to be licensed in at least some situations, and the corresponding formalization does not follow from classical deontic logic.

The second issue is that of *epistemic contradictions.* The term is Yalcin's [33]; the issue was first discussed by Wittgenstein [31].

#It is not raining but it might be raining.

The above is clearly infelicitous, and appears to be a contradiction in some sense, so we may want something like the following to be in force (here $\Diamond$

[18] Many of the results in this section apply to all of our logics, but since our present purpose is to examine linguistic applications which are most relevant for $\mathbf{BSML}^{\mathbb{W}}$, we will frame most results in terms of $\mathbf{BSML}^{\mathbb{W}}$ only.

[19] See also [12] for an account of wide-scope FC using $\vee$ and a modality distinct from ours.

is the epistemic "might"-modality):

$$(\neg r \wedge \Diamond r) \to \bot$$

The challenge in accounting for epistemic contradictions is to obtain something like the above without endorsing the following as a validity:

$$\Diamond r \to r$$

Aloni defines two state properties:

- R is *indisputable* in (M, s) if and only if for all $w, v \in s : R[w] = R[v]$.
- R is *state-based* in (M, s) if and only if for all $w \in s : R[w] = s$.

If R is indisputable in (M, s), we will also say that (M, s) is indisputable for short, and similarly for state-basedness. Note that state-basedness implies indisputability.

Wide-scope FC can then be accounted for in **BSML** by noting that for all indisputable (M, s), we have it that $M, s \vDash (\Diamond\phi \vee \Diamond\psi)^+$ implies $M, s \vDash \Diamond\phi \wedge \Diamond\psi$—so wide-scope FC inferences are predicted to be licensed in indisputable pointed state models.

If one thinks of the distinguished state s in a pointed state model as representing the information state of the speaker that is being modelled, one may conceive of indisputability as representing this speaker being fully informed about the accessibility relation in the state. For instance, if the relation represents permissibility and obligation as in deontic logic, if what is permissible is the same in every world in the speaker's information state (i.e. if $R[w] = R[v]$ for all $w, v \in s$), then the speaker is fully informed about what is permissible (and what is obligatory). In line with this, Aloni conjectures that for the deontic modality, R is indisputable precisely when the speaker is taken to be knowledgeable concerning what is permissible, and she therefore predicts that wide-scope deontic FC inferences are only drawn when this is the case. This prediction is supported by some preliminary experimental data—see [19].

For an example of speaker ignorance cancelling FC effects, consider the following adaptation of an example by Zimmermann [39]:

> You may go to the beach or go to the cinema, but I forget which.
> $\not\rightsquigarrow$ You may go the beach and you may go to the cinema.

Note that while this may appear to be a case of narrow-choice FC, and hence not cancellable by speaker ignorance as per Aloni's predictions, it has been

been argued [9] that the sluice in the above (the ellipsis effectuated by the "which") is only compatible with an interpretation in which the disjunction has scope over "may". If this is the case, the sluice (presumably "I forget which (you may go to)") forces the wide-scope interpretation "(You may go to the beach) or (you may go the cinema)".

As for epistemic contradictions, we have it that for all state-based (M, s), $M, s \vDash \neg\phi \wedge \Diamond\phi$ implies $M, s \vDash \bot$; and $M, s \vDash \Diamond\phi$ does not imply $M, s \vDash \phi$ even if (M, s) is state-based.

Aloni assumes that for the epistemic "might"-modality, R should be taken to be state-based (and hence indisputable), and she therefore predicts both that wide-scope FC-inferences are always drawn for this modality, and that epistemic contradictions are accounted for as above.

As Aloni points out, a state-based accessibility relation is appropriate for representing epistemic modalities at least in the sense that it leads to the satisfaction of the S5 axioms, along with their traditional epistemic interpretations—if (M, s) is state-based, then (noting that $\Box$ here is the epistemic "must"-modality):

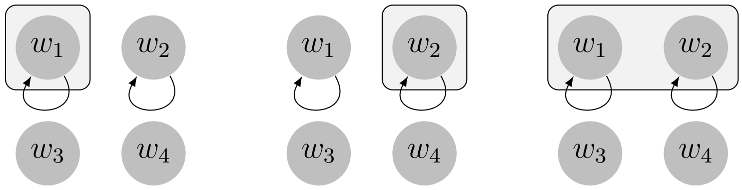
- $M, s \vDash \Box\phi$ implies $M, s \vDash \phi$
- $M, s \vDash \Box\phi$ implies $M, s \vDash \Box\Box\phi$
- $M, s \vDash \neg\Box\phi$ implies $M, s \vDash \Box\neg\Box\phi$

Note that these all hold because they follow from the fact that if (M, s) is state-based, then $M, s \vDash \phi$ if and only if $M, s \vDash \Box\phi$. Recalling that support represents assertability in **BSML**, the equivalence of ϕ and $\Box\phi$ in state-based states might be thought of as representing "'It must be the case that ϕ' is assertible if and only if 'ϕ' is assertible".

In the remainder of this section we will define the classes of indisputable and state-based pointed state models in terms of inference rules. Or, rather, we define classes which serve the same purpose as Aloni intends the indisputable and state-based classes to serve. The issue with the properties as defined above is that they are not closed under bisimulation and hence are not modally definable—consider the example in Figure 3.1 (modified from an example by Del Valle-Inclán (see [3]) to make it suitable for our non-union-closed setting).

In the figure, each of (M, s_1) and (M, s_2) is both state-based and indisputable; (M, s_3) is neither.

If we assume that $M, w_1 \leftrightarrows M, w_2$, then $M, s_1 \leftrightarrows M, s_2$. Then it is easy to see that also $M, s_1 \leftrightarrows M, s_3$ (and $M, s_2 \leftrightarrows M, s_3$) so that $M, s_1 \equiv M, s_3$ (and



(a) (M, s_1) (b) (M, s_2) (c) (M, s_3)

Figure 3.1: Indisputability and state-basedness are not definable

$M, s_2 \equiv M, s_3$). If there were a formula ϕ such that for any $(M, s) \in \mathcal{M}$, $M, s \models \phi$ if and only if (M, s) were indisputable, then we would have $M, s_1 \models \phi$, so that by $M, s_1 \equiv M, s_3$ also $M, s_3 \models \phi$, which would imply that (M, s_3) were indisputable, a contradiction. Similarly for state-basedness, so neither is definable (in any language that is invariant under state bisimulation). Note that this example also shows that the properties are not definable using inference rules (in the sense to be explained below).

In order to correct for this issue we therefore make use of the following variant concepts instead—these are also due to Aloni (personal communication) and will do equally well for her purposes.

Definition 3.3.1. (Indisputability and state-basedness) Let $(M, s) \in \mathcal{M}$.

- R is *indisputable* in (M, s) iff $\forall w, v \in s : M, R[w] \leftrightarrows M, R[v]$.
- R is *state-based* in (M, s) iff $\forall w \in s : M, R[w] \leftrightarrows M, s$.

If R is indisputable / state-based in (M, s), we also say that (M, s) is indisputable / state-based.

Again, state-basedness implies indisputability.

Now note that clearly:

Proposition 3.3.2.

- $\{(M, s) \in \mathcal{M} \mid (M, s)$ is indisputable$\}$ is closed under bisimulation.
- $\{(M, s) \in \mathcal{M} \mid (M, s)$ is state-based$\}$ is closed under bisimulation.

And it is also easy to see that:

Proposition 3.3.3. For any $k \in \mathbb{N}$:

- $\{(M,s) \in \mathcal{M} \mid (M,s)$ is indisputable$\}$ is not closed under k-bisimulation.
- $\{(M,s) \in \mathcal{M} \mid (M,s)$ is state-based$\}$ is not closed under k-bisimulation.

Proof. We prove the results in parallel; assume for contradiction (i) that $\{(M,s) \in \mathcal{M} \mid (M,s)$ is indisputable$\}$ is closed under k-bisimulation for some $k \in \mathbb{N}$ and (ii) that $\{(M,s) \in \mathcal{M} \mid (M,s)$ is state-based$\}$ is also closed under k-bisimulation (we may assume without loss of generality that both are closed for the same k).

Let

$$W := \bigcup_{n \in \mathbb{N}} \{w_n, v_n\}$$
$$R := \{(w_i, w_{i+1}) \mid i \in \mathbb{N}\} \cup \{(v_i, v_{i+1}) \mid i \in \mathbb{N}\}$$

and let V be such that for all $q \in \Phi \backslash \{p\} : V(q) = \varnothing$; and $V(p) = \{v_{k+1}\}$. Let $M := (W, R, V)$.

Then clearly $(M, \{w_0\})$ is state-based and indisputable. It is also easy to see that $M, \{w_0\} \leftrightarroweq_k M, \{w_0, v_0\}$.

By assumption (i), $(M, \{w_0, v_0\})$ is indisputable. Therefore $M, R[w_0] \leftrightarroweq M, R[v_0]$, and so $M, R[w_0] \leftrightarroweq_k M, R[v_0]$. Then by Theorem 3.1.8, $M, R[w_0] \equiv_k M, R[v_0]$. But we also have

$$M, R[v_0] \vDash \overbrace{\Diamond \dots \Diamond}^{k} p \text{ and } M, R[w_0] \nvDash \overbrace{\Diamond \dots \Diamond}^{k} p$$

contradicting $M, R[w_0] \equiv_k M, R[v_0]$.

By assumption (ii), $(M, \{w_0, v_0\})$ is state-based. Therefore $M, R[w_0] \leftrightarroweq M, \{w_0, v_0\}$ and $M, R[v_0] \leftrightarroweq M, \{w_0, v_0\}$ so that by Fact 3.1.4 we have $M, R[w_0] \leftrightarroweq M, R[v_0]$, and then we have a contradiction as above. □

Therefore:

Corollary 3.3.4. There is no formula $\phi \in \mathbf{BSML}^{\mathbb{W}}(\Phi)$ such that

- $||\phi|| = \{(M,s) \in \mathcal{M} \mid (M,s)$ is indisputable$\}$; or such that
- $||\phi|| = \{(M,s) \in \mathcal{M} \mid (M,s)$ is state-based$\}$.

Proof. By Theorem 3.2.13 and Proposition 3.3.3. □

So no specific formula can define the desired properties. However, this does not imply that the properties are undefinable by axioms:

Definition 3.3.5. An axiom $A(p_1,\dots,p_n)$ in the language of a logic L *defines* a property $\mathcal{P} \subseteq \mathcal{M}$ if

$$\mathcal{P} = \{(M,s) \in \mathcal{M} \mid \text{ for all substitution instances } A(\phi_1/p_1,\dots,\phi_n/p_n) \in L \\ (\text{where } \phi_1,\dots,\phi_n \in L) \text{ of } A : M,s \vDash A(\phi_1/p_1,\dots,\phi_n/p_n)\}$$

A property $\mathcal{P} \subseteq \mathcal{M}$ is *definable* (by an axiom) in a logic L if there is an axiom A in the language of L such that A defines $\mathcal{P}$.

As noted above, we will make use of inference rules. Using rules of the form:

$$\frac{A(p_1,\dots,p_n)}{B(p_1\dots,p_n)}$$

allows us, in effect, to also express axioms that utilize the material implication—for any $(M,s) \in \mathcal{M}$:

- the inference rule above is sound in (M,s) for all substitution instances of formulas $\phi_1,\dots,\phi_n \in L$ if and only if
- for all $\phi_1,\dots,\phi_n \in L$: $M,s \vDash A(\phi_1/p_1,\dots,\phi_n/p_n)$ implies $M,s \vDash B(\phi_1/p_1,\dots,\phi_n/p_n)$ if and only if
- for all $\phi_1,\dots,\phi_n \in L$:

  $$M,s \vDash A(\phi_1/p_1,\dots,\phi_n/p_n) \to B(\phi_1/p_1,\dots,\phi_n/p_n)$$

  (where $M,s \vDash \chi \to \nu$ iff $M,s \vDash \chi$ implies $M,s \vDash \nu$).

We may also be able to find axioms in the language of $\mathbf{BSML}^{\mathbb{W}}$ to define the properties. The premises and conclusions used in our inference rules are formulated using the language of $\mathbf{BSML}^{\mathbb{W}}$[20]. Therefore, if the material implication is uniformly definable in $\mathbf{BSML}^{\mathbb{W}}$[21], each of our rules is definable using axioms. Even if this is not the case, particular rules may be definable using axioms.

[20] We do not make use of $\mathbb{W}$, so in fact the rules can also be formulated in **BSML**. Our proofs in this section also do not rely on $\mathbb{W}$ (we require the characteristic formulas but not the normal forms), so the properties are also definable using inference rules formulated in **BSML**.

[21] I.e. if there is some $\phi(p_1,p_2) \in \mathbf{BSML}^{\mathbb{W}}$ such that for all $(M,s) \in \mathcal{M}$ and all $\psi_1,\psi_2 \in \mathbf{BSML}^{\mathbb{W}} : M,s \vDash \psi_1 \to \psi_2$ iff $M,s \vDash \phi(\psi_1/p_1,\psi_2/p_2)$.

Note that by Theorem 3.1.8[22], each substitution instance of an axiom $A(p_1, \ldots, p_n) \to B(p_1, \ldots, p_n)$ (where A and B are in the language of $\mathbf{BSML}^{\mathbb{W}}$) with $\phi_1, \ldots, \phi_n \in \mathbf{BSML}^{\mathbb{W}}$ defines a property

$$\|A(\phi_1/p_1, \ldots, \phi_n/p_n) \to B(\phi_1/p_1, \ldots, \phi_n/p_n)\|$$

which is invariant under k-bisimulation for some $k \in \mathbb{N}$. By Theorem 3.2.13 there is then some formula $\chi \in \mathbf{BSML}^{\mathbb{W}}$ such that

$$\|\chi\| = \|A(\phi_1/p_1, \ldots, \phi_n/p_n) \to B(\phi_1/p_1, \ldots, \phi_n/p_n)\|$$

i.e. this substitution instance property is definable in $\mathbf{BSML}^{\mathbb{W}}$ by a specific formula.

That for a given substitution instance

$$A(\phi_1/p_1, \ldots, \phi_n/p_n) \to B(\phi_1/p_1, \ldots, \phi_n/p_n)$$

we find some $\chi(\phi_1, \ldots, \phi_n) \in \mathbf{BSML}^{\mathbb{W}}$ such that

$$M, s \vDash \chi(\phi_1, \ldots, \phi_n) \text{ iff } M, s \vDash A(\phi_1/p_1, \ldots, \phi_n/p_n) \to B(\phi_1/p_1, \ldots, \phi_n/p_n)$$

does not imply that $\to$ is uniformly definable because $\mathbf{BSML}^{\mathbb{W}}$ is not closed under uniform substitution. We should note here that this holds for all our logics (as well as $\mathbf{PT}^{+}$ [38], for which we also provide a proof system); this means that none of the deduction systems in Chapter 4 will admit a uniform substitution rule.

Fact 3.3.6. For $L \in \{\mathbf{PT}^{+}, \mathbf{SML}^{\mathbb{W}}, \mathbf{SGML}^{\mathbb{W}}, \mathbf{BSML}^{\mathbb{W}}\}$, L is not closed under uniform substitution: there are formulas $\phi(p_1, \ldots, p_n)$, $\psi(p_1, \ldots, p_n)$, $\chi_1, \ldots, \chi_n$ in the language of L such that

$$\phi(p_1, \ldots, p_n) \vDash \psi(p_1, \ldots, p_n) \text{ but } \phi(\chi_1/p_1, \ldots, \chi_n/p_n) \nvDash \psi(\chi_1/p_1, \ldots, \chi_n/p_n).$$

Consider, for instance:

$$p \vee p \vDash p \text{ but } (p \mathbin{\mathbb{W}} \neg p) \vee (p \mathbin{\mathbb{W}} \neg p) \nvDash (p \mathbin{\mathbb{W}} \neg p)$$

This example is from [38] (note that it clearly also holds with $\lnot$ in place of $\neg$); see [38] also for a precise definition of uniform substitution and more discussion on the failure of closure under uniform substitution in team/state-based semantics.

We now define the properties. We will use the following fact, which follows from the properties of characteristic formulas:

[22] As well as the fact that support for $\to$ is invariant under k-bisimulation in the same manner as support for the other connectives: if $M, s \leftrightarrows_k M', s'$ and $M, s \vDash \phi \to \psi$, where $k = md(\phi \to \psi) = max\{md(\phi), md(\psi)\}$, then $M', s' \vDash \phi \to \psi$.

Proposition 3.3.7. For any $(M,s),(M',s') \in \mathcal{M}$, if $M,s \not\rightleftharpoons M',s'$, then there is a formula $\phi \in \mathbf{BSML}^{\mathbb{W}}$ such that $M,s \models \phi$ and $M',s' \not\models \phi$.

Proof. If (M,s) and (M',s') are as specified, then since $M,s \not\rightleftharpoons M',s'$, there is some $k \in \mathbb{N}$ such that $M,s \not\rightleftharpoons_k M',s'$; fix such a k. Then by Proposition 3.2.8, $M,s \models \Theta^k_s$ and $M',s' \not\models \Theta^k_s$, so Θ^k_s is as required. □

We get that the following inference rules correspond to indisputability and state-basedness:

Proposition 3.3.8. For any $(M,s) \in \mathcal{M}(\Phi)$:

(i) R is indisputable in (M,s)

a) iff for all formulas $\phi \in \mathbf{BSML}^{\mathbb{W}}$, the following inference is sound in (M,s):

$$\frac{(\Box\phi \wedge \textsc{ne}) \vee \top\!\!\!\top}{\Box\phi}$$

(i.e. if and only if for all formulas $\phi \in \mathbf{BSML}^{\mathbb{W}}$, $M,s \models (\Box\phi \wedge \textsc{ne}) \vee \top\!\!\!\top$ implies $M,s \models \Box\phi$.)

b) iff for all formulas $\phi \in \mathbf{BSML}^{\mathbb{W}}$, the following inference is sound in (M,s):

$$\frac{(\Diamond\phi \wedge \textsc{ne}) \vee \top\!\!\!\top}{\Diamond\phi}$$

(ii) R is state-based in (M,s)

a) iff for all formulas $\phi \in \mathbf{BSML}^{\mathbb{W}}$, the following inference is sound in (M,s):

$$\frac{\phi}{\Box\phi}$$

b) iff for all formulas $\phi \in \mathbf{BSML}^{\mathbb{W}}$, the following inference is sound in (M,s):

$$\frac{(\Box\phi \wedge \textsc{ne}) \vee \top\!\!\!\top}{\phi}$$

Proof. (i) a) $\Rightarrow$: Assume that R is indisputable in (M,s). Fix $\phi \in \mathbf{BSML}^{\mathbb{W}}$.

Assume $M,s \models (\Box\phi \wedge \textsc{ne}) \vee \top\!\!\!\top$. If $s = \varnothing$, then $M,s \models \Box\phi$. If $s \neq \varnothing$, fix some $w \in s$. Since $M,s \models (\Box\phi \wedge \textsc{ne}) \vee \top\!\!\!\top$, there are t

and t' such that $s = t \cup t'$; $M, t \models \Box\phi \wedge \text{NE}$; and $M, t' \models \text{Ⅱ}$. Then $M, t \models \text{NE}$ so we can fix some $v \in t \subseteq s$. Then since $M, t \models \Box\phi$, we have $M, R[v] \models \phi$. By indisputability, $M, R[w] \leftrightarroweq M, R[v]$, so by Corollary 3.1.9, $M, R[w] \models \phi$. Since w was arbitrary, we have $M, s \models \Box\phi$. So either way we have $M, s \models \Box\phi$.

$\Leftarrow$: Assume that R is not indisputable in (M, s). We show that there is some formula χ such that $M, s \models (\Box\chi \wedge \text{NE}) \vee \text{Ⅱ}$ and $M, s \not\models \Box\chi$; the desired conclusion then follows by contraposition.

By our assumption there are some $w, v \in s$ such that $M, R[w] \not\leftrightarroweq M, R[v]$. By Proposition 3.3.7, there is some $\phi \in \mathbf{BSML}^{\mathbb{W}}$ such that $M, R[w] \models \phi$ and $M, R[v] \not\models \phi$.

We then have it that $M, \{w\} \models \Box\phi \wedge \text{NE}$ and $M, s \setminus \{w\} \models \text{Ⅱ}$, and therefore $M, s \models (\Box\phi \wedge \text{NE}) \vee \text{Ⅱ}$. Since $M, R[v] \not\models \phi$ we have $M, s \not\models \Box\phi$, so ϕ is a formula as desired.

b) $\Rightarrow$: Assume that R is indisputable in (M, s). Fix $\phi \in \mathbf{BSML}^{\mathbb{W}}$.

Assume $M, s \models (\Diamond\phi \wedge \text{NE}) \vee \text{Ⅱ}$. If $s = \varnothing$, then $M, s \models \Diamond\phi$. If $s \neq \varnothing$, fix some $w \in s$. Since $M, s \models (\Diamond\phi \wedge \text{NE}) \vee \text{Ⅱ}$, there are t and t' such that $s = t \cup t'$; $M, t \models \Diamond\phi \wedge \text{NE}$; and $M, t' \models \text{Ⅱ}$. Since $M, t \models \text{NE}$, we can fix some $v \in t \subseteq s$. Then since $M, t \models \Diamond\phi$, there is some non-empty $u \subseteq R[v]$ such that $M, u \models \phi$. By indisputability, $M, R[w] \leftrightarroweq M, R[v]$, and therefore $M, R[w] \leftrightarroweq_k M, R[v]$ for $k = md(\phi)$. Then by Proposition 3.1.7, there is a $u' \subseteq R[w]$ such that $M, u \leftrightarroweq_k M, u'$. Then by Theorem 3.1.8, $M, u' \models \phi$. Since u is non-empty and $M, u \leftrightarroweq_k M, u'$, we have it that u' is non-empty. So since w was arbitrary, we have $M, s \models \Diamond\phi$. So either way $M, s \models \Diamond\phi$.

$\Leftarrow$: Assume that R is not indisputable in (M, s). We show that there is some formula χ such that $M, s \models (\Diamond\chi \wedge \text{NE}) \vee \text{Ⅱ}$; and $M, s \not\models \Diamond\chi$; the result then follows by contraposition.

By our assumption there are some $w, v \in s$ such that $M, R[w] \not\leftrightarroweq M, R[v]$. Then there is some $k \in \mathbb{N}$ such that $M, R[w] \not\leftrightarroweq_k M, R[v]$; fix such a k. By Proposition 3.1.7, there is either some $t \subseteq R[w]$ such that for all $t' \subseteq R[v] : M, t \not\leftrightarroweq_k M, t'$, or some $t' \subseteq R[v]$ such that for all $t \subseteq R[w] : M, t \not\leftrightarroweq_k M, t'$. Assume the former with no loss of generality. Then by Proposition 3.2.8, $M, t \models \Theta^k_t$ and for all $t' \subseteq R[v] : M, t' \not\models \Theta^k_t$.

