Graph Aggregation

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Abstract

Suppose a number of agents each provide us with a directed graph over a common set of vertices. Graph aggregation is the problem of computing a single graph that best represents the information inherent in this profile of individual graphs. We introduce a simple formal framework for graph aggregation and then focus on the notion of collective rationality, which asks whether a given property of graphs, such as transitivity, can be guaranteed to hold for the collective graph whenever it is satisfied by all individual graphs. We refine the ultrafilter method for proving impossibility theorems in social choice theory to arrive at a clear picture relating axiomatic properties of aggregation procedures, properties of graphs with respect to which we want to ensure collective rationality, and properties of ultrafilters.

1 Introduction

Suppose a group of agents each supply us with a particular piece of information and we want to aggregate this information into a collective view to obtain a good representation of the individual views provided. In classical social choice theory the objects of aggregation have been preference orders on a set of alternatives (Arrow, 1963). More recently, the same methodology has also been applied to other types of information, notably beliefs (Konieczny and Pino Pérez, 2002), judgments (List and Puppe, 2009), ontologies (Porello and Endriss, 2011), and rankings provided by Internet search engines (Dwork et al., 2001).

In this paper, we introduce the problem of *graph aggregation*, i.e., the problem of devising methods to aggregate the information inherent in a profile of individual (directed) graphs, one for each agent, into a single collective graph. Given that a preference order is a special kind of directed graph, graph aggregation may be viewed as a direct generalisation of classical preference aggregation. This is a useful generalisation, because also several other problem domains in which aggregation is relevant are naturally modelled as graphs, e.g.:

- In abstract argumentation (Dung, 1995), collections of arguments available for a debate are modelled as a graph (with an edge from A to B if argument A *attacks* B). The question of how to integrate several such argument graphs naturally arises in this context. Recent work of Coste-Marquis et al. (2007) has addressed this question.
- Social and economic networks are often modelled as graphs (Jackson, 2008).[1] We might want to merge the information from several such networks (e.g., the network of work relations in a community, the network of friends in the same community, etc.).
- It is not always reasonable to take the classical assumptions of economic theory (according to which preferences are transitive and complete orders) for granted when modelling an agent's preferences. The work of Pini et al. (2009) goes in this direction by studying aggregation of preferences modelled as incomplete orders; but we might want to go further and also allow for cycles and so forth.

Special instances of the graph aggregation problem we shall consider have previously been studied in work on the aggregation of judgments regarding causal relations between variables

[1] While social networks are usually modelled as undirected graphs, here we shall work with directed graphs (but note that we can model an undirected graph as a directed graph that is symmetric).

(Bradley et al., 2011) and the design of voting agendas for multi-issue elections based on individually reported preferential dependencies between issues (Airiau et al., 2011).

While graph aggregation is more general than preference aggregation, it is less general than the frameworks of judgment aggregation (List and Puppe, 2009) or binary aggregation (Dokow and Holzman, 2010; Grandi and Endriss, 2010): just like classical preference aggregation, graph aggregation can—in principle—be embedded into these frameworks. For a given problem domain, it is important to find the right level of abstraction, and graphs appear to be a particularly useful level of abstraction for a wide range of problems.

In Section 2, we define a framework for graph aggregation and adapt well-known axioms from the literature to express natural desiderata for such aggregators. We also suggest concrete aggregators, including both adaptations from other areas of social choice theory and the novel class of *successor-approval rules*.

Our main interest will then be in the notion of *collective rationality*. In Section 3, we define collective rationality of a given aggregator F wrt. a given property of graphs P (such as transitivity) as the guarantee that, whenever each of the individual graphs satisfies P, so does the collective graph we obtain when we apply F to those individual graphs. That is, assuming that each individual agent is "rational" in the sense of respecting the property P under consideration, we ask whether we can be sure that the collective (as defined by our aggregator F) will be rational as well. This is a well-known concept: in classical preference aggregation, P corresponds to the conjunction of the properties that define a weak order (Arrow, 1963); in judgment aggregation, P corresponds to logical consistency (List and Puppe, 2009); and in our own previous work on binary aggregation, P corresponds to an integrity constraint expressed in a propositional language (Grandi and Endriss, 2010).

We first prove a series of simple results that identify certain (classes of) aggregators that are collectively rational wrt. certain properties of graphs. Our main technical contribution is a refinement of the *ultrafilter method* for proving impossibility theorems in social choice theory (see, e.g., Kirman and Sondermann, 1972; Herzberg and Eckert, 2011). One way of proving Arrow's classical impossibility theorem (Arrow, 1963) is to show that the collection of winning coalitions of individuals (determining which pieces of information need to be accepted by an aggregator) is an ultrafilter (Davey and Priestley, 2002). We will show how each of the conditions defining an ultrafilter corresponds directly to the requirement of collective rationality wrt. a certain graph property. For example, any property that, given the acceptance of two particular edges, forces the acceptance of a third edge can be used to establish the ultrafilter condition of being closed under intersections. This means that we can replace transitivity in the statement of an Arrow-like theorem by, say, the Church-Rosser property or the Euclidean property (see Table 1 for definitions of these three properties, all of which have the general template indicated before). We use our technique to prove several variants of Arrow's Theorem for graph aggregation.

Section 4 concludes with a discussion of related research and directions for future work.