Note that if $t = \varnothing$, then $M, t \leftrightarroweq_k M, \varnothing$ with $\varnothing \subseteq R[v]$, a contradiction. So $t \neq \varnothing$. Therefore $M, \{w\} \models \Diamond\Theta^k_t \wedge \text{NE}$ so that since $M, s \setminus \{w\} \models \text{Ⅱ}$, we have $M, s \models (\Diamond\Theta^k_t \wedge \text{NE}) \vee \text{Ⅱ}$. Since for all $t' \subseteq R[v] : M, t' \not\models \Theta^k_t$, we also have $M, s \not\models \Diamond\Theta^k_t$, so Θ^k_t is a formula as desired.

(ii) a) $\Rightarrow$: Assume that R is state-based in (M, s). Fix $\phi \in \mathbf{BSML}^{\mathbb{W}}$. Assume $M, s \models \phi$. If $s = \varnothing$, then $M, s \models \Box\phi$. If $s \neq \varnothing$, fix some $w \in s$. By state-basedness, $M, R[w] \Leftrightarrow M, s$, so by Corollary 3.1.9, $M, R[w] \models \phi$. w was arbitrary, so $M, s \models \Box\phi$. So either way we have $M, s \models \Box\phi$.

$\Leftarrow$: Assume that R is not state-based in (M, s). We show that there is some formula χ such that $M, s \models \chi$ and $M, s \not\models \Box\chi$; the result then follows by contraposition.

By our assumption there is some $w \in s$ such that $M, R[w] \not\Leftrightarrow M, s$. By Proposition 3.3.7, there is some $\phi \in \mathbf{BSML}^{\mathbb{W}}$ such that $M, s \models \phi$ and $M, R[w] \not\models \phi$.

Then $M, s \not\models \Box\phi$, and so ϕ is a formula as desired.

b) $\Rightarrow$: Assume that R is state-based in (M, s). Fix $\phi \in \mathbf{BSML}^{\mathbb{W}}$. Assume $M, s \models (\Box\phi \land \text{NE}) \lor \top\!\!\!\top$. Then there are t and t' such that $s = t \cup t'$; $M, t \models \Box\phi \land \text{NE}$; and $M, t' \models \top\!\!\!\top$. Since $M, t \models \text{NE}$, we can fix some $w \in t \subseteq s$. Since $M, t \models \Box\phi$, we have $M, R[w] \models \phi$. By state-basedness, $M, R[w] \Leftrightarrow M, s$, so by Corollary 3.1.9, $M, s \models \phi$.

$\Leftarrow$: Assume that R is not state-based in (M, s). We show that there is some formula χ such that $M, s \models (\Box\chi \land \text{NE}) \lor \top\!\!\!\top$ and $M, s \not\models \chi$; the result then follows by contraposition.

By our assumption there is some $w \in s$ such that $M, R[w] \not\Leftrightarrow M, s$. By Proposition 3.3.7, there is some $\phi \in \mathbf{BSML}^{\mathbb{W}}$ such that $M, R[w] \models \phi$ and $M, s \not\models \phi$.

Then $M, \{w\} \models \Box\phi \land \text{NE}$ and $M, s \setminus \{w\} \models \top\!\!\!\top$, so $M, s \models (\Box\phi \land \text{NE}) \lor \top\!\!\!\top$, and therefore ϕ is as desired. $\square$

The second rule given for indisputability clearly allows us to derive wide-scope FC inferences in the way Aloni intends. The first rule relates to what Aloni calls Zimmermann's problem (it first arose for Zimmermann in [39])—for indisputable (M, s), we have it that $M, s \models \Box(\phi \land \text{NE}) \lor \Box(\psi \land \text{NE})$ implies $M, s \models \Box\phi \land \Box\psi$. Consider:

You must go to the beach or you must go to the cinema.
$\not\rightsquigarrow$ You must go the beach and you must go to the cinema.

This does not appear to be an inference which is licensed. Aloni's system is forced to predict, therefore, that either the first statement in the above has to be interpreted with the modality having scope over the disjunction (as $\Box(\phi \vee \psi)$), or as felicitous only if the speaker is taken not to be knowledgeable about what is permissible (and hence only if the state is not indisputable). By what we showed above, indisputable states are precisely the states in which wide-scope FC is accounted for as Aloni intends (the states in which the second rule for indisputability is sound), and also precisely the states in which Zimmermann's problem arises (the states in which the first rule for indisputability is sound). So if one is to account for wide-scope FC as Aloni does, one cannot, in this setting, avoid Zimmermann's problem by finding another type of state in which the wide-scope FC rule is sound.

We lastly show that epistemic contradictions are modelled in state-based states (given the new definition of state-basedness) in the way Aloni intends. We make use of the following relationship between support and anti-support:

Proposition 3.3.9. For any formula $\phi \in \mathbf{BSML}^{\mathbb{W}}$ and any $(M, s) \in \mathcal{M}(\Phi)$, if $M, s \vDash \phi$, then for any state $t \subseteq W$, if $M, t \dashv \phi$, then $s \cap t = \varnothing$ (and in particular, if $M, s \vDash \phi$ and $M, s \dashv \phi$, then $s = \varnothing$).

Proof. By induction on the complexity of ϕ (noting that we may assume that ϕ is in negation normal form):

- $\phi = p$. If $M, s \vDash p$ and $M, t \dashv p$ for some state t, then for all $w \in s$ we have $w \in V(p)$ and for all and $w \in t$ we have $v \notin V(p)$, so clearly $s \cap t = \varnothing$.

- $\phi = \neg p$. This case is analogous to that for $\phi = p$.

- $\phi = \text{NE}$. If $M, s \vDash \text{NE}$ and $M, t \dashv \text{NE}$ for some state t, then $t = \varnothing$ so $s \cap t = \varnothing$.

- $\phi = \neg\text{NE}$. If $M, s \vDash \neg\text{NE}$ and $M, t \dashv \neg\text{NE}$ for some state t, then $s = \varnothing$ so $s \cap t = \varnothing$.

- $\phi = \psi \wedge \chi$. If $M, s \vDash \psi \wedge \chi$, then $M, s \vDash \psi$ and $M, s \vDash \chi$. If for some state t we have $M, t \dashv \psi \wedge \chi$, then there are $u, u' \subseteq t$ such that $t = u \cup u'$, $M, u \dashv \psi$ and $M, u' \dashv \chi$. By the induction hypothesis, $s \cap u = s \cap u' = \varnothing$, and therefore $s \cap (u \cup u') = s \cap t = \varnothing$.

- $\phi = \psi \vee \chi$. If $M, s \vDash \psi \vee \chi$, then there are $u, u' \subseteq s$ such that $s = u \cup u'$, $M, u \vDash \psi$ and $M, u' \vDash \chi$. If for some state t we have $M, t \dashv \phi \vee \psi$, then $M, t \dashv \psi$ and $M, t \dashv \chi$. Then by the induction hypothesis, $u \cap t = u' \cap t = \varnothing$, so $s \cap t = (u \cup u') \cap t = \varnothing$.

- $\phi = \psi \vee\!\!\!\vee \chi$. If $M, s \models \psi \vee\!\!\!\vee \chi$, then $M, s \models \psi$ or $M, s \models \chi$. If for some state t we have $M, t \dashv \phi \vee\!\!\!\vee \psi$, then $M, t \dashv \psi$ and $M, t \dashv \chi$. Then by the induction hypothesis, $s \cap t = \varnothing$ or $s \cap t = \varnothing$, so either way $s \cap t = \varnothing$.

- $\phi = \Diamond\psi$. If $M, s \models \Diamond\psi$, then for each $w \in s$, there is some non-empty $u \subseteq R[w]$ such that $M, u \models \psi$. If for some state t we have $M, t \dashv \Diamond\psi$, then for each $w \in t$, $M, R[w] \dashv \psi$. Assume for contradiction that $s \cap t \neq \varnothing$ and fix some $w \in s \cap t$. By the above $M, R[w] \dashv \psi$, and we can fix some non-empty $u \subseteq R[w]$ such that $M, u \models \psi$. Then by the induction hypothesis, $u \cap R[w] = u = \varnothing$, a contradiction. So $s \cap t = \varnothing$.

- $\phi = \Box\psi$. If $M, s \models \Box\psi$, then for each $w \in s$ we have $M, R[w] \models \psi$. If for some state t we have $M, t \dashv \Box\psi$, then for each $w \in t$, there is a non-empty $u \subseteq R[w]$ such that $M, u \dashv \psi$. Assume for contradiction that $s \cap t \neq \varnothing$ and fix some $w \in s \cap t$. By the above $M, R[w] \models \psi$, and we can fix some non-empty $u \subseteq R[w]$ such that $M, u \dashv \psi$. Then by the induction hypothesis, $u \cap R[w] = u = \varnothing$, a contradiction. So $s \cap t = \varnothing$. □

Then:

Proposition 3.3.10. For any $(M, s) \in \mathcal{M}$, if R is state-based in (M, s), then for all $\phi \in \mathbf{BSML}^{\vee\!\!\!\vee}$, the following inference is sound in (M, s):

$$\frac{\Diamond\phi \wedge \neg\phi}{\bot}$$

Proof. Assume that R is state-based in (M, s). Fix $\phi \in \mathbf{BSML}^{\vee\!\!\!\vee}$.

Assume $M, s \models \Diamond\phi \wedge \neg\phi$. Assume for contradiction that $s \neq \varnothing$. Fix some $w \in s$. Since $M, s \models \Diamond\phi$, there is some non-empty $t \subseteq R[w]$ such that $M, t \models \phi$. By state-basedness, $M, R[w] \leftrightarrows M, s$, and therefore $M, R[w] \leftrightarrows_k M, s$ for $k = md(\phi)$. By Proposition 3.1.7, there is some $t' \subseteq s$ such that $M, t \leftrightarrows_k M, t'$. By Theorem 3.1.8, $M, t' \models \phi$. Then since $M, s \models \neg\phi$ and $M, t' \dashv \neg\phi$, we have by Proposition 3.3.9 that $s \cap t' = t' = \varnothing$. But since $t \neq \varnothing$ and $M, t \leftrightarrows_k M, t'$ we have $t' \neq \varnothing$, a contradiction. So $s = \varnothing$, and therefore $M, s \models \bot$. □

And clearly $M, s \models \Diamond\phi$ does not imply $M, s \models \phi$ even if (M, s) is state-based.

Chapter 4

Axiomatizations

This chapter presents the natural deduction systems and soundness proofs.

Each of our systems makes use of Yang and Väänänen's axiomatization of $\mathbf{PT}^+$ (recall that this is the non-modal fragment of $\mathbf{SML}^{\mathbb{W}}$ and $\mathbf{SGML}^{\mathbb{W}}$); we first introduce this shared subsystem and use it prove some useful results which will then hold for each of our systems.

Recall Fact 3.3.6: none of the logics considered here is closed under uniform substitution. Therefore, our systems will not admit the uniform substitution rule

$$\frac{\phi(p_1, \ldots, p_n)}{\phi(\psi_1/p_1, \ldots, \psi_n/p_n)}\ \mathsf{Sub}$$

Note in particular that occurrences of α and β in the rules refer, as before, exclusively to arbitrary classical formulas (formulas in $\mathbf{ML}$, $\mathbf{ML}^{\circledcirc}$ or $\mathbf{ML}^{**}$, depending on the system), and so they may not be substituted by non-classical formulas.

4.1 $\mathbf{PT}^+$

The rules used here have been adapted from [34]; for a published version, more details and discussion, see [38].

Definition 4.1.1. (Natural deduction system for $\mathbf{PT}^+$) The following rules comprise a natural deduction system for $\mathbf{PT}^+$. We also call the system $\mathbf{PT}^+$.

Rules for $\neg$:

$\neg$ introduction

$$\begin{array}{c}[\alpha]\\ D^*\\ \dfrac{\bot}{\neg\alpha}\ \neg\mathsf{I}(*)\end{array}$$

Reductio ad absurdum

$$\begin{array}{c}[\neg\alpha]\\ D^*\\ \dfrac{\bot}{\alpha}\ \mathsf{RAA}(*)\end{array}$$

$\neg$ elimination

$$\dfrac{\begin{array}{cc}D_1 & D_2\\ \alpha & \neg\alpha\end{array}}{\beta}\ \neg\mathsf{E}$$

$(*)$ The undischarged assumptions in D^* do not contain NE.

Rules for $\wedge$:

$\wedge$ introduction

$$\dfrac{\begin{array}{cc}D_1 & D_2\\ \phi & \psi\end{array}}{\phi\wedge\psi}\ \wedge\mathsf{I}$$

$\wedge$ elimination

$$\dfrac{\begin{array}{c}D\\ \phi\wedge\psi\end{array}}{\phi}\ \wedge\mathsf{E} \qquad \dfrac{\begin{array}{c}D\\ \phi\wedge\psi\end{array}}{\psi}\ \wedge\mathsf{E}$$

Rules for $\mathbin{\vee\mkern-10mu\vee}$:

$\mathbin{\vee\mkern-10mu\vee}$ introduction

$$\dfrac{\begin{array}{c}D\\ \phi\end{array}}{\phi\mathbin{\vee\mkern-10mu\vee}\psi}\ \mathbin{\vee\mkern-10mu\vee}\mathsf{I} \qquad \dfrac{\begin{array}{c}D\\ \psi\end{array}}{\phi\mathbin{\vee\mkern-10mu\vee}\psi}\ \mathbin{\vee\mkern-10mu\vee}\mathsf{I}$$

$\mathbin{\vee\mkern-10mu\vee}$ elimination

$$\dfrac{\begin{array}{ccc} & [\phi] & [\psi]\\ D & D_1 & D_2\\ \phi\mathbin{\vee\mkern-10mu\vee}\psi & \chi & \chi\end{array}}{\chi}\ \mathbin{\vee\mkern-10mu\vee}\mathsf{E}$$

Rules for $\vee$:

$\vee$ weak introduction

$$\dfrac{\begin{array}{c}D\\ \phi\end{array}}{\phi\vee\psi}\ \vee\mathsf{I}(**)$$

$\vee$ weakening

$$\dfrac{\begin{array}{c}D\\ \phi\end{array}}{\phi\vee\phi}\ \vee\mathsf{W}$$

$\vee$ weak elimination

$$\dfrac{\begin{array}{ccc} & [\phi] & [\psi]\\ D & D_1^* & D_2^*\\ \phi\vee\psi & \alpha & \alpha\end{array}}{\alpha}\ \vee\mathsf{E}(*)$$

$\vee$ weak substitution

$$\dfrac{\begin{array}{cc} & [\psi]\\ D & D_1^*\\ \phi\vee\psi & \chi\end{array}}{\phi\vee\chi}\ \vee\mathsf{Sub}(*)$$

∨ commutativity

$$\frac{\begin{matrix} D \\ \phi \vee \psi \end{matrix}}{\psi \vee \phi}\ \mathsf{Com}\vee$$

∨ associativity

$$\frac{\begin{matrix} D \\ (\phi \vee \psi) \vee \chi \end{matrix}}{\phi \vee (\psi \vee \chi)}\ \mathsf{Ass}\vee$$

(∗) The undischarged assumptions in D_1^*, D_2^* do not contain NE.

(∗∗) Where ψ does not contain NE.

∨⩔ Distributivity:

$$\frac{\begin{matrix} D \\ \phi \vee (\psi \vvee \chi) \end{matrix}}{(\phi \vee \psi) \vvee (\phi \vee \chi)}\ \mathsf{Distr} \vee \vvee$$

⊥ Elimination:

$$\frac{\begin{matrix} D \\ \phi \vee \bot \end{matrix}}{\phi}\ \bot\mathsf{E}$$

Rules for ⫫:

⫫ elimination

$$\frac{\begin{matrix} D \\ \perp\!\!\!\perp \end{matrix}}{\phi}\ \perp\!\!\!\perp \mathsf{E}$$

⫫ contraction

$$\frac{\begin{matrix} D \\ \perp\!\!\!\perp \vee \phi \end{matrix}}{\psi}\ \perp\!\!\!\perp \mathsf{Ctr}$$

Rules for NE:

NE introduction

$$\frac{}{\bot \vvee \textsc{ne}}\ \textsc{ne}\mathsf{I}$$

NE contraction

$$\frac{\begin{matrix} D \\ \textsc{ne} \vee \textsc{ne} \end{matrix}}{\textsc{ne}}\ \textsc{ne}\mathsf{Ctr}$$

∨NE elimination

$$\frac{\begin{matrix} \\ D \\ \phi \vee \psi \end{matrix} \quad \begin{matrix} [\phi] \\ D_1 \\ \chi \end{matrix} \quad \begin{matrix} [\psi] \\ D_2 \\ \chi \end{matrix} \quad \begin{matrix} [(\phi \wedge \textsc{ne}) \vee (\psi \wedge \textsc{ne})] \\ D_3 \\ \chi \end{matrix}}{\chi}\ \vee\textsc{ne}\mathsf{E}$$

We give a general definition of the notion of provability $\vdash_S$ for a proof system S:

Definition 4.1.2. For any set of formulas $\Gamma \cup \{\phi, \phi_1, \ldots, \phi_n, \psi\}$ in the language of a logic L with a proof system S:

- If ψ is derivable from the elements of Γ in S, we write $\Gamma \vdash_S \psi$. If $\{\phi\} \vdash_S \psi$, we also write $\phi \vdash_S \psi$. If $\{\phi_1, \ldots, \phi_n\} \vdash_S \psi$, we also write $\phi_1, \ldots, \phi_n \vdash_S \psi$.
- ϕ and ψ are *provably equivalent (in S)*, written $\phi \dashv\vdash_S \psi$ if $\phi \vdash_S \psi$ and $\psi \vdash_S \phi$.
- We write $\vdash_S \phi$ if ϕ can be derived from the empty set of formulas, i.e. $\varnothing \vdash_S \phi$.

We will drop the subscript S whenever it is clear from context which system is being applied. We also say informally that a derivation of ϕ from Γ is a derivation of $\Gamma \vdash \phi$.

Given the importance of this system for $\mathbf{PT}^+$ for our axiomatizations and the fact that the rules we have given are different from those for the published system, we include a soundness proof here.

Theorem 4.1.3. (Soundness of $\mathbf{PT}^+$ rules) For any $\Gamma \cup \{\phi\} \in \mathbf{L}$, we have $\Gamma \vdash_{\mathbf{PT}^+} \phi \Rightarrow \Gamma \models \phi$.

Proof. By induction on the length of possible derivations $D = (R_1, \ldots, R_k)$ of $\Gamma \vdash \phi$.

- Base case: $k = 1$. This implies $\phi \in \Gamma$, in which case $\Gamma \models \phi$; or $\phi = \bot \mathbin{\text{⩔}} \mathrm{NE}$, in which case $\models \bot \mathbin{\text{⩔}} \mathrm{NE}$ and so again $\Gamma \models \phi$.
- Inductive case. Assume the result holds for all derivations of length $\leq k$. We consider different possibilities of the final rule used in the derivation of $\Gamma \vdash \phi$:
  - $\wedge$I, $\wedge$E, ⩔I, ⩔E, $\vee$W, Com$\vee$, Ass$\vee$, Distr $\vee$ ⩔, $\bot$E, ⫫ E, ⫫ Ctr, NEI, NECtr: For each of these rules, the conclusion follows immediately from the relevant support conditions.
  - $\neg$I: Assume D^* is a derivation of $\Gamma, \alpha \vdash \bot$ of length $\leq k$ such that for all $\gamma \in \Gamma$, γ does not contain NE. By the induction hypothesis, $\Gamma \cup \{\alpha\} \models \bot$. We show $\Gamma \models \neg\alpha$.

    Assume that for all $\gamma \in \Gamma$ we have $M, s \models \gamma$.

    * If $s = \varnothing$, then clearly $M, s \models \neg\alpha$.

  * If $s \neq \varnothing$, then fix some $w \in s$. Since each $\gamma \in \Gamma$ does not contain NE, each such γ is downward closed by Proposition 2.2.8 and therefore we also have $M, \{w\} \vDash \Gamma$. Clearly either $M, \{w\} \vDash \alpha$ or $M, \{w\} \vDash \neg\alpha$. If the former, then $M, \{w\} \vDash \Gamma \cup \{\alpha\}$, and therefore, by hypothesis, $M, \{w\} \vDash \bot$, a contradiction since $\{w\} \neq \varnothing$. So $M, \{w\} \vDash \neg\alpha$. Since w was arbitrary, we have for all $w \in s$, $M, \{w\} \vDash \neg\alpha$. Since α is classical, it is union closed by Corollary 2.2.9 and so $M, s \vDash \neg\alpha$.

  Either way then, $M, s \vDash \neg\alpha$.

- RAA: analogous to the case for $\neg$I.
- $\neg$E: Assume that D_1 and D_2 are derivations of length $\leq k$ of $\Gamma_1 \vdash \alpha$ and $\Gamma_2 \vdash \neg\alpha$, respectively. By the induction hypothesis, $\Gamma_1 \vDash \alpha$ and $\Gamma_2 \vDash \neg\alpha$. We show that $\Gamma_1 \cup \Gamma_2 \vDash \beta$.

  Assume that for all $\gamma \in \Gamma_1 \cup \Gamma_2$, $M, s \vDash \gamma$. Then $M, s \vDash \alpha$ and $M, s \vDash \neg\alpha$. Since α is classical, it is flat by Corollary 2.2.9, and so for all $w \in s : M, \{w\} \vDash \alpha$. By $M, s \vDash \neg\alpha$, we also have that for all $w \in s : M, \{w\} \nvDash \alpha$; therefore $s = \varnothing$. Since β is classical, it has the empty state property by Corollary 2.2.9, so $M, s \vDash \beta$.
- $\vee$I: Assume that D is a derivation of $\Gamma \vdash \phi$. By the induction hypothesis, $\Gamma \vDash \phi$. We show $\Gamma \vDash \phi \vee \psi$, where ψ does not contain NE.

  Assume $M, s \vDash \phi$. Since ψ does not contain NE, it has the empty state property by Proposition 2.2.8, and so we have $M, \varnothing \vDash \psi$. Then $s = s \cup \varnothing$; $M, s \vDash \phi$; and $M, \varnothing \vDash \psi$, so $M, s \vDash \phi \vee \psi$.
- $\vee$E: Assume D, D_1^*, and D_2^* are derivations of length $\leq k$ of $\Gamma \vdash \phi \vee \psi$; $\Gamma_1, \phi \vdash \alpha$; and $\Gamma_2, \psi \vdash \alpha$, respectively, such that for all $\gamma \in \Gamma_1 \cup \Gamma_2$, γ does not contain NE. By the induction hypothesis, $\Gamma \vDash \phi \vee \psi$, $\Gamma_1 \cup \{\phi\} \vDash \alpha$ and $\Gamma_2 \cup \{\psi\} \vDash \alpha$. We show $\Gamma \cup \Gamma_1 \cup \Gamma_2 \vDash \alpha$.