2 A Formal Framework of Graph Aggregation

Fix a finite set of *vertices* V. A (directed) *graph* $G = \langle V, E\rangle$ based on V is defined by a set of *edges* $E \subseteq V^2$. Let $\mathcal{G}$ be the set of all such graphs (for our fixed choice of V). Let $\mathcal{N}$ be a finite set of (two or more) *individuals* (or *agents*). Each individual $i \in \mathcal{N}$ provides a graph $G_i = \langle V, E_i\rangle$ with some set of edges E_i. This gives rise to a *profile* of graphs $\boldsymbol{G} = (G_1, \ldots, G_n)$, which we shall also write as $\boldsymbol{G} = \langle V, (E_1, \ldots, E_n)\rangle$. An *aggregator* is a function $F : \mathcal{G}^{\mathcal{N}} \to \mathcal{G}$ mapping any such profile into a single collective graph.

We require a few further pieces of *notation:* First, $E(x) := \{y \in V \mid (x, y) \in E\}$ is the set of successors of a vertex x in a set of edges E; and $E^{-1}(y) := \{x \in V \mid (x, y) \in E\}$ is the

Property	First-order Condition
Reflexivity	$\forall x.xEx$
Irreflexivity	$\neg\exists x.xEx$
Symmetry	$\forall xy.(xEy \to yEx)$
Antisymmetry	$\forall xy.(xEy \wedge yEx \to x = y)$
Transitivity	$\forall xyz.(xEy \wedge yEz \to xEz)$
Euclidean property	$\forall xyz.(xEy \wedge xEz \to yEz)$
Church-Rosser property	$\forall xy.[xEy \wedge xEz \to \exists w.(yEw \wedge zEw)]$
Seriality	$\forall x.\exists y.xEy$
Functionality	$\forall xyz.(xEy \wedge xEz \to y = z)$
Completeness	$\forall xy.[x \neq y \to (xEy \vee yEx)]$
Strong completeness	$\forall xy.(xEy \vee yEx)$
Connectedness	$\forall xyz.[xEy \wedge xEz \to (yEz \vee zEy)]$
Negative transitivity	$\forall xyz.[xEy \to (xEz \vee zEy)]$

Table 1: Common Properties of Directed Graphs.

set of predecessors of y. Second, given an edge e, we sometimes write $e \in G$ instead of $e \in E$ when $G = \langle V, E\rangle$. Third, xEy is a shorthand for $(x, y) \in E$. Fourth, $N_e^{\boldsymbol{G}} := \{i \in \mathcal{N} \mid e \in E_i\}$ is the set of individuals accepting edge e under profile $\boldsymbol{G}$.

A few fundamental *properties* of directed graphs (and, more generally speaking, of binary relations) are shown in Table 1. Recall that a *weak order* is a binary relation that is reflexive, transitive and complete, while a *linear order* is irreflexive, transitive and complete.

2.1 Properties of Graph Aggregators

We now introduce a number of axioms that define certain desirable properties of aggregators. The first such axiom is an independence condition that requires that the decision of whether or not a given edge e should be part of the collective graph should only depend on which of the individual graphs include e. This corresponds to the well-known independence axioms in preference aggregation (Arrow, 1963) and judgment aggregation (List and Puppe, 2009).

Definition 1 (IIE)**.** *F is independent of irrelevant edges if $N_e^{\boldsymbol{G}} = N_e^{\boldsymbol{G'}}$ implies $e \in F(\boldsymbol{G}) \Leftrightarrow e \in F(\boldsymbol{G'})$.*

That is, if exactly the same individuals accept e under profiles $\boldsymbol{G}$ and $\boldsymbol{G'}$, then e should be part of either both or none of the corresponding collective graphs. Note that above definition applies to *all* edges $e \in V^2$ and *all* pairs of profiles $\boldsymbol{G}, \boldsymbol{G'} \in \mathcal{G}^{\mathcal{N}}$. We shall leave this kind of universal quantification implicit also in later definitions.

While very much a standard axiom, we might be dissatisfied with IIE for not making reference to the fact that edges are defined in terms of vertices. Our next axiom is much more graph-specific and does not have a close analogue in preference or judgment aggregation. It requires that the decision of whether or not to collectively accept a given edge $e = (x, y)$ should only depend on which edges with the same source x are accepted by the individuals. Below we abuse notation and write $F(\boldsymbol{G})(x)$ for the set of successors of x in the set of edges in the collective graph $F(\boldsymbol{G})$ (and similarly $F(\boldsymbol{G})^{-1}(y)$ for the predecessors of y in $F(\boldsymbol{G})$).

Definition 2 (IIS)**.** *F is independent of irrelevant sources if $E_i(x) = E'_i(x)$ for all individuals $i \in \mathcal{N}$ implies $F(\boldsymbol{G})(x) = F(\boldsymbol{G'})(x)$.*

Similarly, the next axiom requires that collective acceptance of an edge $e = (x, y)$ should only depend on the pattern of individual acceptance for those edges with the same target y.

Definition 3 (IIT)**.** *F is independent of irrelevant targets if $E_i^{-1}(y) = {E'}_i^{-1}(y)$ for all individuals $i \in \mathcal{N}$ implies $F(\boldsymbol{G})^{-1}(y) = F(\boldsymbol{G'})^{-1}(y)$.*

Note that both IIS and IIT are strictly weaker than IIE. The precise relative strength of our independence axioms is illustrated by the following fact, which is easy to verify.

Fact 1. *An aggregator is IIE iff it is both IIS and IIT.*

The fundamental economic principle of *unanimity* requires that an edge should be accepted by the collective if all individuals accept it.