  Assume that for all $\gamma \in \Gamma \cup \Gamma_1 \cup \Gamma_2$ we have $M, s \vDash \gamma$. Then $M, s \vDash \phi \vee \psi$, so there are some t, u such that $s = t \cup u$; $M, t \vDash \phi$; and $M, u \vDash \psi$. Since each $\gamma \in \Gamma_1 \cup \Gamma_2$ does not contain NE, each such γ is downward closed by Proposition 2.2.8, and we therefore have that for all $\gamma_1 \in \Gamma_1$: $M, t \vDash \gamma_1$, and for all $\gamma_2 \in \Gamma_2$: $M, u \vDash \gamma_2$. Therefore $M, t \vDash \alpha$ and $M, u \vDash \alpha$. Since α is classical, it is union closed by Corollary 2.2.9, and so $M, s \vDash \alpha$.
- $\vee$Sub: Assume D and D_1^* are derivations of length $\leq k$ of $\Gamma \vdash \phi \vee \psi$ and $\Gamma_1, \psi \vdash \chi$, respectively, such that for all $\gamma \in \Gamma_1$, γ does

not contain NE. By the induction hypothesis, $\Gamma \vDash \phi \vee \psi$ and $\Gamma_1 \cup \{\psi\} \vDash \chi$. We show $\Gamma \cup \Gamma_1 \vDash \phi \vee \chi$.

Assume that for all $\gamma \in \Gamma \cup \Gamma_1$ we have $M, s \vDash \gamma$. Then $M, s \vDash \phi \vee \psi$, so there are some t, u such that $s = t \cup u$; $M, t \vDash \phi$; and $M, u \vDash \psi$. Since each $\gamma \in \Gamma_1$ does not contain NE, each such γ is downward closed by Proposition 2.2.8, and so $M, u \vDash \gamma$. Therefore $M, u \vDash \chi$. So $s = t \cup u$; $M, s \vDash \phi$; and $M, u \vDash \chi$; therefore $M, s \vDash \phi \vee \chi$.

– $\vee$NEE: Assume D, D_1, D_2 and D_3 are derivations of length $\leq k$ of $\Gamma \vdash \phi \vee \psi$; $\Gamma_1, \phi \vdash \chi$; $\Gamma_2, \psi \vdash \chi$; and $\Gamma_3, (\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE}) \vdash \chi$ respectively. By the induction hypothesis, $\Gamma \vDash \phi \vee \psi$, $\Gamma_1 \cup \{\phi\} \vDash \chi$, $\Gamma_2 \cup \{\psi\} \vDash \chi$ and $\Gamma_3 \cup \{(\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE})\} \vDash \chi$. We show $\Gamma \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \vDash \chi$.

Assume that for all $\gamma \in \Gamma \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ we have $M, s \vDash \gamma$. By $\Gamma \vDash \phi \vee \psi$ we have $M, s \vDash \phi \vee \psi$, so there are some t, u such that $s = t \cup u$; $M, t \vDash \phi$; and $M, u \vDash \psi$.

* If $u = \varnothing$, then $s = t$. Then $M, s \vDash \phi$, and therefore (since $\Gamma_1 \cup \{\phi\} \vDash \chi$) $M, s \vDash \chi$.
* If $t = \varnothing$, then $s = u$. Then $M, s \vDash \psi$, and therefore (since $\Gamma_2 \cup \{\psi\} \vDash \chi$) $M, s \vDash \chi$.
* If $t \neq \varnothing$ and $u \neq \varnothing$, then $M, t \vDash \phi \wedge \text{NE}$ and $M, u \vDash \psi \wedge \text{NE}$. Therefore $M, s \vDash (\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE})$, and so (since $\Gamma_3 \cup \{(\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE})\} \vDash \chi$) $M, s \vDash \chi$.

In any case, then, $M, s \vDash \chi$. □

We now list some useful derivable rules.

The following proposition lists the commutativity, associativity and distributivity laws provable in the system. These are easy to derive; see [38] for some of the derivations. Note that $\vee$ commutativity and associativity are included as basic rules of the system and are not listed below. The $\vdash$ direction of Distr$\vee\mathbin{\mathrm{w}}$ is also a basic rule but it is listed below together with $\dashv$ to prevent confusion.

Proposition 4.1.4. The following are derivable with the rules for PT^{+}:

$\phi \wedge \psi$	$\dashv\vdash$	$\psi \wedge \phi$	(Com$\wedge$)
$\phi \mathbin{\mathbb{W}} \psi$	$\dashv\vdash$	$\psi \mathbin{\mathbb{W}} \phi$	(Com$\mathbb{W}$)
$\phi \wedge (\psi \wedge \chi)$	$\dashv\vdash$	$(\phi \wedge \psi) \wedge \chi$	(Ass$\wedge$)
$\phi \mathbin{\mathbb{W}} (\psi \mathbin{\mathbb{W}} \chi)$	$\dashv\vdash$	$(\phi \mathbin{\mathbb{W}} \psi) \mathbin{\mathbb{W}} \chi$	(Ass$\mathbb{W}$)
$\phi \wedge (\psi \mathbin{\mathbb{W}} \chi)$	$\dashv\vdash$	$(\phi \wedge \psi) \mathbin{\mathbb{W}} (\phi \wedge \chi)$	(Distr $\wedge$ $\mathbb{W}$)
$\phi \mathbin{\mathbb{W}} (\psi \wedge \chi)$	$\dashv\vdash$	$(\phi \mathbin{\mathbb{W}} \psi) \wedge (\phi \mathbin{\mathbb{W}} \chi)$	(Distr $\mathbb{W}$ $\wedge$)
$\alpha \wedge (\psi \vee \chi)$	$\dashv\vdash$	$(\alpha \wedge \psi) \vee (\alpha \wedge \chi)$	(Distr* $\wedge$ $\vee$)
$\phi \vee (\psi \wedge \chi)$	$\vdash$	$(\phi \vee \psi) \wedge (\phi \vee \chi)$	(Distr $\vee$ $\wedge$)
$\phi \mathbin{\mathbb{W}} (\psi \vee \chi)$	$\vdash$	$(\phi \mathbin{\mathbb{W}} \psi) \vee (\phi \mathbin{\mathbb{W}} \chi)$	(Distr $\mathbb{W}$ $\vee$)
$\phi \vee (\psi \mathbin{\mathbb{W}} \chi)$	$\dashv\vdash$	$(\phi \vee \psi) \mathbin{\mathbb{W}} (\phi \vee \chi)$	(Distr $\vee$ $\mathbb{W}$)

Some more useful derivable rules:

Proposition 4.1.5. The following are derivable with the rules for PT^{+}:

(i) $\phi \vee (\psi \wedge \textsc{ne}) \dashv\vdash (\phi \vee (\psi \wedge \textsc{ne})) \wedge \textsc{ne}$

(ii) $\textsc{ne}, \bigvee_{i \in I} \phi_i \vdash \bigWvee_{\varnothing \neq J \subseteq I} \bigvee_{j \in J} (\phi_j \wedge \textsc{ne})$

(iii) $\textsc{ne} \wedge \bigvee_{i \in I} \alpha_i \dashv\vdash \bigWvee_{\varnothing \neq J \subseteq I} \bigvee_{j \in J} (\alpha_j \wedge \textsc{ne})$

(iv) $\bigvee_{i \in I} \phi_i \vdash \bigWvee_{J \subseteq I} \bigvee_{j \in J} (\phi_j \wedge \textsc{ne})$

(v) $\bigvee_{i \in I} \alpha_i \dashv\vdash \bigWvee_{J \subseteq I} \bigvee_{j \in J} (\alpha_j \wedge \textsc{ne})$

(vi) $(\alpha \wedge \textsc{ne}) \vee \phi, \neg\alpha \vdash \bot\!\!\!\bot$

Proof. (i) $\dashv$ follows by $\wedge\mathsf{E}$. For $\vdash$:[23]

$$
\begin{array}{lll}
 & \phi \vee (\psi \wedge \text{NE}) & \\
\vdash & (\phi \vee (\psi \wedge \text{NE})) \wedge (\bot \mathbin{\vee\!\!\vee} \text{NE})) & \text{NE}\mathsf{I} \\
\vdash & ((\phi \vee (\psi \wedge \text{NE})) \wedge \bot) \mathbin{\vee\!\!\vee} ((\phi \vee (\psi \wedge \text{NE})) \wedge \text{NE}) & \mathsf{Distr} \wedge \mathbin{\vee\!\!\vee} \\
\vdash & ((\phi \wedge \bot) \vee ((\psi \wedge \text{NE}) \wedge \bot)) \mathbin{\vee\!\!\vee} ((\phi \vee (\psi \wedge \text{NE})) \wedge \text{NE})) & \mathsf{Distr}^{*} \wedge \vee \\
\vdash & ((\phi \wedge \bot) \vee (\psi \wedge \perp\!\!\!\perp)) \mathbin{\vee\!\!\vee} ((\phi \vee (\psi \wedge \text{NE})) \wedge \text{NE}) & \\
\vdash & ((\phi \vee (\psi \wedge \text{NE})) \wedge \text{NE}) \mathbin{\vee\!\!\vee} ((\phi \vee (\psi \wedge \text{NE})) \wedge \text{NE}) & \perp\!\!\!\perp\ \mathsf{Ctr} \\
\vdash & (\phi \vee (\psi \wedge \text{NE})) \wedge \text{NE} &
\end{array}
$$

(ii) By induction on the size k of I.

- $k = 0$. $\bigvee \varnothing = \bot$, and by $\perp\!\!\!\perp\ \mathsf{E}$ we have $\text{NE} \wedge \bot \vdash \psi$ for any ψ.
- $k = 1$. We have $\bigvee\{\phi\} = \phi$. $\phi, \text{NE} \vdash \phi \wedge \text{NE}$ by $\wedge\mathsf{I}$, and $\mathbb{W}\{\bigvee\{\phi \wedge \text{NE}\}\} = \phi \wedge \text{NE}$.
- $k = 2$. Denote $\chi := (\phi \wedge \text{NE}) \mathbin{\vee\!\!\vee} (\psi \wedge \text{NE}) \mathbin{\vee\!\!\vee} ((\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE}))$.

$$
\dfrac{\phi \vee \psi \qquad \dfrac{\dfrac{[\phi] \quad \text{NE}}{\phi \wedge \text{NE}}\,\wedge\mathsf{I}}{\chi}\,\mathbin{\vee\!\!\vee}\mathsf{I} \qquad \dfrac{\dfrac{[\psi] \quad \text{NE}}{\psi \wedge \text{NE}}\,\wedge\mathsf{I}}{\chi}\,\mathbin{\vee\!\!\vee}\mathsf{I} \qquad \dfrac{[(\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE})]}{\chi}\,\mathbin{\vee\!\!\vee}\mathsf{I}}{\chi}\,\vee\text{NE}\mathsf{E}
$$

- $k + 1$. By the induction hypothesis,

$$
(\dagger) \quad \text{NE}, \bigvee_{i \in (I \setminus \{x\})} \phi_i \vdash \mathop{\mathbb{W}}_{\varnothing \neq J \subseteq (I \setminus \{x\})} \bigvee_{j \in J} (\phi_j \wedge \text{NE})
$$

[23]For readibility we will only note the most notable rules/derivable rules used for each step in the natural deduction proofs; it should be easy to see which other rules are being used.

where $x \in I$. Then:

$$
\begin{array}{lll}
 & \mathrm{NE}, \bigvee_{i \in I} \phi_i & \\
\vdash & \mathrm{NE} \wedge (\bigvee_{i \in (I \setminus \{x\})} (\phi_i) \vee \phi_x) & \\
\vdash & (\bigvee_{i \in (I \setminus \{x\})} (\phi_i) \wedge \mathrm{NE}) \mathbin{\vee\!\!\vee} (\phi_x \wedge \mathrm{NE}) \mathbin{\vee\!\!\vee} & \\
 & ((\bigvee_{i \in (I \setminus \{x\})} (\phi_i) \wedge \mathrm{NE}) \vee (\phi_x \wedge \mathrm{NE})) & \text{Case } k = 2 \\
\vdash & (\bigvee\!\!\!\bigvee_{\varnothing \neq J \subseteq (I \setminus \{x\})} \bigvee_{j \in J} (\phi_j \wedge \mathrm{NE})) \mathbin{\vee\!\!\vee} (\phi_x \wedge \mathrm{NE}) \mathbin{\vee\!\!\vee} & \\
 & ((\bigvee\!\!\!\bigvee_{\varnothing \neq J \subseteq (I \setminus \{x\})} \bigvee_{j \in J} (\phi_j \wedge \mathrm{NE})) \vee (\phi_x \wedge \mathrm{NE})) & \dagger \\
\vdash & (\bigvee\!\!\!\bigvee_{\varnothing \neq J \subseteq (I \setminus \{x\})} \bigvee_{j \in J} (\phi_j \wedge \mathrm{NE})) \mathbin{\vee\!\!\vee} (\phi_x \wedge \mathrm{NE}) \mathbin{\vee\!\!\vee} & \\
 & (\bigvee\!\!\!\bigvee_{\varnothing \neq J \subseteq (I \setminus \{x\})} \bigvee_{j \in J} (\phi_j \wedge \mathrm{NE}) \vee (\phi_x \wedge \mathrm{NE})) & \mathsf{Distr} \vee \mathbin{\vee\!\!\vee} \\
\vdash & \bigvee\!\!\!\bigvee_{\varnothing \neq J \subseteq I} \bigvee_{j \in J} (\phi_j \wedge \mathrm{NE}) &
\end{array}
$$

(iii) $\vdash$ follows from (ii).

For $\dashv$, note that for any non-empty $J \subseteq I$, we have $\bigvee_{j \in J}(\alpha_j \wedge \mathrm{NE}) \vdash \mathrm{NE}$ by (i) and $\wedge\mathsf{E}$ (if $|J| > 1$) or simply by $\wedge\mathsf{E}$ (if $|J| = 1$).

For any such J we also have that for any $j \in J$, $\alpha_j \wedge \mathrm{NE} \vdash \bigvee_{i \in I} \alpha_i$ by $\wedge\mathsf{E}$ and $\vee\mathsf{I}$. Therefore $\bigvee_{j \in J}(\alpha_j \wedge \mathrm{NE}) \vdash \bigvee_{i \in I} \alpha_i$ by $\vee\mathsf{E}$.

Since both of these are the case for any such J, we have $\bigvee\!\!\!\bigvee_{\varnothing \neq J \subseteq I} \bigvee_{j \in J}(\alpha_j \wedge \mathrm{NE}) \vdash \mathrm{NE} \wedge \bigvee_{i \in I} \alpha_i$ by $\mathbin{\vee\!\!\vee}\mathsf{E}$.

(iv)

$$
\begin{array}{lll}
 & \bigvee_{i \in I} \phi_i & \\
\vdash & (\bot \mathbin{\vee\!\!\vee} \mathrm{NE}) \wedge \bigvee_{i \in I} \phi_i & \mathrm{NE}\mathsf{I} \\
\vdash & (\bot \wedge \bigvee_{i \in I} \phi_i) \mathbin{\vee\!\!\vee} (\mathrm{NE} \wedge \bigvee_{i \in I} \phi_i) & \mathsf{Distr} \wedge \mathbin{\vee\!\!\vee} \\
\vdash & \bot \mathbin{\vee\!\!\vee} \bigvee\!\!\!\bigvee_{\varnothing \neq J \subseteq I} \bigvee_{j \in J} (\phi_j \wedge \mathrm{NE}) & \text{(ii)} \\
\vdash & \bigvee\!\!\!\bigvee_{J \subseteq I} \bigvee_{j \in J} (\phi_j \wedge \mathrm{NE}) & \bigvee \varnothing = \bot
\end{array}
$$

(v) $\vdash$ follows from (iv). For $\dashv$:

$$
\begin{array}{lcll}
\bigvee\!\!\!\!\bigvee_{J\subseteq I}\bigvee_{j\in J}(\alpha_j \land \textsc{ne}) & \vdash & \bot \mathbin{\bigvee\!\!\!\!\bigvee} \bigvee\!\!\!\!\bigvee_{\varnothing\neq J\subseteq I}\bigvee_{j\in J}(\alpha_j \land \textsc{ne}) & \\
 & \vdash & \bot \mathbin{\bigvee\!\!\!\!\bigvee} (\textsc{ne} \land \bigvee_{i\in I}\alpha_i) & \text{(iii)} \\
 & \vdash & \bigvee_{i\in I}\alpha_i & \mathbin{\bigvee\!\!\!\!\bigvee}\mathsf{E}, \neg\mathsf{E}
\end{array}
$$

(vi)

$$
\begin{array}{lll}
 & ((\alpha \land \textsc{ne}) \lor \phi) \land \neg\alpha & \\
\vdash & ((\alpha \land \textsc{ne}) \land \neg\alpha) \lor (\phi \land \neg\alpha) & \mathsf{Distr}^* \land \lor \\
\vdash & (\bot \land \textsc{ne}) \lor (\phi \land \neg\alpha) & \neg\mathsf{E} \\
\vdash & \bot\!\!\!\bot & \bot\!\!\!\bot\ \mathsf{Ctr}
\end{array}
$$

□

4.2 SML$^{\mathbb{W}}$ and BSML$^{\mathbb{W}}$

For **SML**$^{\mathbb{W}}$, we extend the system for **PT**$^{+}$ with some of the rules concerning modalities from **MD**$^{\mathbb{W}}$ (with the flat modalities in place of the global modalities used in that system), and new rules for the interaction of the connectives and NE:

Definition 4.2.1. (Natural deduction system for SML$^{\mathbb{W}}$) The following rules comprise a natural deduction system for **SML**$^{\mathbb{W}}$. We also call the system **SML**$^{\mathbb{W}}$.

1. The rules for PT^{+}.

2. The following modality rules:

$\Diamond$ monotonicity

$$
\dfrac{\begin{array}{c}[\phi]\\ D'\\ \psi\end{array} \qquad \begin{array}{c}D\\ \Diamond\phi\end{array}}{\Diamond\psi}\ \Diamond\mathsf{Mon}(*)
$$

$\Box$ monotonicity

$$
\dfrac{\begin{array}{c}[\phi_1]\dots[\phi_n]\\ D'\\ \psi\end{array} \qquad \begin{array}{c}D_1\\ \Box\phi_1\end{array} \quad \dots \quad \begin{array}{c}D_n\\ \Box\phi_n\end{array}}{\Box\psi}\ \Box\mathsf{Mon}(*)
$$

□◇ interaction

$$\dfrac{\dfrac{D}{\neg\Box\alpha}}{\Diamond\neg\alpha}\ \mathsf{Inter}\,\Box\,\Diamond \quad \text{(double line)}$$

(∗) D' does not contain undischarged assumptions.

n may be 0 in $\Box$Mon, in which case it functions as the necessitation rule.

3. The following rules governing the interaction of the connectives and NE**:**

◇ ⩔ ∨ conversion

$$\dfrac{\dfrac{D}{\Diamond(\phi \mathbin{\mathbb{W}} \psi)}}{\Diamond\phi \vee \Diamond\psi}\ \mathsf{Conv}\,\Diamond\,\mathbb{W}\vee \quad \text{(double line)}$$

□ ⩔ ∨ conversion

$$\dfrac{\dfrac{D}{\Box(\phi \mathbin{\mathbb{W}} \psi)}}{\Box\phi \vee \Box\psi}\ \mathsf{Conv}\,\Box\,\mathbb{W}\vee \quad \text{(double line)}$$

◇ separation

$$\dfrac{\dfrac{D}{\Diamond(\phi \vee (\psi \wedge \mathrm{NE}))}}{\Diamond\psi}\ \Diamond\mathsf{Sep}$$

◇ join

$$\dfrac{\dfrac{D_1}{\Diamond\phi} \qquad \dfrac{D_2}{\Diamond\psi}}{\Diamond(\phi \vee \psi)}\ \Diamond\mathsf{Join}$$

◇NE introduction

$$\dfrac{\dfrac{D}{\Diamond\phi}}{\Diamond(\phi \wedge \mathrm{NE})}\ \Diamond\mathrm{NE}\mathsf{I}$$

□ instantiation

$$\dfrac{\dfrac{D}{\Box(\phi \wedge \mathrm{NE})}}{\Diamond\phi}\ \Box\mathsf{Inst}$$

□◇ join

$$\dfrac{\dfrac{D_1}{\Box(\phi \vee \psi)} \qquad \dfrac{D_2}{\Diamond\psi}}{\Box(\phi \vee (\psi \wedge \mathrm{NE}))}\ \Box\,\Diamond\,\mathsf{Join}$$

The disjunction conversion rules $\mathsf{Conv}\Diamond\mathbb{W}\vee$ and $\mathsf{Conv}\Box\mathbb{W}\vee$ are similar to a rule in Ciardelli's axiomatization of **InqBK** [5] whereby $\Box(\phi \mathbin{\mathbb{W}} \psi)$ implies $\Box\phi \vee_C \Box\psi$, where $\vee_C$ is the classical disjunction defined in terms of conjunction and intuitionistic negation: $\phi \vee_C \psi := \neg_\varnothing(\neg_\varnothing\phi \wedge \neg_\varnothing\psi)$ (see Section 2.2 for the definition of $\neg_\varnothing$). $\vee$ and $\vee_C$ are equivalent on classical formulas, and since $\Box\phi$ clearly defines a flat state property for any given ϕ, any instance of the formula is equivalent to a classical formula by Proposition 3.2.15. We therefore have that $\Box\phi \vee \Box\psi \equiv \Box\phi \vee_C \Box\psi$, and so our $\mathsf{Conv}\,\Box$ W$\vee$ and Ciardelli's rule are essentially capturing the same semantic phenomenon.

$\Diamond$Sep clearly reflects the fact that FC inferences may be drawn in the way explained in Section 2.3.

Conceptually, given Aloni's pragmatic enrichment procedure, the necessity instantiation rule $\Box$Inst may be thought of as expressing a version of the Kantian "ought implies can" maxim for pragmatically enriched deontic necessities $(\Box\phi)^+$. Similarly if the modalities are taken to be epistemic: a pragmatically enhanced assertion of "it must be the case that ϕ" will imply "it might be the case that ϕ".

Theorem 4.2.2. (Soundness of $\mathbf{SML}^{\mathsf{W}}$ rules) For any $\Gamma \cup \{\phi\} \in \mathbf{L}$, we have $\Gamma \vdash_{\mathbf{SML}^{\mathsf{W}}} \phi \Rightarrow \Gamma \vDash \phi$.