Definition 4 (Unanimity)**.** *F is unanimous if $F(\boldsymbol{G}) = \langle V, E\rangle$ implies $E \supseteq E_1 \cap \cdots \cap E_n$.*

A requirement that, in some sense, is dual to unanimity is to ask that the collective graph should only include edges that are part of at least one of the individual graphs. In the context of ontology aggregation this axiom has been called *groundedness* (Porello and Endriss, 2011).

Definition 5 (Groundedness)**.** *F is grounded if $F(\boldsymbol{G}) = \langle V, E\rangle$ implies $E \subseteq E_1 \cup \cdots \cup E_n$.*

The remaining axioms are all standard and closely modelled on their counterparts in judgment aggregation (List and Puppe, 2009).

Definition 6 (Anonymity)**.** *F is anonymous if $F(G_1, \ldots, G_n) = F(G_{\pi(1)}, \ldots, G_{\pi(n)})$ for any permutation $\pi : \mathcal{N} \to \mathcal{N}$.*

Definition 7 (Neutrality)**.** *F is neutral if $N_e^{\boldsymbol{G}} = N_{e'}^{\boldsymbol{G}}$ implies $e \in F(\boldsymbol{G}) \Leftrightarrow e' \in F(\boldsymbol{G})$.*

Definition 8 (Monotonicity)**.** *F is monotonic if $e \in F(\boldsymbol{G})$ implies $e \in F(\boldsymbol{G'})$ whenever $\boldsymbol{G'}$ is obtained from $\boldsymbol{G}$ by having one additional individual accept the edge e.*

That is, anonymity and neutrality are basic symmetry requirements wrt. individuals and edges, respectively, while monotonicity requires that additional support for an edge should never reduce that edge's chances of being collectively accepted.

An extreme form of violating anonymity is to use an aggregator that is *dictatorial* in the sense that a single individual can determine the shape of the collective graph.

Definition 9 (Dictatorships)**.** *F is dictatorial if there exists an individual $i^\star \in \mathcal{N}$ (the dictator) such that $e \in F(\boldsymbol{G}) \Leftrightarrow e \in G_{i^\star}$ for every edge $e \in V^2$.*

Aggregators that are not dictatorial are called *nondictatorial*.

Sometimes we shall only be interested in the properties of an aggregator as far as the *nonreflexive* edges $e = (x, y)$ with $x \neq y$ are concerned. Specifically, we call F *NR-neutral* if $N_{(x,y)}^{\boldsymbol{G}} = N_{(x',y')}^{\boldsymbol{G}}$ implies $(x, y) \in F(\boldsymbol{G}) \Leftrightarrow (x', y') \in F(\boldsymbol{G})$ for all $x \neq y$ and $x' \neq y'$; and we call F *NR-nondictatorial* if there exists no $i^\star \in \mathcal{N}$ such that $(x, y) \in F(\boldsymbol{G}) \Leftrightarrow (x, y) \in G_{i^\star}$ for all $x \neq y$. That is, NR-neutrality is slightly weaker than neutrality and NR-nondictatoriality is slightly stronger than nondictatoriality.

2.2 Aggregators

Next, we define several concrete aggregators. Under a *quota rule*, an edge will be included in the collective graph if the number of individuals accepting it meets a certain quota. If that quota is the same for every edge, then we have a *uniform* quota rule.

Definition 10 (Quota rules)**.** *A quota rule is an aggregator F_q defined via a function $q : V^2 \to \{0, 1, \ldots, n{+}1\}$ by stipulating $F_q(\boldsymbol{G}) := \langle V, E\rangle$ with $E = \{e \in V^2 \mid |N_e^{\boldsymbol{G}}| \geqslant q(e)\}$. F_q is called uniform if q is a constant function.*

The class of uniform quota rules includes several interesting special cases:

- The (strict) *majority rule* accepts an edge if more than half of the individuals do. This is the uniform quota rule with $q = \lceil \frac{n+1}{2} \rceil$.
- The *union rule* is the aggregator that maps any given profile of graphs to their union: $\langle V, E_1 \cup \cdots \cup E_n \rangle$. This is the uniform quota rule with $q = 1$.
- The *intersection rule* is the aggregator that maps any given profile of graphs to their intersection: $\langle V, E_1 \cap \cdots \cap E_n \rangle$. This is the uniform quota rule with $q = n$.

We call the uniform quota rules with $q = 0$ and $q = n{+}1$ the *trivial* quota rules; $q = 0$ means that *all* edges will be included in the collective graph and $q = n{+}1$ means that *no* edge will be included. Quota rules have also been studied in judgment aggregation (Dietrich and List, 2007).

Another important class of aggregators, familiar from both judgment aggregation and belief merging, are the *distance-based aggregators* (Konieczny and Pino Pérez, 2002), which in our context amount to selecting a collective graph that satisfies certain properties and that minimises the distance to the individual graphs (for a suitable notion of distance and a suitable form of aggregating such distances). While of great practical importance, we shall not consider distance-based aggregators here, because they ensure that the collective graph meets the required properties "by design", i.e., the question of collective rationality does not arise for these rules. Distance-based rules also violate several attractive axioms (Lang et al., 2011) and are of high complexity (Endriss et al., 2010).

Inspired by *approval voting* (Brams and Fishburn, 2007), we now introduce a new class of aggregators specifically for graphs. Imagine we associate with each vertex an election in which all the possible successors of that vertex are the candidates (and in which there may be more than one winner). Each individual votes by stating which vertices they consider acceptable successors. We might then elect those vertices that receive the most support or that receive above average support. We might also give each voter a certain weight, which could be inversely proportional to the number of successors they propose, and so forth.