Proof. By induction on the length of possible derivations $D = (R_1, \ldots, R_k)$ of $\Gamma \vdash \phi$.

- Base case: $k = 1$. As in the proof of Theorem 4.1.3.
- Inductive case. Assume the result holds for all derivations of length $\leq k$. We consider different possibilities of the final rule used in the derivation of $\Gamma \vdash \phi$. Most cases are as in the proof of Theorem 4.1.3; we show the remaining cases:
  - $\Diamond$Mon: Assume D and D' are derivations of length $\leq k$ of $\Gamma \vdash \Diamond\phi$ and $\phi \vdash \psi$, respectively. By the induction hypothesis, $\Gamma \vDash \Diamond\phi$ and $\phi \vDash \psi$. We show $\Gamma \vDash \Diamond\psi$.

    Assume that for all $\gamma \in \Gamma$ we have $M, s \vDash \gamma$. Then $M, s \vDash \Diamond\phi$.
    * If $s = \varnothing$, then clearly $M, s \vDash \Diamond\psi$.
    * If $s \neq \varnothing$, let $w \in s$. Then by $M, s \vDash \Diamond\phi$ there is some $t \subseteq R[s]$ such that $t \neq \varnothing$ and $M, t \vDash \phi$. Therefore $M, t \vDash \psi$, and so, since w was arbitrary, $M, s \vDash \Diamond\psi$.
  - $\Box$Mon: Assume $D_1, \ldots, D_n$ and D' are derivations of length $\leq k$ of $\Gamma_1 \vdash \Box\phi_1, \ldots, \Gamma_n \vdash \Box\phi_n$ and $\phi_1, \ldots, \phi_n \vdash \psi$ respectively. By the induction hypothesis, $\Gamma_1 \vDash \Box\phi_1, \ldots, \Gamma_n \vDash \Box\phi_n$ and $\{\phi_1, \ldots, \phi_n\} \vDash \psi$. We show $\Gamma_1 \cup \ldots \cup \Gamma_n \vDash \Box\psi$.

    Assume that for all $\gamma \in \Gamma_1 \cup \ldots \cup \Gamma_n$ we have $M, s \vDash \gamma$. Then $M, s \vDash \Box\phi_1, \ldots, M, s \vDash \Box\phi_n$.
    * If $s = \varnothing$, then clearly $M, s \vDash \Box\psi$.

- ∗ If $s \neq \varnothing$, let $w \in s$. Then for all $i \in \{1, \ldots n\}$ we have $M, R[w] \vDash \phi_i$. Therefore $M, R[w] \vDash \psi$, and so, since w was arbitrary, $M, s \vDash \Box\psi$.

– Inter $\Box \Diamond$:

$\Downarrow$: Assume D is a derivation of length $\leq k$ of $\Gamma \vdash \neg \Box \alpha$. By the induction hypothesis, $\Gamma \vDash \neg \Box \alpha$. We show $\Gamma \vDash \Diamond\neg\alpha$.

Assume that for all $\gamma \in \Gamma$ we have $M, s \vDash \gamma$. Then $M, s \vDash \neg \Box \alpha$.

- ∗ If $s = \varnothing$, then clearly $M, s \vDash \Diamond\neg\alpha$.
- ∗ If $s \neq \varnothing$, fix $w \in s$. Then $M, \{w\} \nvDash \Box\alpha$ so $M, R[w] \nvDash \alpha$. α is classical and is therefore flat by Corollary 2.2.9. Assume for contradiction that for all $v \in R[w]$: $M, \{v\} \vDash \alpha$ (note that this holds also if $R[w] = \varnothing$). Then by flatness $M, R[w] \vDash \alpha$, contradicting $M, R[w] \nvDash \alpha$. So there must be some $v \in R[w]$ such that $M, \{v\} \nvDash \alpha$; fix such a v. Then $M, \{v\} \vDash \neg\alpha$, so there is a non-empty $t \subseteq R[w]$ such that $M, t \vDash \neg\alpha$. Since w was arbitrary, we have $M, s \vDash \Diamond\neg\alpha$.

$\Uparrow$: Assume D is a derivation of length $\leq k$ of $\Gamma \vdash \Diamond\neg\alpha$. By the induction hypothesis, $\Gamma \vDash \Diamond\neg\alpha$. We show $\Gamma \vDash \neg \Box \alpha$.

Assume that for all $\gamma \in \Gamma$ we have $M, s \vDash \gamma$. Then $M, s \vDash \Diamond\neg\alpha$.

- ∗ If $s = \varnothing$, then clearly $M, s \vDash \neg \Box \alpha$.
- ∗ If $s \neq \varnothing$, fix $w \in s$. Then there is a non-empty $t \subseteq R[w]$ such that $M, t \vDash \neg\alpha$. Assume for contradiction that $M, \{w\} \vDash \Box\alpha$. Then $M, R[w] \vDash \alpha$, and since α is downward closed by Corollary 2.2.9, $M, \{v\} \vDash \alpha$ for all $v \in R[v]$. By $M, t \vDash \neg\alpha$ we have that for all $v \in t : M, \{v\} \nvDash \alpha$. Then since $t \subseteq R[v]$, we have $t = \varnothing$, a contradiction. So $M, \{w\} \nvDash \Box\alpha$. Since w was arbitrary, we have $M, s \vDash \neg \Box \alpha$.

– Conv $\Diamond \mathbb{W}\vee$:

$\Downarrow$: Assume D is a derivation of length $\leq k$ of $\Gamma \vdash \Diamond(\phi \mathbin{\mathbb{W}} \psi)$. By the induction hypothesis, $\Gamma \vDash \Diamond(\phi \mathbin{\mathbb{W}} \psi)$. We show $\Gamma \vDash \Diamond\phi \vee \Diamond\psi$.

Assume that for all $\gamma \in \Gamma$ we have $M, s \vDash \gamma$. Then $M, s \vDash \Diamond(\phi \mathbin{\mathbb{W}} \psi)$.

- ∗ If $s = \varnothing$, then clearly $M, s \vDash \Diamond\phi \vee \Diamond\psi$.
- ∗ If $s \neq \varnothing$, fix some $w \in s$. Then there is a non-empty $t \subseteq R[w]$ such that $M, t \vDash \phi \mathbin{\mathbb{W}} \psi$—i.e. such that $M, t \vDash \phi$ or $M, t \vDash \psi$. Since w was arbitrary, this is the case for all $w \in s$, so that

letting

$$s_1 := \{w \in s \mid \exists t \subseteq R[w] : t \neq \varnothing \text{ and } M, t \vDash \phi\}$$
$$s_2 := \{w \in s \mid \exists t \subseteq R[w] : t \neq \varnothing \text{ and } M, t \vDash \psi\}$$

we have $s = s_1 \cup s_2$. Clearly $M, s_1 \vDash \Diamond\phi$ and $M, s_2 \vDash \Diamond\psi$, so $M, s \vDash \Diamond\phi \vee \Diamond\psi$.

$\Uparrow$: Assume D is a derivation of length $\leq k$ of $\Gamma \vdash \Diamond\phi \vee \Diamond\psi$. By the induction hypothesis, $\Gamma \vDash \Diamond\phi \vee \Diamond\psi$. We show $\Gamma \vDash \Diamond(\phi \mathbin{w} \psi)$.

Assume that for all $\gamma \in \Gamma$ we have $M, s \vDash \gamma$. Then $M, s \vDash \Diamond\phi \vee \Diamond\psi$, so there are some s_1, s_2 such that $s = s_1 \cup s_2$; $M, s_1 \vDash \Diamond\phi$; and $M, s_2 \vDash \Diamond\psi$.

* If $s = \varnothing$, then clearly $M, s \vDash \Diamond(\phi \mathbin{w} \psi)$.
* If $s \neq \varnothing$, fix some $w \in s$. If $w \in s_1$, then there is a non-empty $t \subseteq R[w]$ such that $M, t \vDash \phi$. Then also $M, t \vDash \phi \mathbin{w} \psi$. If $w \in s_2$, then there is a non-empty $t \subseteq R[w]$ such that $M, t \vDash \psi$. Again $M, t \vDash \phi \mathbin{w} \psi$. In either case, then, there is a non-empty $t \subseteq R[w]$ such that $M, t \vDash \phi \mathbin{w} \psi$. Since w was arbitrary, $M, s \vDash \Diamond(\phi \mathbin{w} \psi)$.

- Conv $\Box$ w$\vee$: Analogous to the case for Conv $\Diamond$ w$\vee$.
- $\Diamond$Sep: Assume D is a derivation of length $\leq k$ of $\Gamma \vdash \Diamond(\phi \vee (\psi \wedge \text{NE}))$. By the induction hypothesis, $\Gamma \vDash \Diamond(\phi \vee (\psi \wedge \text{NE}))$. We show $\Gamma \vDash \Diamond\psi$.

  Assume that for all $\gamma \in \Gamma$ we have $M, s \vDash \gamma$. Then $M, s \vDash \Diamond(\phi \vee (\psi \wedge \text{NE}))$.

  * If $s = \varnothing$, then clearly $M, s \vDash \Diamond\psi$.
  * If $s \neq \varnothing$, fix some $w \in s$. Then there is a non-empty $t \subseteq R[w]$ such that $M, t \vDash \phi \vee (\psi \wedge \text{NE})$. Therefore, there are some t_1, t_2 such that $t = t_1 \cup t_2$; $M, t_1 \vDash \phi$; and $M, t_2 \vDash \psi \wedge \text{NE}$. Then $t_2 \neq \varnothing$ and $M, t_2 \vDash \psi$, and note $t_2 \subseteq R[w]$. Since w was arbitrary, we have $M, s \vDash \Diamond\psi$.

- $\Diamond$Join: Assume D_1 and D_2 are derivations of length $\leq k$ of $\Gamma_1 \vdash \Diamond\phi$ and $\Gamma_2 \vdash \Diamond\psi$, respectively. By the induction hypothesis, $\Gamma_1 \vDash \Diamond\phi$ and $\Gamma_2 \vDash \Diamond\psi$. We show $\Gamma_1 \cup \Gamma_2 \vDash \Diamond(\phi \vee \psi)$.

  Assume that for all $\gamma \in \Gamma_1 \cup \Gamma_2$ we have $M, s \vDash \gamma$. Then $M, s \vDash \Diamond\phi$ and $M, s \vDash \Diamond\psi$.

  * If $s = \varnothing$, then clearly $M, s \vDash \Diamond(\phi \vee \psi)$.

  * If $s \neq \varnothing$, fix some $w \in s$. Since $M, s \vDash \Diamond\phi$ and $M, s \vDash \Diamond\psi$, there are non-empty $t_1, t_2 \subseteq R[w]$ such that $M, t_1 \vDash \phi$ and $M, t_2 \vDash \psi$. Therefore $t_1 \cup t_2 \vDash \phi \vee \psi$. Clearly $t_1 \cup t_2$ is non-empty and $t_1 \cup t_2 \subseteq R[w]$; w was arbitrary, so $M, s \vDash \Diamond(\phi \vee \psi)$.

- $\Diamond$NEI: The result follows given that for any $(M, s) \in \mathcal{M}$:

$$\begin{aligned} M, s \vDash \Diamond\phi &\iff \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \ \&\ M, t \vDash \phi \\ &\iff \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \ \&\ M, t \vDash \phi \wedge \textsc{ne} \\ &\iff M, s \vDash \Diamond(\phi \wedge \textsc{ne}) \end{aligned}$$

- $\Box$Inst: Assume D is a derivation of length $\leq k$ of $\Gamma \vdash \Box(\phi \wedge \textsc{ne})$. By the induction hypothesis, $\Gamma \vDash \Box(\phi \wedge \textsc{ne})$. We show $\Gamma \vDash \Diamond\phi$. Assume that for all $\gamma \in \Gamma$ we have $M, s \vDash \gamma$. Then $M, s \vDash \Box(\phi \wedge \textsc{ne})$.
  * If $s = \varnothing$, then clearly $M, s \vDash \Diamond\phi$.
  * If $s \neq \varnothing$, fix some $w \in s$. Then $M, R[w] \vDash \phi \wedge \textsc{ne}$, so $R[w] \neq \varnothing$ and $M, R[w] \vDash \phi$. Since w was arbitrary, this gives us $M, s \vDash \Diamond\phi$.

- $\Box\Diamond$ Join: Assume D_1 and D_2 are derivations of length $\leq k$ of $\Gamma_1 \vdash \Box(\phi \vee \psi)$ and $\Gamma_2 \vdash \Diamond\psi$, respectively. By the induction hypothesis, $\Gamma_1 \vDash \Box(\phi \vee \psi)$ and $\Gamma_2 \vDash \Diamond\psi$. We show $\Gamma_1 \cup \Gamma_2 \vDash \Box(\phi \vee (\psi \wedge \textsc{ne}))$.

  Assume that for all $\gamma \in \Gamma_1 \cup \Gamma_2$ we have $M, s \vDash \gamma$. Then $M, s \vDash \Box(\phi \vee \psi)$ and $M, s \vDash \Diamond\psi$.
  * If $s = \varnothing$, then clearly $M, s \vDash \Box(\phi \vee (\psi \wedge \textsc{ne}))$.
  * If $s \neq \varnothing$, fix some $w \in s$. Then since $M, s \vDash \Box(\phi \vee \psi)$, we have $M, R[w] \vDash \phi \vee \psi$, so that there are r_1, r_2 such that $R[w] = r_1 \cup r_2$; $M, r_1 \vDash \phi$; and $M, r_2 \vDash \psi$. And since $M, s \vDash \Diamond\psi$, there is some non-empty $t \subseteq R[w]$ such that $M, t \vDash \psi$.
    · If $r_2 \neq \varnothing$, then $M, r_1 \vDash \phi$ and $M, r_2 \vDash \psi \wedge \textsc{ne}$, and so $M, R[w] \vDash \phi \vee (\psi \wedge \textsc{ne})$.
    · If $r_2 = \varnothing$ and $r_1 \neq \varnothing$, then $R[w] = r_1$ and $t \subseteq r_1$. Then since $M, r_1 \vDash \phi$; $M, t \vDash \psi \wedge \textsc{ne}$; and $R[w] = r_1 = r_1 \cup t$, we have $M, R[w] \vDash \phi \vee (\psi \wedge \textsc{ne})$.
    · The case in which $r_1 = r_2 = \varnothing$ is not possible since this implies $R[w] = \varnothing$, contradicting the fact that $t \neq \varnothing$ and $t \subseteq R[w]$.

In any case, then, $M, R[w] \vDash \phi \vee (\psi \wedge \text{NE})$. Since w was arbitrary, $M, s \vDash \Box(\phi \vee (\psi \wedge \text{NE}))$. □

For $\mathbf{BSML}^{\mathbb{W}}$, we use $\neg$ in place of $\neg$; generalize the rule relating $\Box$ and $\Diamond$ (note that we also switch the modalities around); and add a rules relating $\neg\text{NE}$ and $\bot$ as well as rules for De Morgan's Laws (as set out in Fact 2.2.5) and double negation.

Definition 4.2.3. (Natural deduction system for $\mathbf{BSML}^{\mathbb{W}}$)
The following rules comprise a natural deduction system for $\mathbf{BSML}^{\mathbb{W}}$. We also call the system $\mathbf{BSML}^{\mathbb{W}}$.

1. The rules for $\mathbf{PT}^{+}$ (with $\neg$ in place of $\neg$, and excluding RAA).

2. ◇Mon, □Mon and

◇□ interaction

$$\cfrac{\cfrac{D}{\neg \Diamond \phi}}{\Box \neg \phi}\ \mathsf{Inter}\Diamond\Box$$

3. Conv ◇ W∨, Conv □ W∨, ◇Sep, ◇Join, ◇NEI, □Inst, and □ ◇ Join.

4. The following rules for $\neg$:

¬NE elimination

$$\cfrac{\cfrac{D}{\neg\text{NE}}}{\bot}\ \neg\text{NE}\mathsf{E}$$

Double ¬ elimination

$$\cfrac{\cfrac{D}{\neg\neg\phi}}{\phi}\ \mathsf{DN}$$

De Morgan 1

$$\cfrac{\cfrac{D}{\neg(\phi \wedge \psi)}}{\neg\phi \vee \neg\psi}\ \mathsf{DM}_1$$

De Morgan 2

$$\cfrac{\cfrac{D}{\neg(\phi \vee \psi)}}{\neg\phi \wedge \neg\psi}\ \mathsf{DM}_2$$

De Morgan 3

$$\cfrac{\cfrac{D}{\neg(\phi \mathbin{\mathbb{W}} \psi)}}{\neg\phi \wedge \neg\psi}\ \mathsf{DM}_3$$

Theorem 4.2.4. (Soundness of $\mathbf{BSML}^{\mathbb{W}}$ rules) For any $\Gamma \cup \{\phi\} \in \mathbf{L}$, we have $\Gamma \vdash_{\mathbf{BSML}^{\mathbb{W}}} \phi \Rightarrow \Gamma \vDash \phi$.

Proof. By induction on the length of possible derivations $D = (R_1, \ldots, R_k)$ of $\Gamma \vdash \phi$.

- Base case: $k = 1$. As in the proof of Theorem 4.1.3.

- Inductive case. Assume the result holds for all derivations of length $\leq k$. We consider different possibilities of the final rule used in the derivation of $\Gamma \vdash \phi$. Most cases are as in the proofs of Theorems 4.1.3 and 4.2.2 or follow from those cases and Proposition 2.2.15; we show the remaining cases:
  - Inter $\Diamond\Box$: The result follows given that for any $(M, s) \in \mathcal{M}$ (as in the proof of Fact 2.2.6):

$$\begin{array}{lccc} & M, s \models \neg \Diamond \phi & \iff & M, R[s] \dashv \phi \\ \iff & M, R[s] \models \neg\phi & \iff & M, R[s] \dashv \neg\neg\phi \\ \iff & M, s \models \neg \Diamond \neg\neg\phi & \iff & M, s \models \Box\neg\phi \end{array}$$

  - $\neg$NEE: The result follows given that for any $(M, s) \in \mathcal{M}$:

$$M, s \models \neg\text{NE} \iff M, s \dashv \text{NE} \iff s = \varnothing \iff M, s \models \bot$$

  - $\mathsf{DN}, \mathsf{DM}_1, \mathsf{DM}_2, \mathsf{DM}_3$: The result follows from Fact 2.2.5. □

We will use the NE-connective interaction rules to derive some equivalence laws whose form resembles that of free choice inferences.[24] We make use of these in the completeness proof.

Proposition 4.2.5. The following are derivable with the rules for **SML**$^{\mathbb{W}}$ and with the rules for **BSML**$^{\mathbb{W}}$:

$$\begin{array}{lcll} \Diamond((\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE})) & \dashv\vdash & \Diamond\phi \wedge \Diamond\psi & \mathsf{FC} \\ \Box(\phi \vee (\psi \wedge \text{NE})) & \dashv\vdash & \Box(\phi \vee \psi) \wedge \Diamond\psi & \Box\mathsf{FC} \end{array}$$

Proof. FC: For $\vdash$, we have:

$$\begin{array}{lll} & \Diamond((\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE})) & \\ \vdash & \Diamond((\phi \wedge \text{NE}) \vee \psi) \wedge \Diamond\psi & \Diamond\mathsf{Sep} \\ \vdash & \Diamond\phi \wedge \Diamond\psi & \Diamond\mathsf{Sep} \end{array}$$

For $\dashv$:

$$\begin{array}{lll} & \Diamond\phi \wedge \Diamond\psi & \\ \vdash & \Diamond(\phi \wedge \text{NE}) \wedge \Diamond(\psi \wedge \text{NE}) & \Diamond\text{NE}\mathsf{I} \\ \vdash & \Diamond((\phi \wedge \text{NE}) \vee (\psi \wedge \text{NE})) & \Diamond\mathsf{Join} \end{array}$$

[24] Note that the added rules for $\Diamond$ which do not involve $\mathbb{W}$ ($\Diamond$Sep, $\Diamond$NEI and $\Diamond$Join) may be thought of as expressions of different aspects of the FC-equivalence, and similarly for $\Box$. Since (ignoring $\mathbb{W}$) these are all the rules required above and beyond the classical modality rules, this FC-type functioning may be thought of as encapsulating the modal behaviour of the logics, and so it is also in this sense that these are "logics of free choice".

$\Box$FC: For $\vdash$:

	$\Box(\phi \vee (\psi \wedge \textsc{ne}))$	
$\vdash$	$\Box(\phi \vee \psi) \wedge \Box(\phi \vee (\psi \wedge \textsc{ne}))$	$\wedge$I, $\vee$Sub, $\Box$Mon
$\vdash$	$\Box(\phi \vee \psi) \wedge \Box((\phi \vee (\psi \wedge \textsc{ne})) \wedge \textsc{ne})$	Prop 4.1.5 (i), $\Box$Mon
$\vdash$	$\Box(\phi \vee \psi) \wedge \Diamond(\phi \vee (\psi \wedge \textsc{ne}))$	$\Box$Inst
$\vdash$	$\Box(\phi \vee \psi) \wedge \Diamond\psi$	$\Diamond$Sep

We get $\dashv$ immediately from $\Box\Diamond$ Join. □

4.3 SGML$^{⩖}$

For **SGML**$^{⩖}$, we use the full set of rules concerning modalities from **MD**$^{⩖}$ [35], and a different set of rules for the interactions between NE and the connectives.

Definition 4.3.1. (Natural deduction system for SGML$^{⩖}$) The following rules comprise a natural deduction system for **SGML**$^{⩖}$. We also call the system **SGML**$^{⩖}$.

1. The rules for PT^{+}.

2. ◈Mon, ⊡Mon and Inter ⊡ ◈.
($\Diamond$Mon, $\Box$Mon and Inter $\Box\Diamond$ with ◈ in place of $\Diamond$ and ⊡ in place of $\Box$.)

3. The following rules governing the interaction of the connectives and NE: (where Distr ◈ ⩖ and Distr ⊡ ⩖ are from **MD**$^{⩖}$)

◈⩖ distributivity

$$\dfrac{\begin{matrix} D \\ ◈(\phi \mathbin{⩖} \psi)\end{matrix}}{◈\phi \mathbin{⩖} ◈\psi}\ \text{Distr ◈ ⩖}$$

⊡⩖ distributivity

$$\dfrac{\begin{matrix} D \\ ⊡(\phi \mathbin{⩖} \psi)\end{matrix}}{⊡\phi \mathbin{⩖} ⊡\psi}\ \text{Distr ⊡ ⩖}$$

◈$\vee$ distributivity

$$\dfrac{\begin{matrix} D \\ ◈(\phi \vee \psi)\end{matrix}}{◈\phi \vee ◈\psi}\ \text{Distr ◈ }\vee$$

NE◈ distributivity

$$\dfrac{\begin{matrix} D \\ ◈\phi \wedge \textsc{ne}\end{matrix}}{◈(\phi \wedge \textsc{ne})}\ \text{DistrNE◈}$$

$\boxdot$ instantiation

$$\dfrac{\begin{matrix} D \\ \boxdot\phi \end{matrix}}{\diamonddot\phi \vee \boxdot\bot}\ \boxdot\mathsf{Inst}$$

$\boxdot\diamonddot$ join

$$\dfrac{\begin{matrix} D_1 \\ \boxdot(\phi \vee \psi) \end{matrix} \qquad \begin{matrix} D_2 \\ (\diamonddot\psi \wedge \textsc{ne}) \vee \chi \end{matrix}}{\boxdot(\phi \vee (\psi \wedge \textsc{ne}))}\ \boxdot\diamonddot\ \mathsf{Join}$$

For the $\boxdot$-analogue of $\mathsf{Distr}\diamonddot\vee$, we note that $\boxdot\phi \vee \boxdot\psi \vDash \boxdot(\phi \vee \psi)$ (so this direction is derivable in our complete system), but $\boxdot(\phi \vee \psi) \nvDash \boxdot\phi \vee \boxdot\psi$.

Similarly, for the $\boxdot$-analogue of $\mathsf{Distr}\textsc{ne}\diamonddot$, we have $\boxdot(\phi \wedge \textsc{ne}) \vDash \boxdot\phi \wedge \textsc{ne}$, but $\boxdot\phi \wedge \textsc{ne} \nvDash \boxdot(\phi \wedge \textsc{ne})$.

Theorem 4.3.2. (Soundness of SGML$^{\mathbb{W}}$ rules) For any $\Gamma \cup \{\phi\} \in \mathbf{L}$, we have $\Gamma \vdash_{\mathbf{SGML}^{\mathbb{W}}} \phi \Rightarrow \Gamma \vDash \phi$.

Proof. By induction on the length of possible derivations $D = (R_1, \ldots, R_k)$ of $\Gamma \vdash \phi$.

- Base case: $k = 1$. As in the proof of Theorem 4.1.3.

- Inductive case. Assume the result holds for all derivations of length $\leq k$. We consider different possibilities of the final rule used in the derivation of $\Gamma \vdash \phi$. Most cases are as in the proof of Theorem 4.1.3; we show the remaining cases:

  - $\diamonddot\mathsf{Mon}$: Assume D and D' are derivations of length $\leq k$ of $\Gamma \vdash \diamonddot\phi$ and $\phi \vdash \psi$, respectively. By the induction hypothesis, $\Gamma \vDash \diamonddot\phi$ and $\phi \vDash \psi$. We show $\Gamma \vDash \diamonddot\psi$.

    Assume that for all $\gamma \in \Gamma$ we have $M, s \vDash \gamma$. Then $M, s \vDash \diamonddot\phi$, so there is some t such that sRt and $M, t \vDash \phi$. Then by $\phi \vDash \psi$ we have $M, t \vDash \psi$, and therefore $M, s \vDash \diamonddot\psi$.

  - $\boxdot\mathsf{Mon}$: Assume $D_1, \ldots, D_n$ and D' are derivations of length $\leq k$ of $\Gamma_1 \vdash \boxdot\phi_1$, …, $\Gamma_n \vdash \boxdot\phi_n$ and $\phi_1, \ldots, \phi_n \vdash \psi$, respectively. By the induction hypothesis, $\Gamma_1 \vDash \boxdot\phi_1$, …, $\Gamma_n \vDash \boxdot\phi_n$ and $\{\phi_1, \ldots, \phi_n\} \vDash \psi$. We show $\Gamma_1 \cup \ldots \cup \Gamma_n \vDash \boxdot\psi$.

    Assume that for all $\gamma \in \Gamma$ we have $M, s \vDash \gamma$. Then $M, s \vDash \boxdot\phi_1, \ldots, M, s \vDash \boxdot\phi_n$. Therefore $M, R[s] \vDash \phi_1, \ldots, M, R[s] \vDash \phi_n$. By $\{\phi_1, \ldots, \phi_n\} \vDash \psi$, then, $M, R[s] \vDash \psi$. Therefore $M, s \vDash \boxdot\psi$.

  - $\mathsf{Inter}\ \boxdot\ \diamonddot$: The result follows from the soundness of $\mathsf{Inter}\ \Box\ \Diamond$ (Theorem 4.2.2) and Proposition 2.2.13.

– Distr $\Diamond \mathbin{\text{w}}$: The result follows given that for any $(M, s) \in \mathcal{M}$ we have:

$$
\begin{aligned}
& M, s \vDash \Diamond(\phi \mathbin{\text{w}} \psi) \\
\iff & \exists t : sRt \text{ and } M, t \vDash \phi \mathbin{\text{w}} \psi \\
\iff & \exists t : sRt \text{ and } (M, t \vDash \phi \text{ or } M, t \vDash \psi) \\
\iff & \exists t_1 : sRt_1 \text{ and } M, t_1 \vDash \phi \text{ or } \exists t_2 : sRt_2 \text{ and } M, t_2 \vDash \phi \\
\iff & M, s \vDash \Diamond\phi \text{ or } M, s \vDash \Diamond\psi \\
\iff & M, s \vDash \Diamond\phi \mathbin{\text{w}} \Diamond\psi
\end{aligned}
$$

– Distr $\boxdot \mathbin{\text{w}}$: Analogous to the case for Distr $\Diamond \mathbin{\text{w}}$.

– Distr $\Diamond \vee$:

$\Downarrow$: Assume D is a derivation of length $\leq k$ of $\Gamma \vdash \Diamond(\phi \vee \psi)$. By the induction hypothesis, $\Gamma \vDash \Diamond(\phi \vee \psi)$. We show $\Gamma \vDash \Diamond\phi \vee \Diamond\psi$. Assume that for all $\gamma \in \Gamma$ we have $M, s \vDash \gamma$. Then $M, s \vDash \Diamond(\phi \vee \psi)$, so there is some t such that sRt and $M, t \vDash \phi \vee \psi$. Then there are some t_1, t_2 such that $t = t_1 \cup t_2$; $M, t_1 \vDash \phi$; and $M, t_2 \vDash \psi$.

Let $s_1 := \{w \in s \mid \exists v \in t_1 : wRv\} = R^{-1}[t_1] \cap s$ and $s_2 := \{w \in s \mid \exists v \in t_2 : wRv\} = R^{-1}[t_2] \cap s$. Then:

* $s_1 R t_1$:
  · $t_1 \subseteq R[s_1]$: Let $v \in t_1$. Since $t_1 \subseteq t \subseteq R[s]$, there is some $w \in s$ such that wRv. Then $w \in s_1$, and so $v \in R[s_1]$.
  · $s_1 \subseteq R^{-1}[t_1]$: $s_1 = R^{-1}[t_1] \cap s \subseteq R^{-1}[t_1]$.
* Similarly $s_2 R t_2$.
* $M, s_1 \vDash \Diamond\phi$: Immediate from $s_1 R t_1$ and $M, t_1 \vDash \phi$.
* $M, s_2 \vDash \Diamond\psi$: Immediate from $s_2 R t_2$ and $M, t_2 \vDash \psi$.
* $s = s_1 \cup s_2$: Let $w \in s$. Since $s \subseteq R^{-1}[t]$, there is some $v \in t$ such that wRv. Then $v \in t_1$ or $v \in t_2$ (or both); if $v \in t_1$, then $w \in s_1$, and if $v \in t_2$, then $w \in s_2$. So $w \in s_1 \cup s_2$. w was arbitrary, so $s \subseteq s_1 \cup s_2$. Clearly $s_1 \cup s_2 \subseteq s$, so $s = s_1 \cup s_2$.

Given all of the above, then, $M, s \vDash \Diamond\phi \vee \Diamond\psi$.

$\Uparrow$: Assume D is a derivation of length $\leq k$ of $\Gamma \vdash \Diamond\phi \vee \Diamond\psi$. By the induction hypothesis, $\Gamma \vDash \Diamond\phi \vee \Diamond\psi$. We show $\Gamma \vDash \Diamond(\phi \vee \psi)$. Assume that for all $\gamma \in \Gamma$ we have $M, s \vDash \gamma$. Then $M, s \vDash \Diamond\phi \vee \Diamond\psi$, so there are some s_1, s_2 such that $s = s_1 \cup s_2$; $M, s_1 \vDash \Diamond\phi$; and $M, s_2 \vDash \Diamond\psi$. So there are t_1, t_2 such that $s_1 R t_1$; $M, t_1 \vDash \phi$; $s_2 R t_2$; and $M, t_2 \vDash \psi$.

Then:

* $sR(t_1 \cup t_2)$:
    - $t_1 \cup t_2 \subseteq R[s]$: Let $v \in t_1 \cup t_2$. If $v \in t_1$, then since $t_1 \subseteq R[s_1]$, there is some $w \in s_1$ such that wRv. Since $s_1 \subseteq s$, we have $w \in s$, so $v \in R[s]$. The case in which $v \in t_2$ is similar, giving $v \in R[s]$.
    - $s \subseteq R^{-1}[t_1 \cup t_2]$. Let $w \in s$. If $w \in s_1$, then since $s_1 \subseteq R^{-1}[t_1]$, there is some $v \in t_1 \subseteq t_1 \cup t_2$ such that wRv, so $w \in R^{-1}[t_1 \cup t_2]$. If $w \in s_2$, we get $w \in R^{-1}[t_1 \cup t_2]$ in a similar way.
* $M, t_1 \cup t_2 \models \phi \vee \psi$.

Therefore, $M, s \models \Diamond(\phi \vee \psi)$.

– DistrNE$\Diamond$: The result follows given that for any $(M, s) \in \mathcal{M}$:

$$\begin{array}{rlrl} & M, s \models \Diamond\phi \wedge \textsc{ne} & \Leftrightarrow & s \neq \varnothing \ \& \ \exists t : sRt \ \& \ M, t \models \phi \\ \Leftrightarrow & \exists t \neq \varnothing : sRt \ \& \ M, t \models \phi & \Leftrightarrow & \exists t : sRt \ \& \ M, t \models \phi \wedge \textsc{ne} \\ \Leftrightarrow & M, s \models \Diamond(\phi \wedge \textsc{ne}) & & \end{array}$$

– $\boxdot$Inst: Assume D is a derivation of length $\leq k$ of $\Gamma \vdash \boxdot\phi$. By the induction hypothesis, $\Gamma \models \boxdot\phi$. We show $\Gamma \models \Diamond\phi \vee \boxdot\bot$.

Assume that for all $\gamma \in \Gamma$ we have $M, s \models \gamma$. Then $M, s \models \boxdot\phi$. Therefore $M, R[s] \models \phi$.

Let $t := \{w \in s \mid \exists v : wRv\}$. Then:

* $tR(R[s])$: $R[s] = R[t]$ and clearly $tR(R[t])$.
* $M, s\backslash t \models \boxdot\bot$: Let $w \in s\backslash t$. Then there is no v such that wRv. w was arbitrary, so $R[s\backslash t] = \varnothing$. Therefore $M, R[s\backslash t] \models \bot$, so $M, s\backslash t \models \boxdot\bot$.
* $s = t \cup (s\backslash t)$.

Given $tR(R[s])$ and $M, R[s] \models \phi$, we get $M, t \models \Diamond\phi$. Then given $M, s\backslash t \models \boxdot\bot$ and $s = t \cup (s\backslash t)$ we have $M, s \models \Diamond\phi \vee \boxdot\bot$.

– $\boxdot \Diamond$ Join: Assume D_1 and D_2 are derivations of length $\leq k$ of $\Gamma_1 \vdash \boxdot(\phi \vee \psi)$ and $\Gamma_2 \vdash (\Diamond\psi \wedge \textsc{ne}) \vee \chi$, respectively. By the induction hypothesis, $\Gamma_1 \models \boxdot(\phi \vee \psi)$ and $\Gamma_2 \models (\Diamond\psi \wedge \textsc{ne}) \vee \chi$. We show $\Gamma_1 \cup \Gamma_2 \models \boxdot(\phi \vee (\psi \wedge \textsc{ne}))$.

Assume that for all $\gamma \in \Gamma_1 \cup \Gamma_2$ we have $M, s \models \gamma$.

By $\Gamma_1 \models \boxdot(\phi \vee \psi)$, we have $M, s \models \boxdot(\phi \vee \psi)$. Therefore $M, R[s] \models \phi \vee \psi$, so there are some r_1, r_2 such that $R[s] = r_1 \cup r_2$; $M, r_1 \models \phi$; and $M, r_2 \models \psi$.

By $\Gamma_2 \vDash (◈\psi \wedge \textsc{ne}) \vee \chi$, we have $M, s \vDash (◈\psi \wedge \textsc{ne}) \vee \chi$, so there are some s_1, s_2 such that $s = s_1 \cup s_2$; $M, s_1 \vDash ◈\psi \wedge \textsc{ne}$; and $M, s_2 \vDash \chi$. By $M, s_1 \vDash ◈\psi \wedge \textsc{ne}$ there is some t such that $s_1 R t$ and $M, t \vDash \psi$. We also have $M, s_1 \vDash \textsc{ne}$, so $s_1 \neq \varnothing$ and therefore (by $s_1 R t$) $t \neq \varnothing$ so that $M, t \vDash \textsc{ne}$. So $M, t \vDash \psi \wedge \textsc{ne}$. Note also that clearly $t \subseteq R[s]$.

We can now show that $M, R[s] \vDash \phi \vee (\psi \wedge \textsc{ne})$:

* If $r_2 \neq \varnothing$, then $M, r_1 \vDash \phi$ and $M, r_2 \vDash \psi \wedge \textsc{ne}$, and so $M, R[s] \vDash \phi \vee (\psi \wedge \textsc{ne})$.
* If $r_2 = \varnothing$ and $r_1 \neq \varnothing$, then $R[s] = r_1$ and $t \subseteq r_1$. Then since $M, r_1 \vDash \phi$; $M, t \vDash \psi \wedge \textsc{ne}$; and $R[s] = r_1 = r_1 \cup t$, we have $M, R[s] \vDash \phi \vee (\psi \wedge \textsc{ne})$
* The case in which $r_1 = r_2 = \varnothing$ is not possible since this implies $R[s] = \varnothing$, contradicting the fact that $t \neq \varnothing$ and $t \subseteq R[s]$.

In any case, then, $M, R[s] \vDash \phi \vee (\psi \wedge \textsc{ne})$. Therefore $M, s \vDash \boxdot(\phi \vee (\psi \wedge \textsc{ne}))$. □

As we did for $\mathbf{SML}^{\mathbb{W}}$ and $\mathbf{BSML}^{\mathbb{W}}$, we prove some equivalences involving $\vee$, NE and the modalities. Note that the ◈C-rule shows how ◈ behaves in a situation in which $\Diamond$ allows us to draw FC inferences.

Proposition 4.3.3. The following are derivable with the rules for $\mathbf{SGML}^{\mathbb{W}}$:

$$
\begin{array}{lcll}
◈((\phi \wedge \textsc{ne}) \vee (\psi \wedge \textsc{ne})) & \dashv\vdash & (◈\phi \wedge \textsc{ne}) \vee (◈\psi \wedge \textsc{ne}) & ◈\,\mathsf{C} \\
\boxdot(\phi \vee (\psi \wedge \textsc{ne})) & \dashv\vdash & \boxdot(\phi \vee \psi) \wedge (◈\phi \vee (◈\psi \wedge \textsc{ne}) \vee \boxdot\bot) & \boxdot\,\mathsf{C}
\end{array}
$$

Proof. ◈C:

$$
\begin{array}{lcl}
 & ◈((\phi \wedge \textsc{ne}) \vee (\psi \wedge \textsc{ne})) & \\
\dashv\vdash & ◈(\phi \wedge \textsc{ne}) \vee ◈(\psi \wedge \textsc{ne}) & \mathsf{Distr}\,◈\vee \\
\dashv\vdash & (◈\phi \wedge \textsc{ne}) \vee (◈\psi \wedge \textsc{ne}) & \mathsf{Distr}\textsc{ne}◈
\end{array}
$$

⊡C: For $\vdash$:

$$
\begin{array}{lll}
 & \boxdot(\phi \vee (\psi \wedge \textsc{ne})) & \\
\vdash & \boxdot(\phi \vee \psi) \wedge \boxdot(\phi \vee (\psi \wedge \textsc{ne})) & \wedge\mathsf{I}, \vee\mathsf{Sub}, \boxdot\mathsf{Mon} \\
\vdash & \boxdot(\phi \vee \psi) \wedge (◈(\phi \vee (\psi \wedge \textsc{ne})) \vee \boxdot\bot) & \boxdot\mathsf{Inst} \\
\vdash & \boxdot(\phi \vee \psi) \wedge ((◈\phi \vee ◈(\psi \wedge \textsc{ne})) \vee \boxdot\bot) & \mathsf{Distr}\,◈\vee \\
\vdash & \boxdot(\phi \vee \psi) \wedge (◈\phi \vee (◈\psi \wedge \textsc{ne}) \vee \boxdot\bot) & \mathsf{Distr}\textsc{ne}◈
\end{array}
$$

$\dashv$ is immediate by ⊡◈Join. □

Chapter 5

Completeness

To show the weak completeness of a natural deduction system for logic L, we prove that any L-formula is provably equivalent in the system to an L-formula in normal form, with completeness then following from the semantic properties of formulas in normal form and the rules for $\mathbin{⩔}$. Establishing provable equivalence is the longest and most detailed part of this process; we first simply assume that this equivalence holds and examine the other parts of the proof so as to provide the reader with a clear understanding of how the entire argument functions before delving into the details. Accordingly, Section 5.1 presents the bulk of the proof, and the provable equivalence results are shown in Section 5.2.

The proof here is essentially that given for $\mathbf{PT}^{+}$ in [38] and [34] adapted to the modal setting.

In Section 5.3, we prove strong completeness. This follows from weak completeness and compactness; with compactness following from the strong completeness of Lück's system for modal team logic (**MTL**) (which implies that **MTL** is compact) and the fact that **MTL** and our logics have the same expressive power.

5.1 Weak Completeness

Throughout this chapter, many of our results will apply to all logics and all natural deduction systems; we will specify when this is not the case. We use $\vdash$ and $\dashv\vdash$ to refer to provability in all systems.

The key semantic fact about formulas in normal form is the following:

Proposition 5.1.1. For any finite non-empty state properties $\mathcal{F}, \mathcal{G} \subseteq \mathcal{M}$ and any $k \in \mathbb{N}$, the following are equivalent:

(i) $\bigvee\!\!\!\!\bigvee_{(M,s)\in\mathcal{F}} \Theta^k_s \vDash \bigvee\!\!\!\!\bigvee_{(M',s')\in\mathcal{G}} \Theta^k_{s'}$

(ii) For each $(M,s) \in \mathcal{F}$, there is some $(M',s') \in \mathcal{G}$ such that $M,s \leftrightarroweq_k M',s'$.

Proof. (i) ⇒ (ii): Fix some $(M,s) \in \mathcal{F}$. By Proposition 3.2.8 (and $M,s \leftrightarroweq_k M,s$), we have $M,s \vDash \Theta^k_s$ and therefore $M,s \vDash \bigvee\!\!\!\!\bigvee_{(M,s)\in\mathcal{F}} \Theta^k_s$. By (i) we have $M,s \vDash \bigvee\!\!\!\!\bigvee_{(M',s')\in\mathcal{G}} \Theta^k_{s'}$. Then $M,s \vDash \Theta^k_{s'}$ for some $(M',s') \in \mathcal{G}$, so by Proposition 3.2.8, $M,s \leftrightarroweq_k M',s'$.

(ii) ⇒ (i): Fix some $(N,t) \in \mathcal{M}$ such that $N,t \vDash \bigvee\!\!\!\!\bigvee_{(M,s)\in\mathcal{F}} \Theta^k_s$. Then $N,t \vDash \Theta^k_s$ for some $(M,s) \in \mathcal{F}$. By Proposition 3.2.8, we have $N,t \leftrightarroweq_k M,s$. By (ii), there is some $(M',s') \in \mathcal{G}$ such that $M,s \leftrightarroweq_k M',s'$. By Fact 3.1.4, $N,t \leftrightarroweq_k M,s$ and $M,s \leftrightarroweq_k M',s'$ imply $N,t \leftrightarroweq_k M',s'$. By Proposition 3.2.8, $N,t \vDash \Theta^k_{s'}$ so that $N,t \vDash \bigvee\!\!\!\!\bigvee_{(M',s')\in\mathcal{G}} \Theta^k_{s'}$. (N,t) was arbitrary, so $\bigvee\!\!\!\!\bigvee_{(M,s)\in\mathcal{F}} \Theta^k_s \vDash \bigvee\!\!\!\!\bigvee_{(M',s')\in\mathcal{G}} \Theta^k_{s'}$. □

We next want to show the following about our characteristic formulas:

Proposition 5.1.2. For any $(M,s),(M',s') \in \mathcal{M}$ and any $k \in \mathbb{N}$, if $M,s \leftrightarroweq_k M',s'$, then $\Theta^k_s \dashv\vdash \Theta^k_{s'}$.

In order to do this, we note that our systems are complete for classical modal logic. That this is the case is easy to see and is shown by Yang for her system for $\mathbf{MD}^{\mathbb{W}}$ in [35]. This system is essentially our system for $\mathbf{SGML}^{\mathbb{W}}$ without the rules concerning NE, and since Yang's result only concerns the restriction of her system to classical ($\mathbf{ML}^{\otimes}$) formulas, the same result follows for $\mathbf{SGML}^{\mathbb{W}}$. Then given that the restrictions of each of our systems to their respective classical fragments prove the same formulas modulo the classical-equivalence-invariant mappings $*$ and $**$, the result follows for all of our systems. We now make these claims somewhat more explicit, but for more details see [35].

Yang uses the Hilbert-style system of classical modal logic **K** as a proxy for classical provability. Recall that:[25]

[25]These details are not our focus here, but note that **K** (defined for the language of **ML** here) has the following axioms:

1. The axioms of classical propositional logic
2. $\Box(p \to q) \to (\Box p \to \Box q)$
3. $\Diamond p \leftrightarrow \neg\Box\neg p$

and the following rules:

1. Modus Ponens: $\alpha, \alpha \to \beta \Rightarrow \beta$

Theorem 5.1.3. (Classical completeness of K) For any $\mathrm{B}\cup\{\alpha\} \subseteq \mathbf{ML}$: $\mathrm{B} \vDash \alpha \iff \mathrm{B} \vdash_{\mathbf{K}} \alpha$.