Definition 11 (Successor-approval rules)**.** *A successor-approval rule is an aggregator F_v defined via a function $v : (2^V)^{\mathcal{N}} \to 2^V$ by stipulating $F(\langle V, E_1, \ldots, E_n \rangle) := \langle V, E \rangle$ with $E = \{(x, y) \in V^2 \mid y \in v(E_1(x), \ldots, E_n(x))\}$.*

We call v the *choice function* associated with F_v. We shall only be interested in choice functions v that are *anonymous* and *neutral*, i.e., that satisfy $v(S_1, \ldots, S_n) = v(S_{\pi(1)}, \ldots, S_{\pi(n)})$ for any permutation $\pi : \mathcal{N} \to \mathcal{N}$ and for which $\{i \in \mathcal{N} \mid e \in S_i\} = \{i \in \mathcal{N} \mid e' \in S_i\}$ entails $e \in v(S_1, \ldots, S_n) \Leftrightarrow e' \in v(S_1, \ldots, S_n)$.

2.3 Characterisations

A simple adaptation of a result by Dietrich and List (2007) yields:

Fact 2. *An aggregator is a quota rule iff it is anonymous, IIE and monotonic.*

If we add the axiom of neutrality, then we obtain the class of uniform quota rules. If we furthermore impose unanimity and groundedness, then this excludes the trivial quota rules. Similarly, IIS characterises the class of successor-approval rules:

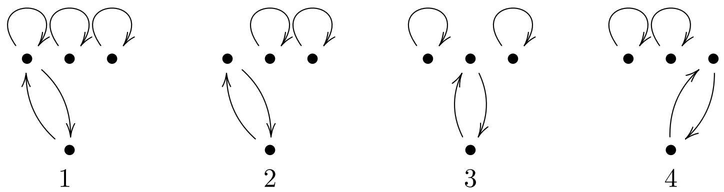
Fact 3. *An aggregator is a successor-approval rule (with an anonymous and neutral choice function) iff it is anonymous, neutral and IIS.*

3 Collective Rationality

We now analyse to what extent aggregators can ensure that a given property that is satisfied by each of the individual graphs is preserved when we move to the corresponding collective graph. This is known as *collective rationality.* For instance, in preference aggregation we may ask whether an aggregator can guarantee that the structure it will produce as output, when all the input structures are transitive and complete preference orders, will also be transitive and complete. Arrow's Theorem shows that the answer to this question is negative for any "reasonable" aggregator (Arrow, 1963). Much of this line of work has concentrated on properties that are natural to consider in the context of preference modelling. In our own previous work on binary aggregation, we have concentrated on properties that can be expressed in simple logical languages (Grandi and Endriss, 2010). Here, instead, we focus on fundamental properties of binary relations and graphs.

Definition 12 (Collective rationality)**.** *An aggregator F is collectively rational (CR) wrt. a property P if $F(\boldsymbol{G})$ satisfies P whenever each of the individual graphs in the profile $\boldsymbol{G}$ do.*

Example 1. *Suppose four individuals each provide a graph over the same set of vertices:*



If we apply the strict majority rule, we obtain a graph where the only edges are those connecting the upper three worlds with themselves. That is, this rule is not CR wrt. seriality, because each of the individual graphs is serial, while the collective graph computed is not. Symmetry, on the other hand, is preserved. There also is no violation of collective rationality wrt. reflexivity, because the individual graphs are not reflexive to begin with. A rule that does preserve seriality is the simple successor-approval rule that accepts an edge if it is (tied for being) most often accepted amongst those with the same source.

3.1 Basic Results

We begin with two very simple positive results, showing how a basic aggregation axiom can guarantee the preservation of a simple graph property:

Proposition 4. *Any unanimous aggregator is CR wrt. reflexivity.*

Proof. Immediate: If every individual graph includes all edges of the form (x, x), then unanimity ensures that the same is true for the collective graph. □

Proposition 5. *Any grounded aggregator is CR wrt. irreflexivity.*

Proof. Immediate: If no individual graph includes the edge (x, x), then groundedness guarantees the same for the collective graph. □

Symmetry is more demanding a property and unanimity alone does not suffice to preserve it. However, if we restrict attention to uniform quota rules, we obtain the following result:

Proposition 6. *Any uniform quota rule is CR wrt. symmetry.*

Proof. Immediate: If each individual respects symmetry, then the number of individual graphs including edge (x, y) will always equal the number of individual graphs including (y, x). Hence, either both or neither will meet the uniformly imposed quota. □

Note that uniformity is a necessary condition for Proposition 6 to hold. Transitivity is yet again more demanding a property:

Proposition 7. *The intersection rule is CR wrt. transitivity. It is the only nontrivial uniform quota rule with that property for $|V| \geqslant 3$.*

Proof. First, it is easy to verify that the intersection rule preserves transitivity. Now consider any nontrivial uniform quota rule F_q with a quota $q < n$. Take a profile in which $q - 1$ individuals accept (x, y), (y, z) and (x, z); one individual accepts only (x, y); and one individual accepts only (y, z). This profile is transitive (as far as the edges under consideration here are concerned). But when we aggregate using F_q, then we obtain a graph that includes the edges (x, y) and (y, z), but not (x, z). Hence, transitivity is not preserved. □

The constant rules corresponding to the trivial quota rules with $q = 0$ and $q = n{+}1$ vacuously ensure collective rationality wrt. transitivity. Another demanding property is seriality:

Proposition 8. *The union rule is CR wrt. seriality. It is the only nontrivial uniform quota rule with that property for $|V| \geqslant n$.*

Proof. Clearly, the union rule (with $q = 1$) will preserve seriality. To see that no uniform quota rule with $1 < q \leqslant n$ does, it suffices to consider a scenario where each of the edges emanating from a particular source x is accepted by (at most) one individual. Note that this construction requires $|V| \geqslant n$. Otherwise, there always is an outgoing edge accepted by more than one individual (when each individual respects seriality), and therefore also some quotas $q > 1$ will work. □

Amongst the trivial uniform quota rules only the one with $q = 0$ ensures seriality. If we move away from quota rules (satisfying IIE) and are content with using successor-approval rules (only satisfying IIS), then we have a wider choice of aggregators available that will preserve seriality (e.g., the simple successor-approval rule of Example 1).

Above we have seen that certain properties will be preserved by certain quota rules. However, if we want to preserve several such properties, those possibility results quickly turn into impossibilities. Let us begin with an immediate corollary of our earlier results:

Corollary 9. *If $|V| \geqslant n$, then no nontrivial uniform quota rule is CR wrt. both transitivity and seriality.*

Proof. Immediate from Proposition 7 and Proposition 8 and the fact that union and intersection differ for $n > 1$. □

Rather surprisingly, in some cases we obtain an impossibility already when only collective rationality wrt. a *single* property is required:

Proposition 10. *If $|V| \geqslant 3$, then no nontrivial uniform quota rule is CR wrt. connectedness.*

Proof. First, the intersection rule (with quota $q = n$) does not preserve connectedness. To see this, consider a scenario where all individuals accept (x, y) and (x, z), half of them accept (y, z), and the other half (z, y). Second, for any uniform quota rule with $0 < q < n$, construct a counterexample as follows: Suppose a group of q individuals accept (x, y), a different group of q individuals accept (x, z), and their intersection accept (y, z), while nobody accepts (z, y). Then (x, y) and (x, z) are part of the collective graph, but neither (y, z) or (z, y) are. This violates connectedness, even though the individual graphs satisfy it. □

Note that both of the trivial uniform quota rules ensure connectedness (because both the complete and the empty graph are connected). If we swap connectedness for completeness, then we obtain the following characterisation:

Proposition 11. *If $|V| \geqslant 2$, then a uniform quota rule F_q is CR wrt. completeness (or strong completeness) iff $q \leqslant \lfloor \frac{n+1}{2} \rfloor$.*

Proof. By the pigeon hole principle, if all individual graphs are complete, then one of (x, y) and (y, x) will always have at least $\lfloor \frac{n+1}{2} \rfloor$ individuals accepting it. Hence, if (and only if) the quota is at most $\lfloor \frac{n+1}{2} \rfloor$ we can ensure that that edge will be collectively accepted. □

While most of our examples so far have been restricted to quota rules, they already give some insight into the close connections between collective rationality and standard axiomatic requirements. In the sequel, we shall explore this connection in much more depth.

3.2 Impossibility Theorems

In view of Fact 2 and the remarks following it, we can reformulate Proposition 10 as saying that there exists no anonymous, neutral, unanimous, grounded, IIE and monotonic aggregator that is CR wrt. connectedness. This closely resembles classical impossibility theorems in social choice theory. For instance, Arrow's Theorem in its form for linear orders (i.e., irreflexive, transitive, and complete preference orders) can be stated as saying that there exists no nondictatorial, unanimous, and IIE aggregator that is CR wrt. irreflexivity, transitivity, and completeness. We shall soon prove the following variant of Arrow's Theorem:[2]

Theorem 12. *If $|V| \geqslant 3$, then there exists no NR-nondictatorial, unanimous, grounded and IIE aggregator that is CR wrt. both transitivity and completeness.*

For now, we want to see whether Arrow's impossibility persists when we move away from properties typically associated with preferences. The central axiom the impossibility feeds on is IIE. Observe that an aggregator F satisfies IIE iff for each edge $e \in V^2$ there exists a set of *winning coalitions* $\mathcal{W}_e \subseteq 2^{\mathcal{N}}$ such that $e \in F(\boldsymbol{G}) \Leftrightarrow N_e^{\boldsymbol{G}} \in \mathcal{W}_e$. Imposing additional axioms on F corresponds to restrictions on the associated family of winning coalitions, e.g.:

- If F is unanimous, then $\mathcal{N} \in \mathcal{W}_e$ for any edge e.
- If F is grounded, then $\emptyset \notin \mathcal{W}_e$ for any edge e.
- If F is (NR-)neutral, then $\mathcal{W}_e = \mathcal{W}_{e'}$ for any two (nonreflexive) edges e and e'.

Recall that neutrality does not feature in Arrow's Theorem. As we shall see next, the reason is that the same restriction on winning coalitions is already enforced by collective rationality wrt. transitivity (at least for nonreflexive edges). This is a surprising and interesting link between a specific collective rationality requirement and a specific axiom. This link is related to the so-called *Contagion Lemma* (Sen, 1986), but we have not seen it noted in the literature in this form before. The same kind of result can also be obtained for other graph properties with a similar structure; besides transitivity, we state it here for the Euclidean property.