(Recall also Fact 2.2.17: for classical formulas, entailment as classically defined coincides with our notion of entailment.)

Adapting Yang's result for $\mathbf{MD}^{\mathbb{W}}$ gives us:

Proposition 5.1.4. For any $\mathrm{B} \cup \{\alpha\} \subseteq \mathbf{ML}$:

$$B \vdash_{\mathbf{K}} \alpha \iff \mathrm{B} \vdash_{\mathbf{SML}^{\mathbb{W}}} \alpha \iff \mathrm{B}^{*} \vdash_{\mathbf{SGML}^{\mathbb{W}}} \alpha^{*} \iff \mathrm{B}^{**} \vdash_{\mathbf{BSML}^{\mathbb{W}}} \alpha^{**}$$

The proof of the above is the same as Yang's proof of her Lemma 2.12 [35], where she shows that her system for $\mathbf{MD}^{\mathbb{W}}$ restricted to classical ($\mathbf{ML}^{\circledast}$) formulas admits of all the axioms and rules of **K** (defined for $\mathbf{ML}^{\circledast}$). The system for $\mathbf{MD}^{\mathbb{W}}$ restricted to classical formulas is essentially[26] our system for $\mathbf{SGML}^{\mathbb{W}}$ restricted to classical formulas. This again is essentially the same as $\mathbf{SML}^{\mathbb{W}}$ and $\mathbf{BSML}^{\mathbb{W}}$ restricted to classical formulas (formulas in **ML** and formulas in $\mathbf{ML}^{**}$) modulo the change in notation mediated by the maps $*$ and $**$.[27] Showing that the systems restricted to classical formulas admit of the axioms and rules of **K** gives us the implication(s) from provability in **K** to provability in any of our systems. Provability in any of our systems implies provability in **K** by the soundness of our systems, Fact 2.2.17 and the completeness of **K**; all of the implications in the proposition then follow.

Given the above and the classical completeness of **K**, we have:

2. Necessitation: $\alpha \Rightarrow \Box\alpha$

3. Uniform Substitution: $\alpha(p) \Rightarrow \alpha(\beta/p)$ (each occurrence of p in α is replaced by an occurrence of β)

[26] There are two rules in the classical fragment of Yang's system for $\mathbf{MD}^{\mathbb{W}}$ not present in $\mathbf{SGML}^{\mathbb{W}}$:

$$\frac{\neg\neg\alpha}{\alpha}\,\mathsf{DN} \qquad\qquad \frac{\bot}{\phi}\,\mathsf{Ex\ falso}$$

These are clearly derivable for the classical fragment of $\mathbf{SGML}^{\mathbb{W}}$. RAA is not present in the system for $\mathbf{MD}^{\mathbb{W}}$; this, similarly, is derivable in $\mathbf{MD}^{\mathbb{W}}$.

[27] To see why this is the case for $\mathbf{BSML}^{\mathbb{W}}$, note that we can derive DN and De Morgan's laws for classical formulas as they are usually derived in $\mathbf{SML}^{\mathbb{W}}$ and $\mathbf{SGML}^{\mathbb{W}}$. The only rule in the classical fragments of these two systems whose analogue is not present in the classical fragment of $\mathbf{BSML}^{\mathbb{W}}$ is RAA; the $\neg$-analogue of RAA is clearly derivable using $\neg$I and DN.

Proposition 5.1.5. (Classical completeness of $\mathbf{SML}^{\mathrm{w}}$, $\mathbf{SGML}^{\mathrm{w}}$, $\mathbf{BSML}^{\mathrm{w}}$) For any $\mathrm{B} \cup \{\alpha\} \subseteq \mathbf{ML}$:

$$\begin{array}{cccc} & \mathrm{B} \vDash \alpha & \iff & \mathrm{B} \vdash_{\mathbf{SML}^{\mathrm{w}}} \alpha \\ \iff & \mathrm{B}^* \vDash \alpha^* & \iff & \mathrm{B}^* \vdash_{\mathbf{SGML}^{\mathrm{w}}} \alpha^* \\ \iff & \mathrm{B}^{**} \vDash \alpha^{**} & \iff & \mathrm{B}^{**} \vdash_{\mathbf{BSML}^{\mathrm{w}}} \alpha^{**} \end{array}$$

We now have what is required. First note:[28]

Proposition 5.1.6. For any $(M, w), (M', w')$ and any $k \in \mathbb{N}$, if $M, w \leftrightarrows_k M', w'$, then $\chi^k_w \dashv\vdash \chi^k_{w'}$.

Proof. We show that if $M, w \leftrightarrows_k M', w'$, then $\chi^k_w \equiv \chi^k_{w'}$; the conclusion then follows from Proposition 5.1.5.

Fix some $(M, s) \in \mathcal{M}$ and assume $M, s \vDash \chi^k_w$. By Proposition 2.2.16, $M, v \vDash \chi^k_w$ for all $v \in s$. By $M, w \leftrightarrows_k M', w'$ and Proposition 2.4.8, $M, v \vDash \chi^k_{w'}$ for all $v \in s$, so that by Proposition 2.2.16, $M, s \vDash \chi^k_{w'}$. (M, s) was arbitrary, so $\chi^k_w \vDash \chi^k_{w'}$. The other direction is similar, so $\chi^k_w \equiv \chi^k_{w'}$. □

We can now show the analogous result for the characteristic formulas of states:

Proof of Proposition 5.1.2. Assume $M, s \leftrightarrows_k M', s'$.

If $s = \varnothing$, then clearly $s' = \varnothing$, so that $\Theta^k_s = \Theta^k_{s'} = \bot$ and so $\Theta^k_s \dashv\vdash \Theta^k_{s'}$.

If $s \neq \varnothing$, then by $M, s \leftrightarrows_k M', s'$ and Proposition 2.4.8 we have that for each $w \in s$ there is a $w' \in s'$ such that $M, w \leftrightarrows_k M', w'$, and for each $w' \in s'$ there is a $w \in s$ such that $M, w \leftrightarrows_k M', w'$. So by Proposition 5.1.6, we have that (†) for each $w \in s$ there is a $w' \in s'$ such that $\chi^k_w \dashv\vdash \chi^k_{w'}$, and for each $w' \in s'$ there is a $w \in s$ such that $\chi^k_w \dashv\vdash \chi^k_{w'}$.

Note that (‡) if $\alpha \dashv\vdash \beta$, then $\alpha \wedge \textsc{ne} \dashv\vdash \beta \wedge \textsc{ne}$ by $\wedge\mathsf{E}$ and $\wedge\mathsf{I}$.

We now prove the following similar claim: (††) if $\alpha \dashv\vdash \beta$, then $\alpha \wedge \textsc{ne} \dashv\vdash (\beta \wedge \textsc{ne}) \vee (\beta \wedge \textsc{ne})$.

For $\alpha \wedge \textsc{ne} \vdash (\beta \wedge \textsc{ne}) \vee (\beta \wedge \textsc{ne})$:

$$\begin{array}{lll} & \alpha \wedge \textsc{ne} & \\ \vdash & \beta \wedge \textsc{ne} & \wedge\mathsf{E}, \alpha \vdash \beta, \wedge\mathsf{I} \\ \vdash & (\beta \wedge \textsc{ne}) \vee (\beta \wedge \textsc{ne}) & \vee\mathsf{W} \end{array}$$

[28]We now suppress mention of the maps $*$ and $**$ and the difference in notation of classical formulas. By what we have shown, our results apply to all variants of the relevant formulas.

For $(\beta \wedge \text{NE}) \vee (\beta \wedge \text{NE}) \vdash \alpha \wedge \text{NE}$:

	$(\beta \wedge \text{NE}) \vee (\beta \wedge \text{NE})$	
$\vdash$	$((\beta \wedge \text{NE}) \vee (\beta \wedge \text{NE})) \wedge \text{NE}$	Prop 4.1.5 (i)
$\vdash$	$\alpha \wedge \text{NE}$	$\wedge\mathsf{E}, \vee\mathsf{E}, \beta \vdash \alpha$

Given this, we now have:

$$\begin{array}{llllll} \Theta^k_s & = & \bigvee_{w \in s} (\chi^k_w \wedge \text{NE}) & & & \\ & \dashv\vdash & \bigvee_{w' \in s'} (\chi^k_{w'} \wedge \text{NE}) & = & \Theta^k_{s'} & \vee\,\mathsf{Sub}, \dagger, \ddagger, \dagger\dagger \end{array}$$

If s and s' are of the same size, we use only † and ‡. If s and s' are of different sizes, we use †, ‡ and ††. □

The final remaining component required for weak completeness is the provable equivalence result. As noted above, we simply assume this for now:

Proposition 5.1.7. (Normal form provable equivalence) For each $\phi \in \mathbf{L}(\Phi)$ and each $n \geq md(\phi)$, there is some finite $\mathcal{F} \subseteq \mathcal{M}(\Phi)$ such that

$$\phi \dashv\vdash \bigvee\!\!\!\bigvee_{(M,s) \in \mathcal{F}} \Theta^n_s.$$

Then:

Theorem 5.1.8. (Weak completeness of $\mathbf{SML}^{\mathbb{W}}$, $\mathbf{SGML}^{\mathbb{W}}$, $\mathbf{BSML}^{\mathbb{W}}$) For any $\phi, \psi \in L$ where L is $\mathbf{SML}^{\mathbb{W}}(\Phi)$, $\mathbf{SGML}^{\mathbb{W}}(\Phi)$ or $\mathbf{BSML}^{\mathbb{W}}(\Phi)$, we have $\phi \vDash \psi \Rightarrow \phi \vdash \psi$. In particular, $\vDash \phi \Rightarrow \vdash \phi$.

Proof. Assume $\phi \vDash \psi$. Let $m := max\{md(\phi), md(\psi)\}$.

By Proposition 5.1.7, we have:

$$(\dagger) \quad \phi \dashv\vdash \bigvee\!\!\!\bigvee_{(M,s) \in \mathcal{F}} \Theta^m_{M,s} \text{ and } \psi \dashv\vdash \bigvee\!\!\!\bigvee_{(M',s') \in \mathcal{G}} \Theta^m_{M',s'}$$

for some finite $\mathcal{F}, \mathcal{G} \subseteq M$.

By †, by $\phi \vDash \psi$, and by the relevant soundness theorem (Theorem 4.2.2, 4.2.4 or 4.3.2), we have:

$$(\ddagger) \quad \bigvee\!\!\!\bigvee_{(M,s) \in \mathcal{F}} \Theta^m_{M,s} \vDash \bigvee\!\!\!\bigvee_{(M',s') \in \mathcal{G}} \Theta^m_{M',s'}$$

Then:

- If $\mathcal{F} = \varnothing$, then by † (recall that $\mathbb{W}\varnothing = \perp\!\!\perp$) $\phi \dashv\vdash \perp\!\!\perp$. Then $\phi \vdash \psi$ by $\perp\!\!\perp$E.
- If $\mathcal{G} = \varnothing$, then $\psi \dashv\vdash \perp\!\!\perp$. By soundness and ‡, we have $\mathbb{W}_{(M,s)\in\mathcal{F}} \Theta^m_s \vDash \perp\!\!\perp$. This implies that $\mathcal{F} = \varnothing$, so that again $\phi \dashv\vdash \perp\!\!\perp$ and therefore $\phi \vdash \psi$.
- If $\mathcal{F}, \mathcal{G} \neq \varnothing$, fix some $(M, s) \in \mathcal{F}$. By ‡ and Proposition 5.1.1 there is some $(M', s') \in \mathcal{G}$ such that $M, s \leftrightarroows_m M', s'$. By Proposition 5.1.2, $\Theta^m_s \dashv\vdash \Theta^m_{s'}$, and so by $\mathbb{w}$I we have $\Theta^m_s \vdash \mathbb{W}_{(M',s')\in\mathcal{G}} \Theta^m_{s'}$. Repeating this argument for each $(M, s) \in \mathcal{F}$, we get $\mathbb{W}_{(M,s)\in\mathcal{F}} \Theta^m_s \vdash \mathbb{W}_{(M',s')\in\mathcal{F}} \Theta^m_{s'}$ by $\mathbb{w}$E, and therefore $\phi \vdash \psi$. □

5.2 Normal Form Provable Equivalence

We first show that all classical formulas are provably equivalent to normal form formulas. In order to do this, we note the following simple fact:

Fact 5.2.1. For any $n \in \mathbb{N}$ and any finite set of n-th Hintikka formulas $\{\chi^n_{M_i,w_i} \mid i \in I\}$, there is some $(M, s) \in \mathcal{M}$ such that $\Theta^n_{M,s} = \bigvee_{i\in I}(\chi^n_{w_i} \wedge \text{NE})$.

Proof. Let $M := \biguplus\{M_i \mid i \in I\}$ (see Definition 2.4.9) and $s := \{w_i \mid i \in I\}$. Then (M, s) is as required by Proposition 2.4.10. □

Proposition 5.2.2. (Normal form provable equivalence for classical formulas) For each $\alpha \in \mathbf{CML}(\Phi)$ and each $n \geq md(\alpha)$, there is some finite $\mathcal{F} \subseteq \mathcal{M}(\Phi)$ such that

$$\alpha \dashv\vdash \mathbb{W}_{(M,s)\in\mathcal{F}} \Theta^n_s.$$

Proof. Let $n \geq md(\alpha)$ and $\mathrm{A} := \{\chi^n_{M,w} \mid M, w \vDash \alpha\}$. By Fact 2.4.7, we can find some finite $\mathrm{X}_\alpha \subseteq \mathrm{A}$ such that for each $\chi^n_w \in \mathrm{A}$, there is some $\chi^n_{w'} \in \mathrm{X}_\alpha$ such that $\chi^n_w \equiv \chi^n_{w'}$.

We show that $\alpha \equiv \bigvee \mathrm{X}_\alpha$.

First assume $M, s \vDash \alpha$. If $s = \varnothing$, then $M, s \vDash \bigvee \mathrm{X}_\alpha$ because $\bigvee \mathrm{X}_\alpha$ has the empty state property by Corollary 2.2.9. If $s \neq \varnothing$, fix $w \in s$. By Proposition 2.2.16, we have $M, w \vDash \alpha$. Then there is a $\chi^n_{w'} \in \mathrm{X}_\alpha$ such that $\chi^n_w \equiv \chi^n_{w'}$. By Proposition 2.4.8 (and $M, w \leftrightarroows_k M, w$) we have $M, w \vDash \chi^n_w$ and so $M, w \vDash \chi^n_{w'}$. w was arbitrary, so for each $w \in s$, we can find a $\chi^n_{w'} \in \chi_\alpha$ such that $M, w \vDash \chi^n_{w'}$—this implies $M, s \vDash \bigvee \mathrm{X}_\alpha$.[29] So either way $M, s \vDash \bigvee \mathrm{X}_\alpha$; therefore $\alpha \vDash \bigvee \mathrm{X}_\alpha$.

[29] To see why this implication holds, it may be helpful to note that each formula in X_α has the empty state property.

Conversely, assume $M, s \vDash \bigvee \mathrm{X}_\alpha$. Then there are $t_1, \dots, t_m \subseteq s$ such that $s = t_1 \cup \dots \cup t_m$; $M, t_1 \vDash \chi^n_{M_1,w_1}, \dots, M, t_m \vDash \chi^n_{M_m,w_m}$; and $\mathrm{X}_\alpha = \{\chi^n_{w_1}, \dots, \chi^n_{w_m}\}$. If $s = \varnothing$, then $M, s \vDash \alpha$ because α has the empty state property by Corollary 2.2.9. If $s \neq \varnothing$, fix some $w \in s$. Then there is some $i \in \{1, \dots, m\}$ such that $w \in t_i$. Since $M, t_i \vDash \chi^n_{w_i}$ we have by Proposition 2.2.16 that $M, w \vDash \chi^n_{w_i}$. Then by Proposition 2.4.8, $M, w \equiv_n M_i, w_i$. Since $\chi^n_{w_i} \in \mathrm{X}_\alpha \subseteq \mathrm{A}$, we have $M_i, w_i \vDash \alpha$. Then by $n \geq md(\alpha)$ and $M, w \equiv_n M_i, w_i$, we have $M, w \vDash \alpha$. Since w was arbitrary, we have $M, s \vDash \alpha$ by Proposition 2.2.16. So either way, $M, s \vDash \alpha$; therefore $\bigvee \mathrm{X}_\alpha \vDash \alpha$.

And so $\alpha \equiv \bigvee \mathrm{X}_\alpha$, and therefore by Proposition 5.1.5, $\alpha \dashv\vdash \bigvee \mathrm{X}_\alpha$.

Then by Proposition 4.1.5 (v):[30]

$$\alpha \dashv\vdash \bigvee \mathrm{X}_\alpha \dashv\vdash \bigvee\!\!\!\!\bigvee_{J \subseteq \mathrm{X}_\alpha} \bigvee_{\chi^n_w \in J} (\chi^n_w \wedge \mathrm{NE})$$

By Fact 5.2.1, we can find for each $J \subseteq \mathrm{X}_\alpha$ some $(M_J, s_J) \in \mathcal{M}$ such that $\Theta^k_{s_J} = \bigvee_{\chi^n_w \in J} (\chi^n_w \wedge \mathrm{NE})$ so that:

$$\alpha \dashv\vdash \bigvee \mathrm{X}_\alpha \dashv\vdash \bigvee\!\!\!\!\bigvee_{J \subseteq \mathrm{X}_\alpha} \Theta^n_{s_J}$$

□

By the foregoing, if we are able to show that some given formula $\phi \in \mathbf{L}$ is provably equivalent to a classical formula, then it is provably equivalent to a formula in normal form. We use this below.

We prove one more lemma before moving on to the main results.

Proposition 5.2.3. For any $k \in \mathbb{N}$ and any $(M, s), (M', s') \in \mathcal{M}$, if $M, s \not\rightleftarrows_k M', s'$, then $\Theta^k_s, \Theta^k_{s'} \vdash \perp\!\!\!\perp$.

Proof. If $M, s \not\rightleftarrows_k M', s'$, then either there is a $w \in s$ such that for all $w' \in s'$: $M, w \not\rightleftarrows_k M', w'$, or there is a $w' \in s'$ such that for all $w \in s : M, w \not\rightleftarrows_k M', w'$. Assume the former with no loss of generality, and fix such a w.

We now show that $\bigvee_{w' \in s'} \chi^k_{w'} \vDash \neg\chi^k_w$. Assume that $N, t \vDash \bigvee_{w' \in s'} \chi^k_{w'}$. If $t = \varnothing$, then $N, t \vDash \neg\chi^k_w$. If $t \neq \varnothing$, let $v \in t$, and assume for contradiction that $N, v \vDash \chi^k_w$. By $N, t \vDash \bigvee_{w' \in s'} \chi^k_{w'}$, there is some $w' \in s'$ and some $u \subseteq t$ such that $N, u \vDash \chi^k_{w'}$ and $v \in u$. Then by Proposition 2.2.16, $N, v \vDash \chi^k_{w'}$. By Proposition 2.4.8, $N, v \vDash \chi^k_w$ gives us $N, v \rightleftarrows_k M, w$, and $N, v \vDash \chi^k_{w'}$ gives us $N, v \rightleftarrows_k M', w'$. Then by Fact 3.1.4, $M, w \rightleftarrows_k M', w'$, a contradiction. So

[30] Note that in the special case in which $\alpha \equiv \bot$, we have $A = \varnothing$ so $\bigvee \mathrm{X}_\alpha = \bigvee \varnothing = \bot$ and $\bigvee\!\!\!\!\bigvee_{J \subseteq \mathrm{X}_\alpha} \bigvee_{\chi^n_w \in J} (\chi^n_w \wedge \mathrm{NE}) = \bigvee\!\!\!\!\bigvee \{\bigvee \varnothing\} = \bigvee\!\!\!\!\bigvee \{\bot\} = \bot$.

$N, v \not\models \chi^k_w$. Since v was arbitrary, we have $N, t \models \neg\chi^k_w$. Either way, then, $N, t \models \neg\chi^k_w$, and so $\bigvee_{w' \in s'} \chi^k_{w'} \models \neg\chi^k_w$.[31]

Therefore by Proposition 5.1.5, $\bigvee_{w' \in s'} \chi^k_{w'} \vdash \neg\chi^k_w$ (†). And so:

$$\begin{array}{lcll}
\Theta^k_s \wedge \Theta^k_{s'} & = & \Theta^k_s \wedge \bigvee_{w' \in s'} (\chi^k_{w'} \wedge \text{NE}) & \\
& \vdash & \Theta^k_s \wedge \bigvee_{w' \in s'} \chi^k_{w'} & \\
& \vdash & \Theta^k_s \wedge \neg\chi^k_w & \dagger \\
& \vdash & ((\chi^k_w \wedge \text{NE}) \vee \Theta^k_{s \setminus \{w\}}) \wedge \neg\chi^k_w & \\
& \vdash & \perp\!\!\!\perp & \text{Prop 4.1.5 (vi)}
\end{array}$$

□

Then:[32]

Proposition 5.2.4. For each $\phi \in \mathbf{SML}^{\mathbb{W}}(\Phi)$ and each $n \geq md(\phi)$, there is some finite $\mathcal{F} \subseteq \mathcal{M}(\Phi)$ such that

$$\phi \dashv\vdash_{\mathbf{SML}^{\mathbb{W}}} \mathbb{W}_{(M,s) \in \mathcal{F}} \Theta^n_s.$$

Proof. We prove the result by induction on the complexity of ϕ. Let $n \geq md(\phi)$. Note that in the non-modal cases we only make use of the rules of $\mathbf{PT}^+$ and Proposition 5.1.5.

- $\phi = p$. The result follows from Proposition 5.2.2.
- $\phi = \text{NE}$.

$$\begin{array}{lcll}
\text{NE} & \dashv\vdash & \text{NE} \wedge (p \vee \neg p) & \text{Prop 5.1.5} \\
& \dashv\vdash & \text{NE} \wedge \mathbb{W}_{(M,s) \in \mathcal{F}} \Theta^n_s & \text{Prop 5.2.2} \\
& \dashv\vdash & \text{NE} \wedge (\perp \mathbb{W} \mathbb{W}_{(M,s) \in \mathcal{G} \subseteq \mathcal{F}} \Theta^n_s) & \dagger \\
& \dashv\vdash & (\text{NE} \wedge \perp) \mathbb{W} \mathbb{W}_{(M,s) \in \mathcal{G}} \Theta^n_s & \text{Distr} \wedge \mathbb{W} \\
& \dashv\vdash & \mathbb{W}_{(M,s) \in \mathcal{G}} \Theta^n_s & \perp\!\!\!\perp \text{E}, \mathbb{W}\text{E}, \mathbb{W}\text{I}
\end{array}$$

For † in the above, $\mathcal{G} = \mathcal{F} \setminus \{(M,s) \in \mathcal{F} \mid s = \varnothing\}$. Note that clearly $\mathcal{G} \neq \varnothing$, since $\top\!\!\!\top = p \vee \neg p \equiv \mathbb{W}_{(M,s) \in \mathcal{F}} \Theta^n_s$.