Lemma 13. *If $|V| \geqslant 3$, then any unanimous and IIE aggregator that is CR wrt. transitivity or the Euclidean property must be NR-neutral.*

Proof. Let F be an aggregator that is unanimous and IIE, and let $\{\mathcal{W}_e\}_{e \in V^2}$ be the associated family of winning coalitions. We need to show that there exists a unique $\mathcal{W} \subseteq 2^{\mathcal{N}}$

[2]Theorem 12 implies both the standard variant of Arrow's Theorem for linear orders and its standard variant for weak orders. (1) For linear orders: First, by Proposition 5 we can add irreflexivity to the CR requirements without changing the logical strength of the theorem. Second, groundedness can be dropped as it follows from unanimity together with CR wrt. completeness. Third, on irreflexive profiles NR-nondictatoriality and nondictatoriality coincide. (2) For weak orders: First, by Proposition 4 we can add reflexivity to the CR requirements. Second, the Pareto condition (i.e., unanimity wrt. the strict part of the preference relation) implies both unanimity (when used with IIE) and groundedness (when used with CR wrt. completeness). Third, on reflexive profiles NR-nondictatoriality and nondictatoriality coincide.

such that $\mathcal{W} = \mathcal{W}_e$ for any nonreflexive edge e. Note that the $\mathcal{W}_e$ are not empty (due to unanimity). Consider any three vertices x, y, z and any coalition $C \in \mathcal{W}_{(x,y)}$. We will employ collective rationality to show that C must also be a winning coalition for each of the other five edges between these three vertices. A simple inductive argument then suffices to show that C will in fact have to be a winning coalition for *all* nonreflexive edges.

Now suppose F is CR wrt. transitivity. Let us first see how to prove that $C \in \mathcal{W}_{(z,x)}$: Consider a scenario in which (x, y) and (z, x) are accepted by the individuals in C and only those (i.e., by definition of C, (x, y) is collectively accepted) and in which (y, z) is accepted by all individuals (i.e., by unanimity, (y, z) is also collectively accepted). Then, by collective transitivity, (z, x) must be collectively accepted. Hence, C must be a winning coalition for (z, x), i.e., $C \in \mathcal{W}_{(z,x)}$. We can use a similar argument for the other edges: e.g., to show $C \in \mathcal{W}_{(z,y)}$ consider the case with C accepting all of (z, x), (x, y) and (z, y); to show $C \in \mathcal{W}_{(y,x)}$ consider the case with everyone accepting (y, z) and C accepting (z, x) and (y, x); and so forth.

The proof in case transitivity is replaced by the Euclidean property is similar. We omit the details in the interest of space. □

Note that Lemma 13 does not hold for $|V| = 2$: the aggregator that accepts (x, y) whenever agent 1 does and that accepts (y, x) whenever agent 2 does is a counterexample.

Also note that full neutrality does not follow from the conditions of Lemma 13. The reason is that, while C being a winning coalition for (x, y) entails C also being a winning coalition for (x, x), the converse is not true. For example, the aggregator that accepts nonreflexive edges only when all individuals do, but that always accepts all reflexive edges (thereby violating neutrality), is unanimous, IIE, and CR wrt. transitivity.

We now prove a result similar to Arrow's Theorem, but replacing completeness by seriality. We do this by proving that the set of winning coalitions corresponding to any aggregator that meets the conditions stated in the theorem is an *ultrafilter* (Davey and Priestley, 2002).

Definition 13 (Ultrafilters.)**.** *An ultrafilter $\mathcal{W}$ on a set $\mathcal{N}$ is a collection of subsets of $\mathcal{N}$ that satisfies the following three conditions:*

(*i*) $\emptyset \notin \mathcal{W}$;
(*ii*) $C_1, C_2 \in \mathcal{W}$ *implies* $C_1 \cap C_2 \in \mathcal{W}$ *(i.e., $\mathcal{W}$ is closed under intersections); and*
(*iii*) C *or* $\mathcal{N} \backslash C$ *is in* $\mathcal{W}$ *for any* $C \subseteq \mathcal{N}$ *(i.e., $\mathcal{W}$ is maximal).*

We are now ready to state and prove our result:

Theorem 14. *If $|V| \geqslant 3$, then there exists no NR-nondictatorial, unanimous, grounded, and IIE aggregator that is CR wrt. both transitivity and seriality.*

Proof. Let F be a unanimous, grounded and IIE aggregator that is CR wrt. transitivity and seriality. By Lemma 13, F is NR-neutral, i.e., there is set of winning coalitions $\mathcal{W} \subseteq 2^{\mathcal{N}}$ with $e \in F(\boldsymbol{G}) \Leftrightarrow N_e^{\boldsymbol{G}} \in \mathcal{W}$ for any nonreflexive edge e. We shall prove that $\mathcal{W}$ is an ultrafilter. Condition (*i*) holds, because F is grounded. Condition (*ii*) follows from collective rationality wrt. transitivity: Suppose $C_1, C_2 \in \mathcal{W}$. Consider a scenario where coalition C_1 accepts (x, y) and C_2 accepts (y, z). Then, by transitivity, *at least* coalition $C_1 \cap C_2$ must accept (x, z). Suppose it is exactly the individuals in $C_1 \cap C_2$ who do. As C_1 and C_2 are winning coalitions, (x, y) and (y, z) are part of the collective graph. To achieve collective rationality wrt. transitivity, we must also have (x, z) be part of the collective graph, and thus we must have $C_1 \cap C_2 \in \mathcal{W}$. Condition (*iii*), finally, follows from collective rationality wrt. seriality: Take an arbitrary coalition $C \in \mathcal{W}$. Consider a scenario where exactly the individuals in C accept (x, y), exactly those in $\mathcal{N} \backslash C$ accept (x, z), and no individual accepts any of the other edges emanating from x. Due to groundedness, of all the edges emanating from x, only (x,y) and (x,z) can possibly be part of the collective graph. Due to collective rationality wrt. seriality at least one of them has to be, i.e., $C \in \mathcal{W}$ or $\mathcal{N} \setminus C \in \mathcal{W}$.

Recall that $\mathcal{N}$ is required to be finite. An ultrafilter $\mathcal{W}$ on a set $\mathcal{N}$ is called *principal* if it is of the form $\mathcal{W} = \{C \in 2^{\mathcal{N}} \mid i^\star \in C\}$ for some fixed $i^\star \in \mathcal{N}$. In our setting, principality of $\mathcal{W}$ corresponds to F being dictatorial (with dictator $i^\star$) on nonreflexive edges. Now, it is a well-known fact that any ultrafilter on a finite set must be principal (Davey and Priestley, 2002), which shows that F cannot be NR-nondictatorial. □

We can obtain a proof of Arrow's Theorem, in our rendering as Theorem 12, using the very same approach. Above, we used seriality only to establish condition (*iii*). We can use completeness, featuring in Theorem 12, instead: simply consider a scenario in which all individuals in C accept (x,y) and all those in $\mathcal{N} \setminus C$ accept (y,x). Then one of C and $\mathcal{N} \setminus C$ must be a winning coalition to ensure completeness for the collective graph. This observation completes the proof of Arrow's Theorem (Theorem 12).

To demonstrate the versatility of our approach, let us state one more impossibility:

Theorem 15. *If $|V| \geqslant 3$, then there exists no NR-nondictatorial, unanimous, grounded, and IIE aggregator that is CR wrt. both the Euclidean property and seriality.*

Proof. In our proof of Theorem 14, we used collective rationality wrt. transitivity twice: to invoke Lemma 13 and to establish ultrafilter condition (*ii*). Lemma 13 still applies when we use the Euclidean property instead of transitivity. So we only need to prove condition (*ii*): Suppose only agents in C_1 accept (x,y), only those in C_2 accept (x,z), and only those in $C_1 \cap C_2$ accept (y,z). That is, all individual graphs satisfy the Euclidean property (wrt. x, y and z). If both C_1 and C_2 are winning coalitions, then the collective graph will include (x,y) and (x,z). To satisfy the Euclidean property, it will also have to include (y,z). Hence, $C_1 \cap C_2$ must also be a winning coalition. □

How interesting Theorems 14 and 15 are is open to debate. Certainly, neither of them has the immediate intuitive appeal of Arrow's Theorem, which speaks about a class of graphs that can be interpreted as preference orders. On the other hand, these results indicate a generic technique for proving impossibility results in the style of Arrow's Theorem by explicitly linking (a) specific properties wrt. which we want to impose collective rationality and (b) specific conditions on ultrafilters. We obtain the following general picture:

(1) The *condition of closure-under-intersections* of an ultrafilter ($C_1, C_2 \in \mathcal{W} \Rightarrow C_1 \cap C_2 \in \mathcal{W}$) is derivable from collective rationality wrt. to any one of the following graph properties: *transitivity*, the *Euclidean property*, and the *Church-Rosser property*.[3] What these properties have in common is that they force the acceptance of one edge (or two, in the case of Church-Rosser) given the acceptance of two other edges. Any other graph property with this feature can be applied to the same effect.

(2) The *condition of maximality* of an ultrafilter ($C \in \mathcal{W}$ or $\mathcal{N} \setminus C \in \mathcal{W}$) is derivable from collective rationality wrt. to any one of the following graph properties: *completeness*, *strong completeness*, *connectedness*, *negative transitivity*, and *seriality*. What they have in common is that they force the acceptance of at least one out of a set of several (usually two) edges, possibly given the acceptance of some other edges (for connectedness and negative transitivity). Any other graph property with this feature can be applied to the same effect.

(3) Collective rationality wrt. graph properties that either do not create dependencies between edges (such as reflexivity or irreflexivity) or that do not force the acceptance of

[3] Church-Rosser requires 4 (rather than just 3) vertices to be applicable.

at least one edge (such as symmetry, antisymmetry, or functionality) cannot contribute to establishing the ultrafilter conditions.

The ultrafilter method itself becomes applicable once we assume IIE (needed to make winning coalitions applicable in the first place), neutrality (needed to show that all edges have the same winning coalitions), and unanimity (needed to show that the collection of winning coalitions is not empty). Groundedness is needed for the first ultrafilter condition. That is, we obtain an Arrovian impossibility for graph aggregation as soon as we accept these four axioms and postulate collective rationality wrt. one property from the first group above and one property from the second property above. As we have seen in Lemma 13, rather than accepting neutrality from the outset, we can also derive it as a consequence of collective rationality wrt. certain graph properties.

4 Conclusions, Related Work and Future Directions

We have argued that graph aggregation is an important problem with several potential applications and we have introduced a simple formal framework to study this problem. We have defined quota rules and successor-approval rules as interesting aggregators and we have stated several natural axiomatic requirements. Finally, we have argued that collective rationality is of central importance in the study of (not only!) this type of aggregation problem. Our main technical contribution has been a refinement of the ultrafilter method, allowing us to approach the proof of Arrovian impossibilities in a highly modular manner, clearly relating axiomatic properties, rationality properties, and ultrafilter properties.