[31] Note that the proof of this claim used the semantics of $\neg$, but the claim now also implies $(\bigvee_{w' \in s'} \chi^k_{w'})^{**} \models {-}(\chi^k_w)^{**}$ by what we have proved before.

[32] We make use of the derivable distributivity laws in these proofs. Recall that these can be found in Proposition 4.1.4.

- $\phi = \neg\alpha$. The result follows from Proposition 5.2.2.

For the induction cases involving subformulas ψ and χ of ϕ, since $n \geq md(\phi) \geq md(\psi), md(\chi)$, we have by the induction hypothesis that there are some finite $\mathcal{H}, \mathcal{I} \subseteq \mathcal{M}$ such that $\psi \dashv\vdash \mathbb{W}_{(M,s)\in\mathcal{H}} \Theta^n_s$ and $\chi \dashv\vdash \mathbb{W}_{(M',s')\in\mathcal{I}} \Theta^n_{s'}$.

- $\phi = \psi \wedge \chi$. We have:

$$\begin{aligned}
\psi \wedge \chi \quad &\dashv\vdash \quad \mathop{\mathbb{W}}_{(M,s)\in\mathcal{H}} \Theta^n_s \wedge \mathop{\mathbb{W}}_{(M',s')\in\mathcal{I}} \Theta^n_{s'} \\
&\dashv\vdash \quad \mathop{\mathbb{W}}_{(M,s)\in\mathcal{H}} \mathop{\mathbb{W}}_{(M',s')\in\mathcal{I}} (\Theta^n_s \wedge \Theta^n_{s'}) && \mathsf{Distr} \wedge \mathbb{W} \\
&\dashv\vdash \quad \mathop{\mathbb{W}}_{(M,s)\in\mathcal{H}\cap\mathcal{I}} \Theta^n_s \mathbin{\mathbb{W}} \mathop{\mathbb{W}}_{(M,s)\in(\mathcal{H}\cup\mathcal{I})\setminus(\mathcal{H}\cap\mathcal{I})} \perp\!\!\!\perp && \text{Prop 5.2.3}, \perp\!\!\!\perp \mathsf{E} \\
&\dashv\vdash \quad \mathop{\mathbb{W}}_{(M,s)\in\mathcal{H}\cap\mathcal{I}} \Theta^n_s && \mathbb{W}\mathsf{E}, \perp\!\!\!\perp \mathsf{E}, \mathbb{W}\mathsf{I}
\end{aligned}$$

  Note here that if $\mathcal{H} \cap \mathcal{I} = \varnothing$, then $\psi \wedge \chi \dashv\vdash \mathbb{W}\varnothing = \perp\!\!\!\perp$; $\perp\!\!\!\perp$ is also a formula in normal form.

- $\phi = \psi \vee \chi$. We have:

$$\begin{aligned}
&\psi \vee \chi \\
\dashv\vdash\ & \mathop{\mathbb{W}}_{(M,s)\in\mathcal{H}} \Theta^n_s \vee \mathop{\mathbb{W}}_{(M',s')\in\mathcal{I}} \Theta^n_{s'} \\
\dashv\vdash\ & \mathop{\mathbb{W}}_{(M,s)\in\mathcal{H}} \mathop{\mathbb{W}}_{(M',s')\in\mathcal{I}} (\Theta^n_s \vee \Theta^n_{s'}) && \mathsf{Distr} \vee \mathbb{W} \\
\dashv\vdash\ & \mathop{\mathbb{W}}_{(M,s)\in\mathcal{H}} \mathop{\mathbb{W}}_{(M',s')\in\mathcal{I}} \Big(\bigvee_{w^*\in t} (\chi^n_{w^*} \wedge \textsc{ne}) \vee \bigvee_{w^*\in u} (\chi^n_{w^*} \wedge \textsc{ne}) \vee \bigvee_{w^*\in u} (\chi^n_{w^*} \wedge \textsc{ne}) \Big) && \dagger \\
\dashv\vdash\ & \mathop{\mathbb{W}}_{(M,s)\in\mathcal{H}} \mathop{\mathbb{W}}_{(M',s')\in\mathcal{I}} \Big(\bigvee_{w^*\in t} (\chi^n_{w^*} \wedge \textsc{ne}) \vee \bigvee_{w^*\in u} ((\chi^n_{w^*} \wedge \textsc{ne}) \vee (\chi^n_{w^*} \wedge \textsc{ne})) \Big) \\
\dashv\vdash\ & \mathop{\mathbb{W}}_{(M,s)\in\mathcal{H}} \mathop{\mathbb{W}}_{(M',s')\in\mathcal{I}} \Big(\bigvee_{w^*\in t} (\chi^n_{w^*} \wedge \textsc{ne}) \vee \bigvee_{w^*\in u} (\chi^n_{w^*} \wedge (\textsc{ne} \vee \textsc{ne})) \Big) && \mathsf{Distr}{*} \wedge \vee \\
\dashv\vdash\ & \mathop{\mathbb{W}}_{(M,s)\in\mathcal{H}} \mathop{\mathbb{W}}_{(M',s')\in\mathcal{I}} \Big(\bigvee_{w^*\in t} (\chi^n_{w^*} \wedge \textsc{ne}) \vee \bigvee_{w^*\in u} (\chi^n_{w^*} \wedge \textsc{ne}) \Big) && \textsc{ne}\mathsf{Ctr}, \vee\mathsf{W} \\
\dashv\vdash\ & \mathop{\mathbb{W}}_{(M,s)\in\mathcal{H}} \mathop{\mathbb{W}}_{(M',s')\in\mathcal{I}} \bigvee_{w^*\in t\cup u} (\chi^n_{w^*} \wedge \textsc{ne})
\end{aligned}$$

  For †: $u = \{w^* \in s \cap s' \mid (M, w^*) = (M', w^*)\}$ and $t = (s \cup s')\setminus u$.

  By Fact 5.2.1, for each $(M, s) \in \mathcal{H}$ and each $(M', s') \in \mathcal{I}$, there is some $(M^*, s^*) \in \mathcal{M}$ such that $\Theta^n_{s^*} = \bigvee_{w^*\in t\cup u}(\chi^n_{w^*} \wedge \textsc{ne})$, so we are done.

- $\phi = \psi \mathbin{\mathbb{W}} \chi$. We are immediately done by the induction hypothesis.

For the induction cases where $\phi = \Diamond\psi$ or $\phi = \Box\psi$, since $n - 1 \geq md(\phi) - 1 \geq md(\psi) + 1 - 1 = md(\psi)$, by the induction hypothesis there is some finite $\mathcal{F} \subseteq \mathcal{M}$ such that $\psi \dashv\vdash \mathbb{W}_{(M,s)\in\mathcal{F}} \Theta_s^{n-1}$. Denote $k = n - 1$ so that $\psi \dashv\vdash \mathbb{W}_{(M,s)\in\mathcal{F}} \Theta_s^k$.

- $\phi = \Diamond\psi$.

$$
\begin{array}{lll}
\Diamond \psi & \dashv\vdash & \Diamond \mathbb{W}_{(M,s)\in\mathcal{F}} \Theta_s^k \\
& \dashv\vdash & \bigvee_{(M,s)\in\mathcal{F}} \Diamond \Theta_s^k \qquad \textsf{Conv} \Diamond \mathbb{W}\vee \\
& \dashv\vdash & \bigvee_{(M,s)\in\mathcal{F}\setminus\mathcal{J}} \Diamond \Theta_s^k \vee \bigvee_{\mathcal{J}} \Diamond \bot \qquad \dagger \\
& \dashv\vdash & \bigvee_{(M,s)\in\mathcal{F}\setminus\mathcal{J}} \bigwedge_{w\in s} \Diamond \chi_w^k \vee \bigvee_{\mathcal{J}} \Diamond \bot \qquad \ddagger
\end{array}
$$

For $\dagger$, $\mathcal{J} = \{(M,s) \in \mathcal{F} \mid s = \varnothing\}$.

The final formula is classical and its modal depth is $\leq k + 1 = n$, so if we show $\ddagger$, we are done by Proposition 5.2.2 (applied with respect to n).

For $\ddagger$, it suffices to show

$$\Diamond\Theta_s^k \dashv\vdash \bigwedge_{w\in s} \Diamond\chi_w^k$$

for each $(M,s) \in \mathcal{M}$ (where $s \neq \varnothing$). We do so by induction on the size m of s.

  - $m = 1$. Let $s = \{v\}$. Then $\bigwedge_{w\in s} \Diamond\chi_w^k = \Diamond\chi_v^k$ and we have:

$$
\begin{array}{lll}
\Diamond \Theta_s^k & = & \Diamond (\chi_v^k \wedge \text{NE}) \\
& \vdash & \Diamond ((\chi_v^k \wedge \text{NE}) \vee \bot) \qquad \vee\textsf{I}, \Diamond\textsf{Mon} \\
& \vdash & \Diamond \chi_v^k \qquad \Diamond\textsf{Sep}
\end{array}
$$

    and $\Diamond\chi_v^k \vdash \Diamond(\chi_v^k \wedge \text{NE}) = \Diamond\Theta_s^k$ by $\Diamond\text{NE}\textsf{I}$.

  - $m + 1$. By the induction hypothesis we can choose some $v \in s$ such that

$$(\dagger\dagger) \qquad \Diamond \Theta_{s\setminus\{v\}}^k \dashv\vdash \bigwedge_{w\in s\setminus\{v\}} \Diamond \chi_w^k$$

so

$$\begin{array}{lll}
& \Diamond\, \Theta^k_s & \\
\dashv\vdash & \Diamond\, (\bigvee_{w\in s\setminus\{v\}} (\chi^k_w \wedge \text{NE}) \vee (\chi^k_v \wedge \text{NE})) & \\
\dashv\vdash & \Diamond\, ((\bigvee_{w\in s\setminus\{v\}} (\chi^k_w \wedge \text{NE}) \wedge \text{NE}) \vee (\chi^k_v \wedge \text{NE})) & \text{Prop 4.1.5 (i)} \\
\dashv\vdash & \Diamond \bigvee_{w\in s\setminus\{v\}} (\chi^k_w \wedge \text{NE}) \wedge \Diamond\chi^k_v & \mathsf{FC} \\
\dashv\vdash & \bigwedge_{w\in s\setminus\{v\}} \Diamond\, \chi^k_w \wedge \Diamond\chi^k_v & \dagger\dagger \\
\dashv\vdash & \bigwedge_{w\in s} \Diamond\, \chi^k_w &
\end{array}$$

- $\phi = \Box\psi$.

$$\begin{array}{llll}
\Box\,\psi & \dashv\vdash & \Box \bigvee\!\!\!\bigvee_{(M,s)\in\mathcal{F}} \Theta^k_s & \\
& \dashv\vdash & \bigvee_{(M,s)\in\mathcal{F}} \Box\,\Theta^k_s & \mathsf{Conv}\,\Box\,\bigvee\!\!\!\bigvee\vee \\
& \dashv\vdash & \bigvee_{(M,s)\in\mathcal{F}\setminus\mathcal{J}} \Box\,\Theta^k_s \vee \bigvee_{\mathcal{J}} \Box\,\bot & \dagger \\
& \dashv\vdash & \bigvee_{(M,s)\in\mathcal{F}\setminus\mathcal{J}} (\Box \bigvee_{w\in s} \chi^k_w \wedge \bigwedge_{w\in s} \Diamond\, \chi^k_w) \vee \bigvee_{\mathcal{J}} \Box\,\bot & \ddagger
\end{array}$$

For †, $\mathcal{J} = \{(M, s) \in \mathcal{F} \mid s = \varnothing\}$.

The final formula is classical and its modal depth is $\le k + 1 = n$, so if we show ‡, we are done by Proposition 5.2.2.

For ‡, it suffices to show

$$\Box\Theta^k_s \dashv\vdash \Box \bigvee_{w\in s} \chi^k_w \wedge \bigwedge_{w\in s} \Diamond\chi^k_w$$

for each $(M, s) \in \mathcal{M}$ (where $s \neq \varnothing$).

If s is of size 1, let $s = \{v\}$. Then

$$\Box \bigvee_{w\in s} \chi^k_w \wedge \bigwedge_{w\in s} \Diamond\chi^k_w = \Box\chi^k_v \wedge \Diamond\chi^k_v.$$

We have:

$$\begin{array}{llll}
\Box\,\Theta^k_s & = & \Box\,(\chi^k_v \wedge \text{NE}) & \\
& \dashv\vdash & \Box\,(\bot \vee (\chi^k_v \wedge \text{NE})) & \vee\mathsf{I}, \bot\mathsf{E}, \Box\mathsf{Mon} \\
& \dashv\vdash & \Box\,(\bot \vee \chi^k_v) \wedge \Diamond\chi^k_v & \Box\mathsf{FC} \\
& \dashv\vdash & \Box\,\chi^k_v \wedge \Diamond\chi^k_v & \vee\mathsf{I}, \bot\mathsf{E}, \Box\mathsf{Mon}
\end{array}$$

Otherwise, let $\Theta^k_s = (\chi^k_{w_1} \wedge \text{NE}) \vee \ldots \vee (\chi^k_{w_m} \wedge \text{NE})$ where $m \geq 2$. We have:

$$
\begin{array}{lll}
 & \Box\, \Theta^k_s & \\
= & \Box \bigvee_{w\in s} (\chi^k_w \wedge \text{NE}) & \\
\dashv\vdash & \Box\, (\bigvee_{i=1}^{m-1} (\chi^k_{w_i} \wedge \text{NE}) \vee \chi^k_{w_m}) \wedge \Diamond \chi^k_{w_m} & \Box\mathsf{FC} \\
\dashv\vdash & \Box\, (\bigvee_{i=1}^{m-2} (\chi^k_{w_i} \wedge \text{NE}) \vee \chi^k_{w_m} \vee (\chi^k_{w_{m-1}} \wedge \text{NE})) \wedge \Diamond \chi^k_{w_m} & \vee\mathsf{Com} \\
\dashv\vdash & \Box\, (\bigvee_{i=1}^{m-2} (\chi^k_{w_i} \wedge \text{NE}) \vee \chi^k_{w_m} \vee \chi^k_{w_{m-1}}) \wedge \Diamond \chi^k_{w_{m-1}} \wedge \Diamond \chi^k_{w_m} & \Box\mathsf{FC} \\
 & \vdots & \\
\dashv\vdash & \Box\, ((\chi^k_{w_1} \wedge \text{NE}) \vee \bigvee_{w\in s\setminus\{w_1\}} \chi^k_w) \wedge \bigwedge_{w\in s\setminus\{w_1\}} \Diamond\, \chi^k_w & \Box\mathsf{FC} \\
\dashv\vdash & \Box \bigvee_{w\in s} \chi^k_w \wedge \bigwedge_{w\in s} \Diamond \chi^k_w & \Box\mathsf{FC}
\end{array}
$$

□

Proposition 5.2.5. For each $\phi \in \mathbf{SGML}^{\mathbb{W}}(\Phi)$ and each $n \geq md(\phi)$, there is some finite $\mathcal{F} \subseteq \mathcal{M}(\Phi)$ such that

$$\phi \dashv\vdash_{\mathbf{SGML}^{\mathbb{W}}} \mathbb{W}_{(M,s)\in\mathcal{F}} \Theta^n_s.$$

Proof. By induction on the complexity of ϕ. Let $n \geq md(\phi)$. Most cases are exactly as in the proof of Proposition 5.2.4. We show the remaining cases.

These are the cases $\phi = \diamondsuit\psi$ and $\phi = \boxdot\psi$. Since $n - 1 \geq md(\phi) - 1 \geq md(\psi) + 1 - 1 = md(\psi)$, by the induction hypothesis there is some finite $\mathcal{F} \subseteq \mathcal{M}$ such that $\psi \dashv\vdash \mathbb{W}_{(M,s)\in\mathcal{F}} \Theta^{n-1}_s$. Denote $k = n - 1$ so that $\psi \dashv\vdash \mathbb{W}_{(M,s)\in\mathcal{F}} \Theta^k_s$.

First consider the case in which $\phi = \diamondsuit\psi$. We have:

$$
\begin{array}{llll}
\diamondsuit\, \psi & \dashv\vdash & \diamondsuit \mathbb{W}_{(M,s)\in\mathcal{F}} \bigvee_{w\in s} (\chi^k \wedge \text{NE}) & \\
 & \dashv\vdash & \mathbb{W}_{(M,s)\in\mathcal{F}} \diamondsuit \bigvee_{w\in s} (\chi^k \wedge \text{NE}) & \mathsf{Distr}\,\diamondsuit\,\mathbb{W} \\
 & \dashv\vdash & \mathbb{W}_{(M,s)\in\mathcal{F}} \bigvee_{w\in s} \diamondsuit\, (\chi^k_w \wedge \text{NE}) & \mathsf{Distr}\,\diamondsuit\,\vee \\
 & \dashv\vdash & \mathbb{W}_{(M,s)\in\mathcal{F}} \bigvee_{w\in s} (\diamondsuit \chi^k_w \wedge \text{NE}) & \mathsf{Distr}\text{NE}\diamondsuit
\end{array}
$$

We now fix a $(M, s) \in \mathcal{F}$ and show that $\bigvee_{w \in s}(\Diamond\chi_w^k \wedge \text{NE})$ is provably equivalent to a normal form formula of degree $k+1$. If this is the case, the formula above is provably equivalent to a normal form formula of degree $k+1 = n$ by the induction case for $\varovee$.

$$
\begin{array}{lll}
 & \bigvee\limits_{w \in s}(\Diamond\chi_w^k \wedge \text{NE}) & \\
\dashv\vdash & \bigvee\limits_{w \in s}(\bigvee\!\!\!\!\bigvee\limits_{(M',s') \in \mathcal{A}_w} \Theta_{s'}^{k+1} \wedge \text{NE}) & \text{Prop 5.2.2} \\
\dashv\vdash & \bigvee\limits_{w \in s}((\bigvee\!\!\!\!\bigvee\limits_{(M',s') \in \mathcal{A}_w \setminus \mathcal{D}_w} \Theta_{s'}^{k+1} \varovee \bigvee\!\!\!\!\bigvee\limits_{\mathcal{D}_w} \bot) \wedge \text{NE}) & \dagger \\
\dashv\vdash & \bigvee\limits_{w \in s}(\bigvee\!\!\!\!\bigvee\limits_{(M',s') \in \mathcal{A}_w \setminus \mathcal{D}_w} (\Theta_{s'}^{k+1} \wedge \text{NE}) \varovee \bigvee\!\!\!\!\bigvee\limits_{\mathcal{D}_w} \bot\!\!\!\bot) & \mathsf{Distr} \wedge \varovee \\
\dashv\vdash & \bigvee\limits_{w \in s}(\bigvee\!\!\!\!\bigvee\limits_{(M',s') \in \mathcal{A}_w \setminus \mathcal{D}_w} \Theta_{s'}^{k+1} \varovee \bigvee\!\!\!\!\bigvee\limits_{\mathcal{D}_w} \bot\!\!\!\bot) & \text{Prop 4.1.5 (i)}, \bot\!\!\!\bot\,\mathsf{E}
\end{array}
$$

For $\dagger$, $\mathcal{D}_w = \{(M', s') \in \mathcal{A}_w \mid s = \varnothing\}$.

By the induction cases for $\varovee$ and $\vee$, this is provably equivalent to a normal form formula of degree $k+1$ (recall that $\bot\!\!\!\bot$ is in normal form).

Now consider the case in which $\phi = \boxdot\psi$. We have:

$$
\begin{array}{lll}
 & \boxdot\,\psi & \\
\dashv\vdash & \boxdot \bigvee\!\!\!\!\bigvee\limits_{(M,s) \in \mathcal{F}} \Theta_s^k & \\
\dashv\vdash & \bigvee\!\!\!\!\bigvee\limits_{(M,s) \in \mathcal{F}} \boxdot\,\Theta_s^k & \mathsf{Distr} \boxdot \varovee \\
\dashv\vdash & \bigvee\!\!\!\!\bigvee\limits_{(M,s) \in \mathcal{F}} (\boxdot \bigvee\limits_{w \in s} \chi_s^k \wedge (\bigvee\limits_{w \in s}(\Diamond\chi_w^k \wedge \text{NE}) \vee \boxdot\bot)) & \dagger \\
\dashv\vdash & \bigvee\!\!\!\!\bigvee\limits_{(M,s) \in \mathcal{F}} (\bigvee\!\!\!\!\bigvee\limits_{(M',s') \in \mathcal{A}_s} \Theta_{s'}^{k+1} \wedge (\bigvee\limits_{w \in s}(\Diamond\chi_w^k \wedge \text{NE}) \vee \bigvee\!\!\!\!\bigvee\limits_{(M',s') \in \mathcal{B}} \Theta_{s'}^1)) & \text{Prop 5.2.2}
\end{array}
$$

By what we showed in the induction case for $\Diamond$, for every $(M, s) \in \mathcal{F}$ the formula $\bigvee_{w \in s}(\Diamond\chi_w^k \wedge \text{NE})$ is provably equivalent to a normal form formula of degree $k+1$. The final formula above is therefore provably equivalent to a normal form formula of degree $k+1$ by the induction cases for $\wedge$, $\vee$ and $\varovee$.

For $\dagger$, it suffices to show that for each $(M, s) \in \mathcal{M}$:

$$\boxdot\Theta_s^k \dashv\vdash \boxdot \bigvee_{w \in s} \chi_w^k \wedge (\bigvee_{w \in s}(\Diamond\chi_w^k \wedge \text{NE}) \vee \boxdot\bot)$$

If s is of size 0, then:

$$
\begin{array}{lll}
 & \boxdot \bigvee_{w\in s} \chi_w^k \wedge (\bigvee_{w\in s} (\Diamonddot \chi_w^k \wedge \text{NE}) \vee \boxdot\bot) & \\
= & \boxdot \bigvee \varnothing \wedge (\bigvee \varnothing \vee \boxdot\bot) & \\
= & \boxdot \bot \wedge (\bot \vee \boxdot\bot) & \\
\dashv\vdash & \boxdot \bot & \vee\mathsf{I}, \bot\mathsf{E} \\
= & \boxdot \Theta_s^k &
\end{array}
$$

If s is of size 1, let $s = \{v\}$. Then:

$$
\begin{array}{llll}
\boxdot \Theta_s^k & = & \boxdot (\chi_v^k \wedge \text{NE}) & \\
 & \dashv\vdash & \boxdot ((\chi_v^k \wedge \text{NE}) \vee \bot) & \vee\mathsf{I}, \bot\mathsf{E}, \boxdot\mathsf{Mon} \\
 & \dashv\vdash & \boxdot (\chi_v^k \vee \bot) \wedge ((\Diamonddot \chi_v^k \wedge \text{NE}) \vee \Diamonddot\bot \vee \boxdot\bot) & \boxdot\mathsf{C} \\
 & \dashv\vdash & \boxdot (\chi_v^k \vee \bot) \wedge ((\Diamonddot \chi_v^k \wedge \text{NE}) \vee \bot \vee \boxdot\bot) & \text{Prop 5.1.5} \\
 & \dashv\vdash & \boxdot \chi_v^k \wedge ((\Diamonddot \chi_v^k \wedge \text{NE}) \vee \boxdot\bot) & \vee\mathsf{I}, \bot\mathsf{E}, \boxdot\mathsf{Mon} \\
 & = & \boxdot \bigvee_{w\in s} \chi_w^k \wedge (\bigvee_{w\in s} (\Diamonddot \chi_w^k \wedge \text{NE}) \vee \boxdot\bot) &
\end{array}
$$

Otherwise, let $\Theta_s^k = (\chi_{w_1}^k \wedge \text{NE}) \vee \ldots \vee (\chi_{w_m}^k \wedge \text{NE})$ where $m \geq 2$. We have:

$$
\begin{array}{lll}
 & \boxdot \Theta_s^k & \\
= & \boxdot \bigvee_{w\in s} (\chi_w^k \wedge \text{NE}) & \\
\dashv\vdash & \boxdot \bigvee_{i=1}^{m-1} (\chi_{w_i}^k \wedge \text{NE}) \vee \chi_{w_m}^k) \wedge & \\
 & ((\Diamonddot \chi_{w_m}^k \wedge \text{NE}) \vee \Diamonddot \bigvee_{i=1}^{m-1} (\chi_{w_i}^k \wedge \text{NE}) \vee \boxdot\bot) & \boxdot\mathsf{C} \\
\dashv\vdash & \boxdot (\bigvee_{i=1}^{m-1} (\chi_{w_i}^k \wedge \text{NE}) \vee \chi_{w_m}^k) \wedge & \\
 & ((\Diamonddot \chi_{w_m}^k \wedge \text{NE}) \vee \bigvee_{i=1}^{m-1} (\Diamonddot \chi_{w_i}^k \wedge \text{NE}) \vee \boxdot\bot) & \ddagger \\
\dashv\vdash & \boxdot (\bigvee_{i=1}^{m-1} (\chi_{w_i}^k \wedge \text{NE}) \vee \chi_{w_m}^k) \wedge (\bigvee_{w\in s} (\Diamonddot \chi_w^k \wedge \text{NE}) \vee \boxdot\bot) &
\end{array}
$$

‡ follows by $\mathsf{Distr}\,\Diamonddot\vee$ and $\mathsf{Distr}\text{NE}\Diamonddot$ (see the induction case for $\Diamonddot$).

Taking the first conjunct of the formula in the last line above, we have

(††):

$$
\begin{aligned}
& \boxdot\Big(\bigvee_{i=1}^{m-1}(\chi^k_{w_i}\wedge \mathrm{NE})\vee \chi^k_{w_m}\Big) && \\
\dashv\vdash\quad & \boxdot\Big(\bigvee_{i=1}^{m-2}(\chi^k_{w_i}\wedge \mathrm{NE})\vee (\chi^k_{w_{m-1}}\wedge \mathrm{NE})\vee \chi^k_{w_m}\Big) && \\
\dashv\vdash\quad & \boxdot\Big(\bigvee_{i=1}^{m-2}(\chi^k_{w_i}\wedge \mathrm{NE})\vee \chi^k_{w_m}\vee (\chi^k_{w_{m-1}}\wedge \mathrm{NE})\Big) && \mathsf{Com}\vee \\
\dashv\vdash\quad & \boxdot\Big(\bigvee_{i=1}^{m-2}(\chi^k_{w_i}\wedge \mathrm{NE})\vee \chi^k_{w_m}\vee \chi^k_{w_{m-1}}\Big)\wedge\big((\Diamond\chi^k_{w_{m-1}}\wedge \mathrm{NE})\vee && \\
& \Diamond\Big(\bigvee_{i=1}^{m-2}(\chi^k_{w_i}\wedge \mathrm{NE})\vee \chi^k_{w_m}\Big)\vee\boxdot\bot\big) && \boxdot\mathsf{C} \\
\dashv\vdash\quad & \boxdot\Big(\bigvee_{i=1}^{m-2}(\chi^k_{w_i}\wedge \mathrm{NE})\vee \chi^k_{w_m}\vee \chi^k_{w_{m-1}}\Big)\wedge\big((\Diamond\chi^k_{w_{m-1}}\wedge \mathrm{NE})\vee && \\
& \Diamond\bigvee_{i=1}^{m-2}(\chi^k_{w_i}\wedge \mathrm{NE})\vee \Diamond\chi^k_{w_m}\vee\boxdot\bot\big) && \mathsf{Distr}\,\Diamond\vee \\
\dashv\vdash\quad & \boxdot\Big(\bigvee_{i=1}^{m-2}(\chi^k_{w_i}\wedge \mathrm{NE})\vee \chi^k_{w_m}\vee \chi^k_{w_{m-1}}\Big)\wedge\big((\Diamond\chi^k_{w_{m-1}}\wedge \mathrm{NE})\vee && \\
& \bigvee_{i=1}^{m-2}(\Diamond\chi^k_{w_i}\wedge \mathrm{NE})\vee \Diamond\chi^k_{w_m}\vee\boxdot\bot\big) && \ddagger \\
\dashv\vdash\quad & \boxdot\Big(\bigvee_{i=1}^{m-2}(\chi^k_{w_i}\wedge \mathrm{NE})\vee \chi^k_{w_m}\vee \chi^k_{w_{m-1}}\Big)\wedge && \\
& \Big(\bigvee_{i=1}^{m-1}(\Diamond\chi^k_{w_i}\wedge \mathrm{NE})\vee \Diamond\chi^k_{w_m}\vee\boxdot\bot\Big) &&
\end{aligned}
$$

‡ again follows by $\mathsf{Distr}\,\Diamond\vee$ and $\mathsf{Distr}\mathrm{NE}\Diamond$ (see the induction case for $\Diamond$).

Now note that (‡‡)

$$\bigvee_{w\in s}(\Diamond\chi^k_w\wedge \mathrm{NE})\vee\boxdot\bot\vdash\bigvee_{i=1}^{m-1}(\Diamond\chi^k_{w_i}\wedge \mathrm{NE})\vee\Diamond\chi^k_{w_m}\vee\boxdot\bot$$

by $\vee\mathsf{Sub}$ and $\wedge\mathsf{E}$.

Therefore, returning to what is provably equivalent to $\boxdot\Theta^k_s$, we have:

$$
\begin{aligned}
& \boxdot\big(\bigvee_{i=1}^{m-1}(\chi^k_{w_i}\wedge \text{NE})\vee\chi^k_{w_m}\big)\wedge\big(\bigvee_{w\in s}(\diamondsuit\chi^k_w\wedge \text{NE})\vee\boxdot\bot\big) \\
\dashv\vdash\quad & \boxdot\big(\bigvee_{i=1}^{m-2}(\chi^k_{w_i}\wedge \text{NE})\vee\chi^k_{w_{m-1}}\vee\chi^k_{w_m}\big)\wedge\big(\bigvee_{w\in s}(\diamondsuit\chi^k_w\wedge \text{NE})\vee\boxdot\bot\big) \\
& \wedge\big(\bigvee_{i=1}^{m-1}(\diamondsuit\chi^k_{w_i}\wedge \text{NE})\vee\diamondsuit\chi^k_{w_m}\vee\boxdot\bot\big) \qquad \dagger\dagger \\
\dashv\vdash\quad & \boxdot\big(\bigvee_{i=1}^{m-2}(\chi^k_{w_i}\wedge \text{NE})\vee\chi^k_{w_{m-1}}\vee\chi^k_{w_m}\big)\wedge\big(\bigvee_{w\in s}(\diamondsuit\chi^k_w\wedge \text{NE})\vee\boxdot\bot\big) \qquad \wedge\mathsf{E},\ddagger\ddagger
\end{aligned}
$$

Iterating this process, we finally get that the following is provably equivalent to $\boxdot\Theta^k_s$:

$$
\begin{aligned}
& \boxdot\big((\chi^k_{w_1}\wedge \text{NE})\vee\bigvee_{w\in s\setminus\{w_1\}}\chi^k_w\big)\wedge\big(\bigvee_{w\in s}(\diamondsuit\chi^k_w\wedge \text{NE})\vee\boxdot\bot\big) \\
\dashv\vdash\quad & \boxdot\bigvee_{w\in s}\chi^k_w\wedge\big((\diamondsuit\chi^k_{w_1}\wedge \text{NE})\vee\diamondsuit\bigvee_{w\in s\setminus\{w_1\}}\chi^k_w\vee\boxdot\bot\big) \\
& \wedge\big(\bigvee_{w\in s}(\diamondsuit\chi^k_w\wedge \text{NE})\vee\boxdot\bot\big) \qquad \boxdot\mathsf{C} \\
\dashv\vdash\quad & \boxdot\bigvee_{w\in s}\chi^k_w\wedge\big(\bigvee_{w\in s}(\diamondsuit\chi^k_w\wedge \text{NE})\vee\boxdot\bot\big)
\end{aligned}
$$

(For the final step we again perform part of the iteration—as with ‡‡ we have

$$\bigvee_{w\in s}(\diamondsuit\chi^k_w\wedge \text{NE})\vee\boxdot\bot\vdash(\diamondsuit\chi^k_{w_1}\wedge \text{NE})\vee\diamondsuit\bigvee_{w\in s\setminus\{w_1\}}\chi^k_w\vee\boxdot\bot$$

and then the final line follows analogously to how we used ‡‡.)

We now have what was required for †, so we are done. □

For $\mathbf{BSML}^{\mathbb{W}}$, the result follows by the proof of Proposition 5.2.4 once we show that any formula is provably equivalent to a formula in negation normal form (see Fact 2.2.6).

Proposition 5.2.6. Each $\phi\in\mathbf{BSML}^{\mathbb{W}}(\Phi)$ is provably equivalent to a formula in negation normal form for $\mathbf{BSML}^{\mathbb{W}}$.

Proof. By induction on the complexity of ϕ.

- $\phi = p$ or $\phi = \text{NE}$. ϕ is already in negation normal form.
- $\phi = \psi\wedge\chi$. The result follows by the induction hypothesis applied to ψ and χ, and $\wedge\mathsf{E}$ and $\wedge\mathsf{I}$.

- $\phi = \psi \vee \chi$. The result follows by the induction hypothesis applied to ψ and χ, and $\vee$Sub.

- $\phi = \psi \mathbin{⩔} \chi$. The result follows by the induction hypothesis applied to ψ and χ, and ⩔E and ⩔I.

- $\phi = \Diamond\psi$. The result follows by the induction hypothesis applied to ψ, and $\Diamond$Mon.

- $\phi = \neg\psi$. We consider different cases:

  - $\phi = \neg p$ or $\phi = \neg\text{NE}$. ϕ is already in negation normal form.
  - $\phi = \neg\neg\psi$. We have $\phi \dashv\vdash \psi$ by DN, and so the result follows by the induction hypothesis applied to ψ.
  - $\phi = \neg(\psi \wedge \chi)$. We have $\phi \dashv\vdash \neg\psi \vee \neg\chi$ by DM_1, and so the result follows by the induction hypothesis applied to $\neg\psi$ and $\neg\chi$, and $\vee$Sub.
  - $\phi = \neg(\psi \vee \chi)$ or $\phi = \neg(\psi \mathbin{⩔} \chi)$. We have $\phi \dashv\vdash \neg\psi \wedge \neg\chi$ by DM_2 or DM_3, and so the result follows by the induction hypothesis applied to $\neg\psi$ and $\neg\chi$, and $\wedge$E and $\wedge$I.
  - $\phi = \neg\Diamond\psi$. We have $\phi \dashv\vdash \Box\neg\psi$ by Inter $\Diamond\Box$. By the induction hypothesis applied to $\neg\psi$, there is some χ in negation normal form such that $\neg\psi \dashv\vdash \chi$. We then have that $\phi \dashv\vdash \Box\chi$ by $\Box$Mon, and $\Box\chi$ is in negation normal form. ☐

Proposition 5.2.7. For each $\phi \in \mathbf{BSML}^{⩔}(\Phi)$ and each $n \geq md(\phi)$, there is some finite $\mathcal{F} \subseteq \mathcal{M}(\Phi)$ such that

$$\phi \dashv\vdash_{\mathbf{BSML}^{⩔}} \bigvee\!\!\!\bigvee_{(M,s)\in\mathcal{F}} \Theta^n_s.$$

Proof. By induction on the complexity of ϕ. Let $n \geq md(\phi)$. By Proposition 5.2.6 we may assume that ϕ is in negation normal form. Most cases are exactly as in the proof of Proposition 5.2.4. We show the remaining cases.

- $\phi = \neg p$: The result follows from Proposition 5.2.2.

- $\phi = \neg\text{NE}$: $\neg\text{NE} \dashv\vdash \bot$ by $\neg$NEE, and then the result follows from Proposition 5.2.2. ☐

And so now we have:

Proof of Proposition 5.1.7. By Propositions 5.2.4, 5.2.5 and 5.2.7. ☐

5.3 Strong Completeness

The strong completeness of our logics follows from weak completeness and the fact that the logics are compact. Compactness in turn follows from the compactness of modal team logic (**MTL**) [25], and the fact that **MTL** and our logics have the same expressive power. The compactness of **MTL** follows from the fact that **MTL** is strongly complete. We do not make use of the details of **MTL**, so we will simply list the results required. First (from [25]):

Theorem 5.3.1. (Strong completeness of MTL) There is a system **MTL** such that for any set of formulas $\Gamma \cup \{\phi\}$ in the language of **MTL**, $\Gamma \vDash_{\mathbf{MTL}} \phi \iff \Gamma \vdash_{\mathbf{MTL}} \phi$.

So that:

Corollary 5.3.2. (Compactness of MTL) For any set of formulas $\Gamma \cup \phi$ in the language of **MTL**, if $\Gamma \vDash_{\mathbf{MTL}} \phi$, then there is a finite $\Psi \subseteq \Gamma$ such that $\Psi \vDash_{\mathbf{MTL}} \phi$.

Proof. Assume that for all finite $\Psi \subseteq \Gamma$ we have $\Psi \nvDash_{\mathbf{MTL}} \phi$. By soundness (Theorem 5.3.1) we have that $\Psi \nvdash_{\mathbf{MTL}} \phi$ for all such Ψ. Therefore, by the fact that derivations are finite, $\Gamma \nvdash_{\mathbf{MTL}} \phi$, so that by strong completeness (Theorem 5.3.1), $\Gamma \nvDash_{\mathbf{MTL}} \phi$. The result follows by contraposition. □

And ([23]):[33]

Theorem 5.3.3. (Characterization theorem for MTL) MTL characterizes the set

$$\mathbb{B} = \{\mathcal{P} \subseteq \mathcal{M}(\Phi) \mid \mathcal{P} \text{ is closed under } k\text{-bisimulation for some } k \in \mathbb{N}\}.$$

Then:

Corollary 5.3.4. (Compactness of SML$^{\mathbb{W}}$, SGML$^{\mathbb{W}}$, BSML$^{\mathbb{W}}$) For any $\Gamma \cup \{\phi\} \subseteq L$, where L is **SML**$^{\mathbb{W}}(\Phi)$, **SGML**$^{\mathbb{W}}(\Phi)$ or **BSML**$^{\mathbb{W}}(\Phi)$, if $\Gamma \vDash \phi$, then there is a finite $\Psi \subseteq \Gamma$ such that $\Psi \vDash \phi$.

Proof. By Theorems 3.2.13 and 5.3.3, we have (using L and **MTL** to refer to the set of formulas of L and **MTL**, respectively):

$$\begin{aligned}&\{\mathcal{P} \subseteq \mathcal{M} \mid \mathcal{P} = \{(M,s) \in \mathcal{M} \mid M,s \vDash \phi\} \;\&\; \phi \in L\}\\ =&\{\mathcal{P} \subseteq \mathcal{M} \mid \mathcal{P} = \{(M,s) \in \mathcal{M} \mid M,s \vDash_{\mathbf{MTL}} \phi\} \;\&\; \phi \in \mathbf{MTL}\}\end{aligned}$$

[33] This could now also be proved using our characterization result for **SGML**$^{\mathbb{W}}$; the atoms and connectives of **SGML**$^{\mathbb{W}}$ are easy to define in **MTL**.

So we can define a one-to-one map $\circ : L \to \mathbf{MTL}$ such that for all $(M, s) \in \mathcal{M} : M, s \vDash \phi$ iff $M, s \vDash_{\mathbf{MTL}} \phi^\circ$. For $\Gamma \subseteq L$ we denote $\Gamma^\circ := \{\gamma^\circ \mid \gamma \in \Gamma\}$.

Assume $\Gamma \vDash \phi$. Then clearly $\Gamma^\circ \vDash_{\mathbf{MTL}} \phi^\circ$, so by Corollary 5.3.2 there is a finite $\Psi^\circ \subseteq \Gamma^\circ$ such that $\Psi^\circ \vDash_{\mathbf{MTL}} \phi^\circ$. Then $\Psi \vDash \phi$. □

Theorem 5.3.5. (Strong completeness of $\mathbf{SML}^{\mathbb{W}}$, $\mathbf{SGML}^{\mathbb{W}}$, $\mathbf{BSML}^{\mathbb{W}}$) For any $\Gamma \cup \{\phi\} \subseteq L$ where L is $\mathbf{SML}^{\mathbb{W}}(\Phi)$, $\mathbf{SGML}^{\mathbb{W}}(\Phi)$ or $\mathbf{BSML}^{\mathbb{W}}(\Phi)$, we have $\Gamma \vDash \phi \Rightarrow \Gamma \vdash \phi$.

Proof. Assume $\Gamma \vDash \phi$. Then by Corollary 5.3.4 there is a finite $\Psi \subseteq \Gamma$ such that $\Psi \vDash \phi$. Then by Theorem 5.1.8 we have $\bigwedge \Psi \vdash \phi$. Therefore $\Psi \vdash \phi$ by $\wedge$I and since $\Psi \subseteq \Gamma$ we have $\Gamma \vdash \phi$. □

Chapter 6

Conclusion

We studied three state-based modal logics with the non-emptiness atom NE and the global disjunction $\mathbin{⩔}$:

- state-based modal logic with $\mathbin{⩔}$ ($\mathbf{SML}^{⩔}$), which employs the flat modalities $\Diamond$ and $\Box$;
- globally state-based modal logic with $\mathbin{⩔}$ ($\mathbf{SGML}^{⩔}$), which employs the global modalities $\Diamond\!\!\!\!\cdot$ and $\boxdot$; and
- bilateral state-based modal logic with $\mathbin{⩔}$ ($\mathbf{BSML}^{⩔}$), which employs the flat modalities $\Diamond$ and $\Box$ and the bilateral negation $\neg$.

We established entailment relationships between the modalities contingent on the type of formula within their scope (Proposition 2.2.10), and showed that all three logics are conservative extensions of classical modal logic (Propositions 2.2.13 and 2.2.15).

We proved that each of the logics is expressively complete for the set of state properties closed under k-bisimulation for some $k \in \mathbb{N}$ (Theorem 3.2.13).

Building on the systems for $\mathbf{PT}^{+}$ [38] and $\mathbf{MD}^{⩔}$ [35], we provided sound and complete natural deduction systems for each of our logics (soundness was proved in Theorems 4.2.2, 4.2.4 and 4.3.2). Our weak completeness proof adapted the completeness-via-normal-forms strategy employed in [38] to the modal setting (Sections 5.1 and 5.2). Strong completeness followed from weak completeness and compactness. Compactness followed from the compactness of **MTL** and the fact that our logics and **MTL** have the same expressive power (Section 5.3).

We confirmed that both the flat modality $\Diamond$ and the bilateral negation $\neg$ are required to account for FC in the way Aloni [1, 3] outlines (Section

2.3). We characterized the properties of indisputability and state-basedness using inference rules and demonstrated that Aloni's analysis of wide-scope FC and epistemic contradictions functions as intended with the revised definitions of these notions (Section 3.3).

Let us conclude with some suggestions for further work.

The rules for the axiomatizations were chosen with the completeness proof and its provable equivalence lemma in mind—it may be possible to find rules which are simpler or otherwise more elegant. Relatedly, some of our rules may be derivable from other rules in the system and hence not necessary to include as basic.

We proved strong completeness via an indirect argument making use of compactness and weak completeness. We similarly proved compactness indirectly by making use of the compactness of **MTL** and the fact that **MTL** and our logics have the same expressive power. Finding a direct proof of strong completeness (one which does not rely on compactness and weak completeness) and a direct proof of compactness (one which does not rely on **MTL**) would provide us a better understanding of these logics and proof systems.

As mentioned in the introduction, Yang and Väänänen [38] have devised a method for adapting their completeness proof involving characteristic formulas and $\mathbb{W}$ to a logic which does not make use of $\mathbb{W}$. We might be able to apply this strategy to the proof in this thesis to axiomatize **BSML**. The adaptation would involve conceptualizing finite sets of characteristic formulas $\{\Theta^k_{M,s} \mid (M,s) \in \mathcal{F}\}$ (rather than global disjunctions over such sets) as a "weak" normal form for formulas and adding rules which simulate the rules for $\mathbb{W}$. Note that this strategy, since it involves simulating the behaviour of $\mathbb{W}$, is indeed best thought of as an adaptation of a proof involving $\mathbb{W}$. Now that we have provided the proof involving $\mathbb{W}$, at least some of the conceptual work required for a possible $\mathbb{W}$-free proof has already been done.

As also mentioned in the introduction, it would be interesting to examine the expressive power of **BSML** and the other $\mathbb{W}$-free fragments of our logics. If these are not expressively complete for the set of all union-closed state properties closed under k-bisimulation, investigating how the logics need to be extended to attain this might yield some interesting connectives.

Finally, we do not know whether in our logics the flat modalities are uniformly definable in terms of the global ones or vice versa. (Ciardelli [5] conjectures that $\Box$ is not uniformly definable in $\mathbf{ML}^{\diamondsuit}$ with $\mathbb{W}$.) Nor do we know whether they are interdefinable in **MTL** (i.e. in the presence of the Boolean negation $\sim$). It may be possible to find some such definition, or to show that no such definition is possible (Yang proves some negative uniform definability results for propositional dependence logic in [37]; we may be able to adapt this approach.)

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