Our approach is also helpful in interpreting a recent result by Pini et al. (2009), who prove a variant of Arrow's Theorem for preorders, i.e., for preferences that need not be complete. To be able to prove their result, these authors require the collective preference order to have one element that is weakly preferred (or dispreferred) to all other elements. This may be interpreted as a (very weak) form of completeness. Indeed, that such a condition would be needed is exactly what we would expect in view of our analysis above (without it, we cannot obtain the third ultrafilter condition) and it is not hard to see how to adapt our proof of Theorem 12 to provide a new simple proof of the result of Pini et al. (2009).

In related work on belief merging, Maynard-Zhang and Lehmann (2003) suggest an approach to circumvent Arrow's Theorem by (a) replacing completeness by negative transitivity (which they call "modularity") and (b) weakening the independence axiom. In the discussion of their result, these authors stress the significance of both of these changes. However, our analysis easily shows that replacing completeness by negative transitivity alone has no effect on Arrow's impossibility: the maximality condition of an ultrafilter can easily be derived using collective rationality wrt. negative transitivity (just consider a scenario where everyone in C accepts (x, z) and everyone else accepts (z, y)). Hence, the crucial source for the possibility result of Maynard-Zhang and Lehmann must be their modification of the independence axiom (and, indeed, this modification is rather substantial as it allows for independence to be violated whenever not doing so would lead to a "conflict").

Other related work includes our own work on collective rationality in binary aggregation, where we link axiomatic properties and structural properties of the integrity constraints used to define rationality assumptions in a propositional language (Grandi and Endriss, 2010), and so-called *agenda characterisation theorems* in judgment aggregation, linking axiomatic properties and collective rationality wrt. logical consistency (List and Puppe, 2009).

An interesting direction for future work that we have begun to explore is to study collective rationality wrt. the truth of a formula in *modal logic* evaluated over a directed graph. A basic result here shows that an aggregator F is grounded iff F is CR wrt. any modal formula not involving a $\Diamond$-operator (or a $\Box$-operator within the scope of a negation).

References

S. Airiau, U. Endriss, U. Grandi, D. Porello, and J. Uckelman. Aggregating dependency graphs into voting agendas in multi-issue elections. In *Proc. 22nd International Joint Conference on Artificial Intelligence (IJCAI-2011)*, 2011.

K. J. Arrow. *Social Choice and Individual Values.* John Wiley and Sons, 2nd edition, 1963.

R. Bradley, F. Dietrich, and C. List. Aggregating causal judgments. Working paper, London School of Economics, 2011.

S. J. Brams and P. C. Fishburn. *Approval Voting.* Springer, 2nd edition, 2007.

S. Coste-Marquis, C. Devred, S. Konieczny, M.-C. Lagasquie-Schiex, and P. Marquis. On the merging of Dung's argumentation systems. *Artificial Intelligence*, 171(10–15):730–753, 2007.

B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order.* Cambridge University Press, 2nd edition, 2002.

F. Dietrich and C. List. Judgment aggregation by quota rules: Majority voting generalized. *Journal of Theoretical Politics*, 19(4):391–424, 2007.

E. Dokow and R. Holzman. Aggregation of binary evaluations. *Journal of Economic Theory*, 145 (2):495–511, 2010.

P. M. Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artificial Intelligence*, 77(2):321–358, 1995.

C. Dwork, R. Kumar, M. Naor, and D. Sivakumar. Rank aggregation methods for the web. In *Proc. 10th International World Wide Web Conference (WWW-2001)*, 2001.

U. Endriss, U. Grandi, and D. Porello. Complexity of winner determination and strategic manipulation in judgment aggregation. In *Proc. 3rd International Workshop on Computational Social Choice (COMSOC-2010)*, 2010.

U. Grandi and U. Endriss. Lifting rationality assumptions in binary aggregation. In *Proc. 24th AAAI Conference on Artificial Intelligence (AAAI-2010)*, 2010.

F. Herzberg and D. Eckert. The model-theoretic approach to aggregation: Impossibility results for finite and infinite electorates. *Mathematical Social Sciences*, 2011. In press.

M. O. Jackson. *Social and Economic Networks.* Princeton University Press, 2008.

A. Kirman and D. Sondermann. Arrow's theorem, many agents, and invisible dictators. *Journal of Economic Theory*, 5(2):267–277, 1972.

S. Konieczny and R. Pino Pérez. Merging information under constraints: A logical framework. *Journal of Logic and Computation*, 12(5):773–808, 2002.

J. Lang, G. Pigozzi, M. Slavkovik, and L. van der Torre. Judgment aggregation rules based on minimization. In *Proc. 13th Conference on Theoretical Aspects of Rationality and Knowledge (TARK-XIII)*, 2011.

C. List and C. Puppe. Judgment aggregation: A survey. In *Handbook of Rational and Social Choice.* Oxford University Press, 2009.

P. Maynard-Zhang and D. J. Lehmann. Representing and aggregating conflicting beliefs. *Journal of Artificial Intelligence Research (JAIR)*, 19:155–203, 2003.

M. S. Pini, F. Rossi, K. B. Venable, and T. Walsh. Aggregating partially ordered preferences. *Journal of Logic and Computation*, 19(3):475–502, 2009.

D. Porello and U. Endriss. Ontology merging as social choice. In *Proc. 12th International Workshop on Computational Logic in Multiagent Systems (CLIMA-2011)*. Springer, 2011.

A. K. Sen. Social choice theory. In *Handbook of Mathematical Economics*, volume 3. North-Holland, 1986.

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